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ALGEBRAS OF EXTENDIBLE UNBOUNDED OPERATORS

Pawel Kruszyński*

Eindhoven University of Technology

Department of Mathematics and Computing Science

Abstract. Algebras of operators acting continuously on an initial space and its dual are introduced and studied.

Spectral theory in these algebras is discussed.

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* On leave from the Department of Mathematical Methods in Physics, University of Warsaw, Poland.

Introduction

The theory of topological algebras of operators arose from quantum theory (cf. [5], [18]).

Their topological and algebraic properties were studied by many authors. (cf. [1-6], [10], [11][13], [14][16], [17-18], [19]).

Operators belonging to these alebras are defined on common domains. Frequently the domains are dense subspaces of a Hilbert space. The theory of analyticity and trajectory spaces [7-10], [12], which is a part of the theory of generalised functions, also leads to algebras of, so called, extendible operators. The starting point of this theory is a description of the underlying domain, here called an initial space. Then, in a natural way, algebras of unbounded operators appear.

In the present paper we study a general situation starting from an abstract locally convex topological vector space S, called an initial state space or, simply, initial space. In particular, in order to be both as general as possible and as close as possible to known examples, we assume that the initial space is semireflexive and bornological with bornological strong dual.

In the frame of our approach the generalized function theory appears by means of a so called positive embedding of S into S'. Both S and S' have identical topological properties and up to this point each of them could be equivalently chosen as 'an ordinary

function space'. Only the existence of an embedding $j : S \subset\!\!\rightarrow S'$ gives an indication in which direction the generalization of functions occurs. For this reason the initial space can not always be regarded as a test function space.

It is clear that such an embedding leads to Gelfand triples and Hilbertian subspaces theory of L. Schwartz [22], as well as to some connections with the theory of regular operators [3] and Op^*-algebras [17,18].

The theory of analyticity spaces [7,12] and its generalization [10] is also based on the existence of an intermediate Hilbert space H, with $S \subset H \subset S'$.

Our main interest is concentrated on so called extendible maps and their algebras. For a given embedding $j : S \subset\!\!\rightarrow S'$ we consider continuous linear maps acting in S which can be extended to strongly continuous maps acting in S'.

Such a concept appears already in papers of L. Schwartz [22], J.P. Antoine and F. Mathot [3]. Our present approach is modelled after results of S.J.L. van Eijndhoven, J. de Graaf and the author [7-10], where topological algebras of extendible maps are introduced.

The aim of the present paper is to unify and generalize certain ideas which appear in the above references. Here we present an introduction to our investigations which lead, as we hope, to

physical applications. For this reason the paper contains not only basic definitions and facts but also conjectures and problems.
In particular we expext that some mathematical aspects of Dirac's formalism of quantum mechanics can be included into the frame of our theory, as it is indicated in [9]. Quantum statistics also can be described within this theory. Although we do not consider these problems in the present paper we should mention here that the idea of extendibility can be connected with the concept of Dirac's observables which act simultaneously both on ket and bra spaces. The most difficult problem in such a physical interpretation is that the algebras of extendible maps depend on the choice of the embedding j i.e. on the choice of the representation. Hence a need of invariant definition of extendibility arises.
Another possible extension of the present theory is an investigation of groups of transformations in the Hilbert space H which leave the initial space invariant.
Considering the 'extremal' case of a Hilbert space as an initial space we see that our definition of the algebra of extendible maps leads to the algebra of all bounded operators. Hence we see a generalization of the theory of algebras of bounded operatos in considering *-sub-algebras of algebras of extendible maps. (See e.g. V^*-algebras theory [4], [11]).

We begin our paper with basic definitions and properties of initial spaces, contained in Sect. 1. In Sect. 2 we introduce the embedding $j : S \subset\to S'$ and discuss its properties. It turns out that it can be studied equivalently in terms of the theory of Hilbertian subspaces developed by L. Schwartz in [22].

In Sect. 3 we introduce the notion of extendible maps which depends on the choice of the embedding j. In Sect. 5 we discuss relations between extendibility of an operator with respect to different embeddings. We also study algebras of extendible maps as topological *-algebras.

Sect. 4 contains an interesting example of an initial space, namely the space ϕ of all finite sequences of complex numbers [15]. It provides us with a lot of illuminating observations.

An attempt to develop a topological spectral theory in the algebras of extendible maps is made in Sect. 6. We follow here some ideas of G.R. Allan [1,2]. We should mention however that our algebras are not GB^*-algebras. We prove that for the case of the space ϕ the usual spectral theory of Hilbert spaces gives the same spectra of bounded normal operators as the spectral theory formulated for extendible maps.

In Sect. 7 we give some ideas on positive elements in algebras of extendible operators. It opens a way to further investigations on positive functionals, quantum statistics and Dirac's formalism.

SECTION 1. PRELIMINARIES AND NOTATION

In [10] the locally convex topological vector spaces $S_{\Phi(A)}$ and $T_{\Phi(A)}$ have been constructed. Under suitable assumptions the pair $S_{\Phi(A)}$, $T_{\Phi(A)}$ becomes a dual pair of barreled, bornological, reflexive, complete locally convex spaces. Moreover, in many cases, $T_{\Phi(A)}$ or $S_{\Phi(A)}$ is a Fréchet space.

Inspired by some ideas contained in [3], [7], [17], [22] we will consider algebras of linear mappings defined on such l.c.t.v spaces which a priori have the above topological properties of $S_{\Phi(A)}$ and $T_{\Phi(A)}$.

The space corresponding to $S_{\Phi(A)}$ we will denote by S and call an initial space.

We need an explanation of the standard terminology and notation we use. (cf. [21]).

By τ we denote the locally convex topology that we originally impose on the space S. We assume that it is generated by a family of seminorms $P = \{p\}$. The topological dual of $S = S_\tau$ is denoted by S'. The strong topology $\beta = \beta(S',S)$ on S' is generated by the family $Q = \{q\}$ of all seminorms of the form: $q_B(s') = \sup_{s\in B} |< s'|s >|$, where $s' \in S'$, $B \subset S$ is weakly bounded and $<\bullet|\bullet>$ denotes the bilinear duality between S and S'.

The space S is semi-reflexive if its second strong dual $S'' = (S'_\beta)'$

is identical with the set S. It means that the natural embedding $S \subset\rightarrow S''$ is onto. The space S is reflexive if this embedding is a homeomorphism i.e. $S_\tau = (S'_\beta)'_\beta$. The space S is bornological if every circled, convex subset $A \subset S$ that absorbs every bounded set in S is a neighborhood of 0, or equivalently S consists of all bounded linear forms on its strong dual S'_β.

The space S is barreled if every circled convex and closed subset $B \subset S$ that absorbs every finite set in S (a barrel) is a neighborhood of 0. The space is infrabarreled if every circled, convex and closed subset $B \subset S$ that absorbs every bounded set in S is a neighborhood of 0. The Mackey topology $\tau(S,S')$ on S is the finest locally convex topology on S for which the toplogical dual of $S_{\tau(S,S')}$ is identical with S'. The space S_τ is called a Mackey space if its original topology τ is Mackey i.e. $\tau = \tau(S,S')$. Any bornological or barreled l.c.t.v. space is Mackey.

The weak topology with respect to the duality S and S' is denoted by $\sigma(S,S')$. The topology $\sigma(S',S)$ on S' is called *-weak.

1.1. Definition

A locally convex topological vector space S over the field of complex numbers is called an initial (state) space if S is:

i) bornological

ii) semi-reflexive

iii) the strong dual S'_β of S is bornological.

It is obvious that the space $S_{\Phi(A)}$ (cf. [10]) is an initial space. Also the spaces of functions such as $\mathcal{D}(\Omega)$, $E(\Omega)$, $S(\mathbb{R}^n)$, where $\Omega \subset \mathbb{R}^n$, which appear in the theory of distributions, are initial spaces.

It follows that an initial space and its strong dual have nice topological properties:

1.2. Proposition

Let S be an initial space and S'_β its strong dual. Then:

i) S and S'_β are barreled

ii) S and S'_β are complete

iii) S and S'_β are Mackey spaces

iv) S and S'_β are reflexive.

Proof

By [15] Ch. I Sec. 8 Prop. 8.7 S is infrabarreled and by [21] Ch. IV § 5 Corollary 5.5 S is semi-reflexive hence by [21] Ch. IV § 5 Coroll. 5.3 S is barreled. Again because S is semi-reflexive and barreled it is reflexive (cf. [21] Ch. IV § 5 Thm. 5.6)). It follows that also S'_β is reflexive and barreled and both S and S'_β are Mackey (cf. Ch. IV § 3. 3.4, § 5. 5.7 of [21]). Further S and S'_β are complete since they are strong duals of bornological spaces (Cf. Ch. IV § 6 6.1 of [21]).

Q.E.D.

In certain interesting cases considered in [7] and [10] the strong dual of the space S is a Fréchet space. In order to obtain this property it is enough to assume that the locally convex topological space S is the strong dual of a metrizable, semi-reflexive locally convex topological vector space T.

1.3. Proposition

Let a l.c.t.v. space E be the strong dual of a metrizable, semi-reflexive l.c.t.v. space T.

Then

i) E and T are barreled

ii) E and T are bornological

iii) E and T are complete

iv) E and T are Mackey

v) E and T are reflexive

vi) $T = E'_\beta$ is Fréchet.

Proof

The space T is bornological since it is metrizable (cf. [21] Ch. II § 8. 8.1). Hence T is infrabarreled. So from the semi-reflexivity of T the reflexivity of T and E follows (Cf. [21] Ch. IV § 5. Thm. 5.6 Coroll. 1, [15] I Sec. 8 Prop. 8.7). By [21] Ch. IV § 6 6.5 Coroll. 1 T is complete since it is metrizable and reflexive. Hence T is Fréchet.

From the metrizability of T the completeness of $E = T'_\beta$ follows (Cf. [21] IV § 6. 6.1). In virtue of [21] IV § 6. 6.6 Coroll. 1 E is bornological because it is the strong dual of the reflexive Fréchet space T. In this case it is equivalent to barrelledness of E (Cf. [21] IV § 6. 6.6).

Because every bornological 1.c.t.v.s. is Mackey S and T are Mackey (Cf. [21] IV § 3. 3.4).

Because T is reflexive and Mackey it is barreled (Cf. [21] IV § 5.7).

Q.E.D.

1.4. Corollary ([21] Ch. IV 5.7, p. 145; 6.1 p. 148).

If a reflexive 1.c.t.v. space E is barreled, bornological, complete and Mackey then

i) $E'_\tau = E'_\beta$ i.e. $\tau(E',E) = \beta(E',E)$

ii) $E_\tau = E_\beta$ i.e. $\tau(E,E') = \beta(E,E')$

iii) E'_τ is barreled

iv) E'_β is complete.

1.5. Proposition ([21], Ch. III Thm. 4.2, p 83).

Let E, F be 1.c.t.v. spaces, E barreled, L(E,F) the set of all continuous linear maps from E into F. Then every pointwise bounded subset M of L(E,F) is equicontinuous i.e. for each neighborhood V of 0 in F there exists a neighborhood U of 0 in E such that $m(U) \subset V$ for all $m \in M$.

In particular we have the following generalized version of uniform boundedness principle.

1.6. Corollary (Uniform boundedness principle)

If E, F are l.c.t.v. spaces then every pointwise bounded subset M of L(E,F) is uniformly bounded on all convex, circled, bounded and complete subsets of E.

i.e. For all seminorms q in F and $B \subset E$ such that $\sup_{s \in B} p(s) < \infty$ for each seminorm p in E we have:

$$\sup_{m \in M} \sup_{s \in B} q(m(s)) < \infty$$

1.7. Remark

If E and F are complete then M is uniformly bounded on closed bounded convex circled subsets in E.

1.8. Remark

Taking E = S F = S' we have a version of the Banach-Steinhaus theorem: Each *-weakly bounded (i.e. $\sigma(S',S)$ bounded) subset B_1 in S' is strongly bounded.

We recall that the *-weak topology is the topology $\sigma(S',S)$ given by the family of seminorms

$$s' \to | \langle s',s \rangle | \qquad s' \in S' , \qquad s \in S.$$

We are interested in spaces of continuous linear mappings acting in S_τ and S'_β . We denote these spaces by $L(S)$ and $L(S')$.

Note that we can replace the Mackey topology τ by the strong topology β. Then $L(S_\tau) = L(S_\beta)$ and $L(S'_\beta) = L(S'_\tau)$ in virtue of Corollary 1.4.

We recall here a useful characterization of elements of $L(S)$ and $L(S')$.

1.9. Remark ([21] Ch. II 8.3, p. 62)

Let E be a bornological l.c.t.v. space, F be a l.c.t.v space and let $u : E \to F$ be a linear map. Then the following conditions are equivalent.

i) u is continuous

ii) The sequence $\{u(x_n)\}$ is a null sequence for every null sequence $\{x_n\}$ in E.

iii) The set $u(B) \subset F$ is bounded for every bounded subset B in E.

The above remark is applicable to the bornological spaces S and S' by Proposition 1.2.

We introduce the following l.c. topologies on the space $L(S)$:

1.10. Definition ([10], [7], [24])

i) The uniform topology τ_s ('strong' in [7]) is given by the family of all seminorms:

$$q_{p,B} : x \to \sup_{s \in B} p(xs)$$

where $x \in L(S)$, B is a bounded subset of S and $p \in \mathcal{P}$ is a semi-norm in S.

τ_s is equivalently given by the seminorms:

$$q_{B,B'} : x \to \sup_{f \in B'} \sup_{s \in B} | \langle f \mid xs \rangle |$$

where B' is a bounded subset of S' and B a bounded subset of S.

ii) The pointwise (or 'strong spatial') topology τ_p is given by the family of all seminorms of the form:

$$q_{p,s}: x \to p(xs) \quad \text{where } p \in P \text{ is a seminorm in } S, \quad s \in S.$$

or equivalently by the seminorms:

$$q_{B',s}: x \to \sup_{f \in B'} | \langle f \mid xs \rangle | \quad \text{where } s \in S$$

and B' is a bounded subset of S'.

iii) The weak topology τ_w is given by the family of all seminorms of the form:

$$q_\psi : x \to |\psi(x)|$$

where ψ is a linear functional on L(S) continuous with respect to the uniform topology τ_s i.e. $\psi \in L(S)'$.

iv) The weak spatial or 'weak pointwise' ([7]) topology τ_{wp} is given by all seminorms of the form:

$$q_{f,s} : x \to |\langle f \mid xs \rangle|$$

where $f \in S'$, $s \in S$.

1.11. Proposition

The following relations hold:

i) $\tau_s > \tau_w > \tau_{wp}$

ii) $\tau_s > \tau_p > \tau_{wp}$.

If S is a dense domain in a Hilbert space H (see e.g. [17]) then usually other topologies on L(S) are introduced. However we do not consider them here because those topologies refer to an explicit embedding of S into H. We shall discuss this problem later on. Similarly we can introduce the following l.c. topologies in the space L(S'):

1.12. Definition ([10], [7], [24])

i) The uniform ('strong' [7]) topology σ_s is given by the family of all seminorms of the form

$$q_{q,B} : y \to \sup_{f \in B} q(yf)$$

where $y \in L(S')$ and B is a bounded subset of S', $q \in Q$ is a seminorm in S', or equivalently by

$$q_{BB'} : y \to \sup_{f \in B'} \sup_{s \in B} |\langle yf \mid s \rangle|$$

ii) The pointwise (or 'strong spatial') topology σ_p is given by the family of all seminorms of the form

$$q_{f,q} : y \to q(yf)$$

where $q \in Q$ is a seminorm in S' and $f \in S'$, or equivalently, by the seminorms

$$q_{f,B} : y \to \sup_{s \in B} |\langle yf \mid s \rangle|$$

where $f \in S'$ and B is a bounded subset of S.

iii) The weak topology σ_w is generated by the family of all seminorms of the form:

$$q_\omega : y \to |\omega(y)|$$

where ω is a linear functional on $L(S')$ continuous with respect to the uniform topology σ_s i.e. $\omega \in L(S')'$

iv) The weak pointwise (or 'weak spatial') topology σ_{wp} is generated by the following family of seminorms:

$$q_{s,f} : y \to |\langle yf \mid s \rangle| \qquad f \in S' \quad s \in S \quad y \in L(S')$$

1.13. Proposition

The following relations hold:

i) $\sigma_s > \sigma_p > \sigma_{wp}$

ii) $\sigma_s > \sigma_w > \sigma_{wp}$.

Bounded subsets of $L(S)$ and $L(S')$ are bounded in all topologies defined above. We have:

1.14. Proposition

Let B be a subset of $L(S)$ (or $L(S')$ respectively). Then the following conditions are equivalent:

i) B is τ_s-bounded (resp. σ_s-bounded)

ii) B is τ_p-bounded (resp. σ_p-bounded)

iii) B is τ_w-bounded (resp. σ_w-bounded)

iv) B is τ_{wp}-bounded (resp. σ_{wp}-bounded)

Proof

We recall the standard proof only for $B \subset L(S)$.

In virtue of Propositions 1.11 and 1.13 it is easy to see that i) $\Rightarrow$ ii) $\Rightarrow$ iv) and i) $\Rightarrow$ iii) $\Rightarrow$ iv). Hence it is enough to prove that iv $\Rightarrow$ i).

Let $B \subset L(S)$ be τ_{wp}-bounded, i.e.

$$\sup_{x \in B} |\langle f \mid xs \rangle| < \infty$$

for all $f \in S'$ and $s \in S$. Hence for a fixed $s \in S$ the family $\{m_x\}_{x \in B}$ of linear β-continuous maps:

$$S' \ni f \to m_x(f) = \langle f \mid xs \rangle \in \mathbb{C}^1$$

is pointwise bounded. In virtue of Proposition (1.5) the family $\{m_x\}_{x \in B}$ is equicontinuous and by Corollary (1.6) it is uniformly bounded on bounded sets in S'. So, for any bounded set $B' \subset S'$ we have

$$\sup_{f \in B'} \sup_{x \in B} | m_x(f)| = \sup_{x \in B} \sup_{f \in B'} |\langle f \mid xs \rangle| < \infty$$

It follows that the set B is τ_p bounded. Thus we have shown the

implication: iv) ⇒ ii).

Suppose now that $B \subset L(S)$ is τ_p bounded, i.e. for any seminorm p on S and any $s \in S$ $\sup_{x \in B} p(xs) < \infty$. Hence the family of τ-continuous linear maps B is pointwise bounded. So it is also equicontinuous (Proposition (1.5)). By Corollary 1.6 the set B is τ_s-bounded.

Thus ii) ⇒ i) and it follows that iv) ⇒ i).

Q.E.D.

We formulate now the following useful result:

1.15. Lemma

The space S (and S') is weakly (*-weakly resp.) sequentially complete.

Proof.

Let $\{s_n\}$ be a weak Cauchy sequence in S. It follows that $\{s_n\}$ is weakly bounded in $S'' = S$. Hence it is uniformly bounded. The weak limit of s_n, say s, is bounded on every bounded subset B' of S'. Hence, in virtue of Remark 1.9, $s \in S'' = S$.

The same arguments applied to the space S' prove its *-weak sequential completeness.

Q.E.D.

By standard arguments we also have

1.16. Proposition

The spaces L(S) and L(S') are sequentially complete with respect to

uniform, pointwise, weak and weak spatial topologies.

Proof

It is sufficient to prove the result for the weakest topology among the given four, i.e. for the weak spatial topology.

Let us consider a sequence $\{x_n\} \subset L(S)$ such that for each $s \in S$ the sequence $\{x_n s\}$ has a weak limit, say xs.

The mapping $s \to xs$ is linear, and τ-bounded on τ-bounded sets, hence τ-continuous, i.e. $x \in L(S)$. Here we have used Corollary 1.6 and Proposition 1.14.

Q.E.D.

The transposition map $x \to x'$ from $L(S)$ into linear maps in S' is defined by the relation: $\langle x'f \mid s\rangle = \langle f \mid xs\rangle$ for all $f \in S'$, $s \in S$. We have $x' \in L(S)$ by Thm. 2 Ch. VII § 1. in [24]. It follows also that $x'' = x$.

1.17. Proposition

The transposition map $L(S) \ni x \to x' \in L(S')$ is continuous if:

i) $L(S')$ is endowed with the topology σ_{wp} and $L(S)$ with one of the topologies τ_s, τ_p, τ_w or τ_{wp} (see 1.10).

ii) $L(S')$ is endowed with one of the topologies (1.12): σ_s, σ_p, σ_w or σ_{wp} and $L(S)$ is endowed with the topology τ_s.

Proof

Clearly the map $x \to x'$ is (τ_{wp}, σ_{wp})-continuous hence it is (τ, σ_{wp})-

continuous for all l.c. topologies τ on $L(S)$ stronger than τ_{wp}

ii) The transposition is obviously (τ_s,σ_s) continuous hence it is continuous with respect to topologies (τ_s,σ), where σ is a l.c. topology on $L(S')$ weaker than σ_s.

Q.E.D.

1.18. Corollary

The map $x \to x'$ is a homeomorphism between $L(S)$ and $L(S')$ endowed with topologies τ_{wp} and σ_{wp} or τ_s and σ_s respectively

1.19. Remark

One should not expect continuity of the transposition with respect to other pairs of topologies because it does not hold even if the space S is a Hilbert space.

§ 2. EMBEDDING $S \hookrightarrow S'$

In this section we study the notion of embedding of S into S'.

2.1. Definition

An antilinear map $j : S \to S'$ is called a positive embedding if

i) j is continuous as a map from S_τ into S'_β.

ii) j is positive i.e. $\forall s \in S$

$$\langle j(s) \mid s \rangle > 0 \Leftrightarrow s \neq 0.$$

2.2. Proposition

i) j is injective

ii) j^{-1} is closed on $D(j^{-1}) \overset{df}{=} j(S)$

iii) the scalar product on S defined by

(2.3) $$(s \mid z) = \langle j(s) \mid z \rangle$$

is hermitian and non-degenerate.

iv) j(S) is β-dense in S'

v) j is continuous if we endow S and S' with weak and *-weak topologies, respectively.

Proof.

i) If $j(s) = 0$ then by positivity $s = 0$

ii) Let $\{S'_\alpha\} \subset j(S)$ be a β-Cauchy net in S' which tends to $s' \in S'$, and suppose that $j^{-1}(S'_\alpha)$ is τ-convergent in S to some $s \in S$.

Then by the continuity of j the net $s'_\alpha = j(j^{-1}(s'_\alpha))$ tends to $j(s) = s' \in j(S)$.

iii) It follows from the polarization formula for the sesquilinear form $\langle j(s) \mid z \rangle$.

iv) By contradiction: Suppose that there exists $s' \in S'$ such that $s' \notin \overline{j(S)}$ - β-closure of $j(S)$ in S', $s' \neq 0$. Then there exists in $S = S''$ a β-continuous functional $s_0 \neq 0$ such that $\langle s' \mid s_0 \rangle \neq 0$ and $\langle j(s) \mid s_0 \rangle = 0$. In particular we have $\langle j(s_0) \mid s_0 \rangle = 0$, so $s_0 = 0$.

The contradiction.

v) The result follows from the equality:

$$\langle j(s) \mid z \rangle = \overline{\langle j(z) \mid s \rangle}.$$

Q.E.D.

By the Schwarz inequality we have:

2.4. Remark

A continuous embedding $j : S \to S'$ is a positive embedding iff one of the following conditions holds:

i) j is injective and nonnegative i.e.: for each $s \in S$

$$\langle j(s) \mid s \rangle \geq 0.$$

ii) j is non-negative and $j(S)$ is dense in S'.

In virtue of iii) Proposition 2.2 the completion of the pre-Hilbert space $(S,(\ |\ ))$ is the Hilbert space $H = \overline{S}^{\|\ \|}$ with the norm: $\|\ \|$ extending the norm $S \ni s \to \|s\| = (\langle j(s) \mid s\rangle)^{\frac{1}{2}}$. The embedding $S \subset H$ is τ-$\|\ \|$ continuous i.e. the norm topology induced on S by H is weaker than the original topology τ on S.

2.5. Proposition

The positive embedding $j : S \to S'$ can be extended to the injection:

$$\hat{j} : H \to S'$$

which is continuous with respect to the norm topology in H and β-topology in S'.

Moreover for each $s \in S$, $h \in H$

$$(h \mid s) = \langle \hat{j}(h) \mid s\rangle .$$

Proof

Observe that the map $j : S \to S'_\beta$ is norm continuous because for $s,z \in S$ $|\langle j(s) \mid z\rangle| \leq \|s\|\|z\|$, hence it can be uniquely extended to the $\|\ \|$-β continuous map $\hat{j}$ ([21] III § 1.). The formula $(h \mid s) = \langle \hat{j}(h) \mid s\rangle$ follows from the norm continuity of j on S.

Q.E.D.

For further applications the problem of existence of a positive embedding for a given initial space is very important. At first we observe that the existence of a positive embedding $j : S \to S'$ is

equivalent to the existence of a non-degenerate separately continuous positive sesqui-linear form on S.

2.6. Proposition

Let S be an initial space. Then there exists a positive embedding $j : S \to S'$ if and only if there exists in the space S a non-degenerate positive separately continuous sesqui-linear form.

Proof

The 'only if' part is contained in Proposition 2.2. For the 'if part' let $B : S \times S \to \mathbb{C}^1$ be a sesqui-linear, positive, non-degenerate form, which is separately continuous.

For each $z \in S$ the map:

$$S \ni s \to B(z,s)$$

is a linear continuous form on S. Hence the formula

$j(z) := B(z,\cdot)$ gives a non-negative embedding $j : S \to S'$.

To prove its continuity we should notice that it is τ-*-weakly continuous. By standard arguments applied to the bornological space S it follows that j is (τ,β)-continuous. Thus $j(z) = B(z,\cdot)$ is the desired positive embedding.

Q.E.D.

2.7. Corollary

Any separately continuous sesqui-linear form on S is jointly continuous.

Proof

It is enough to notice that the map $S \ni s \to |B(s,s)|^{\frac{1}{2}}$ is bounded on bounded subsets of S, hence it is a continuous seminorm on the bornological space S.

Q.E.D.

It is easy to see that the correspondence between positive embeddings and Hilbert spaces is one-to-one. (see [22]).

2.8. Proposition

For a given test space S there exists a positive embedding $j : S \to S'$ if and only if there exists in S' a dense Hilbert subspace H, continuously embedded into S'.

Proof

The 'only if' part follows from the previous result 2.5 and [22] Prop. 1, if we put $H = \hat{j}(H)$.

Now for the 'if' part let $H \hookrightarrow S'$. Taking strong duals we have $S = S'' \hookrightarrow H'$ and this natural embedding is (strongly-) continuous. Hence the desired positive embedding is given by $j(s) := (s \mid \cdot)$, i.e. by the anti-isomorphic Fréchet-Riesz map applied to the elements of S. The positivity of j follows from its injectivity and from the positivity of the scalar product in H.

Q.E.D.

The above proposition gives a description of positive embeddings by means of Hilbertian sub-spaces of S' in the sense of L. Schwartz [22]. We have then a one-to-one correspondence between positive embeddings and dense Hilbertian subspaces of S'. Thus the problem of the existence of a positive embedding has been transformed into the problem of the existence of Hilbertian subspaces of S'. For certain results in this direction we refer to [22].

For our purpose however it is more natural to consider the theory of embeddings rather than the theory of Hilbertian subspaces. Nevertheless the Hilbert space associated with an embedding will play an important role in the present approach.
We should mention also that the topological aspects of our theory are independent of the particular choice of the embedding j.

§ 3. Extendible maps

For a given positive embedding $j : S \to S'$ and for an element x of $L(S)$ we consider the mapping:

$$j(S) \ni j(s) \to j(xs) \in j(S) \ .$$

This map is linear and densely defined in S'. Moreover if it is β-continuous in $j(S)$ then the element x is called an extendible map. More precisely:

3.1. Definition

An element x of $L(S)$ is called a j-extendible map iff there exists $\tilde{x} \in L(S')$, such that the following diagram is commutative:

$$\begin{array}{ccc} S & \xrightarrow{x} & S \\ {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle j} \\ S' & \xrightarrow{\tilde{x}} & S' \end{array}$$

i.e. $j \circ x = \tilde{x} \circ j$.

The set of all j-extendible elements is denoted by A_j. (We usually drop the index j if no confusion is likely to arise.)

The above definition is an abstract formulation of extendibility related to the spaces $S_{X,A}$ and $T_{X,A}$, as described in [7,10,12]. Also the regular operators defined in [3] and elements of Op^*-algebras [18] can be considered as a particular case of extendible

maps. We can trace the idea of extendible maps back to the paper by L. Schwartz [22], although they are defined there for somewhat different purpose. In particular they are connected there with Neumann and Dirichlet problems. For this the assumption on density of $j(S)$ in S' should be dropped. We will not consider this case here.

3.2. Lemma

If $x \in A$ then $x'(j(S)) \subset j(S)$ and $x' \circ j = j \circ (\tilde{x})'$.

Proof

By the reflexivity of S, we have for all $s,z \in S$:

$$\langle x'j(s) \mid z\rangle = \langle j(s) \mid xz\rangle = \overline{\langle j(xz) \mid s\rangle} =$$

$$= \overline{\langle \tilde{x}j(z) \mid s\rangle} = \langle j(\tilde{x}'s) \mid z\rangle \ .$$

Hence

$$x'j(s) = j(\tilde{x}'s)$$

i.e.

$$x' \circ j = j \circ (\tilde{x})' .$$

Q.E.D.

3.3. Proposition

The set $A \subset L(S)$ of extendible maps is an involutive algebra with involution given by $x^+ \overset{df}{=} j^{-1}x'j$.

Proof

It is easy to see that the map $x \to \tilde{x}$ is an injective antilinear homomorphism, well defined on A. In particular if $x,y \in A$ then $\tilde{x} \cdot \tilde{y}$ is the extension of $x \cdot y$ so $x \cdot y \in A$.

We have to show that x^+ is a well defined element of $L(S)$ and that $x^+ \in A$. Because $\tilde{x}' \in L(S)$ we see that $\tilde{x}' \in A$ and $((\tilde{x})')^{\sim} = x'$.

Moreover we have $x^+ = j^{-1} \circ x' \circ j = j^{-1} \circ j \circ \tilde{x}' = \tilde{x}'$.

Thus $x^+ \in A$. This completes the proof of Proposition 3.3.

Q.E.D.

Now we collect some useful algebraic rules fulfilled by the maps $x \to \tilde{x}$, $x \to x'$, $x \to x^+$.

3.4. Corollary

Let $x,y \in A$, $\lambda \in \mathbb{C}^1$. Then

i) $x^+ = j^{-1} \circ x' \circ j$

ii) $x^+ = (\tilde{x})'$, $(x^+)' = \tilde{x}$

iii) $((\tilde{x})')^{\sim} = x'$

iv) $(x^+)^{\sim} = x'$

v) $x^{++} = x'' = x$.

vi) $(xy)^+ = y^+x^+$, $(x + y)^+ = x^+ + y^+$, $(\lambda x)^+ = \bar{\lambda} x^+$.

vii) $\widetilde{xy} = \tilde{x} \cdot \tilde{y}$, $(x + y)^{\sim} = \tilde{x} + \tilde{y}$, $(\lambda x)^{\sim} = \bar{\lambda}\tilde{x}$.

In [7] the involution $x \to x^c$ was introduced on the whole of $L(S)$. The embedding $S_{X,A} \subset T_{X,A}$ was considered as a natural one because of the canonical choice of the Hilbert space X. The 'algebra of extendible maps' E_A was preserved by the map $x \to x^c$.
Also in the present scheme we can formulate a similar result:

3.5. Proposition

i) Let $x \in L(S)$. Then $x \in A$ iff $x'j(S) \subset j(S)$.

ii) Let $y \in L(S')$. Then $y = \tilde{x}$ for some $x \in A$ iff $y' \in A$.

Proof

i) Let $x'j(S) \subset j(S)$. Then $j^{-1}x'j \in L(S)$ in virtue of Proposition 2.2 ii) and of the continuity of $x' \circ j : S \to S'$.
Hence it can be easily seen that $(j^{-1}x'j)' \in L(S')$ is the extension of x, i.e. $x \in A$.
On the other hand by Corollary 3.4 iv) if $x \in A$ then $x' = (x^+)^\sim$ and the result follows.

ii) Suppose that $y = \tilde{x}$ for some $x \in A$. Then $y' = (\tilde{x})' = x^+ \in A$.
On the other hand if $y' \in A$ then $y = y'' = ((y')^+)^\sim$ maps $j(S)$ into $j(S)$.

Q.E.D.

Remark

In general $y \in L(S')$ and $yj(S) \subset j(S)$ does not imply that $y = \tilde{x}$ for some $x \in A$.

The set S can be considered as a dense domain in the Hilbert space H. Then the elements of L(S) can be regarded as unbounded operators with common invariant dense domain S. It is easy to see that $x^+ = x^*|_S$, where $x \in A$ and where x^* is the operator in H adjoint to x. As $S \subset D(x^*)$ for every $x \in A$ we see that A consists of closable operators in H. In this way we have a connection with the theory of Op^*-algebras (see [18]). However, we consider a more general description of the topology on S and A than given in [18].
As we will see further on the hermitian elements x of A, i.e. $x = x^+$, need not be essentially self-adjoint in H. Also closability of an operator x such that $xS \subset S$, $x^*S \subset S$, does not imply its extendibility. Up to now A has been considered from the algebraic standpoint. Next we consider topological properties of it.

3.6. Lemma

i) The antilinear map $L(S) \ni x \to x^+ \in L(S)$ defined on the domain A is closed with respect to each of the topologies τ_ξ in L(S), (ξ = s,w,wp,p).

ii) The antilinear map $L(S) \ni x \to x \in L(S')$ defined on the domain A is closed with respect to each of the topologies τ_ξ and σ_η in L(S) and L(S') respectively. (ξ,η = s,w,wp,p).

Proof

i) Let $\{x_\alpha\}_{\alpha\in J} \subset A$ and $x_\alpha \to x$, $x_\alpha^+ \to y$, $x,y \in L(S)$ in a toplogy τ_ξ.

In virtue of Proposition 1.17 i) $x'_\alpha \underset{\alpha\in J}{\to} x'$ in $L(S')$ with topology σ_{wp}. Hence for each $s \in S$ we have $jys = w - \lim_{\alpha\in J} jx_\alpha^+ s = w^* - \lim_{\alpha\in J} x'_\alpha js = x' \cdot js$. (2.2. v)). It means that $y \in A$ and $\tilde{y} = x'$.

By Corollary 3.4 ii) $y^+ = x$ hence $x \in A$.

ii) Let $\{x_\alpha\}_{\alpha\in J} \subset A$, $x_\alpha \underset{\alpha}{\to} x$, $x \in L(S)$ and $\tilde{x}_\alpha \underset{\alpha}{\to} z$, $z \in L(S')$ in τ_ξ and σ_η topologies respectively. Then we see that $jxs = w\text{-}\lim_\alpha jx_\alpha s = w^* - \lim_\alpha \tilde{x}_\alpha js = zjs$ for all $s \in S$. Hence $x \in A$ and $\tilde{x} = z$.

Q.E.D.

The above results give us a clue how to introduce a natural topology on A, which makes A a topological *-algebra.

3.7. Corollary

Let ρ be a locally convex topology on A which is stronger than one of the topologies τ_ξ, $\xi = s,w,wp$.

Then A is sequentially ρ-closed if either the map $A \ni x \to x^+ \in L(S)$ is $\rho - \tau_\xi$ continuous or the map $A \ni x \to \tilde{x} \in L(S')$ is $\rho - \sigma_\xi$ continuous.

Proof

The map $x \to x^+$ is τ_ξ closed. Hence it is $\rho - \tau_\xi$ closed as by assumption $\rho > \tau_\xi$. Because this map is continuous its domain A must be sequentially ρ-closed. The latter part of the statement follows similarly.

We used here the sequential completeness of $L(S)$ and $L(S')$ respectively (cf. Proposition 1.16).

Q.E.D.

Now we introduce natural *-topologies for A.

3.8. Definition (see [7])

The locally convex topology ρ_ξ on A is the weakest l.c. topology on A which is stronger than the topology τ_ξ and such that the map $A \ni x \to \tilde{x} \in L(S')$ is still $\rho_\xi - \sigma_\xi$ continuous. (By ξ we denote here any of the indices s,wp,p,w)

The existence of topologies ρ is given in the following:

3.9. Remark

i) The topology ρ_s is given by all seminorms of the form:

$$\rho_{pB} : x \to \sup_{b \in B} p(xb)$$

$$\rho^+_{qB} : x \to \sup_{b \in B'} q(\tilde{x}b)$$

where $B \subset S$ and $B' \subset S'$ are bounded subsets, p and q are seminorms in S and S' respectively.

ii) The topology ρ_p is given by all seminorms of the form:

$$\rho_p : x \to p(xs)$$

$$\rho^+_q : x \to q(\tilde{x}s')$$

where $s \in S$, $s' \in S'$ and p, q are seminorms in S and S' respectively.

iii) The topology ρ_{wp} is given by the seminorms:

$$\rho^{+}_{f,s} : x \to |\langle f \mid xs\rangle|$$

$$\rho^{+}_{f,s} : x \to |\langle \tilde{x}f \mid s\rangle|$$

where $f \in S'$, $s \in S$.

iv) The topology ρ_w is given by the seminorms:

$$\rho_\psi : x \to |\psi(x)|$$

$$\rho^{+}_{\phi} : x \to |\phi(\tilde{x})|$$

where $\psi \in L(S)'$, $\phi \in L(S')'$.

3.10. Proposition

The map $\mathring{A} \ni x \to x^{+} \in A$ is continuous in the topologies ρ_s and ρ_{wp}.

Proof

For all $f \in S'$, $s \in S$, $x \in A$, we have:

$$\langle f \mid x^{+}s\rangle = \langle (x^{+})'f \mid s\rangle = \langle \tilde{x}f \mid s\rangle \quad .$$

(Corollary 3.4 ii)). From this relation the ρ_{wp}-continuity follows.

Observe now that for any pair of bounded subsets $B \subset S$, $B' \subset S'$ we have:

$$\sup_{f\in B'} \sup_{s\in B} |\langle f \mid x^{+}s\rangle| = q_{B,B'}(x^{+}) = \sup_{f\in B'} \sup_{s\in B} |\langle \tilde{x}f \mid s\rangle| = q_{B'B}(\tilde{x}) \quad .$$

Thus the ρ_s-continuity follows.

Q.E.D.

3.11. Remark

In general the map $x \to x^+$ is not ρ_w, ρ_p-continuous, for instance if S is a Hilbert space. However there are examples (see below $S = S_{X,A}$, or $S = \phi$) in which this map is ρ_w-continuous.

3.12. Proposition

Endowed with any of the topologies ρ_ξ, $\xi = s,w,p,wp$ the algebra $A \subset L(S)$ of extendible maps is a locally convex sequentially complete topological algebra. Moreover in the topologies ρ_s and ρ_{wp} A is a l.c. involutive topological algebra.

Proof

We have to notice only that the multiplication is separately ρ-continuous in A. It follows easily from its continuity in each of the topologies τ_ξ and the assumed $\rho_\xi - \sigma_\xi$ continuity of the homomorphism $x \to \tilde{x}$.

Q.E.D.

§ 4. Example

We refer to [10] for a general description of spaces of type $S_{\Phi(A)}$, $T_{\Phi(A)}$.

The simplest example of this construction is the space ϕ of all finite sequences of complex numbers. We discuss it here.

The space $S = \phi$ with the inductive limit topology generated by the increasing family of spaces $\{\mathbb{C}^n, n \in \mathbb{N}\}$ fulfils all assumptions imposed on an initial space. A representation of its dual is the space ω of all sequences of complex numbers, endowed with the Fréchet topology

generated by the countable family of seminorms:

$$\omega \ni \{s(n)\} \to p_k(s) = \left(\sum |s(n)|^2\right)^{\frac{1}{2}} .$$

Following [10] we can describe the inductive limit topology of ϕ by the family of seminorms:

$$(4.1) \qquad q_f(s) = \left(\sum_{n=1}^{\infty} |f(n)s(n)|^2\right)^{\frac{1}{2}} = \|fs\|_{\ell_2}$$

where $s \in \phi$, $f \in \omega$.

The strong topology β in $\phi' = \omega$ is described by the seminorms

$$(4.2) \qquad q_B(s') = \sup_{b \in B} |\langle s' \mid b \rangle| = \sup_{b \in B} \left| \sum_{n=1}^{\infty} s'(n)b(n) \right|$$

where B is a weakly bounded subset of ϕ. We recall that a subset of ϕ is bounded iff it is in a finite dimensional space $\mathbb{C}^n$ and bounded in the euclidean norm of $\mathbb{C}^n$.

A subset B' of $S' = \omega$ is β-bounded if and only if all its finite dimensional projections into $\mathbb{C}^n$ are bounded, (i.e. $B' \subset \omega$ is bounded iff for each $n \in \mathbb{N}$ $B' \cap \mathbb{C}^n$ is bounded in $\mathbb{C}^n$).

We consider the natural embedding $j : S \hookrightarrow S'$ realized by $j(s) = \{\overline{s(n)}\}_{n \in \mathbb{N}} \in \omega$ where $s = \{s(n)\}_{n \in \mathbb{N}} \in \phi$. It is positive.

The Hilbert space associated with it is just ℓ_2.

It is easy to see ([20]) that continuous linear maps from $H = \ell_2$ into $S = \phi$ are of a matrix form. Moreover we have the following description of them:

4.3. Lemma

If $a : H \to S$ is a continuous linear map, $H = \ell_2$, $S = \phi$, then the columns of its matrix $\{a(i,j)\}_{i,j=1,2,..}$ are equally limited i.e. there exists a number $n_0 \in \mathbb{N}$ such that $a(i,j) = 0$ for all $j = 1,2,...$ and all $i > n_0$.

Proof

It follows from the theory of sequence spaces [20] that a continuous map $a : \ell_2 \to \phi \subset \ell_2$ has a matrix representation $\{a(i,j)\}$. Moreover it is easy to see that any matrix mapping from ℓ_2 into the space ϕ must have finite columns. Suppose then that the columns of $\{a(i,j)\}$ are not equally finite. It means that for each $m \in \mathbb{N}$ there exists $n_m \geq m$ such that for some $i(n_m) \in \mathbb{N}$ a $(n_m, i(n_m)) \neq 0$. Then the sequence $s_m = \frac{1}{m} \cdot e_{i(n_m)}$, where $e_k = (0,0,...1,0...) = \{\delta_k(n)\}$, belongs to ℓ_2. Clearly $s_m \underset{m\to\infty}{\to} 0$ in ℓ_2 but $\{as_m\}_{m\in\mathbb{N}} = \{\{\frac{1}{m} a(n_k, i(n_k)\}_{k\in\mathbb{N}}\}_{m\in\mathbb{N}}$ is not bounded in ϕ. Hence the sequence $\{as_m\}_{m\in\mathbb{N}}$ is not convergent $((a\, s_m)(m) = \frac{1}{m} a(n_m, i(n_m)) \neq 0)$.

Contradiction.

Q.E.D.

4.4. Corollary

The above description of continuous mappings from H into S shows that the space $S = \phi$ is not of the type $S_{X,A}$ because there is no continuous injection from ℓ_2 into ϕ (such as e^{-tA} in the $S_{X,A}$ case).

4.5. Corollary ([20])

i) For $S = \phi$, $x \in L(S)$ if and only if it has a matrix form with finite columns.

ii) $y \in L(S')$, where $S = \phi$, $S' = \omega$, if and only if y has a matrix form with finite rows.

4.6. Proposition

An element $x \in L(S)$ is j-extendible for the natural embedding $j : \phi \subset\!\!\rightarrow \omega$ if and only if it has a matrix form with finite rows and columns.

Proof

Let us notice that if x has a matrix representation then its j-extension, if it exists, has also the same matrix form, with complex conjugate entries.

Because $x : \phi \to \phi$, it has finite columns. On the other hand because $\tilde{x} : \omega \to \omega$ so $\tilde{x}$ has finite rows ([20]). Since $\tilde{x} = \{\overline{x(i,j)}\}_{i,j=1,2}$, where $x = \{x(i,j)\}_{i,j}$, the result follows.

Q.E.D.

The description of the j-extendible map algebra A is very simple here: A consists of all matrices of finite rows and columns. The involution is just the hermitian adjoint of matrices.

The topologies ρ_s, ρ_{wp} and ρ_p are equivalent because they are described by seminorms on finite dimensional subspaces.

A sequence $\{x_n\} \subset A$ is ρ_s, ρ_p, ρ_{wp}-convergent if the sequences $\{P_k\ x_n\ P_k\}_{n\in\mathbb{N}}$ and $\{P_k\ x_n^+\ P_k\}_{n\in\mathbb{N}}$ are convergent in the algebra of $k \times k$ matrices $M_{k\times k}$, for all $k \in \mathbb{N}$.

The above example is interesting as a source of counterexamples in the general theory.

4.7. Remark

The assumption that an element $A \in A$ is hermitian, i.e. $A^+ = A$, is not sufficient for essential self-adjointness of A as an operator in H with the domain $D(A) = S$.

Proof

It is easy to see this in the general case of an initial space. The well known example of $S = D(\Omega)$, $\Omega \subset \mathbb{R}^+$ and the operator $A = i\frac{d}{dx}$ shows that an operator can be symmetric on S but not essentially selfadjoint.

In order to show this in the case of the space ϕ of all finite seuqences we consider the following matrix:

$$A = i\begin{bmatrix} 0, & -a_1, & 0 & & \\ a_1, & 0, & -a_2 & & \\ 0, & a_2, & 0 & & \\ \cdots\cdots\cdots\cdots & & a_n & 0 & -a_{n+1} \\ & & \cdots\cdots\cdots\cdots & & \\ & & & & \cdots\cdots \end{bmatrix}$$

where $a_n = (n+1)!\ \sum_{k=0}^{n} \frac{1}{k!}$. Clearly $A : \phi \to \phi$, $A \in L(S)$, $A^+ = A$.

However, A, as an operator in ℓ_2 with the domain $D(A) = \phi$, is not essentially selfadjoint. Indeed, the vector $x = (x_1, x_2, \ldots)$ defined by $x_1 = 1$, $x_n = \frac{1}{a_{n-1}} x_{n-1} + \frac{a_{n-1}}{a_{n-2}} x_{n-2}$ where $x_0 = 0$, $a_0 = 1$, fulfils the following:

$$x \in D(A^*) \quad A^* x = Ax = -ix \ .$$

To see that $x \in \ell_2$ we observe that

$$|x_n| \leq \max(x_{n-1}, x_{n-2}) \cdot \left(\frac{1}{a_{n-1}} + \frac{a_{n-1}}{a_{n-2}}\right) = \max(x_{n-1}, x_{n-2}) \cdot \frac{1}{n} \ .$$

By induction it follows that $|x_n| < \frac{1}{n}$.

Now for any $s \in \phi$ $\langle x \mid As \rangle = \langle A^+ x \mid s \rangle = \langle Ax \mid s \rangle = i\langle x \mid s \rangle$, hence $x \in D(A^*)$.

Q.E.D.

We notice that the above operator A has a cyclic vector $e_1 = (1,0,0,\ldots)$. Namely for each $n \in \mathbb{N}$ $e_n = P_n(A)e_1$ where P_n are polynomials fulfilling the recurrence: $P_{n+1}(A) = \frac{a_{n-1}}{a_n} P_{n-1}(A) - \frac{i}{a_n} A\, P_n(A)$, with $P_1(A) = 1$, $P_2(A) = -\frac{i}{4} A$.

It is also easy to see that the operator A has selfadjoint extensions. Let C be the conjugation in H defined by $C\lambda e_k = (-1)^{k-}\bar{\lambda} e_k$. We have $C : \phi \to \phi$ and $CA = AC$, thus using the theorem of von Neumann we obtain the existence of self-adjoint extensions of A.

4.8. Remark

No element of A is closed on $S = \phi$ as an operator in the Hilbert space $H = \ell_2$.

Proof. Let $x \in A$. Then its matrix form has finite rows and columns. Let for each $j \in \mathbb{N}$ $N(j)$ denote the length of j-th column. Let us choose a sequence of non-zero columns of x with indices j_k such that the j_k-th column begins below the end of j_{k-1}-th column, i.e. $x(i,j_k) = 0$ for all $i \leq N(j_{k-1})$ where $x(i,j_{k-1}) = 0$ for all $i > N(j_{k-1})$. Let us consider a vector $h \in H = \ell_2$ such that $h(j) = 0$ for $j \neq j_1, j_2, \ldots$. The action of x on h can be reduced to the action of the matrix $\hat{x}$ consisting only of j_k-th columns of x and such that its rows contain at most one non-zero element of each j-th column. So we may consider only those $x \in A$ which already have this form.

Let $x(j) \overset{df}{=} \max_{N(j-1)<i\leq N(j)} |x(i,j)|$.

Put

$$h(j) \overset{df}{=} \frac{1}{x(j)+1} \cdot \frac{1}{N(j)^{\frac{1}{2}}} \cdot \frac{1}{2^{j/2}} .$$

Then $h \in H = \ell_2$, because $|h(j)|^2 \leq \frac{1}{2^j}$. To show that $xh \in H$ we notice that:

$$(xh)(i) = x(i,j_i)h(i)$$

where j_i is the index of this column of x to which the only non-zero element of the i-th row belongs.

We have

$$j_{N(j_i)} = \ldots = j_{i-1} = j_i = \ldots = j_{N(j_i-1)}$$

and so

$$\|xh\|^2 = \sum_{i=1}^{\infty} |(xh)(i)|^2 \leq \sum_{i=1}^{\infty} \frac{1}{N(j_i)} \frac{1}{2^{j_i}} \leq \sum_{k=1}^{\infty} N(k) \frac{1}{N(k)} \cdot \frac{1}{2^k} < \infty$$

i.e. $xh \in H = \ell_2$.

Now let us take the sequence $\{h_n\} \subset \phi$ defined by:

$$h_n(i) = \begin{cases} h(i) & i \leq n \\ 0 & i > n \end{cases}$$

The sequence $\{h_n\}$ converges in H to h.

Moreover $xh_n \to xh$ in H, but $h \notin \phi = D(x)$, i.e. x is not closed on $D(x)$.

Q.E.D.

§ 5. Relations between embeddings and extendibility

In the previous sections we have considered a fixed embedding $j : S \hookrightarrow S'$ which gave rise to the algebra of extendible maps in $L(S)$. However this construction essentially depends on the choice of j.

The following example shows that in general for two different embeddings the corresponding algebras of extendible maps are essentially different.

5.1. Example

Let $S = \phi$, $S' = \omega$ as in Sect. 4. Let j denote the natural embedding $\phi \subset \omega$. Let

$$\sigma_0 = \begin{bmatrix} 1 & 1 & 1 & 1 & \dots \\ 1 & 2 & 0 & 0 & \dots \\ 1 & 0 & 2^2 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \\ & & & & 2^n \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

be an infinite matrix. Then the associated linear map $\sigma_0 : \phi \to \omega$ is (τ,β)-continuous. Let $\sigma(s) = \{(\sigma_0 \bar{s})(n)\}$ be the anti-linear embedding connected with σ_0. The antilinear mapping $\sigma : \phi \to \omega$ is a positive embedding, fulfilling definition (2.1), because $\det(P_n \sigma_0 P_n) = 2^{\frac{n(n-1)}{2}} > 0$ for all $n \in \mathbb{N}$ and $\langle \sigma(s) \mid s \rangle > 0$ iff $s \neq 0$.

We can construct the algebra of σ-extendible maps $A_\sigma \subset L(S)$.

Let us take $a \in L(S)$ of the form

$$a = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

i.e. the projection on e_1 in ℓ_2. Clearly a is j-extendible. Suppose that $\sigma \circ a = \tilde{a} \circ \sigma$ for some $\tilde{a} \in L(S')$. Then $\tilde{a}$ is of a matrix form ([20], Prop. 1.2 p.98) as a continuous map $\tilde{a} : \omega \to \omega$.

Let $\tilde{a} = \{\tilde{a}(i,j)\}_{i,j\in\mathbb{N}}$. Then for each $s \in S$ $\tilde{a} \circ \sigma(s) = \{(\tilde{a}\sigma_0\bar{s})(n)\}_{n\in\mathbb{N}}$ i.e. $\tilde{a} \circ \sigma$ is described by the matrix $\tilde{a} \circ \sigma_0$. But

$$\tilde{a} \circ \sigma_0 = \begin{bmatrix} \sum_{i=1}^{\infty} \tilde{a}(1,i), & \tilde{a}(1,1) + 2\tilde{a}(1,2), \ldots, \tilde{a}(1,1) + 2^{n-1}\tilde{a}(1,n), \ldots \\ \sum_{i=1}^{\infty} \tilde{a}(2,i), & \tilde{a}(2,1) + 2\tilde{a}(2,2), \ldots \ \ldots \\ & \text{etc.} \end{bmatrix}$$

and

$$\sigma_0 a = \begin{bmatrix} 1 & 0 & 0 & 0 & \ldots \\ 1 & 0 & 0 & \ldots & \\ 1 & & & & \\ 1 & & & \text{etc.} & \end{bmatrix}$$

Hence

$$1 = \sum_{i=1}^{\infty} \tilde{a}(1,i) = \tilde{a}(1,1)\left(1 - \sum_{n=1}^{\infty} \frac{1}{2^n}\right) = 0,$$

which is a contradiction. It means that a is not σ-extendible.

This example shows the need of studying relations between different embeddings.

The following result gives us an useful approximation of extendible maps:

5.2. Proposition

Let $j : S \subset\!\!\rightarrow S'$ be a positive embedding, H_j the Hilbert space associated with j, $A_j \subset L(S)$ the algebra of j-extendible elements. Let P_s be the orthogonal projection on a vector $s \in S$ in H_j. Then

i) for each $s \in S$ $P_s \in A_j$.

ii) the set $\{P_s : s \in S\}$ is total in A_j with respect to the topology ρ_{wp}.

Proof

i) For any $s \in S$, $h \in H_j$ we have

$$P_s h = (s \mid h)s \ .$$

In particular for $h \in S$ we have

$$P_s h = \langle j(s) \mid h \rangle s \in S .$$

It is clear that $P_s \in L(S)$.

Now consider $(j \circ P_s)(h) = \overline{\langle j(s) \mid h \rangle} j(s) = \langle j(h) \mid s \rangle j(s) = (\tilde{P}_s \circ j)(h)$

where $\tilde{P}_s z = \langle z \mid s \rangle j(s)$ for all $z \in S'$.

We have $\tilde{P}_s \in L(S')$ and thus $P_s \in A_j$.

ii) Let $a \in A_j$. Any ρ_{wp}-neighbourhood of a is determined by finite sequences $\{s_1,\ldots,s_n\} \subset S$ and $\{f_1,\ldots,f_n\} \subset S'$.

Consider the at most 3n dimensional subspace X of H_j spanned by $\{s_1,\ldots,s_n, as_1,\ldots,as_n, a^+s_1,\ldots,a^+s_n\}$.

Let $S_n = \text{span}\{s_1,\ldots,s_n\}$. Then $a\big|_{S_n} : S_n \to X$ and the operator $a\big|_{S_n} \oplus 0\big|_{X\setminus S_n}$ has a representation in the Hilbert space X of H_j in the form of a finite linear combination of one-dimensional projections onto some elements of X i.e. for any $z \in S_n$ $az = \sum\sigma_k P_{z_k} z$ $z_k \in X$. In particular $\langle f_i \mid (a - \sum\sigma_k P_{z_k})s_i\rangle = 0$ $\langle f_i \mid (a^+ - \sum\bar{\sigma}_k P_{z_k})s_i\rangle = 0$.

Q.E.D.

Let u be a densely defined linear map from S' into S, with β-dense domain $D(u) \subset S'$.

Let $D(u^t) := \{g \in S' : f \to \langle g \mid uf\rangle$ is β-continuous $f \in D(u)\}$.

We can define the transposition u^t of u by $\langle f \mid u^t g\rangle = \langle g \mid uf\rangle$ for all $f \in D(u)$ $g \in D(u^t)$ as a linear map $u^t : D(u^t) \to S$.

If $D(u^t)$ is dense in S' then we have $u \subset u^{tt}$ i.e. for all $h \in D(u)$, $f \in D(u^t)$

$$\langle f \mid uh\rangle = \langle f \mid u^{tt}h\rangle = \langle h \mid u^t f\rangle.$$

Similarly we define transposition of maps from S into S'. As a particular case we will consider the maps $j : S \to S'$ and $j^{-1} : j(S) \to S$.

We define the antilinear transposed map, $j^t : S \to S'$ of j by:

$\langle j^t s \mid z\rangle \overset{df}{=} \overline{\langle jz \mid s\rangle}$.

Thus we have $\langle j^t(s) \mid z\rangle = \langle j(s) \mid z\rangle$ i.e. $j^t = j$.

Defining $D(j^{-1}) = j(S)$ we have $D((j^{-1})^t) = \{g \in S' : f \to \langle g \mid j^{-1}f\rangle$ is β-continuous for $f \in j(S)\}$.

Let $z,h \in j(S)$ and let $h = j(s)$ for some $s \in S$. Then

$$\langle h \mid j^{-1}(z)\rangle = \langle j(S) \mid j^{-1}(z)\rangle = \overline{\langle j(j^{-1}(z)) \mid s\rangle} = \overline{\langle z \mid s\rangle}.$$

Thus we see that the map $j(S) \ni z \to \langle h \mid j^{-1}(z)\rangle$ is β-continuous i.e. $h \in D((j^{-1})^t)$. Hence $j(S) \subset D((j^{-1})^t)$.

On the other hand let $h \in D((j^{-1})^t)$ and let $v = (j^{-1})^t h$. Consider the map $j(S) \ni z \to \langle h \mid j^{-1}(z)\rangle$. It is β-continuous in virtue of the definition of $D((j^{-1})^t)$. So $\langle h \mid j^{-1}(z)\rangle = \overline{\langle z \mid v\rangle}$ for all $z \in j(S)$. For each $s \in S$ we have $\langle j(v) \mid s\rangle = \langle j((j^{-1})^t h) \mid s\rangle =$ $= \overline{\langle j(s) \mid (j^{-1})^t h\rangle} = \langle h \mid j^{-1}j(s)\rangle = \langle h \mid s\rangle$. Thus $h = j(v) \in j(S)$ i.e. $D((j^{-1})^t) \subset j(S)$ and $(j^{-1})^t = j^{-1}$.

In this way we proved

5.3. Lemma

If $j : S \to S'$ is a positive embedding then $j^t = j$ and $(j^{-1})^t = j^{-1}$.

Let us now consider two positive embeddings $j_1 : S \to S'$ and $j_2 : S \to S'$, and associated with them algebras of extendible maps A_1, A_2. We discuss now certain technical results concerning extendibility with respect to these two embeddings.

5.4. Lemma

Let j_1, j_2 be two positive embeddings of S into S'. Then the following conditions are equivalent:

i) For each $s \in S$ the orthogonal projection P_s on the vector s in the Hilbert space H_{j_1} is j_2-extendible..

ii) $j_1(S) \subset j_2(S)$.

iii) The map $\Delta_{12} = j_1 \circ j_2^{-1}$ is β-continuous on $j_2(S)$ in S'.

Proof

In virtue of lemma 5.3 ii) is equivalent to the condition

$j_1(S) \subset D((j_2^{-1})^t)$.

i) $\Rightarrow$ ii). Let $s \in S$ and P_s be j_2-extendible then there exists $\tilde{P}_s \in L(S')$ such that: $j_2 \circ P_s = \tilde{P}_s \circ j_2$. For each $s,z \in S$ we have:

$$(5.5) \qquad j_2 P_s z = \overline{\langle j_1(s) \mid z\rangle} j_2(s) = \overline{\langle j_1(s) \mid j_2^{-1} j_2(z)\rangle} j_2(s) = \tilde{P}_s j_2(z).$$

From the assumed β-continuity of $\tilde{P}_s$ on $j_2(S)$ we have

$j_1(s) \in D((j_2^{-1})^t) = j_2(S)$.

ii) $\Rightarrow$ i). We have $j_2^{-1} = (j_2^{-1})^{tt}$. Then using again the equalities 5.5 we obtain continuity of the map:

$$j_2(S) \ni h \to \overline{\langle h \mid (j_2^{-1})^t j_1(s)\rangle} j_2(s) = j_2 P_s j_2^{-1}(h) = \tilde{P}_s h .$$

In this way, by the continuity, $\tilde{P}_s$ extends to the whole S' and we have $\tilde{P}_s \in L(S')$. Hence P_s is j_2-extendible.

ii) $\Rightarrow$ iii). By the definition of $D((j_2^{-1})^t) = j_2(S)$ the following map is β-continuous for each $s \in S$:

$$j_2(S) \ni h \to \overline{\langle j_1(s) \mid j_2^{-1}(h)\rangle} = \overline{\langle j_1(j_2^{-1}h) \mid s\rangle} .$$

It means that the linear map

$$\Delta_{12} = j_1 \circ j_2^{-1} : j_2(S) \to S'$$

is β-*-weak continuous. Its unique β-*-weak continuous extension on S', denoted also by Δ_{12}, is a β-continuous map in S' i.e. $\Delta_{12} \in L(S')$. (see Lemma 1.15, Remark 1.9).

iii) ⇒ ii) Let $\Delta_{12} \in L(S')$ and let $s \in S$. Then the map

$$j_2(S) \ni h \to \langle j_1(s) \mid j_2^{-1}(h) \rangle$$

is β-continuous because

$$\langle j_1(s) \mid j_2^{-1}(h) \rangle = \overline{\langle j_1(j_2^{-1}h) \mid s \rangle} = \overline{\langle \Delta_{12}h \mid s \rangle}$$

It means that for each $s \in S$ $j_1(s) \in D((j_2^{-1})^t) = j_2(S)$.

Q.E.D.

5.6. Proposition

Let j_1, j_2 be two positive embeddings. Let A_1, A_2 be the corresponding algebras of j_1 and j_2-extendible maps. Suppose that $A_1 \subset A_2$.
Then $j_1(S) \subset j_2(S)$.

Proof

By Proposition 5.2 i) every one-dimensional projection P_s on a vector $s \in S$ in H_{j_1} is j_1-extendible. By the assumption it is j_2-extendible and the result follows by Lemma 5.4 ii).

Q.E.D.

There arises the natural problem whether the inclusion $j_1(S) \subset j_2(S)$ implies the inclusion $A_1 \subset A_2$. Although the set $P_1 = \text{lin.span}\{P_s \in B(H_{j_1}); P_s$ is the projection on $s \in S\}$ is ρ_{wp}-dense in A_1, in general it is not

sequentially ρ_{wp}-dense in A_1. On the other hand P_1 is τ_p-dense in $L(S)$. Hence A_1 is ρ_p-dense in A_2, independently of a relation between embeddings.

Let $\tilde{a}$ be the j_1-extension of $a \in A_1$. Then the formal equality:

$$(5.7) \qquad j_2 a = j_2 j_1^{-1} \tilde{a} j_1 j_2^{-1} j_2 = \Delta_{21}\, \tilde{a}\, \Delta_{12}\, j_2$$

shows that the following holds:

5.8. Corollary

$$j_1(S) = j_2(S) \quad \text{iff} \quad A_1 = A_2$$

as sets.

Proof

By Lemma 5.4 the maps $\Delta_{12} = j_1 \circ j_2^{-1}$, $\Delta_{21} = j_2 \circ j_1^{-1}$ are β-continuous, $\tilde{a}$ is β-continuous by definition, thus by (5.7) $\Delta_{21}\, \tilde{a}\, \Delta_{12} \in L(S')$ is the j_2-extension of a. i.e. $A_1 \subset A_2$. The inverse is also true by the same argument.

The converse implication follows from Proposition 5.6.

Q.E.D.

Let us observe that if $j_1(S) \subset j_2(S)$ then it is sufficient to assume continuity of the map $\Delta_{21} = j_2 \circ j_1^{-1}$ on $j_1(S)$ to obtain $j_1(S) = j_2(S)$, i.e. $A_1 = A_2$. Hence the continuity condition imposed on $j_2 \circ j_1^{-1}$ seems to be too strong. Although we know that $j_1(S) = j_2(S)$ if and only if $A_1 = A_2$, we do not know whether $A_1 \subset A_2$ if $j_1(S) \subset j_2(S)$.

We conclude this section with a natural definition of equivalence between embeddings.

5.9. Definition

Two positive embeddings j_1 and j_2 are equivalent iff $j_1(S) = j_2(S)$ (or $A_1 = A_2$).

§ 6. Spectral theory

In this section we present an approach to spectral theory in the algebras of extendible maps on initial spaces.

Considering the particular case that $j(S)$ equals S', we see that all elements of A_j are bounded as operators in the Hilbert space H_j. Moreover we have $A_j = B(H_j)$ i.e. in this case an operator in H_j is j-extendible if and only if it is bounded. It easily follows from the closed graph theorem [21] IV § 8 applied to the map $\hat{j}^{-1} : S'_\beta \to H_j$ that $S = H_j$ as topological vector spaces.

In this case our theory reduces itself to the theory of bounded operators in Hilbert space. The topologies τ_s, τ_{wp}, τ_p, τ_w defined on $L(S) = B(H_j)$ become the norm topology, weak operator and strong operator topologies and weak topology on $B(H_j)$ respectively. The topologies ρ on A_j are now the *-topologies on $B(H_j)$.

This shows that from the topological point of view the theory of algebras of bounded operators in a Hilbert space is a very special case of our theory.

This suggests that we have to look for such a spectral theory of extendible maps which would be compatible with the theory in Hilbert space. We can use some elements of the spectral theory of GB*-algebras [1,2], although the extendible map algebras are not GB*-algebras in general. (see Remark 4.7.)

We will discuss Allan's concept of spectra applied to our case and

compare it with the usual definition of spectra of operators in Hilbert space. In this context we consider also possible definitions of positive elements in A. Henceforth we assume that for a given initial space S we have fixed a positive embedding j. Hence the Hilbert space H and the algebra $A = A_j$ are fixed too.
First we notice an interesting relation between closedness of an operator in H and its continuity in S.

6.1. Lemma

Let S be an initial space for which the general closed graph theorem holds (e.g. S is metrizable or its dual S'_β is metrizable). Let a be a closable operator in H, $S \subset D(a)$ and $a : S \to S$. Then $a \in L(S)$.
If moreover $S \subset D(a^*)$ and $a^* : S \to S$ then $a \in A$.

Proof

At first we will show that a is closed as a mapping acting in S. Let $\{s_\alpha\} \subset S$ be a net converging in S to $s \in S$, and let $\{as_\alpha\}$ converge in S to $h \in S$. Then by Proposition 2.2 $s_\alpha \to s$ in H and $\{as_\alpha\}$ is convergent to h.
By closability of a we have h = as i.e. a is closed map in S. It is defined on the whole S. Thus by Theorem 8.5 [21] Ch. IV we have $a \in L(S)$. Applying this result to a^* we have $a^*|_S \in L(S)$ and then applying Proposition 3.5 to $a^*|_S$ we have $a' = a^*|_S \in L(S')$ i.e. $a \in A$.

Q.E.D.

We recall the definition of bounded elements in topological algebras.

6.2. Definition ([2])

Let A be a topological algebra. Then an element y of A is called bounded if there exists a complex number $\xi \neq 0$ such that the set $\{(\xi y)^n\}_{n\in\mathbb{N}}$ is bounded in A. The subset of all bounded elements of A is denoted by A_0.

In general the set A_0 is neither an algebra nor a vector space. However we have the following result:

6.3. Proposition

Let A be the algebra of extendible maps acting in an initial space S. Let A_0 be its bounded part in the topology ρ_s. Let $x,y \in A_0$ and $xy = yx$. Then:

i) $xy \in A_0$.

ii) $x + y \in A_0$.

Proof

i) Consider xy, let $\xi_1,\xi_2 \in \mathbb{C}^1$ be such that the sets $\{(\xi_1 x)^n\}_{n\in\mathbb{N}}$, $\{(\xi_2 x)^n\}_{n\in\mathbb{N}}$ are bounded in A.

Let $B \subset S$ be a bounded set, then for any seminorm p in S and for each $n \in \mathbb{N}$:

$$\sup_{s\in B} p(\xi_1^n\xi_2^n(xy)^n s) = \sup_{s\in B} p((\xi_1 x)^n(\xi_2 y)^n s) \ .$$

Observe that $B_1 \overset{df}{=} \bigcup_{k=1}^{\infty} ((\xi_2 y)^k B)$ is bounded. Indeed for any seminorm p_1 in S

$$\sup_{t\in B_1} p_1(t) = \sup_{s\in B, n\in\mathbb{N}} p_1((\xi_2 y)^n s) < \infty$$

since $y \in A_0$.

In particular, because $x \in A_0$, we have:

$\sup_{n\in\mathbb{N}} \sup_{t\in B_1} p((\xi_1 x)^n t) < \infty$. It means that xy is a τ_s-bounded element of L(S).

Applying a similar argument to $\tilde{x},\tilde{y} \in L(S')$ we have eventually $\tilde{\tilde{xy}}$ σ_s bounded, hence $xy \in A_0$.

ii) Assume for simplicity that $\xi_1 = \xi_2 = 1$. We have

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

and for any seminorm p in S, $s \in S$

$$p(2^{-n}(x + y)^n s) \leq 2^{-n} \sum_{k=0}^{n} \binom{n}{k} p((x^{n-k}y^k)s) \leq$$

$$\leq \max_{k\leq n} p(x^{n-k}y^k s) \leq \sup_{t\in B_1} \sup_{m\in\mathbb{N}} p(x^m t) \ .$$

Here $B_1 \overset{df}{=} \bigcup_{k=0}^{\infty} y^k B$ and B is any bounded set in S containing s.

Therefore for a given bounded set $B \subset S$ we have:

$$\sup_{s\in B,\ n\in\mathbb{N}} p(((x + y)/2)^n s) < \infty .$$

Applying the same argument to $\tilde{x},\tilde{y} \in L(S')$ we see that the set $\{((x + y)/2)^n\}_{n\in\mathbb{N}}$ is bounded in A, i.e. $x + y \in A_0$.

Q.E.D.

It turns out that the normal elements of A_0 are bounded also as operators in the Hilbert space H:

6.4. Lemma

If $x \in A_0$, $xx^+ = x^+x$ then the closure $\bar{x}$ of the operator x with the domain $D(x) = S$ is bounded in H, i.e. $\bar{x} \in B(H)$.

Proof

Let $\xi \in \mathcal{C}^1$ be such that $\{(\xi x)^n\}_{n\in\mathbb{N}}$ is ρ_s-bounded in A. i.e. for any seminorm p in S and a bounded subset $B \subset S$ there exists a constant $M_{p,B} > 0$ such that

$$\sup_{n\in\mathbb{N}} \sup_{s\in B} p((\xi x)^n s) \leq M_{p,B} < \infty .$$

In particular for each $s \in S$ such that $\|s\| = 1$

$$\sup_{n\in\mathbb{N}} \|(\xi x)^n s\| \leq M_s < \infty .$$

Because from $\overline{x^+x} \in B(H)$ it follows that $\bar{x} \in B(H)$ and because

$$\|\xi xs\|^2 = |\xi|^2\|xs\|^2 \leq |\xi|^2\|x^+xs\|$$

we can assume that $x = x^+$ by Proposition 6.3.

Put $z = \xi x$. Then

$$\|zs\|^2 \leq \|z^2s\| \leq \|z^4s\|^{\frac{1}{2}} \leq \ldots \leq \|z^{2^{n+1}}s\|^{\frac{1}{2^n}} \leq M_s^{\frac{1}{2^n}}.$$

As the LHS does not depend on n we have: $|\xi|\|xs\| \leq 1$ for all $s \in S$, $\|s\| \leq 1$. Thus by continuity $\|\bar{x}\| \leq \frac{1}{|\xi|}$ and $\bar{x} \in B(H)$.

Q.E.D.

6.5. Remark

i.) The assumption $xy = yx$ in Proposition 6.3 is essential as the following example will show. Let $S = \phi$ (see § 4) and put

$x_n = \begin{pmatrix} 0 & 0 \\ n & 0 \end{pmatrix}$ $n \in \mathbb{N}$.

Put $x = \sum^{\oplus} x_n$. Then clearly $x \in A_0$ as $x^2 = 0$ but $\overline{x + x^+} \notin A_0$ because $(x_n + x_n^+)^2 = \begin{pmatrix} n^2 & 0 \\ 0 & n^2 \end{pmatrix}$. Observe that $\overline{x + x^+} \notin B(H)$ either.

ii) The converse to the Lemma 6.4 is not true: Take $S = \phi$ and $x \in B(H)$ such that $xe_n = e_{n+1}$. Then clearly $x \in A$ but $x \notin A_0$. Indeed: put $f = \{\exp(n^2)\}_{n\in\mathbb{N}} \in \omega$. Then it gives rise to the continuous seminorm p_f in S ([10]):

$$p_f(s) = \|\{\exp(n^2)s(n)\}_{n\in\mathbb{N}}\|_{\ell_2}$$

but for instance:

$$\|fx^ne_1\|_{\ell_2} = \exp(n+1)^2 \not\leq \xi^n \text{ for any } \xi > 0.$$

In this case $x + x^+$ is a normal element of A, $\overline{x + x^+} \in B(H)$ but $x + x^+ \notin A_0$.

6.6. Corollary

If $x \in A_0$ and $x = x^+$ then $\bar{x}^* = \bar{x} \in B(H)$.

Proof

By Lemma 6.4 $\bar{x} \in B(H)$ and it is symmetric on the domain $D(x) = S$ which is dense in H.

Q.E.D.

We recall now Allan's definition of regular set and of spectrum of an element of the topological algebra A.

6.7. Definition

Let A be a topological algebra, A_0 its bounded part, $\mathbb{C}^*$ the extended complex plane, $x \in A$. Then the set:

i) $\rho_0(x) = \{\lambda \in \mathbb{C}^* \mid (\lambda - x)^{-1} \in A_0\}$

is called the "regular set of x".

ii) $\sigma_0(x) = \{\lambda \in \mathbb{C}^* \mid (\lambda - x)^{-1} \notin A_0\} = \mathbb{C}^* - \rho_0(x)$

is called "the spectrum of x with respect to A_0".

iii) $\rho_A(x) = \{\lambda \in \mathbb{C}^* \mid (\lambda - x)^{-1} \in A\}$

is called "the A-regular set of x".

iv) $\sigma_A(x) = \mathbb{C}^* - \rho_A(x)$

is called the "A-spectrum of x".

If $x \in A$ then the regular set and the spectrum of its unique closure $\bar{x}$ in the Hilbert space H will be denoted by $\rho(\bar{x})$ and $\sigma(\bar{x})$ respectively.

The resolvent of $x \in A$ is the map: $\mathbb{C}^* \ni \lambda \to R_\lambda = (\lambda - x)^{-1}$.

6.8. Proposition ([2] Theorem 3.8)

i) If $\mathbb{C}^* \ni \lambda \to R_\lambda \in A$ is weakly holomorphic in a point $\mu \in \mathbb{C}^*$ then $\mu \in \rho_0(x)$.

ii) ([2], Corollary 3.9). $\sigma_0(x) \neq \emptyset$ and it is closed in $\mathbb{C}^*$.

iii) ([2] Corollary 3.11). Let $K \subset \rho_0(x)$ be a compact subset of $\mathbb{C}^*$. Then there exists a bounded subset B of A such that for each $\lambda \in K$

$$R_\lambda \in \{\xi b \mid \xi \in \mathbb{C}^1, \quad b \in B\} = A(B).$$

The set B is absolutely convex, closed and moreover $B^2 \subset B$, $\P \in B$.

iv) ([2] Theorem 3.12). Let $x \in A$ and let

$$\beta(x) \stackrel{df}{=} \inf\{\xi > 0 \mid \{(x/\xi)^n\}_{n \in \mathbb{N}} \text{ is bounded in } A\}$$

be "the radius of boundedness" and

$$r(x) \stackrel{df}{=} \sup\{|\zeta| \mid \zeta \in \sigma_0(x)\}$$

be "the spectral radius" of x.

Then $\beta(x) = r(x)$.

6.9. Remark

In [2] it has been assumed that A is pseudo-complete. The algebra

A of extendible maps is pseudo-complete because it is ρ_s-sequentially complete (see Proposition 3.12 and [2] Proposition 2.6).

6.10. Corollary

i) $x \in A_0$ iff $\sigma_0(x)$ is bounded.

ii) For each $x \in A$ $\sigma_0(x^+) = \overline{\sigma_0(x)}$ (complex conjugate).

Proof

i) follows from Proposition 6.8 i.v).

ii) follows from the equality:

$$((\lambda - x)^{-1})^+ = (\bar{\lambda} - x^+)^{-1} \quad \text{for} \quad \lambda \in \rho_0(x).$$

Q.E.D.

6.11. Corollary ([2] Theorem 1.4)

If the map $A \ni x \to x^{-1} \in A$ is ρ_s continuous then $\sigma_0(x) = \overline{\sigma_A(x)}^{cl}$ where the closure is taken in $\mathbb{C}^*$.

As we mentioned before one should not expect that elements of A were closed as operators in H with domain S (See Remark 4.8). On the other hand all elements of A are closable and the spectra of their closures are well defined ([23]).

6.12. Theorem

Let A be the algebra of extendible maps in a test space S. Then

i) For each $x \in A$ $\sigma_A(x) \subset \sigma_0(x)$.

ii) If $x^+x = xx^+$ then $\sigma(\bar{x}) \subset \sigma_0(x)$.

Proof

i) follows from the inclusion $\rho_0(x) \subset \rho_A(x)$.

ii) Let $\lambda \in \rho_0(x)$. Then $(\lambda - x)^{-1} \in A_0$.

Since $(\lambda - x)^{-1}$ is normal in A and since x is closable in H, it follows from Lemma 6.4 that $(\lambda - \bar{x})^{-1} \in B(H)$. (See [23] Theorem 5.2 p. 90). and so $\lambda \in \rho(\bar{x})$.

Hence $\sigma(\bar{x}) \subset \sigma_0(x)$.

Q.E.D.

The relation $\sigma(\bar{x}) \subset \sigma_0(x)$ shows that the description of x as an element of A gives too little information about its Hilbert space spectral properties. However it seems to be interesting whether there exist conditions giving the relation $\sigma(\bar{x}) = \sigma_0(x)$.
In the particular case of the initial space ϕ (see § 4) we have found such conditions:

6.13. Theorem

Let $S = \phi$ (see § 4) and let Λ be the algebra of extendible maps associated with the natural embedding $\phi \subset \omega$. Then:

i) $x \in A_0$ if $x \in A$, $\bar{x} \in B(H)$ and $\sigma_0(x) \subset \sigma(\bar{x})$.

ii) $\bar{x} \in B(H)$ and $\sigma(\bar{x}) = \sigma_0(x)$ if $x \in A_0$ and $xx^+ = x^+x$.

Proof

i) We have $\rho(\bar{x}) \subset \rho_0(x)$.

We can choose a compact set $K \subset \rho(\bar{x}) - \{0\}$ with non-empty interior and such that the series $(\lambda - \bar{x})^{-1} = \sum (\frac{1}{\lambda})^{n+1} (\bar{x})^n$ converges in norm in $B(H)$ for all $\lambda \in K$.

By the assumption $K \subset \rho_0(x)$ hence for each $\lambda \in K$, $s \in S$ $R_\lambda s = (\lambda - x)^{-1} s = \sum (\frac{1}{\lambda})^{n+1} x^n s$ and $R_\lambda \in A_0$. Moreover by Proposition 6.8 iii) there exists the bounded set $B_k \subset A$ such that $R_\lambda \in A(B_k)$ for all $\lambda \in K$. It follows that for any $p \in \mathbb{N}$ there exists $m(p) \in \mathbb{N}$ such that $R_\lambda s \in \mathbb{C}^{m(p)}$ for all $\lambda \in K$ and $s \in \mathbb{C}^p$.

For any $z,s \in S$ the function

$$\xi \to (z \mid R_{\frac{1}{\xi}} s)_{\ell_2} = \sum_{k=0}^{\infty} \xi^{n+1} (z \mid x^n s) \overset{df}{=} F_{z,s}(\xi)$$

is defined and analytic on $K^{-1} = \{\xi \in \mathbb{C}^1 \mid \frac{1}{\xi} \in K\}$ and it can be analytically continued on some open disc containing 0.

Taking $z \in S, z \perp C^{m(p)}$ in ℓ_2 we have $F_{z,s}(\xi) = 0$ for all $\xi \in K^{-1}$ and hence for each $n \in \mathbb{N}$ $(z \mid x^n s)_{\ell_2} = 0$. It means that for each $n \in \mathbb{N}$, $s \in \mathbb{C}^p$ we have $x^n s \in \mathbb{C}^{m(p)}$.

By a similar argument and Corollary 6.10 ii) we have also: $(x^+)^n \in \mathbb{C}^{m(p)}$.

Because each bounded subset $B_1 \subset S$ is contained in some $\mathbb{C}^p$, for $p \in \mathbb{N}$, we see that for all $f \in \omega$.

$$q_{fB_1}\left(\left(\frac{x^{\#}}{\|x\|}\right)^n\right) = \sup_{s\in B_1} \|f\left(\frac{x^{\#}}{\|x\|}\right)^n s\|_{\ell_2} \leq \max_{j\leq m(p)} |f(j)| \sup_{s\in B_1} \|s\|_{\ell_2} < \infty.$$

Here $x^{\#}$ denotes x or x^+.

It follows that $x \in A_0$.

ii) Suppose that $x \in A_0$ and $xx^+ = x^+x$.

Then by Lemma 6.4 $\bar{x} \in B(H)$.

Let $\lambda \in \rho(\bar{x})$ with $|\lambda|$ sufficiently large. Then the series $\frac{1}{\lambda}\sum_{n=0}^{\infty}(\frac{\bar{x}}{\lambda})^n = (\lambda - \bar{x})^{-1}$ is uniformly convergent in $B(H)$. For a given $p \in \mathbb{N}$ there exists $m(p) \in \mathbb{N}$ such that for each $s \in \mathbb{C}^p$, $n \in \mathbb{N}$ we have $(\frac{x}{\lambda})^n s \in \mathbb{C}^{m(p)}$. Hence we have $(\lambda - x)^{-1}s \in \mathbb{C}^{m(p)}$. By an argument of analyticity we have $(\lambda - x)^{-1}s \in \mathbb{C}^{m(p)}$ for all $\lambda \in \rho(\bar{x})$. Similarly we can show that for all $\lambda \in \rho(\bar{x})$, all $m \in \mathbb{N}$ and each $s \in \mathbb{C}^p$ we have $(\lambda - x)^{-m} \in \mathbb{C}^{m(p)}$.

Using an analogous argument for $x^+ = x^*|_S$ we have $(\bar{\lambda} - x^*)^{-m} \in \mathbb{C}^{m(p)}$.

It is clear that $(\lambda - x)^{-m} : S \to S$ and $(\bar{\lambda} - x^+)^{-1} = (\bar{\lambda} - x^*)^{-m}|_S : S \to S$.

These operators are bounded in H. Hence by Lemma 6.1 $(\lambda - x)^{-m} \in A$.

Let $x^{\#} = x$ or x^+. Then for any $f \in \omega$ and any bounded subset $B \subset S$ with $B \subset \mathbb{C}^p$ we have:

$$\sup_m q_{f,B}\left(\frac{(\lambda - x^{\#})^{-m}}{\|(\lambda - x^{\#})^{-1}\|^m}\right) = \sup_m \sup_{s\in B} \|f \frac{(\lambda - x^{\#})^{-m}}{\|(\lambda - x^{\#})^{-1}\|^m} s\| \leq$$

$$\leq \sup_{j\leq m(p)} |f(j)| \sup_{s\in B} s < \infty.$$

We conclude that $(\lambda - x)^{-1} \in A_0$ i.e. $\lambda \in \rho_0(x)$.

Q.E.D.

6.14. Corollary

For the initial space $S = \phi$ we have:

i) $x \in A$, $\bar{x} \in B(H)$, $xx^+ = x^+x$ and $\sigma_0(x) = \sigma(\bar{x})$ if and only if $x \in A_0$ and $xx^+ = x^+x$.

ii) If $x \in A_0$ $x^+x = xx^+$ then $\sigma_A(x) \subset \sigma(\bar{x})$.

§ 7. Concluding remarks

It seems worthwhile to introduce some order structure in the algebra A, which would be a counterpart of the order structure in von Neumann algebras. However we meet here some ambiguity in defining positive elements. In the case of von Neumann algebras the spectral definition and the definition by means of expectation values are equivalent. That is not the case here. By the example in § 4, Remark 4.7 we see that there exist hermitian elements in A which do not have real spectrum. On the other hand we have the following result for $S = \phi$.

7.1. Proposition

If $x \in A_0$ and $\langle j(s) \mid xs \rangle \geq 0$ for all $s \in S = \phi$, then $\sigma_0(x) \subseteq \mathbb{R}^+$.

Proof

It is obvious that $x = x^+$ hence $\bar{x} \in B(H)$ and $\bar{x}^* = \bar{x}$ by Lemma 6.4 and Corollary 6.6. By Theorem 6.13 ii) we have $\sigma_0(x) = \sigma(\bar{x}) \subset \mathbb{R}$. Positivity follows from the continuity of the scalar product in H.

Q.E.D.

On the other hand the following general result is a corollary to Theorem 6.12.

7.2. Proposition

If $x \in A$, $\sigma_0(x) \subseteq \mathbb{R}^+$ and $x = x^+$ then $\bar{x}^* = \bar{x}$ and $\langle j(s) \mid xs \rangle \geq 0$ for all $s \in S$.

We can say that an element $x \in A$ is "positive" if $\langle j(s) \mid xs \rangle \geq 0$, for all $s \in S$ and it is "spectrally positive" if it is hermitian in A and $\sigma_0(x) \geq 0$.

Hence the spectral positivity implies positivity and in particular cases e.g. $S = \phi$ it is equivalent to the positivity.

We conclude with the following simple result.

7.3. Proposition

Let $x \in A$ then

i) x^+x and xx^+ are positive.

ii) if x is positive then $x^+ = x$.

iii) if $x = x^+$ then $x_+ = \frac{1}{2}(1 + x)^2$ and $x_- = \frac{1}{2}(1 + x)^2 - x$ are positive and $x = x_+ - x_-$.

From the above Proposition it follows that there are many positive elements in A.

Each $x \in A$ can be decomposed into a linear combination of four of them.

This leads to the following open problems.

7.4. Problems

1. In the nuclear case (see [7,8]) continuous linear functionals on A are fully characterized. The same would be useful in the general case of initial spaces.

2. We can define positive elements of $L(S)$, not only of A. How does depend this definition on the choice of the embedding j?

3. Dirac's formalism as formulated in [9] is based on a family of positive maps from S' into S. It is desirable to develop such a scheme in our theory.

4. Physical interpretation can be based on Dirac's idea of an observable, adapted to our scheme. What a physical meaning has the embedding j? Is it possible to characterize those embeddings for which a given algebra of observables is the algebra of extendible maps? What is a characterization of subalgebras of $L(S)$ which are exactly the algebras of extendible maps associated with some positive embedding?

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