Hardy spaces on Lie groups of polynomial growth

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Abstract

We give several characterizations of Hardy spaces associated with complex, second-order, subelliptic operators on Lie groups with polynomial growth.

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1 Introduction

The theory of functions plays an important role in the classical theory of harmonic analysis. Because of this certain function spaces, the Hardy spaces, denoted by H^p, have been studied extensively on $\mathbf{R}^n$ (see [AuR], [CoW], [Ste] and the references therein for details). When $p > 1$ the spaces H^p and L_p are essentially the same. When $p \leq 1$, however, the space H^p is much better adapted to problems arising in the theory of Fourier series, PDE etc. In this note we discuss the characterization of Hardy spaces on Lie groups of polynomial growth.

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. One can associate with each fixed algebraic basis $a_1, \ldots, a_{d'}$ of $\mathfrak{g}$ a subelliptic distance $(g, h) \mapsto d'(g\,; h)$. For all $i \in \{1, 2, \ldots, d'\}$, let $A_i = dL(a_i)$ denote the generator of left translations in the direction a_i. This distance has the characterization

$$d'(g\,; h) = \sup\{|\psi(g) - \psi(h)| : \psi \in C_c^\infty(G),\ \sum_{i=1}^{d'} |(A_i\psi)|^2 \leq 1\} \quad ,$$

(cf. [Rob], Lemma IV.2.3). Let $g \mapsto |g| = d'(g\,; e)$, where e is the identity element of G, denote the corresponding modulus. Moreover, denote by $|B(g\,; r)|$ the Haar measure of the subelliptic ball $B(g\,; r) = \{h \in G : |gh^{-1}| < r\}$ and set $V(r) = |B(e\,; r)|$. We assume throughout that G has polynomial growth, and is not compact, i.e., one has bounds

$$c^{-1} r^D \leq V(r) \leq c r^D$$

for some $c > 0$ and integer $D \geq 1$, uniformly for all $r \geq 1$. These bounds automatically imply that G is unimodular and that $V(r) = |B(g\,; r)|$ is independent of g. For all $j \in \mathbf{N}$ let $\mathfrak{g}'_j$ denote the span of the multiple commutators of order less than or equal to j in the basis elements $a_1, \ldots, a_{d'}$. Then $\mathfrak{g}'_1 \subset \ldots \subset \mathfrak{g}'_r = \mathfrak{g}$, where r is the rank of the algebraic basis. This gives a corresponding direct sum decomposition $\mathfrak{g} = V'_1 \oplus \ldots \oplus V'_r$ of the Lie algebra. The local dimension D' is defined as

$$D' = \sum_{j=1}^{r} j \dim(\mathrm{V'_j}) \quad .$$

Then $V(r) \asymp r^{D'}$ for all $0 < r \leq 1$ (see, for example, [NSW], Theorem 1).

It is easy to verify that the Lie group G with the distance $d'(\,\cdot\,; \cdot\,)$ and bi-invariant Haar measure dg is a space of homogeneous type in the sense of Coifman and Weiss [CoW]. The measure has the doubling property. More specifically, there exists a $c > 0$ such that

$$V(\lambda r) \leq c\, \lambda^{\overline{D}} V(r)$$

for all $r > 0$ and $\lambda \geq 1$ where $\overline{D} = \max(D, D')$.

Consider the second order subelliptic operator $H = -\sum_{i,j=1}^{d'} c_{ij} A_i A_j$, where $C = (c_{ij})$ is a $d' \times d'$ matrix of complex coefficients. Assume

$$2^{-1}(C + C^*) \geq \mu I \tag{1}$$

for some $\mu > 0$. Then it follows from [ElR2] that the closure of the subelliptic operator H generates a holomorphic contraction semigroup S which has a smooth kernel K. Moreover there exist $a, b > 0$, such that

$$|K_t(g)| + \sum_{i=1}^{d'} t^{1/2}|(A_i K_t)(g)| \le a\,V(t)^{-1/2} e^{-b|g|^2 t^{-1}} \tag{2}$$

for all $t > 0$ and all $g \in G$ (see [DER] for details or [Dun] for a simpler derivation).

The operator H gives rise to various natural notions of Hardy space on G. First, for each tempered distribution φ over G one can define $S_t\varphi$ pointwise by convolution with the kernel K_t, i.e., $S_t\varphi = K_t * \varphi$. Secondly, for all $\alpha > 0$ one may define the (nontangential) maximal function $\varphi^*_{\alpha,H}: G \to [0, \infty]$ by

$$\varphi^*_{\alpha,H}(g) = \sup_{\{(h,t)\in G\times\langle 0,\infty\rangle \,:\, |gh^{-1}|<\alpha t^{1/2}\}} |(S_t\varphi)(h)| \quad .$$

For simplicity we set $\varphi^*_H = \varphi^*_{1,H}$. Thirdly, for all $p \in \langle 0, \infty\rangle$ one defines the maximal Hardy space $H^p_{\mathrm{max},H}(G) = \{\varphi : \varphi^*_H \in L_p(G)\}$ with norm $\|\varphi\|_{H^p_{\mathrm{max},H}(G)} = \|\varphi^*_H\|_p$.

Similarly if $\gamma \in \langle 0, 1\rangle$ then one may define the fractional power H^γ of H by various standard algorithms (see, for example, [Yos]). Then H^γ generates a holomorphic contraction semigroup S^γ and this semigroup has an $L_1(G)$-kernel K^γ. But one can extend the definition of S^γ_t to the bounded tempered distributions φ over G (see [Ste], page 89) and for all $\alpha > 0$ define $\varphi^*_{\alpha,H^\gamma}: G \to [0, \infty]$ by

$$\varphi^*_{\alpha,H^\gamma}(g) = \sup_{\{(h,t)\in G\times\langle 0,\infty\rangle \,:\, |gh^{-1}|<\alpha t^{1/(2\gamma)}\}} |(S^\gamma_t\varphi)(h)| \quad .$$

We set $\varphi^*_{H^\gamma} = \varphi^*_{1,H^\gamma}$. Then define $H^p_{\mathrm{max},H^\gamma}(G) = \{\varphi :\, \varphi^*_{H^\gamma} \in L_p(G)\}$ with $\|\varphi\|_{H^p_{\mathrm{max},H^\gamma}(G)} = \|\varphi^*_{H^\gamma}\|_p$ for all $p \in \langle 0, \infty\rangle$.

Next we introduce the atomic Hardy spaces. If $p \in \langle 0, \infty\rangle$ then a function a is defined to be a p-atom if the following three conditions are valid.

i. The support of a is contained in a ball $B(g_0\,; r)$.

ii. $|a| \le (V(r))^{-1/p}$ almost everywhere.

iii. $\int dg\, a(g) = 0$.

Then we define the atomic Hardy space $H^p_{\mathrm{atom}}(G)$, for all $p \in \langle 0, \infty\rangle$, to be the space of tempered distributions φ admitting an atomic decomposition $\varphi = \sum_{j=0}^\infty \lambda_j a_j$, where the a_j are p-atoms and $\sum_{j=0}^\infty |\lambda_j|^p < \infty$. The norm $\|\cdot\|_{p,\mathrm{atom}}$ is then defined by

$$\|\varphi\|_{p,\mathrm{atom}} = \inf \Big(\sum_{j=0}^\infty |\lambda_j|^p\Big)^{1/p}$$

where the infimum is over all possible atomic decompositions. (In fact these definitions are only appropriate if $p > \overline{D}/(\overline{D}+1)$ and the definition of the atoms has to be modified for smaller p [Ste]).

Finally we need some information on spaces of bounded mean oscillation, BMO-spaces. Let ψ be a locally integrable function. For any ball B we define $\psi_B = |B|^{-1}\int_B dg\,\psi(g)$. We say ψ belongs to $\mathrm{BMO}(G)$ if there exists a finite constant c such that

$$\frac{1}{|B|}\int_B dg\,|\psi(g)-\psi_B| \le c \tag{3}$$

holds for any ball B. Let $N(\psi)$ denote the infimum of all c for which (3) holds. Then we define the BMO(G) semi-norm by

$$\|\psi\|_{\mathrm{BMO}(G)} = N(\psi) \quad .$$

Then $(H^1_{\mathrm{atom}}(G))^* = \mathrm{BMO}(G)$ (see [CoW]).

Our main result is the following characterization of Hardy spaces with $p = 1$.

Theorem 1.1 *Let $\varphi \in L_1(G)$. The following statements are equivalent.*

I. *There exists a $\gamma \in [2^{-1}, 1]$ such that $\varphi \in H^1_{\max,H^\gamma}(G)$.*

II. *$\varphi \in H^1_{\max,H^\gamma}(G)$ for all $\gamma \in [2^{-1}, 1]$.*

III. *$\sup_{t>0} |S_t\varphi| \in L_1(G)$.*

IV. *$\sup_{t>0} |S^\gamma_t\varphi| \in L_1(G)$ for all $\gamma \in [2^{-1}, 1]$.*

V. *$\varphi \in H^1_{\mathrm{atom}}(G)$.*

Moreover, if the coefficients c_{ij} of H are real then the conditions are equivalent to the following.

III′. *There exists a $\gamma \in [2^{-1}, 1]$ such that $\sup_{t>0} |S^\gamma_t\varphi| \in L_1(G)$.*

The theorem is a natural generalization of results for the Laplacian on $\mathbf{R}^d$ insofar it characterizes the atomic Hardy space in terms of the maximal functions associated with the heat equation and the wave equation. Since the atomic space is defined independently of the operator H it follows immediately that the various maximal spaces are independent, up to equivalence, of the choice of H.

It is quite possible that Condition III′ of the theorem is equivalent to the other conditions for the general case of complex coefficients. Clearly $\mathrm{I}_\gamma \Rightarrow \mathrm{III}'_\gamma$ where I_γ and III'_γ denote the first and last condition for the fixed value $\gamma \in [2^{-1}, 1]$, respectively. Our proof that $\mathrm{III}'_{1/2} \Rightarrow \mathrm{I}_{1/2}$ is, however, only valid for real coefficients although symmetry of the coefficients is not necessary.

2 Hardy spaces

The proof of Theorem 1.1 depends on several lemmas. In the sequel we adopt the convention that c denotes a positive constant whose value may change line by line but is independent of all crucial variables.

For every Borel measurable function $\Phi\colon G \times \langle 0, \infty\rangle \to \mathbf{C}$ define $\Phi^*_{\alpha,\gamma}\colon G \to [0, \infty]$ by

$$\Phi^*_{\alpha,\gamma}(g) = \sup_{\{(h,t)\in G\times\langle 0,\infty\rangle\,:\,|gh^{-1}|<\alpha t^\gamma\}} |\Phi(h,t)|$$

for all $\alpha > 0$ and $\gamma \in \langle 0, 1]$.

Lemma 2.1 *For all $p \in \langle 0,\infty\rangle$ there exists a $c > 0$ such that*

$$\|\Phi^*_{\alpha,\gamma}\|_p \leq \|\Phi^*_{\beta,\gamma}\|_p \leq c\,(1+\alpha^{-1}\beta)^{2\overline{D}/p}\|\Phi^*_{\alpha,\gamma}\|_p$$

for all $0 < \alpha \leq \beta < \infty$, $\gamma \in \langle 0,1]$ and Borel measurable functions $\Phi \colon G \times \langle 0,\infty\rangle \to \mathbf{C}$.

Proof First if $0 < \alpha \leq \beta < \infty$ then $\|\Phi^*_{\alpha,\gamma}\|_p \leq \|\Phi^*_{\beta,\gamma}\|_p$ by definition.

Secondly, by the Hardy–Littlewood maximal function theorem there exists a $c_1 > 0$ such that

$$\|M_{HL}(\varphi)\|_2 \leq c_1\,\|\varphi\|_2$$

for all $\varphi \in L_2(G)$. Moreover, let $c_2 \geq 1$ be as in the volume doubling property, i.e.,

$$V(\lambda r) \leq c_2\,\lambda^{\overline{D}} V(r)$$

for all $\lambda \geq 1$ and $r > 0$.

Let $\alpha,\beta \in \langle 0,\infty\rangle$, $\gamma \in \langle 0,1]$ and $\Phi \colon G\times\langle 0,\infty\rangle \to \mathbf{C}$ Borel measurable. Define $\Phi^{**}_\alpha \colon G \to [0,\infty]$ by

$$\Phi^{**}_\alpha(g) = \sup_{h\in G,\, t>0} |\Phi(h,t)| \Big(\frac{\alpha t^\gamma}{|gh^{-1}| + \alpha t^\gamma}\Big)^{2\overline{D}/p} \quad .$$

Then

$$\Phi^{**}_\alpha(g) \geq \sup_{\{(h,t)\,:\,|gh^{-1}|<\beta t^\gamma\}} |\Phi(h,t)| \Big(\frac{1}{1+\alpha^{-1}\beta}\Big)^{2\overline{D}/p} = \Big(\frac{1}{1+\alpha^{-1}\beta}\Big)^{2\overline{D}/p} \Phi^*_{\beta,\gamma}(g)$$

for all $g \in G$. So $\Phi^*_{\beta,\gamma} \leq (1+\alpha^{-1}\beta)^{2\overline{D}/p}\Phi^{**}_\alpha$ and $\|\Phi^*_{\beta,\gamma}\|_p \leq (1+\alpha^{-1}\beta)^{2\overline{D}/p}\|\Phi^{**}_\alpha\|_p$.

Let $g,h \in G$ and $t \in \langle 0,\infty\rangle$. Then $|\Phi(h,t)| \leq \Phi^*_{\alpha,\gamma}(k)$ for all $k \in B(h\,;\alpha t^\gamma)$. Moreover, $B(h\,;\alpha t^\gamma) \subset B(g\,;|gh^{-1}| + \alpha t^\gamma)$. Therefore

$$\begin{aligned}
|\Phi(h,t)|^{p/2} &\leq \frac{1}{V(\alpha t^\gamma)} \int_{B(h;\alpha t^\gamma)} dk\,(\Phi^*_{\alpha,\gamma}(k))^{p/2} \\
&\leq \frac{V(|gh^{-1}| + \alpha t^\gamma)}{V(\alpha t^\gamma)} M_{HL}((\Phi^*_{\alpha,\gamma})^{p/2})(g) \\
&\leq c_2 \Big(\frac{|gh^{-1}| + \alpha t^\gamma}{\alpha t^\gamma}\Big)^{\overline{D}} M_{HL}((\Phi^*_{\alpha,\gamma})^{p/2})(g) \quad .
\end{aligned}$$

Hence

$$(\Phi^{**}_\alpha)^{p/2} \leq c_2 M_{HL}((\Phi^*_{\alpha,\gamma})^{p/2}) \quad .$$

Then

$$\|\Phi^{**}_\alpha\|_p \leq c_2^{2/p}\|M_{HL}((\Phi^*_{\alpha,\gamma})^{p/2})\|_2^{2/p} \leq c_1^2\,c_2^{2/p}\,\|\Phi^*_{\alpha,\gamma}\|_p \quad .$$

Combining these estimates completes the proof of the lemma. □

As a consequence of Lemma 2.1 one obtains many implications in Theorem 1.1.

Lemma 2.2

$$\begin{array}{ccccccc}
\mathrm{I}_1 & \Rightarrow & \mathrm{II} & \Rightarrow & \mathrm{I}_\gamma & \Rightarrow & \mathrm{I}_{1/2} \\
\Downarrow & & & & & & \Downarrow \\
\mathrm{III} & \Leftrightarrow & \mathrm{IV} & \Rightarrow & \mathrm{III}'_\gamma & \Rightarrow & \mathrm{III}'_{1/2}
\end{array}$$

for all $\gamma \in [2^{-1},1]$.

Proof Note that all the implications in the lemma are valid for complex coefficients.

The implications $\mathrm{I}_\gamma \Rightarrow \mathrm{III}'_\gamma$, $\mathrm{II} \Rightarrow \mathrm{I}_\gamma$, $\mathrm{IV} \Rightarrow \mathrm{III}$ and $\mathrm{IV} \Rightarrow \mathrm{III}'_\gamma$ are trivial.

The relation between the semigroups S^γ and S is given by

$$S^\gamma_t = \int_0^\infty ds\, \mu^\gamma_t(s)\, S_s$$

where μ^γ_t is a positive smooth function with the scaling property $\mu^\gamma_t(s) = t^{-1/\gamma}\mu^\gamma_1(st^{-1/\gamma})$ for all $s, t > 0$ (see, for example, [Rob] Section II.5). Therefore one calculates that

$$|(S^\gamma_t\varphi)(h)| \leq \int_0^\infty ds\, \mu^\gamma_1(s)\, |(S_{st^{1/\gamma}}\varphi)(h)| \leq \int_0^\infty ds\, \mu^\gamma_1(s)\, \varphi^*_{s^{-1/2},H}(g)$$

for all $g, h \in G$ and $t > 0$ with $|hg^{-1}| < t^{1/(2\gamma)}$. Therefore

$$\varphi^*_{H^\gamma}(g) \leq \int_0^\infty ds\, \mu^\gamma_1(s)\, \varphi^*_{s^{-1/2},H}(g)$$

for all $g \in G$. Hence

$$\|\varphi\|_{H^1_{\mathrm{max},H^\gamma}} \leq c \int_0^\infty ds\, \mu^\gamma_1(s)\, \|\varphi^*_{s^{-1/2},H}\|_1$$

and, by Lemma 2.1,

$$\|\varphi\|_{H^1_{\mathrm{max},H^\gamma}} \leq c \int_0^\infty ds\, \mu^\gamma_1(s)\, (1 + s^{-1/2})^{2\overline{D}} \|\varphi\|_{H^1_{\mathrm{max},H}} \quad .$$

But

$$\int_0^\infty ds\, \mu^\gamma_1(s)\, (1 + s^{-1/2})^{2\overline{D}} < \infty$$

by the sixth property of the μ^γ_t given in [Rob] Section II.5. This establishes that $\mathrm{I}_1 \Rightarrow \mathrm{I}_\gamma$. Hence one also has $\mathrm{I}_1 \Rightarrow \mathrm{II}$.

Using the identity $H^{1/2} = (H^\gamma)^{1/(2\gamma)}$, a similar argument establishes that $\mathrm{I}_\gamma \Rightarrow \mathrm{I}_{1/2}$ Then the first row of implications has been proved. The proof of the second row is slightly easier. □

Therefore, in order to prove the first statement of Theorem 1.1 it suffices to prove $\mathrm{I}_{1/2} \Rightarrow \mathrm{V} \Rightarrow \mathrm{III} \Rightarrow \mathrm{I}_1$. The hardest proof is the first implication, on which we first concentrate. It needs a lot of preparation. Let P be the holomorphic contraction semigroup generated by $H^{1/2}$.

If $\varphi \in L_2(G)$ then, as in [AuR], one has

$$\int_G dg\, \overline{\psi(g)}\, \varphi(g) = 4 \int_G \int_0^\infty \frac{dg\, dt}{t}\, \overline{t(\partial_t P^*_t\psi)(g)}\, t(\partial_t P_t\varphi)(g) \tag{4}$$

for all $\psi \in C_c(G)$. By [CMS] Theorem 1 and Proposition 4 there is a $c > 0$ such that

$$\int_{G\times\langle 0,\infty\rangle} \frac{dg\, dt}{t}\, |\Phi(g,t)\, \Psi(g,t)| \leq c \int_G dg\, (\mathcal{A}\Phi)(g)\, (\mathcal{C}\Psi)(g) \quad .$$

for all Borel measurable $\Phi, \Psi\colon G \times \langle 0,\infty\rangle \to \mathbf{C}$ with $\mathcal{A}\Phi \in L_1$ and $\mathcal{C}\Psi \in L_\infty$, where for all Borel measurable $\Phi\colon G \times \langle 0,\infty\rangle \to \mathbf{C}$ we define $\mathcal{A}\Phi, \mathcal{C}\Phi\colon G \to [0,\infty]$ by

$$(\mathcal{A}\Phi)(g) = \Big(\int_{\Gamma(g)} \frac{dh\, dt}{t\, V(t)}\, |\Phi(h,t)|^2 \Big)^{1/2}$$

and

$$(\mathcal{C}\Phi)(g) = \sup_{B\ni g}\Big(\frac{1}{|B|}\int_{\hat{B}}\frac{dh\,dt}{t}\,|\Phi(h,t)|^2\Big)^{1/2} \quad ,$$

with $\Gamma(g) = \{(h,t) \in G \times \langle 0,\infty\rangle : |gh^{-1}| < t\}$ and for any ball B the set $\widehat{B} = \{(h,t) \in G \times \langle 0,\infty\rangle : d'(h,B^c) \geq t\}$ is the tent over B. Hence

$$\Big|\int_G dg\,\overline{\psi(g)}\,\varphi(g)\Big| \leq c\int_G dg\Big(\int_{\Gamma_1(g)}\frac{dh\,dt}{t\,V(t)}\,|t(\partial_t P_t\varphi)(h)|^2\Big)^{1/2} \cdot$$

$$\cdot \sup_{g\in G}\Big(\sup_{B\ni g}\frac{1}{|B|}\int_{\widehat{B}}\frac{dh\,dt}{t}\,|t(\partial_t P_t^*\psi)(h)|^2\Big)^{1/2} \tag{5}$$

for all $\varphi \in L_2(G)$ and $\psi \in C_c(G)$. We estimate both factors on the right hand side.

A key ingredient are the following Poisson bounds for the kernel p of P, together with its derivative.

Lemma 2.3 *There exists a $c > 0$ such that*

$$t|(\partial_t p_t)(g)| + |p_t(g)| \leq c\frac{t}{(t+|g|)\,V(t+|g|)}$$

for all $t > 0$ and $g \in G$.

Proof By the subordination formula

$$P_t = \frac{1}{\sqrt{\pi}}\int_0^\infty ds\,e^{-s}s^{-1/2}S_{\frac{t^2}{4s}} \tag{6}$$

one deduces that

$$\begin{aligned}|p_t(g)| &\leq \frac{t}{2\sqrt{\pi}}\int_0^\infty du\,e^{-\frac{t^2}{4u}}u^{-3/2}|K_u(g)|\\
&\leq ct\int_0^\infty du\,u^{-3/2}e^{-\frac{t^2}{4u}}V(u)^{-1/2}e^{-b\frac{|g|^2}{u}}\\
&\leq ct\int_0^\infty du\,u^{-3/2}e^{-br^2/u}V(u)^{-1/2} = c\frac{t}{r}\int_0^\infty ds\,s^{-3/2}e^{-b/s}V(sr^2)^{-1/2}\end{aligned}$$

where $r^2 = t^2 + |g|^2$. By the doubling property one has

$$V(sr^2)^{-1/2} \leq c\,V(r^2)^{-1/2}(1+s^{-\overline{D}})^{1/2} \quad . \tag{7}$$

Therefore

$$|p_t(g)| \leq c\frac{t}{r}\int_0^\infty ds\,s^{-3/2}e^{-b/s}V(r^2)^{-1/2}(1+s^{-\overline{D}})^{1/2} \leq c\frac{t}{(t+|g|)\,V(t+|g|)}$$

for all $t > 0$ and $g \in G$.

The bound on the derivative follows from the identity

$$t(\partial_t p_t)(g) = p_t(g) - \frac{t^3}{4\pi}\int_0^\infty du\,e^{-\frac{t^2}{4u}}u^{-5/2}\,K_u(g)$$

which is also valid for all $t > 0$ and $g \in G$. $\square$

The next lemma is a Lie group version of a standard estimate in $\mathbf{R}^d$. Using the Poisson kernel bounds of Lemma 2.3 we can estimate the second factor on the right hand side of (5).

Lemma 2.4 *There exists a $c > 0$, such that for every ball B in G*

$$\Big(\frac{1}{|B|}\int_{\widehat{B}} \frac{dh\,dt}{t}\,|t(\partial_t P_t\varphi)(h)|^2\Big)^{1/2} \le c\,\|\varphi\|_{\mathrm{BMO}}$$

for all $\varphi \in \mathrm{BMO}(G)$.

Proof Set $\psi_{\{B\}} = \varphi - \varphi_{2B}$, where $2B$ denotes the ball $B(g\,;2r)$ if $B = B(g\,;r)$. Since $\partial_t P_t$ anhilates the constants one has

$$\begin{aligned}\Big(\frac{1}{|B|}\int_{\widehat{B}} \frac{dh\,dt}{t}\,|t(\partial_t P_t\varphi)(h)|^2\Big)^{1/2} &= \Big(\frac{1}{|B|}\int_{\widehat{B}} \frac{dh\,dt}{t}\,|t(\partial_t P_t\psi_{\{B\}})(h)|^2\Big)^{1/2}\\ &\le \Big(\frac{1}{|B|}\int_{\widehat{B}} \frac{dh\,dt}{t}\,|t(\partial_t P_t(\psi_{\{B\}}1\!\!1_{2B}))(h)|^2\Big)^{1/2}\\ &\quad + \Big(\frac{1}{|B|}\int_{\widehat{B}} \frac{dh\,dt}{t}\,|t(\partial_t P_t(\psi_{\{B\}}1\!\!1_{(2B)^c}))(h)|^2\Big)^{1/2} = I_1 + I_2 \quad .\end{aligned}$$

Next, H is injective and is maximal accretive. Therefore H has an H_∞-holomorphic calculus by [ADM], Theorem G. Hence H has square integral inequalities by [McI], Section 8, and one estimates that

$$I_1 \le \Big(\frac{1}{|B|}\int_0^\infty \frac{dt}{t}\,\|t\partial_t P_t(\psi_{\{B\}}1\!\!1_{2B})\|_2^2\Big)^{1/2} \le c\,\Big(\frac{1}{|B|}\int_{2B} dh\,|\psi_{\{B\}}(h)|^2\Big)^{1/2} \le c\,\|\varphi\|_{\mathrm{BMO}} \quad .$$

The third inequality is the John–Nirenberg inequality (see, for example, [Ste] Section IV.1.3).

To estimate I_2 we use Lemma 2.3 and the volume doubling property to deduce that

$$\begin{aligned}|t\Big(\partial_t P_t(\psi_{\{B\}}1\!\!1_{(2B)^c})\Big)(h)| &\le c\int_{(2B)^c} dk\,\frac{t\,|\psi_{\{B\}}(k)|}{(t+|hk^{-1}|)\,V(t+|hk^{-1}|)}\\ &= c\sum_{n=1}^\infty \int_{2^{n+1}B\backslash 2^nB} dk\,\frac{t\,|\psi_{\{B\}}(k)|}{(t+|hk^{-1}|)\,V(t+|hk^{-1}|)}\\ &\le c\sum_{n=1}^\infty \frac{t}{V(2^nB)}(2^nr)^{-1}\int_{2^{n+1}B} dk\,|\psi_{\{B\}}(k)|\\ &\le c\frac{t}{r}\sum_{n=1}^\infty \frac{n}{2^n}\|\varphi\|_{\mathrm{BMO}} \le c\frac{t}{r}\|\varphi\|_{\mathrm{BMO}}.\end{aligned}$$

Thus

$$I_2 \le c\Big(\frac{1}{|B|}\int_{\widehat{B}} dh\,dt\,\frac{t}{r^2}\Big)^{1/2}\|\varphi\|_{\mathrm{BMO}} \le c\|\varphi\|_{\mathrm{BMO}}$$

and Lemma 2.4 is proved. □

Next we turn to the first factor on the right hand side of (5).

For every bounded tempered distribution φ over G define $P\varphi \in C^\infty(G \times \langle 0,\infty\rangle)$ by $(P\varphi)(g,t) = (P_t\varphi)(g)$ (see [Ste], page 90.) For all $\alpha > 0$, $0 < \varepsilon < R < \infty$ and each bounded tempered distribution φ over G define $A_\alpha\varphi, A_\alpha^{\varepsilon,R}\varphi\colon G \to [0,\infty]$ by

$$(A_\alpha\varphi)(g) = \Big(\int_{\Gamma_\alpha(g)} \frac{dh\,dt}{t\,V(t)}\,|t(\overline{\nabla}P\varphi)(h,t)|^2\Big)^{1/2} \quad ,$$

and

$$(A_\alpha^{\varepsilon,R}\varphi)(g) = \Big(\int_{\Gamma_\alpha^{\varepsilon,R}(g)} \frac{dh\,dt}{t\,V(t)}\,|t(\overline{\nabla}P\varphi)(h,t)|^2\Big)^{1/2} \quad ,$$

where $(\overline{\nabla}\Phi)(g,t) = ((\nabla\Phi)(g,t), (\partial_t\Phi)(g,t))$,

$$\Gamma_\alpha(g) = \{(h,t) \in G \times \langle 0,\infty\rangle : |gh^{-1}| < \alpha t\}$$

and $\Gamma_\alpha^{\varepsilon,R}(g)$ is the truncated cone defined by

$$\Gamma_\alpha^{\varepsilon,R}(g) = \{(h,t) \in G \times \langle \varepsilon,R\rangle : |gh^{-1}| < \alpha t\} \quad .$$

Since $|(\partial_t P_t\varphi)(g)| \leq |(\overline{\nabla}P\varphi)(g,t)|$ for every bounded tempered distribution φ one deduces that

$$\int_G dg\Big(\int_{\Gamma_1(g)} \frac{dh\,dt}{t\,V(t)}\,|t(\partial_t P_t\varphi)(h)|^2\Big)^{1/2} \leq \|A_1\varphi\|_1 \tag{8}$$

for every bounded tempered distribution φ. So the first factor on the right hand side of (5) is bounded by $\|A_1\varphi\|_1$.

Lemma 2.5 *For all $\alpha < 1$ there exists a $c > 0$ such that*

$$A_\alpha^{\varepsilon,R}\varphi \leq c(1 + |\log(R/\varepsilon)|)\varphi^*_{H^{1/2}}$$

for all $\varphi \in L_2$ and $0 < \varepsilon < R < \infty$.

Proof The proof is similar to the proof of Lemma 7 in [AuR], but now using the Caccioppoli inequality in [ElR1]. □

The next lemma uses ideas from [AuR] and [CMS]. In particular it is a Lie group version of Proposition 8 in [AuR] and the proof follows closely the arguments of [AuR] Lemmas 9 and 10 which in turn are based on arguments of [CMS].

Lemma 2.6 *There exists a $c > 0$ such that*

$$\|A_1\varphi\|_1 \leq c\|\varphi\|_{H^1_{\max,H^{1/2}}}$$

for all $\varphi \in H^1_{\max,H^{1/2}}(G)$.

Proof The proof relies on Lemma 2.5 and a 'good λ' inequality. We shall prove the following statement.

There exists a $c > 0$ such that

$$|\{g \in G : (A_{1/20}^{\varepsilon,R}\varphi)(g) > 2\lambda \text{ and } \varphi^*_{H^{1/2}} \leq \gamma\lambda\}| \leq c\gamma^2|\{g \in G : (A_{1/2}^{\varepsilon,R}\varphi)(g) > \lambda\}| \tag{9}$$

for all $0 < \gamma \leq 1$, $\lambda > 0$, $0 < \varepsilon < R < \infty$ and $\varphi \in H^1_{\max,H^{1/2}} \cap L_2(G)$.

Define $\mathcal{O} = \{g \in G : (A^{\varepsilon,R}_{1/2}\varphi)(g) > \lambda\}$. Let $\mathcal{O} = \bigcup_{k=1}^\infty \mathcal{O}_k$ be a Whitney decomposition of $\mathcal{O}$, such that $\mathcal{O}_k \subset \mathcal{O}$ but $3\mathcal{O}_k \cap \mathcal{O}^c \neq \emptyset$ for all k. Since $\{g \in G : (A^{\varepsilon,R}_{1/20}\varphi)(g) > 2\lambda\} \subset \{g \in G : (A^{\varepsilon,R}_{1/2}\varphi)(g) > \lambda\}$ it is enough to show that

$$|\{g \in \mathcal{O}_k : (A^{\varepsilon,R}_{1/20}\varphi)(g) > 2\lambda \text{ and } \varphi^*_{H^{1/2}} \leq \gamma\lambda\}| \leq c\,\gamma^2|\mathcal{O}_k| \quad . \tag{10}$$

From now on fix k and denote by r the radius of $\mathcal{O}_k$.

If $g \in \mathcal{O}_k$ then $(A^{\max(10r,\varepsilon),R}_{1/20}\varphi)(g) \leq \lambda$. Indeed, pick $g_k \in 3\mathcal{O}_k \cap \mathcal{O}^c$. Let $h \in \Gamma^{\max(10r,\varepsilon),R}_{1/20}(g)$. Then $|gh^{-1}| < t/20$ and $t \geq \max(10r,\varepsilon)$. Hence one has $|g_kh^{-1}| < t/2$ and $h \in \Gamma^{\max(10r,\varepsilon),R}_{1/2}(g_k)$. Therefore

$$(A^{\max(10r,\varepsilon),R}_{1/20}\varphi)(g) \leq (A^{\max(10r,\varepsilon),R}_{1/2}\varphi)(g_k) \leq (A^{\varepsilon,R}_{1/2}\varphi)(g_k) \leq \lambda \quad .$$

If $\varepsilon \geq 10r$ then (10) is obviously valid.

If $\varepsilon < 10r$, using $(A^{\varepsilon,R}_{1/20}\varphi)(g) \leq (A^{\varepsilon,10r}_{1/20}\varphi)(g) + (A^{10r,R}_{1/20}\varphi)(g)$, it remains to prove that

$$|\{g \in \mathcal{O}_k \cap \Omega : l(g) > \lambda\}| \leq c\gamma^2|\mathcal{O}_k|, \tag{11}$$

where $l(g) = (A^{\varepsilon,10r}_{1/20}\varphi)(g)$ and $\Omega = \{g \in G : \varphi^*_{H^{1/2}}(g) \leq \gamma\lambda\}$.

To prove (11) we only need to prove

$$\int_{\mathcal{O}_k\cap\Omega} dg\, l(g)^2 \leq c\gamma^2\lambda^2|\mathcal{O}_k| \quad . \tag{12}$$

If $\varepsilon \geq 5r$, then by Lemma 2.5

$$\int_{\mathcal{O}_k\cap\Omega} dg\, l(g)^2 \leq c\int_{\mathcal{O}_k\cap\Omega} dg\,(\varphi^*_{H^{1/2}}(g))^2 \leq c\gamma^2\lambda^2|\mathcal{O}_k\cap\Omega| \quad .$$

If $\varepsilon < 5r$ then

$$\begin{aligned}\int_{\mathcal{O}_k\cap\Omega} dg\, l(g)^2 &= \int_{\mathcal{O}_k\cap\Omega} dg \int_{\Gamma^{\varepsilon,10r}_{1/20}(g)} \frac{dh\,dt}{t\,V(t)}|t(\overline{\nabla}P\varphi)(h,t)|^2 \\ &\leq \int_{\mathcal{R}} dh\,dt\,t|(\overline{\nabla}P\varphi)(h,t)|^2\end{aligned}$$

where $\mathcal{R} = \{(h,t) \in G \times (\varepsilon, 10r) : d'(h\,;\mathcal{O}_k\cap\Omega) < t/20\}$.

If μ is the smallest eigenvalue of the real part of $C = (c_{ij})$ then

$$\int_{\mathcal{R}} dh\,dt\,t|(\overline{\nabla}P\varphi)(h,t)|^2 \leq \mu^{-1}\,\mathrm{Re}\int_{\mathcal{R}} dh\,dt\,t(B\overline{\nabla}P\varphi)(h,t)\cdot\overline{(\overline{\nabla}P\varphi)(h,t)}$$

where B is the $(d'+1)\times(d'+1)$ block diagonal matrix with components C and I. Since $P\varphi$ satisfies the equation $\overline{\nabla}\cdot B\overline{\nabla}P\varphi = 0$ we may integrate by parts and deduce that

$$\begin{aligned}&\int_{\mathcal{R}} dh\,dt\,t(B\overline{\nabla}P\varphi)(h,t)\cdot\overline{(\overline{\nabla}P\varphi)(h,t)} \\ &\quad = -\int_{\mathcal{R}} dh\,dt\,(\partial_tP_t\varphi)(h)\,\overline{(P_t\varphi)(h)} + \int_{\partial\mathcal{R}} d\sigma(h,t)\,t(B\overline{\nabla}P\varphi)(h,t)\cdot N(h,t)\,\overline{(P_t\varphi)(h)} \quad ,\end{aligned}$$

where $N(h,t)$ is the unit normal vector outward $\mathcal{R}$ and $d\sigma$ is the surface measure over $\partial\mathcal{R}$. Moreover, integrating by parts again gives

$$\mathrm{Re}\int_{\mathcal{R}} dh\,dt\,(\partial_t P_t\varphi)(h)\,\overline{(P_t\varphi)(h)} = 2^{-1}\int_{\partial\mathcal{R}} d\sigma(h,t)\,|(P_t\varphi)(h)|^2\Big(N(h,t)\cdot(0,\ldots,0,1)\Big) \quad .$$

Finally,

$$\int_{\mathcal{R}} dh\,dt\,t|(\overline{\nabla}P\varphi)(h,t)|^2 \le c\int_{\partial\mathcal{R}} d\sigma(h,t)\,|(P_t\varphi)(h)|^2$$
$$+\,c\int_{\partial\mathcal{R}} d\sigma(h,t)\,t|(P_t\varphi)(h)|\,|(\overline{\nabla}P\varphi)(h,t)| \quad .$$

Since $|(P_t\varphi)(h)| \le \lambda\gamma$ for all $(h,t) \in \partial\mathcal{R}$ we find

$$\int_{\partial\mathcal{R}} d\sigma(h,t)\,|(P_t\varphi)(h)|^2 \le \lambda^2\gamma^2\int_{\partial\mathcal{R}} d\sigma(h,t) \le c\,\lambda^2\gamma^2|\mathcal{O}_k| \quad .$$

The last estimate follows by a crude estimate of the surface area of the truncated cone $\mathcal{R}$ since r is the radius of $\mathcal{O}_k$. Moreover,

$$\begin{aligned}
&\int_{\partial\mathcal{R}} d\sigma(h,t)\,t|(P_t\varphi)(h)|\,|(\overline{\nabla}P\varphi)(h,t)| \\
&\qquad\le \lambda\gamma\int_{\partial\mathcal{R}} d\sigma(h,t)\,t\,|(\overline{\nabla}P\varphi)(h,t)| \\
&\qquad\le \lambda\gamma\Big(\int_{\partial\mathcal{R}} d\sigma(h,t)\,t^2|(\overline{\nabla}P\varphi)(h,t)|^2\Big)^{1/2}\Big(\int_{\partial\mathcal{R}} d\sigma(h,t)\Big)^{1/2} \\
&\qquad\le c\lambda\gamma|\mathcal{O}_k|^{1/2}r\Big(\int_{\partial\mathcal{R}} d\sigma(h,t)\,|(\overline{\nabla}P\varphi)(h,t)|^2\Big)^{1/2} \\
&\qquad\le c\lambda\gamma|\mathcal{O}_k|^{1/2}\Big(\int_{\partial\mathcal{R}} d\sigma(h,t)\,|(P_t\varphi)(h)|^2\Big)^{1/2} \le c\lambda^2\gamma^2|\mathcal{O}_k| \quad .
\end{aligned}$$

The penultimate estimate follows from a covering argument and an application of Caccioppoli's inequality (see [AuR], proof of Lemma 9). Combining these estimates we obtain (12). Hence we have proved (9).

Then Lemma 2.6 follows by standard reasoning (see, for example, [Ste] Chapter IV). Again details of an almost identical argument can be found in [AuR]. □

Proof of $\mathbf{I}_{1/2} \Rightarrow \mathbf{V}$ It follows from (5) and (8) that

$$\begin{aligned}
\Big|\int_G dg\,\overline{\psi(g)}\,\varphi(g)\Big| &\le c\,\|A_1\varphi\|_1 \sup_{g\in G}\Big(\sup_{B\ni g}\frac{1}{|B|}\int_{\widehat{B}}\frac{dh\,dt}{t}\,|t(\partial_t P_t^*\psi)(h)|^2\Big)^{1/2} \\
&\le c\,\|\varphi\|_{H^1_{\max,H^{1/2}}(G)}\|\psi\|_{\mathrm{BMO}(G)}
\end{aligned}$$

for all $\varphi \in H^1_{\max,H^{1/2}}(G)$ and $\psi \in C_c(G)$, where the last step follows from Lemmas 2.6 and 2.4. Hence by duality

$$\|\varphi\|_{H^1_{\mathrm{atom}(G)}} \le c\,\|\varphi\|_{H^1_{\max,H^{1/2}}(G)}$$

for all $\varphi \in L_2(G) \cap H^1_{\max,H^{1/2}}(G)$. Since $H^1_{\max,H^{1/2}}(G) \cap L_2(G)$ is dense in $H^1_{\max,H^{1/2}}(G)$ the proof of the implication $\mathrm{I}_{1/2} \Rightarrow \mathrm{V}$ in Theorem 1.1 is complete. $\square$

Proof of V $\Rightarrow$ III This is a special case of the following lemma.

Lemma 2.7 *If $p \in \langle (\overline{D}+1)^{-1}\overline{D}, 1]$ then $H^p_{\mathrm{atom}}(G) \subset H^p_{\max,H}(G)$.*

Proof Again set $\varphi^+_H = \sup_{t>0} |S_t\varphi|$ for every tempered distribution φ. It suffices to prove that there is a $c > 0$ such that $a^+_H \in L_p(G)$ and $\|a^+_H\|_p \leq c$ uniformly for every p-atom a on G. (Cf. [Ste], page 107.)

Let a be a p-atom on G. We may, without loss of generality, suppose that $\operatorname{supp} a \subset B(e\,;r)$ for some $r > 0$. Write $B^* = B(e\,;2r)$. Then

$$\int_G (a^+_H)^p = \int_{B^*} (a^+_H)^p + \int_{G\setminus B^*} (a^+_H)^p \quad .$$

For the first term one estimates

$$\int_{B^*} (a^+_H)^p \leq |B^*|\,\|a^+_H\|^p_\infty \leq |B^*|\,\|a\|^p_\infty \leq \frac{V(2r)}{V(r)} \leq c$$

where c is a constant independent of a.

To estimate the second term one needs a pointwise estimation of a^+_H. If $g \in G \setminus B^*$ and $t > 0$ then it follows from (2) that

$$\begin{aligned}
|(S_t a)(g)| &= \Big| \int_G dh\, K_t(gh^{-1})\, a(h)\Big| \\
&= \Big| \int_G dh\, \Big(K_t(gh^{-1}) - K_t(g)\Big) a(h)\Big| \\
&\leq c \int_G dh\, t^{-1/2}|h|\, V(t)^{-1/2}\, e^{-b|g|^2 t^{-1}}\, |a(h)| \\
&\leq c t^{-1/2} r V(r)^{1-p^{-1}}\, V(t)^{-1/2}\, e^{-b|g|^2 t^{-1}} \quad .
\end{aligned}$$

Setting $s^{-1} = |g|^2\, t^{-1}$ one has by (7)

$$\begin{aligned}
(a^+_H)(g) &\leq c\, r\, V(r)^{1-p^{-1}} |g|^{-1} \sup_{s>0} V(s|g|^2)^{-1/2} s^{-1/2} e^{-bs^{-1}} \\
&\leq c\, r\, V(r)^{1-p^{-1}} |g|^{-1} V(|g|^2)^{-1/2} \quad .
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_{G\setminus B^*} (a^+_H)^p &\leq c \int_{G\setminus B^*} dg\, r^p\, V(r)^{p-1} |g|^{-p}\, V(|g|^2)^{-p/2} \\
&\leq c \sum_{k=1}^{\infty} V(2^{k+1} r)\, V(r)^{p-1}\, 2^{-pk}\, V(2^k r)^{-p} \quad .
\end{aligned}$$

We now consider two cases.

Case 1. If $r \geq 1$ then

$$\int_{G\backslash B^*} (a_H^+)^p \leq c \sum_{k=1}^{\infty} (2^k r)^D \, r^{D(p-1)} \, 2^{-pk} \, (2^k r)^{-Dp} = c \sum_{k=1}^{\infty} 2^{k(D-p-Dp)} \leq c$$

where we used $p > (D+1)^{-1}D$.

Case 2. If $0 < r \leq 1$, let k_0 be the integer such that $2r^{-1} < 2^{k_0} \leq r^{-1}$. Then

$$\int_{G\backslash B^*} (a_H^+)^p \leq \Big(\sum_{k=1}^{k_0} + \sum_{k=k_0+1}^{\infty} \Big) V(2^{k+1}r) \, V(r)^{p-1} \, 2^{-pk} \, V(2^k r)^{-p} = I_1 + I_2 \quad .$$

The estimate of I_1 is similar to Case 1 except that D is replaced by the local dimension D'. Next,

$$\begin{aligned} I_2 &\leq c \sum_{k=k_0+1}^{\infty} (2^k r)^D \, r^{D'(p-1)} \, 2^{-pk} \, (2^k r)^{-Dp} \\ &= c \sum_{k=k_0+1}^{\infty} 2^{k(D-p-Dp)} \, r^{D+D'(p-1)-Dp} \\ &\leq c \, 2^{k_0(D-p-Dp)} \, r^{(D-D')(1-p)} \leq c \, r^{-D'+D'p+p} \leq c \end{aligned}$$

where we used that $p > (D+1)^{-1}D$ and that $p > (D'+1)^{-1}D'$.

Combining these estimates one finds $\|a_H^+\|_p \leq c$ with c independent of a. This completes the proofs of Lemma 2.7 and the implication V $\Rightarrow$ III in Theorem 1.1. □ □

Proof of III $\Rightarrow$ I$_1$ This is a special case of the following lemma.

Lemma 2.8 *If φ is a bounded tempered distribution over G and $p \in \langle 0, \infty \rangle$ then $\varphi_H^* \in L_p(G)$ if and only if $\sup_{t>0} |S_t \varphi| \in L_p(G)$.*

Proof The 'only if' part is obvious. To prove the converse define $\varphi^+ = \sup_{t>0} |S_t \varphi|$. Then we need to prove that $\|\varphi_H^*\|_p \leq c \, \|\varphi^+\|_p$.

Fix $N \in \mathbf{N}$ with $Np > \overline{D}$. Introduce $\varphi_\varepsilon^*, \Delta_\varepsilon : G \to [0, \infty]$ by

$$\varphi_\varepsilon^*(g) = \sup_{\{(h,t) \in G \times \langle 0, \infty \rangle \, : \, |gh^{-1}| < t^{1/2} < \varepsilon^{-1}\}} \Big(\frac{t^{1/2}}{t^{1/2} + \varepsilon} \Big)^N (1 + \varepsilon |h|)^{-N} \, |(S_t \varphi)(h)|$$

and

$$\Delta_\varepsilon(g) = \sup_{\substack{h \neq h', \, t \\ |gh^{-1}| \vee |gh'^{-1}| < t^{1/2} < \varepsilon^{-1}}} \Big(\frac{t^{1/2}}{|h'h^{-1}|} \Big) \Big(\frac{t^{1/2}}{t^{1/2} + \varepsilon} \Big)^N (1 + \varepsilon |h|)^{-N} \, |(S_t \varphi)(h) - (S_t \varphi)(h')|$$

for all $\varepsilon > 0$.

First we prove that there is a $c > 0$ such that $\|\Delta_\varepsilon\|_p \leq c \, \|\varphi_\varepsilon^*\|_p$ uniformly for all $\varepsilon > 0$. Since the derivatives of the kernel satisfy Gaussian bounds [DER], Lemma V.2.10, there are $b, c_1 > 0$ such that

$$|K_{t/2}(h) - K_{t/2}(h')| \leq c_1 \frac{|h'h^{-1}|}{t^{1/2}} V(t)^{-1/2} e^{-5b|h|^2 t^{-1}} \tag{13}$$

for all $h, h' \in G$ and $t > 0$ with $|h'h^{-1}| \le 2t^{1/2}$. Let $\varepsilon > 0$. Fix $g \in G$, let $h, h' \in G$, $t > 0$ and suppose that $h \neq h'$, $|gh^{-1}| < t^{1/2} < \varepsilon^{-1}$ and $|gh'^{-1}| < t^{1/2}$. Then the semigroup property of S gives

$$|(S_t\varphi)(h) - (S_t\varphi)(h')| \le \int_G dk \left|K_{t/2}(hk^{-1}) - K_{t/2}(h'k^{-1})\right| |(S_{t/2}\varphi)(k)| = \sum_{n=0}^{\infty} I_n$$

where

$$I_n = \int_{G_n} dk \, |K_{t/2}(hk^{-1}) - K_{t/2}(h'k^{-1})| \, |(S_{t/2}\varphi)(k)|$$

for all $n \in \mathbf{N}_0$ with

$$G_0 = \{k \in G : |hk^{-1}| \le t^{1/2}\}$$

and

$$G_n = \{k \in G : 2^{n-1}t^{1/2} < |hk^{-1}| \le 2^n t^{1/2}\}$$

for all $n \in \mathbf{N}$. Then it follows from (13) that

$$\Big(\frac{t^{1/2}}{|h'h^{-1}|}\Big) \int_{G_n} dk \, |K_{t/2}(hk^{-1}) - K_{t/2}(h'k^{-1})| \le c_2 \, e^{-b2^{2n}}$$

for all $n \in \mathbf{N}_0$, where $c_2 = c_1 e^b \sup_{s>0} \int_G dl \, V(s)^{-1/2} e^{-b|l|^2 s^{-1}}$. Next $(1 + \varepsilon|k|)^N \le (1 + \varepsilon|h|)^N (1 + 2^n)^N$ for all $n \in \mathbf{N}$ and $k \in G$, since $\varepsilon t^{1/2} < 1$. Therefore

$$\begin{aligned}
&\Big(\frac{t^{1/2}}{|h'h^{-1}|}\Big)\Big(\frac{t^{1/2}}{t^{1/2} + \varepsilon}\Big)^N (1 + \varepsilon|h|)^{-N} I_n \\
&\qquad \le c_2 \, e^{-b2^{2n}} (1 + 2^n)^N \Big(\frac{t^{1/2}}{t^{1/2} + \varepsilon}\Big)^N \sup_{k \in G_n} (1 + \varepsilon|k|)^{-N} |(S_{t/2}\varphi)(k)| \\
&\qquad \le c_2 \, e^{-b2^{2n}} (1 + 2^n)^N \, \Phi^*_{2^{n+1},1/2}(g)
\end{aligned}$$

where $\Phi \colon G \times \langle 0, \infty\rangle \to \mathbf{C}$ is defined by

$$\Phi(k, s) = (S_s\varphi)(k) \, (1 + \varepsilon|k|)^{-N} \Big(\frac{s^{1/2}}{s^{1/2} + \varepsilon}\Big)^N \mathbb{1}_{[0,1]}(\varepsilon s^{1/2}) \quad .$$

Hence

$$\Delta_\varepsilon(g) \le c_2 \sum_{n=0}^{\infty} e^{-b2^{2n}} (1 + 2^n)^N \, \Phi^*_{2^{n+1},1/2}(g) \quad .$$

Therefore

$$\|\Delta_\varepsilon\|_p \le c_2 \sum_{n=0}^{\infty} e^{-b2^{2n}} (1 + 2^n)^N \|\Phi^*_{2^{n+1},1/2}\|_p \le c \, \|\Phi^*_{1,1/2}\|_p = c \, \|\varphi^*_\varepsilon\|_p \tag{14}$$

by Lemma 2.1. The value of c is independent of ε.

Now we are ready to prove that $\|\varphi^*_H\|_p \le c\|\varphi^+\|_p$.

Set $B = 2^{1/p}c$ where $c > 0$ is as in (14) and assume $\varphi^+ \in L_p$. Then since $(S_t\varphi)(y) = \int_{\mathbf{R}^d} dz \, K_{t/2}(y\,;z)(S_{t/2}\varphi)(z)$ and K satisfies Gaussian bounds one verifies that $\varphi^*_\varepsilon \in L_p$. Next define

$$G_\varepsilon = \{g \in G : \Delta_\varepsilon(g) \le B \, \varphi^*_\varepsilon(g)\}$$

Then one has

$$\int_{G_\varepsilon^c} dg\, \varphi_\varepsilon^*(g)^p \leq \frac{1}{B^p}\int_{G_\varepsilon^c} dg\, \Delta_\varepsilon(g)^p \leq \frac{1}{2}\int_G dg\, \varphi_\varepsilon^*(g)^p \quad .$$

In particular

$$\int_{G_\varepsilon^c} dg\, \varphi_\varepsilon^*(g)^p \leq \int_{G_\varepsilon} dg\, \varphi_\varepsilon^*(g)^p$$

and

$$\int_G dg\, \varphi_\varepsilon^*(g)^p \leq 2\int_{G_\varepsilon} dg\, \varphi_\varepsilon^*(g)^p \quad . \tag{15}$$

Now fix $g \in G_\varepsilon$. Since $\varphi_\varepsilon^* \in L_p$ we may assume that $\varphi_\varepsilon^*(g) < \infty$. Then there exist $h \in G$ and $t > 0$ such that $|gh^{-1}| < t^{1/2} < 1/\varepsilon$ and

$$\Big(\frac{t^{1/2}}{t^{1/2}+\varepsilon}\Big)^N (1+\varepsilon|h|)^{-N}|(S_t\varphi)(h)| \geq 2^{-1}\varphi_\varepsilon^*(g) \quad . \tag{16}$$

Therefore for all $k \in G$ with $|kg^{-1}| < t^{1/2}$ one has

$$\begin{aligned}
&\frac{t^{1/2}}{|hk^{-1}|}\Big(\frac{t^{1/2}}{t^{1/2}+\varepsilon}\Big)^N (1+\varepsilon|h|)^{-N}|(S_t\varphi)(k)-(S_t\varphi)(h)| \\
&\qquad\qquad \leq \Delta_\varepsilon(g) \leq B\,\varphi_\varepsilon^*(g) \leq 2B\Big(\frac{t^{1/2}}{t^{1/2}+\varepsilon}\Big)^N (1+\varepsilon|h|)^{-N}|(S_t\varphi)(h)| \quad .
\end{aligned}$$

Hence, $\frac{t^{1/2}}{|hk^{-1}|}|(S_t\varphi)(k)-(S_t\varphi)(h)| \leq 2B|(S_t\varphi)(h)|$. It follows that $|(S_t\varphi)(k)| \geq 2^{-1}|(S_t\varphi)(h)|$ for all $k \in \Omega = \{k : |kg^{-1}| < t^{1/2} \text{ and } |kh^{-1}| < \frac{t^{1/2}}{4B}\}$. Therefore, for all $k \in \Omega$ one has

$$|(S_t\varphi)(k)| \geq 2^{-1}\Big(\frac{t^{1/2}}{t^{1/2}+\varepsilon}\Big)^N (1+\varepsilon|h|)^{-N}|(S_t\varphi)(h)| \geq 4^{-1}\varphi_\varepsilon^*(g)$$

by (16).

Next define

$$M_q(g) = \sup_{Q\ni g}\Big(\frac{1}{V(Q)}\int_Q dh\, \varphi^+(h)^q\Big)^{1/q}$$

where $q = 2^{-1}p$. Then

$$\begin{aligned}
M_q(g)^q &\geq \frac{1}{V(2t^{1/2})}\int_{B(g;2t^{1/2})} dk\, \varphi^+(k)^q \\
&\geq \frac{1}{V(2t^{1/2})}\int_{B(g;2t^{1/2})} dk\, |(S_t\varphi)(k)|^q \\
&\geq \frac{|\Omega|}{V(2t^{1/2})}(4^{-1}\varphi_\varepsilon^*(g))^q \geq \frac{V((4c)^{-1}t^{1/2})}{V(2t^{1/2})}(4^{-1}\varphi_\varepsilon^*(g))^q \geq c'\,\varphi_\varepsilon^*(g)^q
\end{aligned}$$

by the volume doubling property. Using (15) one immediately deduces that

$$\int_G dg\, \varphi_\varepsilon^*(g)^p \leq 2\int_{G_\varepsilon} dg\, \varphi_\varepsilon^*(g)^p \leq c\int_G dg\, M_q(g)^p \leq c\int_G dg\, \varphi^+(g)^p$$

where c is independent of ε. Letting $\varepsilon \to 0$ yields $\|\varphi^*\|_p \leq c\|\varphi^+\|_p$ by the monotone convergence theorem. $\Box$ $\Box$

At this stage we have established the first statement of Theorem 1.1. Lemma 2.2 also establishes that the first five conditions of the theorem imply Condition III$'$. Therefore to complete the proof of the theorem it suffices to prove that $\mathrm{III}'_{1/2} \Rightarrow \mathrm{I}_{1/2}$ for operators with real coefficients. From now on the coefficients c_{ij} are real and all functions are real valued. For this we follow an argument of [Lu] based on the mean value theorem.

Proposition 2.9 *Suppose that $D' \leq D$. Let*

$$L = -\partial_t^2 + H = -\partial_t^2 - \sum_{i,j=1}^{d'} c_{ij} A_i A_j \quad .$$

Then for all $p > 0$ there is a $C > 0$ such that

$$\max_{(g,t)\in B(\rho)} u(g,t) \leq c\Big(\frac{1}{|B(2\rho)|}\int_{B(2\rho)} |u|^p\Big)^{1/p} \quad .$$

for all $\rho \in \langle 0, \infty\rangle$ and any non-negative subsolution u of $Lu = 0$ in a ball $B(9\rho) \subset G \times \mathbf{R}$.

Proof Let $\rho \in \langle 0,\infty\rangle$ and u a non-negative subsolution of $Lu = 0$ in a ball $B(9\rho) \subset G\times\mathbf{R}$. For all $p > 0$ and $r \in \langle 0, 9\rho\rangle$ define

$$\varphi(p,r) = \Big(\frac{1}{|B(r)|}\int_{B(r)} |u|^p\Big)^{1/p} \quad \text{and} \quad \varphi(\infty,r) = \max_{(g,t)\in B(r)} u(g,t) \quad .$$

Hence we need to prove that there is a $c > 0$, independent of ρ and u, such that

$$\varphi(\infty,\rho) \leq c\varphi(p,2\rho) \quad .$$

We first prove that

$$\varphi(\infty,r) \leq c\lambda^{-(D'+1)/p}\varphi(p,(1+\lambda)r) \tag{17}$$

uniformly for all $p \in [2,\infty\rangle$, ρ, u, $r \in \langle 0, 2\rho]$ and $\lambda \in \langle 0, 1]$.

Let $\overline{\nabla} u = (A_1 u, \ldots, A_{d'}u, \partial_t u)$ and

$$M = (\sum_{j=1}^{d'} c_{1j}A_j u, \ldots, \sum_{j=1}^{d'} c_{d'j}A_j u, \partial_t u) \quad .$$

Then one easily finds $\overline{\nabla} u \cdot M \geq \mu|\overline{\nabla} u|^2$ and $|M| \leq a\,|\overline{\nabla} u|$, where $a = \max_{i,j}|c_{ij}|$ and μ is the ellipticity constant.

Next let $q \geq 1$, $r \in \langle 0, 4\rho]$ and $\xi \in C_c^\infty(B(2r))$ with $\xi \geq 0$. Set $\psi = \xi^2\, u^q$. Then

$$\overline{\nabla}\psi = q\,\xi^2\,u^{q-1}\overline{\nabla} u + 2\xi\,u^q\,\overline{\nabla}\xi \quad .$$

Since u is a subsolution one has $\int_{B(9\rho)} \overline{\nabla}\psi \cdot M \leq 0$. Therefore

$$q\int_{B(2r)} \xi^2 \cdot u^{q-1}\,\overline{\nabla} u \cdot M + 2\int_{B(2r)} \xi\,u^q\,\overline{\nabla}\xi \cdot M \leq 0 \quad .$$

Then

$$\begin{aligned}
\int_{B(2r)} \xi^2\, u^{q-1}\, |\overline{\nabla} u|^2 &\leq \mu^{-1} \int_{B(2r)} \xi^2\, u^{q-1}\, \overline{\nabla} u \cdot M \\
&\leq -2(\mu q)^{-1} \int_{B(2r)} \xi\, u^q\, \overline{\nabla} \xi \cdot M \\
&= -2(\mu q)^{-1} \int_{B(2r)} \xi\, u^{(q-1)/2}\, (M \cdot \overline{\nabla} \xi)\, u^{(q+1)/2} \\
&\leq 2a(\mu q)^{-1} \Big(\int_{B(2r)} \xi^2\, u^{q-1}\, |\overline{\nabla} u|^2 \Big)^{1/2} \Big(\int_{B(2r)} u^{q+1}\, |\overline{\nabla} \xi|^2 \Big)^{1/2} \\
&\leq \frac{1}{2} \int_{B(2r)} \xi^2\, u^{q-1}\, |\overline{\nabla} u|^2 + \frac{2a^2}{\mu^2 q^2} \int_{B(2r)} u^{q+1}\, |\overline{\nabla} \xi|^2 \quad .
\end{aligned}$$

Therefore,

$$\int_{B(2r)} \xi^2\, u^{q-1}\, |\overline{\nabla} u|^2 \leq c\, q^{-2} \int_{B(2r)} u^{q+1}\, |\overline{\nabla} \xi|^2 \quad .$$

Set $v = u^{(q+1)/2}$. Then $|\overline{\nabla} v|^2 = 4^{-1}(q+1)^2\, u^{q-1}\, |\overline{\nabla} u|^2$. Hence

$$\|\, \xi\, |\overline{\nabla} v|\, \|_{L_2(B(2r))} \leq c(q+1)\, q^{-1}\, \|\, v\, |\overline{\nabla} \xi|\, \|_{L_2(B(2r))} \leq c \|\, v\, |\overline{\nabla} \xi|\, \|_{L_2(B(2r))} \tag{18}$$

since $q \geq 1$.

Next we use the Sobolev inequality on $G \times \mathbf{R}$,

$$\Big(\frac{1}{|B|} \int_B |f|^{2\nu} \Big)^{1/(2\nu)} \leq c\, r(B) \Big(\frac{1}{|B|} \int_B |\nabla f|^2 \Big)^{1/2} \quad ,$$

where $\nu = (D'+1)(D'-1)^{-1} > 1$. Here we use that $D' \leq D$. Then one obtains from the foregoing estimate

$$\begin{aligned}
\|\xi v\|_{L_{2\nu}(B(2r))} &\leq c\rho\, |B(2r)|^{\frac{1}{2\nu} - \frac{1}{2}}\, \|\overline{\nabla}(v\xi)\|_{L_2(B(2r))} \\
&\leq c\rho\, |B(2r)|^{\frac{1}{2\nu} - \frac{1}{2}}\, \|v \overline{\nabla} \xi + \xi \overline{\nabla} v\|_{L_2(B(2r))} \leq c\rho\, |B(2r)|^{\frac{1}{2\nu} - \frac{1}{2}}\, \|\, v\, |\overline{\nabla} \xi|\, \|_{L_2(B(2r))} \quad ,
\end{aligned}$$

where we used (18) in the last step.

Let $R \in \langle r, 2r]$, select ξ as a cut-off function such that $\xi = 1$ on $B(r)$ and $\xi = 0$ outside $B(R)$ and $|\overline{\nabla} \xi| \leq c(R-r)^{-1}$. Then

$$\|v\|_{L_{2\nu}(B(r))} \leq c(R-r)^{-1} r\, |B(2r)|^{\frac{1}{2\nu} - \frac{1}{2}} \|v\|_{L_2(B(R))} \quad ,$$

or, equivalently,

$$\Big(\frac{1}{|B(r)|} \int_{B(r)} u^{(q+1)\nu} \Big)^{1/(2\nu)} \leq c\, (R-r)^{-1} r \Big(\frac{1}{|B(R)|} \int_{B(R)} |u|^{q+1} \Big)^{1/2} \quad .$$

Taking the (q+1)/2-th root of both sides one finds

$$\varphi(p\nu, r) \leq \Big(c(R-r)^{-1} r \Big)^{2/p} \varphi(p, R) \tag{19}$$

with $p = q + 1$.

Now fix a $p_0 \in [2, \infty\rangle$, $\lambda \in \langle 0, 1]$ and $\rho_0 \in \langle 0, 2\rho]$, and for all $j \in \{0, 1, \ldots\}$ define $p_j = p_0 \nu^j$ and $r_j = (1 + 2^{-j}\lambda)\rho_0$. Then it follows by iteration from (19) applied with $p = p_j$, $r = r_{j+1}$ and $R = r_j$ that

$$\begin{aligned}\varphi(p_{j+1}, r_{j+1}) &\leq \Big(c(2^{-(j+1)}\lambda\rho_0)^{-1} 2\rho_0\Big)^{2p_0^{-1}\nu^{-j}} \varphi(p_j, r_j)\\ &\leq \ldots \leq (2c\lambda^{-1})^{2p_0^{-1}\sum_{k=0}^{j}\nu^{-k}}\, 2^{2p_0^{-1}\sum_{k=0}^{j}(k+1)\nu^{-k}} \varphi(p_0, (1+\lambda)\rho_0)\end{aligned}$$

for all $j \in \mathbf{N}$. In the limit $j \to \infty$ one deduces that

$$\varphi(\infty, \rho_0) \leq c\, \lambda^{-(D'+1)/p} \varphi(p_0, (1+\lambda)\rho_0) \quad .$$

This proves (17). In particular it follows by setting $p_0 = 2$ that

$$\varphi(\infty, r) \leq c\, \lambda^{-\alpha} \varphi(2, (1+\lambda)r)$$

for all $\lambda \in \langle 0, 1]$ and $r \in \langle 0, 2\rho]$, where $\alpha = (D'+1)/2 > 0$.

Now fix $p \in \langle 0, 2\rangle$. Set $t = 1 - p/2 \in \langle 0, 1\rangle$. Then

$$\varphi(2, r)^2 = \frac{1}{|B(r)|}\int_{B(r)} |u|^2 \leq \Big(\frac{1}{|B(r)|}\int_{B(r)} |u|^p\Big)\Big(\sup_{(g,t)\in B(r)} u(g,t)^{2-p}\Big) = \varphi(p, r)^p\, \varphi(\infty, r)^{2t}$$

for all $r \in \langle 0, 9\rho\rangle$. Therefore

$$\begin{aligned}\varphi(\infty, r) &\leq c\, \lambda^{-\alpha} \varphi(p, (1+\lambda)r)^{p/2}\, \varphi(\infty, (1+\lambda)r)^t\\ &\leq c\, \lambda^{-\alpha} \varphi(p, 2\rho)^{p/2}\, \varphi(\infty, (1+\lambda)r)^t\end{aligned}$$

for all $r \in \langle 0, \rho\rangle$ and $\lambda \in \langle 0, 2^{-1}]$. Choosing $\lambda = 2^{-j}$ with $j \in \mathbf{N}$ one deduces that

$$\begin{aligned}\varphi(\infty, \rho) &\leq c\, 2^{\alpha} \varphi(p, 2\rho)^{p/2}\, \varphi(\infty, (1 + \tfrac{1}{2})\rho)^t\\ &\leq c\, 2^{\alpha} \varphi(p, 2\rho)^{p/2} \Big(c\, 4^{\alpha} \varphi(p, 2\rho)^{p/2} \varphi(\infty, (1 + \tfrac{1}{2} + \tfrac{1}{4})\rho)^t\Big)^t\\ &\leq \ldots \leq \Big(c\, \varphi(p, 2\rho)^{p/2}\Big)^{\sum_{k=0}^{j-1} t^k} 2^{\alpha\sum_{k=0}^{j-1}(k+1)t^k} \Big(\varphi(\infty, \sum_{k=0}^{j} 2^{-k}\rho)\Big)^{t^j}\\ &\leq \Big(c\, \varphi(p, 2\rho)^{p/2}\Big)^{\sum_{k=0}^{j-1} t^k} 2^{\alpha\sum_{k=0}^{j-1}(k+1)t^k} \Big(\varphi(\infty, 2\rho)\Big)^{t^j}\end{aligned}$$

for all $j \in \mathbf{N}$. Since $t < 1$ the last factor tends to 1 if $j \to \infty$. Moreover, $\sum_{k=0}^{\infty} t^k = 2/p$. So $\varphi(\infty, \rho) \leq c\, \varphi(p, 2\rho)$ and the proof of the proposition is complete. □

Proof of $\mathrm{III}'_{1/2} \Rightarrow \mathrm{I}_{1/2}$ This is a special case of the next proposition.

Proposition 2.10 *There is a $c > 0$ such that, for any φ satisfying $\sup_{t>0} |S_t^{1/2}\varphi| \in L_1(G)$ one has $\varphi \in H^1_{\max, H^{1/2}}(G)$ and*

$$\|\varphi\|_{H^1_{\max, H^{1/2}}} \leq c\| \sup_{t>0} |S_t^{1/2}\varphi|\, \|_1 \quad .$$

Proof First suppose that $D' \leq D$. Fix $p \in \langle 0,1\rangle$. Note that the function $u\colon (g,t) \mapsto |(e^{-tH^{1/2}}\varphi)(g)|$ is a non-negative subsolution of $Lu = 0$, where L is as in Proposition 2.9. Let $t > 0$ and $g \in G$. Then for all h with $|gh^{-1}| \leq t$ one has $(h,t) \in B((g\,;t)\,;t)$. Therefore

$$|(e^{-tH^{1/2}}\varphi)(h)| \leq c\Big(\frac{1}{t\,|B(g\,;2t)|}\int_0^{2t} ds \int_{B(g\,;2t)} dh'\,|(e^{-sH^{1/2}}\varphi)(h')|^p\Big)^{1/p}$$

by Proposition 2.9. Hence

$$\begin{aligned}\varphi^*_{H^{1/2}}(g)^p &\leq c\sup_{t>0}\frac{1}{t\,|B(g\,;2t)|}\int_0^{2t} ds\int_{B(g\,;2t)} dh'\,|(e^{-sH^{1/2}}\varphi)(h')|^p\\ &\leq c\sup_{t>0}\frac{1}{|B(g\,;2t)|}\int_{B(g\,;2t)} dh'\sup_{s>0}|(S_s^{1/2}\varphi)(h')| \leq cM_{H-L}(\sup_{t>0}|S_t^{1/2}\varphi|^p)\quad .\end{aligned}$$

The statement of the proposition follows immediately if $D' \leq D$.

Now we consider the case $D' > D$. Define $G' = H^{D'-D} \times \mathbb{R}^3$ where H is the three-dimensional Heisenberg group. Let $\widetilde{G} = G \times G'$. Choose $\widetilde{H} = H \otimes I + I \otimes \Delta'$ where Δ' is the full Laplacian on G'. Then $\widetilde{S}_t = S_t \otimes S'_t$, and $\widetilde{P}_t = P_t \otimes P'_t$, where $P_t = S_t^{1/2}$. Moreover, choose $\varphi' \in C_c^\infty(G')$ such that $\varphi' \geq 0$ and $\int_{G'} dg'\,\varphi(g') = 1$. Then $\int_{G'} dg'\,(P'_t\varphi')(g') = 1$

Now let $\varphi \in L_1(G)$ and suppose that $\sup_{t>0}|P_t\varphi| \in L_1(G)$. Then

$$\sup_{t>0}|(\widetilde{P}_t(\varphi\otimes\varphi'))(g,g')| = \sup_{t>0}|(P_t\varphi)(g)|\cdot|(P'_t\varphi')(g')| \leq \sup_{t>0}|(P_t\varphi)(g)|\,\sup_{t>0}|(P'_t\varphi')(g')|\quad .$$

It follows that $\sup_{t>0}|(\widetilde{P}_t(\varphi\otimes\varphi'))| \in L_1(\widetilde{G})$, and $(\varphi\otimes\varphi')^*_{\widetilde{H}^{1/2}} \in L_1(\widetilde{G})$ by the first part of the proof. Therefore,

$$\begin{aligned}\int_G dg\,\varphi^*_{H^{1/2}}(g) &= \int_G dg \sup_{|gh^{-1}|<t}\int_{G'} dg'\,|(P_t\varphi)(h)\,(P'_t\varphi')(g')|\\ &\leq \int_G dg\int_{G'} dg' \sup_{|gh^{-1}|<t}|(P_t\varphi)(h)\,(P'_t\varphi')(g')|\\ &\leq \int_G dg\int_{G'} dg' \sup_{|(g,g')(h,h')^{-1}|<t}|(\widetilde{P}_t(\varphi\otimes\varphi')(h,h')|\quad .\end{aligned}$$

It follows that $\varphi^*_{H^{1/2}} \in L_1(G)$, i.e., $\mathrm{III}'_{1/2} \Rightarrow \mathrm{I}_{1/2}$ for the general case. $\Box\;\Box$

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