Analytical approach to transient heat conduction in cooling load calculations

ANALYTICAL APPROACH TO TRANSIENT HEAT CONDUCTION IN COOLING LOAD CALCULATIONS

Michal Duška[1], Martin Barták[1], František Drkal[1] and Jan Hensen[2]

[1]Department of Environmental Engineering, CTU in Prague
166 07 Prague 6, Czech Republic

[2]Center for Building & Systems TNO - TU/e, TU Eindhoven
5600 MB Eindhoven, Netherlands
e-mail: michal.duska@fs.cvut.cz

ABSTRACT

The paper briefly reviews four existing analytical methods for the solution of the Fourier's equation in cooling load calculations. The performance of nine different procedures (the four methods and their modifications) is presented on an example of heat transmission through a heavy wall under realistic external conditions. For the current case it was proved that the admittance method gives results similar to those obtained by the periodic response factors (PRF) approach, which is normally considered to be more advanced. Possible errors were found in the previously published coefficients for one of the PRF modifications.

INTRODUCTION

Current pressure to lower the consumption of primary energy sources has recently emphasized the issue of cooling load calculation accuracy in the Czech Republic and other European countries. This leads to revisions of existing methods and efforts to define standards for required accuracy in cooling load calculations (e.g. prEN 15255).

Cooling load calculations account for a number of interrelated processes that are difficult to describe precisely. One of the most complicated components is the thermal storage effect which is important when converting instantaneous heat gains for a given room into its cooling load. The thermal storage capacity of constructions that enclose a room affects (a) the heat conduction of radiant energy absorbed on internal surfaces and (b) the heat conduction due to external temperature and radiant excitation. The latter is a subject of the present study.

The spatial and temporal distribution of temperature $t(x, \tau)$ in a homogeneous wall subject to one dimensional heat flow is given by the Fourier's equation:

$$\frac{\partial t}{\partial \tau} = \frac{\lambda}{\rho \cdot c} \left(\frac{\partial^2 t}{\partial x^2} \right) \qquad (1)$$

The Fourier's equation is linear and time invariant which enables to find its analytical solutions even for complicated boundary conditions.

Four analytical methods for the solution of the Fourier's equation are reviewed and compared regarding their accuracy and complexity in the calculation of heat transmission through an external wall:

- thermal response factors (TRF),
- conduction transfer functions (CTF),
- periodic response factors (PRF),
- admittance method (AM).

The first two methods can be found in computer-added energy performance analyses while the latter two are usually applied in manual cooling load calculations.

METHODOLOGY

Thermal Response Factors

The TRF concept is a simplification of wall boundary conditions (i.e. temperature variations) to the series of linear functions, for which the Fourier's equation can be solved analytically. The mostly used functions in the form of triangular pulses (Mitalas and Stephenson, 1967) can be combined into a piece-wise linear approximation of the required temperature profile as presented in Fig. 1.
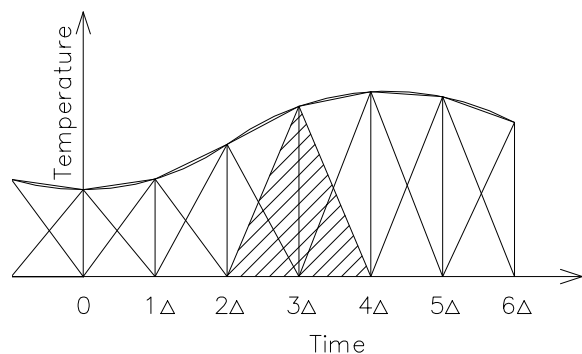

Fig. 1: Linear approximation of input by triangular pulses

The linearity of the equation (1) allows superposing reactions to individual temperature pulses imposed on a wall surface. This means that instead of solving the equation (1) directly for a complicated input we can superpose the solutions for a number of very simple inputs (e.g. triangular pulses) to evaluate the wall response to continuously changing boundary conditions. An example of the heat flux response induced by a unit triangular temperature step is shown in Fig. 2.
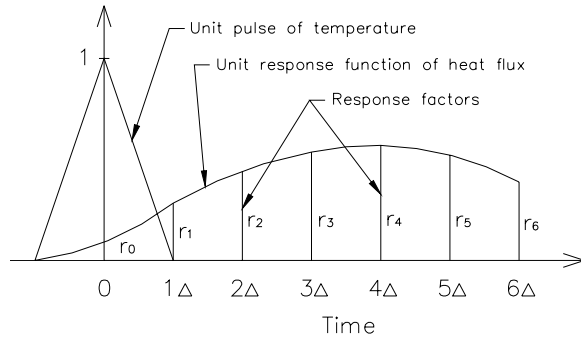

Fig. 2: Response to a unit triangular pulse

The heat flux response q_j at time step j to the initial temperature pulse of an arbitrary magnitude t is

$$q_j = r_j \cdot t \tag{2}$$

where r_j are the response factors obtained from the response function illustrated in Fig. 2.

The heat flux at the surface A and time step n, derived as a response to the series of triangular temperature pulses acting on both surfaces of the wall, reads

$$q_n^A = \sum_{j=0}^{\infty} r_j^{A-A} \cdot t_{n-j}^A - \sum_{j=0}^{\infty} r_j^{B-A} \cdot t_{n-j}^B \tag{3}$$

where A, B denote the surfaces, n and j are the time index variables, t_{n-j} is the surface temperature at time step j ahead of n, r^{A-A} and r^{B-A} are the response factors at the surface A corresponding to the unit temperature pulses acting on the surface A and B, respectively.

Similarly, the heat flux at the surface B is

$$q_n^B = \sum_{j=0}^{\infty} r_j^{A-B} \cdot t_{n-j}^A - \sum_{j=0}^{\infty} r_j^{B-B} \cdot t_{n-j}^B \tag{4}$$

where j determines the number of temperatures and response factors which must be considered prior to the step n. The maximum of j depends on the type of a wall and required accuracy (ideally j would be infinite). The identity $r^{A-B} = r^{B-A}$ applies for any wall.

Conduction Transfer Functions
This method assumes that the relation between input and output signals for a multi-layer wall can be described as

$$O_n \cdot b_0 + O_{n-1} \cdot b_1 + O_{n-2} \cdot b_2 + ... + O_{n-P} \cdot b_P$$
$$= I_n \cdot a_0 + I_{n-1} \cdot a_1 + I_{n-2} \cdot a_2 + ... + I_{n-J} \cdot a_J \tag{5}$$

where a_j and b_j are the coefficients of the z-transfer function $K(z)$ (Stephenson and Mitalas, 1971)

$$K(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} + ... + a_J z^{-J}}{b_0 + b_1 z^{-1} + b_2 z^{-2} + ... + b_P z^{-P}} \tag{6}$$

where O_{n-j} is the output signal (heat flux) and I_{n-j} is the input signal (temperature) at time step n–j. The input signal is a series of discrete pulses replacing the continuous temperature excitation (see Fig. 3).
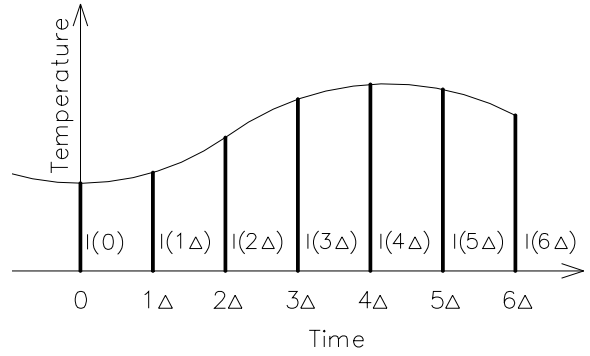

Fig. 3: Approximation of input by discrete pulses

Rewriting the equation (5) for a wall with surfaces A and B yields:

$$q_n^A = \sum_{j=0}^{J} t_{n-j}^A \cdot a_j^A - \sum_{j=0}^{J} t_{n-j}^B \cdot a_j^B - \sum_{j=1}^{P} q_{n-j}^A \cdot b_j^A \tag{7}$$

$$q_n^B = \sum_{j=0}^{J} t_{n-j}^B \cdot a_j^B - \sum_{j=0}^{J} t_{n-j}^A \cdot a_j^A - \sum_{j=1}^{P} q_{n-j}^B \cdot b_j^B \tag{8}$$

Finding the z-transfer function coefficients is more difficult task than solving the response factors. The z-transfer approach in heat conduction analysis was probably introduced by Stephenson and Mitalas (1971). Their direct root finding procedure was later improved by Hittle and Bishop (1983) and used for the calculation of z-transfer function coefficients in Harris and McQuiston (1988) and ASHRAE Handbook of Fundamentals (1989, 1993, 1997). Spitler and Fisher (1999) pointed out that all the previously published coefficients for heavy walls had been erroneous. Other possibilities to determine the coefficients are the time-domain method (Davies, 1996), state-space method (Jiang, 1982) and frequency-domain regression method (Chen and Wang, 2001).

Periodic Response Factors
The periodic response factors were defined for use in the Radiant Time Series method developed by Spitler et al. (1997). This method assumes periodic variations of external sol-air temperature and constant internal air temperature for the computation of the heat flux through a wall at time step n:

$$q_n^B = \sum_{j=0}^{23} r_{Pj}^{A-B} \cdot \left(t_{n-j}^A - t^B \right) \tag{9}$$

where A and B are the external and internal surfaces, r_{Pj}^{A-B} is the periodic response factor, t_{n-j}^{A} is the sol-air temperature j hours ago and, t^{B} is the constant room air temperature. The PRF values can be obtained using TRF (Spitler, 1997) or CTF (Spitler and Fisher, 1999).

Admittance Method

This approach is based on the assumption that time variations of temperature or heat flow at a wall surface are sinusoidal. The response to sinusoidal excitation on one surface is again sinusoidal within the wall and on its other surface. The excitation and response signals differ in amplitude and phase.

The relations between the variables on surfaces A and B are usually presented in matrix form:

$$\begin{bmatrix} \hat{t}^{B} \\ \hat{q}^{B} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \cdot \begin{bmatrix} \hat{t}^{A} \\ \hat{q}^{A} \end{bmatrix} \qquad (10)$$

where $\hat{t}$ and $\hat{q}$ denote the cyclic (sinusoidal) temperatures and heat fluxes. The complex elements of the transmission matrix are given as:

$$a_{11} = a_{22} = \cosh(p + \mathrm{i}p) \qquad (11)$$

$$a_{12} = \frac{L \cdot \sinh(p + \mathrm{i}p)}{\lambda \cdot (p + \mathrm{i}p)} \qquad (12)$$

$$a_{21} = \frac{\lambda \cdot (p + \mathrm{i}p) \cdot \sinh(p + \mathrm{i}p)}{L} \qquad (13)$$

where L is the wall thickness and the parameter p for a 24-hour cycle ($24 \times 3600 = 86400$ seconds) is:

$$p = \sqrt{\frac{\pi \cdot L^{2} \cdot \rho \cdot c}{86400 \cdot \lambda}} \qquad (14)$$

The wall response to periodic excitations can be described by three factors implemented in AM-based cooling load calculations: admittance, decrement factor and surface factor (Milbank and Lynn, 1974). The admittance is the magnitude of the ratio of the cyclic heat flux to the cyclic temperature at the same surface. The surface factor is the ratio of the cyclic heat flow readmitted to the space from the surface to the cyclic heat flow absorbed by the same surface.

The decrement factor f is the magnitude of the ratio of the cyclic transmittance $\hat{U}^{A-B}$ to the steady-state U value:

$$f^{A-B} = \left| \hat{f}^{A-B} \right| = \left| \frac{\hat{U}^{A-B}}{U} \right| \qquad (15)$$

The cyclic transmittance is defined using the equations (10) and (12) as:

$$\hat{U}^{A-B} = \frac{\hat{q}^{B}}{\hat{t}^{A}} = \frac{1}{a_{12}} \qquad (16)$$

The time lag ω in hours between $\hat{t}^{A}$ and $\hat{q}^{B}$ is:

$$\omega = \frac{12}{\pi} \cdot \arctan \left[\frac{\mathrm{Im}(\hat{f}^{A-B})}{\mathrm{Re}(\hat{f}^{A-B})} \right] \qquad (17)$$

The mean temperatures over 24-hour cycle at the surfaces A and B can be obtained from

$$\bar{t}^{A} = \frac{\sum_{n=0}^{23} t_{n}^{A}}{24} \quad \text{and} \quad \bar{t}^{B} = \frac{\sum_{n=0}^{23} t_{n}^{B}}{24} \qquad (18a,b)$$

The swing in temperature at the surface A and time step n is

$$\widetilde{t}_{n}^{A} = t_{n}^{A} - \bar{t}^{A} \qquad (19)$$

and the heat flux on the opposite surface B at the same time step n reads

$$q_{n}^{B} = \widetilde{q}_{n}^{B} + \bar{q}^{B} = U \cdot f^{A-B} \cdot \widetilde{t}_{n-\omega}^{A} + U(\bar{t}^{A} - \bar{t}^{B}) \qquad (20)$$

where $\widetilde{t}_{n-\omega}^{A}$ denotes the swing in temperature at the surface A and time step ω ahead of n (assuming one-hour steps).

EXAMPLE AND ANALYSIS

The above methods were applied in the calculations of transient heat conduction through a heavy external wall defined as the ASHRAE Wall Group 37 (ASHRAE, 1997). The WG37 is 505 mm thick and consists of four layers (from outside to inside): face brick, lightweight concrete, insulation and plaster. The thermal resistance at the external and internal surfaces is 0.06 m²·K·W⁻¹ and 0.12 m²·K·W⁻¹, respectively.

The external boundary conditions were defined by sol-air temperature varying in time. Realistic external conditions (air temperature, solar and longwave radiation flux) were taken from the BESTEST database (Judkoff, 1995). The indoor air temperature was set constant at 20 °C. The present analysis was elaborated for a single summer day.

The thermal response factors were adopted from the study by Chen et al. (2006) which provides not only the TRF values but also their verification. In the current case of WG37, 144 TRF values are necessary for an acceptable accuracy (Chen et al., 2006). This means that input values of the sol-air temperature must be provided for a period starting six days prior to the design day. The input sol-air data are presented in Fig. 4.
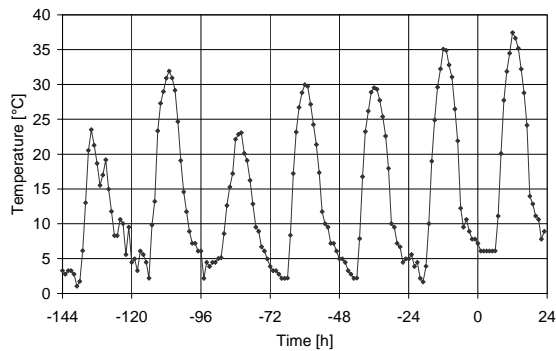
Fig. 4: Sol-air temperature (design day: 0 – 23 h)

The version of PRF values published by Spitler and Fisher (1999) is indicated as PRF/ASHRAE. We calculated another set of periodic response factors denoted as PRF/FDR using TRF and the procedure introduced by Spitler et al. (1997).

The PRF- and AM-based manual cooling load calculations assume that the design day has been preceded by an infinite number of identical days. Therefore the temperatures just from the design day are needed as input. The results of this standard procedure are noted as "cyclic" in the current study. In order to use boundary conditions that are realistic and comparable with the TRF and CTF methods, another set of PRF and AM calculations was produced using the real sol-air temperatures from the 24-hour interval ahead of the design day.

Two sets of the coefficients for the CTF method were applied in this study. Those adopted from Harris and McQuiston (1988) are indicated as CTF/ASHRAE. The set developed by Chen et al. (2006) is denoted as CTF/FDR. The CTF values for WG37 cover the current time step plus six preceding hours. However, the heat flux at the beginning of the above period is crucial and not known a priori. Therefore the present CTF calculations included a start-up period of six days to obtain an accurate value of the heat flux six hours before the design day begins.

RESULTS

The comparison of the wall response characteristics calculated by different methods is presented as time variation plots of the heat flux at the internal surface and the external sol-air temperature for the design day in Fig. 5. Note that the response curves take account of the preceding period not shown in the graph, except those denoted with the suffix "cyclic".
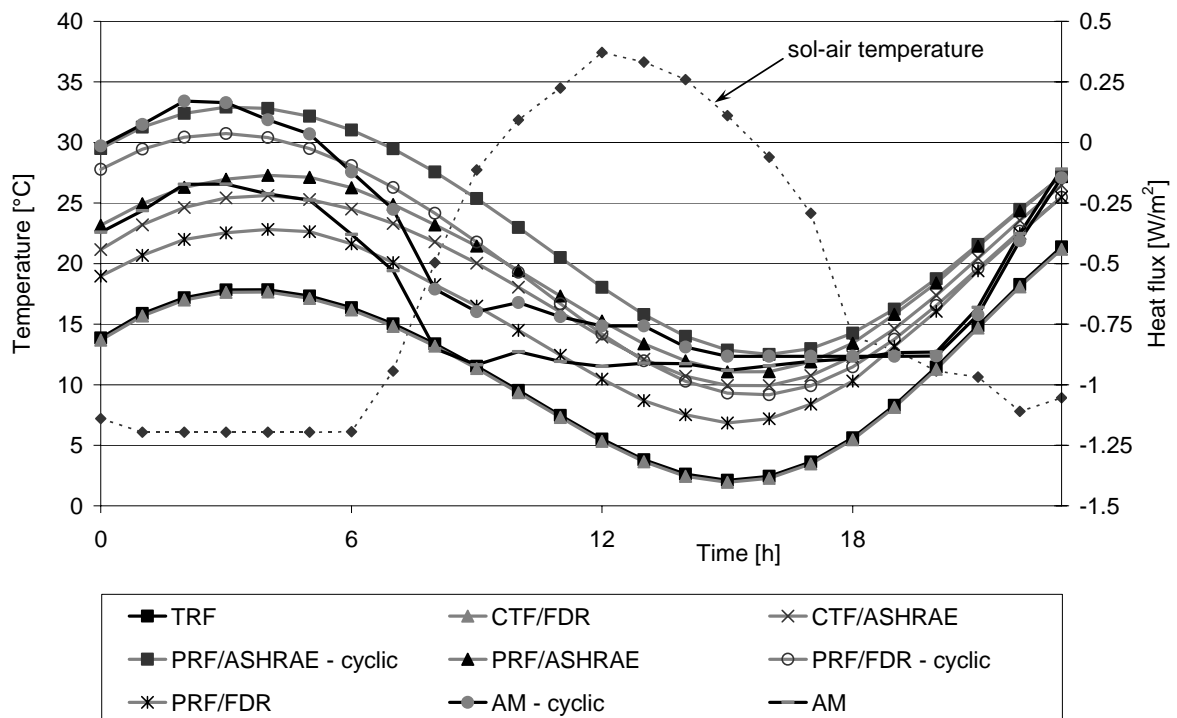

Fig. 5: Time-variation of the heat flux based on different calculation methods

The results obtained by the TRF technique are considered as reference because fully documented and verified factors for this method were available from Chen et al. (2006). The results from other methods are compared with the reference in Fig. 6 using the standard deviation calculated over the design day. The standard deviation for each method was defined as

$$STD^{method} = \sqrt{\frac{\sum_{j=0}^{23}\left(q_j^{method} - q_j^{TRF}\right)^2}{24}} \qquad (21)$$

where q_j^{method} and q_j^{TRF} denote the heat fluxes calculated at the same time step j according to the assessed method and TRF, respectively.
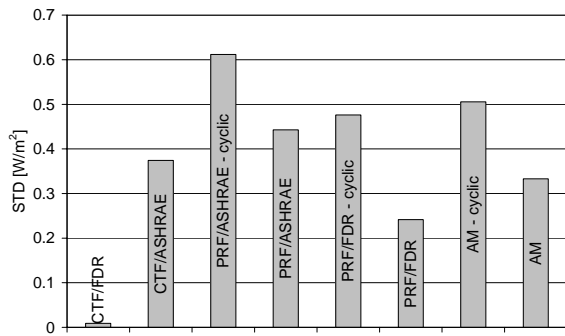

Fig. 6: Standard deviation from TRF results

DISCUSSION

Comparison of the results in Fig. 6 shows that even though the CTF/FDR method requires only 6 preceding time steps for the calculation, it performs comparably to the TRF technique which needs input data for 143 preceding time steps. The results from the CTF/ASHRAE procedure confirm the incorrect coefficients for WG37 as pointed out by Spitler and Fisher (1999).

The PRF/ASHRAE procedure produced results with surprisingly high STD comparing to the PRF/FDR method. Further analysis revealed that the PRF/ASHRAE coefficients published by Spitler a Fisher (1999, Table A-3) are not correct because they do not follow the summation rule (Chen et al., 2006):

$$\sum_{j=0}^{23} r_{Pj}^{A-B} = U \qquad (22)$$

where $U = 0.226$ W·m^{-2}·K^{-1} for WG37.

The control sums for the methods where the summation rule is applicable are listed in Table 1.

Another important outcome of the current analysis is that both PRF and AM techniques produce very similar results when assuming identical cycles of the design day, although the AM is based on a rather crude simplification of boundary conditions.

It is obvious that the use of a cyclic design day substantially deteriorates the potential of the PRF/FDR and AM methods.

Table 1: Control sums of coefficients

Method	Summation rule	Control sum
TRF	$\sum_{j=0}^{143} r_j^{A-B} = U$	0.226
CTF/FDR	$\sum_{j=0}^{6} a_j^A / \sum_{j=0}^{6} b_j^B = U$	0.225
CTF/ASHRAE	$\sum_{j=0}^{6} a_j^A / \sum_{j=0}^{6} b_j^B = U$	0.158
PRF/ASHRAE	$\sum_{j=0}^{23} r_{Pj}^{A-B} = U$	0.159
PRF/FDR	$\sum_{j=0}^{23} r_{Pj}^{A-B} = U$	0.226

CONCLUSIONS

Coefficients for the TRF, CTF and PRF techniques should be carefully revisited and those already published should be used with caution.

Manual cooling load calculations based on simplified boundary conditions may produce results comparable to results obtained by more complicated methods.

Current attempts to define standards for the cooling load calculation accuracy (e.g. prEN 15255) should be aware of achievable accuracy for individual components participating in cooling load. The present study performed only for a single type of wall indicates that this could be a difficult issue. Future work in this area is needed to analyse properly different modes of heat gain in buildings.

ACKNOWLEDGEMENTS

REFERENCES

CIBSE Guide A: Environmental design, Chartered Institution of Building Services Engineers, London, 1999.

Davies M. G., "A time-domain estimation of wall conduction transfer function coefficients", ASHRAE Transactions, v. 102, no. l, pp. 328-343, 1996.

ASHRAE Handbook of Fundamentals, American Society of Heating, Refrigerating and Air-conditioning Engineers, Atlanta (GA), 1989.

ASHRAE Handbook of Fundamentals, American Society of Heating, Refrigerating and Air-conditioning Engineers, Atlanta (GA), 1993.

ASHRAE Handbook of Fundamentals, American Society of Heating, Refrigerating and Air-conditioning Engineers, Atlanta (GA), 1997.

Harris, S. M., F. C. McQuiston, "A study to categorize walls and roofs on the basis of thermal response", ASHRAE Transactions, v. 94, no. 2, pp. 688-715, 1988.

Hittle D. C., R. Bishop, "An improved root-finding procedure for use in calculating transient heat flow through multilayered slabs", International Journal of Heat and Mass Transfer, v. 26, no.1, pp. 1685–1694, 1983.

Chen Y. M., J. Zhou, J. D. Spitler, "Verification for transient heat conduction calculation of multilayer building constructions", Energy and Buildings, v. 38, no. 4, pp. 340-348, 2006

Chen Y. M., S. W. Wang, "Frequency domain regression method for estimating CTF model of building multi-layer walls", Applied Mathematical Modeling, v. 25, no. 7, pp 579–592, 2001.

Jiang Y., "State-space method for the calculation of air-conditioning loads and the simulation of thermal behavior of the room", ASHRAE Transactions v. 88, no. 2, pp. 122–138, 1982.

Judkoff R., J. Neymark. "International Energy Agency building energy simulation test (BESTEST) and diagnostic method", National Renewable Energy Laboratory, Golden (CO), 1995.

Milbank, N. O., J. H. Lynn, "Thermal response and the admittance procedure", Building Services Engineer, pp. 38-50, 1974.

Mitalas, G. P., D. G. Stephenson, "Room thermal response factors", ASHRAE Transactions, v.73, no.1, pp. 1-10, 1967.

prEN 15255 "Thermal performance of buildings – sensible room cooling load calculation - General criteria and validation procedures", European Committee for Standardization (under approval).

Seem J. E., S. A. Klein, W. A. Beckman, J. W. Mitchell, "Transfer functions for efficient calculation of multidimensional transient heat transfer", Journal of Heat Transfer, v. 11, pp. 5–12, 1989.

Spitler J. D., D. E. Fisher, "Development of periodic response factors for use with the radiant time series method", ASHRAE Transactions, v 105, no 2, pp. 491-502, 1999.

Spitler J. D., D. E. Fisher, C. O. Pedersen, "The radiant time series cooling load calculation procedure", ASHRAE Transactions, v. 103, pp. 503-515, 1997.

Stephenson, D. G., G. P. Mitalas, "Calculation of heat conduction transfer functions for multi-layer slabs", ASHRAE Transactions, v.77, no.2, pp.117-126, 1971.

NOMENCLATURE

c specific heat capacity [J·kg^{-1}·K^{-1}]
L wall thickness [m]
q heat flux [W·m^{-2}]
t temperature [°C]
U steady-state heat transmittance [W·m^{-2}·K^{-1}]
x spatial coordinate [m]
λ thermal conductivity [W·m^{-1}·K^{-1}]
ρ density [kg·m^{-3}]
τ time [s]