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TAIL PROCESSES UNDER HEAVY RANDOM CENSORSHIP WITH APPLICATIONS

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We consider a type of heavy random censoring where the number of uncensored observations still tends to infinity. Under natural conditions the life distribution can be locally analyzed by generalizing tail empirical processes to the heavily censored case. A uniform central limit theorem for the tail product-limit process and the tail empirical cumulative hazard process is established. Statistical applications include a local confidence band for the cumulative life distribution and a test concerning the value of its density at the origin.

Key words: heavy censoring, tail product-limit and tail empirical cumulative hazard process, uniform central limit theorem.

Running title: Heavy random censorship.

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1. INTRODUCTION AND PRELIMINARIES

Most extensively studied in the random censoring literature is the situation where the distribution of the censoring variables is fixed. A fundamental result on the weak convergence of empirical processes under fixed censoring was first obtained in Breslow & Crowley (1974). Besides this standard case extremely heavy censoring is considered in Wellner (1985) where the distribution of the censoring variables tends to the degenerate distribution at 0 at such a high rate that the number of uncensored observations remains bounded and the usual asymptotics cannot be performed. In this note we consider a type of censoring that still will be called "heavy" in the sense that the censoring distribution is again degenerate at 0 in the limit but at a sufficiently slow rate to ensure that the number of uncensored observations tends to infinity so that asymptotic considerations remain possible.

Practically such censoring might provide a realistic alternative to Wellner's (1985) model when items are tested for a certain defect that is only likely to occur in the long run and when a relatively short amount of time is available for testing. Mathematically the ensuing theory is based on the tail empirical process (see, e.g., Einmahl (1992)) generalized so as to allow for censored observations. The weak convergence of the thus obtained processes will be presented in Section 2. Two statistical applications are considered in Section 3. First we construct a confidence band for the cumulative life distribution function of interest, locally, near 0. Asymptotically the data contain information about the density of the life distribution only at the origin. Next we consider estimation of the density at the origin and the related problem of testing on the slope of the cumulative life distribution at the origin. We conclude this introduction with a specification of the assumptions.

For each $n \in I\!N$ let $(X_1, Y_{n1}), \ldots, (X_n, Y_{nn})$ be independent random vectors with $X_i \perp\!\!\!\perp Y_{ni}$ for each $i = 1, \ldots, n$. We observe the random variables $Z_{ni} \wedge Y_{ni}, \delta_{ni} = 1\!\!1_{\{X_i \leq Y_{ni}\}}$, $i = 1, \ldots, n$. The X_i represent the life time of interest and are consequently nonnegative; moreover, they are i.i.d. with common continuous cumulative distribution function (c.d.f.) F that does not depend on n. For one of the applications in Section 3 it will be required that F has a continuous second derivative in a right neighborhood of 0. The censoring variables Y_{ni} are also nonnegative and i.i.d. but with a common continuous c.d.f. G_n that *does* depend on n. More specifically we assume the existence of a continuous c.d.f. G such that

$$(1.1) \qquad G_n(s) = G(s/a_n) \;\; \textit{for} \;\; 0 \leq s \leq a_n, G(1) < 1 \; ,$$

for some sequence of strictly positive numbers $(a_n)_{n \in I\!N}$ satisfying

$$(1.2) \qquad a_n \to 0 \;\; \textit{and} \;\; nF(a_n) \to \infty, \;\; \textit{as} \;\; n \to \infty \; .$$

A further condition, rather natural in this context, is that F be regularly varying at 0, meaning that

$$(1.3) \qquad \lim_{x \downarrow 0} \frac{F(xt)}{F(x)} = \widetilde{F}(t), t \geq 0 \; .$$

In addition we want $\widetilde{F}$ to be a c.d.f. on $[0,1]$ which necessarily entails that

$$(1.4) \qquad \widetilde{F}(t) = t^{\gamma}, 0 \leq t \leq 1, \;\; \textit{for some} \;\; \gamma > 0 \; .$$

2. WEAK CONVERGENCE OF CENSORED TAIL PROCESSES

We need to start this section with a short review of some basic concepts and relations and some further notation for which the reader is referred to Shorack & Wellner (1986). We are interested in estimating the left-hand tail of F. For this purpose we may use the product-limit or Kaplan-Meier estimator

$$(2.1) \qquad \widehat{F}_n(t) = 1 - \prod_{i:\widetilde{Z}_{ni} \le t} \left(1 - \frac{1}{n-i+1}\right)^{\widetilde{\delta}_{ni}}, t \ge 0 ,$$

where the $\widetilde{Z}_{ni}$ denote the ordered Z_{ni} and $\widetilde{\delta}_{ni}$ the corresponding δ_{ni}. This estimator is not in general unbiased. Another option is indirect estimation via the cumulative hazard function, of interest in its own right, defined by

$$(2.2) \qquad \Lambda(t) = \int_0^t \frac{1}{1-F^-(s)} dF(s), \ t \ge 0 ,$$

where for any right-continuous function with left-hand limits $\Psi : [0,\infty) \to \mathbb{R}$ we write Ψ^- for the left-continuous version. The c.d.f. F can be recovered from Λ according to $F(t) = 1 - \exp(-\Lambda(t))$, $t \ge 0$. Hence an estimator of Λ yields an estimator of F.
In order to describe a natural estimator of Λ let us introduce the notation

$$(2.3) \qquad H_n(t) = \mathbb{P}\{Z_{ni} \le t\}, \ \widehat{H}_n(t) = \tfrac{1}{n}\sum_{i=1}^n \mathbb{1}_{[0,t]}(Z_{ni}), \ t \ge 0 ,$$

$$(2.4) \qquad H_n^*(t) = \mathbb{P}\{Z_{ni} \le t, \delta_{ni} = 1\}, \ \widehat{H}_n^*(t) = \tfrac{1}{n}\sum_{i=1}^n \delta_i \mathbb{1}_{[0,t]}(Z_{ni}), \ t \ge 0 ,$$

and the empirical processes

$$(2.5) \qquad U_n(t) = \sqrt{n}(\widehat{H}_n(t) - H_n(t)), \ U_n^*(t) = \sqrt{n}(\widehat{H}_n^*(t) - H_n^*(t)), \ t \ge 0 .$$

Since

$$(2.6) \qquad \Lambda(t) = \int_0^t \frac{1}{1-H_n^-(s)} dH_n^*(s), \ t \ge 0 ,$$

let us consider the estimator

$$(2.7) \qquad \widehat{\Lambda}_n(t) = \int_0^t \frac{1}{1-\widehat{H}_n^-(s)} d\widehat{H}_n^*(s), \ t \ge 0 .$$

Note that, indeed, $\widehat{H}_n$ and $\widehat{H}_n^*$ are known from the data.
We are now ready to introduce the tail product-limit process

(2.8) $$\xi_n(t) = \sqrt{\frac{n}{F(a_n)}}(\widehat{F}_n(ta_n) - F(ta_n)),\ 0 \le t \le 1\ ,$$

and the tail empirical cumulative hazard process

(2.9) $$\beta_n(t) = \sqrt{\frac{n}{F(a_n)}}(\widehat{\Lambda}_n(ta_n) - \Lambda(ta_n)),\ 0 \le t \le 1\ .$$

Furthermore let

(2.10) $$D(t) = \int_0^t \frac{1}{1 - G(s)} ds^{\gamma},\ 0 \le t \le 1\ ,$$

denote a suitable time transformation, well defined because $G(1) < 1$, and W a standard Wiener process. Although the weak convergence of the processes β_n, established below, is a basic ingredient for statistical inference about the cumulative hazard function and the hazard rate, here we will only use the result to conveniently deal with the related but more complicated processes ξ_n. For statistical application we restrict ourselves to those related to the latter processes.

THEOREM 2.1. *Under assumptions* (1.1)-(1.4) *there exist a special construction of the processes* $\xi_n, \beta_n (n \in I\!N)$ *and* W*, defined on one and the same probability space such that*

(2.11) $$\sup_{0 \le t \le 1} |\xi_n(t) - W \circ D(t)| \to_p 0,\ as\ n \to \infty\ ,$$

(2.12) $$\sup_{0 \le t \le 1} |\beta_n(t) - W \circ D(t)| \to_p 0,\ as\ n \to \infty\ .$$

PROOF. Let us first prove (2.12). It is immediate from Shorack & Wellner (1986, formula (16) of Section 7.1, Theorem 1 of Section 7.2), and (2.6) that β_n can be rewritten as

(2.13) $$\beta_n(t) = \frac{U_n^*(ta_n)}{\sqrt{F(a_n)}(1 - H_n^-(ta_n))} - \int_0^{ta_n} \frac{U_n^*(s)}{\sqrt{F(a_n)}} d\frac{1}{1 - \widehat{H}_n^-(s)}$$

$$+ \int_0^{ta_n} \frac{U_n^-(s)}{\sqrt{F(a_n)}(1 - \widehat{H}_n^-(s))} d\Lambda(s),\ 0 \le t \le 1\ ,$$

provided that $a_n \le \widetilde{Z}_{nn}$. The obvious relation $H_n = 1 - (1 - F)(1 - G_n)$ and (1.1) and (1.2) entail that $H_n(t) = G(t/a_n) + F(t)(1 - G(t/a_n))$ and hence that $H_n(ta_n) = G(t) + F(ta_n)(1 - G(t)) \to G(t), 0 \le t \le 1$, as $n \to \infty$. Hence it follows from standard empirical process theory that $\{U_n^-(ta_n), 0 \le t \le 1\}$ converges weakly to $\{B \circ G(t),\ 0 \le t \le 1\}$ on $D[0,1]$, as $n \to \infty$, where B is a standard Brownian bridge. This yields

(2.14) $$\sup_{0 \le t \le 1} \left| \int_0^{ta_n} \frac{U_n^-(s)}{\sqrt{F(a_n)}(1 - \widehat{H}_n^-(s))} d\Lambda(s) \right|$$

$$= \sup_{0\le t\le 1}\left|\int_0^t \frac{U_n^-(sa_n)}{1-\widehat{H}_n^-(sa_n)}d\frac{\Lambda(sa_n)}{\sqrt{F(a_n)}}\right|$$

$$= \frac{\Lambda(a_n)}{\sqrt{F(a_n)}}O_p(1) = \sqrt{F(a_n)}O_p(1) = o_p(1),\ as\ n\to\infty\ ,$$

employing (2.2).

So we can focus on the first two terms on the right in (2.13). Another well-known relation, $H_n^*(t) = \int_0^t (1-G_n(s))dF(s), 0\le t\le 1$, yields $H_n^*(ta_n)/F(a_n) \to \int_0^t (1-G(s))ds^\gamma = \widetilde{D}(t)$, $0\le t\le 1$, as $n\to\infty$. It has been shown in, e.g., Einmahl & Koning (1992) that $U_n^*(ta_n)$ behaves like an ordinary empirical process based on n i.i.d. observations from the c.d.f. $H_n^*(ta_n)$, which entails that $U_n^*(ta_n)/\sqrt{F(a_n)}$ has variance $H_n^*(ta_n)(1-H_n^*(ta_n))/F(a_n)\to \widetilde{D}(t)$, as $n\to\infty$. This explains that there exist a special construction such that

$$\text{(2.15)}\qquad \sup_{0\le t\le 1}\left|\frac{U_n^*(ta_n)}{\sqrt{F(a_n)}} - \widetilde{W}\circ\widetilde{D}(t)\right| \to_p 0,\ as\ n\to\infty\ ,$$

where $\tilde{W}$ is a standard Wiener process. This implies

$$\text{(2.16)}\qquad \sup_{0\le t\le 1}\left|\frac{U_n^*(ta_n)}{\sqrt{F(a_n)}(1-\widehat{H}_n^-(ta_n))} - \frac{\widetilde{W}\circ\widetilde{D}(t)}{1-G(t)}\right| \to_p 0,\ as\ n\to\infty\ ,$$

because

$$\text{(2.17)}\qquad \sup_{0\le t\le 1}|\widehat{H}_n^-(ta_n) - G(t)| \to_p 0, \text{ as } n\to\infty\ .$$

Using (2.15) again we see that

$$\text{(2.18)}\qquad \sup_{0\le t\le 1}\left|\int_0^{ta_n}\frac{U_n^*(s)}{\sqrt{F(a_n)}}\, d\,\frac{1}{1-\widehat{H}_n^-(s)} - \int_0^t \widetilde{W}\circ\widetilde{D}(s)d\frac{1}{1-\widehat{H}_n^-(sa_n)}\right| \to_p 0,\ as\ n\to\infty\ .$$

Moreover, subtle application of the Helly-Bray theorem (cf. Shorack & Wellner (1986, p. 309)) yields

$$\text{(2.19)}\qquad \sup_{0\le t\le 1}\left|\int_0^t \widetilde{W}\circ\widetilde{D}(s)d\left(\frac{1}{1-\widehat{H}_n^-(sa_n)} - \frac{1}{1-G(s)}\right)\right| \to_p 0,\ as\ n\to\infty\ .$$

Combining (2.13), (2.14), (2.16), (2.18), and (2.19) we obtain

$$\text{(2.20)}\qquad \sup_{0\le t\le 1}\left|\beta_n(t) - \left(\frac{\widetilde{W}\circ\widetilde{D}(t)}{1-G(t)} - \int_0^t \widetilde{W}\circ\widetilde{D}(s)d\frac{1}{1-G(s)}\right)\right| \to_p 0,\ as\ n\to\infty\ .$$

Finally, routine considerations show that the process

$$(2.21) \qquad W\circ D(t) = \frac{\widetilde{W}\circ\widetilde{D}(t)}{1-G(t)} - \int_0^t \widetilde{W}\circ\widetilde{D}(s) d\frac{1}{1-G(s)}, \quad 0 \le t \le 1 ,$$

is a zero mean Wiener process with covariance function $D(s) \wedge D(t)$ for $s, t \in [0,1]^2$ which entails (2.12) using (2.20).
Next let us consider (2.11). We start with the well-known identity

$$(2.22) \qquad \xi_n(t) = (1 - F(ta_n)) \int_0^t \frac{1 - \widehat{F}_n^-(sa_n)}{1 - F(sa_n)} d\beta_n(s), \ 0 \le t \le 1 ;$$

see, e.g., Shorack & Wellner (1986, Proposition 1 of Section 7.2). Using (1.2) and the result (2.12) that we just proved, it suffices to show that

$$(2.23) \qquad \sup_{0\le t\le 1} \left| \int_0^t \frac{1 - \widehat{F}_n^-(sa_n)}{1 - F(sa_n)} d\beta_n(s) - \beta_n(t) \right| \to_p 0, \ as\ n \to \infty .$$

This is equivalent with showing that

$$(2.24) \qquad \sup_{0\le t\le 1} \left| \int_0^t \frac{F(sa_n) - \widehat{F}_n^-(sa_n)}{1 - F(sa_n)} d\beta_n(s) \right| \to_p 0, \ as\ n \to \infty .$$

Integration by parts shows that the expression on the left in (2.24) is bounded by

$$(2.25) \qquad \sup_{0\le t\le 1} \left| \frac{F(ta_n) - \widehat{F}_n^-(ta_n)}{1 - F(ta_n)} \right| |\beta_n(t)|$$

$$+\sup_{0\le t\le 1} \int_0^t |\beta_n(s)| \, d\frac{\widehat{F}_n^-(sa_n)}{1 - F(sa_n)} + \sup_{0\le t\le 1} \int_0^t |\beta_n(s)| \, d\frac{F(sa_n)}{1 - F(sa_n)} .$$

First let us note that

$$(2.26) \qquad \sup_{0\le t\le 1} |\widehat{F}_n(ta_n) - F(ta_n)| \to_p 0, \ as\ n \to \infty .$$

To see this observe that $\widehat{F}_n(ta_n) - F(ta_n)$ equals the expression on the right in (2.22) with $\beta_n(t)$ replaced by $\beta_n^*(t) = \sqrt{\frac{F(a_n)}{n}}\beta_n(t)$. From (2.12) we obtain that the three terms in (2.25) with β_n replaced by β_n^* converge to 0 in probability, so that (2.23) holds true with β_n^* instead of β_n, and hence (2.26) follows.
Now (2.24), and hence (2.11), easily follows by combining (2.26) and the facts that $\sup_{0\le t\le 1}|\beta_n(t)| = O_p(1)$, and $F(a_n) \to 0$, as $n \to \infty$. Q.E.D.

3. A CONFIDENCE BAND AND A TEST

For the construction of a confidence band for F near 0 it turns out that we need to estimate the value of the function D in (2.10) at the point $t = 1$. For this purpose we estimate the c.d.f. $G(s) = G_n(sa_n)$ by $\widehat{G}_n(sa_n), 0 \le s \le 1$, where $\widehat{G}_n$ is the product-limit estimator of G_n. This estimator is obtained by formally considering the Y_{ni} as the variables censored by the X_i, in other wordt $\widehat{G}_n$ is obtained from the expression on the right in (2.1) by replacing the $\widetilde{\delta}_{ni}$ with $1 - \widetilde{\delta}_{ni}$. Estimating γ is essentially the problem of estimating the extreme value index of a c.d.f. in the domain of min-attraction of a c.d.f. of Weibull type. Various choices for the estimator are possible, a particularly simple one being

$$\text{(3.1)} \qquad \widehat{\gamma}_n = \log\ \widehat{F}_n(a_n)/\log\ a_n\ .$$

It is not overly hard to show that

$$\text{(3.2)} \qquad \widehat{\gamma}_n \to_p \gamma,\ \textit{as}\ n \to \infty\ .$$

For us $\widehat{\gamma}_n$ is just *any* estimator satisfying (3.2). Finally we propose

$$\text{(3.3)} \qquad \widehat{D}_n(1) = \int_0^1 \frac{1}{1 - \widehat{G}_n(sa_n)} ds^{\widehat{\gamma}_n}\ ,$$

as an estimator of $D(1)$. As before let W be a standard Wiener process and let $c = c(\alpha)$ be such that

$$\text{(3.4)} \qquad I\!P\{\sup_{0 \le t \le 1} |W(t)| \ge c\} = \alpha,\ 0 < \alpha < 1\ .$$

THEOREM 3.1. *Under assumptions* (1.1)-(1.4) *and* (3.2) *we have*

$$\text{(3.5)} \qquad \lim_{n\to\infty} I\!P \left\{ \widehat{F}_n(t) - c\sqrt{\frac{\widehat{F}_n(a_n)\widehat{D}_n(1)}{n}} < F(t) \right.$$

$$\left. < \widehat{F}_n(t) + c\sqrt{\frac{\widehat{F}_n(a_n)\widehat{D}_n(1)}{n}},\ 0 \le t \le a_n \right\} = 1 - \alpha\ .$$

PROOF. From (2.11) it is immediate that

$$\text{(3.6)} \qquad \sqrt{\frac{n}{F(a_n)}} \sup_{0 \le t \le a_n} |\widehat{F}_n(t) - F(t)| \to_d \sup_{0 \le t \le 1} |W \circ D(t)|,\ \textit{as}\ n \to \infty\ .$$

Noticing that D is an increasing function, mapping [0,1] onto $[0, D(1)]$, it follows that $\sup_{0 \le t \le 1} |W \circ D(t)| = \sup_{0 \le t \le D(1)} |W(t)| =_d \sqrt{D(1)} \sup_{0 \le t \le 1} |W(t)|$ and hence (3.6) implies

$$\text{(3.7)} \qquad \sqrt{\frac{n}{F(a_n)D(1)}} \sup_{0 \le t \le a_n} |\widehat{F}_n(t) - F(t)| \to_d \sup_{0 \le t \le 1} |W(t)|,\ \textit{as}\ n \to \infty\ .$$

The theorem follows if it can be shown that

$$(3.8)\qquad \frac{\widehat{F}_n(a_n)}{F(a_n)} \to_p 1, \textit{ and } \frac{\widehat{D}_n(1)}{D(1)} \to_p 1, \textit{ as } n \to \infty .$$

The first of these statements is immediate from (2.11). For the second one, observe that

$$(3.9)\qquad \widehat{D}_n(1) - D(1) = \int_0^1 \left(\frac{1}{1-\widehat{G}_n(sa_n)} - \frac{1}{1-G_n(sa_n)} \right) ds^{\widehat{\gamma}_n}$$

$$+ \int_0^1 \frac{1}{1-G(s)} d(s^{\widehat{\gamma}_n} - s^{\gamma}) .$$

The product-limit estimator $\widehat{G}_n$ is very close to the empirical c.d.f. for uncensored Y_{ni} and it is easy to prove that $\sup_{0 \le t \le a_n} |\widehat{G}_n(t) - G_n(t)| \to_p 0$, as $n \to \infty$, which entails

$$(3.10)\qquad \left| \int_0^1 \left(\frac{1}{1-\widehat{G}_n(sa_n)} - \frac{1}{1-G_n(sa_n)} \right) ds^{\widehat{\gamma}_n} \right|$$

$$\le \int_0^1 \left| \frac{\widehat{G}_n(sa_n) - G_n(sa_n)}{(1-\widehat{G}_n(sa_n))(1-G(s))} \right| ds^{\widehat{\gamma}_n} = o_p(1) \int_0^1 ds^{\widehat{\gamma}_n} = o_p(1), \textit{ as } n \to \infty .$$

Consequently, to prove the second part of (3.8) it suffices to show that for arbirary $\varepsilon > 0$ and n sufficiently large

$$(3.11)\qquad \mathbb{P} \left\{ \left| \int_0^1 \frac{1}{1-G(s)} d(s^{\widehat{\gamma}_n} - s^{\gamma}) \right| \ge \varepsilon \right\} \le \varepsilon .$$

Without loss of generality we may and will assume that $\widehat{\gamma}_n > 0$. Integration by parts yields

$$(3.12)\qquad \left| \int_0^1 \frac{1}{1-G(s)} d(s^{\widehat{\gamma}_n} - s^{\gamma}) \right| = \int_0^1 |s^{\widehat{\gamma}_n - \gamma} - 1| s^{\gamma} d \frac{1}{1-G(s)} .$$

Let us first note the simple fact that $\sup_{0<\eta\le s\le 1} |s^a - 1| \le |\eta^a - 1|$ for each $a \in \mathbb{R}$. Now let us choose $\delta = \varepsilon / \{4 \int_0^1 s^{\gamma} d((1-G(s))^{-1})\}$ for ε sufficienlty small to ensure that $\delta \le \frac{1}{2}$, and let us define $\widehat{\eta}_n = (1-\delta)^{1/|\widehat{\gamma}_n - \gamma|}$. We have

$$(3.13)\qquad \int_{\widehat{\eta}_n}^1 |s^{\widehat{\gamma}_n - \gamma} - 1| s^{\gamma} d \frac{1}{1-G(s)} \le \int_0^1 |(1-\delta)^{(\widehat{\gamma}_n - \gamma)/|\widehat{\gamma}_n - \gamma|} - 1| s^{\gamma} d \frac{1}{1-G(s)}$$

$$\leq \int_0^1 \{\max_{i=\pm 1} |(1-\delta)^i - 1|\} s^\gamma d\frac{1}{1-G(s)} \leq 2\delta \int_0^1 s^\gamma d\frac{1}{1-G(s)} = \varepsilon/2 \,.$$

Since $\widehat{\eta}_n \to_p 0$, as $n \to \infty$, for any $0 < \delta < 1$ we also have with arbitrary high probability for sufficiencly large n that

$$\int_0^{\widehat{\eta}_n} |s^{\widehat{\gamma}_n - \gamma} - 1| s^\gamma d\frac{1}{1-G(s)} \leq \int_0^{\widehat{\eta}_n} d\frac{1}{1-G(s)} = \frac{G(\widehat{\eta}_n)}{1-G(\widehat{\eta}_n)} \leq \varepsilon/2 \,. \tag{3.14}$$

This completes the proof of (3.11) and hence of (3.5). Q.E.D.

The class of c.d.f.'s F satisfying (1.3) and (1.4) contains c.d.f.'s with derivative $F'(0) = f(0)$ which is either zero, finite nonzero, or infinite, partly depending on γ. Henceforth we will restrict ourselves to the subclass $\mathcal{F}$ of c.d.f.'s with the following properties: (1.3) and (1.4) are fulfilled: a continuous second derivative exists on $(0, \varepsilon]$, for some $\varepsilon > 0$; $\lim_{t \downarrow 0} f(t)$ exists ($f = F'$) and equals $f(0)$, say, where $f(0) = \infty$ is admitted; $\lim_{t \downarrow 0} f'(t)$ is finite if $f(0)$ is finite. It should be noted that finiteness of $f(0)$ entails that $F'(0) = f(0)$ so that F' is continuous on $[0, \varepsilon]$ with continuous bounded derivative f' on $(0, \varepsilon]$ in that case.
For practical purposes it is interesting to know that failure is unlikely to occur immediately. Consequently for some $0 < c < \infty$ we are interested in testing the null hypothesis $H_0 : f(0) \geq c$ (including $f(0) = \infty$) versus the alternative $H_1 : f(0) < c$ (including $f(0) = 0$). Furthermore we introduce a kernel $K : \mathbb{R} \to \mathbb{R}$ which is of bounded variation on $[0,1]$, zero outside $[0,1]$, and which satisfies $\int_0^1 K(t)dt = 1$. As a test statistic we introduce

$$\widehat{f}_n(0) = \frac{1}{a_n} \int_{-\infty}^{\infty} K\left(\frac{t}{a_n}\right) d\widehat{F}_n(t) \,, \tag{3.15}$$

which is an estimator for the density f at the point zero. Write also (cf. (3.3)):

$$\widehat{D}_n(t) = \int_0^t \frac{1}{1-\widehat{G}_n(sa_n)} ds^{\widehat{\gamma}_n}, \quad 0 \leq t \leq 1 \,. \tag{3.16}$$

THEOREM 3.2. *In addition to* (1.1) - (1.4) *and* (3.2), *let us assume that* $na_n^3 \to 0$, *as* $n \to \infty$. *An asymptotically size* $\alpha \in (0,1)$ *test for testing* $H_0 : f(0) \geq c \in (0, \infty)$ *versus* $H_1 : f(0) < c$ *is obtained when we reject* H_0 *if*

$$\sqrt{\frac{na_n^2}{\widehat{F}_n(a_n) \int_0^1 \widehat{D}_n(t) dK^2(t)}} (\widehat{f}_n(0) - c) \leq \Phi^{-1}(\alpha) \,, \tag{3.17}$$

where Φ is the standard normal c.d.f. The test is consistent against any alternative covered by H_1.

PROOF. Let us introduce $f_n(0) = (1/a_n) \int_{-\infty}^{\infty} K(t/a_n)dF(t)$, and note that via integration by parts (2.11) entails

$$\sqrt{\frac{n}{F(a_n)}}a_n(\hat{f}_n(0) - f_n(0)) \rightarrow_d - \int_0^1 W \circ D(t)dK(t), \text{ as } n \rightarrow \infty \ , \tag{3.18}$$

since integration with respect to K is a continuous functional on $D[0,1]$. The random variable on the right in (3.18) is normal with mean 0 and variance $\int_0^1 D(t)dK^2(t)$. A weakly consistent estimator for this variance is given by $\int_0^1 \hat{D}_n(t)dK^2(t)$. Since $\sup_{0\le t\le 1}|\hat{D}_n(t) - D(t)|$ is bounded by the sum of the expressions on the right in (3.10) and (3.12) it follows at once from the last part of the proof of Theorem 3.1 that $\sup_{0\le t\le 1}|\hat{D}_n(t) - D(t)| \rightarrow_p 0$, as $n \rightarrow \infty$. Hence we have $| \int_0^1 \hat{D}_n(t)dK^2(t) - \int_0^1 D(t)dK^2(t)| \le \sup_{0\le t\le 1} M|\hat{D}_n(t) - D(t)| \rightarrow_p 0$, as $n \rightarrow \infty$, where M is the mass assigned to $[0,1]$ by the total variation measure determined by K^2. Jointly with the first statement in (3.8) this yields

$$\sqrt{\frac{na_n^2}{\hat{F}_n(a_n)\int_0^1 \hat{D}_n(t)dK^2(t)}}(\hat{f}_n(0) - f_n(0)) \rightarrow_d \mathcal{N}(0,1), \text{ as } n \rightarrow \infty \ . \tag{3.19}$$

Next let us replace $f_n(0)$ with c in the expression on the left in (3.19) and first show that

$$\sqrt{\frac{n}{F(a_n)}}\left\{\int_0^1 K(t)dF(ta_n) - ca_n\right\} \rightarrow \begin{cases} \infty & , f(0) > c \\ 0 & , f(0) = c, \\ -\infty & , f(0) < c \end{cases} \text{ as } n \rightarrow \infty \ . \tag{3.20}$$

If $f(0) = c$ we have

$$\int_0^1 K(t)dF(ta_n) - ca_n = - \int_0^1 F(ta_n)dK(t) - ca_n \tag{3.21}$$

$$= - \int_0^1 (ca_nt + O(a_n^2))dK(t) - ca_n = O(a_n^2), \text{ as } n \rightarrow \infty \ .$$

Because $\sqrt{n/F(a_n)} = O(\sqrt{n/a_n})$ it follows that the expression on the left in (3.20) is of order $O(\sqrt{na_n^3}) = o(1)$, as $n \rightarrow \infty$. The remaining two cases can be dealt with in a similar manner. It is clear that (3.8), (3.19), and (3.20) imply the claims of the theorem. Q.E.D.

4. SOME REMARKS

4.1. Tail empirical processes are of theoretical interest in their own right and the same could be said about the present generalization to the censored case. For censored tail empirical processes it turns out that only the kind of heavy censoring that we consider here makes sense.

4.2. Theorem 2.1 can be extended to the case where the processes ξ_n, β_n and $W \circ D$ are divided by a weight function; also functional laws of the iterated logarithm for ξ_n and β_n are readily derived similarly, cf. Einmahl (1992).

4.3. An admissible choice for the kernel K in (3.15) is the indicator function $\mathbb{1}_{[0,1]}$ in which case $\widehat{f}_n(0)$ reduces to $\widehat{F}_n(a_n)/a_n$.

4.4. An alternative estimator of $D(t)$ in the variance $\int_0^1 D(t)dK^2(t)$ of the random variable on the right in (3.18) is given by $\widehat{\widehat{D}}_n(t) = \int_0^t (1 - \widehat{G}_n(sa_n))^{-1} ds$, since $\gamma = 1$ under H_0.

4.5. The condition $na_n^3 \to 0$, as $n \to \infty$, in Theorem 3.2 is in fact a restriction on the model. If the condition is not fulfilled a result like the one in Theorem 3.2 does not exist. Further smoothness of F, however, can be used to relax this restriction.

4.6. Local confidence bands for Λ and tests on $\lambda(0) = \Lambda'(0)$ follow along similar lines from (2.12), but since $\lambda(0) = f(0)$, Theorem 3.2 can also be used directly for testing on $\lambda(0)$.

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