

On the sliding-window representation in digital signal processing

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On the Sliding-Window Representation in Digital Signal Processing

MARTIN J. BASTIAANS

Abstract—The short-time Fourier transform of a discrete-time signal, which is the Fourier transform of a “windowed” version of the signal, is interpreted as a sliding-window spectrum. This sliding-window spectrum is a function of two variables: a discrete time index, which represents the position of the window, and a continuous frequency variable. It is shown that the signal can be reconstructed from the sampled sliding-window spectrum, i.e., from the values at the points of a certain time-frequency lattice. This sampling lattice is rectangular, and the rectangular cells occupy an area of 2π in the time-frequency domain. It is shown that an elegant way to represent the signal directly in terms of the sample values of the sliding-window spectrum, is in the form of Gabor’s signal representation. Therefore, a reciprocal window is introduced, and it is shown how the window and the reciprocal window are related. Gabor’s signal representation then expands the signal in terms of properly shifted and modulated versions of the reciprocal window, and the expansion coefficients are just the values of the sampled sliding-window spectrum.

INTRODUCTION

SHORT-TIME Fourier analysis [1] of discrete-time signals is of considerable interest in a number of signal-processing applications. In order to study spectral properties of speech signals, for instance, the concept of a short-time Fourier transform of the signal is very convenient [1], [2]. Such a short-time Fourier transform is usually constructed by first multiplying the signal by a window function that is “slided” to a certain position, and then Fourier transforming the “windowed” signal. Therefore, we like to consider the short-time Fourier transform as a *sliding-window representation* of the signal. There are other interpretations of the short-time Fourier transform, including a well-known filter bank interpretation [1]. However, for the purpose of this paper, we find the sliding-window interpretation to be the most appropriate, and, to emphasize this, we shall call the short-time Fourier transform the sliding-window spectrum of the signal.

The sliding-window representation of a signal, which is a signal description in time and frequency simultaneously, is complete in the sense that the signal can be reconstructed from its sliding-window spectrum [1]. However, to reconstruct the signal, we need not know the entire sliding-window spectrum. In this paper we show that it suffices to know the values of the sliding-window spectrum only at the points of a certain rectangular lattice in the time-frequency domain, and we describe how the signal

can be expressed directly in terms of the values of this sampled sliding-window spectrum. This will lead us in a natural way to *Gabor’s representation* [3] of a signal as a superposition of properly shifted and modulated versions of a function that is related to the window. We show a way to determine this function from the knowledge of the window, and we elucidate this with some simple examples of window functions.

SLIDING-WINDOW REPRESENTATION OF DISCRETE-TIME SIGNALS

Let $x(n)$ ($n = \dots, -1, 0, 1, \dots$) denote a one-dimensional discrete-time signal and let $w(n)$ represent a *window sequence*; the signal and the window may take complex values and they need not have a finite extent. We multiply the signal by a shifted and complex conjugated version of the window and take the Fourier transform of the product, thus constructing the function [cf. [1], (6.1)]

$$f(\Omega, n) = \sum_{m=-\infty}^{\infty} x(m) w^*(m - n) \exp[-j\Omega m]. \quad (1)$$

Unlike (6.1) in [1], (1) uses a complex conjugated version of the window; moreover, the window has not been time-reversed. The only reason for doing this is to get more elegant formulas in the remainder of the paper.

We shall call $f(\Omega, n)$ the *sliding-window spectrum* [cf. [4], Section 4.1] of the discrete-time signal; it is clearly a function of two variables: the time index n , which is discrete and represents the position of the window, and the frequency variable Ω , which is continuous. Of course, as in the case of normal Fourier transforms of discrete-time signals, the sliding-window spectrum $f(\Omega, n)$ is periodic in Ω with period 2π . Two choices of window sequences are of special interest. If $w(n)$ vanishes for $n \neq 0$, then $f(0, n)$ is proportional to the signal $x(n)$; the sliding-window spectrum thus reduces to a pure time representation of the signal. If, on the other hand, $w(n)$ does not depend on n , then $f(\Omega, 0)$ is proportional to the Fourier transform of $x(n)$; the sliding-window spectrum then reduces to a pure frequency representation of the signal. In general, however, the sliding-window spectrum is an intermediate signal description between the pure time and the pure frequency representation.

We can reconstruct the signal $x(n)$ from its sliding-window spectrum $f(\Omega, n)$ in the usual way [cf. [1], (6.6)] by inverse Fourier transforming and taking $m = n$, which

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yields the inversion formula

$$x(n) = \frac{1}{w^*(0)} \frac{1}{2\pi} \int_{2\pi} d\Omega \cdot f(\Omega, n) \exp [j\Omega n], \quad (2)$$

in which $\int_{2\pi} d\Omega \cdot$ represents integration over one period 2π ; of course, the rather mild requirement that $w(0)$ be nonzero should be satisfied. There exists another way of reconstructing the signal from its sliding-window spectrum, viz. by means of the inversion formula [cf. [5], (2) and [6], (27.12.1.5)]

$$x(m) = \frac{1}{\sum_{n=-\infty}^{\infty} |w(n)|^2} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi} d\Omega \cdot f(\Omega, n) w(m - n) \exp [j\Omega m], \quad (3)$$

which represents the signal as a *linear combination of shifted and modulated versions of the window*. However, this linear combination is not unique [cf. [6], Section 27.12.1]; indeed, there are many kernels $p(\Omega, n)$, periodic in Ω with period 2π , that satisfy the relationship

$$x(m) = \frac{1}{\sum_{n=-\infty}^{\infty} |w(n)|^2} \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi} d\Omega \cdot p(\Omega, n) w(m - n) \exp [j\Omega m]. \quad (4)$$

The representation (3), i.e., choosing the kernel $p(\Omega, n)$ in (4) equal to the sliding-window spectrum $f(\Omega, n)$, is the best possible one in the sense that for this choice the L^2 -norm of $p(\Omega, n)$ takes its minimum value. To see this, we multiply both sides of (3) and (4) by $x^*(m)$, sum up over all m , and conclude from the equivalence of the right-hand sides of the resulting equations that $f(\Omega, n)$ and $p(\Omega, n) - f(\Omega, n)$ are orthogonal in the sense

$$\sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi} d\Omega \cdot \{p(\Omega, n) - f(\Omega, n)\} f^*(\Omega, n) = 0; \quad (5)$$

hence, the relationship

$$\begin{aligned} & \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi} d\Omega \cdot |p(\Omega, n)|^2 \\ &= \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi} d\Omega \cdot |f(\Omega, n)|^2 \\ &+ \sum_{n=-\infty}^{\infty} \frac{1}{2\pi} \int_{2\pi} d\Omega \cdot |p(\Omega, n) - f(\Omega, n)|^2 \end{aligned} \quad (6)$$

holds. It will be clear that the L^2 -norm of $p(\Omega, n)$, i.e., the left-hand side of (6), takes its minimum value if we choose the kernel $p(\Omega, n)$ equal to the sliding-window spectrum $f(\Omega, n)$.

We can reconstruct the signal from its sliding-window spectrum via the inversion formulas (2) or (3). However, in order to reconstruct the signal, we need not know the entire sliding-window spectrum; it suffices to know its

values at the points of a certain lattice in the Ω - n domain. This will be shown in the next section.

SIGNAL RECONSTRUCTION FROM ITS SAMPLED SLIDING-WINDOW SPECTRUM

Let N be a positive integer, let the sliding-window spectrum $f(\Omega, n)$ be known at the points $\{\Omega = k(2\pi/N), n = mN\}$ ($k, m = \dots, -1, 0, 1, \dots$), and let the values at these points be denoted by f_{km} ; hence,

$$\begin{aligned} f_{km} &= f\left(k \frac{2\pi}{N}, mN\right) \\ &= \sum_{n=-\infty}^{\infty} x(n) w^*(n - mN) \exp \left[-jk \frac{2\pi}{N} n\right]. \end{aligned} \quad (7)$$

Of course, the array of coefficients f_{km} is periodic in k with period N . Note that the sampling lattice $\{\Omega = k(2\pi/N), n = mN\}$ is rectangular, and that the rectangular cells occupy an area of 2π in the time-frequency domain. We shall now demonstrate how the signal can be found when we know the values f_{km} of the sampled sliding-window spectrum (cf. [4, Section 4.2]).

We first define the function $\tilde{f}(n, \omega)$ by a Fourier series with coefficients f_{km} ,

$$\tilde{f}(n, \omega) = \sum_{m=-\infty}^{\infty} \sum_{k=\langle N \rangle} f_{km} \exp \left[-j\left(\omega mN - k \frac{2\pi}{N} n\right)\right] \quad (8)$$

where $\Sigma_{k=\langle N \rangle}$ represents summation over one period N . Note that the function $\tilde{f}(n, \omega)$ is periodic in n and ω , with periods N and $2\pi/N$, respectively. The inverse relationship has the form

$$\begin{aligned} f_{km} &= \frac{1}{N} \sum_{n=\langle N \rangle} \frac{N}{2\pi} \int_{2\pi/N} d\omega \\ &\cdot \tilde{f}(n, \omega) \exp \left[j\left(\omega mN - k \frac{2\pi}{N} n\right)\right]. \end{aligned} \quad (9)$$

Furthermore, we define the function $\tilde{x}(n, \omega)$ by

$$\tilde{x}(n, \omega) = \sum_{m=-\infty}^{\infty} x(n + mN) \exp [-j\omega mN]. \quad (10)$$

Note that the function $\tilde{x}(n, \omega)$ is periodic in ω , with period $2\pi/N$, and *quasi-periodic* in n , with quasi-period N :

$$\tilde{x}(n + N, \omega) = \tilde{x}(n, \omega) \exp [j\omega N]. \quad (11)$$

Equation (10) provides a means of representing a one-dimensional discrete-time signal $x(n)$ by a two-dimensional time-frequency function $\tilde{x}(n, \omega)$ on a rectangle with *finite* area 2π . The inverse relationship has the form

$$x(n + mN) = \frac{N}{2\pi} \int_{2\pi/N} d\omega \cdot \tilde{x}(n, \omega) \exp [j\omega mN]. \quad (12)$$

It will be clear that the variable n in (12) can be restricted

to an interval of length N , with m taking on all integer values.

With the help of the functions $\tilde{f}(n, \omega)$, $\tilde{x}(n, \omega)$ and a similar function $\tilde{w}(n, \omega)$ associated with the window $w(n)$, (7) can be rewritten as

$$\tilde{f}(n, \omega) = N\tilde{x}(n, \omega) \tilde{w}^*(n, \omega). \quad (13)$$

In fact, we have now solved the problem of reconstructing the signal from its sampled sliding-window spectrum:

1) from the sample values f_{km} we determine the function $\tilde{f}(n, \omega)$ via (8);

2) from the window $w(n)$ we derive the associated function $\tilde{w}(n, \omega)$ by (10);

3) under the assumption that division by $\tilde{w}^*(n, \omega)$ is allowed, the function $\tilde{x}(n, \omega)$ can be found with the help of (13); and

4) finally, the signal follows from $\tilde{x}(n, \omega)$ by means of the inversion formula (12).

A simpler reconstruction method will be derived in the next section.

Problems may arise in the case that $\tilde{w}(n, \omega)$ has zeros. In that case *homogeneous solutions* $\tilde{h}(n, \omega)$ may occur, for which the relation

$$N\tilde{h}(n, \omega) \tilde{w}^*(n, \omega) = 0 \quad (14)$$

holds. Equation (14), which is similar to (13) with $\tilde{f}(n, \omega) = 0$, can be transformed into the relation

$$\sum_{n=-\infty}^{\infty} h(n) w^*(n - mN) \exp \left[-jk \frac{2\pi}{N} n \right] = 0, \quad (15)$$

which is similar to (7) with $f_{km} = 0$. Equation (15) shows that the sliding-window spectrum of a homogeneous solution $h(n)$ vanishes at the sampling points $\{\Omega = k(2\pi/N), n = mN\}$. We conclude that the existence of homogeneous solutions makes the reconstruction of the signal from its sampled sliding-window spectrum nonunique: if $x(n)$ is a possible reconstruction, then $x(n) + h(n)$ is a possible reconstruction, too.

RECONSTRUCTION VIA GABOR'S SIGNAL REPRESENTATION

In the previous section we showed a way to reconstruct the signal from its sampled sliding-window spectrum; in this section we shall elaborate this a little further, and show a different way of signal reconstruction [cf. [4], Section 4.3]. We therefore introduce a *reciprocal window sequence* $g(n)$, say, which is defined via its associated function $\tilde{g}(n, \omega)$ by

$$N\tilde{g}(n, \omega) \tilde{w}^*(n, \omega) = 1. \quad (16)$$

On substituting from (16) into (13) we get

$$\tilde{x}(n, \omega) = \tilde{f}(n, \omega) \tilde{g}(n, \omega), \quad (17)$$

which relationship can be transformed into

$$x(n) = \sum_{m=-\infty}^{\infty} \sum_{k=\langle N \rangle} f_{km} g(n - mN) \exp \left[jk \frac{2\pi}{N} n \right], \quad (18)$$

by expressing $\tilde{f}(n, \omega)$ in terms of f_{km} via (8), expressing $\tilde{g}(n, \omega)$ in terms of $g(n)$ via (10), and using the inversion relationship (12). Note the strong resemblance between (18) and the inversion formula (3). Equation (18), which expresses the signal as a combination of properly shifted and modulated versions of the reciprocal window, is in the form of *Gabor's signal representation* [cf. [3], (1.29)]. Gabor's signal representation thus provides a way to express the signal directly in terms of the sample values of the sliding-window spectrum.

Equation (16) can be transformed into

$$\sum_{n=-\infty}^{\infty} g(n) w^*(n - mN) \exp \left[-jk \frac{2\pi}{N} n \right] = \begin{cases} 1 & \text{for } k = m = 0 \\ 0 & \text{elsewhere.} \end{cases} \quad (19)$$

From (19) we conclude that the discrete set of shifted and modulated versions of the window, $w(n - mN) \exp [jk(2\pi/N)n]$, and the corresponding set of versions of the reciprocal window, $g(n - mN) \exp [jk(2\pi/N)n]$, are, in a certain sense, *biorthonormal*.

Gabor's signal representation may be nonunique in the case that $\tilde{g}(n, \omega)$ has zeros. In that case functions $\tilde{z}(n, \omega)$ may occur, for which the relation

$$\tilde{z}(n, \omega) \tilde{g}(n, \omega) = 0 \quad (20)$$

holds. Equation (20), which is similar to (17) with $\tilde{x}(n, \omega) = 0$, can be transformed into the relation

$$\sum_{m=-\infty}^{\infty} \sum_{k=\langle N \rangle} z_{km} g(n - mN) \exp \left[jk \frac{2\pi}{N} n \right] = 0, \quad (21)$$

which is similar to (18) with $x(n) = 0$. Equation (21) shows that certain arrays of nonzero coefficients in Gabor's signal representation may yield a zero result. We conclude that Gabor's signal representation may be nonunique: if the array of coefficients f_{km} yields the signal $x(n)$, then $f_{km} + z_{km}$ yields the same signal.

It is easy to formulate *Parseval's energy theorem*

$$\sum_{m=-\infty}^{\infty} |x(m)|^2 = \sum_{n=\langle N \rangle} \frac{N}{2\pi} \int_{2\pi/N} d\omega \cdot |\tilde{x}(n, \omega)|^2, \quad (22)$$

which follows directly from (10) or (12). When we apply Parseval's energy theorem to the reciprocal window $g(n)$ and substitute from (16), we get the relationship

$$\sum_{m=-\infty}^{\infty} |g(m)|^2 = \sum_{n=\langle N \rangle} \frac{N}{2\pi} \int_{2\pi/N} d\omega \cdot \frac{1}{|N\tilde{w}(n, \omega)|^2}. \quad (23)$$

From (23) we conclude that in the case that $\tilde{w}(n, \omega)$ has zeros, the reciprocal window may not be quadratically summable. This consequence of the occurrence of zeros

in $\tilde{w}(n, \omega)$ is even worse than the fact that homogeneous solutions may be present; it may cause a very bad convergence of Gabor's signal representation.

We conclude this section with an interpolation formula that enables us to express the sliding-window spectrum $f(\Omega, n)$ directly in terms of its sample values f_{km} . On substituting from (18) into (1), we get indeed the relationship

$$f(\Omega, n) = \sum_{m=-\infty}^{\infty} \sum_{k=\langle N \rangle} f_{km} q \left(\Omega - k \frac{2\pi}{N}, n - mN \right) \cdot \exp [-j\Omega mN], \tag{24}$$

where we have introduced the *interpolation function*

$$q(\Omega, n) = \sum_{m=-\infty}^{\infty} g(m) w^*(m - n) \exp [-j\Omega m], \tag{25}$$

which is, in fact, the sliding-window spectrum of the reciprocal window.

SPECIAL CASES: MAXIMUM AND MINIMUM OVERLAP

The cases of maximum and minimum overlap deserve special attention. Maximum overlap occurs for $N = 1$: in that case there is maximum overlap between the window $w(n)$ and its direct neighbors $w(n \pm N)$. In the maximum-overlap case, the formulas of the previous two sections can be simplified. Without losing any information, we can take $k \equiv 0$ in (7), (15), (18), (19), and (21), and take $n \equiv 0$ in (8), (9), (10), (11), (12), (13), (14), (16), (17), (20), (22), and (23). Equation (7), for instance, then reduces to a simple correlation,

$$f_{0m} = \sum_{n=-\infty}^{\infty} x(n) w^*(n - m), \tag{26}$$

and so do (15) and (19); note that, moreover, the coefficients f_{0m} become real when the signal $x(n)$ and the window $w(n)$ are real. Equation (18), on the other hand, reduces to a simple convolution,

$$x(n) = \sum_{m=-\infty}^{\infty} f_{0m} g(n - m), \tag{27}$$

and so does (21). Furthermore, (8) and (9) then constitute a normal Fourier transform pair, and so do (10) and (12). Note that if in the maximum-overlap case the window $w(n)$ vanishes for $n \neq 0$, then (7) or (26) tell us that the array of coefficients f_{0m} is proportional to the signal $x(m)$, as can be expected.

Minimum overlap occurs when the window has a *finite* extent and the shifting distance N is chosen equal to this finite extent. In that case the formulas of the previous two sections again simplify drastically, and the relationship between the window $w(n)$ and the reciprocal window $g(n)$ takes the simple form

$$g(n) = \begin{cases} \frac{1}{Nw^*(n)} & \text{inside the extent of } w(n) \\ 0 & \text{outside the extent of } w(n). \end{cases} \tag{28}$$

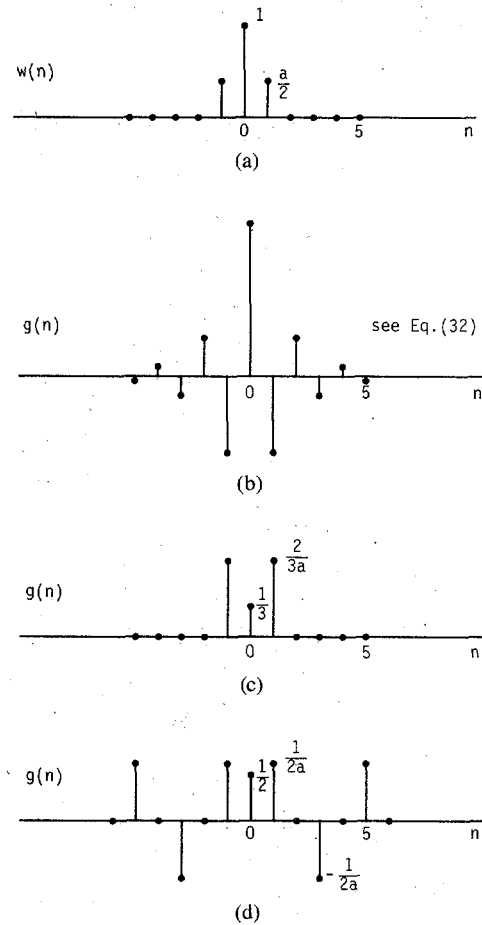


Fig. 1. Sketches of (a) a three-point, symmetrical window $w(n)$, and its corresponding reciprocal window $g(n)$ in the case of (b) maximum overlap, (c) minimum overlap, and (d) partial overlap.

It will be clear that inside its finite extent, the window $w(n)$ should take nonzero values. Note that if in the minimum-overlap case the window is uniform inside its finite extent, and if the signal $x(n)$ vanishes outside the extent of the window, then (7) tells us that the array of coefficients f_{k0} is proportional to the *discrete* Fourier transform of the signal, as can be expected.

EXAMPLES

To elucidate the concepts of this paper, we consider two simple examples of window sequences, and determine the corresponding reciprocal window sequences for different values of the shifting distance N . Our first example is the three-point, symmetrical window [see Fig. 1(a)]

$$w(n) = \begin{cases} 1 & \text{for } n = 0 \\ \frac{a}{2} & \text{for } n = \pm 1 \quad (0 < a^2 < 1) \\ 0 & \text{elsewhere;} \end{cases} \tag{29}$$

note that for $a = 0.16$, we are dealing with a three-point Hamming window. For the maximum-overlap case ($N = 1$), we find

$$\bar{w}(0, \omega) = 1 + a \cos \omega, \quad (30)$$

$$\bar{g}(0, \omega) = \frac{1}{1 + a \cos \omega}, \quad (31)$$

and, hence, the reciprocal window $g(n)$ takes the form [see Fig. 1(b)]

$$g(n) = \frac{1}{\sqrt{1-a^2}} \left(\frac{\sqrt{1-a^2}-1}{a} \right)^{|n|}. \quad (32)$$

For the minimum-overlap case ($N = 3$) the reciprocal window becomes [see Fig. 1(c)]

$$g(n) = \begin{cases} \frac{1}{3} & \text{for } n = 0 \\ \frac{2}{3a} & \text{for } n = \pm 1 \\ 0 & \text{elsewhere.} \end{cases}$$

For the case of partial overlap ($N = 2$) we find

$$\begin{cases} \bar{w}(0, \omega) = 1 \\ \bar{w}(1, \omega) = \frac{a}{2} (1 + \exp [j2\omega]), \end{cases} \quad (34)$$

$$\begin{cases} 2\bar{g}(0, \omega) = 1 \\ 2\bar{g}(1, \omega) = \frac{2}{a} \frac{1}{1 + \exp [-j2\omega]}, \end{cases} \quad (35)$$

and the reciprocal window now takes the form [see Fig. 1(d)]

$$\begin{cases} g(2m) = \begin{cases} \frac{1}{2} & \text{for } m = 0 \\ 0 & \text{for } m \neq 0 \end{cases} \\ g(2m+1) = \begin{cases} \frac{1}{2a} (-1)^m & \text{for } m \geq 0 \\ \frac{-1}{2a} (-1)^m & \text{for } m < 0. \end{cases} \end{cases} \quad (36)$$

Note that in the case of partial overlap, the function $\bar{w}(n, \omega)$ has zeros for $\omega = \pi/2 + r\pi$ ($r = \dots, -1, 0, 1, \dots$), and hence a homogeneous solution $h(n)$ arises. Its associated function $\bar{h}(n, \omega)$ is given by

$$\begin{cases} 2\bar{h}(0, \omega) = 0 \\ 2\bar{h}(1, \omega) = \pi h \sum_{r=-\infty}^{\infty} \delta \left(\omega - \frac{\pi}{2} - r\pi \right), \end{cases} \quad (37)$$

where $\delta(\cdot)$ represents the Dirac delta function. The homogeneous solution $h(n)$ thus takes the form

$$\begin{cases} h(2m) = 0 \\ h(2m+1) = (-1)^m h. \end{cases} \quad (38)$$

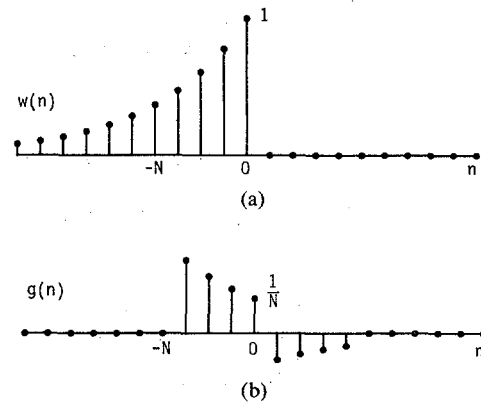


Fig. 2. Sketches of (a) a one-sided, exponential window $w(n)$, and (b) its corresponding reciprocal window $g(n)$.

Our second example is the one-sided, exponential window [see Fig. 2(a)]

$$w(n) = \begin{cases} \exp [\alpha n] & \text{for } n \leq 0 \quad (\alpha > 0) \\ 0 & \text{for } n > 0. \end{cases} \quad (39)$$

In the interval $-(N-1) \leq n \leq 0$, the associated function $\bar{w}(n, \omega)$ takes the form

$$\bar{w}(n, \omega) = \exp [\alpha n] \frac{1}{1 - \exp [-(\alpha - j\omega)N]}, \quad (40)$$

and the function $N\bar{g}(n, \omega)$ in this interval thus reads

$$N\bar{g}(n, \omega) = \exp [-\alpha n] \{1 - \exp [-(\alpha + j\omega)N]\}. \quad (41)$$

The reciprocal window $g(n)$ now takes the form [see Fig. 2(b)]

$$g(n) = \begin{cases} \frac{1}{N} \exp [-\alpha n] & \text{for } -(N-1) \leq n \leq 0 \\ \frac{-1}{N} \exp [-\alpha n] & \text{for } 1 \leq n \leq N \\ 0 & \text{elsewhere.} \end{cases} \quad (42)$$

We use this example to show the possible nonuniqueness of Gabor's signal representation. In the limiting case $\alpha = 0$, the function $\bar{g}(n, \omega)$ has zeros for $\omega = r(2\pi/N)$ ($r = \dots, -1, 0, 1, \dots$), and an array of coefficients z_{km} arises whose associated function $\bar{z}(n, \omega)$ in the interval $0 \leq n \leq N-1$, say, is given by

$$\bar{z}(n, \omega) = \frac{2\pi}{N} z_n \sum_{r=-\infty}^{\infty} \delta \left(\omega - r \frac{2\pi}{N} \right). \quad (43)$$

The array z_{km} thus takes the form

$$z_{km} = \frac{1}{N} \sum_{n=0}^{N-1} z_n \exp \left[-jk \frac{2\pi}{N} n \right] = z_{k0}, \quad (44)$$

and yields a zero result when substituted in Gabor's signal representation.

CONCLUSION

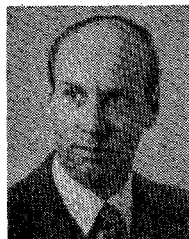
In this paper we have studied the short-time Fourier transform of a discrete-time signal, or, as we prefer to call

it, the sliding-window spectrum. This sliding-window spectrum is a function of two variables: a discrete time index, which represents the position of the window, and a continuous frequency variable. We have shown that the signal can be reconstructed from the sliding-window spectrum, when we know its values at the points of a certain time-frequency lattice. This lattice is rectangular, and the rectangular cells occupy an area of 2π ; hence, the coarser the sampling in time, the finer the sampling in frequency, and vice versa.

The most elegant form to represent the signal in terms of the sample values of the sliding-window spectrum, is by means of Gabor's signal representation. We therefore had to introduce a reciprocal window, and we have shown how the window and the reciprocal window are related. Gabor's signal representation then expresses the signal as a superposition of properly shifted and modulated versions of the reciprocal window.

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