# Model predictive control for stochastic systems by randomized algorithms

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Ivo Batina

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Dit proefschrift is goedgekeurd door de promotoren:

prof.dr. A.A. Stoorvogel
en
prof.dr.ir. M.L.J. Hautus

Copromotor:
dr. S. Weiland

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Batina, Ivo

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Eerste promotor: prof.dr. A. Stoorvogel

Tweede promotor: prof.dr.ir. M.L.J. Hautus

Copromotor: dr. S. Weiland

Kerncommissie:

prof.dr. D.Q. Mayne
dr.ir. T.J.J. van den Boom

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# *Contents*

# 1

# Introduction

The goals of this chapter are to motivate a study of control systems with constraints and to give an overview of available methods for control of such systems. Furthermore, the objectives of the thesis are defined and the outline of the thesis is given.

## 1.1 Constraints in systems

A successfully designed control system has to be able to control a plant in spite of requirements that are contradictory, in many cases. A constant push towards higher quality of products with lower manufacturing costs is an example of contradictory requirements. Usually, the requirements that a control system has to meet are associated with direct costs, like energy costs, but also with environmental and safety demands. This requirements are expressed as constraints that have to be respected. It is often claimed (see [39, 119]), that a control strategy that allows operation of the plant close to constraints has advantages in applications today. The reason is that the most profitable operation of the industrial plant is often obtained when the process is running at a constraint boundary.

Consider for example ( [91]), an industrial unit in which the product is manufactured in a unit that requires heating. The energy cost can be optimized by keeping the amount of heat supplied as small as possible but just large enough to obtain the desired quality of the product. The optimum is on the constraint boundary, in this case the constraint on the quality of the product is the one to be respected.

Constraints can be present in the system to be controlled in different ways. Constraints on inputs of the system are commonly present. Valves have a finite range of adjustment, a maximum value of the flow rate in hydraulic systems is determined by pipe diameters, etc. These constraints are saturation or rate constraints. Typical example of rate constraints are valves and other actuators with limited opening rates. Consider for example a semi-batch reactor (see [49, 111]) depicted in figure 1.1. A reactant enters the reactor on the left. In the reactor, an exothermic reaction occurs (roughly speaking, an exotherimic reaction is a chemical reaction which is characterized by development of heat). To control the behaviour in the reactor, the temperature inside the reactor has to be kept on (or close) to a set-point. The reactor needs cooling, otherwise the temperature in the reactor would rise without control. The cooling is performed by

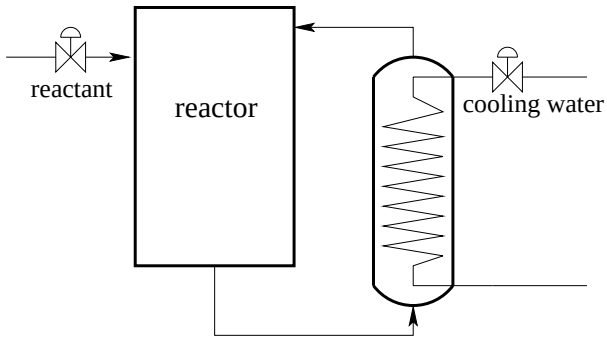


Figure 1.1: Semi-batch reactor

a flow of water which is cooled by the heat exchanger shown on the right of figure 1.1. The reactant flow rate and the flow rate of the cooling water are controlled by valves. These flaws can be adjusted only between certain minimum and maximum values which are determined by minimum and maximum openings of the valves. In other words, the dynamics of the process is subject to input constraints.

Additional constraints that are usually imposed to processes like this semi-batch reaction are determined by the economic objectives. For instance, to keep the reactor at an economically profitable operating point, to have a quick change-over of product specification, to minimize environmental damage, to minimize off-spec production time, to minimize product variations etc. The economic objective can be to finish the batch as quickly as possible which is usually equivalent to keeping the reactant flow as large as possible. The amount of heat from the reaction is proportional to the reactant flow and the ability to cool the reactor is limited by the maximal flow of the cooling water. As a consequence, if the reactant flow is above a certain value, the dynamics of the reaction becomes unstable, because it is not possible to provide sufficient amount of cooling. Thus, the economic objective of having a large reactant flow needs to be compromised with the ability to cool the reaction. This is a typical example of difficulties that constraints on the input impose to a control system.

In most cases, the product of an industrial process has to meet certain quality requirements. Quality requirements can be seen as constraints on the output of the system, in this case the system is the industrial process. These constraints have to be respected as much as possible i.e. violation of quality constraints can be tolerated in some cases although it is not desirable. In this case we talk about *soft* constriants. Constraints on the output of the system can be of different origins, however. Constraints on output are often imposed by safety or environmental considerations, in most cases these constraints have to be respected absolutely. These constraints are known as *hard* constraints. Constraints on the output mentioned so far are constraints that are not in-

herent to the dynamics of the systems but are imposed by economical, environmental or safety reasons.

There are many examples in which constraints naturally occur in the process dynamics. As a simple example of such a system, consider an ideal diode. There are two variables that describe behavior of the diode. These variables are electrical current through the diode, denoted with $i_d$ and the electrical voltage across the diode, denoted with $u_d$. The main role of a diode in an electrical circuit is to regulate the "direction" of the electrical current flow. When the voltage across the diode is positive, the diode does not have any resistance and when the voltage across the diode is negative the diode has an infinite resistance. The behavior of the diode can be described by the following equation.

$$i_d(u_d) = \begin{cases} i_d = 0 & \text{if} \quad u_d < 0 \\ u_d = 0 & \text{if} \quad i_d > 0. \end{cases}$$

The value of the current through the diode is constrained from below and this constraint is inherent to the behavior of the diode. An electrical network with an ideal diode will have two modes of operation depending on the direction of the voltage across the diode. Because of its multimode characteristic, this is an example of a hybrid system (see [4, 22, 26, 67]).

In real world, control systems are effected with disturbances from various sources. To control a plant in a desired manner the presence of undesired disturbances is one of the most important objectives that a control system has to meet. In many cases disturbances are not known exactly and they are random in nature. Mathematically, such disturbances are described by a stochastic model. A system in which stochastic disturbances play an important role is a problem of Aircraft Conflict Detection in Air Traffic Management Systems (ATMS) (see [110, 147]). A sketch of ATMS structure is shown on figure 1.2. The primary concern of all ATMS is to guarantee safety. Safety is typically quantified in terms of the number of conflicts, i.e. situations where two aircraft come closer than a certain distance to one another. The safety distance is encoded by means of minimum allowed *horizontal separation* and minimum allowed *vertical separation*. This quantities are constraints that have to be respected.

One of the difficulties in predicting aircraft positions is modeling the perturbations influencing their motion. The motion of the aircraft is affected by uncertainty, wind force, errors in tracking, navigation and control. Since most of these effects have a random, unpredictable behaviour, the resultant deviation from the nominal trajectory can often be modeled by stochastic disturbance inputs.

The position of the aircraft is known exactly but the control has to be based not just on the known position and desired flight path of the aircraft that is led by the air traffic control but also on the *prediction* of the future air traffic movements. That is the only way to guarantee that a necessary flight path correction will be scheduled in time to the aircraft. Here, there are several important elements in the control problem. These are: the dynamics of the aircraft, predicitons of future air traffic situation in relation

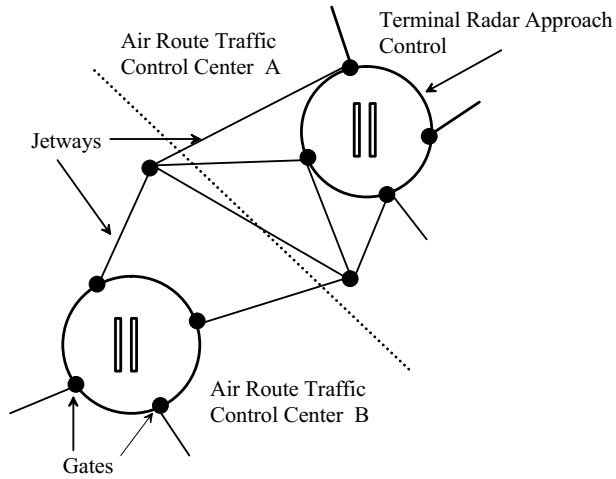


Figure 1.2: A sketch of ATMS structure

to position of aircraft, constraints in terms of minimum safety distances and random disturbances. The difficulty is that with the presence of the random disturbance it is not possible to guarantee that the constraints in future will be respected absolutely. A feasible approach is to minimize the possibility of the constraint violation in the predicted future by an appropriate choice of the current control action. Note that this choice has to be based on the predicted control action in future, not just on the model of the system and the stochastic model of the disturbance.

Given the known position of the aircraft, known aerodynamical characteristics of the aircraft and the stochastic model of the disturbance, the control strategy in this example is to compute the flight path correction that will be ordered to the aircraft in a finite time slot, ranging from the current time to some time in the future, so that the possibility of the constraint violation over this time slot is minimized. The first one of computed flight path corrections is actually ordered to the aircraft, in the next time instant the procedure is repeated based on the new information about the position of the aircraft. Conceptually, this control strategy is known as the model predictive control technique.

Despite the overall presence of constraints in real world control problems and their importance in today control practice there is only a few techniques available in the literature that can be used in a design of constrained control systems. This is especially true for systems that are subject to stochastic disturbances. Each one of the techniques that deal with constrained systems has its own merits and drawbacks. In the following subsections we give an overview of various available approaches to the control of constrained systems.

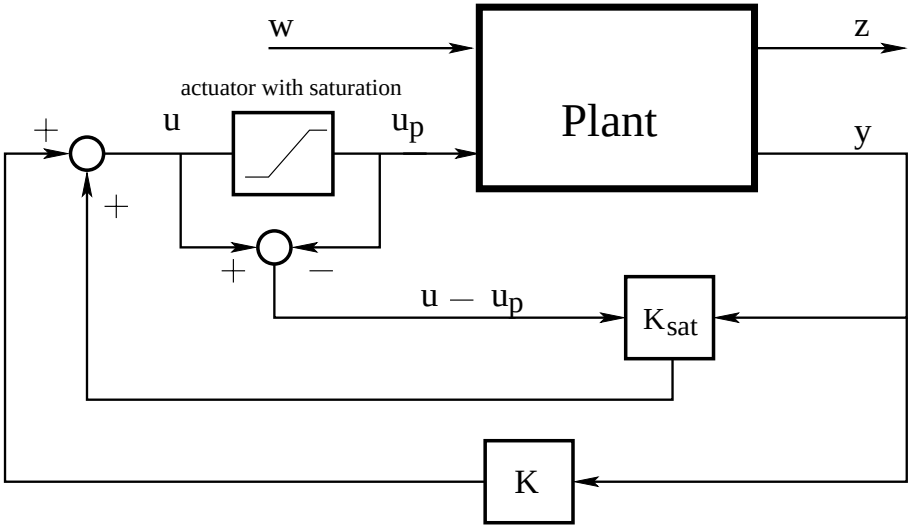


Figure 1.3: Anti-windup structure

## 1.2 Anti-windup designs

As examples in section 1.1 show, systems with control input constraints are often encountered in the control engineering practice. Control systems with saturated inputs were encountered early in the development of control engineering, when control of dynamical systems was more a craft than a science. Anti-windup designs have roots in early attempts of control practitioners to deal with the problems that are posed by constrained control inputs. Essentially, in an anti-windup design we look at the control input saturation as a change in mode in which the plant operates. For example, a linear plant with input saturation is in its linear mode if its actuators do not saturate. When saturation occurs, the mode of operation switches and the plant is no longer in the linear mode. If the plant is controlled by a linear controller, the actual input to the plant will be different from the control input set by the controller when a change in mode occurs. As a result, the states of the controller are wrongly updated. This effect is called *controller windup*. An idea implemented in an anti-windup design is that each mode of operation has a linear controller designed to satisfy the performance objective corresponding to that mode. In this way, controller windup is avoided. The structure of an anti-windup design is shown on the figure 1.3. The difference $u - u_p$ is an indication of operational mode of the plant. When $u - u_p$ is equal to zero, the plant is in its linear mode, controlled with controller $K$. When the difference $u - u_p$ becomes larger than zero it indicates that saturation has occurred and the control structure changes so that controller $K_{sat}$ determines the input to the plant,

also. For details and different anti-windup structures we refer to [79] and [6].

The main drawback of anti-windup designs is that they can be dealing only with input constraints. More precisely, anti-windup designs are suitable for a specific type of input constraints and that is actuator saturation. Actuator saturation is frequently encountered in control engineering practice but there are also other types of input saturation, for example actuators with rate constraints. For these, more general input constraints, anti-windup designs are becoming quite complex in structure. When the system is not saturated, the control system has linear behavior. When the input saturates, the overall system is not linear anymore and the performance of the system can be poor, because the design of anti-windup is aimed to preserve stability. In short: anti-windup is suitable for the plant with saturated actuators when the system does not saturate often. When the system does not saturate often, a linear controller controls the plant most of the time and it determines the overall system performance. The saturation is viewed as a rare exception and the anti-windup is designed to handle that exceptional case. When the system saturates more often or the state of the system is subject to constraints itself, anti-windup designs are not suitable for handling constraints.

## 1.3 Control of linear systems with input constraints

In many practical applications, an important limiting factor for control is the saturation of the actuators. That is, there is a significant discrepancy between a demand signal $u$ for an actuator and the signal $\sigma(u)$ that is actually fed in the system. Here $\sigma$ is a function $\sigma : \mathbb{R} \to \mathbb{R}$, usually continuous, often scaled to have $\sigma(0) = 0$. Some examples of the saturation function are depicted on the figure 1.4. The importance of saturated actuators in applications as well as difficulties that arise when one aims to analyze a more general setup with general constraints on inputs and states motivated researchers to look at the simple extension of the traditional linear, time invariant, state space setup. This simple extension uses a vector-valued saturation function $\sigma$ defined as

$$\sigma(s) = \begin{bmatrix} \bar{\sigma}(s_1) \\ \bar{\sigma}(s_2) \\ \vdots \\ \bar{\sigma}(s_m) \end{bmatrix} \qquad \text{with} \qquad \bar{\sigma}(s) = \begin{cases} s & \text{if } |s| \leq 1 \\ -1 & \text{if } s < -1 \\ 1 & \text{if } s > 1. \end{cases}$$

and is in the form of the following system

$$\begin{aligned} \dot{x} &= Ax + B\,\sigma(u) \\ y &= Cx \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$. This extension treats one aspect of constraints,

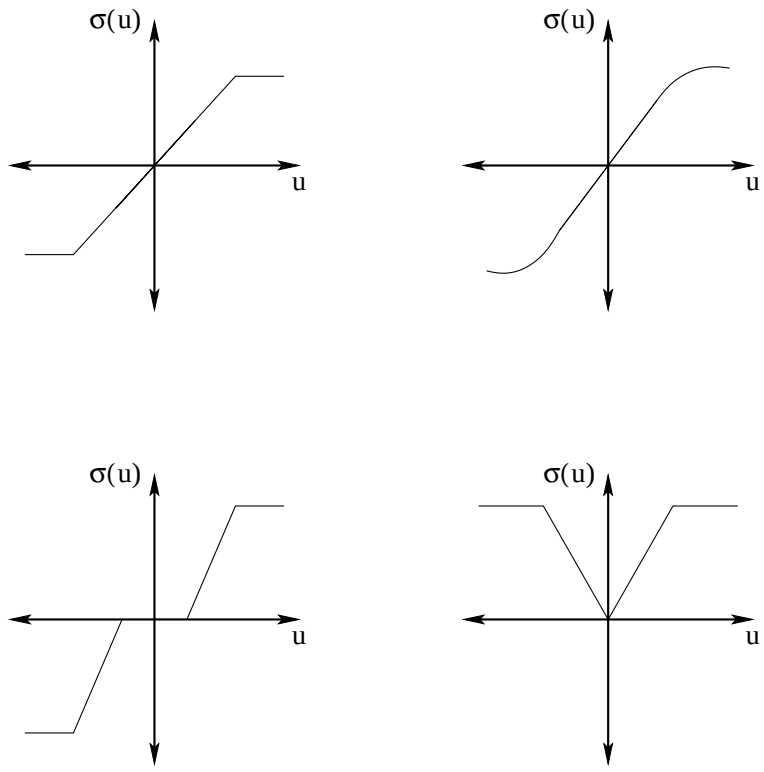


Figure 1.4: Some examples of the saturation function

namely the saturation, and does not treat hysteresis, rate limits and other static nonlinearities. Analysis of the system (1.1) showed some important facts about systems that are subject to input constraints. First results are concerned with the issue of internal stabilization of the system (1.1). An important notion in this context is the notion of *null controllability*. A state $x \in \mathbb{R}^n$ of the system (1.1) is called null controllable if it is possible to steer the system (1.1) to the origin of the state space from the state $x$. The set of all null controllable points in the state space is called a *recoverable set*. If the system (1.1) is such that all states in $\mathbb{R}^n$ are null controllable we say that the system (1.1) is *globally asymptotically stabilizable*. It has been shown in [135] that a linear continuous-time system subject to amplitude saturation is globally asymptotically stabilizable if and only if the matrix pair $(A, B)$ is stabilizable and all eigenvalues of the matrix $A$ lie in the closed left half of the complex plane. Another important result can be found in [59] and [142] which states that, in general, a linear feedback control law can not be used for global asymptotic stabilization of the system (1.1).

This result initiated research where a non-linear feedback is proposed to deal with global asymptotic stabilization of the linear system (1.1) subject to input saturation (see for example [144] where such a design is proposed for a certain class of the linear systems).

An other view point has been taken in [88, 89]. In these papers, instead of searching for a controller that will achieve a global stabilization, the authors proposed a *semi-global stabilization* approach. In the semi-global stabilization setting, one seeks for a family of feedbacks $u = f_\varepsilon(x)$ such that for any compact set $\mathcal{X} \subset \mathbb{R}^n$, there exists $\varepsilon^*$ such that $u = f_\varepsilon(x)$ with $\varepsilon < \varepsilon^*$ is such that the set $\mathcal{X}$ is contained in the recoverable set for the system (1.1). It is shown that, under the assumption that the system (1.1) is globally asymptotically stabilizable, the feedback that achieve this can be chosen to be linear and that the set $\mathcal{X}$ can be made arbitrary large provided that the gain of the linear feedback is sufficiently small. A linear feedback $u = f_\varepsilon(x)$ that will achieve semi global stabilization for the system (1.1) can be found via so called low-gain design. The low-gain linear feedback for (1.1) is given by

$$u = f_\varepsilon(x) := -B^T P(\varepsilon)x \tag{1.2}$$

where $P(\varepsilon) > 0$ is the solution of the parameterized Riccati equation defined as

$$0 = A^T P(\varepsilon) + P(\varepsilon)A - P(\varepsilon)BB^T P(\varepsilon) + Q(\varepsilon)$$

with a continuously differentiable matrix-valued function $Q : (0, 1] \to \mathbb{R}^{n\times m}$ such that $Q(\varepsilon) > 0$, $\frac{dQ(\varepsilon)}{d\varepsilon} > 0$ for any $\varepsilon \in (0, 1]$ and $\lim_{\varepsilon\to 0} Q(\varepsilon) = 0$. For a given set $\mathcal{X}$, the low-gain feedback (1.2) stabilizes the system (1.1) for all states in $\mathcal{X}$, with $\varepsilon$ small enough. When the system (1.1) is not globally asymptotically stabilizable, the set $\mathcal{X}$ can not be chosen arbitrary. In general however, the size of the recoverable set will grow as $\varepsilon$ is getting smaller.

The main feature of the low-gain design is that the controller (1.2) avoids saturation of the actuator and the overall system therefore remains linear. The main drawback of the low-gain design is a slow response for large recoverable sets, because of the low gain in the controller that is necessary when the recoverable set is large.

To cope with the drawback of a slow response a low-and-high design technique is proposed in [90]. The low-and-high gain feedback is given by

$$u = -(1 + \rho)B^T P(\varepsilon)x, \qquad \rho > 0. \tag{1.3}$$

With the feedback (1.3) the control signal gets saturated as $\rho$ increases thus the controller takes advantage of the available input "power". However, the overall system is nonlinear which makes analysis more difficult, when compared with the low-gain design. For details about low-gain and low-and-high-gain design as well as for in-depth view of the approach described in this subsection we refer to [125].

## 1.4 Minimization of the maximum peak-to-peak gain: $\ell_1$ optimal control problem

The effect of constraints can be expressed in terms of time-domain bounds on the amplitude of signals. This observation motivated a formulation of a new optimization problem in [150], the so called $\ell_1$ optimal control problem. First results (see [46–48]) brought considerable attention to the problem of $\ell_1$ optimization.

To present the $\ell_1$ optimal control problem, we first introduce the $\ell_\infty$ norm on the space of all real vector-valued sequences of dimension $n$. Suppose that $x = (x(t))_{t=0}^{\infty}$ with $x(t) \in \mathbb{R}^n$ is such a sequence. Then, its $\ell_\infty$ norm is defined as

$$\|x\|_\infty = \sup_t \max_i |x_i(t)| \qquad i \in [1, n].$$

and we say that $x$ belongs to $\ell_\infty$ whenever $\|x\|_\infty$ is finite. Note that for scalar valued signals ($n = 1$), $\|x\|_\infty$ expresses the maximal amplitude of the signal. This makes a bound on this norm a natural choice when the magnitude of the signal has to be limited because of constraints. Consider the plant given with the state space system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Ew(t) \\ y(t) &= C_y x(t) + D_y u(t) \\ z(t) &= C_z x(t) + D_z u(t) \end{aligned} \tag{1.4}$$

where $u$ is the control input with $u(t) \in \mathbb{R}^m$ and $x$ is the state with $x(t) \in \mathbb{R}^n$. The second equation describes the measured output $y$ with $y(t) \in \mathbb{R}^d$. The output to be controlled is $z$ with $z(t) \in \mathbb{R}^p$. The disturbance $w$ with $w \in \mathbb{R}^q$ belongs to $\ell_\infty$.

In the standard $\ell_1$ optimal control setup, the plant (1.4) is assumed to be controlled by a feedback controller

$$u = Ky. \tag{1.5}$$

The controller $K \in \mathcal{K}$ is a linear, time invariant operator. Suppose that we are faced with the problem of keeping the regulated output vector $z$ constrained. The objective can be expressed as a requirement on the $\ell_\infty$ norm of the signal $z$

$$\|z\|_\infty \leq k_z. \tag{1.6}$$

It can be shown that objective (1.6) can be expressed as a condition on the $\ell_1$ operator norm.

Especially for discrete-time systems, there is a solid theory available for the design of linear controllers that achieve (1.6) (see for example [52]). The approach taken to solve $\ell_1$ control problem is based on linear programming. Note that we give here a simple case where only constraints on output are considered. The approach can be extended so that the constraints on input and/or state are taken into account as well (see [45]).

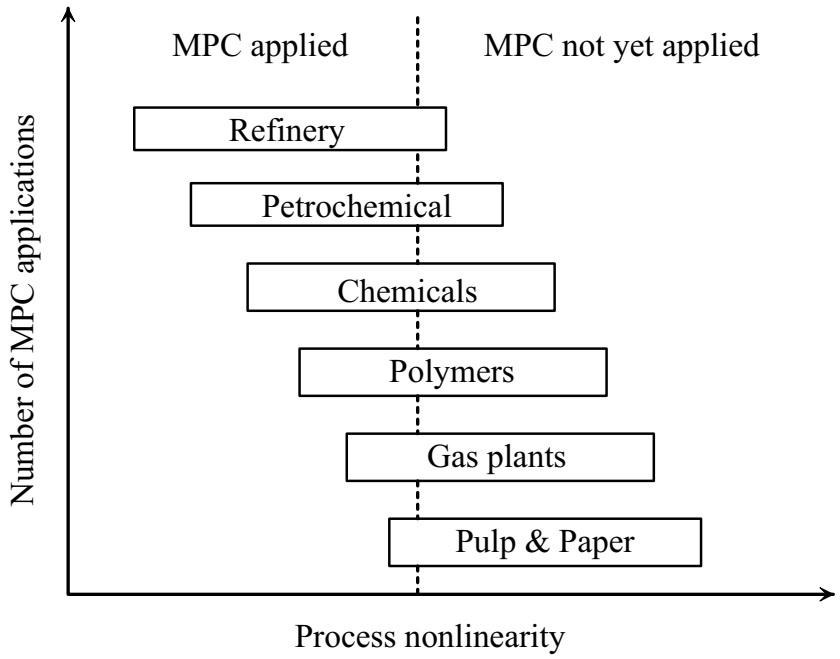


Figure 1.5: Industrial applications of model predictive control, see [114]

A problem with the approach based on $\ell_1$ operator norm is that the available methods for solving such a problem are essentially constrained to linear controllers. When one seeks a solution of $\ell_1$ control problem among linear controllers only, the result is a controller with a very high order in discrete-time case or a controller with an infinite order in the continuous case. The optimal controllers for $\ell_1$ control problem are nonlinear (see [138]). Unfortunately, there is no available theory that will teach us how to obtain nonlinear optimal controllers for $\ell_1$ control problem in general case.

## 1.5 Model predictive control

The origin of model predictive control is in attempt of control engineers to provide a control technique that is adequate for handling constraints and be able to cope with problems raised by nonlinearities and uncertainty. Despite the lack of solid theoretical foundations of the early model predictive controllers, model predictive control has had a significant and widespread impact on industrial process control [114]. There are many proposals for model predictive control but all of them have common ingredients. The first of these ingredients is the model for predicting the “future” behavior of the

plant to be controlled. The prediciton is performed over a finite time interval starting with the current time and extending to some time in the "future", usually called the control horizon. Next, the performance of the system over the control horizon is accessed by computing the cost. Model predictive controller has to solve the optimization problem in which it has to find an input sequence over the control horizon so that the cost is minimized and the constraints on input and the state are respected. In model predictive controller the optimization described above is performed at each time step but only the first input of the obtained optimal input sequence is applied to the plant. In the next time step, a new measurement is collected and the optimization is repeated over the control horizon that is "shifted" for one step. This implementation is called the *receding horizon* implementation.

Model predictive control framework plays an important role in the work that is presented in this thesis. More detailed overview of model predictive control is given in chapter 2.

## 1.6 Goals of the thesis

In this thesis we deal with control of constrained systems that are subject to stochastic disturbances. The main motivation for dealing with control of such systems is that there is no method available that adequately deals with this problem, despite the fact that stochastic, constrained systems are often encountered in real world problems. Goals of the thesis are to

1. Formulate a mathematical problem for the synthesis of a controller that will achieve desired performance of the controlled system. More precisely, to minimize a performance measure that captures desired performance while respecting constraints in the face of stochastic disturbances.

2. Deduce verifiable conditions under which the problem formulated in 1. is solvable.

3. Formulate a solution concept for the problem in 1. that is based on the model predictive control technique.

4. Create feasible computational algorithms for the synthesis of controllers that solve control problems from 1. within the solution setup from 3.

5. Investigate convergence properties of the approximate solutions obtained by computational algorithms from 4.

The first goal is concerned with the fundamental basis of the work. The problem setup has to be general enough to capture the true nature of the challenge that constraints are posing to the control of real world dynamical systems but not too complex for a

mathematical analysis and a synthesis of a feasible controller. The plant to be controlled is assumed to be from a class of linear, time invariant systems and it is subject to disturbances and constraints. Note that constraints are essentially making the control problem nonlinear, even when the plant is linear. Since it is our aim to derive computational algorithms that will make possible to implement controller on some kind of digital computational device, it is convenient to use a discrete time system as a prototype of the plant to be controlled. Although simple, the discrete, linear, time invariant system with a disturbance model has a wide range of applications in control theory and practice. The model/plant mismatch and uncertainty in the knowledge of the true plant dynamics can be viewed as disturbances that are included in the model.

In practice, the controller has to keep the plant in the desired working regime despite the disturbances and/or mismatch between the model and the plant or to ensure that all important variables in the process evolve according to some desired profile. Mathematically, with suitable extensions and modification of the linear, time invariant model with a disturbance model, it is possible to capture these requirements as an objective of steering the plant to the equilibrium point. The steering has to be performed by taking into consideration various objectives like a minimum use of energy (i.e economic objective) and required demands on the quality of product which can be viewed as constraints on the state of the model. These demands are captured in the performance measure that is a function of the model's state and the input to the model. Then, a controller has to be designed that will respect constraints on the state as much as possible and that will keep the performance measure as low as possible despite constraints on the input and disturbances, thus giving the best possible performance of the overall system. The model, constraints on the input and the state of the model, a performance measure and the optimization problem of designing the controller that will control the plant optimally with respect to the performance measure are the elements of the problem setup that is proposed in this thesis.

It is natural to design a controller that will respect constraints on the state as much as possible. A constrained input limits ability to control the plant and disturbances are posing additional difficulties to the control problem. It is possible that demands on the control system are too harsh for the problem at hand and that no solution exists. Given the system, the performance measure and the level of stochastic disturbances it is natural to ask how well constraints on the state can be respected. To answer this question it is necessary to derive a solvability condition within the mathematical problem setup, which is the second goal of the thesis. With a solvability condition it is possible to compute the limit of performance that can be achieved for the given problem, which is necessary information when one judges how successfully a design of the controller can be performed.

The third and fourth goals of the thesis are concerned with the solution of the optimization problem from goal 1. The only feasible approach is the model predictive control technique. The difficulty is that design methods for model predictive controllers that are available in the literature are not suitable for dealing with constrained

systems that are subject to stochastic disturbances. Therefore, it is necessary to develop a new solution concept that is based on the predictive control technique but can handle constrained systems that are subject to stochastic disturbances better than proposals available in the literature. This is the third goal of this thesis. The approach reported in the thesis is based on optimization in closed loop and the standard algorithms for model predictive control that are performing optimization in open loop via quadratic programming algorithms can not be applied. Therefore, we set the fourth goal of the thesis to develop algorithms by which model predictive controllers based on the solution concept from goal 3. can be implemented.

## 1.7 Outline of the thesis

**Chapter 2.** Model predictive control is the main tool used in the thesis. In chapter 2 we give an overview of available results in model predictive control. The focus in this chapter is on two major issues: techniques that are currently used to ensure stability of model predictive controllers and techniques by which disturbances are handled in model predictive control schemes. The main conclusion of chapter 2 is that none of the techniques is adequate to deal with constrained systems that are subject to stochastic disturbances. That shows that a new approach to predictive control of such systems is necessary and this gives a strong motivation for the work reported in this thesis.

**Chapter 3.** An optimal control problem for constrained systems that are subject to stochastic disturbances is a complex problem when one considers constraints on the input and the state together. Therefore, a better approach is to look at the simpler case first. For the problem of optimal control of constrained systems the simpler case is the case in which only constrained inputs are considered and all states are assumed to be measured. Synthesis of model predictive controllers for linear systems with constrained input and stochastic disturbances is given in chapter 3. In this chapter, we propose a problem setup (goal 1.) that consists of a linear, time invariant, discrete time model of the plant with a Gaussian white noise disturbance. The input is, of course, assumed to be constrained. The performance measure is a quadratic cost function. It is shown that the model predictive controller has to be designed in the closed loop to deal with the constrained systems that are subject to stochastic disturbances. A solution (goal 3.) based on the model predictive technique that deploy such an approach is proposed in chapter 3 for linear stochastic systems with the constrained input. Since the cost is stochastic because of the stochastic disturbance, an expectation of the cost has to be computed recursively and the optimal feedback strategy has to be determined in each time step in the control horizon.

The algorithm for the practical implementation of the controller developed in the chapter 3 is based on the empirical mean, because a computation of the true expectation is difficult for all but very simple systems. The solution obtained by the empirical mean is an approximate one, and the accuracy of the solution depends on the number of

samples taken to compute empirical mean. Convergence properties of the solution obtained by the algorithm are investigated and a convergence of the result to the optimal solution is established. At the end of the chapter, we give two numerical examples, one of them is based on the model of an ill-conditioned plant, for which feasibility of the approach is shown.

**Chapter 4.** In chapter 4 the problem setup and the solution concept based on model predictive control is extended to the general case in which a stochastic system with constrained input and constraints on the state is considered. As is shown in chapter 4, it is not possible to find a solution when one aims to respect the state constraints as hard constraints i.e. to find a controller that will ensure that constraints are respected with probability one. Therefore, it is necessary to modify the problem setup to handle the state constraints in addition to the constraints on the input. We modify the cost function so that the performance of the controlled system is preserved when the state satisfies constraints and is far away from the boundary of the state constraint set but when the system gets close to the constraint boundary or vioaltes the constraints the cost function penalizes a probability of the constraint violation on the first place. It is natural to keep this probability as low as possible but because of the stochastic disturbance, the probability of the constraint violation can not be arbitrary small. We derive a solvability condition (goal 2.) that shows how large a penalty on constraint violation can be imposed with respect to the level of the Gaussian white noise disturbance before the cost function becomes unbounded. By using this result, it is possible to find the largest possible penalty on the constraint violation, thus to keep the probability of state constraint violation as low as possible for the problem in hand.

Because of the modifications, the cost function can not be quadratic as in the case with only input constraints, and results from chapter 3 can not be directly applied to the problem setup extended to handles the state constraints in addition to constraints on input. We develop a new algorithm to find a solution within this extended problem setup. The algorithm is based on predictive control techniques, computation of the empirical mean and optimization in closed loop. A convergence result for the new algorithm is derived and reported in chapter 4. Finally, we give a simulation study that shows feasibility and benefits of the approach.

**Chapter 5.** The approach presented in chapters 3 and 4 is extended in chapter 5 to the measurement feedback case. We remove the assumption that the state of the system is available for feedback and show how algorithms from the previous chapters can be used in the measurement feedback framework. A design is based on the use of Kalman filter for state estimation. The optimization problem is then solved with the cost function that uses the estimated state instead of the true state of the controlled plant. The chapter is concluded with simulation experiments in which the algorithm from the chapter is applied to the double integrator system.

**Chapter 6.** The thesis is concluded with chapter 6 where we give a summary of contributions made in the thesis and the outline of topics for further research.

Contributions of the thesis are twofold. The first set of contributions is made with regard to the model predictive control of constrained, stochastic systems. In this thesis, we develop a novel approach to the model predictive control of such systems, that is based on the optimization in closed loop over the control horizon and stochastic sampling of the disturbance.

The second set of contributions has been made in more general framework of the optimal control of stochastic systems that are subject to input and state constraints. We present a novel problem setup for control of such systems and give initial results that are concerned with solvability of the posed optimization problem.

# 2

# *Model predictive control: an overview*

The goal of this chapter is to give a detailed overview of model predictive control technique. Beside the formulation of model predictive control, we focus on two important issues: available techniques for stabilizing a model predictive control scheme and techniques for incorporating the disturbance.

## 2.1 A standard formulation of Model Predictive Control

Model predictive control or predictive control is a control technique in which the current control action is obtained by minimizing a cost criterion, defined on a finite time interval, ranging from the current time to some future time instant. The current state of the plant is used as an initial state for the optimization and the optimization yields an optimal control sequence from which the first element is applied to the plant. At the next time instant the procedure is repeated. The development of this control technique was initiated by needs and concerns of industry. Predictive control has been seen as one of few suitable methods that are able to handle constraints. Early publications are mainly concerned with different models for the prediction and ad hoc methods for constraints handling. In those publications there is a number of proposals for predictive control such as IDCOM (identification and command), DMC (dynamic matrix control), quadratic matrix control (QDMC) etc. (see [44, 60, 94, 113, 120, 121] for the development of model predictive control techniques reported in the process control literature). The success of the model predictive control in industry has inspired intensive academic research where issues like stability are addressed directly. Today, there exists a vast literature dealing with model predictive control. Here we give a reference to the review papers [61, 84, 85, 96, 97, 113, 116, 121] that describe its historical development. There exist several books dealing with different aspects of model predictive control. An early attempt to give a sound theoretical foundation to model predictive control without constraints can be found in the book [20]. In [134], the relationship between different predictive control techniques has been investigated. The books [28] and [91] give a recent overview of model predictive control techniques.

In the model predictive control literature the plant to be controlled is usually described

in terms of a difference equation of the form:

$$x(t+1) = f(x(t), u(t)) \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state and $u(t) \in \mathbb{R}^m$ is the input at time $t$, $t \in \mathbb{Z}_+$. The function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function with

$$f(0, 0) = 0.$$

The input $u(t) \in \mathbb{R}^m$ and the state $x(t) \in \mathbb{R}^n$ are constrained in that

$$u(t) \in \mathbf{U}$$

and

$$x(t) \in \mathbf{X}$$

for all time instances $t$, where $\mathbf{U}$ is a closed, convex subset of $\mathbb{R}^m$ and $\mathbf{X}$ is a closed, convex subset of $\mathbb{R}^n$ with $0 \in \mathbf{U}$ and $0 \in \mathbf{X}$.

Typically, the objective of a model predictive controller is to steer the initial state $x$ to the origin or to an equilibrium state in a desirable way. Performance is expressed via a performance measure, usually called the cost, and a "desirable way"means that the plant has to be controlled so that the performance measure is minimized. Other objectives like reference trajectory tracking or transition of the system to the set point state can be translated to the objective of steering the system state to the origin by a suitable extension of the model or a choice of the performance measure.

The performance is computed over a finite interval $T := [0, N]$ where $N > 0$ and this interval is usually called the *control horizon* with length $N$. The performance is calculated by means of a prediction of the state variable on the horizon $T$. Specifically, the *predicted state* $x_N : \{0, \cdots, N+1\} \to \mathbb{R}^n$ is defined by the recursion according to

$$x_N(k+1) = f(x_N(k), v_k) \tag{2.2}$$

with an initial condition $x_N(0) := x$ where $v_k \in \mathbf{U}$ is the input to the model (2.2) at $k \in T$. We consider a set of input sequences $v := (v_k)_{k=0}^N$ denoted as $\mathcal{V}$. It is important to observe that the predicted state $x_N$ is generated by the model (2.2) where $v$ is an open-loop strategy, i.e. a strategy that only depends on time, not on other measured variables.

Conceptually, at time $t$ the initial condition $x_N(0)$ in (2.2) is set as $x_N(0) = x(t)$ where $x(t)$ is the measured state of the system (2.1) at time $t$. The predicted state at $k$, $x_N(k)$, is prediction of the state $x(t+k)$ and the input $v_k$ is the "future" input $u(t+k)$. Note however that the model (2.2) is time invariant. Therefore, the current time can be set to zero, without loss of generality. Variables involved in the design of a predictive controller can be defined as functions of $k$, $k \in T$ rather than the functions of the current time.

Next, consider the cost $J : \mathbb{R}^n \times \mathcal{V} \to \mathbb{R}$:

$$J(x, v) = \sum_{k \in T} g(x_N(k), v_k) + G\big(x_N(N+1)\big) \quad x \in \mathbb{R}^n \tag{2.3}$$

subject to $x_N(0) = x = x(t)$. The function $g : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ is a convex and nonnegative function with $g(0, 0) = 0$. Also $G : \mathbb{R}^n \to \mathbb{R}$, $G(0) = 0$ is convex, nonnegative and referred to as the *end point penalty*.

This typical model predictive setup is depicted on figure 2.1. The optimization prob-

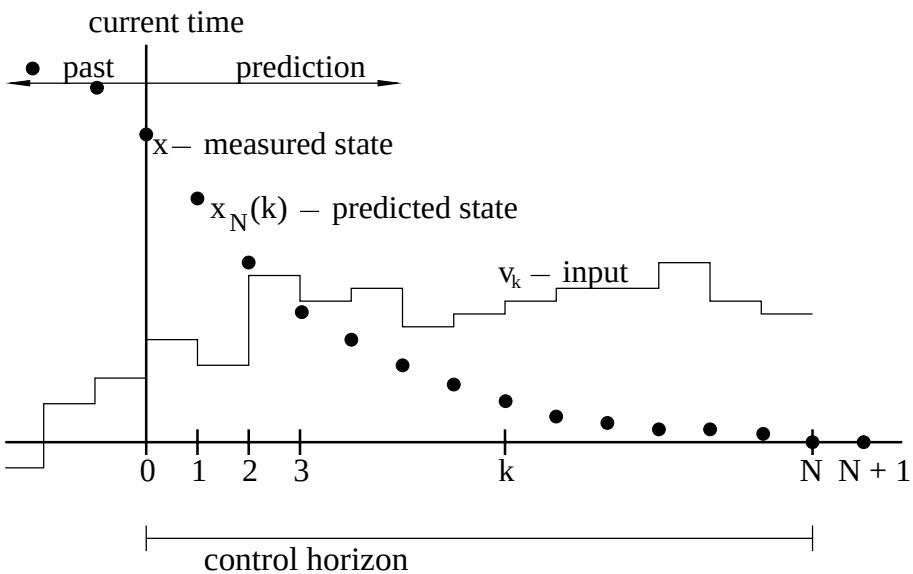


Figure 2.1: Optimization setup in model predictive control

lem to be solved by a predictive controller is given next.

**Problem 2.1.1** Suppose that, at time $t$, the measured state is $x$. Find an optimal input $v^* \in \mathcal{V}$ such that

$$J(x, v^*) \leq J(x, v) \tag{2.4}$$

for all $v \in \mathcal{V}$. In addition determine the optimal cost given by:

$$V(x) := \inf_{v \in \mathcal{V}} J(x, v).$$

If problem 2.1.1 admits a solution, it yields an optimal input $v^* = \big(v_k^*\big)_{k=0}^N \in \mathcal{V}$ that depends on the current state $x$. Only the input $v_0^*$ is applied to the plant i.e. we set

$$u(t) = v_0^*$$

as the input (2.1) at time $t$. By (2.1) this yields the next state $x(t+1)$. At the next time instant $t+1$, the optimization problem 2.1.1 is solved for the new state $x(t+1)$. This on-line computation of the optimal input is called a *receding horizon optimization*. We can say that the optimization problem 2.1.1, implemented in a receding horizon manner, when ranging over all possible conditions $x \in \mathbb{R}^n$, implicitly defines a time invariant model predictive control law that associates with the measured state $x$ the input $v_0^*$. That is, it defines a map $\eta : \mathbb{R}^n \to \mathbf{U}$ as

$$\eta(x) = v_0^*. \tag{2.5}$$

The optimal cost is given by

$$V(x) = J(x, v^*). \tag{2.6}$$

When the model predictive controller (2.5) is applied to the plant (2.1), the closed loop system is

$$x(t+1) = f\big(x(t), \eta(x(t))\big). \tag{2.7}$$

Typically, the optimization problem 2.1.1 is solved numerically. In the context of numerical optimization, an important requirement is convexity of the optimization problem. The optimization problem 2.1.1 is convex if the cost function (2.3) is convex and sets $\mathbf{U}$ and $\mathbf{X}$ are convex. A convex function has only global minima and standard algorithms exist for minimization of convex functions (see [58, 107, 122] for general introduction to convex optimization and [24] for convex optimization in control).

There are several types of cost functions that can be found in the model predictive control literature. A much used criterion is a quadratic cost where the function $g$ is chosen as:

$$g(x, u) = \|x\|_Q^2 + \|u\|_R^2 \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \tag{2.8}$$

where $\|x\|_Q^2 := \langle x, Qx\rangle$, $Q \in \mathbb{R}^{n\times n}$ and $\|u\|_R^2 = \langle u, Ru\rangle$, $R \in \mathbf{R}^{m\times m}$, $Q \geq 0$, $R \geq 0$. Matrices $Q$ and $R$ are known as the weighting matrices. With the quadratic cost, polyhedral set $\mathcal{V}$ and the plant (2.1) defined as the linear, time invariant system

$$f(x(t), u(t)) = Ax(t) + Bu(t), \quad A \in \mathbb{R}^{n\times n} \text{ and } B \in \mathbb{R}^{m\times m} \tag{2.9}$$

the optimization problem 2.1.1 can be rewritten as a quadratic programming problem (see example 2.1). There exist standard algorithms for solving a quadratic programming problem (see [23, 149] for example). The quadratic programming problem is a convex optimization problem. This feature makes this formulation of model predictive control very attractive from the implementation point of view and a large portion of the model predictive control literature is devoted to this special case.

Another example that can be found in literature is the case where the $\ell_\infty$ norm is used in the cost function (see [30] where $\ell_\infty$ norm in the cost is proposed in the disturbance rejection context). In its simplest form the function $g$ is chosen as:

$$g(x) = \|x\|_\infty \qquad x \in \mathbb{R}^n \tag{2.10}$$

where $\|x\|_\infty$ denotes $\ell_\infty$ norm of $x$ defined as $\|x\|_\infty := \max_i |x_i|$. The cost with (2.10) minimizes the error between the state and the origin (equilibrium state) measured in the sense of the amplitude of $x$. The control effort is not taken into account in the cost (2.10) which may result in a type of so called "bang - bang" control where the input is "switching" very quickly between its constraint boundaries. This problem is usually handled by introducing additional weight on the control input. When the model is linear and the cost is based on the infinity norm the solution of the optimization problem 2.1.1 can be found by a linear programming algorithm. Linear programming is a standard and well documented optimization technique.

In [1] the $\ell_1$ norm in the cost function (2.3) has been proposed. The function $g$ is then chosen as:

$$g(x, u) = \|x\|_1 + \lambda\|u\|_1 \tag{2.11}$$

where

$$\|x\|_1 := \sum_{i=1}^{n} |x_i|$$

denotes $\ell_1$ norm of a vector $x$. The optimization problem with the cost in which $g$ is chosen to be (2.11) can be solved as a linear program. In the literature, it is often claimed that a main reason for adopting a model predictive control formulation based on the linear programming is that linear programming problems can be solved more quickly than quadratic programming problems. With the available computational power today, the issue of difference in the computational load between linear and quadratic programs is no longer relevant. Nevertheless, the quadratic cost (2.8) that allows to solve the optimization problem 2.1.1 as a quadratic program is used in the most applications.

**Example 2.1** To illustrate a standard approach to model predictive control, we consider the problem of steering a cart to a prescribed position. The cart is shown in figure 2.2 where $s(t)$ denotes its position from some set point at time $t$. The objective is to steer the cart from an initial position $s(0) = 10$ where the cart is at rest, to the set point $s(K) = 0$ in finite time $K$ by applying some force $u$ to the cart. The constraint to be respected is:

$$s(t) \geq 0.$$

In addition, we assume that the input force $u(t)$ is constrained. The constraints on input are given by:

$$-u_0 < u(t) < u_0$$

where $u_0 > 0$.

To simplify the exposition, we assume that all involved dimensions are normalized. The differential equation that describes the motion of the cart is given by

$$\frac{d^2 s}{dt^2} = u. \tag{2.12}$$

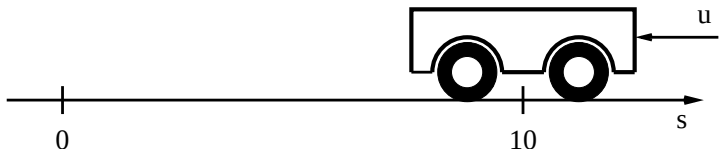


Figure 2.2: Steering the cart

A discrete time, state space representation of (2.12) obtained with the unit sample time and with the input assumed to be constant between the samples is given by

$$\begin{aligned} x(k+1) &= Ax(k) + Bu(k) \\ s(k) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x(k) \end{aligned} \tag{2.13}$$

where

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We design a model predictive controller for the cart, based on the model (2.13) and the following specifications.

- The length of the control horizon is $N = 4$.
- The cost function is quadratic. The function $g$ is chosen as (2.8) with

$$Q = \begin{pmatrix} 0.7 & 0 \\ 0 & 0.7 \end{pmatrix} \qquad R = 1.$$

- The end point penalty is chosen as:

$$G(x) = \|x\|^2_{Q_{end}}$$

  where $\|x\|^2_{Q_{end}} := \langle x, Q_{end} x \rangle$ with

$$Q_{end} = \begin{pmatrix} 1.6 & 0.9 \\ 0.9 & 1.33 \end{pmatrix}.$$

Given a state $x$ and an input $v$, the predicted state $x_N$ can be computed as:

$$x_N = Fx + Hv$$

with

$$F = \mathrm{col}\begin{pmatrix} A^0 & A & A^2 & A^3 & A^4 \end{pmatrix}$$

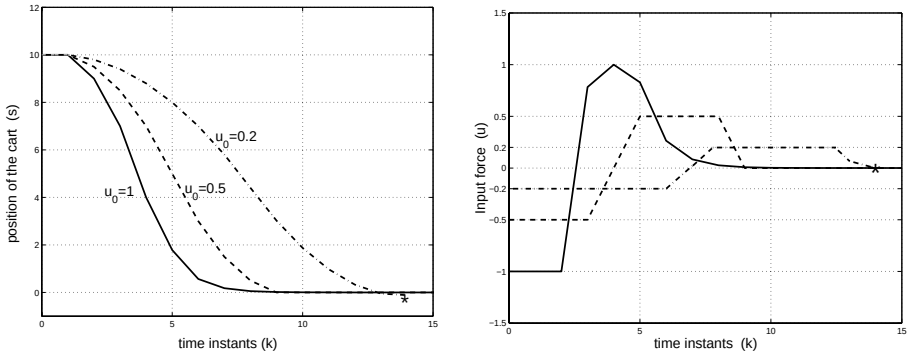


Figure 2.3: Cart is controlled by standard model predictive controller

and

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 \\ AB & B & 0 & 0 \\ A^2B & AB & B & 0 \\ A^3B & A^2B & AB & B \end{pmatrix}.$$

Then, the cost (2.3) can be written in a compact form as:

$$J(x, v) = v^T \Gamma v + 2x^T X_u v + x^T (Y + Q)x \tag{2.14}$$

with:

$$\Gamma = H^T \bar{Q} H + \bar{R} \qquad X_u = F^T \bar{Q} H \qquad Y = F^T \bar{Q} F$$

$$\bar{Q} = \begin{pmatrix} Q & 0 & 0 & 0 & 0 \\ 0 & Q & 0 & 0 & 0 \\ 0 & 0 & Q & 0 & 0 \\ 0 & 0 & 0 & Q & 0 \\ 0 & 0 & 0 & 0 & Q_{end} \end{pmatrix} \qquad \bar{R} = \begin{pmatrix} R & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & R \end{pmatrix}.$$

Problem 2.1.1 with cost (2.14) can be solved as a standard quadratic program. At each time $t \in \mathbb{Z}_+$ we solve the optimization problem 2.1.1 with cost (2.14) and with a state of the plant at $t$ as an initial condition. In this way, the controller (2.5) for the cart is implemented in a receding horizon fashion.

We test the design by simulations for different values of input constraint $u_0$. Results of the simulations are shown in figure 2.3. For $u_0 = 1$ and $u_0 = 0.5$ the controller is able to steer the cart to the origin while respecting constraints on input and the state. Note that as the amount of input force that can be applied is getting smaller the response of the system is getting slower. With the input constraint $u_0 = 0.2$, the maximum amount of the input force is not large enough to slow down the cart to the rest in only

4 time steps from the moment when the constraint violation is predicted by the model predictive controller. Therefore, at $k = 14$ a constraint violation occurs. Simulation stops because it is not possible to find an input sequence that minimizes (2.14) without violating the constraints. This is known as the "infeasibility". •

This example points out an often encountered difficulty with the optimization problem 2.1.1. When constraints are "hard" i.e. no constraint violation is allowed, it is possible that for some of the initial states no solution exists. Evidently, this is not a desirable outcome. The methodology that is often used to avoid this problem is so called constraint "softening" (see [50, 58, 131]).

Essentially, the softening of constraints is performed by adding new variables, so called “slack variables”, to the optimization problem 2.1.1 which are defined in such a way that they are non-zero only if the constraints are violated. The cost function (2.3) is then modified so that the value of the “slack variables” is heavily penalized. In that way, the value of the “slack variables” will be kept as small as possible, therewith respecting the constraints as much as possible.

An intrinsic feature of the optimization problem 2.1.1 is that the optimization over the control horizon is performed in open loop. On the other hand, the model predictive controller (2.5) is a feedback controller. Therefore, there is a discrepancy between the assumption of an open loop control that is used in the prediction model (2.2) and the actual control of the plant which is in closed loop. Instead of determining “off line” an optimal control law, in model predictive control the optimal control problem is solved “on line” for the current state of the plant.

## 2.2 Stability issues in model predictive control

It is well known that under the assumptions of stabilizability and detectability a standard linear quadratic, infinite horizon optimal control problem yields an optimal controller that is also stabilizing (see [148] for a recent exposition). A model predictive controller (2.5), that solves a finite horizon optimization problem 2.1.1 is not necessarily stabilizing even if the cost function (2.3) is quadratic and the model (2.2) is linear.

In the industrial practice, the issue of stability of model predictive controllers is addressed mainly in an “ad hoc” manner. The closed loop stability is achieved by tuning the parameters involved in the design (the length of the control horizon, the choice of weighting matrices etc.). Experience gained over years has been collected in “tuning rules” which serve as a guideline for tuning the model predictive controller. On the other hand, academic research has addressed the stability of the predictive controller, leading to more comprehensive results. There are two basic modifications of the original model predictive control formulation that have been proposed to yield a stabilizing model predictive controller. The first one is to add an end point penalty

at the end of the control horizon. This is one of the earliest modifications proposed (see [20, 97, 117]) and today it becomes a standard ingredient of the model predictive control. The second proposal that can be found in the literature is to constrain the state at the end of the control horizon in the *terminal constraint set* (see [37, 100, 130]).

Conceptually, the end point penalty will increase the cost if the state at the end of the control horizon is not in the origin. Depending on the "shape" of the end point penalty we can make the weight on the end point state arbitrary large. For example, consider the end point penalty of the form:

$$G(x) = \begin{cases} 0 & \text{if } x = 0 \\ \infty & \text{if } x \neq 0. \end{cases} \tag{2.15}$$

The cost (2.3) with the end point penalty (2.15) is finite only if the end point state is in the origin. If there exists a controller (2.5) that solves the optimization problem 2.1.1 with the cost function (2.3) that contains the end point penalty (2.15) then, this controller is also stabilizing for the system (2.1), under some mild conditions. Having an infinite end point penalty is not desirable in practice, because it often leads to undesirable transient behavior. Typically, the end point is chosen to be a quadratic function of the state:

$$G(x) = \|x\|^2_{Q_{\text{end}}} \tag{2.16}$$

where $\|x\|^2_{Q_{\text{end}}} := \langle x, Q_{\text{end}} x \rangle$ with $Q_{\text{end}} \geq 0$, called the end-point "weight". By varying the weight $Q_{\text{end}}$ one can try to "tune" the controller (2.5) so that closed loop stability is achieved (by "increasing the weight") or to improve a transient behavior (by "decreasing the weight"). In this context, an interesting problem that is posed in [141, 158] is to investigate the stability properties of the controlled system as a function of the end point penalty.

The second proposal that can be found in literature is to constrain the state at the end of the control horizon in the *terminal constraint set* (see [37, 100, 130]). The requirement that is added to the model predictive cost (2.3) is

$$x_N(N+1) \in \mathbf{X}_c \tag{2.17}$$

where $\mathbf{X}_c$ denotes the terminal constraint set and $N$ is the length of the control horizon. Usually, the terminal constraint set satisfies the following assumption.

**Assumption 2.2.1** The terminal constraint set is a subset of $\mathbf{X}$ ($\mathbf{X}_c \subset \mathbf{X}$), it is closed and contains the origin in its interior ($0 \in \operatorname{int} \mathbf{X}_c$).

Inside the terminal constraint set $\mathbf{X}_c$ a local stabilizing controller

$$u = \eta_c(x) \qquad u \in \mathbb{R}^m \quad x \in \mathbb{R}^n$$

is applied. Features that are usually required from the local stabilizing controller are listed in the following assumptions.

**Assumption 2.2.2** A local stabilizing controller $\eta_c$ satisfies the input constraint in the terminal constraint set, i.e.:

$$\eta_c(x) \in \mathbf{U} \qquad \text{for all} \quad x \in \mathbf{X}_c.$$

**Assumption 2.2.3** The terminal constraint set $\mathbf{X}_c$ is *controlled invariant* under the controller $\eta_c$ i.e.:

$$f(x, \eta_c(x)) \in \mathbf{X}_c \qquad \text{for all} \quad x \in \mathbf{X}_c.$$

The stability analysis of the model predictive control with terminal constraint set is based on the observation that under certain conditions the optimal cost (2.6) can be seen as a Lyapunov function for the controlled system (2.7). A survey of stability analysis methods for model predictive control can be found in the paper [97] (see also [95].)

Lyapunov functions are used to investigate the stability properties of the autonomous differential equations. Consider the system (2.1) controlled by controller (2.5):

$$x(t+1) = f(x(t), \eta(x(t))). \tag{2.18}$$

The system (2.18) is autonomous system with the origin as an equilibrium point. Next, we give a definition of the Lyapunov function (see [132]).

**Definition 2.2.4** Consider the autonomous system (2.18). A function $W : \mathbb{R}^n \to \mathbb{R}$ is called a Lyapunov function for the system (2.18) in a neighborhood $\mathcal{M}(x^*) \subset \mathbf{R}^n$ of an equilibrium point $x^*$ if

1. $W$ is continuous at $x^*$.
2. $W$ attains a strong local minimum at $x^*$, i.e. there exists a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ which is continuous, strictly increasing, with $\alpha(0) = 0$, such that

   $$W(x) - W(x^*) \geq \alpha(\|x - x^*\|)$$

   for all $x \in \mathcal{M}(x^*)$.
3. $W$ is monotone non-increasing along all solutions of (2.18) with $x(0) \in \mathcal{M}(x^*)$ i.e.

   $$W\big(f(x, \eta(x))\big) \leq W(x)$$

   for all $x \in \mathbb{R}^n$ along solutions of (2.18) with $x(0) \in \mathcal{M}(x^*)$.

If the system (2.18) is such that it is possible to find a Lyapunov function $W$ in a neighborhood $\mathcal{M}(x^*)$ of an equilibrium point $x^*$ then the equilibrium point $x^*$ is a stable point. If $W$ is strictly decreasing along solutions of (2.18) with $x(0) \in \mathcal{M}(x^*)$ then $x^*$ is an asymptotically stable equilibrium point.

It is easy to see that the optimal cost $V$ (2.6) satisfies condition (1) from definition 2.2.4 at the origin. Condition (2) is satisfied at the origin because the optimal cost $V$ is a convex function in $x$ with $V(0) = 0$. It remains to be shown that the closed loop optimal cost $V$ satisfies condition (3). To show this, we need additional assumptions on the end point penalty function $G$. Note that an asymptotic stability proof requires a strict inequality in condition (3). For additional assumptions on the end point penalty function $G$ that are necessary for asymptotic stability, as well as for an in-depth overview of stability issues in model predictive control we refer to the papers [95, 97]. Here, we would like to present main ideas in stability proofs for model predictive control that utilize Lyapunov stability theory and in order to avoid technical details we consider only stability. To show this, we need the following assumption on the end point penalty function $G$.

**Assumption 2.2.5** The end point penalty function $G$ is a local Lyapunov function in the terminal constraint set i.e. it satisfies conditions (1) and (2) from definition 2.2.4 at origin as equilibrium point and $G$ is monotone non-increasing along all solutions of (2.18) for all initial conditions $x(0) \in \operatorname{int} \mathbf{X}_c$.

Moreover, the end point penalty function $G$ bounds the *infinite* horizon cost i.e for all $x(0) \in \operatorname{int} \mathbf{X}_c$

$$G(x(0)) \geq \sum_{k=0}^{\infty} g\big(x(k), \eta_c(x(k))\big)$$

where the state $x$ is defined with the recursion

$$x(k+1) = f(x(k), \eta_c(x(k))$$

for $k = 0, 1, \ldots$.

The stability result is given in the following theorem.

**Theorem 2.2.6** *Consider the system* (2.1) *and the cost function* (2.3) *with an additional requirement* (2.17)*. Assume that the system* (2.1) *is controlled with the model predictive controller* (2.5)*. Moreover, assume that a set of initial conditions for which the optimization problem 2.1.1 is solvable is not empty and denote that set by* $\mathbf{X}_0$*. Then, under assumptions 2.2.1, 2.2.2, 2.2.3 and 2.2.5 the closed loop system* (2.18) *is a stable system for all* $x(0) \in \mathbf{X}_0$*.*

**Proof:** In order to prove the theorem, we have to show that the optimal cost $V$ is a Lyapunov function for the system (2.18). Since the conditions (1) and (2) from the definition 2.2.4 are trivially satisfied, it remains to be shown that $V$ satisfies condition (3) i.e.

$$V\big(f(x, \eta(x))\big) \leq V(x) \qquad x \in \mathbb{R}^n$$

along all solutions of (2.18) with $x(0) \in \mathbf{X}_0$. To show that, suppose that for a given $x(0) \in \mathbf{X}_0$ we obtain the state trajectory $x$ of the system (2.18). On the state trajectory $x$, consider the state $x(t)$ at time $t \in \mathbb{Z}_+$.

With $v^* = (v_i^*)_{i=0}^N$ we denote an input sequence that solves the optimization problem (2.1.1) with $x_N(0) = x(t)$. The optimal predicted state is denoted with $x_N = (x_N(i))_{i=0}^{N+1}$. The predicted state $x_N$ is obtained by solving recursion (2.1) with $x(t)$ as the initial state and the input sequence $v^*$.

At time $t$ the input $\eta(x(t)) = v_0^*$ is applied to the plant and the state in the next time instant $t+1$ is determined by (2.1). The model predictive optimization problem 2.1.1 has to be solved again, with the new initial state $x(t+1)$.

Instead of an optimal input over the control horizon that will result in the optimal cost $V(x(t+1))$ we will construct a feasible input sequence over the control horizon and denote it with $\tilde{v}$. By a feasible input sequence we mean that the input $\tilde{v} = (\tilde{v}_i)_{i=0}^N$ respects the constraint on input i.e.

$$\tilde{v}_i \in \mathbf{U} \qquad \text{for all} \quad i \in T \tag{2.19}$$

and that the state trajectory $\tilde{x}_N$ predicted with the model (2.2), an initial state $\tilde{x}_N(0) = x(t+1)$ and the input $\tilde{v}$ respects the constraint on the state:

$$\tilde{x}_N(i) \in \mathbf{X} \qquad \text{for all} \quad i \in T. \tag{2.20}$$

Also, the state $\tilde{x}_N(N+1)$ has to be in the terminal constraint set:

$$\tilde{x}_N(N+1) \in \mathbf{X}_c$$

for the input $\tilde{v}$ to be feasible. Note that the cost $J(x(t+1), \tilde{v})$ is an upper bound for the optimal cost $V(x(t+1))$ since the input sequence $\tilde{v}$ is suboptimal.

To obtain the feasible input sequence $\tilde{v}$ we first note that $x(t+1) = x_N(1)$ and that $x_N(N+1) \in \mathbf{X}_c$. Because of that a choice

$$\tilde{v} = \left((v_i^*)_{i=1}^N \tilde{v}_N\right), \qquad \tilde{v}_N = \eta_c(\tilde{x}_N(N))$$

results in $\tilde{x}_N(i) = x_N(i+1)$, $i = 0, \cdots, N-1$ and $\tilde{x}_N(N) \in \mathbf{X}_c$. Assumptions 2.2.2 and 2.2.3 ensure that with

$$\tilde{v}_N = \eta_c(\tilde{x}_N(N))$$

it is satisfied that

$$\tilde{v}_N \in \mathbf{U} \qquad \text{and} \qquad \tilde{x}_N(N+1) \in \mathbf{X}_c$$

i.e. $\tilde{v}$ is a feasible input sequence for the optimization problem 2.1.1.

Now, we can write:

$$\begin{aligned} J(x(t+1), \tilde{v}) = V(x(t)) - g(x(t), \eta(x(t)) - G(\tilde{x}_N(N)) \\ + G(\tilde{x}_N(N+1)) + g(\tilde{x}_N(N), \eta(\tilde{x}_N(N)) \end{aligned}$$

As a consequence of assumption 2.2.5

$$-G(\tilde{x}_N(N)) + G(\tilde{x}_N(N+1)) + g(\tilde{x}_N(N), \eta(\tilde{x}_N(N)) \leq 0$$

thus

$$J(x(t+1), \tilde{v}) \leq V(x(t))$$

which together with

$$V(x(t+1)) \leq J(x(t+1), \tilde{v})$$

gives

$$V(x(t+1)) \leq V(x(t)) \tag{2.21}$$

By repeating the argument in the proof it is easy to show that (2.21) is satisfied for all $t \in \mathbb{Z}_+$. ■

As a typical example to illustrate the stability result in theorem 2.2.6 consider the linear system (see [143])

$$f(x, u) = Ax + Bu \quad x \in \mathbb{R}^n \quad u \in \mathbb{R}^m$$

and the cost function (2.3) with

$$g(x, u) = \|x\|_Q^2 + \|u\|_R^2.$$

The end point function $G$ is chosen to be a quadratic function of the state (2.16) with the end point weighting matrix

$$Q_{\text{end}} = P$$

where $P$ is the unique non negative solution to the following matrix Riccati equation

$$P = A^T PA + Q - A^T PB(R + B^T PB)^{-1} B^T PA.$$

Let the controller $\eta_c$ be the optimal controller for the unconstrained, infinite horizon optimal control problem i.e. standard LQ problem. That is, $\eta_c(x) = Fx$ where

$$F = -(R + B^T PB)^{-1} B^T PA. \tag{2.22}$$

It can be easily verified that this end point penalty function satisfies assumption 2.2.5 so by a straightforward application of the result in theorem 2.2.6 the closed loop stability of this model predictive control scheme can be verified. In this way, the standard model predictive control problem can be seen as the infinite horizon LQ control problem for the system with the constraints on the input and the state.

There are two problems that arise in the application of theorem 2.2.6 and other stability proofs that are based on Lyapunv stability theory. The first one is the characterization of the controlled invariant set $\mathbf{X}_c$ when constraints are present. This characterization depends on the class from which a local stabilizing controller $\eta_c(x)$ has been chosen.

In general, a controlled invariant set may not admit linear controllers. We refer to the survey paper [21] which deals with the issue of set invariance in control.

Another issue that is usually not addressed in the model predictive control literature is the characterization of the set of feasible initial conditions $\mathbf{X}_0$. The problem 2.1.1 is solvable only for those initial states in $\mathbf{X}$ for which it is possible to find an input $v$ that will steer the state to the terminal constraint set $\mathbf{X}_c$ while respecting constraints on input and/or states.

In general, the set of feasible initial conditions $\mathbf{X}_0$ is a subset of the *recoverable set* of the system (2.1). To define the recoverable set, assume that the system (2.1) is controlled by a *static feedback controller* i.e. at each time $t$, the input $u(t)$ is a function of the state $x(t)$. Precisely, a feasibile state feedback controller is a function $\varphi \in \Psi$ where $\Psi$ is the set of continuous maps $\varphi : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbf{U}$ that map the origin of the state space into the zero input i.e.

$$\varphi(0, t) = 0 \quad \text{for all} \quad t \in \mathbb{Z}_+.$$

Suppose that at time $t = 0$ system (2.1) has an initial state $x(0) = x_0$, $x_0 \in \mathbb{R}^n$. Starting at time $t = 0$, the state $x$ and the output $y$ are generated by (2.1) with the input:

$$u = \big(\varphi(x(t), t)\big)_{t=0}^{\infty} \tag{2.23}$$

with the initial condition $x(0) = x_0$ and with $\varphi \in \Psi$.

It is well known, that a constrained input limits our ability to control the linear plant. Suppose that the state $x$ is generated by (2.1) with input (2.23) and with an initial state $x(0) = x_0$, $x_0 \in \mathbb{R}^n$. If there exists a controller $\varphi \in \Psi$ such that:

$$x(t) \to 0 \quad \text{as} \quad t \to \infty$$

we say that the state $x_0$ is a *null controllable point* in the state space. All null controllable points define a set in the state space which is known as the *recoverable set*, here denoted as $\underline{\mathbf{X}}$. In general, the recoverable set is a subset of the state space. If the recoverable set contains all points in the state space we say that the system (2.1) is *globally asymptotically stabilizable*. Obviously, global asymptotic stability is a very desirable property. Unfortunately, it can be achieved only for a very restricted class of systems (see [125] for details).

## 2.3 Model predictive control and disturbances

It is often claimed that disturbances and model uncertainties are the essential reason for using feedback control of dynamical systems. A control system is often judged by its ability to reject disturbances and to control the plant in a prescribed manner, despite the plant/model mismatch. Model predictive control uses a predicted behavior of the

plant to determine the control input to the plant. In the case that the disturbances are known and measured, it is easy to extend the model of the plant with the disturbance model and to use the extended model in the prediction. Therefore, dealing with known and measured disturbances is not difficult and we will not discuss this issue further. When the disturbance is unknown, the standard model predictive controller is faced with a difficulty. It is based on the *prediction* of the future behavior of the plant and it is necessary to make assumptions about the disturbance in the prediction. If these assumptions are too far from the real nature of the disturbance, closed loop performance can be poor with likely violations of the constraints (see [116]). In general, we would like to take advantage of any information that we have about the disturbance. An example that is very often encountered in the literature is the example with an unknown but bounded disturbance where bounds on the "magnitude" of the disturbance are available information and there are various proposals how to include this information in the prediction.

In this section, we will assume that the plant is described in terms of a difference equation of the form

$$x(t+1) = f(x(t), u(t), w(t)) \tag{2.24}$$

where $x(t) \in \mathbf{X} \subset \mathbb{R}^n$ is the state and $u(t) \in \mathbf{U} \subset \mathbb{R}^m$ is the input. The disturbance $w$ is an unknown disturbance with $w(k) \in \mathbf{W} \subset \mathbb{R}^l$. The function $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}^n$ is a continuous function with

$$f(0, 0, 0) = 0.$$

Typically, to include the disturbance in the prediction, the prediction model is extended in the following way:

$$x_N(k+1) = f(x_N(k), v_k, w_N(k)) \tag{2.25}$$

where $x_N(k)$ and $v_k$ are defined in a same way as in the model (2.2) and $w_N : T \to \mathbf{W}$ denotes the disturbance on the horizon $T$. An obvious approach to model predictive control of the system (2.24) is to ignore the disturbance and to design a model predictive controller as described in section 2.1. The resulting model predictive control law (2.5) ignores the effects of possible future changes in disturbance and closed loop performance can be poor when disturbances are neglected in the design of the controller. The presence of constraints makes the problem more complicated. By ignoring the disturbance it is not possible to predict the constraint violation caused by it, therefore the resulting model predictive controller will likely cause violations of the constraints. Another issue is the issue of the closed loop stability. With the disturbance acting on the plant assumption 2.2.3 no longer applies so that the stability result from Theorem 2.2.6 can not be applied to the plant (2.24) controlled by model predictive controller (2.5), i.e. the closed loop stability can not be guaranteed.

First attempts to deal with problems that unknown but bounded disturbances are posing in model predictive control paradigm can be found in [35, 100] and from the robustness point of view in [92]. The approach that is used is commonly known as the

*min-max* optimization. The main feature of this approach is that the optimization over the control horizon is performed with an assumption that the disturbance over the control horizon is the worst possible in the sense that it maximizes the cost. The results reported in the literature can be distinguished by the nature of the optimization over the control horizon. We will make an overview of the approaches in the following two subsections.

### 2.3.1 Min-max optimization in open loop

One way of dealing with the unknown but bounded disturbance $w$ is to design a controller that perform well for all possible realization of the disturbance. It is assumed that the model (2.25) is subject to the disturbance $w_N : T \to \mathbf{W}$ on the horizon $T$. We denote the set of all disturbances $w_N$ as $\mathbf{W}_N$ i.e. $w_N \in \mathbf{W}_N$. The predicted state $(x_N(k))_{k=0}^{N+1}$ is generated by (2.25), with an initial condition $x_N(0) := x$, with the disturbance $w_N(k) = w(t+k)$, $k \in T$ and the input $v = (v_k)_{k=0}^{N} \in \mathcal{V}$. Note that the predicted state $x_N$ is a function of the disturbance $w_N$ and the input $v$.

For a disturbance $w_N$ and the input $v = (v_k)_{k=0}^{N}$ the cost acquired is given by

$$J(x, v, w_N) = \sum_{k \in T} g(x_N(k), v_k) + G(x_N(N+1)) \qquad x \in \mathbf{R} \tag{2.26}$$

with the initial state $x_N(0) = x$. For fixed $x$ and $v$, each one of the disturbances in $\mathbf{W}_N$ gives a different predicted state trajectory and hence a different cost $J(x, v, w_N)$. The maximal cost is defined by

$$J_{\max}(x, v) := \max_{w_N \in \mathbf{W}_N} J(x, v, w_N) \tag{2.27}$$

The optimization problem that is posed in this setting is the problem of minimization of the maximal cost (2.27), so called min-max optimization. We formalize the optimization problem next.

**Problem 2.3.1** Given the measured state $x$ at time $t$, find an optimal input $v^\star \in \mathcal{V}$ such that

$$J_{\max}(x, v^\star) \leq J_{\max}(x, v) \tag{2.28}$$

for all $v \in \mathcal{V}$.

If problem 2.3.1 admits a solution, it yields an optimal input $v^\star = (v_k^\star)_{k=0}^{N}$ that depends on the current state $x$. At time $t$, only the input $v_0^\star$ is applied to the plant (2.25) i.e.

$$u(t) = v_0^\star. \tag{2.29}$$

This input is fed in (2.25) to result in the next state $x(t+1)$. In the next time instant, the optimization problem 2.3.1 is solved for the state $x(t+1)$, according to the receding

horizon paradigm. In this way, the optimization problem 2.3.1 implicitly defines a time invariant model predictive control law $\mu : \mathbf{R}^n \to \mathbf{U}$ such that for a given $x \in \mathbf{R}^n$

$$\mu(x) = v_0^\star. \tag{2.30}$$

The optimal cost is given by

$$V_{\text{min-max}}(x) = J_{\max}(x, v^\star). \tag{2.31}$$

Stability results reported in the literature constrain the state at the end of the horizon in the terminal constraint set, leading to the results similar to the one in theorem 2.2.6. The theorem 2.2.6 can not be applied directly to conclude on the stability of the closed loop system

$$x(k+1) = f(x(k), \mu(x), w(k)). \tag{2.32}$$

The problem arises when one seeks for a feasible input sequence (2.19). A choice (2.20) does not ensure that the condition 2.2.5 is satisfied when the disturbance is present and therefore it is not possible to conclude on the closed loop stability of the overall system (2.32). There are different modifications of the optimization problem 2.3.1 and the conditions 2.2.1, 2.2.2, 2.2.3 and 2.2.5 that are proposed in the literature (see for example [8, 100]) to recover the closed loop stability.

The essential problem with the model predictive controllers described in this section is the open loop nature of the optimization problem 2.3.1. In the problem 2.3.1 we seek for a single input $v^\star$ over all possible disturbance realizations and we do not include the feedback that is present in the receding horizon implementation. Because of this, the predicted and true behavior of the plant differ significantly when the disturbance is present which results in poor disturbance rejection and "conservative" control with respect to constraints (i.e. constraints are respected but the "steady" state of the controlled system is not close to the constraint boundary).

### 2.3.2 Min-max optimization in closed loop

A modification of the optimization problem described in subsection 2.3.1 is proposed in [129]. The optimization is based on the min-max paradigm but the control over the control horizon is assumed to be in closed loop. In this section we briefly outline this approach.

The main difference between model predictive schemes with the optimization in open loop (section 2.1 and subsection 2.3.1) and the model predictive scheme described in this section is in the type of controller of the plant over the control horizon. While in open loop formulations we assume input $v : T \to \mathbf{U}$, here we define a set of feedback control laws $\Pi$ where $\pi \in \Pi$ is a vector $(\pi_k)_{k=0}^N$ such that for any $k \in T$, the map $\pi_k : \mathbb{R}^n \to \mathbf{U}$ is continuous. A feedback controller on $T$ is therefore a sequence of continuous maps $\pi_k : \mathbb{R}^n \to \mathbf{U}$, defining the control action as a feedback at all $k \in T$.

As in section 2.3.1, it is assumed that the model (2.25) is subject to the disturbance $w_N : T \to \mathbf{W}$. The predicted state $(x_N(k))_{k=0}^{N+1}$ is generated by (2.25), with an initial condition $x_N(0) := x(t)$, with the disturbance $w_N(k) = w(t+k)$ and the input $v_k = \pi_k(x_N(k))$. The predicted state $x_N$ is a function of the measured state $x(t)$, the feedback control laws in the vector $\pi$ and the disturbance $w$.

With a disturbance $w_N$ and the input $\pi = (\pi_k)_{k=0}^{N}$ the cost acquired with the predicted state $x_N$ is given by:

$$J^{\text{fb}}(x, \pi, w_N) = \sum_{k \in T} g(x_N(k), \pi_k) + G(x_N(N+1)) \qquad x \in \mathbf{R} \tag{2.33}$$

with $x_N(0) = x$ and $x_N(N+1) \in \mathbf{X}_c$. Each one of the disturbances in $\mathbf{W}_N$ gives a different predicted state trajectory and hence a different cost (2.33). The cost to be minimized over all $\pi \in \Pi$ is defined by

$$J^{\text{fb}}_{\max}(x, \pi) := \max_{w_N \in \mathbf{W}_N} J^{\text{fb}}(x, \pi, w_N). \tag{2.34}$$

The optimization problem that is posed in this setting is the problem of minimization of the maximal cost (2.34), so called min-max optimization. We define the optimization problem next.

**Problem 2.3.2** Given the initial state $x \in \mathbb{R}$, find an optimal feedback $\pi^\star \in \Pi$ such that

$$J^{\text{fb}}_{\max}(x, \pi^\star) \leq J^{\text{fb}}_{\max}(x, \pi) \tag{2.35}$$

for all $\pi \in \Pi$.

The optimal cost is given by

$$V^{\text{fb}}_{\text{min-max}}(x) = J^{\text{fb}}_{\max}(x, \pi^\star). \tag{2.36}$$

Problem 2.3.2 is an infinite dimensional optimization problem. Practical implementation of the controller appears possible only in an approximate sense, through quantization of the disturbance. However, due to linearity of the process and convexity of the constraints and cost, this problem can be resolved. It is shown in [129] that if $\mathbf{W}$ is a polytope in $\mathbb{R}^l$ it is sufficient to consider the disturbance realizations that take values at the vertices of $\mathbf{W}$ in order to find an optimal feedback $\pi^\star$.

An interesting issue is the relationship between the optimal costs (2.31) and (2.36). In the following theorem we show that by the optimization in open loop one can achieve the performance that is at most as the performance in the closed loop.

**Theorem 2.3.3** *Consider the min-max optimization problem in open loop 2.3.1 and the min-max optimization problem in closed loop 2.3.2. If optimal costs* $V_{\text{min-max}}$ (2.31) *and* $V^{\text{fb}}_{\text{min-max}}$ (2.36) *exist then*

$$V_{\text{min-max}}(x) \geq V^{\text{fb}}_{\text{min-max}}(x)$$

*for all* $x \in \mathbb{R}$.

**Proof:** Suppose that for a given $x \in \mathbb{R}$ solution to the optimization problems 2.3.1 and 2.3.2 exist. The optimal cost for the optimization problem 2.3.1 is given by

$$V_{\text{min-max}}(x) = J_{\max}(x, v^\star)$$

where $v^\star \in \mathcal{V}$ is the input that solves the optimization problem 2.3.1. Next, choose $\pi_k : \mathbb{R}^n \to \mathbf{U}$ as $\pi_k(x) = v_k^\star$ for all $k \in T$. In this way we form a sequence of the feedback maps $\pi = (\pi_k)_{k=0}^N \in \Pi$. By construction

$$J_{\max}(x, v^\star) = J_{\max}^{\text{fb}}(x, \pi).$$

Because $\pi$ is a suboptimal feedback for the optimization problem 2.3.2

$$J_{\max}^{\text{fb}}(x, \pi) \geq V_{\text{min-max}}^{\text{fb}}(x)$$

where $V_{\text{min-max}}^{\text{fb}}(x)$ is the optimal cost for the optimization problem 2.3.2. Therefore

$$V_{\text{min-max}}(x) \geq V_{\text{min-max}}^{\text{fb}}(x).$$

■

In the next example we compare two model predictive controllers, one of them is based on the min-max optimization in open loop (section 2.3.1) and the second one is based on the optimization in closed loop (section 2.3.2).

**Example 2.2** In this example we consider the problem of the steering of a cart (figure 2.2) to a prescribed position, as in example 2.1. Here we assume that the disturbance $w \in \mathbb{R}$ is acting on the cart. The disturbance is an additional periodic force in the same direction as the input force. As in the example 2.1 the objective is to steer the cart from an initial position $s(0) = 10$ to the position $s(K) = 0$ in a finite time $K$ with the constraint

$$s(t) \geq 0$$

for all $t \in \mathbb{R}$. The input is constrained with

$$-0.5 < u(t) < 0.5.$$

for all $t \in \mathbb{R}$. The differential equation that describes the motion of the cart with the disturbance is given by

$$\frac{d^2 s}{dt} = u + w.$$

As in example 2.1, the system is sampled with the unit sample time so as to obtain a discrete time, state space representation given by

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Ew(t) \\ s(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x(t) \end{aligned} \tag{2.37}$$

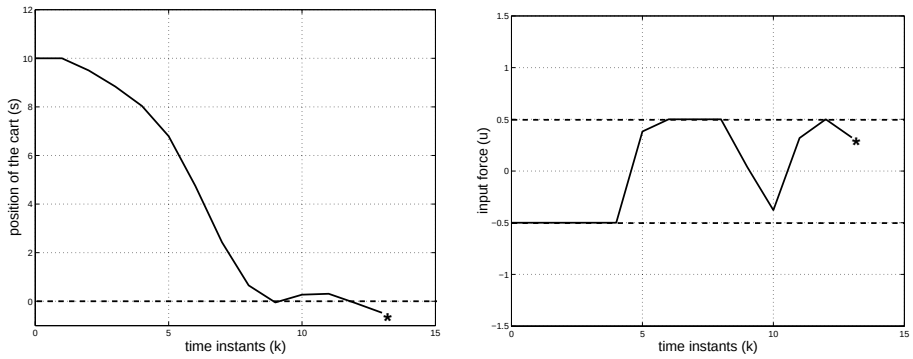


Figure 2.4: Standard model predictive controller

where:

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad E = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Suppose that the disturbance is periodic and given by

$$w(t) = 0.4 \sin(t) .$$

Obviously, the disturbance is bounded by its amplitude, i.e.

$$-0.4 \leq w(t) \leq 0.4 \qquad \text{for all} \quad t \in \mathbb{R}. \tag{2.38}$$

We will assume that the only available information about the disturbance is its amplitude as defined by (2.38). The straightforward approach would be to ignore the disturbance in the prediction and to design the controller as in example 2.1. This approach does not give satisfactory results in the presence of the disturbance $w$. When the cart is close to the constraint boundary (i.e. close to $s = 0$) the disturbance "pushes" the cart over the constraint boundary. Results of the simulation are shown in figure 2.4. At $t = 13$ the cart is in the region $s \leq 0$ because of the disturbance which is not taken into account by the controller. The control design stops because of infeasibility.

Next, we compare two predictive controllers described in this section. The first one is based on the optimization in open loop (subsection 2.3.1) and the second one is based on the optimization in closed loop (subsection 2.3.2). It can be observed from the results of the simulation shown in figure 2.5, that the controller based on the min-max optimization in closed loop controls the plant closer to the constraint and rejects the disturbance better. The price of the improvement is in added computational complexity. The total number of inputs computed at each time step is 16 ($2^4$) with the min-max optimization in closed loop while the min-max optimization in open loop requires only 4. •

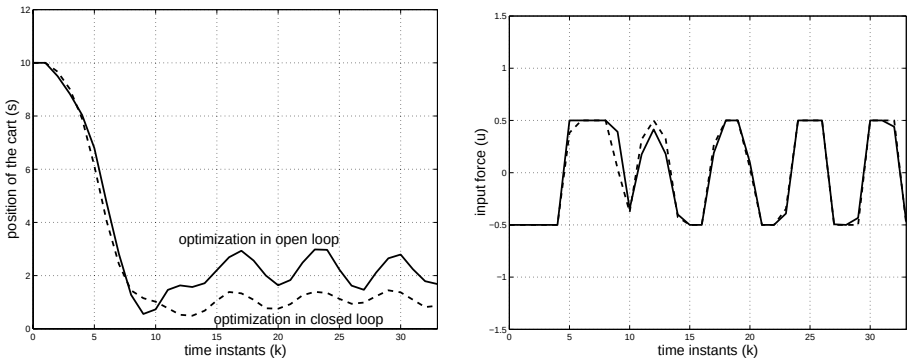


Figure 2.5: Model predictive controllers based on min-max optimization

The synthesis of a model predictive controller based on the optimization in closed loop is computationally more expensive than model predictive controller based on the optimization in open loop (section 2.3.1). A total number of controls that are computed with the optimization in closed loop depends on the length of the control horizon as well as on the number of vertices of the set $\mathbf{W}$. With the length of the control horizon denoted by $N$ and the set $\mathbf{W}$ with $m$ vertices, a total number of computed control inputs is equal to $m^N$. It grows exponentially with the control horizon.

### 2.3.3 Stochastic disturbances in model predictive control

Deterministic, worst case approach to the disturbance rejection in model predictive control described in the previous two subsections has one obvious drawback. Performance of the control system is determined by the most excessive disturbance realization. It can be too conservative due to the over-bounding of the disturbance. An alternative approach to the disturbance rejection is the stochastic approach. The main advantage of the stochastic approach is that the performance of the control system can be improved by a considerable degree, at the expense of a small risk of the constraint violation. If the application at hand allows that risk, the potential benefit in the form of the performance enhancement might be large. This is already well known in the robust control community and it motivated a probabilistic view on robustness of uncertain control systems (see [27, 78, 118, 136, 145, 146]).

To approach to the problem of the stochastic disturbance in model predictive control formally, we assume that the model (2.25) is subject to a disturbance $w(k) \in \mathcal{N}(0, Q_w)$, i.e. the disturbance $w(k)$ is a member of the family of normally distributed random variables denoted by $\mathcal{N}$, with zero mean and a covariance matrix $Q_w \in \mathbb{R}^l \times \mathbb{R}^l$. Moreover, for $k \neq j$, $w(k)$ and $w(j)$ are independent. In other words, the disturbance $w$ is a Gaussian white noise.

Next, assume that the predicted state $(x_N(k))_{k=0}^{N+1}$ is generated by (2.25), with an initial condition $x_N(0) := x$, with the disturbance $w_N(k) = w(t+k), k \in T$ and the input $v = (v_k)_{k=0}^{N} \in \mathcal{V}$. Thus, the disturbance over the control horizon is assumed to be stochastic, from the same family as the disturbance $w$. Since the predicted state $x_N$ is a function of the disturbance $w_N$ and the input $v$ it is also stochastic.

In [83], the authors propose a model predictive controller to deal with the stochastic disturbance $w_N$ over the control horizon. The optimization in the model predictive controller is assumed to be in the open loop. We will briefly outline the approach reported in [83].

With the stochastic disturbance, the cost (2.3) is a stochastic quantity. Because of that, it is necessary to consider the expected value of the cost (2.3) in the optimization problem that has to be solved at each time step by a model predictive controller. With the stochastic disturbance over the control horizon consider the following optimization problem.

**Problem 2.3.4** Given the initial state $x = x(t)$, find an optimal input $v^* \in \mathcal{V}$ such that

$$\mathbb{E}J(x, v^*) \leq \mathbb{E}J(x, v) \tag{2.39}$$

for all $v \in \mathcal{V}$. In addition determine the optimal cost given by:

$$V(x) := \inf_{v \in \mathbf{V}} \mathbb{E}J(x, v)$$

where $\mathbb{E}$ denotes expectation.

If an optimal input $v^* = (v_k^*)_{k=0}^{N}$ exists, then $V(x) = \mathbb{E}J(x, v^*)$. Only the first element of $v^*$ is applied to the plant. At the next time instant the control horizon is shifted forward and the optimization problem 2.3.4 is solved for a new state measurement. The optimization problem 2.3.4 with the receding horizon implementation defines a time invariant stochastic model predictive control law $\xi : \mathbb{R}^n \to \mathbf{U}$ such that for a given $x$

$$\xi(x) = v_0^*. \tag{2.40}$$

A model predictive controller (2.40), derived in [83], is based on an optimization in open loop. As already mentioned, there is a discrepancy between control in open loop over the control horizon and the actual control of the plant which is performed by the feedback law given by (2.40). When stochastic disturbances are present, this discrepancy causes a significant difference between predicted and the true behaviour of the system. The following example illustrates this difference.

**Example 2.3** Consider a first order system:

$$x(k+1) = x(k) + u(k) + w(k)$$

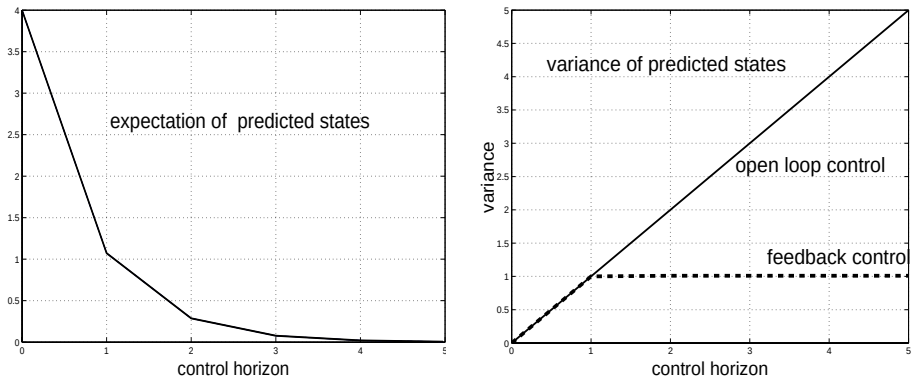


Figure 2.6: Prediction with a stochastic disturbance

with $x(k)$, $u(k)$ and $w(k)$ all scalar valued, $w(k)$ is a stochastic variable with zero mean and variance $Q_w$. We look at the prediction $x_N : T \to \mathbb{R}$ over the control horizon $T = \{0, \cdots, N\}$. The predicted state $x_N(k)$, $k \in T$ is a stochastic variable. We are interested in the variance of the predicted state for two cases. The first one is the case with an open loop control sequence $v : T \to \mathbb{R}$ and the second one is with feedback control laws $(\pi_k)_{k=0}^N$. For the closed loop case we use a linear feedback of the form:

$$v(k) = -f\, x_N(k), \quad f \in \mathbb{R}, \quad k \in T.$$

The variance for the control in open loop is given by:

$$\mathrm{Var}\,\{x_N(k+1)\} = \mathbb{E}\left\{(x_N(k+1) - \mathbb{E}x_N(k+1))^2\right\} = \mathrm{Var}\,\{x_N(k)\} + Q_w \quad (2.41)$$

and for the closed loop control as:

$$\mathrm{Var}\,\{x_N(k+1)\} = (1-f)^2\ \mathrm{Var}\,(x_N(k)) + Q_w. \quad (2.42)$$

With $f = 1$ the variance (2.42) is equal to $Q_w$. On the other hand, with the open loop control one does not have any control on the growth of the variance in (2.41) as $N \to \infty$. In this example the variance (2.41) will grow without upper bound as $N \to \infty$ (see figure 2.6). •

As example 2.2 shows, a model predictive controller with the optimization in closed loop rejects disturbances better and steer the system closer to constraints. The closed loop optimization described in subsection 2.3.2 consider the disturbance realizations that maximize the cost. Unfortunately, with $w(k) \in \mathcal{N}(0, Q_w)$ the disturbance is no longer bounded so min-max approach to the optimization is not feasible. An ad hoc solution would be to neglect the disturbance realizations that have a small probability to occur and bound the disturbance in this sense. Note that any bound that is not a

polytope in $\mathbb{R}^l$ (but a ball or ellipsoid in $\mathbb{R}^l$) will still yield an infinite dimensional optimization problem in the min-max setting from subsection 2.3.2.

An intrinsic feature of the min-max optimization is that the overall performance of the model predictive controller is determined by the worst case disturbance realizations. In min-max optimization we seek for a controller that will ensure that constraints are respected for all possible disturbance realizations. When the disturbance is stochastic, however, the natural problem setup is to find a controller that will minimize the probability of the constraint violation, so min-max optimization does not capture the true nature of the problem.

## 2.4 Conclusion

In model predictive control the current control action is obtained by solving, at each sampling instant, a finite horizon, convex optimal control problem, using the current state of the plant as the initial state. The optimization yields an optimal control sequence and the first control in this sequence is applied to the plant. This process is repeated at the next sampling time, therewith defining a receding horizon control strategy. Although there are other control techniques available in the control literature to deal with the control constrains, model predictive control is the only one that has a significant and widespread impact on industrial process control.

In this chapter we introduced the standard setting for model predictive control. Since there is a vast portion of the model predictive control literature that deals with the stability of the standard model predictive scheme, we have given an overview of the available approaches to the stability issue. There are two basic modifications of the standard model predictive control that are proposed in the literature to achieve the closed loop stability. The first one of them is to include the end point penalty in the cost function. The end point penalty increases the cost if the state at the end of the control horizon is not in the origin. The rigorous analysis of the closed loop behavior as a function of the end point penalty is difficult and there is not many results available in the literature. Despite that, the end point penalty is a common ingredient of model predictive schemes that is accepted in industrial control practice, where the weight of the end point penalty is seen as a "tuning" parameter. The second modification is to constrain the state at the end of the control horizon in the *terminal constraint set* which is controlled invariant under some known controller. This technique is widespread in the model predictive control literature. Stability results based on the terminal constraint set are nowadays considered classical. When an unconstrained, stochastic disturbance is acting on the plant, an analysis based on the terminal constraint set is impossible.

When an unbounded, stochastic disturbance is present, such as Gaussian white noise, a tuning of the end point penalty is the only available mechanism to achieve the closed loop stability. When the disturbance is bounded, the approaches proposed in the liter-

ature are mostly based on the so called *min-max optimization*. The main feature of the min-max optimization is that the optimization over the control horizon is performed with an assumption that the disturbance over the control horizon is the worst possible in the sense that it maximizes the cost. In this way, the performance of the controlled system is determined by the most excessive disturbance realization. When the disturbance acting on the plant admits a stochastic model, with the worst case optimization the closed loop behavior of the system is likely to be determined by disturbance realizations that have a small probability to occur. Alternative to the worst case paradigm is to include stochastic disturbance model in the optimization and to solve a stochastic optimization problem. In this way, we seek for a controller that will minimize the probability of the constraint violation. In this approach it is necessary to consider optimization in closed loop, since the optimization in open loop gives a prediction with unbounded variance as $N \to \infty$. The optimization in closed loop, however, results in a difficult optimization problem. In the following chapters we will show how this problem can be solved approximately but with arbitrary accuracy.

# 3

# *Model predictive control for stochastic systems with constrained inputs*

This chapter is based on the paper [11], which is submitted for publication. Parts of this paper have been presented at the American Control Conference 2001 [13] and the European Control Conference 2001 [12].

## 3.1 Introduction

It is often claimed that the increasing popularity of Model Predictive Control (MPC) in an industrial environment stems from its capability to allow operation closer to constraint boundaries, when compared with conventional control techniques. This often leads to more profitable operation of the plant (see [91,119]). When disturbances are acting on the plant which one aims to control, then it is evident that the better the control system is dealing with disturbances the closer one can operate the plant to the constraint boundaries.

In the MPC setting, there are three basic approaches for dealing with disturbances that have been suggested in the literature.

The first approach is to assume that the disturbance is known and either zero or constant over the optimization interval. This is known as the classical setting for which there exists a vast literature (see [97]) based on convex on-line optimization. First attempts have been made to obtain a closed-loop solution (see [17, 129]). During the optimization over the chosen control horizon, this approach tends to ignore the effects that disturbances can have on the plant. In particular, the performance limitations imposed by the constraints are underestimated by this approach. Moreover, as the control horizon tends to infinity, optimal performance is not recovered.

The second approach assumes unknown disturbances and is based on a worst case optimization where the minimization is performed over a set of input sequences and maximization over a set of disturbance sequences (see [85]). To be feasible, this approach requires disturbances to be bounded. Since the min-max optimization looks for the worst possible disturbance realization this approach is generally too "pessimistic".

The third approach is a stochastic one. A stochastic view on disturbances in MPC

could be traced back to the early works in the field like Clarke's Generalized Predictive Control (see [41–43]). The classical results are only valid when there are no constraints on the input and/or states as in the many references that follow the same line of thought up to today. A modification of the open loop convex optimization is proposed in [83] for the case of a constrained input and a stochastic disturbance.

We would like to stress that almost none of the model predictive control algorithms currently available incorporate the disturbances in their design. It is only the stability of the feedback scheme that let us conclude that the effect of disturbances will remain bounded. The few papers that handled disturbances either excluded constraints (a relatively easy case) or looked at a worst case analysis. Regarding the latter it is widely accepted that worst case analysis has its limits. Therefore, it is clearly necessary to study the effect of stochastic disturbances in more detail.

The main difficulty with a stochastic disturbance in MPC is that the predicted behavior and the actual behavior of the plant can differ significantly. The standard, convex optimization in open loop does not take the difference into account between actual and predicted behavior of the plant. As a consequence, questions related to achievable performance can not be addressed properly, while the optimization criterion largely ignores the true characteristics of the plant. Hence the input is chosen on the basis of a criterion which does not reflect the true characteristic of the plant. Unfortunately, when a controller is designed in closed loop, constraints make a minimization of the expected value of the cost function over the horizon a very difficult optimization problem. In the case where an analytic solution is not possible and standard computational methods are too complex, Monte Carlo methods have been applied in control theory mostly in connection with robustness (see [10, 36, 118, 136]).

In this chapter we present a disturbance rejection scheme for MPC based on a randomized algorithm which minimizes an empirical mean of the cost function. The optimization at each step is a closed loop optimization. Therefore it takes the effect of disturbances into account. Because we do not impose any a priori parameterization of the feedback laws over the horizon, the algorithm is computationally demanding but it gives a reliable measure of the achievable performance.

In the second algorithm, presented here, the optimization is performed over a class of saturated feedback controllers. A significant reduction in the computational effort is achieved by postulating a controller structure in the closed loop optimization. The result is an algorithm that is computationally less demanding compared to the first one, at the expense of some performance loss.

The chapter is organized as follows. The problem definition is given in section 3.2. The background material on randomized algorithms is presented in section 3.3. The algorithm for solving the problem with an arbitrary accuracy and a convergence proof for the result obtained by the algorithm are given in section 3.4. A simplified algorithm is given in section 3.5. Finally, numerical examples are presented in section 3.6 and conclusions are given in section 3.7.

## 3.2 Problem formulation

In this chapter, we consider a linear time-invariant plant subject to amplitude constraints on the input and stochastic disturbances. The plant is represented with the following state space model:

$$\begin{aligned} \rho x &= Ax + Bu + Ew \\ z &= C_z x + D_z u \end{aligned} \tag{3.1}$$

where $x(k) \in \mathbb{R}^n$ is the state and $u(k) \in \mathbf{U} \subset \mathbb{R}^m$ is the input. The set $\mathbf{U}$ is a compact, convex set which contains an open neighborhood of the origin. Input constraints that we consider are constraints on the amplitude of the input. A typical example of these constraints are constraints imposed by saturating actuators. The forward shift operator $\rho$ is defined by $(\rho x)(k) := x(k+1)$. We assume that a disturbance $w(k) \in \mathbf{W} \subseteq \mathbb{R}$ is a scalar valued white noise stochastic process taking values in the set $\mathbf{W}$ with some known probability distribution. All results presented in this chapter can be easily extended to the general case where $\mathbf{W} \subseteq \mathbb{R}^q$. The assumption of a scalar valued disturbance is for notational convenience, it allows us to expose ideas in the chapter clearly. The second equation describes the controlled output $z(k) \in \mathbb{R}^p$. We assume that the state of the plant is measured and we denote the measured state at time $t$, $t \in \mathbb{Z}_+$ by $x_t$, i.e. $x_t := x(t)$.

A constrained input limits our ability to control the linear plant. To approach this in a more formal way, set $w = 0$ in (3.1):

$$x(k+1) = Ax(k) + Bu(k) \qquad u(k) \in \mathbf{U}. \tag{3.2}$$

Suppose that at time $t = 0$ system (3.2) has an initial state $x(0) = x_0$, $x_0 \in \mathbb{R}^n$. Further, suppose that the system (3.1) is controlled by a *static feedback controller*, i.e., at each $t$, the input $u(t)$ is a function of the state $x(t)$. A controller is an element $\varphi \in \Psi$ where $\Psi$ is the set of continuous maps $\varphi : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbf{U}$ that map the origin of the state space to the zero input:

$$\varphi(0, t) = 0 \quad \text{for all} \quad t \in \mathbb{Z}_+.$$

Starting at time $t = 0$, the state $x$ and the output $z$ are stochastic processes generated by (3.1) with the input:

$$u = \big(\varphi(x(t), t)\big)_{t=0}^{\infty}. \tag{3.3}$$

**Definition 3.2.1** Suppose that the state $x$ is generated by (3.2) with input (3.3) and an initial state $x_0 \in \mathbb{R}$. If there exists a controller $\varphi \in \Psi$ such that

$$x(t) \to 0 \quad \text{as} \quad t \to \infty$$

then the state $x_0$ is a *null controllable point* in the state space.

All null controllable points define a set in the state space which is known as the *recoverable set*, here denoted as $\underline{\mathbf{X}}$. In general, the recoverable set is a subset of the state space. The recoverable set contains all points in the state space if and only if the matrix pair $(A, B)$ is stabilizable and all eigenvalues of the system matrix $A$ lie on or inside the unit circle in which case we say that the system (3.2) is *globally asymptotically stabilizable* (see [125] for details).

When $w$ is a nonzero stochastic process, a Gaussian white noise is a typical example, the recoverable set for the system (3.1) is empty unless the system (3.1) is globally asymptotically stabilizable. Therefore, the following assumption is necessary when one deals with the stabilization of the linear system, subject to input constraints and possibly unbounded disturbances.

**Assumption 3.2.2** The system (3.1) is globally asymptotically stabilizable i.e. the matrix pair $(A, B)$ is stabilizable and all eigenvalues of the matrix $A$ lie on or inside the unit circle. As a consequence $\underline{\mathbf{X}} = \mathbb{R}^n$.

The essential problem in which we are interested is the design of a controller which minimizes some cost function over an infinite horizon. This problem is too complex and therefore we design a controller based on a receding horizon paradigm. The optimization problem is solved at each time instant $t$, $t \in \mathbb{Z}_+$ over an interval $I_t := \{t + k | k \in T\}$ where $T := \{0, \cdots, N\}$ and $N > 0$. The interval $I_t$ is a time dependent interval, it recedes with time.

The model of the plant (3.1) as well as the cost function and optimization problem to be defined later, are time-invariant. Therefore, the current time can be set to 0 without loss of generality. We will refer to the interval $T$ as the *control horizon* with length $N$.

The progression of the *predicted state* over the control horizon is denoted by $x_N : \{0, \cdots, N+1\} \to \mathbb{R}^n$. Over the control horizon, it is assumed that the model (3.1) is subject to the disturbance $w_N : T \to \mathbf{W}$. The input of the plant over the control horizon is optimized in closed loop i.e. the input $u(k)$ is a function of the predicted state $x_N(k)$. Formally, we define the set of feedback control laws $\Pi$ where $\pi \in \Pi$ is a sequence $(\pi_k)_{k=0}^N$ such that for any $k \in T$, the map $\pi_k : \mathbb{R}^n \to \mathbf{U}$ is continuous. The progression of the predicted state $(x_N(k))_{k=0}^{N+1}$ is generated by (3.1), with an initial condition $x_N(0) := x_t$, with the disturbance $w(k) = w_N(k)$ and the input $u(k) = \pi_k(x_N(k))$. Note that the predicted state $x_N$ is a function of the measured state $x_t$, the feedback control laws in the vector $\pi$ and the disturbance $w_N$ and is therefore stochastic.

The cost we consider over the control horizon is given by:

$$J(x_t, \pi, w_N) := \sum_{k \in T} \|C_z x_N(k) + D_z \pi_k(x_N(k))\|^2 \; + \; \|x_N(N+1)\|_Q^2 \tag{3.4}$$

with $x_N(0) = x_t$ and the predicted state $x_N$.

The expression $\|x\|_Q^2 := \langle x, Qx \rangle$ is called an end point penalty with $Q \in \mathbb{R}^{n \times n}$ a non-negative definite, symmetric matrix. An end point penalty is required to achieve stability of the receding horizon controller.

Because of the stochastic disturbance, we minimize the expectation of the cost function (3.4). The optimization problem to be solved is stated next.

**Problem 3.2.3** Find a vector of optimal feedback mappings $\pi^* \in \Pi$ such that

$$\mathbb{E}J(x_t, \pi^*, w_N) \leq \mathbb{E}J(x_t, \pi, w_N)$$

for all $\pi = (\pi_k)_{k=0}^N$, $\pi \in \Pi$ and for all $x_t$ where $\mathbb{E}$ denotes the expectation operator. In addition, determine the *optimal cost* given by:

$$V(x_t) := \inf_{\pi \in \Pi} \mathbb{E}\, J(x_t, \pi, w_N). \tag{3.5}$$

If the vector of optimal feedback mappings $\pi^*$ exists, then $V(x_t) = \mathbb{E}\, J(x_t, \pi^*, w_N)$ and only the first element of $\pi^*$ is significant in the receding horizon implementation. It determines the current input for the plant as a function of the current measurement. In the next time instant, the control horizon is shifted forward and problem 3.2.3 is solved based on a new state measurement. Note that the cost function (3.4) is time invariant. Therefore, the receding horizon controller in the setting described above, is given by:

$$u(t) = \pi_0^*(x_t) \qquad t \in \mathbb{Z}_+ \tag{3.6}$$

where $u(t)$ is the input which is fed to the plant at time $t$.

With an analytical solution of problem 3.2.3 one can implement the controller (3.6) explicitly, without on-line computations. Unfortunately, there are several reasons why the above problem is a difficult one to solve. The main difficulty is that the optimal feedback $\pi^*$ is an element of an infinite dimensional space which makes problem 3.2.3 an infinite dimensional optimization problem except for cases in which the disturbance is taking values from a finite set. In contrast, the standard MPC optimization chooses the input in open loop and only requires a search for an optimal open loop input vector which is an element of a finite dimensional space.

The optimization problem 3.2.3 is related to some well-known control problems as outlined in the following remarks.

**Remark 3.2.4** Without constraints on the input and with a stochastic disturbance which has the expectation equal to zero it is well-known that the optimal feedback is in the class of linear state feedback controllers. In other words $\pi_k$ can, without loss of generality, be chosen as a linear state feedback. The receding horizon controller (3.6) then has the form of a linear state feedback law:

$$u(t) = F_0 x_t \quad t \in \mathbb{Z}_+$$

where $F_0$ is the first element of the vector $(F_k)_{k=0}^N$ defined by the following backwards recursions:

$$P_k = A^T P_{k+1} A + C_z^T C_z - (A^T P_{k+1} B + C_z^T D_z)(B^T P_{k+1} B + D_z^T D_z)^{-1} \times \\ \times (B^T P_{k+1} A + D_z^T C_z) \qquad P_{N+1} := Q \quad (3.7)$$

and

$$F_k = -(B^T P_{k+1} B + D_z^T D_z)^{-1}(B^T P_{k+1} A + D_z^T C_z)$$

where $k \in [0, N]$.

For this case, it is known that the issue of stability of the overall system crucially depends on the choice of the matrix $Q$ in the end point penalty. In general, the more the state at the end point is penalized in (3.4), the more likely it is that a model predictive control law will yield a stable closed loop system. The case is especially simple when $Q$ is chosen as a steady-state solution of the Riccati equation (3.7). The controller (3.6) is then simply equivalent to the infinite-horizon LQ controller for the plant. An interesting problem that is posed in [141] and [158] is to investigate the stability properties of the controlled system as a function of the end point penalty. Results are obtained for the unconstrained case.

When constraints are present, the analysis of the closed loop stability as a function of the end point penalty becomes difficult. It is crucial to note that we need generally a stronger notion of stability when a stochastic system is considered because the variance of the state is required to remain bounded. Note that model predictive control schemes based on a *terminal constraint set* (see [37, 97, 100]) can obviously not be applied when the disturbance is unbounded. Even if the disturbance is bounded it still requires considerable work to establish stability in the stochastic setting with a use of these techniques as a starting point.

The only available approach is to include the end point penalty in the cost function and to see the end point penalty as a "tuning parameter".

**Remark 3.2.5** Another simplification of the problem is the case with input constraints where the disturbance is assumed to be known and constant over the control horizon, i.e. $w_N = w_N^*$ where $w_N^*$ is a fixed disturbance. In that case the progression of the predicted state is a trajectory, denoted as $x_N^*$ which is a function of the state measurement $x_t$, fixed disturbance $w_N^*$ and the input $u(k) = \pi_k\left(x_N^*(k)\right)$. Since at each $k \in T$ one has to compute an optimal map only in states $x_N^*(k)$ an optimization in open loop and an optimization in closed loop yield the same infimum. This observation allows

one to consider input vectors $u_N : T \to \mathbf{U}$ and an optimization problem of finding an optimal input vector $u_N^*$ such that:

$$J(x_t, u_N^*, w_N^*) \leq J(x_t, u_N, w_N^*)$$

for all $u_N : T \to \mathbf{U}$, instead of problem 3.2.3. This problem is a finite dimensional convex optimization problem. The solution can be obtained by a standard quadratic programming algorithm. This is a prototype of the optimization in open-loop that is prevailing in the MPC literature. As shown in [18] the resulting receding horizon controller can be expressed as a feedback which is piecewise linear and continuous. When stochastic disturbances are included in the model, solving the optimization in open loop yields a suboptimal solution to the optimization problem with a considerably larger cost (see [99] and the example in [55] and [56]). The only way to compute the optimal solution is via stochastic dynamic programming which can be a vastly difficult task for all but very simple systems. The predictive controller with the optimization in open loop is a *feedback* controller because of the receding horizon implementation. The difference between a predictive controller and the control law computed via stochastic dynamic programming is difficult to access. Different claims can be found in the literature. Example in [55] and [56] shows small difference but a similar example in [116] shows a considerable difference. These examples are mainly concerned with the difference in the closed loop performance. The prediction is often used as an indication of the future behavior of the system not just as a means to compute the control input. Because of the stochastic nature of the state, the prediction based on the control in open loop and the true behavior of the system can differ significantly.

In this chapter, we solve the optimization problem 3.2.3 approximately but with an arbitrary high accuracy. The method, presented as an algorithm, is based on the use of the empirical mean instead of the expectation in the optimization problem 3.2.3. A computation of the empirical mean is based on a randomized algorithm. The accuracy depends on the number of samples of the disturbance $w$. The main value of the algorithm is that it is able to compute the limit of performance in stochastic disturbance rejection for linear systems with input constraints, a question that is left unanswered in the available literature.

## 3.3 Empirical mean

An analytical computation of the expectation of the cost (3.4) is difficult. An alternative is to compute the empirical mean of the cost in (3.4). The cost for a specific realization of the stochastic disturbance $w$ is easily computed but realizations have to be chosen so that the empirical mean is computed efficiently. It is well known that an estimate based on linear gridding requires a number of samples that is exponential in the dimension of the stochastic variable to preserve accuracy in estimation. A standard

method that overcomes this problem is Monte Carlo simulation. Realizations of the stochastic disturbance are chosen randomly, according to the distribution of $w$. It is well known that bounds on the number of samples needed to preserve accuracy of the estimation can be obtained independent of the underlying distribution of the stochastic process. In the following, the problem of computing the empirical mean is given a formal setting.

Assume a set $\Theta$ and a probability measure $P$ on $\Theta$ are given. Let $f : \Theta \to \Omega$ be a scalar valued function measurable with respect to $P$ where $\Omega$ is an interval on $\mathbb{R}$ (possibly equal to $\mathbb{R}$). The expectation of $f$ can be expressed as:

$$\mathbb{E} f = \int_{\Theta} f(\theta) dP \tag{3.8}$$

Our aim is to approximate (3.8) by drawing $m$ independent, identically distributed (i.i.d) samples $\vartheta = \{\theta_1, \cdots, \theta_m\}$ from $\Theta$ in accordance with $P$ and computing the empirical mean by setting

$$\hat{\mathbb{E}} f := \frac{1}{m} \sum_{j=1}^{m} f(\theta_j) \tag{3.9}$$

The empirical mean (3.9) is a function of a randomly chosen multisample $\vartheta$ and it is obviously stochastic. Such an estimate is useful only if we have an insight in the error given by $|\mathbb{E} f - \hat{\mathbb{E}} f|$. Since (3.9) is stochastic, the error is expressed in a probabilistic confidence interval rather than in the form of a strict bound. We have confidence $\delta$ in the approximation (3.9) with accuracy $\varepsilon$ if $|\mathbb{E} f - \hat{\mathbb{E}} f| < \varepsilon$ with a probability of at least $\delta$. A lower bound for the confidence $\delta$ can be easily derived by using a well known Chebyshev inequality (see [151], for example). The bound then takes the form:

$$\text{Prob}\left(|\hat{\mathbb{E}} f - \mathbb{E} f| < \varepsilon\right) \geq 1 - \frac{(\text{Var } f)^2}{m\varepsilon^2} \tag{3.10}$$

where Var denotes variance.

**Theorem 3.3.1** *The empirical mean* (3.9) converges in probability *to the expectation* $\mathbb{E} f$ *i.e.*

$$\text{Prob}\left(|\hat{\mathbb{E}} f - \mathbb{E} f| < \varepsilon\right) \to 1 \text{ as } m \to \infty,$$

*for all* $\varepsilon > 0$.

**Proof:** The claim of the theorem follows from (3.10) as $m \to \infty$. ■

The lower bound (3.10) can be applied in the general case $\Omega = \mathbb{R}$. When $\varepsilon$ is held constant, the lower bound on probability (3.10) converges to 1 at a polynomial rate as the number of samples $m$ increases. In the literature, lower bounds for the confidence $\delta$ which converge to 1 exponentially are available for some special cases. In [69] Hoeffding's inequality is derived which yields a lower bound for the confidence $\delta$. The

bound converges to 1 exponentially if the random variable has a bounded range. The bound obtained from Hoeffding's inequality does not depend on the probability measure $P$ nor the dimension of the set $\Theta$. In particular, the exponential convergence of the confidence makes its assessment for a large number of samples less conservative than with the inequality (3.10). For details on the application of Hoeffding's inequality to the confidence of the empirical mean (3.9) we refer to [151]. However, this inequality is intrinsically restricted to the case of stochastic variables with a bounded range, where $\Omega$ is a bounded subset of $\mathbb{R}$. For a detailed treatment of convergence issues arising when one consider stochastic processes we refer to [112].

## 3.4 Algorithm 3.1: An approximate but arbitrarily accurate solution

At time instant $s \in T$ the state $x_N(s)$ is a stochastic variable. The system (3.1) is strictly causal so $x_N(s)$ does not depend on the "future" disturbances $(w_N(k))_{k=s}^N$. This allows us to define an optimal cost "to go" at each $s \in T$ as:

$$V_s(x) := \inf_{\pi^s} \mathbb{E}\, J_s(x, \pi^s, w^s) \tag{3.11}$$

where for all $s \in T$ we define:

$$J_s(x, \pi^s, w^s) := \sum_{k=s}^{N} \|C_z x_N(k) + D_z \pi_k(x_N(k))\|^2 \; + \; \|x_N(N+1)\|_Q^2 \tag{3.12}$$

with $x_N(s) = x$, disturbance $w^s := (w_N(k))_{k=s}^N$ and $\pi^s := (\pi_k)_{k=s}^N$ is a vector of feedback mappings $\pi_k : \mathbb{R}^n \to \mathbf{U}$. Note that for $s = 0$ the optimal cost $V_0(x)$ is equal to the optimal cost (3.5) and the cost "to go" $J_0(x, \pi^0, w^0)$ is equal to the cost (3.4). Also, $\pi^0 = \pi$ and $w^0 = w_N$.

By using (3.11), the optimal cost (3.5) can be rewritten as a dynamic program (see [9] for detailed treatment of stochastic dynamic programming) given by:

$$V_s(x) := \inf_{u \in \mathbf{U}} \left\{ \|C_z x + D_z u\|^2 + \mathbb{E}_w V_{s+1}(Ax + Bu + Ew) \right\} \tag{3.13}$$

with an initial condition:

$$V_{N+1}(x) := \|x\|_Q^2$$

and where $\mathbb{E}_{(\cdot)}$ denotes conditional mean with respect to $(\cdot)$. Dynamic program (3.13) has to be solved backwards from $s = N$ to $s = 0$.

The expectation in (3.11) and (3.13) can be easily computed only for the case $s = N$.

For $s = N$, an optimal cost "to go" is given by:

$$V_N(x) = \inf_{u \in \mathbf{U}} \left\{ \|C_z x + D_z u\|^2 + \|Ax + Bu\|_Q^2 \right\} + E^T Q E \operatorname{Var}(w). \tag{3.14}$$

For $s = 0, \cdots N-1$ the optimal cost "to go" does not have a quadratic structure and the computation of the expectation is not straightforward. An alternative that we suggest in this chapter is to compute an empirical mean and use it as an approximation for the expectation. To compute the empirical mean, a number of samples have to be drawn based on an underlying probability distribution, a process usually called sampling.

Suppose that we take $\kappa$ samples of the disturbance $w_N(0)$ at $k = 0$. Given a fixed initial condition $x_t$ and a fixed input $u(0) = \pi_0(x_t)$ there are $\kappa$ possible states $x_N(1)$. For each one of these possible futures we generate $\kappa$ samples of the disturbance $w_N(1)$ which establishes $\kappa^2$ possible future states $x_N(2)$. In this way, the sampling of the disturbance yields $\kappa^N$ samples of $w$. The number of samples of the restricted disturbance sequence $w^s$ is $\kappa^{N-s}$. The number of samples of $w$ grows exponentially with the horizon. The sampling as described is required for a sufficient number of samples of the future disturbance given $x_N(s)$ which is needed to get a good estimate for $V_s$. One might conjecture that we do not need this because a very accurate estimate of $V_s$ is not required. Actually, only a good estimate of $\pi_0$ is needed. However, we have no proof that a restricted set of samples still yields a correct result with a high probability.

Other an approach would be to form a grid on the state space and to estimate $V_s$ on the points of the grid. Note that any kind of linear grid will not reflect a spread of the state around its mean value, resulting in a great number of points in which the state is not likely to be. The sampling procedure described above gives a grid on the state space that is more dense in the region in which the state is more likely to be. Moreover, the number of grid points grows exponentially (in the dimension of the state space) while the number of points required for stochastic sampling is independent of the dimension of the state space.

For all $s \in \{0, \cdots, N\}$ and for each of the $\kappa^{N-s}$ samples of $w^s$ denoted by $w_i^s$, $i \in \{1, \cdots, \kappa^{N-s}\}$ the cost function is given by:

$$J_s(x, \pi^s, w_i^s) = \sum_{k=s}^{N} \|C_z x_N(k) + D_z \pi_k(x_N(k))\|^2 + \|x_N(N+1)\|_Q^2 \tag{3.15}$$

with $x_N(s) = x$.

The empirical mean of the cost function given a vector of feedback mappings $\pi^s = (\pi_k)_{k=s}^N$ is:

$$\hat{\mathbb{E}} J_s(x, \pi^s, w^s) := \frac{1}{\kappa^{N-s}} \sum_{i=1}^{\kappa^{N-s}} J_s(x, \pi^s, w_i^s). \tag{3.16}$$

A direct application of the inequality (3.10), for some vector of feedback mappings $\pi^s$, yields:

$$\text{Prob}\left\{\left|\hat{\mathbb{E}}J_s\left(x,\pi^s,w^s\right)-\mathbb{E}J_s\left(x,\pi^s,w^s\right)\right|<\varepsilon\right\}\to 1$$

for all $s$ and for all $\varepsilon > 0$ as $\kappa \to \infty$. Thus, the empirical mean of the cost "to go" (3.16) converge in probability to its true mean.

An approximation of the optimal cost "to go" (3.11) is given by:

$$\hat{V}_s(x) := \inf_{\pi^s} \hat{\mathbb{E}}J_s(x,\pi^s,w^s). \tag{3.17}$$

We call $\hat{V}_s(x)$ an *empirical optimal cost "to go"*. The empirical optimal cost "to go" needs, in principle, to be computed for all points in the state space. To make the problem computable in finite time, one could suggest to define a grid on the state space and compute the empirical optimal cost "to go" in the points of the grid. However, not only is this approach near impossible when the dimension of the state space is high, but it also ignores the fact that some states are more likely than others. Instead, we look at all of our sampled past disturbances and predict $x_N(s)$. This yields a "grid" of the state space which is not uniform but is instead biased towards those states which are "likely", given past disturbances.

If we consider an arbitrary time instant $s$ in the control $T$ then the number of points for which we evaluate $\hat{V}_s(x)$ is determined by all past disturbance realizations $w(\tau)$, $\tau \in \{0,\cdots,s-1\}$. With disturbances sampled as described previously, the number of points in the state space in which we evaluate the empirical cost "to go" is equal to $\kappa^s$. This yields an exponential growth of the number of points as we move further from $s = 0$ towards the end of the horizon.

The algorithm for an arbitrary accurate solution to the optimization problem 3.2.3 is based on the following theorem.

**Theorem 3.4.1** *Consider the approximation of the optimal cost* (3.5) *given by* (3.17) *with* $s = 0$*:*

$$\hat{V}_0\big(x\big) = \inf_{\pi^0\in\Pi}\{\hat{\mathbb{E}}\,J\big(x,\pi^0,w^0\big)\}. \tag{3.18}$$

*The optimal cost* (3.18) *and the associated optimal vector of feedback mappings* $\pi \in \Pi$ *can be obtained recursively from the following dynamic program:*

$$\hat{V}_s(x) := \inf_{u\in\mathbf{U}}\left\{\|C_z x + D_z u\|^2 + \hat{\mathbb{E}}_w\,\hat{V}_{s+1}(Ax+Bu+Ew)\right\} \tag{3.19}$$

*with an initial condition:*

$$\hat{V}_{N+1}(x) := \|x\|_Q^2$$

*that has to be solved backwards from* $s = N$ *to* $s = 0$ *and where* $\hat{\mathbb{E}}_{(\cdot)}$ *denotes empirical conditional expectation with respect to* $(\cdot)$.

We first present an auxiliary result, used in the proof of the above theorem.

**Lemma 3.4.2** *Consider a random variable* $\omega$ *taking values in* $\mathbb{R}$ *and the set* $\Pi_w$ *of continuous functions* $\pi : \mathbb{R} \to \mathbf{U} \subset \mathbb{R}$ *with* $\mathbf{U}$ *compact. Next, consider a continuous function* $f : \mathbb{R} \times \mathbf{U} \to \mathbb{R}$ *such that for each* $w \in \mathbb{R}$, $f(w, u)$ *is convex in* $u$. *Then*

$$\inf_{\pi \in \Pi_w} \mathbb{E}\{ f\big(\omega, \pi(\omega)\big)\} = \mathbb{E}\{\min_{u \in \mathbf{U}}\{ f(\omega, u)\}\}. \tag{3.20}$$

**Proof:** Assume that a function $\pi \in \Pi_w$ is given. The following inequality follows:

$$\mathbb{E}\{ f\big(\omega, \pi(\omega)\big)\} \geq \mathbb{E}\{\min_{u \in \mathbf{U}}\{ f(\omega, u)\}\}. \tag{3.21}$$

If $f(w, u)$ is strictly convex in $u$ then the minimum with respect to $u$ is unique for each $w$ and it is easily shown that there exists a unique continuous function $\pi \in \Pi_w$ such that:

$$f(\omega, \pi(\omega)) = \min_{u \in \mathbf{U}} f(\omega, u). \tag{3.22}$$

This clearly implies in combination with (3.21) that (3.20) is satisfied.

If $f(w, u)$ is convex in $u$ then $f(x, u) + \varepsilon \|u\|^2$ is strictly convex in $u$ and hence according to the above we find:

$$\inf_{\pi \in \Pi_w} \mathbb{E}\ f\ (\omega, \pi(\omega)) + \varepsilon \|\pi(\omega)\|^2 = \mathbb{E}\ \min_{u \in \mathbf{U}} f(\omega, u) + \varepsilon \|u\|^2. \tag{3.23}$$

Noting that $\mathbf{U}$ is compact and hence $\|u\|^2$ uniformly bounded implies that (3.23) yields (3.20) as $\varepsilon \to 0$. ■

The proof of theorem 3.4.1 is presented next.

**Proof:** Observe that (3.18) can be rewritten as:

$$\inf_{\pi^0} \hat{\mathbb{E}}_{w_N(0)} \hat{\mathbb{E}}_{w_N(1)} \cdots \hat{\mathbb{E}}_{w_N(N)} \Big\{ \sum_{k=0}^{N} \|C_z x_N(k) + D_z \pi_k\big(x_N(k)\big)\|^2 + \|x_N(N+1)\|_Q^2 \Big\}$$

because of the fact that $w_N(0) \cdots w_N(N)$ are independent stochastic variables. Next, we use the causality of the system (3.1) (a "current"state does not depend on "future" disturbances) to rewrite (3.18) again:

$$\inf_{\pi^0} \Big\{ \|C_z x_N(0) + D_z \pi_0(x_N(0))\|^2 + \sum_{k=1}^{N} \hat{\mathbb{E}}_{w_N(0)} \hat{\mathbb{E}}_{w_N(1)} \cdots \hat{\mathbb{E}}_{w_N(k-1)} \|z(k)\|^2 + \\ + \hat{\mathbb{E}}_{w_N(0)} \hat{\mathbb{E}}_{w_N(1)} \cdots \hat{\mathbb{E}}_{w_N(N)} \|x_N(N+1)\|_Q^2 \Big\}$$

where:

$$z(k) = C_z x_N(k) + D_z \pi_k(x_N(k)), \quad k \in [1, N].$$

The causality argument allows to "move" the infimization operator inside the brackets so that we can consider the optimization over feedback maps one by one as follows:

$$\inf_{\pi_0 \cdots \pi_{N-1}} \Bigg\{ \|z(0)\|^2 + \sum_{k=1}^{N-1} \hat{\mathbb{E}}_{w_N(0)} \hat{\mathbb{E}}_{w_N(1)} \cdots \hat{\mathbb{E}}_{w_N(k-1)} \|z(k)\|^2 + \\ + \inf_{\pi_N} \hat{\mathbb{E}}_{w_N(0)} \hat{\mathbb{E}}_{w_N(1)} \cdots \hat{\mathbb{E}}_{w_N(N-1)} \big\{ \|z(N)\|^2 + \hat{\mathbb{E}}_{w_N(N)} \|x_N(N+1)\|^2 \big\} \Bigg\}.$$

By lemma 3.4.2, the last term of this expression can be rewritten as

$$\hat{\mathbb{E}}_{w_N(0)} \hat{\mathbb{E}}_{w_N(1)} \cdots \hat{\mathbb{E}}_{w_N(N-1)} \inf_{u \in \mathbf{U}} \big\{ \|z(N)\|^2 + \hat{\mathbb{E}}_{w_N(N)} \|x_N(N+1)\|^2 \big\}.$$

Define

$$\hat{V}_N(x) := \inf_{u \in \mathbf{U}} \big\{ \|C_z x + D_z u\|^2 + \hat{\mathbb{E}}_w \|Ax + Bu + Ew\|^2 \big\}.$$

This allows to rewrite (3.18) as:

$$\inf_{\pi_0 \cdots \pi_{N-2}} \Bigg\{ \|z(0)\|^2 + \sum_{k=1}^{N-2} \hat{\mathbb{E}}_{w_N(0)} \hat{\mathbb{E}}_{w_N(1)} \cdots \hat{\mathbb{E}}_{w_N(k-1)} \|z(k)\|^2 + \\ + \inf_{\pi_{N-1}} \hat{\mathbb{E}}_{w_N(0)} \hat{\mathbb{E}}_{w_N(1)} \cdots \hat{\mathbb{E}}_{w_N(N-2)} \big\{ \|z(N-1)\|^2 + \hat{\mathbb{E}}_{w_N(N-1)} \hat{V}_N(x_N(N)) \big\} \Bigg\}.$$

As the next step, define:

$$\hat{V}_{N-1}(x) := \inf_{u \in \mathbf{U}} \big\{ \|C_z x + D_z u\|^2 + \hat{\mathbb{E}}_{w_N(N-1)} \hat{V}_N(Ax + Bu + Ew) \big\}$$

and rewrite (3.18) as:

$$\inf_{\pi_0 \cdots \pi_{N-3}} \Bigg\{ \|z(0)\|^2 + \sum_{k=1}^{N-3} \hat{\mathbb{E}}_{w_N(0)} \hat{\mathbb{E}}_{w_N(1)} \cdots \hat{\mathbb{E}}_{w_N(k-1)} \|z(k)\|^2 + \\ + \inf_{\pi_{N-2}} \hat{\mathbb{E}}_{w_N(0)} \hat{\mathbb{E}}_{w_N(1)} \cdots \hat{\mathbb{E}}_{w_N(N-3)} \big\{ \|z(N-2)\|^2 + \hat{\mathbb{E}}_{w_N(N-2)} \hat{V}_{N-1}(x_N(N-1)) \big\} \Bigg\}.$$

By proceeding in this way, the optimization problem (3.18) can be rewritten as the recursion (3.19). ■

The following lemma states an important property of the empirical optimal cost "to go" (3.17).

**Lemma 3.4.3** *The empirical optimal cost to go* (3.17) *is a convex function in* $x$ *for all* $s \in T$.

**Proof:** The empirical optimal cost (3.17) is defined as a minimization over a class $\Pi^s$ where $\pi^s \in \Pi^s$ is a sequence of maps $\pi_k^s : \mathbb{R} \to \mathbf{U}$ such that $u_N(k) = \pi_k^s(x_N(k))$ with $k = s, \ldots, n$. We first extend the class $\Pi^s$ of functions over which optimization is defined. Assume we optimize over $\tilde{\Pi}^s$ where $\tilde{\pi}^s \in \tilde{\Pi}^s$ is a sequence of maps $\tilde{\pi}_k^s : \mathbb{R}^{n(k+1)} \to \mathbf{U}$ such that

$$u_N(k) = \tilde{\pi}_k^s(x_N(0), x_N(1), \ldots, x_N(k)). \tag{3.24}$$

Since the future at time $k$ only depends on $x_N(k)$, it is obvious that this extension of the class of controllers does not change the infimum. Next, note that the feedbacks in the class $\tilde{\Pi}^s$ can be equally represented by the class $\bar{\Pi}^s$ where $\bar{\pi}^s \in \bar{\Pi}^s$ is a sequence of maps $\bar{\pi}_k^s : \mathbb{R}^{n+k} \to \mathbf{U}$ such that

$$u_N(k) = \bar{\pi}_k^s\left(x_N(0), w_N(0), \ldots, w_N(k-1)\right). \tag{3.25}$$

This is based on the fact that

- Given $x_N(0), w_N(0), \ldots, w_N(k-1)$ and $\bar{\pi}^s$ we can recursively construct $x_N(0)$, $x_N(1), \ldots, x_N(k)$ and $\tilde{\pi}^s$ such that

$$\bar{\pi}_k^s\left(x_N(0), w_N(0), \ldots, w_N(k-1)\right) = \tilde{\pi}_k^s(x_N(0), x_N(1), \ldots, x_N(k)) \tag{3.26}$$

- Given $x_N(0), x_N(1), \ldots, x_N(k)$ and $\tilde{\pi}^s$ it is possible to recursively construct $x_N(0)$, $w_N(0), \ldots, w_N(k-1)$ and $\bar{\pi}^s$ such that (3.26) is satisfied. If $E$ is not injective then $w_N(0), \ldots, w_N(k-1)$ are not uniquely determined but it is trivial to see that this does not affect the corresponding infima.

We conclude that the infimum in (3.17) is the same if $\pi^s$ is taken from $\Pi^s$, $\tilde{\Pi}^s$ or $\bar{\Pi}^s$. It therefore suffices to show that

$$\inf_{\bar{\pi}^s \in \bar{\Pi}} \hat{\mathbb{E}} J_s(x, \bar{\pi}^s, w^s)$$

is a convex function in $x$.

To prove that, consider $x^a, x^b \in \mathbb{R}^n$, $x^a \neq x^b$. The corresponding minimizing feedbacks in $\bar{\Pi}^s$ are denoted by $\bar{\pi}_a^s$ and $\bar{\pi}_b^s$, respectively.

The empirical optimal cost to go is computed for a finite number of disturbance realizations $w^s$. Define:

$$u_i^a := \bar{\pi}_a^s(x^a, w_i^s)$$

and:

$$u_i^b := \bar{\pi}_b^s(x^b, w_i^s).$$

The cost (3.12) for fixed $w$ is quadratic in $(x, u)$ and therefore jointly convex in $(x, u)$:

$$J_s(\lambda x^a + (1-\lambda)x^b, \lambda u_i^a + (1-\lambda)u_i^b, w_i^s) \leq \lambda J_s(x^a, u_i^a, w_i^s) + (1-\lambda) J_s(x^b, u_i^b, w_i^s)$$

where $\lambda \in (0, 1)$.

Since the empirical mean $\hat{\mathbb{E}} J_s(x, u, w)$ is defined via operations that preserve convexity, it is also a convex function in $(x, u)$. Clearly:

$$\bar{\pi}_*^s(x, w^s) = \lambda \bar{\pi}_a^s(x^a, w^s) + (1-\lambda)\bar{\pi}_b^s(x^b, w^s)$$

satisfies $\bar{\pi}_*^s \in \bar{\Pi}^s$. Convexity of (3.16) then implies:

$$\begin{aligned}
&\hat{V}_s(\lambda x^a + (1-\lambda)x^b) \\
&\quad \leq \hat{\mathbb{E}} J_s(\lambda x^a + (1-\lambda)x^b, \bar{\pi}_*^s(\lambda x^a + (1-\lambda)x^b, w^s), w^s) \\
&\quad = \hat{\mathbb{E}} J_s(\lambda x^a + (1-\lambda)x^b, \lambda \bar{\pi}_a^s(x_a, w^s) + (1-\lambda)\bar{\pi}_b^s(x_b, w^s), w^s) \\
&\quad = \tfrac{1}{\kappa^s} \sum_{i=1}^{\kappa^s} J_s(\lambda x^a + (1-\lambda)x^b, \lambda \bar{\pi}_a^s(x_a, w_i^s) + (1-\lambda)\bar{\pi}_b^s(x_b, w_i^s), w_i^s) \\
&\quad = \tfrac{1}{\kappa^s} \sum_{i=1}^{\kappa^s} J_s(\lambda x^a + (1-\lambda)x^b, \lambda u_i^a + (1-\lambda)u_i^b, w_i^s) \\
&\quad \leq \tfrac{1}{\kappa^s} \sum_{i=1}^{\kappa^s} \lambda J_s(x^a, u_i^a, w_i^s) + (1-\lambda) J_s(x^b, u_i^b, w_i^s) \\
&\quad = \tfrac{1}{\kappa^s} \sum_{i=1}^{\kappa^s} \lambda J_s(x^a, \bar{\pi}_a^s(x_a, w_i^s), w_i^s) + (1-\lambda) J_s(x^b, \bar{\pi}_a^s(x_b, w_s^i), w_i^s) \\
&\quad = \lambda \hat{V}_s(x^a) + (1-\lambda)\hat{V}_s(x^b)
\end{aligned}$$

for all $x^a, x^b \in \mathbb{R}^n$ and $\lambda \in (0, 1)$. ■

The result presented in lemma 3.4.3 makes it possible to derive an efficient algorithm for minimization of the empirical mean in (3.17). The minimization of the convex empirical mean (3.17) in this algorithm utilizes a convex optimization technique, for example a bisection algorithm.

The algorithm for solving 3.2.3 can now be derived following the dynamic program (3.19). We start by choosing $\kappa$ and the length of the control horizon $N$. With the disturbance sampled as described before we obtain $\kappa^N$ samples of the disturbance $w$. Each of these samples give rise to the one predicted state trajectory. Therefore, at each $s$, $s \in \{0, \cdots N-1\}$ there are $\kappa^s$ possible states denoted by $x_N^i(s)$, $i \in \{1, \cdots, \kappa^s\}$. In the following we present the algorithm.

**Algorithm 3.1**

**Step 1: Initialization**
Take the measurement $x_t$ and set $x_N^1(0) = x_t$. Set $\hat{u}_i(s) = 0$ for $s = 0, 1, \ldots, N$, $i = 1, \ldots, N$. Draw $\kappa^N$ samples for $w$ as described before. Set $V = \infty$. Set accuracy parameter $\varepsilon$. Set $s = N$.

**Step 2: Compute cost at the end of the horizon**
Determine a new $\hat{u}_i(N)$ using (3.14) for each $x_N^i(N)$, $i = 1, \ldots, \kappa^N$. Compute $\hat{V}_N(x_N^i(N))$ for each $i$. Set $s = N - 1$.

**Step 3: Compute cost "to go"**
Determine a new $\hat{u}_i(s)$ by solving (3.19) for each $x_N^i(s)$, $i = 1, \ldots \kappa^s$. Compute $\hat{V}_s(x_N^i(s))$ for each $i$. If $s = 0$ go to **step 4**, otherwise set $s = s - 1$ and go to **step 3**.

**Step 4: Exit condition**
If $|\hat{V}_0(x_N^1(0)) - V| < \varepsilon$ stop. Otherwise: set $V = \hat{V}_0(x_N^1(0))$ and go to **step 2**.

The relation of the solution obtained by algorithm 3.1 and the original problem 3.2.3 is described in the following theorem.

**Theorem 3.4.4** *Assume $D_z$ is injective. For any initial condition $x \in \mathbb{R}^n$, and for all $s = 0, \cdots, N$, the empirical optimal cost to go $\hat{V}_s(x)$, defined in* (3.17)*, converges with probability* 1 *to the optimal cost $V_s(x)$, defined in* (3.11)*, whenever $\kappa \to \infty$. In particular, $\hat{V}_0(x)$ converges with probability* 1 *to $V(x)$, defined in* (3.5)*, as $\kappa \to \infty$.*

**Proof:** We will establish this result recursively. First, we consider $s = N$. In this case, we have:

$$\hat{V}_N(x) = \inf_u \hat{\mathbb{E}} \left( \|C_z x + D_z u\|^2 + \|Ax + Bu + Ew\|_Q^2 \right).$$

Since $D_z$ is injective the function we want to minimize on the right hand side is strictly convex in $u$ and grows at most quadratically in $x$. Then, it follows ( [112], theorem 24, p.p.25) that for any $\varepsilon_N > 0$ and $\delta_N \in (0, 1)$ there exists $\kappa_N^*$ such that for any $\kappa > \kappa_N^*$ we have with probability $(1 - \delta_N)$ that:

$$\left| \hat{V}_N(x) - V_N(x) \right| \leq \varepsilon_N \|x_N\|^2.$$

Next assume that for any $\varepsilon_t > 0$ and $\delta_t \in (0, 1)$ there exists $\kappa_t^*$ such that for any $\kappa > \kappa_t^*$ we have with probability $(1 - \delta_t)$ that:

$$\left| \hat{V}_t(x) - V_t(x) \right| \leq \varepsilon_t \|x\|^2$$

for some $t \in (s, N]$. We have

$$\hat{V}_{t-1}(x) = \inf_{u_{t-1}} \hat{\mathbb{E}} \left( \|C_z x + D_z u\|^2 + \hat{V}_t(Ax + Bu + Ew) \right).$$

But this yields that:

$$\hat{V}_{t-1}(x) \leq \inf_u \hat{\mathbb{E}} \big( \|C_z x + D_z u\|^2 + \hat{V}_t(Ax + Bu + Ew) + \varepsilon_t \|Ax + Bu + Ew\|^2 \big)$$

and similarly we can obtain the lower bound:

$$\hat{V}_{t-1}(x) \geq \inf_u \hat{\mathbb{E}} \big( \|C_z x + D_z u\|^2 + \hat{V}_t(Ax + Bu + Ew) - \varepsilon_t \|Ax + Bu + Ew\|^2 \big).$$

On the other hand,

$$V_{t-1}(x) = \inf_u \hat{\mathbb{E}} \big( \|C_z x + D_z u\|^2 + \hat{V}_t(Ax + Bu + Ew) \big). \tag{3.27}$$

We know that $V_t(x)$ grows at most quadratic in $x$ and is convex. The latter makes the right hand side of (3.27) strictly convex since $D_z$ injective implies that the first term on the right hand side is strictly convex. This implies that again we can be sure that changing the expectation into an empirical mean has, with arbitrary large probability, a negligible effect. In other words, we find that for any $\varepsilon_{t-1} > 0$ and $\delta_{t-1} \in (0, 1)$ there exists $\varepsilon_t > 0$, $\delta_t \in (0, 1)$ small enough and $\kappa^*_{t-1} > \kappa^*_t$ such that for any $\kappa > \kappa^*_{t-1}$ we have with probability $(1 - \delta_{t-1})$ that:

$$\left| \hat{V}_{t-1}(x) - V_{t-1}(x) \right| \leq \varepsilon_{t-1} \|x\|^2.$$

Hence by using this recursion we establish that for any $\varepsilon_s > 0$ and $\delta_s \in (0, 1)$ there exists $\kappa^*_s$ such that for any $\kappa > \kappa^*_s$ we have with probability $(1 - \delta_s)$ that:

$$\left| \hat{V}_s(x) - V_s(x) \right| \leq \varepsilon_s \|x\|^2$$

This clearly implies $\hat{V}_s(x)$ converges with probability 1 to the optimal cost $V(x_s)$ if $\kappa \to \infty$. After all, for any fixed $\delta$ and $\varepsilon$ we can choose $\kappa$ large enough such that for fixed $x$:

$$\left| \hat{V}_s(x) - V_s(x) \right| \leq \varepsilon$$

with probability at least $(1 - \delta)$.

Finally, note that (3.11) coincides with (3.5) for the case $s = 0$. ■

Theorem 3.4.4 states that the empirical optimal cost to go (3.17) converges in probability to the optimal cost. This does not mean that the optimal controller derived by

minimizing the empirical cost converges in probability to the optimal controller, however. Note that our design only determines the controller in certain states determined by the drawn samples. For other states the controller is not defined. Note however that, in a receding horizon framework, we only apply the first input which depends on the current state which is fixed. Future states are unknown due to the stochastic disturbance. Hence we might end up in a state for which the controller is not defined. But since the control action over the full optimization horizon beyond the first step are never implemented in a receding horizon scheme this does not constitute a problem.

If we want to actually determine a controller over the full optimization horizon then we should interpolate the states generated by the sampled disturbances. Since the optimization is strictly convex for $D_z$ injective we know that the optimal controller will be differentiable with a bounded derivative (a bound can actually be computed a priori). Using that we can compute with arbitrary accuracy the controller on a compact subset of the space. The probability that the state gets outside of this compact set can be made arbitrarily small and therefore how we choose our input in these cases has only a negligible effect on the cost.

With algorithm 3.1, the optimization problem 3.2.3 can be solved approximately but with an arbitrary high accuracy. The accuracy of the solution depends on a number of samples of the disturbance $w$ taken for computing the empirical mean. The drawback of the algorithm is the high computational complexity. The number of points in which an empirical mean has to be evaluated grows exponentially with the horizon. The value of the algorithm is in its ability to access the information about achievable performance when one aims to control the plant (3.1), subject to the input constraints and the stochastic disturbance.

The exponential growth in the number of evaluating points is not necessary for the algorithm to work. It is a consequence of a fixed number of disturbance samples $\kappa$ used for evaluating the empirical mean. It can be expected that the accuracy by which one estimates the empirical optimal cost "to go" for time instants $s$ in the horizon further away from $s = 0$ does not have a significant effect on the overall performance of the algorithm. Thus, the number of samples can be smaller for those time instants. However, we have no proof of this. It is the reason we do not give explicit bounds for the number of samples needed. The estimates available in the literature are extremely conservative and all experiments we tried show that we can get away with much smaller numbers (often, by a factor of more that million).

An interesting possibility is to use a neural network (see [66]) as an approximation for the nonlinear map $\pi_0^*$. Neural networks have been used as approximations for the predictive controllers with constraints (see [70]). Two important issues arise in a design of a neural network for the approximation of the controller (3.6). The first one is the choice of an appropriate structure for the neural network. The second issue is the training of the neural network. To obtain a training set one needs an algorithm, such as the one described in this chapter, to obtain pairs of initial states and corresponding

optimal inputs. Although an initial training of the neural network, based on the algorithm given in this chapter, could be time consuming, once the network is trained it can be used for a fast on-line implementation of the controller (3.6). Note however, that a neural network would still be only an approximation of the nonlinear function $\pi_0^*$ unless the number of neurons is very high. The algorithm in this chapter will enable us to evaluate the gap between the optimal performance and the performance of a neural network.

## 3.5 Algorithm 3.2: A computationally less demanding solution

The computational burden involved in algorithm 3.1 can be reduced by trading accuracy against computational load. This can be done by fixing a class of feedback control laws in the optimization problem (3.2.3) rather than optimizing over a general feedback map.

The class of feedback laws that we propose in this section is the class of a linear feedback with saturation. At each time instant of the control horizon we assume that the feedback relation between the predicted state and the input over the horizon is given as:

$$u(k) = \sigma\left(F x_N(k)\right) \quad k \in T \tag{3.28}$$

where $\sigma$ is a saturation function that achieves that $\sigma(u) \in \mathbf{U}$ for all $u \in \mathbb{R}^m$ according to

$$\sigma(u) = \begin{cases} u & \text{if } u \in \mathbf{U} \\ \arg\min_{v \in \mathbf{U}} \|u - v\|_2 & \text{if } u \notin \mathbf{U} \end{cases}$$

and $F$ is a linear feedback control law $F : \mathbb{R}^n \to \mathbb{R}^m$.

With the feedback (3.28), we consider the cost function of the form:

$$J_{\text{fb}}(x_t, F, w_N) := \sum_{k \in T} \|C_z x_N(k) + D_z \sigma\left(F x_N(k)\right)\|^2 + \|x_N(N+1)\|_Q^2.$$

The following optimization problem is considered.

**Problem 3.5.1** Given a fixed state measurement $x_t$ at time $t \in \mathbb{Z}_+$ find a linear feedback control law $F_t^*$ such that

$$\mathbb{E}J(x_t, F_t^*, w_N) \leq \mathbb{E}J(x_t, F, w_N) \quad \forall F : \mathbb{R}^n \to \mathbb{R}^m.$$

In addition, determine the optimal cost given by:

$$V_{\text{fb}}(x_t) := \inf_F \mathbb{E}\, J_{\text{fb}}(x_t, F, w_N) \tag{3.29}$$

If $F_t^*$ exists, then the receding horizon controller, obtained by solving the optimization problem 3.5.1, is given by:

$$u(t) = F_t^* x_t \tag{3.30}$$

where $t \in \mathbb{Z}_+$. That is, only the first time sample is fed to the plant.

Unlike the receding horizon controller (3.6) the feedback control law (3.30) is *time varying*. As in the optimization problem 3.2.3, it is very difficult to obtain an analytical expression for the expectation in (3.29). In this section we propose an algorithm for solving optimization problem 3.5.1 that uses an empirical mean instead of the expectation. The empirical mean is computed by using randomly chosen samples of the disturbance $w_N$. The sampling procedure is different from the one described in section 3.4. The feedback structure in (3.28) is time invariant and finitely parameterized, therefore there is no need for dynamic programming and consequently, there is no need for an exponential growth in the number of samples of the disturbance over the horizon.

For the control horizon $T$ of the length $N$ we choose a number $\kappa$ of disturbance samples at each $k \in T$. The number of samples of $w_N$ is then $\kappa N$. We denote those samples as $w_N^i$, $i \in \{1, \cdots, \kappa N\}$. With the disturbance sampling as described, the empirical mean is given as:

$$\hat{\mathbb{E}} J_F(x_t, F, w_N) := \frac{1}{\kappa N} \sum_{i=1}^{\kappa N} J_F(x_t, F, w_N^i) \tag{3.31}$$

It is easy to see by using (3.10) that the empirical mean (3.31) converge in probability to its true mean.

The algorithm for a solution to the optimization problem 3.5.1 follows.

---

**Algorithm 3.2**

---

**Step 1: Initialization**

Take the measurement $x_t$. Draw $\kappa$ samples for $w$ according to the distribution of $w$. Set $V_0 = \infty$. Set accuracy parameter $\varepsilon$. Set $F = F_{LQ}$ where $F_{LQ}$ is the solution of the unconstrained infinite horizon LQ problem for the system (3.1):

$$F_{LQ} = -(D_z^T D_z + B^T P B)^{-1} B^T P A$$

where $P = P^T \geq 0$ is the solution of:

$$P = A^T P A + C_z^T C_z - (A^T P B + C_z^T D) \\ \times (B^T P B + D_z^T D_z)^{-1} (B^T P A + D_z^T C_z)$$

**Step 2: Compute the cost**
Compute (3.31).

**Step 3: Exit condition**
If $\|\hat{\mathbb{E}} J_F(x_t, F, w_N) - V_0\| < \varepsilon$ set $F_t^* = F$ and stop. Otherwise: set the temporary cost $V_0 = \hat{\mathbb{E}} J_F(x_t, F, w_N)$ and update F according to the numerical algorithm that has been chosen for the numerical minimization of (3.31). Go to **step 2**.

---

The input of the plant at some time $t \in \mathbb{Z}_+$ is then computed according to (3.30) and in the next time instant computations in algorithm 3.2 are repeated.

The optimization problem 3.5.1 is not a convex one. Therefore, the result computed by algorithm 3.2 can be a local minima. In this case a careful choice of an initial "guess" for $F$ in the algorithm is crucial for performance. We choose the solution of the unconstrained infinite horizon LQ problem as the starting point in the optimization. This choice is motivated for the following reason. With a disturbance with mean zero and a small variance acting on the plant, initial states close to the origin, i.e. in the states for which a saturation of inputs is not likely to occur and the $N$ sufficiently large, the optimal $F$ for optimization problem 3.5.1 will be close to $F_{LQ}$. On the other hand, for states far from origin the saturation dominates the performance and the value of the computed $F$ only determines a direction of the optimal input.

We would like to stress the fact that algorithm 3.2 computes the optimum with high accuracy. Therefore, we can evaluate simplifications of this scheme which reduce the computational time and show to what extent performance has been compromised. The parameterization of control laws which we suggest in this chapter is a simple one. If performance loss in this scheme turns out to be too large for a specific application, we may consider more elaborate parameterizations of control laws such as an optimization over a vector of saturated linear feedbacks over the control horizon $(\sigma(F_k x_N(k)))_{k=0}^N$, $F_k : \mathbb{R}^n \to \mathbb{R}^m$.

## 3.6 Numerical examples

### 3.6.1 A single input example

In this subsection we present a very simple example in which we considered a second order plant with a Gaussian white noise disturbance. With the growth of the con-

trol horizon, the number of disturbance samples grows. There are several ways to overcome the problem of an excessive number of samples with a large horizon. One possible approach is to reduce the number of samples toward the end of the control horizon. The "empirical reasoning" behind this idea is that, because of the receding horizon paradigm, only the first element in the vector of computed control laws is implemented and we use a large number of the disturbance samples where the largest accuracy is needed.

We consider the plant with the model of the form (3.1) with:

$$A = \begin{bmatrix} 0.7326 & -0.0861 \\ 0.1722 & 0.9909 \end{bmatrix} \quad B = \begin{bmatrix} 0.0609 \\ 0.0064 \end{bmatrix}$$
$$E = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \quad C_z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_z = \begin{bmatrix} 0.1 \\ 0 \\ 0 \end{bmatrix}$$

See [13] and [18]. For each time instant $t \in \mathbb{Z}_+$, the stochastic disturbance $w(t)$ is assumed to be uniformly distributed on the interval $[-\alpha,\ \alpha]$ where $\alpha$ is chosen in the set $\{0.5, 1, 1.5\}$ and for $k \neq j$, $w(k)$ and $w(j)$ are independent stochastic variables.

Our aim is to regulate the system in the origin (disturbance rejection) while fulfilling the following constraint on the input:

$$-2 \leq u(t) \leq 2 \quad t \in \mathbb{Z}_+$$

As an indication of the achieved level of disturbance rejection we consider the variance of the system state. We compare the disturbance rejection performance of three different receding horizon controllers. The first one, named RHC1 is based on algorithm 3.1, section 3.4. The receding horizon controller RHC2 is based on algorithm 3.2, section 3.5. Finally, RHC3 is the receding horizon controller based on the standard MPC design (see Remark 3.2.5, Section 3.2).

All controllers are designed over a control horizon of length $N = 10$. For RHC1 and RHC2, the number of disturbance samples is set to 10 for the first and the second time instant in the control horizon and 5 for the third time instant in the control horizon. The accuracy parameter $\varepsilon$ is set to $\varepsilon = 0.01$ in both algorithms. Algorithms have been coded in Matlab. Note that actual computation time critically depends on the simulation software that has been used. For the purpose of a comparison, the fact that the average computation time is reduced by a factor 8 with algorithm 3.2 is more interesting. Simulations are performed over an interval of 200 time units. Results are summarized in tables 3.1 and 3.2.

As expected, the variance is the largest when a classical MPC controller (RHC3) is applied. The performance loss of the system controlled by RHC2 and RHC3 are expressed as the relative increase of the variance with respect to the system controlled

Table 3.1: Variance of the state

| $\alpha$ | 0.5 | 1 | 1.5 |
|---|---|---|---|
| $RHC1$ | 0.0233 | 0.1158 | 0.2981 |
| $RHC2$ | 0.0288 | 0.1163 | 0.2995 |
| $RHC3$ | 0.0289 | 0.1293 | 0.3133 |

Table 3.2: Performance loss

| $\alpha$ | 0.5 | 1 | 1.5 |
|---|---|---|---|
| $RHC2$ | 0.5 % | 0.5 % | 0.5 % |
| $RHC3$ | 0.5 % | 13.5 % | 5.1 % |

by RHC1 and given in table 3.2. When the level of the disturbance is small as in the case $\alpha = 0.5$, the performances are comparable. For small $\alpha$, the constraints are not dominating the performance and all controllers yields approximately the same performance. For disturbance levels $\alpha = 1$ and $\alpha = 1.5$, the performance losses of RHC2 are significantly smaller then the losses of the standard MPC controller (RHC3). Note that a further increase of $\alpha$ will result in a behavior of the controlled system dominated by the disturbance, because of the constrained input.

### 3.6.2 A multi-input example

We consider the plant with the model of the form (3.1) with:

$$A = \begin{bmatrix} 0.98676 & 0 \\ 0 & 0.98676 \end{bmatrix} \quad B = \begin{bmatrix} 0.011629 & -0.011444 \\ 0.014331 & -0.014517 \end{bmatrix}$$
$$E = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad C_z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D_z = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \\ 0 & 0 \end{bmatrix}$$

The input to the plant $u$ is assumed to be constrained for all $t \in \mathbb{Z}_+$ within the set $\mathbf{U}$ defined by

$$\mathbf{U} := \{u \in \mathbb{R}^2 : A_u \leq b_u\}$$

with:

$$A_u = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad b_u = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

The plant is the discretized version of the ill-conditioned distillation column model from [133], used also in [83]. The model is linear and therefore a very crude approximation of the distillation column dynamics but its gain is strongly dependent on the input direction which is an essential characteristic of ill-conditioned plants. Our aim is to regulate the system to the origin from a given initial state $x = [x_a \ x_b]^T = [0.1 \ 0.1]^T$.

Here, two model predictive controllers are compared, a controller based on algorithm 3.1 (**stochastic MPC**), section 3.4 and a controller based on the standard MPC algorithm (**standard MPC**).

The standard MPC controller (see section 3.2, remark 3.2.5) is based on the assumption that the disturbance over the control horizon is equal to its expected value i.e. zero, and on the optimization in the open loop. The length of the control horizon is set to $N = 10$. When there is no stochastic disturbance ($w = 0$), the standard MPC controller achieves satisfactory results when applied to the plant (see figure 3.1). From the results shown on figure 3.1, it can be observed how gain of the plant strongly depends on the direction of the input.

Next, we consider the disturbance $w$ which is assumed to be a Gaussian white noise i.e. the disturbance $w$ is a stochastic process with $w(k) \in \mathcal{N}(0, 0.2)$ where $\mathcal{N}(0, 0.2)$ denotes a normally distributed random variables with zero mean and variance 0.2 and for $k \neq j$, $w(k)$ and $w(j)$ are independent stochastic variables.

The stochastic MPC controller is based on the optimization in closed loop and sampling of the stochastic disturbance. The length of the control horizon is set to $N = 10$ as in the standard MPC controller. We use a different number of disturbance samples at different time instants of the horizon. For $s = 0$ we use 15 samples of the disturbance for $s = 1$ 10 samples, for $s = 2$ we use 5 samples. For $2 \leq s \leq 10$ the disturbance is kept fixed and equal as the disturbance in $s = 2$. Thus, the number of samples the disturbance is equal to 750.

For both standard MPC controller and stochastic MPC controller we perform 150 Monte Carlo experiments, each one of them with a different realization of the white noise disturbance $w$. As a result, 150 state trajectories for each controller are obtained. Since stochastic properties of the state is in focus of attention, we compute the empirical mean and the empirical variance of 150 trajectories at each time step $t$. The empirical mean shows central tendency of the state and the empirical variance is a measure of the "dispersion" of the state. It is obvious that a controller that achieves a smaller empirical variance and "keeps" the empirical mean of the state closer to the origin will perform better.

Results are presented on figure 3.2 for the standard MPC controller and on figure 3.3 for the stochastic MPC controller. The empirical mean of trajectories shown on figure 3.2 clearly shows that the standard MPC controller does not stabilize the plant when the stochastic disturbance is present. Simulations with larger control horizons and/or different matrices $C_z$, $D_z$ and $Q$ have the same result. The essential problem with the standard MPC in this example is not to find a suitable values of "tuning parameters"

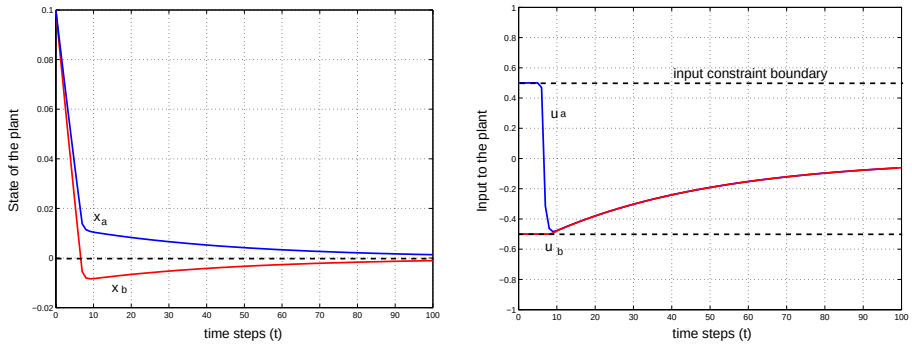


Figure 3.1: Standarad MPC applied to the ill-conditioned plant with disturbance $w = 0$

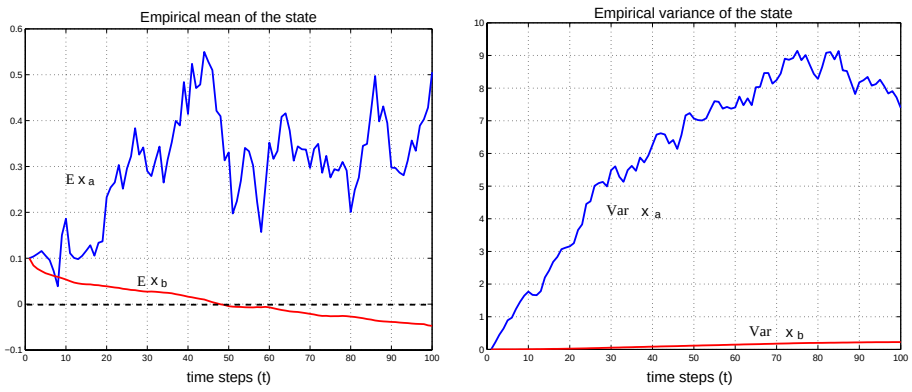


Figure 3.2: Standard MPC applied to the ill-conditioned plant subject to a Gaussian white noise disturbance

(the length of the control horizon, end point penalty, weights in the state and the input) but in inability of the optimization in open loop to capture the true nature of the stochastic disturbance. The empirical variance of state trajectories shown on the figure 3.2 is approximately 45 times larger than the variance of the disturbance which indicates very poor disturbance rejection performance.

The empirical mean and the empirical variance of state trajectories obtained when the plant is controlled by stochastic MPC show a significant improvement in performance, compared with standard MPC. The central tendency of the state converges to zero and the variance of the state is approximately 20 times smaller than the variance of the disturbance, which shows not only that stochastic MPC stabilizes the plant but also has good disturbance rejection performance.

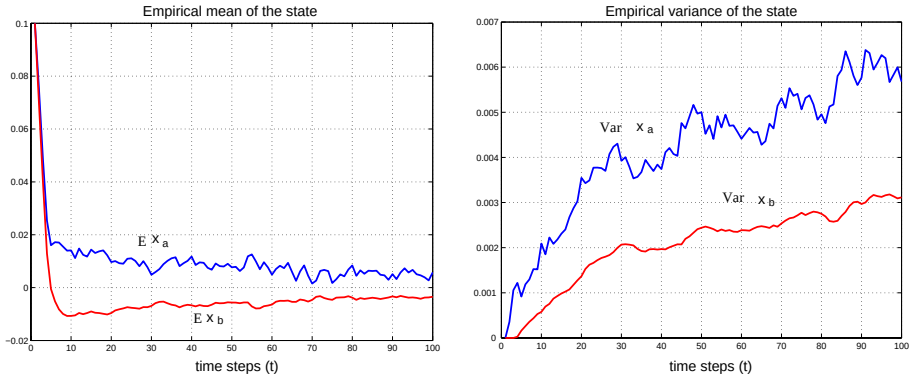


Figure 3.3: Stochastic MPC applied to the ill-conditioned plant subject to a Gaussian white noise disturbance

## 3.7 Conclusion

A constant increase in computing speed and power allows us to explore more elaborate algorithms for predictive control, with the benefit of an increased performance compared to standard schemes.

In this chapter we presented two algorithms for the design of model predictive controllers. Algorithms incorporate the effect of (stochastic) disturbances and constraints on the input. The optimization that arises from the problem formulation is difficult to solve analytically. By exploiting structural properties of the problem, its convexity in the first place, we developed an algorithm by which the optimization problem can be solved with an arbitrary accuracy. The algorithm is computationally demanding but serves as a tool for assessing the achievable performance when one aims to control a plant subject to (stochastic) disturbances and constraint on the input. Also, the algorithm can be used for off-line training of a neural network. The computational burden involved can be reduced in various ways. A reduction in computational complexity is always achieved at the expense of some performance loss, however. One possibility is to choose a class of feedback laws over which the optimization has to be performed. A class of saturated linear feedback laws is proposed in this chapter as a possibility. The second algorithm presented in section 3.5 is based on this assumption. As shown by two numerical examples, the performance loss is marginal when the second algorithm is used but the reduction in computation time is significant. In the examples, both algorithms perform better than a standard MPC controller designed under an assumption that the disturbance over the control horizon is equal to its expected value. The difference in performance is larger in the second example where we use an ill-conditioned plant, notoriously difficult to control.

# 4

# *Model predictive control for stochastic systems with state and input constraints*

In this chapter we consider an optimal control problem for constrained stochastic systems and propose a solution concept that is based on model predictive control technique. A part of this chapter has been presented at the Conference on Decision and Control 2002 [14].

## 4.1 Introduction

In an industrial environment, the ability of a control system to efficiently deal with constraints is of increasing importance. The reason is that the most profitable operation of the industrial plant is often obtained when the process is running at a constraint boundary (see [91]). It is often claimed that the increasing popularity of Model Predictive Control (MPC) in industry stems from its capability to allow operation closer to constraint boundaries, when compared with conventional control techniques. When disturbances are acting on the plant, then it is evident that the better the control system is dealing with disturbances the closer one can operate the plant to the constraint boundaries.

When disturbances acting on the plant have stochastic nature, the classical MPC setting is faced with a difficulty. The difficulty with a stochastic disturbance in MPC is that the predicted behavior and the actual behavior of the plant can differ significantly. The standard, convex optimization in open loop does not take the difference into account between actual and predicted behavior of the plant. As a consequence, questions related to achievable performance can not be addressed properly, while the optimization criterion largely ignores the true characteristics of the plant. Hence the input is chosen on the basis of a criterion which does not reflect the true characteristics of the plant. This may suggest that optimization in closed-loop would be feasible alternative. However, as we will see in this chapter, when a controller is designed in closed loop, constraints make a minimization of the expected value of the cost function over the horizon a very difficult optimization task.

In Chapter 3 we presented a stochastic disturbance rejection scheme for linear, discrete time systems with constrained inputs. The disturbance rejection scheme is based

on model predictive control techniques which utilize a randomized algorithm which minimizes an empirical mean of the cost function. The optimization at each step is a closed loop optimization. Therefore, it takes the effect of disturbances into account. Because we do not impose any a priori parameterization of the feedback laws over the horizon, the algorithm is computationally demanding but it gives a reliable measure of the achievable performance.

In this chapter we extend this research further. The system that we consider is a linear, time invariant, discrete time system with constraints on the input and the state, subject to a stochastic disturbance. We pose our problem as an optimal control problem with a cost function that is not necessarily quadratic and discuss possible approaches to the optimal control problem for the system with stochastic disturbances and constraints on the state and the input (section 4.2). Because of the stochastic nature of the problem, the penalty on the state constraint violation can not be made arbitrary high. We derive a condition on the growth of the state violation cost that has to be satisfied for the optimization problem to be solvable (section 4.3).

In section 4.4 we design a model predictive controller to deal with the optimal control problem of the stochastic system with state and input constraints. The controller is obtained by a receding horizon, convex optimization in closed loop. In section 4.5 we present an algorithm to implement the controller. The algorithm is based on the empirical mean that is computed by use of a number of samples of the disturbance. The accuracy of the algorithm is arbitrary large depending on the length of the control horizon and the number of samples taken to compute the empirical mean. It is shown that the solution obtained by the algorithm converges in probability to the model predictive controller designed in section 4.4.

Finally, in section 4.6 we present an example in which we use the controller designed in this chapter on the problem of steering a cart with a constrained input to the prescribed position, with an additional condition of minimizing the probability of the "overshoot" in the state trajectory.

## 4.2 Optimal control of constrained stochastic systems

We consider a linear, time-invariant plant subject to stochastic disturbances. The plant is described by the following state space model:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Ew(t) \\ z(t) &= C_z x(t) + D_z u(t) \end{aligned} \tag{4.1}$$

where $u$ is the control input with $u(t) \in \mathbf{U} \subseteq \mathbb{R}^m$ and $x$ is the state with $x(t) \in \mathbb{R}^n$. The set $\mathbf{U}$ is a not necessarily bounded, closed, convex set which contains an open neighborhood of the origin. The second equation describes the controlled output $z$ with $z(t) \in \mathbb{R}^p$. Finally, the disturbance $w$ is a stochastic process with $w(t) \in$

$\mathcal{N}(0, Q_w)$ where $\boldsymbol{\mathcal{N}}$ denotes a family of normally distributed random variables with zero mean and covariance matrix $Q_w \in \mathbb{R}^{l \times l}$. Moreover, for $k \neq j$, $w(k)$ and $w(j)$ are independent. Thus, the disturbance $w$ is a Gaussian white noise.

Matrices $A$, $B$, $E$, $C_z$ and $D_z$ are matrices of suitable dimensions with real elements. It is assumed that the matrix pair $(A, B)$ is stabilizable and the matrix pair $(A, C_z)$ observable. We assume that the state of the plant is measured.

The system (4.1) is controlled by a *static feedback controller* i.e. at each $t$, the input $u(t)$ is a function of the state $x(t)$. The class of controllers $\Psi$ that we consider is the set of continuous maps $\varphi : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbf{U}$ that map the origin of the state space into the zero input

$$\varphi(0, t) = 0 \quad \text{for all} \quad t \in \mathbb{Z}_+.$$

Thus, we have

$$u(t) = \varphi(x(t), t) \tag{4.2}$$

for some $\varphi \in \Psi$. Starting at time $t = 0$, the state $x$ and the output $z$ are stochastic processes generated by (4.1) with the input (4.2).

We consider a linear, time invariant system that is subject to the stochastic disturbance, with state constraints and a constrained input. It is well known, that a constrained input limits our ability to control the linear plant. To approach this in more formal way, set $w = 0$ in (4.1) and consider the system:

$$x(t+1) = Ax(t) + Bu(t) \qquad u(t) \in \mathbf{U}. \tag{4.3}$$

Suppose that at time $t = 0$ system (4.3) has an initial state $x(0) = x_0$. The initial condition $x_0 \in \mathbb{R}^n$ is a *null controllable point* if the condition in the definition 3.2.1 is satisfied for the initial state $x_0$. All null controllable points define a set in the state space which is known as the *recoverable set*, here denoted as $\underline{\mathbf{X}}$. In general, the recoverable set is a subset of the state space. If $\mathbf{U}$ is bounded then the recoverable set contains all points in the state space if and only if the matrix pair $(A, B)$ is stabilizable and all eigenvalues of the system matrix $A$ lie on or inside the unit circle in which case we say that the system (4.3) is *globally asymptotically stabilizable* (see [125] for details when $\mathbf{U}$ is bounded). Thus, the following assumption is natural when one deals with the stabilization of a linear system, subject to input constraints and unbounded disturbances.

**Assumption 4.2.1** The system (4.1) is globally asymptotically stabilizable. As a consequence $\underline{\mathbf{X}} = \mathbb{R}^n$.

Next, suppose that constraints on the state $x$ define a convex, closed set $\mathbf{X} \subseteq \mathbb{R}^n$ that contains the origin in its interior. The output $z$ is used to measure performance. Our objective is to control plant (4.1) from an initial state to the origin in such a way that the size of the controlled output $z$ is as small as possible while $x(t) \in \mathbf{X}$ and $u(t) \in \mathbf{U}$ for all $t \geq 0$. Our performance measure, usually called the cost, is a convex

function of the output $z$. A number of efficient algorithms exist to minimize a convex function. However, the dynamic structure of the problem makes this optimization far from trivial. The controlled output $z$ is a stochastic process because it depends on the stochastic disturbance $w$. Thus, we consider the following performance measure for system (4.1):

$$P(x_0, \varphi) := \lim_{T\to\infty} \mathbb{E}\, \frac{1}{T} \sum_{t=0}^{T} g\,(z(t)) \qquad \varphi \in \Psi, x_0 \in \mathbf{X}. \tag{4.4}$$

where:

$$\begin{aligned} x(t+1) &= \ Ax(t) + \ B\varphi(x(t), t) + Ew(t) \qquad x(0) = x_0 \\ z(t) \quad &= C_z x(t) + D_z \varphi(x(t), t) \end{aligned}$$

and where $\mathbb{E}$ denotes expectation. The function $g : \mathbb{R}^p \to \mathbb{R}_+$ is a strictly convex function with $g(0) = 0$. Note that $z$ is a linear function in $x$ and $u$ and therefore the composite function $g$ is strictly convex in $x$ and $u$.

The main optimizaton problem is stated in the following problem formulation.

**Problem 4.2.2** Suppose that at time $t = 0$ system (4.1) has an initial condition $x(0) = x_0$, $x_0 \in \mathbf{X} \cap \underline{\mathbf{X}}$. Under assumption 4.2.1, find an optimal controller $\varphi^* \in \Psi$ such that:

$$x(t) \in \mathbf{X}, \qquad u(t) \in \mathbf{U} \tag{4.5}$$

for all $t \in \mathbb{Z}_+$ and

$$P(x_0, \varphi^*) \leq P(x_0, \varphi)$$

for all other controllers $\varphi \in \Psi$ which guarantee (4.5). In addition, determine the optimal cost:

$$P^*(x_0) := \inf_{\varphi\in\Psi} \left\{ P(x_0, \varphi) \;\middle|\; (4.5) \text{ holds for all } t \in \mathbb{Z}_+ \right\}.$$

Problem 4.2.2 is a stochastic, optimal control problem with constraints on the state and the input. No constraint violation is allowed and the problem resembles what is known as the *hard constraint* approach. The main difficulty with the problem 4.2.2 is that the set of admissible initial conditions

$$\left\{ x_0 \in \mathbf{X} \;\middle|\; P^*(x_0) < \infty \right\} \tag{4.6}$$

is almost always empty for a Gaussian white noise disturbance $w$. An empty set of admissible initial conditions implies that problem 4.2.2 is unsolvable. The reason is that the Gaussian white noise is an unbounded disturbance and it is always possible to find a realization of the disturbance $w$ that violates the conditions (4.5), for any $x_0 \in \mathbf{X}$ and $\varphi \in \Psi$. In the case that the disturbance is bounded, the set of admissible initial conditions (4.6) can be very small, which is too restrictive in many practical

applications. This is an inherent difficulty with the hard constraint approach (problem 4.2.2). The only way to overcome this difficulty is to allow certain performance degradation i.e. to allow constraint violation. The performance degradation should be kept as small as possible.

In this chapter, we propose an approach for dealing with constraints on the state of stochastic systems. An optimal controller should control the plant optimally with respect to the performance measure (4.4) while keeping the state in the constraint set $\mathbf{X}$ "as much as possible". When the state is in the set $\mathbf{X}$ the performance measure (4.4) determines the performance. When there is a probability of a constraint violation, the performance of the system is determined by an additional cost that will penalize the constraint violation. In this way, we have two different objectives for control: minimizing the performance measure (4.4) and minimizing the probability of constraint violation.

In the paper [87], the probability of the constraint violation is explicitly incorporated in the problem formulation and a model predictive algorithm is proposed to deal with the plant with constraints on the state and stochastic disturbances. The optimization over the control horizon is performed in open loop. The variance of the state is significantly larger when control is in open loop, when compared to the variance of the state when control is in closed loop.

In our setting, the task of minimizing the constraint violation is accomplished by an additional cost that will penalize constraint violation.

**Definition 4.2.3** The constraint violation cost is a convex function $h : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ with $h(x) = 0$ for all $x \in \mathbf{X}$.

The state $x$ depends on the stochastic disturbance $w$ and is therefore stochastic itself. We consider the expected value of the constraint violation cost. The performance measure (4.4) and the expected value of the constraint violation cost are added in the cost function to reflect both requirements:

$$\bar{J}(x_0, \varphi) := \lim_{T\to\infty} \mathbb{E}\frac{1}{T}\sum_{t=0}^{T}\Big\{g\big(C_z x(t) + D_z\varphi(x(t), t)\big) + h\big(x(t)\big)\Big\}. \tag{4.7}$$

Consider the following optimization problem:

**Problem 4.2.4** Given an initial condition $x_0 \in \mathbf{X}$, find an optimal controller $\tilde{\varphi} \in \Psi$ such that

$$\bar{J}(x_0, \tilde{\varphi}) \leq \bar{J}(x_0, \varphi)$$

for all $\varphi \in \Psi$. In addition, determine the *optimal cost* given by:

$$\bar{V}(x_0) := \inf_{\varphi\in\Psi} \bar{J}(x_0, \varphi). \tag{4.8}$$

The optimization problem 4.2.4 is an optimal control problem of a linear discrete time system subject to stochastic disturbances and only constraints on the input. The constraints on the state have been incorporated implicitly by the modified cost function.

The constraint violation cost introduces an additional degree of freedom in the design of an optimal controller for the system (4.1). It determines the strategy in dealing with the state constraints. A choice:

$$h(x) = 0, \qquad x \in \mathbb{R}^n$$

would imply an optimal control problem without constraints on the state. Setting $h$ to be:

$$h(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbf{X} \\ \infty & \text{if} \quad x \notin \mathbf{X} \end{cases} \tag{4.9}$$

makes problem 4.2.4 identical to problem 4.2.2 i.e. the hard constraints approach. In between these two extreme cases there is a large number of choices to tailor the cost (4.7) for the application at hand. Note however, that any choice that will make the constraint violation cost infinite in some point even for large $x$ will make the set of admissible initial conditions (4.6) almost always empty, when disturbances are not bounded. The following assumption is therefore necessary.

**Assumption 4.2.5** The constraint violation cost $h$ is from the class of finite valued convex functions i.e. $h : \mathbb{R}^n \to \mathbb{R}_+$ is convex with $h(x) = 0$ for all $x \in \mathbf{X}$, instead of $h : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$.

Assumption 4.2.5 is not very restrictive, simply because the growth of the constraint violation cost $h$ can be made almost arbitrary large with assumption 4.2.5 satisfied. For example, consider the constraint violation cost that satisfies assumption 4.2.5 and has an exponential growth away from the boundary of $\mathbf{X}$

$$h(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbf{X} \\ e^{\gamma\, \mathrm{d}(x,\mathbf{X})} - 1 & \text{if} \quad x \notin \mathbf{X} \end{cases} \tag{4.10}$$

where $\mathrm{d}(x, \mathbf{X})$ denotes the distance between $x$ and the boundary of set $\mathbf{X}$. With $\gamma$ large enough (4.10) can be made arbitrary large. Having a large $\gamma$ will mean a tighter control with respect to the state constraints and is therefore an advantage. In order to compute cost (4.7) for a given initial condition and a given controller we need to compute the expectation of the function of the stochastic state. In general, when $w$ is unbounded expectation does not need not to be finite for all initial conditions in $\mathbf{X}$ and an arbitrary controller $\varphi \in \Psi$ even if the constraint violation cost $h$ is as in assumption 4.2.5. For example, if the constraint violation cost $h$ is as in (4.10) then an question is for which $\gamma$ the optimization problem 4.2.4 still yields a finite cost for all initial conditions in $\mathbf{X}$ and all $\varphi \in \Psi$.

The answer to the question above can be deduced from the conditions that are presented in the following section. They relate the growth of the constraint violation cost $h$ and the decay of the probability density function of the disturbance. That is, we wish to investigate the relation between the growth of the constraint violation cost $h$ and the probability density function of the disturbance that will yield *finite* optimal cost (4.8) for all $x_0 \in \mathbf{X}$.

## 4.3 Solvability conditions

Before presenting results in this section, we rewrite the cost (4.7) in more compact form as

$$\bar{J}(x_0, \varphi) := \lim_{T\to\infty} \mathbb{E}\, \frac{1}{T} \sum_{t=0}^{T} j\big(x(t), \varphi(x(t), t)\big) \tag{4.11}$$

where $x$ is the state that is generated recursively by (4.1) with the disturbance $w$ and the input $u$ given by (4.2) starting at $t = 0$ with an initial condition $x_0 \in \mathbf{X}$. The function $j$ is defined as:

$$j\big(x, u\big) := g\big(C_z x + D_z u\big) + h(x) \quad x \in \mathbb{R}^n \quad u \in \mathbf{U}. \tag{4.12}$$

As already mentioned in section 4.2, the state violation cost $h$ determines the strategy in dealing with the state constraints. The constraint violation cost is defined in definition 4.2.3 as a convex function of the state. We choose the constraint violation cost to be an exponential function of the state. A possibility is for example (4.10). A precise choice of the constraint violation cost has to be done based on the application at hand. Note, however, that an exponential constraint violation cost (such as (4.10)) makes the function $j$ a function of exponential growth in $x$.

Next, we define a class of functions from which we choose the function $j$. This class is a class of functions that have a so called "Polynomial - Exponential Growth".

**Definition 4.3.1** The class of functions $\Theta(R)$ with $R \in \mathbb{R}^{n\times n}$ a positive semidefinite, symmetric matrix is the class of all functions $\theta : \mathbb{R}^n \to \mathbb{R}$ for which there exist nonzero polynomials $q$ and $p$ such that:

$$q(x)e^{\|x\|_R^2} \le \theta(x) \le p(x)e^{\|x\|_R^2}$$

for all $x \in \mathbb{R}^n$. Here, $\|x\|_R^2 := \langle x, Rx\rangle$.

The state of the system (4.1) is a stochastic process. Stochastic properties of the state depend not only on the structure of the system (4.1) but also on the feedback from $\Psi$ that is applied to the plant. For the problem 4.2.4 to be solvable, it is necessary that the cost (4.11) is finite for at least one feedback from $\Psi$. The cost will be finite if the

expectation of the function $j(x(t), \varphi(x(t), t)) \in \Theta(R)$, where $\varphi \in \Psi$ and $x$ is the state of the system (4.1) is finite for all $t = 0, 1, 2 \cdots$. Finiteness of this expectation depends on the relationship between matrix $R$ that defines the exponential growth of the cost and the covariance matrix $Q_w$.

We would like to characterize feedbacks that achieve a finite cost (4.11) with $j(\cdot, u) \in \Theta(R)$ where $x$ is the state of the system (4.1). To this aim we define an auxiliary system as follows

$$x^a(t+1) = Ax^a(t) + B\varphi(x^a(t), t) + E\xi(t) \qquad \xi(t) \in \mathbb{R}^l \tag{4.13}$$

with an initial condition $x^a(0) = x_0$ and $\varphi \in \Psi$. Note that the structure of the system (4.13) is the same as the structure of the plant (4.1). The difference is that the disturbance $w$ is a stochastic process and the disturbance $\xi$ is assumed to be a deterministic signal. As a consequence the state $x$ is a stochastic process and the state $x^a$ is a deterministic signal.

In general, feedbacks that achieve a finite cost (4.11) constitute a subset in $\Psi$. We denote this subset as $\Psi_R$ and, for an easy reference, we give the definition first.

**Definition 4.3.2** The set $\Psi_R$ is the set of all feedbacks $\varphi \in \Psi$ such that for all $t = 1, 2, 3 \cdots$

$$\|x^a(t)\|_R^2 - \frac{1}{2}\|\xi^{t-1}\|_\star^2 < 0 \quad \text{for all} \quad \xi^{t-1} \in \mathbb{R}^l \times \mathbb{R}^{t-1} \setminus \Xi \tag{4.14}$$

where $\Xi$ is a nonempty, bounded subset of $\mathbb{R}^\ell \times \mathbb{R}^{t-1}$ that contains zero, $x^a$ and $\xi$ satisfy (4.13) with an initial condition $x^a(0) \in \mathbf{X}$ and where

$$\|\xi^{t-1}\|_\star^2 := \sum_{i=0}^{t-1} \|\xi(i)\|_{Q_w^{-1}}^2 \tag{4.15}$$

is the square of the weighted $\ell_2$ norm of a vector valued real sequence $\xi^{t-1} := (\xi(i))_{i=0}^{t-1}$ with $\xi(i) \in \mathbb{R}^l$ and $\|\cdot\|_{Q_w^{-1}}^2 := \langle \cdot, Q_w^{-1} \cdot \rangle$.

The following result shows that the state generated by (4.1) and a feedback from $\Psi_R$ achieves a finite expectation for functions of $x$ that are in the class $\Theta(R)$.

**Lemma 4.3.3** *Consider a state $x$ generated recursively by the system* (4.1) *where $w \in (0, Q_w)$ with an initial condition $x(0) \in \mathbf{X}$ and a feedback $\varphi \in \Psi_R$.*

*Next, consider a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ such that for every fixed $u \in \mathbb{R}^m$*

$$f(\cdot, u) \in \Theta(R).$$

*Then,*

$$\mathbb{E}\ f(x(t), \varphi(x(t), t)) < \infty$$

*for all $t = 1, 2, 3 \cdots$.*

**Proof:** Assume that the system (4.1) with the initial condition $x(0)$ is controlled by $u(t) = \varphi(x(t), t)$ with $\varphi \in \Psi_R$. The state of the system (4.1) at $t$, $x(t)$, is given by the recursion defined by (4.1) so it is a function of the initial condition $x(0)$, the disturbance up to $t-1$, denoted as $w^{t-1} := (w(k))_{k=0}^{t-1}$, $w(k) \in \mathcal{N}(0, Q_w)$, for $k \neq j$, $w(k)$ and $w(j)$ are independent stochastic variables, and the input $(\varphi(x(k), k))_{k=0}^{t-1}$.

The disturbance $w^{t-1}$ is a finite sequence of independent, identically distributed random variables. Therefore, the probability density function of $w^{t-1}$ is simply:

$$f_{w^{t-1}}\left(\xi^{t-1}\right) = \prod_{i=0}^{t-1} f_w(\xi(i)) \qquad \xi^{t-1} := (\xi(i))_{i=0}^{t-1}, \quad \xi(i) \in \mathbb{R}^l \tag{4.16}$$

where $f_w$ is the normal probability density function of the variable $w \in \mathcal{N}(0, Q_w)$, $Q_w \in \mathbb{R}^{l \times l}$

$$f_w(\xi) = \frac{1}{(2\pi)^{\frac{\ell}{2}} \sqrt{\det(Q_w)}} e^{-\frac{1}{2}\alpha(\xi)} \tag{4.17}$$

with $\xi \in \mathbb{R}^l$ and

$$\alpha(\xi) := \|\xi\|^2_{Q_w^{-1}}.$$

The probability density function of $w^{t-1}$ can be computed according to (4.16) as

$$f_{w^{t-1}}(\xi^{t-1}) = \frac{1}{(2\pi)^{\frac{\ell(t-1)}{2}} \left(\det(Q_w)\right)^{\frac{\ell}{2}}} e^{-\frac{1}{2}\|\xi^{t-1}\|^2_\star} \tag{4.18}$$

where

$$\|\xi^{t-1}\|^2_\star := \sum_{i=0}^{t-1} \|\xi(i)\|^2_{Q_w^{-1}}$$

is the square of the weighted $\ell_2$ norm of a vector valued real sequence $\xi^{t-1}$. Next, consider the expectation

$$\mathbb{E}\, f\left(x(t), \varphi(x(t), t)\right) = \int_{\mathbb{R}^\ell \times \mathbb{R}^{t-1}} f\left(x^a(t), \varphi(x^a(t), t)\right) f_{w^{t-1}}(\xi^{t-1})\, d\xi^{t-1} \tag{4.19}$$

where $x^a(t)$ is the state at $t$ given by recursion (4.13) with the initial condition $x^a(0) = x_0$.

Expectation (4.19) can be rewritten as

$$\begin{aligned}\mathbb{E}\, f\left(x(t), \varphi(x(t), t)\right) = &\int_{\Xi} f\left(x^a(t), \varphi(x^a(t), t)\right) f_{w^{t-1}}(\xi^{t-1})\, d\xi^{t-1} \\ &+ \int_{\mathbb{R}^\ell \times \mathbb{R}^{t-1} \setminus \Xi} f\left(x^a(t), \varphi(x^a(t), t)\right) f_{w^{t-1}}(\xi^{t-1})\, d\xi^{t-1}\end{aligned} \tag{4.20}$$

where $\Xi$ is a nonempty, bounded subset of $\mathbb{R}^\ell \times \mathbb{R}^{t-1}$ that contains zero.

Because $\Xi$ is bounded, the first summand in (4.20) is always finite. We use (4.18) and the assumption that $f \in \Theta(R)$ from the lemma to find an upper bound for the second summand in (4.20).

$$\int_{\mathbb{R}^\ell \times \mathbb{R}^{t-1} \setminus \Xi} f\big(x^a(t), \varphi(x^a(t), t)\big) f_{w^{t-1}}(\xi^{t-1})\, d\xi^{t-1} \leq$$
$$\frac{1}{(2\pi)^{\frac{\ell(t-1)}{2}} \big(\det(Q_w)\big)^{\frac{\ell}{2}}} \int_{\mathbb{R}^\ell \times \mathbb{R}^{t-1} \setminus \Xi} p\big(x^a(t), \varphi(x^a(t), t)\big)\, e^{\|x^a(t)\|_R^2}\, e^{-\frac{1}{2}\|\xi^{t-1}\|_\star^2}\, d\xi^{t-1}. \tag{4.21}$$

The upper bound in (4.21) is finite since the feedback $\varphi$ achieves the condition (4.14) i.e.

$$\|x^a(t)\|_R^2 - \frac{1}{2}\|\xi^{t-1}\|_\star^2 < 0 \tag{4.22}$$

for all $\xi^{t-1} \in \mathbb{R}^\ell \times \mathbb{R}^{t-1} \setminus \Xi$. Since the feedback $\varphi$ achieves (4.22) for all $t$ the expectation (4.19) is finite for all $t$ so

$$\mathbb{E}\, f\big(x(t), \varphi(x(t), t)\big) < \infty \tag{4.23}$$

for all $t = 1, 2, 3 \cdots$. ■

In the following theorem, we show that the optimization problem 4.2.4 is solvable if the set $\Psi_R$ is a nonempty set i.e. if exists at least one feedback that achieves condition (4.14).

**Theorem 4.3.4** *Consider the optimization problem 4.2.4 for the system* (4.1) *with an initial condition* $x_0 \in \mathbf{X}$. *In addition to assumptions 4.2.1 and 4.2.5 assume that for every fixed* $u \in \mathbb{R}^m$

$$j(\cdot, u) \in \Theta(R). \tag{4.24}$$

*Then,* $\bar{V}(x_0) < \infty$ *if the set* $\Psi_R$ *for the system* (4.1) *with the initial condition* $x(0) = x_0$ *is a nonempty set.*

**Proof:** Assume that the set $\Psi_R$ is a nonempty set and that a feedback $\overline{\varphi} \in \Psi_R$ for the system (4.1) with the initial condition $x_0$ exists and it is given. Consider the cost

$$\bar{J}(x_0, \overline{\varphi}) = \lim_{T \to \infty} \mathbb{E}\, \frac{1}{T} \sum_{t=0}^{T} j\big(x(t), \overline{\varphi}(x(t), t)\big) \tag{4.25}$$

where $x(t)$ is the state of the system (4.1) at time $t$, so it is a function of the initial condition $x(0) = x_0$, the disturbance up to $t-1$ and the input $u(k) = (\overline{\varphi}(x(k), k))$.

Since $j \in \Theta(R)$ and $\overline{\varphi} \in \Psi_R$, lemma 4.3.3 can be applied so

$$\mathbb{E} j\big(x(t), \overline{\varphi}(x(t), t)\big) < \infty$$

for all $t$. This implies that (4.25) is finite. Finally

$$\bar{V}(x_0) \leq \bar{J}(x_0, \overline{\varphi}(x(t), t)\big) < \infty$$

which concludes the proof. ■

Results presented so far in this section, show that the optimization problem 4.2.4 is solvable if there exists a feedback that satisfies condition (4.14). The results are derived for the general case in which the set **U** is a closed, convex and not necessarily bounded set which contains an open neighborhood of the origin. In this case, a set of feedbacks that achieve (4.14) along the state trajectory of the system (4.1) is a subset of the set $\Psi$. The condition (4.14) is not easy to verify, however. At each $t$, it relates a vector norm of the state at $t$, weighted with the matrix $R$ and the $\ell_2$ signal norm of the disturbance, weighted with the covariance matrix $Q_w$.

From the application point of view, a very important case is the case in which **U** is a bounded set. Fortunately, as it will be shown in the sequel, in this case the solvability condition is equivalent to the algebraic condition

$$\big(A^t E\big)^T R\big(A^t E\big) - \frac{1}{2} Q_w^{-1} < 0$$

that has to be satisfied for all $t = 1, 2, 3 \cdots$. This condition is easy to verify. Another important point is that when this algebraic condition is satisfied every feedback in the set $\Psi$ achieves a finite cost (4.11).

**Theorem 4.3.5** *Assume that $B$ is an injective matrix. Then, $\Psi_R \neq \emptyset$ if and only if* **U** *is a bounded set and*

$$\big(A^t E\big)^T R\big(A^t E\big) - \frac{1}{2} Q_w^{-1} < 0 \tag{4.26}$$

*for all $t = 0, 1, 2 \cdots$.*

**Proof:** Assume that **U** is bounded and that condition (4.26) is satisfied for all $t$. To prove the theorem we need to establish condition (4.14) which is equivalent to

$$\begin{aligned}
&\Big\| A^t x_0 + \sum_{i=0}^{t-1} A^i B\varphi(x_i, i) + \sum_{i=0}^{t-1} A^i E\xi(i) \Big\|_R^2 - \frac{1}{2} \sum_{i=0}^{t-1} \|\xi(i)\|_{Q_w^{-1}}^2 \\
&\leq \Big\{ \big\| A^t x_0 \big\|_R + \Big\| \sum_{i=0}^{t-1} A^i B\varphi(x_i, i) \Big\|_R + \Big\| \sum_{i=0}^{t-1} A^i E\xi(i) \Big\|_R \Big\}^2 - \frac{1}{2} \sum_{i=0}^{t-1} \|\xi(i)\|_{Q_w^{-1}}^2 < 0.
\end{aligned}$$

The second term in inequality above can be expanded as

$$2\left\|A^t x_0\right\|_R \left\|\sum_{i=0}^{t-1} A^i B\varphi(x_i,i)\right\|_R + 2\left\|A^t x_0\right\|_R \left\|\sum_{i=0}^{t-1} A^i E\xi(i)\right\|_R +$$
$$2\left\|\sum_{i=0}^{t-1} A^i B\varphi(x_i,i)\right\|_R \left\|\sum_{i=0}^{t-1} A^i E\xi(i)\right\|_R + \left\|A^t x_0\right\|_R^2 + \left\|\sum_{i=0}^{t-1} A^i B\varphi(x_i,i)\right\|_R^2 +$$
$$\left\|\sum_{i=0}^{t-1} A^i E\xi(i)\right\|_R^2 - \frac{1}{2}\sum_{i=0}^{t-1} \|\xi(i)\|_{Q_w^{-1}}^2 < 0. \quad (4.27)$$

Because $\mathbf{U}$ is bounded, for every $x_0 \in \mathbf{X}$ there exists $a_t > 0$ and $b_t > 0$ such that

$$\left\|A^t x_0\right\|_R \leq a_t$$

and

$$\left\|\sum_{i=0}^{t-1} A^i B\varphi(x_i,i)\right\|_R \leq b_t$$

for every $\varphi \in \Psi$.

Therefore, it is possible to upper bound left hand side in (4.27) and it is sufficient to consider

$$\left\|\sum_{i=0}^{t-1} A^i E\xi(i)\right\|_R^2 - \frac{1}{2}\sum_{i=0}^{t-1} \|\xi(i)\|_{Q_w^{-1}}^2 \leq -\left(c_t + d_t\left\|\sum_{i=0}^{t-1} A^i E\xi(i)\right\|_R\right) \quad (4.28)$$

where

$$c_t = (a_t + b_t)^2 \quad \text{and} \quad d_t = 2(a_t + b_t).$$

Observe that the left hand side of (4.28) is a quadratic form for all $i$ and that the right hand side grows at most linearly in $\xi(i)$ for all $i$. Thus, it remains to show that left hand side of (4.28) is a negative definite quadratic form under condition (4.26) because then it is possible to find $\Xi \in \mathbb{R}^\ell \times \mathbb{R}^{t-1}$ such that (4.28) is satisfied for all $\xi^{t-1} \in \mathbb{R}^\ell \times \mathbb{R}^{t-1} \setminus \Xi$. Therefore, consider

$$\left\|\sum_{i=0}^{t-1} A^i E\xi(i)\right\|_R^2 - \frac{1}{2}\sum_{i=0}^{t-1} \|\xi(i)\|_{Q_w^{-1}}^2 < 0.$$

The inequality above can be rewritten as

$$\begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{t-1} \end{pmatrix}^T \left[ P^T R P - \begin{pmatrix} \frac{1}{2}Q_w^{-1} & 0 & \cdots & 0 \\ 0 & \frac{1}{2}Q_w^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \frac{1}{2}Q_w^{-1} \end{pmatrix} \right] \begin{pmatrix} \xi_0 \\ \xi_1 \\ \vdots \\ \xi_{t-1} \end{pmatrix} < 0 \quad (4.29)$$

where

$$P := \begin{bmatrix} E & AE & \cdots & A^{t-1}E \end{bmatrix}.$$

We introduce

$$\tilde{\xi}_i := \frac{1}{\sqrt{2}} Q_w^{-\frac{1}{2}} \, \xi(i)$$

for $i = 0, \cdots, t-1$. Next, we rewrite (4.29) as

$$\begin{pmatrix} \tilde{\xi}_0 \\ \tilde{\xi}_1 \\ \vdots \\ \tilde{\xi}_{t-1} \end{pmatrix}^T \left[ \tilde{P}^T R \tilde{P} - I \right] \begin{pmatrix} \tilde{\xi}_0 \\ \tilde{\xi}_1 \\ \vdots \\ \tilde{\xi}_{t-1} \end{pmatrix} < 0$$

where

$$\tilde{P} := 2 P Q_w^{\frac{1}{2}}$$

and $I$ is an identity matrix. So, (4.29) is satisfied if $\tilde{P}^T R \tilde{P} - I$ is negative definite i.e.

$$\tilde{P}^T R \tilde{P} < I$$

which is equivalent to

$$R^{\frac{1}{2}} \tilde{P} \tilde{P}^T R^{\frac{1}{2}} < I. \tag{4.30}$$

Next, we expand (4.30) as

$$R^{\frac{1}{2}} Q_w^{\frac{1}{2}} \Big[ \sum_{i=0}^{t-1} A^i E (A^i E)^T \Big] Q_w^{\frac{1}{2}} R^{\frac{1}{2}} < I. \tag{4.31}$$

To show that (4.31) is satisfied consider the controlability grammian

$$M_\xi := \sum_{i=0}^{\infty} A^i E (A^i E)^T.$$

Because of assumption 4.2.1

$$M_\xi = I$$

and since

$$\sum_{i=0}^{t-1} A^i E (A^i E)^T < M_\xi$$

it can be concluded that (4.31) is satisfied for all $t$. This completes the necessity part of the proof.

For the sufficiency part assume that $\Psi_R = \Psi$ and observe that (4.27) grows quadratically in $A^i B \varphi(x_i, i)$. Since $B$ is injective, it is always possible to find a $\varphi \in \Psi$ such

that condition (4.14) is violated if $\mathbf{U}$ is unbounded. Thus, (4.14) can be satisfied for all $\varphi \in \Psi$ only if $\mathbf{U}$ is bounded. Next, assume

$$\left(A^t E\right)^T R\left(A^t E\right) - \frac{1}{2} Q_w^{-1} \geq 0$$

for some $t = 0, 1, 2 \cdots$. With this, it is not possible to find $\Xi$ such that (4.27) is satisfied for all $\xi^{t-1} \in \mathbb{R}^{\ell} \times \mathbb{R}^{t-1} \backslash \Xi$. Thus, (4.14) can be satisfied only if

$$\left(A^t E\right)^T R\left(A^t E\right) - \frac{1}{2} Q_w^{-1} < 0$$

■

Note that assuming $B$ is an injective matrix is actually a very weak assumption in the sense that it is naturally satisfied in any practical application.

The result presented in theorem 4.3.5 gives conditions under which is possible to search for the solution of the optimization problem 4.2.4 in the general set of continuous feedbacks $\Psi$, when $\mathbf{U}$ is bounded. This is a considerable simplification when compared to the general case in which the solution has to be searched in a subset of the set $\Psi$. When one deals with the general case in which $\mathbf{U}$ is not necessarily bounded, the essential problem is to characterize the set $\Psi_R$ or a subset of $\Psi_R$ which contains a solution of the optimization problem 4.2.4. Problems of this kind, related to the optimization problem 4.2.4, are topics for further research.

Even when the set $\mathbf{U}$ is bounded, solving the optimization problem 4.2.4 is difficult. A way to tackle the optimization problem 4.2.4 is to design a model predictive controller. The resulting controller will not be the optimal one for the optimization problem 4.2.4 but the approximation with the predictive controller can be arbitrary good, depending on the size of the control horizon. As pointed out in [11–13] the standard, convex optimization in open loop that is prevailing in the MPC literature can not be applied when stochastic disturbances are considered, because it is not possible to control the variance of the state over the control horizon without the control in closed loop. Algorithms presented in the papers [11–13] are based on the computation of the empirical mean. With suitable modifications and extensions the same approach can be used to develop an algorithm that will solve the optimization problem 4.2.4.

## 4.4 Stochastic model predictive controller

The design of a predictive controller is based on an optimization problem that is solved at each time instant $t$, $(t \in \mathbb{Z}_+)$ over an interval $I_t := \{t + k | k \in T\}$, $T := [0, N]$ where $N > 0$. The interval $I_t$ is a fixed length interval which recedes with the time $t$. The model of the plant (4.1) as well as the cost function and optimization problem to be defined later, are time-invariant. Therefore, variables involved in the design of

a predictive controller can be defined as functions of $k \in T$ rather than the functions of the current time, without loss of generality. We will refer to the interval $T$ as the control horizon with the length $N$.

With $x_N : \{0, \cdots, N+1\} \to \mathbb{R}^n$ we denote the progression of the *predicted state* over the control horizon. Over the control horizon, it is assumed that the model (4.1) is subject to the disturbance $\nu : T \to \mathbf{W}$. The input of the plant over the control horizon is optimized in closed loop i.e. the input $u_N(k)$ is a function of the predicted state $x_N(k)$. Formally, we define the set of feedback control laws $\Pi$ where $\pi \in \Pi$ is a vector $(\pi_k)_{k=0}^N$ such that for any $k \in T$, the map $\pi_k : \mathbb{R}^n \to \mathbf{U}$ is continuous. The progression of the predicted state $(x_N(k))_{k=0}^{N+1}$ is generated by (4.1), with an initial condition $x_N(0) := x(t)$, with the disturbance $\nu(k) := w(t+k)$ and the input $u_N(k) = \pi_k(x_N(k))$. Note that the predicted state $x_N$ is a function of the measured state $x(t)$, the feedback control laws in the vector $\pi$ and the disturbance $\nu$ and is therefore stochastic.

The cost that we consider is defined by:

$$J(x, \pi) := \mathbb{E}\Big\{ \sum_{k \in T} \big\{ g(z_N(k)) + h(x_N(k)) \big\} + \|x_N(N+1)\|_Q^2 \Big\} \tag{4.32}$$

subject to $x_N(0) = x$. The expression $\|x\|_Q^2 := \langle x, Qx \rangle$ is called an end point penalty with $Q \in \mathbb{R}^{n \times n}$ a positive definite, symmetric matrix.

The optimization problem to be solved is stated next:

**Problem 4.4.1** Find a vector of optimal feedback mappings $\pi^* \in \Pi$ such that

$$J(x, \pi^*) \leq J(x, \pi)$$

for all $\pi \in \Pi$ and for all $x \in \mathbb{R}^n$. In addition, determine the *optimal cost* given by:

$$V(x) := \inf_{\pi} J(x, \pi). \tag{4.33}$$

If the vector of optimal feedback mappings $\pi^*$ exists, then $V(x) = J(x, \pi^*)$ and only the first element of $\pi^*$ is significant in the receding horizon implementation. It determines the current input for the plant as a function of the current measurement. In the next time instant, the control horizon is shifted forward and optimization problem 4.4.1 is solved based on a new state measurement. Therefore, the receding horizon controller in the setting described above, is given by:

$$u(t) = \pi_0^*(x(t)) \quad \forall\ t \in \mathbb{Z}_+ \tag{4.34}$$

where $u(t)$ is the input to the plant at time $t$.

The optimization problem 4.4.1 is a finite horizon optimization problem. A rigorous analysis of the relationship between the optimization problem 4.4.1 and the optimization problem 4.2.4, which is an infinite horizon optimization problem, is an interesting

but difficult topic for further research. Intuitively, it can be expected that optimization problems 4.2.4 and 4.4.1 yield the same result asymptotically, for large $N$. An asymptotic result is of a limited value in practical applications, however. An additional difficulty in analysis is the end point penalty that is needed for stability of overall system.

Stability of the model predictive control for constrained systems has been addressed in many papers (see [97] for a survey). Available results require the *stability constraint* on the state at the end of the control horizon or the *terminal constraint set* to be incorporated in the cost. A stability constraint requires $x_N(N+1) = 0$ which is an equivalent of an infinite end point penalty in (4.32). An infinite end point penalty is often not desirable in applications of the model predictive control. Results have been reported about the minimum size of the end point penalty that is needed for stability of the overall system in deterministic setting. (See [141, 158].) Since the state is stochastic in the optimization problem 4.4.1, the expectation of the end point penalty is:

$$\mathbb{E}\|x_N(N+1)\|_Q^2 = \|\mathbb{E}x_N(N+1)\|_Q^2 + \text{Trace}(Q_x\, Q)$$

where $Q_x$ is the covariance matrix of the state at $N+1$. The above equation explains the trade-off inherent to the predictive control of the stochastic systems. It is not just the expectation of the state that has to be kept small (close to zero) but also its variance. By setting the end point penalty to be $\|\mathbb{E}x_N(N+1)\|_Q^2$ some of the difficulties in the analysis will be removed but we will lose ability to control the variance of the state at the end of control horizon. The expectation of the state could be made arbitrary small by choosing a large $Q$ with an expense of the large variance.

In the stochastic setting, there are no results that relate the size of the end point penalty and stability of the closed loop system. An additional interesting aspect is a trade off between the size of the end point penalty and the variance of the predicted state. It is a topic for further research and for the time being we look at the choice of the end point penalty as a "tuning parameter".

Model predictive controller (4.34) is a solution to the optimization problem 4.4.1. To solve the optimization problem 4.4.1 one has to overcome several difficulties. The optimal vector of feedback laws $\pi^*$ is an element of an infinite dimensional set, a fact that renders the optimization problem 4.4.1 infinite dimensional optimization problem except for cases in which the disturbance is taking values from a finite set.

**Theorem 4.4.2** *Consider the optimization problem 4.4.1. Under assumptions 4.2.1 and 4.2.5, the optimal cost* (4.33) *and the associate vector of feedback mappings* $\pi$ *can be obtained recursively as follows:*

$$V_s(x) := \inf_{u\in\mathbb{R}^m} \left\{ g(C_z x + D_z u) + h(x) + \mathbb{E}_\nu V_{s+1}(Ax + Bu + E\nu) \right\} \tag{4.35}$$

*with an initial condition:*

$$V_{N+1}(x) := \|x\|_Q^2$$

*that has to be solved backwards from $s = N$ to $s = 0$. The expression $\mathbb{E}_{(\cdot)}$ denotes conditional expectation with respect to $(\cdot)$.*

**Proof:** Two properties are crucial for the proof of the theorem. The first one is the causality of the system (4.1): a state at some $k \in T$ does not depend on disturbances $\nu(j)$ and feedback laws $\pi_j$, $j > k$, $j \in T$. The second one is the fact that $\nu(0) \cdots \nu(N)$ are independent stochastic variables. Because of these two properties, the optimal cost (4.33) can be rewritten as:

$$\inf_{\pi_0 \cdots \pi_{N-1}} \Big\{ g(z_N(0)) + h(x_N(0)) + \\ + \sum_{k=0}^{N-1} \mathbb{E}_{\nu(0)} \mathbb{E}_{\nu(1)} \cdots \mathbb{E}_{\nu(k-1)} \big\{ g(z_N(k)) + h(x_N(k)) \big\} + \\ + \inf_{\pi_N} \mathbb{E}_{\nu(0)} \cdots \mathbb{E}_{\nu(N-1)} \big\{ g(z_N(N)) + h(x_N(N) + \mathbb{E}_{\nu(N)} \|x_N(N+1)\|_Q^2 \big\} \Big\} \quad (4.36)$$

where:

$$z_N(k) = C_z x_N(k) + D_z \pi_k(x_N(k)), \quad k \in [0, N].$$

According to assumption 4.2.5, the state constraints violation cost $h$ is a convex, finite valued function. The last term from the above can be rewritten as follows (see [11], lemma 1):

$$\mathbb{E}_{\nu(0)} \mathbb{E}_{\nu(1)} \cdots \mathbb{E}_{\nu(N-1)} \inf_{u \in \mathbb{R}^m} \big\{ g(z_N(N)) + h(x_N(N)) + \mathbb{E}_{\nu(N)} \|x_N(N+1)\|_Q^2 \big\}.$$

Define:

$$V_N(x) := \inf_{u \in \mathbb{R}^m} \big\{ g(C_z x + D_z u) + h(x) + \mathbb{E}_\nu \|Ax + Bu + E\nu\|_Q^2 \big\}$$

so that (4.36) can be rewritten as:

$$\inf_{\pi_0 \cdots \pi_{N-2}} \Big\{ g(z_N(0)) + h(x_N(0)) + \\ + \sum_{k=0}^{N-2} \mathbb{E}_{\nu(0)} \mathbb{E}_{\nu(1)} \cdots \mathbb{E}_{\nu(k-1)} \big\{ g(z_N(k)) + h(x_N(k)) \big\} + \\ + \inf_{\pi_{N-1}} \mathbb{E}_{\nu(0)} \cdots \mathbb{E}_{\nu(N-2)} \big\{ g(z_N(N-1)) + h(x_N(N) + \mathbb{E}_{\nu(N-1)} V_N(x_N(N)) \big\} \Big\}.$$

Define:

$$V_{N-1}(x) := \inf_{u \in \mathbb{R}^m} \big\{ g(C_z x + D_z u) + h(x) + \mathbb{E}_\nu V_N(Ax + Bu + E\nu) \big\}$$

and rewrite (4.36) as:

$$\inf_{\pi_0\cdots\pi_{N-3}} \Big\{ g(z_N(0)) + h(x_N(0)) + \\ + \sum_{k=0}^{N-3} \mathbb{E}_{\nu(0)}\mathbb{E}_{\nu(1)}\cdots\mathbb{E}_{\nu(k-1)}\big\{g(z_N(k)) + h(x_N(k))\big\} + \\ + \inf_{\pi_{N-3}} \mathbb{E}_{\nu(0)}\cdots\mathbb{E}_{\nu(N-3)}\big\{g(z_N(N-1)) + h(x_N(N) + \mathbb{E}_{\nu(N-2)}V_{N-1}(x_N(N-1))\big\}\Big\}.$$

By proceeding in this way, the optimization problem 4.4.1 can be rewritten as the recursion (4.35). ■

The recursion given by (4.35) can be seen as a nested sequence of optimization problems defined over the input $u \in \mathbf{U}$. As already mentioned, the function $g$ is a strictly convex function in $x$ and $u$ and the function $h$ is a convex function in $x$. To see that optimization problems in (4.35) are convex optimization problems it is necessary to show that the optimal "cost-to-go" $V_s$ is a convex function in $x$ for all $s \in T$.

**Theorem 4.4.3** *The optimal cost to go $V_s$ is a convex function in $x$ for all $s \in T$.*

**Proof:** At each $s \in T$, the optimal cost to go $V_s$ can be written as:

$$V_s(x) = \inf_{\pi^s} J_s(x, \pi^s) \tag{4.37}$$

where $\pi^s \in \Pi^s$ is a sequence of maps $\pi_k^s : \mathbb{R} \to \mathbf{U}$ such that $u_N(k) = \pi_k^s(x_N(k))$ with $k = s, \ldots, N$. The cost to go $J_s$ is defined with:

$$J_s(x, \pi^s) := \mathbb{E}\Big\{ \sum_{k=s}^{N} \big\{ g\big(C_z(x_N(k)) + D_z\pi_k(x_N(k))\big) + h(x_N(k))\big\} + \|x_N(N+1)\|_Q^2 \Big\}. \tag{4.38}$$

where $x_N(s) = x$. The optimal cost to go (4.37) is defined as a minimization over a class $\Pi^s$. We first extend the class $\Pi^s$ of functions over which optimization is defined. Assume we optimize over $\tilde{\Pi}^s$ where $\tilde{\pi}^s \in \tilde{\Pi}^s$ is a sequence of maps $\tilde{\pi}_k^s : \mathbb{R}^{n(k+1)} \to \mathbf{U}$ such that

$$u_N(k) = \tilde{\pi}_k^s(x_N(0), x_N(1), \ldots, x_N(k)). \tag{4.39}$$

Since the future at time $k$ only depends on $x_N(k)$, it is obvious that this extension of the class of controllers does not change the infimum. Next, note that the feedbacks in the class $\tilde{\Pi}^s$ can be equally represented by the class $\bar{\Pi}^s$ where $\bar{\pi}^s \in \bar{\Pi}^s$ is a sequence of maps $\bar{\pi}_k^s : \mathbb{R}^{n+k} \to \mathbf{U}$ such that

$$u_N(k) = \bar{\pi}_k^s\left(x_N(0), \nu(0), \ldots, \nu(k-1)\right). \tag{4.40}$$

This is based on the fact that

- Given $x_N(0), \nu(0), \ldots, \nu(k-1)$ and $\bar{\pi}^s$ we can recursively construct $x_N(0)$, $x_N(1), \ldots, x_N(k)$ and $\tilde{\pi}^s$ such that

$$\bar{\pi}_k^s\left(x_N(0), \nu(0), \ldots, \nu(k-1)\right) = \tilde{\pi}_k^s(x_N(0), x_N(1), \ldots, x_N(k)) \tag{4.41}$$

- Given $x_N(0), x_N(1), \ldots, x_N(k)$ and $\tilde{\pi}^s$ it is possible to recursively construct $x_N(0)$, $\nu(0)$, $\ldots$, $\nu(k-1)$ and $\bar{\pi}^s$ such that (4.41) is satisfied. If $E$ is not injective then $\nu(0), \ldots, \nu(k-1)$ are not uniquely determined but it is trivial to see that this does not affect the corresponding infima.

We conclude that the infimum in (4.37) is the same if $\pi^s$ is taken from $\Pi^s$, $\tilde{\Pi}^s$ or $\bar{\Pi}^s$. It therefore suffices to show that

$$\inf_{\bar{\pi}^s \in \bar{\Pi}} J_s(x, \bar{\pi}^s)$$

is a convex function in $x$.

With the disturbance up to $k-1$ denoted by $\nu^k := (\nu(j))_{j=0}^{k-1}$ we define:

$$j_s^k(x, \bar{\pi}_k, \nu^k) := \begin{cases} g\big(C_z(x_N(k)) + D_z\bar{\pi}_k(x, \nu^k)\big) + h(x_N(k)) & \text{if} \quad k \neq N+1 \\ \|x_N(k)\|_Q^2 & \text{if} \quad k = N+1 \end{cases}$$

with $x_N(s) = x$. The cost to go (4.38) can be written as:

$$J_s(x, \bar{\pi}^s) = \mathbb{E} \sum_{k=s}^{N+1} j_s^k(x_N(k), \bar{\pi}_k, \nu^k)$$

where $x_N(s) = x$.

Next, consider $x^a, x^b \in \mathbb{R}^n$, $x^a \neq x^b$. The corresponding minimizing feedback in $\bar{\Pi}^s$ are denoted by $\bar{\pi}_a^s$ and $\bar{\pi}_b^s$, respectively. Suppose that a realization of the disturbance $\nu$ is denoted by $\nu_i$. A disturbance realization up to $k-1$ is denoted by $\nu_i^k := (\nu_i(j))_{j=0}^{k-1}$. Define:

$$u_i^a := \bar{\pi}_a^s\left(x^a, \nu_i\right)$$

and:

$$u_i^b := \bar{\pi}_b^s\left(x^b, \nu_i\right).$$

Inputs up to time $k$ are denoted by $\bar{u}_i^{ak} := (\bar{u}^a(j))_{j=0}^k$ and $\bar{u}_i^{bk} := \left(\bar{u}^b(j)\right)_{j=0}^k$. The function $g$ is a strictly convex function in $x$ and $u$, the function $h$ is a convex function in $x$ and the cost $j_s^k$ is jointly convex in $(x, u)$, therefore:

$$j_s^k(\lambda x^a + (1-\lambda)x^b, \lambda\bar{u}_i^{ak} + (1-\lambda)\bar{u}_i^{bk}, \nu_i^k) \leq \lambda j_s^k(x^a, \bar{u}_i^{ak}, \nu_i^k) + (1-\lambda) j_s^k(x^b, \bar{u}_i^{bk}, \nu_i^k)$$

where $\lambda \in (0, 1)$. Define:

$$\dot{\bar{\pi}}^s(x, \nu) = \lambda\bar{\pi}_a^s(x^a, \nu) + (1-\lambda)\bar{\pi}_b^s(x^b, \nu)$$

Clearly $\dot{\bar{\pi}}^s \in \bar{\Pi}^s$ and $\dot{\bar{\pi}}^s$ is a sequence of feedback maps:

$$\dot{\bar{\pi}}_k^s(x, \nu^k) = \lambda \bar{\pi}_a^s(x^a, \nu^k) + (1-\lambda)\bar{\pi}_b^s(x^b, \nu^k).$$

Consider:

$$J_s(x, \dot{\bar{\pi}}^s) = \mathbb{E} \sum_{k=s}^{N+1} j_s^k(x_N(k), \dot{\bar{\pi}}_k^s, \nu^k).$$

Because the cost $j_s^k$ is convex, the expectation is a linear operator and therefore preserve convexity we can write:

$$J_s(\lambda x^a + (1-\lambda)x^b, \dot{\bar{\pi}}^s) \leq \lambda J_s(x^a, \bar{\pi}_a^s) + (1-\lambda) J_s(x^b, \bar{\pi}_b^s).$$

Finally:

$$\begin{aligned} V_s(\lambda x^a + (1-\lambda)x^b) &\leq J_s(\lambda x^a + (1-\lambda)x^b, \dot{\bar{\pi}}^s) \\ &\leq \lambda J_s(x_a, \bar{\pi}_a^s) + (1-\lambda) J_s(x_b, \bar{\pi}_b^s) = \lambda V_s(x^a) + (1-\lambda) V_s(x^b) \end{aligned}$$

for all $x^a, x^b \in \mathbb{R}^n$ and $\lambda \in (0, 1)$. ■

## 4.5 The algorithm

The algorithm for solving the optimization problem 4.4.1 has its origin in the result presented in theorem 4.4.2. An analytical computation of the expectation (4.35) is generally a difficult task. An alternative is to compute the empirical mean.

**Definition 4.5.1** Assume a set $\Theta$ and a probability measure $P$ on $\Theta$ are given. Let $f : \Theta \to \Omega$ be a function measurable with respect to $P$ where $\Omega$ is an interval on $\mathbb{R}^n$ (possibly equal to $\mathbb{R}^n$). Suppose that we draw $m$ independent, identically distributed (i.i.d) samples $\vartheta = \{\theta_1, \cdots, \theta_m\}$ from $\Theta$ in accordance with $P$. The empirical mean of the function $f$ is given by:

$$\hat{\mathbb{E}} f := \frac{1}{m} \sum_{j=1}^{m} f(\theta_j). \tag{4.42}$$

For more detailed treatment of the empirical mean and the convergence properties for a large number of samples see [11–13, 69, 151]. To compute the cost by the empirical mean, a number of realizations of the stochastic disturbance $w$ is needed. The cost for a specific realization of the stochastic disturbance $w$ is easily computed but realizations have to be chosen so that the empirical mean is computed efficiently. It is well known that an estimate based on linear gridding requires a number of samples that is exponential in the dimension of the stochastic variable to preserve accuracy in

the estimation. A standard method that is used instead of the linear gridding is the Monte Carlo simulation. Realizations of the stochastic disturbance are chosen randomly, according to the distribution of $w$. It is well known that bounds on the number of samples needed to preserve accuracy of the estimation can be obtained independent of the underlying distribution of the stochastic process. Before we present the algorithm, we explain briefly stochastic sampling of the disturbance.

Suppose that we take $\kappa$ samples of the disturbance $\nu(0)$ at $s = 0$. Given a fixed initial condition $x_0$ and an input $u(0) = \pi_0(x^a(t))$ there are $\kappa$ possible states $x_N(1)$. For each one of these possible futures we generate $\kappa$ samples of the disturbance $\nu(1)$ which establishes $\kappa^2$ possible future states $x_N(2)$. In this way, we obtain $\kappa^N$ samples of the disturbance $\nu$. The number of samples of $\nu$ grows exponentially with the horizon. The sampling as described is required for a good estimate of the optimal cost to go $V_s$ (4.35). One might conjecture that we do not need this because a very accurate estimate of $V_s$ is not required. Actually, only a good estimate of $V_0$ is required, because it determines $\pi_0$. However, we have no proof that a restricted set of samples still yields a correct result with a high probability.

An other approach would be to form a grid on the state space and to estimate the optimal cost "to go" on the points of the grid. Note that any kind of linear grid will not reflect a spread of the state around its mean value, resulting in a great number of points in which the state is not likely to be. The sampling procedure described above gives a grid on the state space that is more dense in the region in which the state is more likely to be. Moreover, the number of grid points grows exponentially (in the dimension of the state space) while the number of points required for stochastic sampling is independent of the dimension of the state space.

The cost computed via an empirical mean is given with:

$$\hat{J}(x, \pi) := \hat{\mathbb{E}}\Big\{ \sum_{k \in T} \big\{ g(z_N(k)) + h(x_N(k)) \big\} + \|x_N(N+1)\|_Q^2 \Big\}. \tag{4.43}$$

The optimization problem 4.4.1 is replaced by the optimization in which we seek for a minimum of the empirical cost (4.43) instead of the cost (4.32). The algorithm is based on the following theorem.

**Theorem 4.5.2** *Consider the optimization problem 4.4.1 in which the empirical cost* (4.43) *is minimized instead of* (4.32)*. Under assumptions 4.2.1 and 4.2.5, the empirical optimal cost:*

$$\hat{V}(x) := \inf_{\pi} \hat{J}(x, \pi) \tag{4.44}$$

*and the associate vector of feedback mappings $\pi$ can be obtained recursively as follows:*

$$\hat{V}_s(x) := \inf_{u \in \mathbb{R}^m} \big\{ g(C_z x + D_z u) + h(x) + \hat{\mathbb{E}}_\nu \hat{V}_{s+1}(Ax + Bu + E\nu) \big\} \tag{4.45}$$

*with an initial condition:*

$$\hat{V}_{N+1}(x) := \|x\|_Q^2$$

*that has to be solved backwards from* $s = N$ *to* $s = 0$. *The expression* $\hat{\mathbb{E}}_{(\cdot)}$ *denotes empirical conditional expectation with respect to* $(\cdot)$. *The empirical optimal cost* (4.44) *is obtained from* (4.45) *by* $\hat{V}(x) = \hat{V}_0(x)$.

**Proof:** The proof is obtained from the proof of theorem 4.4.2 with the difference that the expectation is replaced by the empirical mean. ■

The optimization problems in (4.45) are convex optimization problems, as outlined in the following lemma.

**Lemma 4.5.3** *The empirical optimal cost to go* $\hat{V}_s$ *is a convex function in* $x$ *for all* $s \in T$.

**Proof:** With obvious modifications, the proof of the lemma follows the proof of theorem 4.4.3. ■

With the disturbance sampled as described at each $s$, $s \in \{0, \cdots, N-1\}$ there are $\kappa^s$ possible states denoted by $x_N^i(s)$, $i \in \{1, \cdots, \kappa^s\}$. For initialization of the algorithm we use the result presented in corollary 4.3.4. Denote a feedback controller that achieves condition (4.14) for the problem at hand with $\varphi^0$. Note that the case in which all inputs are constrained or the case of no input constraints can be easily treated as special cases of the result presented in corollary 4.3.4. The algorithm is presented next.

---

**Algorithm 4.1**

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**Step 1: Initialization**

Take the measurement $x_0$ and set $x_N^1(0) = x_0$. Set $\hat{u}_i(s) = \varphi^0(x_N^i(s))$ for $s = 0, 1, \ldots, N$, $i = 1, \ldots, N$. Draw $\kappa^N$ samples for $w$. Set $V = \infty$. Set accuracy parameter $\varepsilon$. Set $s = N$.

**Step 2: Compute cost "to go"**

Determine a new $\hat{u}_i(s)$ by solving (4.45) for each $x_N^i(s)$, $i = 1, \ldots \kappa^s$. Compute $\hat{V}_s(x_N^i(s))$ for each $i$. If $s = 0$ go to **step 4**, otherwise set $s = s - 1$ and go to **step 2**.

**Step 4: Exit condition**

If $|\hat{V}_0(x_N^1(0)) - V| < \varepsilon$ stop. Otherwise: set $V = \hat{V}_0(x_N^1(0))$, set $s = N$ and go to **step 2**.

---

The convergence of the solution obtained by algorithm 4.1 is not a trivial issue. The solution is a function of the disturbance sample that is used in the computation and therefore the convergence of the cost function $\hat{V}_0(x)$ obtained from algorithm 4.1 to the optimal cost $V(x)$, defined in (4.33) need to be accessed in a probabilistic sense. As it will be shown in this section, the convergence in probability of $\hat{V}_0(x)$ to the optimal cost $V(x)$ can be proven under some weak assumptions. The first one of them is that matrix $D_z$ is injective which is necessary for the cost function to be strictly convex. Next assumption is that the cost function has at most exponential growth. As shown in section 4.2, this assumption does not pose limitations on the growth of the cost function. The limitation is posed by the stochastic nature of the disturbance and it is expressed in solvability conditions presented in section 4.3. It turns out that condition (4.26) is necessary to be satisfied for the solution obtained by algorithm 4.1 to converge in probability to the solution of the optimization problem 4.4.1.

To prove the convergence, we need an auxiliary result that is presented next.

**Theorem 4.5.4** *Consider strictly convex functions $f : \mathbb{R}^n \times \mathbf{U} \times \mathbb{R}^l \to \mathbb{R}$ where $\mathbf{U}$ is a bounded subset of $\mathbb{R}^m$ and $V : \mathbb{R}^n \to \mathbb{R}$ related by*

$$f(x, u, \xi) = g(C_z x + D_z u) + h(x) + V(Ax + Bu + E\xi) \tag{4.46}$$

*such that*

$$V, g, h \in \Theta(R).$$

*Next, consider a random variable $w \in \mathcal{N}(0, Q_w)$. Assume that $R$ satisfies condition* (4.26) *with respect to the covariance matrix $Q_w$. Then, for any $\varepsilon > 0$ and $\delta \in (0, 1)$ there exists $\kappa^*$ such that for any $\kappa > \kappa^*$ it is true with probability $(1 - \delta)$ that*

$$\left| \inf_{u \in \mathbf{U}} \hat{\mathbb{E}}_w f(x, u, w) - \inf_{u \in \mathbf{U}} \mathbb{E}_w f(x, u, w) \right| \leq \varepsilon e^{\|x\|_R^2} \tag{4.47}$$

*where*

$$\hat{\mathbb{E}}_w f(x, u, w) := \frac{1}{\kappa} \sum_{i=1}^{\kappa} f(x, u, w_i) \tag{4.48}$$

*is the empirical mean based on $\kappa$ independent samples $w_1, \cdots, w_\kappa$ identically distributed according to $\mathcal{N}(0, Q_w)$.*

**Proof:** The proof of the theorem is motivated by the proof of the theorem 2, page 8 in [112] where the convergence of empirical means is shown for measurable functions of stochastic variable in a class $\mathcal{F}$ when there exists a finite class containing an upper and a lower approximations to each $f$ in $\mathcal{F}$. Because $\mathbf{U}$ is compact, the essence of this proof can be used to prove theorem 4.5.4. This claim will be illustrated for a simple case first.

Let $p : \mathbb{R}^l \times [-1, 1] \to \mathbb{R}$ be a convex function given by:

$$p(w, u) = g(D_z u) + V(Bu + Ew).$$

Note that since $V, g \in \Theta(R)$ and $R$ satisfies condition (4.26) with respect to the covariance matrix $Q_w$, the expectation of $p$ is well defined (theorem 4.3.5). Define functions $p_i$, $i = 1, \cdots, n$ by

$$p_i(w) := p\big(w, -1 + \frac{2i}{n}\big) \qquad i = 0, \cdots, n.$$

Consider

$$\Big| \inf_u \hat{\mathbb{E}}_w \, p(w, u) - \inf_u \mathbb{E}_w \, p(w, u) \Big| \qquad u \in [-1, 1] \tag{4.49}$$

which can be rewritten as

$$\begin{aligned} \Big| \inf_u \hat{\mathbb{E}}_w \, p(w, u) - \inf_i \hat{\mathbb{E}}_w \, p_i(w) + \inf_i \hat{\mathbb{E}}_w \, p_i(w) \\ - \inf_i \mathbb{E}_w \, p_i(w) + \inf_i \mathbb{E}_w \, p_i(w) - \inf_u \mathbb{E}_w \, p(w, u) \Big| \leq \\ \Big| \inf_u \hat{\mathbb{E}}_w \, p(w, u) - \inf_i \hat{\mathbb{E}}_w \, p_i(w) \Big| + \Big| \inf_i \hat{\mathbb{E}}_w \, p_i(w) - \inf_i \mathbb{E}_w \, p_i(w) \Big| + \\ \Big| \inf_i \mathbb{E}_w \, p_i(w) - \inf_u \mathbb{E}_w \, p(w, u) \Big| \end{aligned} \tag{4.50}$$

The second term on the right hand side in inequality (4.50) can be made arbitrary small by choosing $\kappa$ large enough since we infimize over a finite set of functions. Set

$$u_i = -1 + \frac{2i}{n}.$$

Next, note that $p \in \Theta(R)$ since $V, g \in \Theta(R)$. Then there exists $q \in \Theta(R)$ such that there exists $u_i$ with $|u - u_i| < 1$ and $\tilde{u}$ such that

$$\frac{|p(w, u) - p(w, u_i)|}{|u - u_i|} = \frac{|p(w, \tilde{u} + 1) - p(w, \tilde{u})|}{1} \leq q(Bu + Ew). \tag{4.51}$$

Equation (4.51) can be more elegantly expressed in terms of subdifferentials. Therefore, there exists $p, q \in \Theta(R)$ such that for some suitable chosen $u_i$

$$|p(w, u) - p(w, u_i)| \leq q(Bu + Ew) \, |u - u_i|. \tag{4.52}$$

Inequality (4.52) implies that the first term on the right hand side of (4.50) satisfies

$$\Big| \inf_u \hat{\mathbb{E}}_w \, p(w, u) - \inf_i \hat{\mathbb{E}}_w \, p_i(w) \Big| \leq \hat{\mathbb{E}}_w \, q(Bu + Ew) \min_i |u - u_i|$$

and similarly to the term above

$$\Big| \inf_u \mathbb{E}_w \, p(w, u) - \inf_i \mathbb{E}_w \, p_i(w) \Big| \leq \mathbb{E}_w \, q(Bu + Ew) \min_i |u - u_i|.$$

Next, note that by choosing $n$ sufficiently large $\min_i |u - u_i|$ can be made arbitrary small. Since all terms on the right hand side in (4.50) can be made arbitrary small, (4.49) can be made also arbitrary small.

In our case $u \in \mathbf{U}$ instead of $u \in [-1, 1]$ but since $\mathbf{U}$ is a compact set, the proof relies on the same reasoning. Also, in general, $p$ is exponential in $x$ and therefore $q$ becomes an exponential function in $x$, which yields an exponential bound in (4.47). ■

The relation of the solution obtained by algorithm 4.1 and the original optimization problem 4.4.1 is described in the theorem that is given next.

**Theorem 4.5.5** *Assume the following*

1. *matrix $D_z$ is injective*
2. $g(\cdot, u) + h(\cdot) \in \Theta(R)$ *for all* $u \in \mathbf{U}$
3. *$R$ satisfies condition* (4.26) *for* $t = 1$

*Then, for any initial condition $x \in \mathbf{X}$, the cost function $\hat{V}_0(x)$ obtained from algorithm 4.1 converge in probability to the optimal cost $V(x)$, defined in* (4.33)*, as $\varepsilon \to 0$ and $\kappa \to \infty$.*

**Proof:** The result will be established recursively. First, consider the optimal cost to go at $s = N$

$$V_N(x) = \inf_{u \in \mathbf{U}} \mathbb{E}_v \left\{ g(C_z x + D_z u) + h(x) + \|Ax + Bu + Ev\|_Q^2 \right\} \tag{4.53}$$

and the empirical optimal cost to go $s = N$

$$\hat{V}_N(x) = \inf_{u \in \mathbf{U}} \hat{\mathbb{E}}_v \left\{ g(C_z x + D_z u) + h(x) + \|Ax + Bu + Ev\|_Q^2 \right\}. \tag{4.54}$$

Since matrix $D_z$ is injective

$$f_N(x, u, v) := g(C_z x + D_z u) + h(x) + \|Ax + Bu + Ev\|_Q^2$$

is a strictly convex function in $u$. Note that $f_N(x, u, v)$ grows at most exponentially in $x$ (because of assumption (2)) and at most quadratically in $v$. By applying the result of theorem 4.5.4 on $f_N(x, u, v)$ and with $P = 0$ we conclude that for any $\varepsilon_N > 0$ and $\delta_N \in (0, 1)$ there exists $\kappa_N$ such that for any $\kappa > \kappa_N$ we have with probability $(1 - \delta_N)$ that

$$\left| \hat{V}_N(x) - V_N(x) \right| \leq \varepsilon_N e^{\|x\|_R^2} \tag{4.55}$$

Next, consider the empirical optimal cost to go at $s = N - 1$

$$\hat{V}_{N-1}(x) = \inf_{u \in \mathbf{U}} \hat{\mathbb{E}}_v \left\{ g(C_z x + D_z u) + h(x) + \hat{V}_N(Ax + Bu + Ev) \right\}.$$

Inequality (4.55) defines an upper bound for $\hat{V}_{N-1}(x)$

$$\hat{V}_{N-1}(x) \leq \inf_{u\in\mathbf{U}} \hat{\mathbb{E}}_v \left\{ g(C_z x + D_z u) + h(x) + V_N(Ax + Bu + Ev) + \varepsilon_N e^{\|Ax+Bu+Ev\|_R^2} \right\} \tag{4.56}$$

and an lower bound

$$\hat{V}_{N-1}(x) \geq \inf_{u\in\mathbf{U}} \hat{\mathbb{E}}_v \left\{ g(C_z x + D_z u) + h(x) + V_N(Ax + Bu + Ev) - \varepsilon_N e^{\|Ax+Bu+Ev\|_R^2} \right\}. \tag{4.57}$$

Define

$$\overline{V}_{N-1}(x) := \inf_{u\in\mathbf{U}} \hat{\mathbb{E}}_v \left\{ g(C_z x + D_z u) + h(x) + V_N(Ax + Bu + Ev) \right\}. \tag{4.58}$$

Note that (4.56) and (4.57) imply that there exists $\alpha_N > 0$ such that

$$\left| \hat{V}_{N-1}(x) - \overline{V}_{N-1}(x) \right| \leq \varepsilon_N e^{\alpha_N \|x\|_R^2} \tag{4.59}$$

with probability $(1 - \delta_N)$ if $\kappa > \kappa_N$.

Next, consider optimal cost to go at $s = N - 1$

$$V_{N-1}(x) = \inf_{u\in\mathbf{U}} \mathbb{E}_v \left\{ g(C_z x + D_z u) + h(x) + V_N(Ax + Bu + Ev) \right\}.$$

and define

$$f_{N-1}(x, u, v) := g(C_z x + D_z u) + h(x) + V_N(Ax + Bu + Ev).$$

Function $f_{N-1}(x, u, v)$ has an exponential growth in $x$ because of assumption (2) and it grows in $v$ exponentially.

Theorem 4.5.4 can be applied on $f_{N-1}(x, u, v)$ with $P = E^T RE$ and under assumption (3). Thus, for any $\overline{\varepsilon}_{N-1} > 0$ and $\overline{\delta}_{N-1} \in (0, 1)$ there exist $\kappa_N$ and $\kappa_{N-1} > \kappa_N$ such that for any $\kappa > \kappa_{N-1}$ we have with probability $(1 - \overline{\delta}_{N-1})$ that

$$\left| \overline{V}_{N-1}(x) - V_{N-1}(x) \right| \leq \overline{\varepsilon}_{N-1} e^{\|x\|_R^2} \tag{4.60}$$

Bounds (4.59) and (4.60) implies that for any $\varepsilon_{N-1} > 0$ and $\delta_{N-1} \in (0, 1)$ there exist $\kappa_N$ and $\kappa_{N-1} > \kappa_N$ such that with probability $(1 - \delta_{N-1})$

$$\left| \hat{V}_{N-1}(x) - V_{N-1}(x) \right| \leq \varepsilon_{N-1} e^{\alpha_N \|x\|_R^2} \tag{4.61}$$

By repeating arguments that are used to show (4.61), we conclude that for all $s = 0, \cdots, T$, for any $\varepsilon_s > 0$ and $\delta_s \in (0, 1)$ there exist

$$\kappa_N, \kappa_{N-1}, \cdots, \kappa_{s+1} \quad \text{and} \quad \kappa_s > \max\{\kappa_N, \kappa_{N-1}, \cdots, \kappa_{s+1}\}$$

such that for any $\kappa > \kappa_s$ we have with probability $(1 - \delta_s)$ that

$$\left|\hat{V}_s(x) - V_s(x)\right| \leq \varepsilon_s e^{\alpha_s \|x\|_R^2}. \tag{4.62}$$

Finally, note that for $s = 0$, (4.62) proves the claim of the theorem. ■

With algorithm 4.1, the optimization problem 4.4.1 can be solved approximately but with an arbitrary accuracy. The accuracy of the solution depends on a number of samples of the disturbance $w$ taken for computing the empirical mean. With algorithm 4.1 we are able to access the achievable performance when one aims to control the plant (4.1), subject to the state and input constraints with the stochastic disturbance.

Note that the number of the evaluating points grow exponentially with the control horizon. It is a consequence of a fixed number of disturbance samples used to evaluate the empirical cost. As the following numerical example shows, an exponential growth in the number of samples is not needed and significant improvements in overall performance can be achieved with a smaller number of disturbance samples for time instants $s$ in the control horizon further away from $s = 0$. This effect is a consequence of the receding horizon implementation of the controller. In the receding horizon implementation only the control input computed at $s = 0$ is implemented.

## 4.6 Numerical example

In this section we present an example in which we consider a "double integrator" system of the form:

$$\begin{aligned} x(k+1) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w(k) \\ z(k) &= \begin{pmatrix} 0 & 0 \\ 0.7 & 0 \\ 0 & 0.7 \end{pmatrix} x(k) + \begin{pmatrix} 0.33 \\ 0 \\ 0 \end{pmatrix} u(k) \end{aligned} \tag{4.63}$$

The input is constrained as:

$$-0.5 \leq u \leq 0.5 \quad u \in \mathbb{R}.$$

The disturbance is a normally distributed random variable with zero mean and variance 0.2:

$$w \in \mathcal{N}(0, 0.2) \quad w \in \mathbb{R}.$$

The state $x$ is parameterized as:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and we impose a constraint on the state:

$$x_2 \geq 0.$$

It is assumed that the system has an initial state:

$$x(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix}.$$

The task is to steer the system (4.63), subject to the stochastic disturbance, from the initial state to the origin with the constrained input while respecting constraint on the state. With that aim, we design a stochastic model predictive controller described in section 4.4. We choose:

$$g(z) = \|z\|^2 \quad z \in \mathbb{R}^3 \tag{4.64}$$

and

$$h(x) = \begin{cases} 0 & \text{if } x_2 \geq 0 \\ e^{4.5x_2^2} - 1 & \text{if } x_2 < 0 \end{cases} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \tag{4.65}$$

With functions $g$ and $h$ as above, the controller minimizes the expectation of the quadratic cost when the state is away from the constraint $x_2 > 0$. When the state is near or on the boundary of the constraint the exponential constraint violation cost $h$ dominates and the main objective of the controller is to avoid a constraint violation. The constraint violation cost $h$ makes overal cost to be in $\Theta(R)$ class of functions, with

$$R = \begin{pmatrix} 0 & 0 \\ 0 & 4.5 \end{pmatrix}.$$

It can be easily verified that this choice of $R$ satisfies condition (3) of theorem 4.5.4. The length of the control horizon is $N = 5$. The end point penalty in (4.32) is chosen as:

$$Q = \begin{pmatrix} 1.6 & 0.9 \\ 0.9 & 1.33 \end{pmatrix}.$$

We compare two predictive controllers.

**Stochastic MPC.** *The disturbance is sampled. We take* 5 *samples of the disturbance at the first instant in the control horizon and* 3 *at the second instant in the control horizon. The samples are taken according to the distribution of the disturbance* $\mathcal{N}(0, 0.2)$*. The number of samples of the disturbance* $\nu$ *is equal to 15. For each* $k \in \mathbb{Z}_+$ *solve the optimization problem 4.4.1 by algorithm 4.1 with the state of the "double integrator"* (4.63) *at* $k$ *as the initial state. The input to the system* (4.63) *at* $k$ *is the first control from the vector of controls obtained by algorithm 4.1 at* $k$*.*

Stochastic MPC controller takes into account the stochastic nature of the disturbance and the optimization is performed in closed loop, based on a number of disturbance samples.

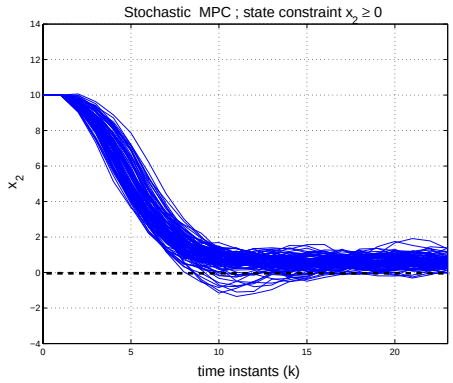


Figure 4.1: "Double integrator" is controlled by stochastic MPC

In the second controller, the stochastic disturbance over the control horizon is set to be equal to its mean value. In this way there is only one predicted state trajectory so the result obtained by algorithm 4.1 is equivalent to the result of the optimization in the open loop.

**Standard MPC.** *Assume that the disturbance over the control horizon $\nu$ is equal to the mean of $w$, i.e.*

$$\nu(k) = 0 \quad \textit{for all} \quad k \in T.$$

*For each $k \in \mathbb{Z}_+$ solve the optimization problem 4.4.1 by algorithm 4.1 with the state of the system* (4.63) *as an initial state and with the function $g$ and the constraint violation cost $h$ as* (4.64) *and* (4.65). *The number of samples $\kappa$ is equal to* 1 *and the sample values are set to zero. The first control from the resulting sequence of controls is applied to the plant at $k$.*

Standard MPC controller represents a standard approach to predictive control of stochastic systems in the model predictive control literature. (see [17] for a design of the standard model predictive controller for the "double integrator" example). We include this controller in the numerical example to compare its performance with stochastic MPC controller where the optimization is performed in the closed loop and a vector of *feedbacks laws* over the control horizon is obtained.

We perform two sets of simulations. In the first set the system (4.63) is controlled by stochastic MPC controller and in the second one with standard MPC controller. In each set of simulations there is a 100 simulations, each one of them performed with the different realization of the disturbance $w$. The resulting trajectories of $x_2$ are plotted on figure 4.1 for stochastic MPC and figure 4.2 for standard MPC. When the system is "far" from the constraint boundary, controllers stochastic MPC and standard MPC show similar performance. When the state of the system is near or on the boundary

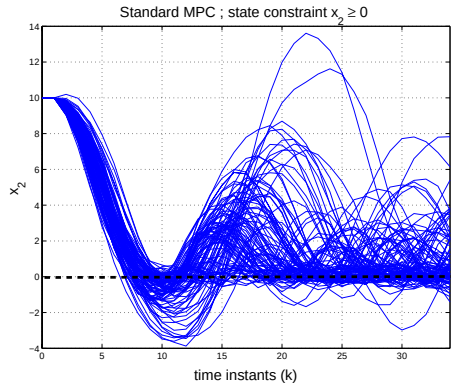


Figure 4.2: "Double integrator" is controlled by standard MPC

of the state constraint, standard MPC controller is not able to realistically predict a possibility of the constraint violation, because of the assumption that the disturbance in the "next time step" over the control horizon is equal to the mean value of the disturbance, in this case zero. A probability that $w$ will be larger than zero is high so for a large number of disturbance trajectories the state constraint is violated. On the contrary, stochastic MPC controller computes an optimal map from the state to the input for a number of points in the state space. Points are determined with the stochastic sampling of the disturbance and therefore there is a large probability that the optimal map for the predicted states is computed in the region in which the state of the system will be. Stochastic MPC controller takes into an account a possibility of the constraint violation when the state of the system is near the boundary of the constraint. This leads to the more realistic "prediction" and the control strategy that respects the state constraints better.

To support that further, we compute a frequency distribution of trajectories at $k = 11$ ("overshoot" region) and the "steady state" region at the end of simulations. The $x_2$ axis is divided in 20 regions between $-4$ and 4 and the number of trajectories passing trough each region is calculated. Dividing the number of trajectories with the total number of simulations gives the frequency distribution of the trajectories. The frequency distribution gives an estimation of the probability distribution of the states. Because of the constraints, the controlled system is nonlinear and the probability distribution of the state is not easy to compute. The frequency distributions are plotted on figure 4.3. Both figures show that the probability of constraint violation is significantly smaller when the system (4.63) is controlled with stochastic MPC controller. It can be observed that the variance of the trajectories around mean value is significantly smaller when the "double integrator" is controlled with stochastic MPC controller. With standard MPC controller, not only that the probability of the constraint violation

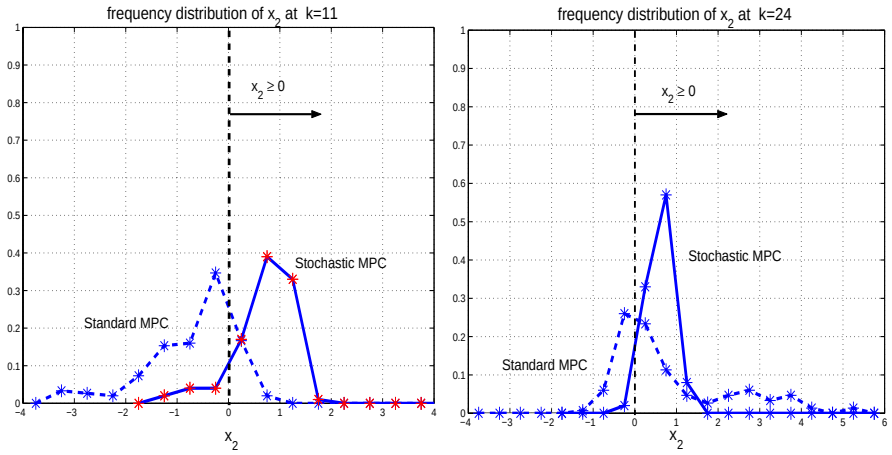


Figure 4.3: Frequency distribution of the state: solid line - stochastic MPC; dashed line - standard MPC

is higher but also the probability that the system state will be away from the equilibrium point in the region $x_2 > 0$.

Typically, to prevent an excessive constraint violation shown on figure 4.2 it is necessary to increase a "set point" to which the plant is to be steered. Figure 4.4 shows results of simulations in which the "double integrator" is controlled by standard MPC controller with an increased set point. The set point is set to 1. The mean value of the state at $k = 24$ is approximately the same for both controllers. Stochastic MPC controller shows better performance with respect to the variance of the state. Note that stochastic MPC controller chooses the set point "automatically" i.e. the "optimal set point" is a result of the optimization performed with algorithm 4.1. Note that the number of disturbance samples taken is smaller than the number that one would expect according to estimates based on the probabilistic bounds available in the literature. In despite of that, the performance of stochastic MPC shows significant improvement over the performance of standard MPC. The reason is that in stochastic MPC the optimization is performed in closed loop and the sampling of the disturbance is based on its probability distribution so we have a large probability of computing the control input for the states in which the system is likely to be in "future". In contrary, standard MPC computes only one control input for all states which is very crude way of approximating the controller, given the fact that the system is subject to stochastic disturbance. In short, a significant improvement of the performance can be achieved even with a small number of disturbance samples when the optimization is in closed loop.

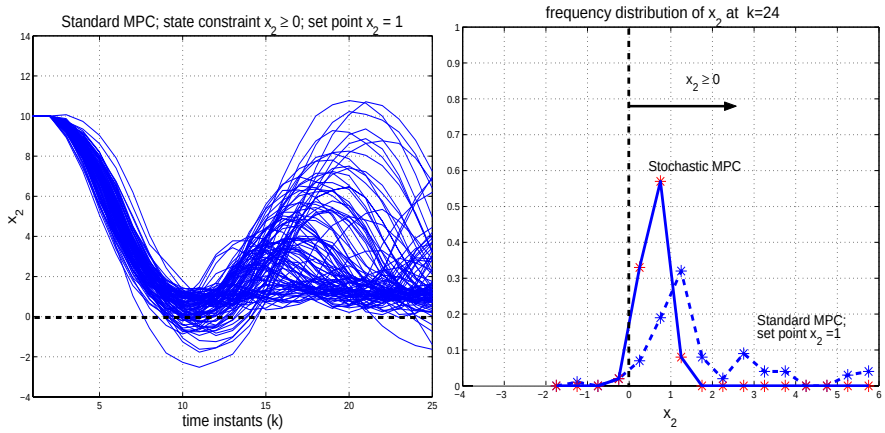


Figure 4.4: "Double integrator" is controlled by standard MPC; set point is increased to $x_2 = 1$.

## 4.7 Conclusion

In this chapter we extend the work presented in chapter 3 by considering state constraints in addition to constraints on the input. Constraints on the state are handled by introducing an additional cost that penalizes constraint violation. When the state of the system is in the constraint set, away from the boundary an optimal controller minimizes the cost that measures the performance. When the state is close to the boundary, the constraint violation cost dominates in the overall cost and the optimal controller minimizes the possibility of constraint violation. In this approach, it is natural to ask for a large penalty so that the state is kept within constraints as much as possible. The penalty can not be arbitrary large, however. We present a condition on the growth of the penalty function.

The optimization problem for the constrained linear system with the stochastic disturbances is difficult problem to solve analytically. There is no method reported in the literature that can be used to obtain an optimal controller for the problem. In this chapter we present an approximate solution to this problem by a predictive controller. A simulation study shows that even with the small number of disturbance samples a significant improvement in the closed loop performance can be achieved when the optimization over the control horizon in model predictive controller is performed in closed loop.

Algorithms presented so far in the thesis deal with the state feedback case. In the next chapter we remove the assumption that the state of the system is available for feedback and consider the measurement feedback case.

# 5

# *Optimal control of stochastic systems by measurement feedback*

In this chapter we extend the approach of the previous chapters to the measurement feedback case. We remove the assumption that the state of the system is available for feedback and show how algorithms from the previous chapters can be used in the measurement feedback case.

## 5.1 Introduction

In chapters 3 and 4 we considered the optimal control of constrained systems that are subject to stochastic disturbances. We derived solvability conditions for the problem but analytical computation of the optimal controller turned out to be extremely difficult task. The feasibile approach is to use model predictive control technique. So far, we have obtained several computational algorithms for model predictive control of constrained systems that are subject to stochastic disturbances. These results have been based on the assumption that all states of the plant are available for feedback.

In this chapter, we consider the more general case in which we assume that output of the plant is measured and available for feedback. In this case, static feedbacks are no longer sufficient and we need to study dynamic feedbacks. In general, the state of the plant is subject to constraints and we have only partial information about the state via output measurements. The standard approach is to use an optimal state observer to estimate the state of the plant. The state observer that we use for the purpose of optimal state estimation is the well known Kalman filter. A measurement feedback controller then have two separate tasks: the state estimation and the computation the optimal input that is based on the static feedback from the estimated state. Within the classical Linear Quadratic Gaussian framework, it is possible to obtain the optimal controller by this approach, according to the well known "separation principle".

In section 5.2 we propose a problem setup for optimal control of systems with the hard constraints on input and possible constraints on the state. When constraints on the state are present, the constraint violation cost is added to the cost function (see section 4.2, chapter 4) which makes the overall cost function non quadratic in general. The problem setup does not fit in the classical LQG framework because of the

input constraints and the possibly non quadratic cost function. The "separation principle" does not necessarily give an optimal controller in this case. In section 5.3 we study this issue and investigate in which cases the solution based on the "separation principle" gives an optimal controller and in which cases we have to find an alternative control structure.

In section 5.4, we design a model predictive controller that uses the optimal state estimate of the plant as an initial state for prediction. The feedback structure that is inherent to the problem (i.e. the estimated state of the plant is used for feedback) is taken into account in the prediction. The difficulty is that the output measurement is not available over the control horizon and the correction of the prediction is not possible as in the standard Kalman filtering algorithm. To overcome this difficulty, we consider the innovation of the prediction as a stochastic process. We present an algorithm for model predictive control of stochastic systems via measurement feedback.

Finally, in section 5.5 we present two examples in which we implement a model predictive controller developed the section 5.4 on the system with constrained input and the double integrator system.

## 5.2 Optimal control of stochastic systems by measurement feedback

We consider the plant given by the discrete time state space equations

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Ew(t) \\ y(t) &= C_y x(t) + \eta(t) \\ z(t) &= C_z x(t) + D_z u(t) \end{aligned} \tag{5.1}$$

where $u$ is the control input with $u(t) \in \mathbf{U} \subseteq \mathbb{R}^m$ and $x$ is the state with $x(t) \in \mathbb{R}^n$. The set $\mathbf{U}$ is a not necessarily bounded, closed, convex set which contains an open neighborhood of the origin. We assume that constraints on the state $x$ are imposed in that $x(t)$ is supposed to belong to a convex, closed set $\mathbf{X} \subseteq \mathbb{R}^n$ that contains the origin in its interior.

The second equation describes the measured output $y$ with $y(t) \in \mathbb{R}^d$. The output to be controlled is $z$ with $z(t) \in \mathbb{R}^p$. The disturbance $w$ and the measurement noise $\eta$ are two mutually independent stochastic processes with $w(t) \in \mathcal{N}(0, Q_w)$ and $\eta(t) \in \mathcal{N}(0, Q_\eta)$ where $\mathcal{N}(0, Q)$ denotes the family of normally distributed random variables with zero mean and covariance matrice $Q$. Moreover, for $k \neq j$, $w(k)$ and $w(j)$ are independent as well as $\eta(k)$ and $\eta(j)$. Note that this implies that also the state $x$, the measurement $y$ and the controlled output $z$ are stochastic processes.

Thus, we consider a linear, time invariant plant, subject to stochastic disturbances with a constrained input and a constrained state variable. The measurement output

$y$ is available for feedback. When the plant is subject to stochastic disturbances, the constrained input limits the ability to control the plant, as already discussed in chapters 3 (page 47) and 4 (page 73). Therefore, the following assumption is necessary.

**Assumption 5.2.1** The system (5.1) is globally asymptotically stabilizable.

As shown in chapter 3, if the system (5.1) is not globally asymptotically stabilizable then no controller will exist that stabilizes this system. In addition to assumption 5.2.1 we assume the following, rather natural assumption when one deals with the stabilization of the system (5.1) via measurement feedback.

**Assumption 5.2.2** The matrix pair $\left(C_y, A\right)$ is detectable.

We consider a problem of choosing $u$ such that the following cost is minimized.

$$J(x, u) := \lim_{T\to\infty} \mathbb{E}\frac{1}{T}\sum_{t=0}^{T} j(x(t), u(t)) \tag{5.2}$$

subject to the state equations (5.1) with $x(0) = x$ where $j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ is a strictly convex function with $j(0, 0) = 0$. The choice of the function $j$ depends on the problem at hand. The case where only the input $u$ is constrained (i.e. $\mathbf{X} = \mathbb{R}^n$) has been treated in chapter 3. In this case, the function $j$ has been chosen as a quadratic function. The general case with constraints on the state and the input has been treated in chapter 4, where we redefined the cost $j$ so as to include an exponential penalty on state violations. Therefore, the structure of the cost (5.2) is general enough to capture different problems.

The control input $u$ has to be chosen such that $u(t)$ is a function of all past measurements. Ultimately, we will wish to implement the controller by means of a digital computational device, which implies that at least 1 time unit will be required to calculate the next control action. Because of this, we assume that at time $t$ measurements $y(\tau)$, $0 < \tau < t$ are used for computation of the input $u(t)$. Thus, the system (5.1) is controlled by means of a *strictly causal dynamic feedback controller* which is assumed to be representable by the state equations

$$\begin{aligned} r(t+1) &= f_{\text{con}}(r(t), y(t)) \\ u(t) &= g_{\text{con}}(r(t)) \end{aligned} \tag{5.3}$$

with the initial condition $r(0) = 0$ and where functions $f_{\text{con}} : \mathbb{R}^g \times \mathbb{R}^d \to \mathbb{R}^g$ and $g_{\text{con}} : \mathbb{R}^g \to \mathbb{R}^m$ are continuous functions with

$$f_{\text{con}}(0, 0) = 0 \quad \text{and} \quad g_{\text{con}}(0) = 0$$

and where $\dim(r)$ is the (undecided) state dimension of the controller. We denote the set of all feedback controllers of the form (5.3) by $\Sigma_{\text{con}}$.

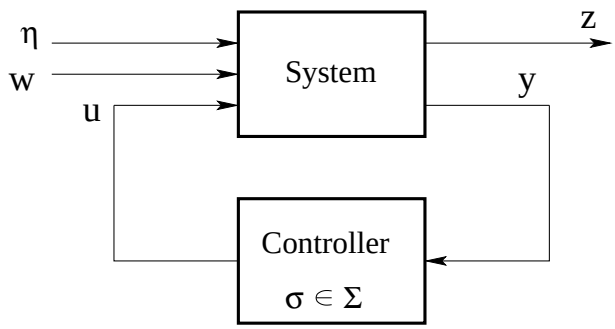


Figure 5.1: The system (5.1) is controlled by strictly causal feedback controller (5.3)

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The control system that consists of the system (5.1) and controller (5.3) is depicted in figure 5.1. Our aim is to find a controller from $\Sigma_{\text{con}}$ such that the cost function (5.2) is minimized. Obviously, the cost (5.2) with the input $u$ subject to (5.3) is a function of the controller from $\Sigma_{\text{con}}$. We will denote this (with some abuse of the notation in (5.2)) as

$$J(x, \sigma), \quad \sigma \in \Sigma_{\text{con}}.$$

That is, $J$ is viewed as a mapping $J : \mathbb{R}^n \times \Sigma_{\text{con}} \to \mathbb{R}$. Observe that if $x$ is stochastic, $J(x, \sigma)$ will be stochastic for any $\sigma \in \Sigma_{\text{con}}$.

Formally we define the following optimization problem.

**Problem 5.2.3 (GBG [1] problem)** Consider the cost (5.2) where $x$ is the state of the system (5.1) with an initial condition $x_0 \in \mathcal{N}(\bar{x}, Q_x)$, $\bar{x} \in \mathbb{R}^n$, $Q_x \in \mathbb{R}^{n\times n}$, $Q_x \geq 0$. Find an optimal controller $\tilde{\sigma} \in \Sigma_{\text{con}}$ such that

$$\mathbb{E}_{x_0} J(x_0, \tilde{\sigma}) \leq \mathbb{E}_{x_0} J(x_0, \sigma) \tag{5.4}$$

for all $\sigma \in \Sigma_{\text{con}}$.

Finding an optimal, dynamic feedback controller $\tilde{\sigma} \in \Sigma_{\text{con}}$ is a difficult task. A possible approach is to first estimate the state $x$ by the Kalman filter (see Appendix for more details) given by

$$x^*(t+1) = Ax^*(t) + Bu(t) + G(t)\left(y(t) - C_y x^*(t)\right) \tag{5.5}$$

with $x^*(0) = \bar{x}$. The *Kalman gain*, denoted by $G(t)$, is given by

$$G(t) = AQ(t)C_y^T \left(Q_\eta + C_y Q(t) C_y^T\right)^{-1}$$

[1] *The Glass Bead Game* abbreviated (see [68] for details)

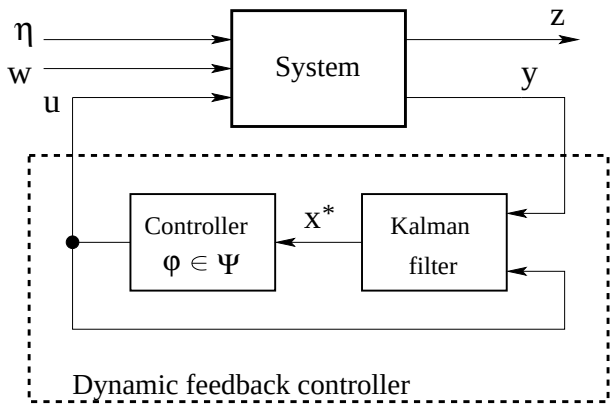


Figure 5.2: The system (5.1) is controlled by the "composite"controller: Kalman filter + static feedback

where $Q(t)$ is the estimation error covariance matrix that satisfies the following recursive relationship

$$Q(t+1) = AQ(t)A^T - AQ(t)C_y^T \left(Q_\eta + C_y Q(t) C_y^T\right)^{-1} C_y Q(t) A^T + E Q_w E^T$$

with $Q(0) = Q_x$.

Note that the plant data, together with the noise assumptions allow to compute $Q(t)$, $G(t)$ and hence (5.5). Then, $u$ is chosen so that $u(t)$ is a function of $x^*(t)$ i.e. the control input is computed by means of a *static feedback controller*

$$u(t) = \varphi\left(x^*(t)\right) \tag{5.6}$$

where $\varphi : \mathbb{R}^n \to \mathbf{U}$ is a continuous function with $\varphi(0) = 0$. We denote the set of all such static feedback controllers by $\Psi$.

By using the control input to the plant given by (5.6) with $\varphi \in \Psi$ where $x^*$ is the state of the Kalman filter (5.5), we actually assemble a strictly causal dynamic controller of the form (5.3). This control structure is shown in figure 5.2. The set of all strictly causal dynamic controllers obtained by the "composition"of the feedback (5.6) with $\varphi \in \Psi$ and the (fixed) Kalman filter (5.5) is denoted by $\Sigma_{\text{kf+sf}}$. Hence, in general

$$\Sigma_{\text{kf+sf}} \subseteq \Sigma_{\text{con}}.$$

Next, we consider the following optimization problem.

**Problem 5.2.4 (CDC [2] problem)** Consider the cost (5.2) where $x$ is the state of the system (5.1) with an initial condition $x_0 \in \mathcal{N}(\bar{x}, Q_x)$, $\bar{x} \in \mathbb{R}^n$, $Q_x \in \mathbb{R}^{n\times n}$, $Q_x \geq 0$. Find an optimal controller $\tilde{\sigma} \in \Sigma_{\text{kf+sf}}$ such that

$$\mathbb{E}_{x_0} J(x_0, \tilde{\sigma}) \leq \mathbb{E}_{x_0} J(x_0, \sigma) \tag{5.7}$$

for all $\sigma \in \Sigma_{\text{kf+sf}}$.

Suppose now that an optimal controller to the CDC problem, $\tilde{\sigma} \in \Sigma_{\text{kf+sf}}$, exists. Then, by construction, there exists $\tilde{\varphi} \in \Psi$ such that the control input produced by $\tilde{\sigma}$ can be represented as

$$u(t) = \tilde{\varphi}\left(x^*(t)\right) \tag{5.8}$$

where $x^*(t)$ is given by Kalman filter (5.5). The design consists of two separate tasks: a design of the Kalman filter and the task of finding an optimal, static feedback controller $\tilde{\varphi} \in \Psi$.

The design of the Kalman filter is straightforward in the problem setting that we consider. After fixing Kalman filter (5.5), the CDC problem amounts to finding $\tilde{\varphi} \in \Psi$ i.e. an optimal static state feedback in the composite controller shown in figure 5.2.

For the classical LQG problem, where the system is assumed to be unconstrained and the cost is assumed to be quadratic, it is well known that the "composite" controller that consists of the LQ optimal static state feedback and the Kalman filter is optimal in the set $\Sigma_{\text{con}}$. Thus, in this case, the optimal static state feedback can be found by considering a separate, static state feedback optimization problem. In general, the same can not be done for the CDC problem because of the saturated input and the possibly non quadratic cost function. To investigate in which cases an optimal $\tilde{\varphi} \in \Psi$ in the composite controller from figure 5.2 can be found as a solution of the static state feedback optimization problem we consider the *auxiliary system*

$$x^a(t+1) = Ax^a(t) + Bu(t) + G(t)\zeta(t) \tag{5.9}$$

where $\zeta$ is a normally distributed stochastic process with zero mean and a time dependent covariance matrix $Q(t)$. Note that with an initial condition $x^a(0)$ set to be equal to the initial condition of the Kalman filter (5.5) the stochastic properties of the state $x^a$ are equal to the stochastic properties of the estimated state $x^*$.

Consider the following cost function

$$J^a(\bar{x}^a, u) := \lim_{T\to\infty} \mathbb{E}\frac{1}{T}\sum_{t=0}^{T} j(x^a(t), u(t)) \tag{5.10}$$

subject to (5.9) with $x^a(0) = \bar{x}$. Here, $u$ is chosen so that $u(t)$ is a function of $x^a(t)$, i.e.

$$u(t) = \varphi\left(x^a(t)\right) \tag{5.11}$$

[2] *The composite dynamic controller* abbreviated

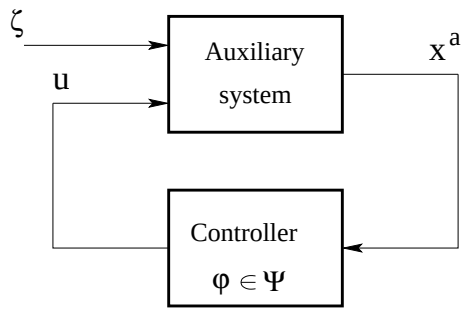


Figure 5.3: The auxiliary system (5.9) with the state feedback

where $\varphi$ is a controller in $\Psi$. Thus we consider static state feedback for the system (5.9). This control structure is shown in the figure 5.3.

Then, we consider the following optimization problem.

**Problem 5.2.5 (FSF [3] problem)** Consider the cost (5.10) where $x^a$ is the state of the auxiliary system (5.9) with an initial condition $x^a(0) = \bar{x}$ and the input $u$ given by (5.11) for some $\varphi \in \Psi$. Find an optimal controller $\varphi^* \in \Psi$ such that

$$J^a(\bar{x}^a, \varphi^*) \leq J^a(\bar{x}^a, \varphi) \tag{5.12}$$

for all $\varphi \in \Psi$.

The reason that the FSF problem has been introduced is that under certain conditions a feedback $\varphi^*$ optimal to the FSF problem, in composition with the Kalman filter as shown in figure 5.2, is optimal to the CDC problem. If this holds, a solution of the CDC problem can be found by solving two separate optimization problems: the optimization problem of finding an optimal state estimator (the solution to this problem is given by the Kalman filter (5.5)) and the FSF optimization problem. This is celebrated "separation principle" that is nowadays a classical topic in stochastic and optimization control theory. Moreover, a controller $\tilde{\sigma} \in \Sigma_{\text{kf+sf}}$ obtained by this separate design, optimal to the CDC problem, is an optimal controller for the GBG problem. In the next section we give a condition on the cost function that has to be satisfied so that this desirable equivalence between optimization problems GBG, CDC and FSF can be established.

The main advantage of the controller design represented by the FSF optimization problem is that it is equivalent to the optimization problem 4.2.4 except for the time varying covariance of the disturbance input. Therefore, the discussion about solvability

[3] *The fictitious state feedback* abbreviated

conditions that is presented in section 4.3 concerns the FSF optimization problem as well.

### 5.2.1 Kalman filter

Consider the system (5.1). It is well known that the optimal estimator of the state $x$ at time $t$ is the conditional expectation of the state with respect to the available information about the plant at time $t$. Because the controller (5.3) is strictly proper, the best we can do is to use the one-step-ahead optimal predictor of the state defined by

$$x^*(t) := \mathbb{E}\{x(t)|y(0), \cdots, y(t-1), u(0), \cdots u(t-1)\}$$

where $\mathbb{E}\{x(t)|(.)\}$ denotes the conditional expectation of $x(t)$ with respect to (.). It can be shown (see [9, 81]) that $x^*$ satisfies the following recursive relationship

$$x^*(t+1) = Ax^*(t) + Bu(t) + G(t)\big(y(t) - C_y x^*(t)\big) \tag{5.13}$$

where

$$G(t) = AQ(t)C_y^T \left(Q_\eta + C_y Q(t) C_y^T\right)^{-1}$$

and

$$Q(t+1) = AQ(t)A^T - AQ(t)C_y^T \left(Q_\eta + C_y Q(t) C_y^T\right)^{-1} C_y Q(t) A^T + E Q_w E^T \tag{5.14}$$

with

$$Q(0) = \mathbb{E}x(0)x(0)^T.$$

We will refer to the predictor (5.13) as the Kalman filter. The matrix $Q(t)$ is the estimation error covariance matrix i.e.

$$Q(t) = \mathbb{E}\big\{ \big(x(t) - x^*(t)\big) \big(x(t) - x^*(t)\big)^T \big\}. \tag{5.15}$$

The *innovation process* in the predictor (5.13) is defined as

$$\tau(t) := y(t) - C_y x^*(t). \tag{5.16}$$

The random vector $\tau(t)$is a normally distributed random vector with zero mean and the covariance given by

$$\mathbb{E}\big(\tau(t)\tau(t)^T\big) = C_y Q(t) C_y^T + Q_\eta. \tag{5.17}$$

Kalman filtering is nowadays a classical topic in stochastic control and estimation theory, for further details and modifications see for instance [3, 9, 63, 73]. Note that the Kalman gain $G(t)$ and the covariance matrix $Q(t)$ in the Kalman filter (5.13) are time dependent matrices. Therefore, it is necessary to perform computations in each

time step. It can be shown that under assumption 5.2.2, the error covariance matrix converges to a fixed matrix $Q$ as $t \to \infty$. The matrix $Q$ satisfies the following algebraic Riccati matrix equation

$$Q = AQA^T - AQC_y^T \left(Q_\eta + C_y Q C_y^T\right)^{-1} C_y Q A^T + E Q_w E^T. \tag{5.18}$$

Moreover, if the matrix pair $(A, E)$ is stabilizable, the matrix $Q$ does not depend on the initial value $Q(0)$. The Kalman gain converge to a fixed value $G$ given by

$$G = AQC_y^T \left(Q_\eta + C_y Q C_y^T\right)^{-1}. \tag{5.19}$$

The Kalman filter (5.13) with the limit value of the Kalman gain $G$ is called asymptotic or steady-state Kalman filter. Because the asymptotic Kalman filter is significantly less computationally demanding, it is used frequently in engineering applications. Note that for small $t$ the asymptotic Kalman filter is not optimal. However the convergence to the limit value $Q$ of the error covariance matrix is usually very fast.

## 5.3 Equivalence condition

The separation principle as described above relates CDC and FSF optimization problems. In this section we will show that this relationship depends on the class of functions from which we choose function $j$ in costs (5.2) and (5.10).

A choice of the function $j$ depends on the problem at hand. In chapter 3 we have considered systems with the constrained input and without constraints on the state. The cost function was a quadratic function. In the next theorem we show that for the case in which $j$ is a quadratic function, the input

$$u(t) = \varphi^* \left(x^*(t)\right)$$

with a controller $\varphi^*$ that solves the FSF problem also minimizes the cost (5.2) provided that $x^*(0) = x^a(0)$ i.e. under this condition the separation principle can be used to solve the CDC optimization problem.

**Theorem 5.3.1** *Let $\varphi^* \in \Psi$ be an optimal controller for the FSF optimization problem. Then*

$$u(t) = \varphi^* \left(x^*(t)\right) \tag{5.20}$$

*with $x^*$ obtained from* (5.5) *is an optimal controller for the CDC optimization problem if the function $j$ is a quadratic function.*

*Moreover, if the function $j$ is a quadratic function controller* (5.20) *is an optimal controller for the GBG optimization problem.*

**Proof:** Suppose that

$$j(x,u) = \|C_z x + D_z u\|^2 \qquad x \in \mathbb{R}^n \quad u \in \mathbf{U}.$$

Consider the following computation of the conditional expectation

$$\begin{aligned}\mathbb{E}\big\{ j(x(t),u(t)) \big| y(0),\cdots,y(t-1),u(0),\cdots,u(t-1)\big\} = \\ = \mathbb{E} j(x^a(t),u(t)) + \mathbb{E}\big\{ C_z(x(t)-x^a(t))(x(t)-x^a(t))^T C_z^T\big\}.\end{aligned}$$

Note that

$$\mathbb{E}\big\{ C_z(x(t)-x^a(t))(x(t)-x^a(t))^T C_z^T\big\} = \operatorname{trace} C_z Q(t) C_z^T$$

where $Q(t)$ is the estimation error covariance matrix (5.15).

Next, we rewrite the cost (5.2) in terms of the auxiliary state $x^a(t)$ instead of $x(t)$ as

$$J(x,u) = \lim_{T\to\infty} \mathbb{E}\frac{1}{T}\sum_{t=0}^{T} j(x^a(t),u(t)) + \lim_{T\to\infty}\frac{1}{T}\sum_{t=0}^{T} \operatorname{trace} C_z Q(t) C_z^T \tag{5.21}$$

where $x^a(0) = \bar{x}$. Equation (5.21) can be rewritten as

$$J(x,u) = J^a(x,u) + c \tag{5.22}$$

where $c$ is the constant term given by

$$c = \frac{1}{T}\sum_{t=0}^{T} \operatorname{trace} C_z Q(t) C_z^T$$

Finally, note that according to (5.22), the input $u$ that minimizes $J^a$ also minimizes $J$.
■

In chapter 4 we considered a more general class of systems with constraints on the state in addition to the constrained input. In that chapter we showed that the cost function needs to be extended with the constraint violation cost.

**Definition 5.3.2** The constraint violation cost is a finite valued convex function $h : \mathbb{R}^n \to \mathbb{R}_+$ with $h(x) = 0$ for all $x \in \mathbf{X}$.

For the general case with constraints on the state and the constrained input, the function $j$ in costs (5.2) and (5.10) is defined with

$$j\big(x,u\big) := g\big(x,u\big) + h(x) \quad x \in \mathbb{R}^n \quad u \in \mathbf{U} \tag{5.23}$$

where $g$ and $h$ are chosen so that $j$ is in the class of functions that have a so called "Polynomial - Exponential Growth" denoted by $\Theta(R)$ (see definition 4.3.1). This choice implies that the cost grows exponentially for large $\|x\|$ i.e

$$j\big(x, u\big) \sim e^{\|x\|_R^2} \qquad \text{as} \quad \|x\| \to \infty.$$

Note that

$$\mathbb{E}\, e^{\|x\|_R^2} = \mathbb{E}\, e^{(x^T Rx + x^{a^T} Rx^a - x^{a^T} Rx^a)} = \mathbb{E}\, e^{(x-x^a)^T R(x-x^a)}\ \mathbb{E}\, e^{\|x^a\|_R^2}$$

where $x$ is the state of the system (5.1) and $x^a$ is the state of the auxiliary system (5.9). Because of this, rewriting the cost (5.2) in terms of the auxiliary state $x^a$ instead of $x$ will not give a linear relationship between costs (5.2) and (5.10) as in (5.22) i.e. the input $u$, obtained by controller that solves the optimization problem 5.2.4 with the function $j$ given by (5.23) does not necessarily minimize $J$.

An alternative is to use the available information about the covariance of the state estimation error, obtained by Kalman filter. The considered cost function is of the form

$$J_e^a(x, u) := \lim_{T\to\infty} \mathbb{E}\frac{1}{T}\sum_{t=0}^{T} j_e(x^a(t), Q(t), u(t)). \tag{5.24}$$

The function $j_e$ has to be chosen so that an input that minimizes cost (5.24) also minimizes cost (5.2).

## 5.4 Model predictive controller by measurement feedback

Within the model predictive framework, an optimization problem is solved at each time instant $t$, $(t \in \mathbb{Z}_+)$ over an interval $I_t := \{t + k | k \in T\}$ where $T$ denotes the control horizon $T := \{0, \cdots, N\}$ and $N > 0$. The model of the plant that is used for the prediction and the optimization problem to be defined later, are time-invariant. Therefore, variables involved in the design of a predictive controller can be defined as functions of $k \in T$ rather than the functions of the current time, without loss of generality.

We assume that the optimization over the control horizon is performed in closed loop. Formally, we define the set of feedback control laws $\Pi$ where $\pi \in \Pi$ is a vector $(\pi_k)_{k=0}^N$ such that for any $k \in T$, the map $\pi_k : \mathbb{R}^n \to \mathbf{U}$ is continuous.

The dynamics of the plant (5.1) over the control horizon is then given by

$$\begin{aligned} x_N(k+1) &= Ax_N(k) + B\,\pi_k(\hat{x}_N(k)) + E\,\nu(k) \\ y_N(k) &= C_y x_N(k) + \upsilon(k) \end{aligned} \qquad k = 0, 1, 2 \cdots N \tag{5.25}$$

The disturbance $\nu$ and the measurement noise $\upsilon$ are mutually independent Gaussian white noise processes with $\nu(k) \in \mathcal{N}(0, Q_w)$ and $\upsilon(k) \in \mathcal{N}(0, Q_\eta)$. The initial condition for the recursion (5.25) is given by

$$x_N(0) \in \mathcal{N}(x_t^*, Q_t)$$

where $x_t^*$ and $P_t$ are the state estimate and the error covariance matrix obtained by the Kalman filter (5.13) at time $t$ i.e. $x_t^* = x^*(t)$ and $Q_t = Q(t)$.

In the problem 5.2.4 we assume that the input is a function of the predicted state according to (5.6). To make the prediction of the behaviour of the controlled plant as realistic as possible, we introduce the *predicted state* over the control horizon $\big(\hat{x}_N(k)\big)_{k=0}^{N+1}$ that is generated by the following one-step-ahead recursive predictor

$$\hat{x}_N(k+1) = A\hat{x}_N(k) + B\,\pi_k(\hat{x}_N(k)) + G(k)\big\{y_N(k) - C_y\hat{x}_N(k)\big\} \qquad k = 0, 1, 2 \cdots N \tag{5.26}$$

where

$$G(k) := AP(k)C_y^T\left(Q_\eta + C_yP(k)C_y^T\right)^{-1}$$

and with $\hat{x}_N(0) = x_t^*$. The matrix $P(k)$ is the covariance matrix of the estimation error. It can be showed that the covariance matrix of the estimation error satisfies the following Riccati equation

$$P(k+1) = AP(k)A^T - AP(k)C_y^T(Q_w + C_yP(k)C_y^T)^{-1}C_yP(k)A^T + EQ_wE^T \tag{5.27}$$

with $P(0) = Q_t$. The difference between the Kalman filter (5.5) and the predictor (5.26) is that in the Kalman filter innovation is performed based on the measurement $y(t)$. The measurement is not available over the control horizon because we predict the "future"of the plant. Therefore we consider the measurement $y_N$ as a stochastic process and define the *innovation process* in the predictor (5.26) as

$$\omega(k) := y_N(k) - C_y\hat{x}_N(k). \tag{5.28}$$

The random vector $\omega(k)$ is a normally distributed random vector with zero mean and the covariance given by

$$\mathbb{E}\left(\omega(k)\omega(k)^T\right) = C_yP(k)C_y^T + Q_\eta. \tag{5.29}$$

The cost acquired over the control horizon is defined by:

$$J(x,\pi) := \mathbb{E}\Big\{\sum_{k\in T}\big\{g\big(C_z\hat{x}_N(k) + D_z\,\pi_k(\hat{x}_N(k))\big) + h(\hat{x}_N(k))\big\} + \|\hat{x}_N(N+1)\|_Q^2\Big\} \tag{5.30}$$

subject to $\hat{x}_N(0) = x$. The expression $\|x\|_Q^2 := \langle x, Qx\rangle$ is called an end point penalty with $Q \in \mathbb{R}^{n\times n}$ a positive definite, symmetric matrix.

The optimization problem to be solved is stated next:

**Problem 5.4.1** Find a vector of optimal feedback mappings $\pi^* \in \Pi$ such that

$$J(x, \pi^*) \leq J(x, \pi)$$

for all $\pi \in \Pi$ and for all $x \in \mathbb{R}^n$. In addition, determine the *optimal cost* given by:

$$V(x) := \inf_{\pi} J(x, \pi). \tag{5.31}$$

If the vector of optimal feedback mappings $\pi^*$ exists, then $V(x) = J(x, \pi^*)$.

Suppose that the solution of the optimization problem 5.4.1, $\pi^*$, exists. According to the receding horizon implementation, only the first element of $\pi^*$ is significant in the receding horizon implementation. The model predictive controller in our setting is given by

$$u(t) = \pi_0^*(x^*(t)) \qquad t \in \mathbb{Z}_+. \tag{5.32}$$

Note that the structure of the predictor given by equation (5.26) is the same as the structure of systems that are used for prediction in the chapters 3 and 4. The difference is that in (5.26) stochastic is determined by innovation process (5.28) instead of the stochastic properties of the disturbance. Because the structure is the same, the results from chapter 4 can be used to derive an algorithm for model predictive controller that solves optimization problem 5.4.1.

**Theorem 5.4.2** *Consider the optimization problem 5.4.1. Under assumptions 5.2.1 and the constraint violation cost as given by definition 5.3.2, the optimal cost* (5.31) *and the associate vector of feedback mappings $\pi$ can be obtained recursively as follows:*

$$V_s(x) := \inf_{u \in \mathbf{U}} \left\{ g(C_z x + D_z u) + h(x) + \mathbb{E}_{\omega(s)} V_{s+1}\left(Ax + Bu + K(s)\omega(s)\right) \right\} \tag{5.33}$$

*with an initial condition:*

$$V_{N+1}(x) := \|x\|_Q^2$$

*that has to be solved backwards from $s = N$ to $s = 0$. The expression $\mathbb{E}_{(\cdot)}$ denotes conditional expectation with respect to $(\cdot)$.*

**Proof:** For the proof we refer to the proof of the theorem 4.4.2 ■

An analytical computation of the expectation in (5.33) is generally difficult task and we use the empirical mean as an alternative.

**Definition 5.4.3** Assume a set $\Theta$ and a probability measure $P$ on $\Theta$ are given. Let $f : \Theta \to \Omega$ be a function measurable with respect to $P$ where $\Omega$ is an interval on $\mathbb{R}^n$ (possibly equal to $\mathbb{R}^n$). Suppose that we draw $m$ independent, identically distributed

(i.i.d) samples $\vartheta = \{\theta_1, \cdots, \theta_m\}$ from $\Theta$ in accordance with $P$. The empirical mean of the function $f$ is given by:

$$\hat{\mathbb{E}} f := \frac{1}{m} \sum_{j=1}^{m} f(\theta_j). \tag{5.34}$$

The cost computed via an empirical mean is given with:

$$\hat{J}(x, \pi) := \hat{\mathbb{E}}\Big\{ \sum_{k \in T} \big\{ g\big(C_z \hat{x}_N(k) + D_z \pi_k(\hat{x}_N(k))\big) + h(\hat{x}_N(k)) \big\} + \|\hat{x}_N(N+1)\|_Q^2 \Big\}. \tag{5.35}$$

The optimization problem 5.4.1 is replaced by the optimization in which we seek for a minimum of the empirical cost (5.35) instead of the cost (5.30). The algorithm is based on the following theorem (see [12]).

**Theorem 5.4.4** *Consider the optimization problem 5.4.1 in which the empirical cost* (5.35) *is minimized instead of* (5.30). *Under assumptions 5.2.1 and the constraint violation cost as defined in definition 5.3.2, the empirical optimal cost:*

$$\hat{V}(x) := \inf_{\pi} \hat{J}(x, \pi) \tag{5.36}$$

*and the associate vector of feedback mappings $\pi$ can be obtained recursively as follows:*

$$\hat{V}_s(x) := \inf_{u \in \mathbf{U}} \big\{ g(C_z x + D_z u) + h(x) + \hat{\mathbb{E}}_\nu \hat{V}_{s+1}(Ax + Bu + K(s)\omega(s)) \big\} \tag{5.37}$$

*with an initial condition:*

$$\hat{V}_{N+1}(x) := \|x\|_Q^2$$

*that has to be solved backwards from $s = N$ to $s = 0$. The expression $\hat{\mathbb{E}}_{(\cdot)}$ denotes empirical conditional expectation with respect to $(\cdot)$. The empirical optimal cost* (5.36) *is obtained from* (5.37) *by $\hat{V}(x) = \hat{V}_0(x)$.*

To compute the empirical mean (5.37), a number of realizations of the innovation process (5.28) is needed. The samples are chosen randomly, according to the distribution of the innovation process $\omega$. This method is called the Monte Carlo simulation. In the following we explain how to perform the sampling of the innovation process.

Firstly, we take $\kappa$ samples of $\omega(0)$ at $s = 0$. The random vector $\omega(0)$ is normally distributed with zero mean and the covariance matrix given by (5.29) for $k = 0$. Given $\kappa$ samples of $\omega(0)$ there are $\kappa$ possible states $\hat{x}_N(1)$. For each one of these possible futures we generate $\kappa$ samples of the disturbance $\omega(1)$ which establishes $\kappa^2$ possible future states $\hat{x}_N(2)$. By proceeding in this way, we obtain $\kappa^N$ samples of the innovation process $\omega$. The number of samples of $\omega$ grows exponentially with the

horizon. The sampling as described is required for a good estimate of the optimal cost to go $V_s$ (5.33). One might conjecture that we do not need this because a very accurate estimate of $V_s$ is not required. Actually, only a good estimate of $V_0$ is required, because it determines $\pi_0$. However, we have no proof that a restricted set of samples still yields a correct result with a high probability.

Next, we present the algorithm by which the optimization problem 5.4.1 can be solved by an use of the empirical mean computed via Monte Carlo simulations.

With the disturbance sampled as described at each $s$, $s \in \{0, \cdots, N-1\}$ there are $\kappa^s$ possible states denoted by $\hat{x}_N^i(s), i \in \{1, \cdots, \kappa^s\}$. Denote a feedback controller that achieves a finite cost (5.35) for the problem at hand with $\varphi^0$. Then, at each time $t = 1, 2, 3 \cdots$ the following algorithm is executed.

---

**Algorithm 5.1**

---

**Step 1: Initialization**

Compute the optimal state estimate $x^*(t)$ and the estimated covariance matrix $P(t)$. Draw $\kappa$ samples of $\omega(0) \in \mathcal{N}(0, C_y P(t) C_y^T + Q_\eta)$. Set $\hat{u}_i(s) = \varphi^0(x_N^i(s))$ for $s = 0, 1, \ldots, N$, $i = 1, \ldots, N$. Draw $\kappa^N$ samples for $\omega$. Set $V = \infty$. Set accuracy parameter $\varepsilon$. Set $s = N$.

**Step 2: Compute cost "to go"**

Determine a new $\hat{u}_i(s)$ by solving (5.37) for each $x_N^i(s)$, $i = 1, \ldots \kappa^s$. Compute $\hat{V}_s(x_N^i(s))$ for each $i$. If $s = 0$ go to **step 4**, otherwise set $s = s - 1$ and go to **step 2**.

**Step 4: Exit condition**

If $|\hat{V}_0(x_N^1(0)) - V| < \varepsilon$ stop. Otherwise: set $V = \hat{V}_0(x_N^1(0))$, set $s = N$ and go to **step 2**.

---

The remaining issue is the convergence of the solution obtained by algorithm 5.1. Note again that the structure of the problem that we consider here is the same as the structure of the problem considered in chapter 4 with the difference that the stochastic of the prediction is determined by stpchastic properties of the innovation process (5.28) instead of the disturbance as in the previous chapter. The convergence result presented in theorem 4.5.5 can be applied to algorithm 5.1 with trivial modifications.

The computations of the covariance and the Monte Carlo simulation procedure in algorithm 5.1 can be significantly simplified by exploiting asymptotic behaviour of the Riccati equation (5.27). It can be shown that the solution of the Riccati equation

converges to the matrix $P$ given by

$$P = APA^T - APC_y^T \left(Q_\eta + C_y PC_y^T\right)^{-1} C_y PA^T + EQ_w E^T. \tag{5.38}$$

as $k \to \infty$, under assumption 5.2.2. Moreover, if the matrix pair $(A, E)$ is stabilizable, the matrix $P$ does not depend on the initial value $Q_t$. The gain in (5.26) converge to a fixed value $K$ given by

$$K = APC_y^T \left(Q_\eta + C_y PC_y^T\right)^{-1}. \tag{5.39}$$

Regarding the fact that the covariance (5.38) and the gain (5.39) are asymptotic quantities, it seems overly optimistic to assume that the covariance and the gain in the estimation structure (5.26) are equal to (5.38) and (5.39) given the finite control horizon $N$. Note however, that the convergence of the covariance defined with (5.27) is in most cases very fast. Moreover, the initial condition for the Riccati (5.27) equation is the covariance matrix of the state estimate obtained from the Kalman filter (5.13) with the covariance given by the Riccati equation (5.14) that also converge to an asymptotic value.

## 5.5 Numerical Examples

### 5.5.1 Stochastic system with constrained input

We consider the plant with the model of the form (5.1) with:

$$A = \begin{pmatrix} 1.1269 & -0.4940 & 0.1129 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -0.3832 \\ 0.5919 \\ 0.5191 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad C_z = \begin{pmatrix} 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0.7 \end{pmatrix} \quad D_z = \begin{pmatrix} 0.33 & 0 & 0 & 0 \end{pmatrix}$$

$$C_y = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

The disturbance and the measurement noise are mutually independent a normally distributed random variables with zero mean and variance 0.4 and 0.2 respectively:

$$w(t) \in \mathcal{N}(0, 0.4) \quad \text{and} \quad \eta(k) \in \mathcal{N}(0, 0.2).$$

The input to the plant $u$ is assumed to be constrained for all $t \in \mathbb{Z}_+$ as

$$-0.2 \leq u(t) \leq 0.2.$$

The state $x$ is parameterized as:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

It is assumed that the system has an initial state:

$$x(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

To steer the system to the origin, we design the model predictive controller based on algorithm 5.1. We use predictor (5.26) with the asymptotic gain $K$ that can be easily computed as

$$K = \begin{pmatrix} 0.3204 \\ 0.5807 \\ 0.8670 \end{pmatrix}.$$

The innovation process $\omega$ defined in (5.28) is a normally distributed stochastic process with the zero mean and the covariance (5.29) that is computed as

$$Q_\omega = 1.2165.$$

Next, we set the length of the control horizon $N = 10$. For the computation of the empirical mean by Monte Carlo simulation we sample the innovation process according to its distribution. We take 10 samples at the first time instant and 5 samples at the second time instant in the control horizon. In this way, we obtain 50 samples of the innovation process over the control horizon. The controller is then obtained by algorithm 5.1. The input to the system at time $t \in \mathbb{Z}_+$ is then the first control obtained from the vector of controls obtained by algorithm 5.1 at time $t$.

To access the performance of the stochastic predictive controller we perform 100 simulations. Each one of them is performed with different realization of the disturbance $w$ and the measurement noise $\eta$. The resulting mean of the state trajectories are plotted on the figure 5.4 and the variance on the figure 5.5.

To compare the performance of the stochastic predictive controller we perform simulations in the same setting but with a standard predictive controller. The standard model predictive controller is designed with assumption that the innovation process takes its mean value over the control horizon i.e. we assume that the $\omega(k) = 0$ for all $k \in N$. The controller is then obtained by executing algorithm 5.1. Note that without sampling of the disturbance the optimization performed by algorithm 5.1 is equivalent with the optimization in the open loop i.e. standard approach to the optimization in the model predictive control context. The mean of the obtained trajectories are plotted on figure 5.4 and the variance the figure 5.5. Note that the mean of the trajectories for both stochastic and standard model predictive controller show similar behaviour. The improvement of the performance can be seen when the variance of the trajectories are compared.

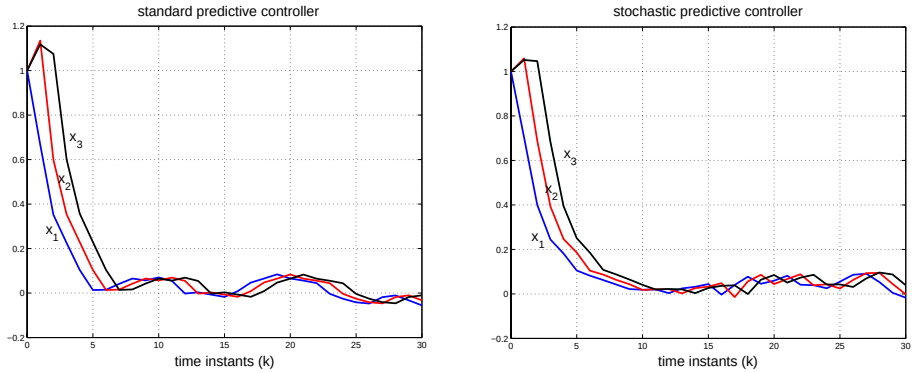


Figure 5.4: The mean of the state trajectories

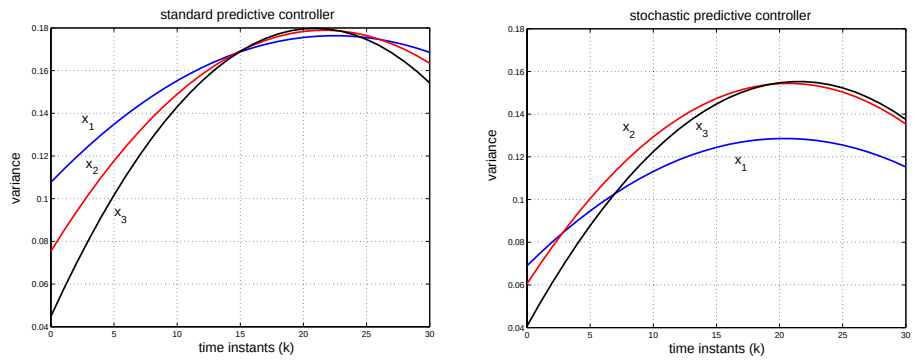


Figure 5.5: The variance of the state trajectories

### 5.5.2 Stochastic system with constrained input and constraint on the state

In this section we present an example in which we consider a "double integrator" system of the form:

$$
\begin{aligned}
x(t+1) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w(t) \\
y(t) &= \begin{pmatrix} 0 & 1 \end{pmatrix} x(t) + \eta(t) \\
z(t) &= \begin{pmatrix} 0 & 0 \\ 0.7 & 0 \\ 0 & 0.7 \end{pmatrix} x(t) + \begin{pmatrix} 0.33 \\ 0 \\ 0 \end{pmatrix} u(t)
\end{aligned}
\tag{5.40}
$$

The input is constrained as:

$$-0.5 \leq u \leq 0.5 \quad u \in \mathbb{R}.$$

The disturbance and the measurement noise are mutually independent normally distributed random variables with zero mean and variance 0.4 and 0.2 respectively:

$$w(k) \in \mathcal{N}(0, 0.4) \quad \text{and} \quad \eta(k) \in \mathcal{N}(0, 0.2).$$

The state $x$ is parameterized as:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and we impose a constraint on the state:

$$x_2 \geq 0. \tag{5.41}$$

It is assumed that the system has an initial state:

$$x(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix}.$$

The task is to steer the system (5.40), subject to the stochastic disturbance, from the initial state to the origin with the constrained input while respecting constraint on the state. Note that the "double integrator"system can be physically interpreted as a system that describes the unit mass under influence of the force. The force is the input to the system and the measurement output is the position of the mass. Initially the mass is at rest at the position of 10 units. The position is the only measured state and the constraint (5.41) is equivalent to

$$y(k) \geq 0 \quad \forall k \in \mathbb{Z}_+.$$

The first task is to design the Kalman filter for the double integrator (5.40). This task is straightforward, given the system and covariances of disturbances and the measurement noise. The remaining task is to approximate the controller (5.6). To fulfill this task we design the model predictive controller based on algorithm 5.1. We use predictor (5.26) with the asymptotic gain $K$ that can be easily computed as

$$K = \begin{pmatrix} 0.5857 \\ 1.4142 \end{pmatrix}.$$

The innovation process $\omega$ defined in (5.28) is a normally distributed stochastic process with the zero mean and the covariance (5.29) that is computed as

$$Q_\omega = 1.1657.$$

We choose:

$$g(z) = \|z\|^2 \quad z \in \mathbb{R}^3 \tag{5.42}$$

and

$$h(x) = \begin{cases} 0 & \text{if } x_2 \geq 0 \\ e^{4.5x_2^2} - 1 & \text{if } x_2 < 0 \end{cases} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \tag{5.43}$$

With functions $g$ and $h$ as above, the controller minimizes the expectation of the quadratic cost when the state is away from the constraint $x_2 > 0$. When the state is near or on the boundary of the constraint the exponential constraint violation cost $h$ dominates and the main objective of the controller is to avoid a constraint violation. The constraint violation cost $h$ makes overall cost to be in $\Theta(R)$ class of functions (see definition 4.3.1), with

$$R = \begin{pmatrix} 0 & 0 \\ 0 & 4.5 \end{pmatrix}.$$

It can be easily verified that this choice of $R$ satisfies solvability condition of theorem 4.3.5.

Next, we set the length of the control horizon $N = 10$. For the computation of the empirical mean by Monte Carlo simulation we sample the innovation process according to its distribution. We take 10 samples at the first time instant and 5 samples at the second time instant in the control horizon. In this way, we obtain 50 samples of the innovation process over the control horizon. The controller is then obtained by algorithm 5.1. The input to the double integrator (5.40) at time $t \in \mathbb{Z}_+$ is then the first control obtained from the vector of controls obtained by algorithm 5.1 at time $t$. In the next time instant, the new measurement is taken and the state estimate obtained by the Kalman filter (5.13). Algorithm 5.1 is then executed again, with the same number of samples but with new initial values, to determine the input to the double integrator.

To access the performance of the stochastic predictive controller we perform 100 simulations. Each one of them is performed with different realization of the disturbance $w$ and the measurement noise $\eta$. The resulting measurement output trajectories $y$ are plotted on the figure 5.6.

To compare the performance of the stochastic predictive controller we perform the simulations in the same setting but with the standard predictive controller. The standard model predictive controller is designed with assumption that the innovation process takes its mean value over the control horizon i.e. we assume that the $\omega(k) = 0$ for all $k \in N$. The controller is then obtained by executing algorithm 5.1. Note that without sampling of the disturbance the optimization performed by algorithm 5.1 is equivalent with the optimization in the open loop i.e. standard approach to the optimization in the model predictive control context. The results of 100 simulations are plotted on figure 5.7.

Note that controllers show very different performance. Standard standard MPC controller is not able to realistically predict a possibility of the constraint violation, because of the assumption that the innovation of the state estimation in the “next time

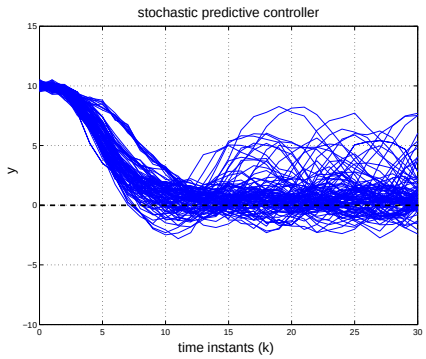


Figure 5.6: "Double integrator"is controlled by **stochastic predictive controller**

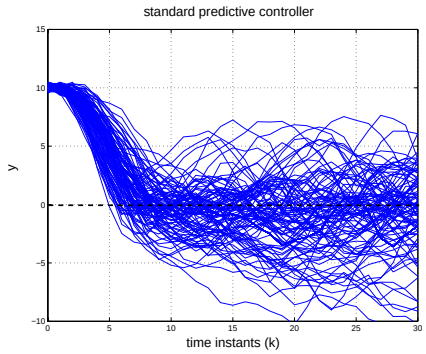


Figure 5.7: “Double integrator” is controlled by **standard predictive controller**

step” over the control horizon is equal to the mean value of the innovation process, in this case zero. A probability that $\omega$ will be larger than zero is high so for a large number of trajectories prediction is not realistic. On the contrary, stochastic MPC controller computes an optimal map from the state to the input for a number of points in the state space. Points are determined with the stochastic sampling of the innovation process and therefore there is a large probability that the optimal map for the predicted states is computed in the region in which the estimated state of the system will be. This leads to the more realistic “prediction” and the control strategy that respects the state constraints better.

On figure 5.8 we plot the mean of obtained trajectories. Note that the mean trajectory obtained by standard model predictive controller is a stable trajectory, but it converges to the point that is in the region $y < 0$, i.e. does not respect the constraint on the output.

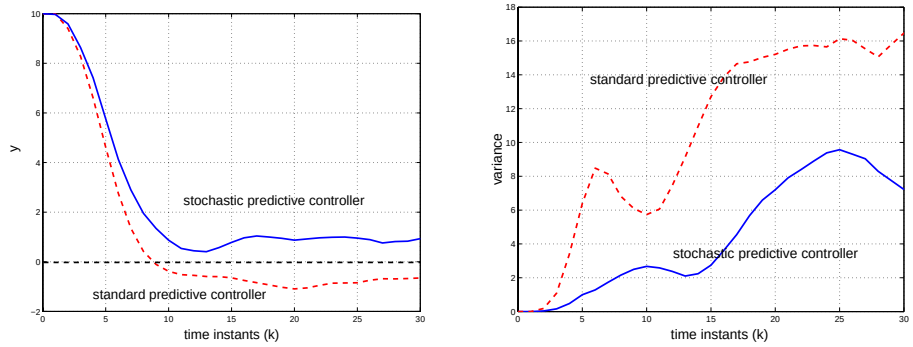


Figure 5.8: The mean and the variance of the trajectories from figures 5.7 and 5.6

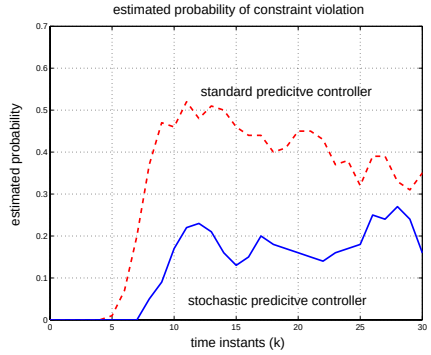


Figure 5.9: Probability of the constraint violation

In contrary, the mean trajectory obtained by stochastic predictive controller converges to the point in the region $y \geq 0$. The point is larger than the set point. Stochastic model predictive controller is designed to minimize the probability of the constraint violation, which is the reason for an increased set point. Finally, on the figure 5.9 we show the estimated probability of the constraint violation for both controllers.

## 5.6 Conclusion

In this chapter we consider optimal control of linear, constrained stochastic systems via measurement feedback. We chose a controller from a set of strictly proper dynamic controllers. The controller has three main tasks: to render the closed loop system stable, to control the system so that constrains on the state are respected as much

as possible and to minimize the performance measure when states are away from the constraint boundary. Since the state is not available for the measurement it is necessary to design a state estimator. The estimation has to be performed optimally, in the sense that the estimation error should have the minimum variance. This estimator is well known Kalman filter. A static feedback controller is than used to determine the input to the system, based on the estimated state.

We show how this controller can be designed within the model predictive control framework. To make prediction in the model predictive controller as realistic as possible, it is necessary to include the fact that we use the estimated state for the feedback and not the true state of the system. Therefore, it is necessary to include the estimation structure in the prediction. A difficulty is that there is no measurement available over the control horizon. The innovation part of the Kalman filtering algorithm is considered as a stochastic process. This fact is taken into an account in optimization that is performed in closed loop.

Finally, we present an example in which we use model predictive controller developed in this chapter on the double integrator system. The simulation results show improved performance compared to the standard model predictive controller even for the relatively small number of samples.

# 6

# *Concluding remarks*

Objectives of the final chapter in the thesis are to present a concise summary of contributions that have been made in the thesis and to give a starting point for dealing with open problems and topics that deserve to attract research attention in the future.

## 6.1 Summary of contributions

The main contributions in this thesis have been made with regard to the model predictive control of stochastic systems that are subject to constraints. Model predictive controllers that have been developed in the previous chapters are aimed to solve a more general problem of optimal control of stochastic, constrained systems. So far, there is no satisfactory methodology proposed in the available literature for dealing with such problems.

### 6.1.1 Model predictive control of constrained, stochastic systems

Stochastic disturbances can not be successfully rejected by standard model predictive control algorithms that are based on the optimization in open loop. In this thesis, we develop a novel approach to the model predictive control of such systems, that is based on the optimization in closed loop over the control horizon and stochastic sampling of the disturbance.

In chapter 3 we consider a case with a stochastic system that is subject to the constrained input. Model predictive controller developed in this chapter is based on the stochastic model of the plant i.e. the stochastic disturbance is taken into account in the prediction. As a consequence, the predicted state is stochastic. The cost function to be minimized by the model predictive controller is a quadratic function and optimization is assumed to be in closed loop i.e. it is a sequence of optimal feedback laws that have to be computed over the control horizon, not a sequence of optimal inputs as in the standard, open loop formulations of the model predictive control.

Finding a sequence of optimal feedback laws is a difficult problem to solve, not just because it is infinite dimensional, but because we do not have a characterization of

the class of optimal feedback laws. Result given in theorem 3.4.1 removes necessity of the characterization of the optimal class of feedback laws, and lemma 3.4.3 shows that a new optimization is a convex optimization problem. The optimization problem is still infinite dimensional, however. A finite dimensionality can be achieved through "quantization"of the state space, an approach commonly known as gridding. Since the state is stochastic and the goal is to compute the empirical mean a grid that is based on the stochastic sampling is more efficient than the grid that is based on, for example, linear gridding. Stochastic sampling is the base of the algorithms that are known in the literature as randomized algorithms. This form the basis of the algorithms 3.1 and 3.2 for model predictive control of stochastic systems with constrained inputs.

Simulation experiments have shown that even with the small number of samples algorithms 3.1 and 3.2 perform better than standard model predictive control algorithms, based on the optimization in open loop. The difference in the performance is in some cases significant, as in the ill-conditioned plant example presented in section 3.6.2. The price is significantly larger computational load, compared to the standard model predictive control algorithms that are based on the optimization in open loop.

A natural extension of the approach presented in chapter 3 is to consider the constraint on the state in addition. A model predictive control algorithm for dealing with this more general case is presented in chapter 4. To deal with the constraint on the state, we add the constraint violation cost (see definition 4.2.3 and assumption 4.2.5). The basic idea behind the constraint violation cost is to penalize the probability of the constraint violation. The model predictive controller from chapter 4 is based on the optimization problem 4.4.1. The algorithm for the model predictive controller presented in this chapter is based on the theorem 4.4.2. Important result is given in the theorem 4.4.3 where we show that the optimization problem based on the theorem 4.4.2 is in fact a convex optimization problem. As in the case with only input constraints, a finite dimensionality of the optimization problem is achieved through stochastic sampling of the disturbance. Simulation examples show a significant improvement of the performance when the model predictive controller based on the algorithm 4.1 is applied instead of the standard model predictive controller. Note that the performance in this case is mainly measured with respect to the ability of a controller to respect the constraint on the state, when the plant is subject to stochastic disturbance. The standard model predictive controller does not exploit all available informations about the stochastic disturbance and therefore it is not able to compete with the controller based on the algorithm 4.1 even for a small number of disturbance samples.

Finally, in chapter 5 we design a model predictive controller within the framework presented in chapters 3 and 4 but without assumption that the state of the plant is available for feedback. Over the control horizon, we use the estimated state for the prediction. In the standard model predictive control the model of the plant is used for prediction and the estimated state is used as an initial state for the prediction. In this way the prediction structure reflects better the way in which the plant is actually controlled: by using estimated state as the true state of the system. The difficulty

with the prediction structure proposed in chapter 5 is that the measurement output is not available in prediction and the estimated state of the model is seen as a stochastic variable. The algorithm (5.1) is different from the algorithms presented in chapters 3 and 4 in the nature of the sampling: instead of the disturbances as in the state feedback case we sample the innovation process of the state estimator.

### 6.1.2 A novel problem formulation for the optimal control of constrained, stochastic systems

Today, stochastic control theory offers a fairly completed treatment for the various control problems that deal with linear, unconstrained systems subject to a stochastic disturbance. When one is faced with the control problems that involve stochastic systems and constraints, there is a very limited number of techniques that are offered in the available literature. In this thesis we propose a novel problem setup for the optimal control of linear, stochastic, constrained systems. This problem setup is given in chapter 4, section 4.2. The problem is posed as an optimal control problem, where a controller has to be found so that the cost 4.7 is minimized. The cost consists of two performance measures. The first one of them, the constraint violation cost, see definition 4.2.3 and assumption 4.2.5, measures a probability of constraint violation, and when this probability is high this measure dominates in the cost. An optimal controller in this case minimizes the probability of constraint violation as a priority. When the (stochastic) state is such that probability of constraint violation is not high, the second measure dominates the cost and the optimal controller will control the plant according to the desired control strategy. The resulting optimization problem is formally defined as problem 4.2.4.

The first question that is posed with regard to the optimization problem 4.2.4 is the question of its solvability. The class of feedbacks that solve the optimization problem 4.2.4 is formally defined in definition 4.3.2. Feedbacks in this class ensure that the expectation of the performance measure with an exponential growth in the state is finite. The limitation on the exponential growth of the performance measure is imposed by the covariance of the disturbance. This is a fundamental limitation when one deals with the control of constrained stochastic systems i.e. it is not possible to achieve an arbitrary small probability of the constraint violation.

Condition (4.14) is not easy to verify in the general setting. The case in which all inputs to the plant are constrained leads to the significant simplification of the condition. In this, from application point of view very important case, the solvability condition is given by a very simple relationship

$$\left(A^t E\right)^T R\left(A^t E\right) - \frac{1}{4} Q_w^{-1} < 0$$

for all $t = 0, 1, 2 \cdots$. This condition relates the growth of the constraint violation cost and the covariance matrix of the Gaussian disturbance.

## 6.2 Outline of topics for further research

### 6.2.1 Optimal control of constrained, stochastic systems

As already mentioned, when the stochastic system is constrained, there is not much that has been written in the available literature. The reason is that constraints make the system nonlinear and analysis becomes difficult. In the model predictive control literature, a stochastic disturbance has been included in the problem setup from the early proposals (see [40–43], for instance) so model predictive control can be seen as one of the rarely available techniques for dealing with constrained, stochastic systems. As shown in a couple of occasions in this thesis, standard model predictive controllers are not suitable for dealing with stochastic, constrained systems, however. Optimization in closed loop is essential but then the resulting optimization problem is difficult to solve.

Model predictive control is just a technique that can be applied. There is obvious necessity for an analysis of constrained stochastic systems in the general sense i.e. independently of the technique that is used for the design of the controller. That is the only way to discover fundamental limitations that a stochastic disturbance is posing to the control of constrained systems. In this thesis, we have made first steps in this direction. In chapter 4, section 4.2 we present a problem setup that consists of a linear state space model of the plant (4.1) with a Gaussian white noise disturbance, a feedback controller (4.34), cost function (4.7) and the optimization problem 4.2.4. This problem setup is general enough to capture a variety of possible practical problems but simple enough to admit a mathematical analysis. Note that it is possible to "penalize"the probability of constraint violation directly in the cost, which is intuitively more natural approach, but that would make mathematical analysis very difficult.

Solvability conditions presented in section 4.3 are derived for the case when all inputs are constrained. This is very important case from the application point of view, but the question of the general case, when the input is partially constrained, is still open and it is a topic for further research.

Another issues are existence and uniqueness of solution in the state feedback case, presented in chapters 3 and 4, and the measurement feedback case presented in chapters 5.

### 6.2.2 Simplification and extensions to other classes of systems

Convergence results for the model predictive control algorithms presented in this thesis are valid when the number of samples is large. Simulation studies, on the other hand, show that even with the small number of samples significant improvements over the standard model predictive control algorithms can be achieved. An interesting topic for further research would be to investigate the convergence properties of

the algorithms with regard to the number of samples of the stochastic disturbance, in the state feedback case, or the number of samples of the innovation process in the measurement feedback case.

A significant simplification of model predictive controllers by randomized algorithms can be achieved by an "a priori" parametrization of the feedback law over the control horizon. This idea is outlined in section 3.5 where we assume that the feedback over the control horizon is of the form of a linear feedback with saturation. The algorithm derived in section 3.5 is significantly less computationally demanding than the original algorithm from section 3.1 but the price is a loss of the performance. The resulting optimization problem is a non-convex optimization problem which is an additional difficulty.

Finally, future research directions for the model predictive control by randomized algorithms are also in the extension of the class of the systems to which it is applied. Application to the time-varying systems, important from the application point of view, can be done almost straightforward. A more difficult but interesting topic for further research is application of the methodology proposed in this thesis to the various classes of hybrid systems.

### 6.2.3 Implementation of model predictive control by randomized algorithms on real-world control problems

A difference in the performance between standard model predictive control and model predictive control algorithms presented in this thesis is small when the disturbance acting on the plant to be controlled is not significant. As simulation results presented in chapters 3, 4 and 5 show, we can expect a significant performance improvement for control problems in which constraints plays an important role and in which the plant to be controlled is subject to a significant level of stochastic disturbances. This gives a general framework in which we can look for real-world control problems that can benefit from the algorithms presented in this thesis.

An application of the theoretical concept to the real-world control problem is never straightforward. Model predictive control technique presented in this thesis gives a number of possibilities to "tune" the optimization problem for the practical application in hand. Therefore, implementation of stochastic model predictive control techniques on real-world control problems is an important and broad topic for further research efforts. In this short subsection we outline the basic "tuning knobs"of the methodology presented in this thesis for a practically oriented researcher.

The first set of parameters that have to be chosen is well known from the standard model predictive control setting. These parameters are the length of the control horizon, a choice of the cost function and the matrix "weight" in the end point penalty. Numerical examples presented in chapters 3, 4 and 5 show that parameters obtained from the well designed model predictive controller are valuable starting points for

choosing the length of the control horizon, the cost function and the end point penalty in the stochastic model predictive control setting.

The second set of parameters that have to be chosen are specific for the model predictive control techniques presented in this thesis. For example, a number of samples of the stochastic disturbance in the randomized algorithm. The larger is the number of samples the larger is the accuracy of the algorithm. A large number of samples gives a computationally demanding algorithm. Therefore, a designer is faced with the tradeoff between a computational load and the performance of the algorithm. Note that computational load does not depend only on the number of samples but also on the length of the control horizon and the number of states of the system. This issues are important topics for further research by itself, as outlined in section 6.2.2.

A constraint violation cost (see definition 4.2.3) is another choice that has to be made by the designer. A rate of the exponential growth in the constraint violation cost determines how well constraints on the state are respected despite the presence of the stochastic disturbance. In the case when the set point is close to the constraint boundary (which is very often the case in applications) it is possible that the constraint violation cost will influence the optimum that is set by the performance measure (4.4). The advice to the designer is to look at the cost (4.7) and to tailor the growth of the constraint violation cost and the constraint set in order to achieve a desired behaviour.

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# *Summary*

The main topic of this thesis is control of dynamic systems that are subject to stochastic disturbances and constraints on the input and the state. The main motivation for dealing with control of such systems is that there is no method available that adequately deals with this problem, despite the fact that stochastic, constrained systems are often encountered in real world problems. For example, in process industry the margins of physical quantities such as temperature, pressure, concentration, velocity and position can be expressed as amplitude constraints in a natural way. Such constraints are usually persistent in that suitable control actions need to be implemented that respect these constraints irrespective of the presence of uncontrolled disturbances that effect the system.

Goals of the thesis are to

1. Formulate a mathematical problem for the synthesis of a controller that will achieve desired performance of the controlled system. More precisely, to minimize a performance measure that captures desired performance while respecting constraints in the face of stochastic disturbances.

2. Deduce verifiable conditions under which the problem formulated in 1. is solvable.

3. Formulate a solution concept for the problem in 1. that is based on the model predictive control technique.

4. Create feasible computational algorithms for the synthesis of controllers that solve control problems from 1. within the solution setup from 3.

5. Investigate convergence properties of the approximate solutions obtained by computational algorithms from 4.

The main tool that is used in the thesis to solve the problem formulated in 1. is the model predictive control technique. Model predictive control has had a significant and widespread impact on industrial process control. When dealing with stochastic systems, however, application of the standard model predictive control algorithms results in a significant loss in the controlled system performance. Therefore, to deal with the problem 1. within the model predictive control framework, it was necessary to develop alternative model predictive control techniques.

Contributions of the thesis are twofold. The first set of contributions is made with regard to the model predictive control of constrained, stochastic systems. In this thesis, we develop a novel approach to the model predictive control of such systems, that

is based on the optimization in closed loop over the control horizon and stochastic sampling of the disturbance i.e. a randomized algorithm.

The second set of contributions has been made in more general framework of the optimal control of stochastic systems that are subject to input and state constraints. We present a novel problem setup for control of such systems and give initial results that are concerned with solvability conditions for the posed optimization problem and the characterization of the optimal solution.

# *Samenvatting*

Dit proefschrift behandelt de regeling (of automatische besturing) van dynamische systemen die beïvloed worden door stochastisch veronderstelde storingen en waarbij amplitude-begrenzingen op zowel actuator- als toestandsvariabelen gerespecteerd dienen te worden. De belangrijkste motivatie voor de bestudering van dit type van besturingssystemen is gelegen in het feit dat er geen geschikte technieken beschikbaar zijn voor de synthese van regelaars voor dit soort regelproblemen. Dit, ondanks het feit dat stochastische systemen en amplitude-begrenzingen op signalen veel voorkomen in toepassingsgebieden. In de procesindustrie worden bijvoorbeeld de marges van fysische grootheden zoals temperatuur, druk, concentratie, snelheid, en positie op een natuurlijke wijze uitgedrukt als amplitude-begrenzingen. Deze begrenzingen hebben veelal een persistent karakter in de zin dat met gecontroleerde regelakties de begrenzingen van fysieke grootheden gerespecteerd dienen te worden ongeacht de aanwezigheid van ongecontroleerde stoorsignalen die op het proces inwerken.

De doelstellingen van dit proefschrift zijn als volgt:

1. Het formuleren van een wiskundig probleem voor de synthese van regelaars die een gewenste prestatie van het geregelde systeem garanderen. Preciezer gezegd, een mathematische formalisering van het probleem om een besturingssysteem te ontwerpen dat een kostenfunctie minimaliseert, amplitude-begrenzingen van fysische grootheden respecteert, en bovendien rekening houdt met de invloed van stochastische verstoringen op het systeem.

2. Het afleiden van verifieerbare voorwaarden waaronder het probleem genoemd in 1. oplosbaar is.

3. Het formuleren van een oplossingsconcept voor het probleem genoemd in 1. dat gebruik maakt van technieken uit de theorie van model voorspellende regelaars (*model predictive control*).

4. Het genereren van bruikbare computer algoritmen voor de synthese van regelaars die het probleem genoemd in 1. oplossen met behulp van technieken genoemd in 3.

5. Het onderzoeken van convergentie eigenschappen van approximatieve oplossingen verkregen met de algoritmen genoemd in 4.

De belangrijkste methodologie die in dit proefschrift gebruikt is om het regelprobleem genoemd in 1 op te lossen is de theorie van model voorspellende regelaars (*model predictive control*). Deze methodologie heeft een significante en grootschalige invloed gehad op de procestechnologie waar het gaat om de automatische besturing van

industriële processen. Echter, onder de veronderstelling dat dynamische processen beïnvloed worden door stochastische storingen, resulteert de toepassing van standaard model voorspellende regelaars in een significante degradatie van het gedrag van het geregeld systeem. Om die reden is het noodzakelijk om alternatieve model voorspellende regeltechnieken te ontwikkelen om het probleem genoemd in 1 op te kunnen lossen.

Dit proefschrift kent hierin twee essentiële bijdragen. De eerste bijdrage betreft een reeks resultaten gerelateerd aan de ontwikkeling van model voorspellende regelaars van stochastische systemen met amplitude-begrenzingen. Dit proefschrift ontwikkelt een nieuwe methodologie voor voorspellende regelaars voor deze klasse van systemen, gebaseerd op de optimalisatie van terugkoppel-wetten in gesloten lus over een eindige regel-horizon en met gebruikmaking van een stochastische bemonstering van verstoringssignalen.

De tweede bijdrage is gerelateerd aan de meer algemene aspecten die betrekking hebben op het optimaal regelen van stochastische systemen met inachtneming van persistente beperkingen op actuator signalen en toestandsvariabelen. Dit proefschrift presenteert een nieuwe opzet voor de bestudering en analyse van dit soort systemen en geeft initiële resultaten die enerzijds betrekking hebben op de oplosbaarheid van het onderliggende optimalisatie probleem en anderzijds de karakterisatie en berekening van optimale oplossingen.

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Ivo Batina

# *Curriculum Vitae*

Ivo Batina was born in Split, Croatia in 1967. After finishing his high school education in 1986 at Technical Gymnasium in Split he enrolled the Faculty of Electrical Engineering, University of Split, Croatia. He graduated in 1991 under supervision of prof.dr.sc Vlasta Zanchi at the Control and System Theory Group.

After graduation he worked as a research and teaching assistant at the Faculty of Maritime Studies, University of Split until 1999. In 1996 he obtained his M.Sc. degree in Electrical Engineering at the Department of Control and Computer Engineering in Automation, University of Zagreb, Croatia under supervision of prof.dr.sc. Zoran Vukić. From 1999 till 2003 he worked towards his Ph.D. degree at the Department of Mathematics and Computer Science, University of Eindhoven in the area of Control Theory.

In 2003 he joined the Information and Communication Theory Group, Delft University of Technology as a post-doctoral researcher.

STELLINGEN

behorende bij het proefschrift

***Model predictive control for stochastic systems by randomized algorithms***

van

Ivo Batina

Eindhoven, 14 January 2004

1. In model predictive control for stochastic systems the variance of the predicted state can not be bounded by open loop control strategies. Optimization over feedback strategies is therefore crucial.

   *Chapter 3, this thesis*

2. It is not possible to control a system that is subject to Gaussian noise so that constraints on the state are violated with zero probability.

   *Chapter 4, this thesis*

3. The *separation principle* is not a principle but a theorem that applies under specific assumptions.

   *Chapter 5, this thesis*

4. New technologies are a product of multi-disciplinary scientific research. Mathematics is the common foundation of technical scientific disciplines. Therefore, an increased focus on mathematics in higher education is necessary.

5. A scientific discipline needs an elite of dedicated scientists in order to grow. However, when only members of the elite are concerned about this growth further development of the scientific discipline loses its *raison d'être*.

6. Consumer behavior is non-symmetric: Buyers of photo cameras would not bother about built-in telephones but buyers of mobile telephones do bother about built-in cameras.

7. There is a simple recipe to make periodic economic crises in the consumer society less painful. Here it is: During the crisis, consumers spend savings which they do not need buying goods which they do not need. After the crisis, consumers stop buying goods which they do not need and save money which they do not need.

8. To encourage a more efficient use of personal cars, a government can introduce a tax scheme that support a shared ownership of vehicles. The tax that each owner pays, denoted as $x$, is given by

$$x = \frac{t}{k^{N-1}N} \tag{1}$$

   where $t$ is the total tax load on a car, $N$ is the number of owners of the car and $k$ is the coefficient of encouragement. A person can have only one car on which this scheme applies. For $N$ sufficiently large this means stimulation of public transport.

9. An affordable system of institutions for day nursery would have benefit not only for the parents but also for the quality of schooling and health care in the country.

10. A productive life has change as its permanent feature.

11. A "life-style" program in elementary education is indispensable for shaping positive moral values of the future generation. Its success, however, is a test for today parents.