

# Bounds and approximations for the fixed-cycle traffic-light queue

**Citation for published version (APA):**

Broek, van den, M. S., Leeuwaarden, van, J. S. H., Adan, I. J. B. F., & Boxma, O. J. (2004). *Bounds and approximations for the fixed-cycle traffic-light queue*. (SPOR-Report : reports in statistics, probability and operations research; Vol. 200418). Technische Universiteit Eindhoven.

**Document status and date:**

Published: 01/01/2004

**Document Version:**

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

**Please check the document version of this publication:**

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

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SPOR-Report 2004-18

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SPOR-Report  
Reports in Statistics, Probability and Operations Research

Eindhoven, December 2004  
The Netherlands

SPOR-Report  
Reports in Statistics, Probability and Operations Research

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ISSN 1567-5211

# BOUNDS AND APPROXIMATIONS FOR THE FIXED-CYCLE TRAFFIC-LIGHT QUEUE

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**December 15, 2004**

## **Abstract**

This paper deals with the fixed-cycle traffic-light (FCTL) queue, where vehicles arrive to an intersection controlled by a traffic light and form a queue. The traffic light alternates between green and red periods, and delayed vehicles are assumed to depart during the green period at equal time intervals. The key performance characteristic in the FCTL queue is the so-called mean overflow, defined as the mean queue length at the end of a green period.

An exact solution for the mean overflow is available, but it has been considered to be of little practical value since it requires some numerical procedures. Therefore, most of the literature on the FCTL queue is about deriving approximations for the mean overflow. In deriving these approximations, most authors first approximate the FCTL queue by a bulk service queue, approximate the mean overflow in the bulk service queue, and use this as an approximation for the mean overflow in the FCTL queue. So far, no quantitative comparison of both models has been given. We compare both models and assess the quality of the approximation for various settings of the parameter values. In this comparison and throughout the whole paper we do not restrict ourselves to Poisson arrivals, but consider a more general arrival process instead.

We discuss the numerical issues that need to be resolved to calculate the exact expression for the mean overflow in both queues and show that clear computational schemes are available. Next, we present several bounds and approximations of the mean overflow that do not require numerical procedures. In particular, we derive a new approximation based on the heavy traffic limit and a scaling argument. We compare the new bounds and approximation with the existing ones. We elaborate on the impact of several parameters, like the length of the green and red period, and the variance of the arrival distribution. Each of these parameters turns out to be crucial.

**Keywords:** traffic light, fixed-time control, overflow, bulk service queue, approximations, bounds, mean delay.

# 1 Introduction

Over the last four decades, there has been a broad effort to obtain good approximations for the average queue length and delay of vehicles at signalized intersections. In this effort, the fixed-cycle traffic-light (FCTL) queue is one of the best-studied models. Vehicles arrive to an intersection controlled by a traffic light and form a queue. The traffic light alternates between green and red periods of durations  $g$  and  $r$ , respectively, and delayed vehicles are assumed to depart during the green period at equal time intervals.

Most of the research devoted to the FCTL queue is based on the simplifying assumption that vehicles arrive to the traffic light according to a Poisson process, with Webster's formula [22] as the most famous result. Webster's formula, which is partially based on simulation results, gives the mean delay of a vehicle. McNeill [13] derived an exact expression for the mean delay up to one unknown: the mean size of the overflow (the mean queue length at the end of a green period). We denote this unknown by  $\mathbb{E}X_g$ . From McNeill's formula, it became clear that obtaining an exact formula for the mean delay was equivalent to obtaining an exact formula for  $\mathbb{E}X_g$ . Darroch [7] provided an exact formula for  $\mathbb{E}X_g$  in 1964, but up to this day, this is not well-known (see e.g. the recent publications Dion et al. [9], Tarko [21]), probably due to the fact that Darroch uses a slightly different setting (discrete-time) and his approach is both analytically and computationally involved. The latter must be why Ohno [19] gives a detailed algorithmic description of the computation of Darroch's rather complicated expression for  $\mathbb{E}X_g$ . Ohno further presents a thorough overview on the research that is based on the Poisson assumption. In particular, Ohno compares the various approximative closed-form formulas for the mean delay, like those of Webster [22], Webster & Cobbe [23], McNeill [13], Miller [15] and Newell [18]. The latter three were obtained by approximating  $\mathbb{E}X_g$ .

Hence, although an exact expression for  $\mathbb{E}X_g$  exists, there has been a desire for deriving approximative closed-form expressions. This is perfectly legitimate for several reasons. First, the FCTL queue is mainly used for the dimensioning of intersections, and for this purpose approximations are sufficient. Secondly, most results were obtained in the 1960's, when the available computational power was much less than nowadays. Since Darroch's solution requires numerical computations, at that time rather difficult, searching for easy-calculable approximations was a natural thing to do. Third, the results of the FCTL queue might serve as the input for a network of intersections. In order to optimize such a network, the expressions for the input should not be computationally cumbersome.

The latter two arguments do not necessarily hold true nowadays. That is, due to improved numerical algorithms and the increased computational power, the numerical determination of Darroch's solution has become a straightforward exercise. The first argument remains true, and an approximation can both be sufficient and provide insight in the impact of the various parameters. Remarkably, most approximations have been derived by first reducing the FCTL queue to a slightly easier model known as the *bulk service queue*. The overflow in the bulk service queue is then approximated and also used as an approximation for  $\mathbb{E}X_g$ . The validity of such a two-stage approximation heavily depends on how well the bulk service queue approximates the FCTL queue. Although one can reason that for heavily loaded intersections both models are much alike, this issue has never been investigated properly.

As mentioned, most of the research on the FCTL queue is devoted to Poisson arrivals. Notable exceptions are Miller [15] and Newell [18], who both derive approximations for  $\mathbb{E}X_g$  using fairly general arguments, and Darroch [7] and McNeill [13] who both consider a compound Poisson process. The assumption of (compound) Poisson arrivals allows the FCTL queue to be modeled at embedded points, namely the times just after the departure of a delayed vehicle, see Darroch [7]. We generalize this by assuming that the number of vehicles that arrive per slot follow some discrete distribution (the Poisson and compound Poisson are also discrete distributions). This allows one to consider distributions with a larger coefficient of variation, distributions with a finite support, or distributions fitted to empirical data.

## 1.1 Our contribution

As mentioned, most approximations for the FCTL queue are derived from approximating the FCTL queue by the discrete bulk service queue. So far, no quantitative comparison of both models has been given. We compare both models and assess the quality of the approximation of the FCTL queue by the bulk service queue for various settings of the parameter values. In this comparison and throughout the paper we do not restrict ourselves to Poisson arrivals, but we consider a more general arrival process instead.

We discuss the numerical issues that are needed to calculate the exact expression for the mean overflow in both the FCTL queue and the bulk service queue. As it turns out, determining the exact expressions requires for both models the numerical calculations of the roots of a certain equation. We will show that clear computational schemes for determining the roots are available. With these schemes, the way to calculate the exact expression for the mean overflow is fully specified. However, from a practical point of view, there is still a need for bounds and approximations of the mean overflow that do not depend on root-finding, to allow for a quick evaluation and give some more insight.

We present several lower and upper bounds on the exact solution of the mean overflow, and an upper bound on the mean overflow in the bulk service queue, which also serves as an upper bound on the mean overflow in the FCTL queue. We further derive a new approximation for the mean overflow, based on the heavy traffic limit and a scaling argument. The approximation holds for a general arrival process. We compare the new approximation with the existing approximations and the newly derived bounds. In particular, we elaborate on the impact of the various parameters like the length of the green and red period, and the variance of the arrival distribution. Each of these parameters turns out to be crucial.

The remainder of this paper is structured as follows. In Sec. 2 we present the exact solutions for the FCTL queue and the bulk service queue, along with a comparison between both models. For the mean overflow in the FCTL queue we present bounds in Sec. 3 and approximations in Sec. 4. The new bounds and approximations are compared with the existing ones in Sec. 5 for various settings. Some conclusions are presented in Sec. 6.

## 2 Exact solutions

In most studies on the FCTL queue that do not rely on the Poisson assumption, e.g. [4, 7, 14, 17], the following two assumptions are made:

**Assumption 2.1.** (discrete-time assumption) The time axis is divided into constant time intervals of unit length, so-called slots, where each slot corresponds to the time needed for a delayed vehicle to depart from the queue. The green and red period, and thus the cycle time  $c$  (with  $c = g + r$ ), are assumed to be fixed multiples of one slot. Hence,  $g$  and  $r$  are integers expressed in slots. Those vehicles that arrive to the queue and are delayed, join the queue at the end of the slot in which they arrive.

**Assumption 2.2.** (independence assumption) Let  $Y_{k,n}$  denote the number of vehicles that arrive to the intersection during slot  $k$  in cycle  $n$ . The random variables  $Y_{k,n}$  are assumed to be independent and identically distributed (i.i.d.) for all  $k, n$ , according to some discrete random variable  $Y$  with probability generating function (pgf)  $Y(z)$ .

The discrete-time assumption together with the independence assumption allow one to model the queue length at the end of slots as a discrete-time Markov chain. We also work under these assumptions and assess their implications at several places in this paper. Note that a Poisson arrival process satisfies the independence assumption. Furthermore, although it might seem rather restrictive, the independence assumption is frequently made and allows for a far larger class of arrival distributions than just the Poisson case.

The following assumption is always made for the FCTL queue:

**Assumption 2.3.** (FCTL assumption) For those cycles in which the queue clears before the green period terminates, all vehicles that arrive during the residual green period pass through the system and experience no delay whatsoever.

The FCTL assumption is legitimate in the sense that the vehicles in question can pass the intersection without slowing down. It does however have some severe consequences for the analysis of the queue length, as discussed next. Let  $X_{k,n}$  denote the queue length at time  $k$  in cycle  $n$  (time expressed in slots). Then, in cycle  $n$ ,  $X_{0,n}$  is the queue length at the beginning of the green period, and  $X_{g,n}$  is the overflow, defined as the queue length at the end of a green period (and thus the beginning of the red period). Let  $A_n$  denote the total number of vehicles that arrive to the intersection in between the two measurements of the overflow  $X_{g,n}$  and  $X_{g,n+1}$ . So  $A_n$  is the number of arrivals from  $X_{g,n}$  onwards in a consecutive red and green period, and  $A_n = \sum_{k=g+1}^c Y_{k,n} + \sum_{k=1}^g Y_{k,n+1}$ . Further,  $A_n = A_n^d + A_n^p$ , where  $A_n^d$  denotes the delayed vehicles and  $A_n^p$  those vehicles that pass without delay on behalf of the FCTL assumption. The overflow queue can then be defined as

$$X_{g,n+1} = \max\{X_{g,n} + A_n^d - g, 0\}. \quad (2.1)$$

The fact that  $A_n^d$  depends on both  $X_{g,n}$  and the exact specification of when the arrivals occur makes (2.1) hard to analyse. The analysis could be simplified if all vehicles would be delayed, so all vehicles arrive while the queue length is at least one, and  $A_n = A_n^d$ . The variables  $A_n^d$  are then i.i.d. and (2.1) reduces to the *classical bulk service queue*, first solved by Bailey [3]. He derived the pgf of the stationary queue length  $X_g$ , defined as  $\lim_{n \rightarrow \infty} X_{g,n}$ , that exists if  $\mathbb{E}A < g$ . The pgf requires the determination of  $g$  (complex-valued) roots on and within the unit circle of some characteristic equation.

Beckmann et al. [4] and Newell [17] assumed a Bernoulli arrival process with  $Y(z) = 1 - \alpha + \alpha z$ . On this assumption of 0 or 1 arrivals per slot, the bulk service assumption holds true ( $A_n = A_n^d$ ), and Bailey's solution can be applied to derive the exact value of  $\mathbb{E}X_g$ . When  $Y$  can take values larger than 1, the bulk service assumption is obviously an approximation and yields an upper bound on the overflow queue. For a compound Poisson process, McNeill [13] used Bailey's solution to derive an upper bound on  $\mathbb{E}X_g$ . Although one would want an upper bound to be easy to compute, McNeill's upper bound is not, since it still requires the numerical calculation of the  $g$  roots.

Using the discrete-time assumption, Darroch [7] derived the solution to the true FCTL queue that is of the same complexity as Bailey's solution to the bulk service queue, again requiring the roots of a characteristic equation. The effort put into determining the roots is in Darroch's case more justifiable, though, since it leads to the exact solution.

Next, we introduce the following definition of delay:

**Definition 2.4.** (delay) The delay  $D$  of a vehicle is defined as the number of slots from the beginning of the first slot after the slot in which the vehicle arrives, until the end of the slot in which the delayed vehicle departs from the queue (see Fig. 1).

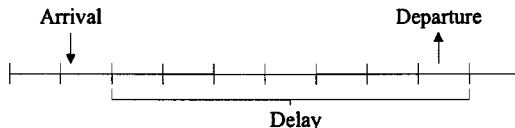


Figure 1: Graphical representation of delay

Using this definition, the following expression for the mean delay can be derived (see Darroch [7]):

$$\mathbb{E}D = \frac{r}{2c\mu_Y(1 - \mu_Y)} \left[ \frac{\sigma_Y^2}{1 - \mu_Y} + r\mu_Y + 2\mathbb{E}X_g \right], \quad (2.2)$$

where we have denoted the mean and variance of  $Y$  by  $\mu_Y$  and  $\sigma_Y^2$ , respectively.

Delay in the FCTL queue can be defined in several ways, depending on how one handles the delay of a vehicle experienced within the slot of its arrival. We use Definition 2.4 which does not include this part of the delay; we assume that the vehicle joins the queue at the beginning of the next slot after its arrival. This is in line with the discrete-time assumption, where we assume that the arrival of vehicles only occurs at the end of a slot; the vehicles then arrive as a batch. In reality, the vehicles arrive one-by-one, each vehicle arriving at some different time point during the slot, and the total delay  $D^T$  satisfies  $D^T = D + D^R$  where  $D$  is defined by Definition 2.4, and  $D^R$  is the residual delay within the slot of arrival,  $D^R \in [0, 1]$ .

For (compound) Poisson arrivals, it is possible to include  $D^R$  (due to the PASTA property), and the mean delay is then given by (see McNeill [13])

$$\mathbb{E}D = \frac{r}{2c\mu_Y(1-\mu_Y)} \left[ \frac{\sigma_Y^2}{1-\mu_Y} + r\mu_Y + 2\mathbb{E}X_g + 1 \right]. \quad (2.3)$$

In comparison with (2.2), Expression (2.3) has an additional term  $r/(2c\mu_Y(1-\mu_Y))$ , which can be easily shown to be  $\mathbb{E}D^R$ . For Poisson arrivals, Ohno [19] gives a comparison between McNeill's expression, Darroch's Expression (2.2) (where Ohno's formula (17) for Darroch's mean delay formula is incorrect), and several other approximations. Ohno shows that the differences are only marginal. To allow for a more general arrival distribution, we take (2.2) as our expression for the mean delay, except when we present results for (compound) Poisson arrivals, in which case we use (2.3). In Appendix B we derive the mean delay (2.3) for Poisson arrivals, where we rely on the PASTA property and Little's law.

## 2.1 Mean overflow

### 2.1.1 The FCTL queue

We now present Darroch's solution to the FCTL queue. Let  $Y$  be any discrete random variable (Darroch assumes  $Y$  to be of compound Poisson type). Clearly, to have stability, it is required that the number of arriving vehicles is less than the maximum number of vehicles that can depart, and hence  $Y$  should satisfy

$$c\mu_Y < g. \quad (2.4)$$

Darroch [7] obtains the following exact expression for the mean overflow:

$$\mathbb{E}X_g = \frac{c\sigma_Y^2 + r^2\mu_Y^2 - g^2(1-\mu_Y)^2}{2(g-c\mu_Y)} - \frac{\sigma_Y^2}{2(1-\mu_Y)} + \frac{1-\mu_Y}{2} + \frac{(1-\mu_Y)^2}{g-c\mu_Y} \sum_{j=0}^{g-1} jq_j, \quad (2.5)$$

where  $q_j$  ( $0 \leq j < g$ ) denotes the probability that the queue is empty at time point  $j$  of the green period. The  $g$  unknowns  $q_0, \dots, q_{g-1}$  can be found using a numerical procedure explained in Appendix A. Also, the following identity holds:

$$\sum_{j=0}^{g-1} q_j = \frac{g-c\mu_Y}{1-\mu_Y} =: \alpha. \quad (2.6)$$

One can think of  $\alpha$  as the mean number of slots in a green period that the queue is empty. Furthermore, (2.6) can be written as

$$g - \sum_{j=0}^{g-1} q_j = \left( c - \sum_{j=0}^{g-1} q_j \right) \mu_Y. \quad (2.7)$$

The right-hand side of (2.7) represents the mean number of delayed vehicles that arrive in a consecutive red and green period (earlier denoted by  $\mathbb{E}A^d$ , see (2.1)). The left-hand side of (2.7) represents the mean number of slots per green period used for the departure of delayed vehicles, and in equilibrium both quantities should be equal.



### 2.1.2 The bulk service queue

The bulk service queue is defined by the recursive relation

$$\bar{X}_{g,n+1} = \max\{\bar{X}_{g,n} + A_n - g, 0\}, \quad A_n = \sum_{k=g+1}^c Y_{k,n} + \sum_{k=1}^g Y_{k,n+1}. \quad (2.8)$$

Here, we have added a bar to the  $X$  for reasons of clarity. Under the stability condition  $c\mu_Y < g$  the stationary queue length  $\bar{X}_g = \lim_{n \rightarrow \infty} \bar{X}_{g,n}$  exists. The mean stationary queue length satisfies

$$\mathbb{E}\bar{X}_g = \frac{c\sigma_Y^2}{2(g - c\mu_Y)} + \frac{g - c\mu_Y}{2} - \sum_{j=0}^{g-1} \frac{x_j(g-j)^2}{2(g - c\mu_Y)}, \quad (2.9)$$

where  $x_j = \mathbb{P}(\bar{X}_g + A = j)$ . It further holds that

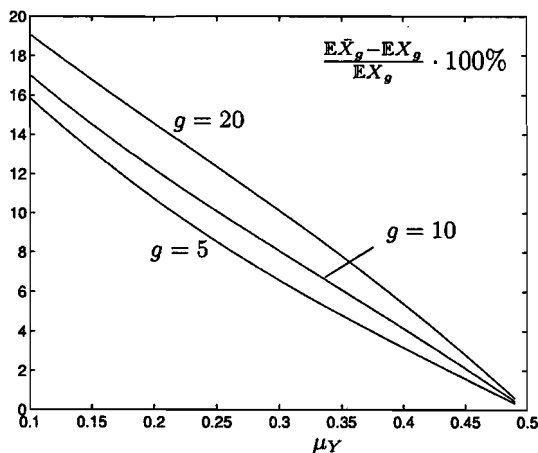
$$\sum_{j=0}^{g-1} x_j(g-j) = g - c\mu_Y = \alpha_2. \quad (2.10)$$

The  $g$  unknowns  $x_0, \dots, x_{g-1}$  can be found using a similar numerical procedure as for determining  $q_0, \dots, q_{g-1}$  and this procedure is explained in Appendix A.

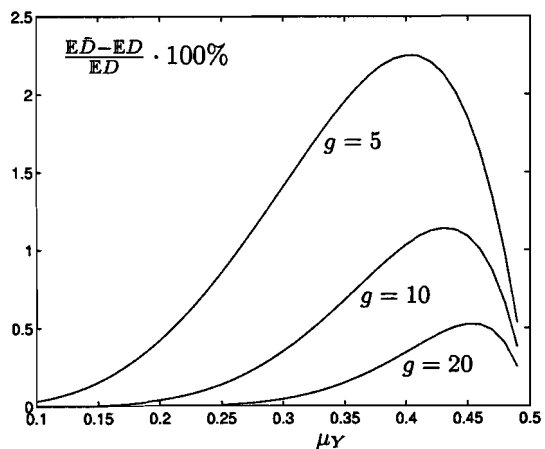
## 2.2 FCTL queue vs bulk service queue

We now aim at investigating the consequences of approximating the mean overflow in the FCTL queue by the mean overflow in the bulk service queue. So how well does the bulk service queue reflect the specific properties of the FCTL queue, and in particular, how severe are the consequences of neglecting the FCTL assumption?

For ease of presentation we assume Poisson arrivals and take the green and red period of equal length. In Figs. 2-3 we compare the mean overflow and mean delay as a function of the mean number  $\mu_Y$  of arriving vehicles per slot for several lengths of the green period.



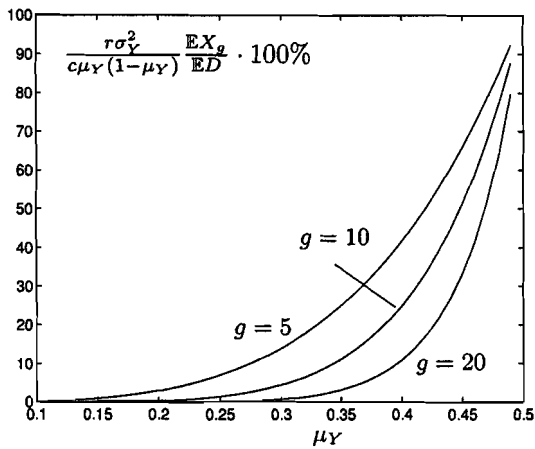
**Figure 2:** The relative difference in mean overflow of the FCTL queue and discrete bulk service queue for  $0.1 \leq \mu_Y < 0.5$  and  $g = 5, 10, 20$ .



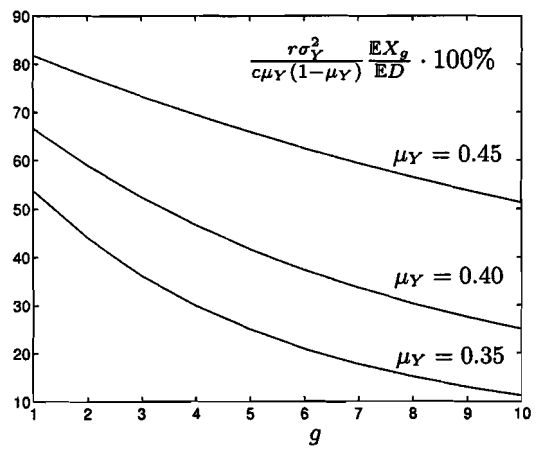
**Figure 3:** The relative difference in mean delay of the FCTL queue and discrete bulk service queue for  $0.1 \leq \mu_Y < 0.5$  and  $g = 5, 10, 20$ .

From Fig. 2 we conclude that the difference in mean overflow decreases for increasing values of  $g$  and  $\mu_Y$ . The latter can be reasoned as follows. When more vehicles arrive to the system,

fewer vehicles will not be delayed when they arrive to the intersection, and hence the impact of the FCTL assumption gets smaller. And it is exactly that assumption that separates the FCTL queue from the discrete bulk service queue. In the limit, when  $\mu_Y \uparrow 1/2$ , the impact of the FCTL assumption diminishes and the two models converge to each other. This also holds for the mean delay in Fig. 3. However, a second observation from Fig. 3 is that, when  $\mu_Y$  ranges from zero to its maximum allowed value, the relative difference in mean delay first increases while after reaching its maximum it again goes to zero when  $\mu_Y \uparrow 1/2$ . The reason for this is the varying impact of the mean overflow on the mean delay. Let us investigate this a bit further. In Figs. 4-5 we have plotted the percentage of the mean delay that is determined by the term in (2.3) that includes the mean overflow for various values of  $\mu_Y$  and  $g$ .



**Figure 4:** The percentage of the mean delay determined by the mean overflow for  $0.1 \leq \mu_Y < 0.5$  and  $g = 5, 10, 20$ .



**Figure 5:** The percentage of the mean delay determined by the mean overflow for  $\mu_Y = 0.35, 0.40, 0.45$  and  $g = 1, 2, \dots, 10$ .

In Fig. 4 we see that the impact of the mean overflow on the mean delay increases for higher values of  $\mu_Y$ . That is, when more vehicles arrive to the system, queues at the end of the green period are more likely to occur and build up. The non-trivial shape of the curves in Fig. 3 can then be explained by two contradicting forces: for increasing values of  $\mu_Y$ , the impact on the mean delay increases while the relative difference between the overflow in the FCTL queue and the discrete bulk service queue decreases (see Fig. 2).

### 3 Bounds

In the previous section we have presented the exact solution of the mean overflow in the FCTL queue. The exact solution requires the numerical determination of the roots of the equation  $z^g = Y(z)^c$  inside the unit circle. In this section we present bounds on the mean overflow that do not require this numerical procedure.

Expression (2.5) shows that finding bounds for  $\mathbb{E}X_g$  (and thus for  $\mathbb{E}D$ ) is tantamount to finding bounds for  $\sum_{j=0}^{g-1} j q_j$ . Darroch [7] derived such bounds. In this section, we extend his approach and present some new bounds. Throughout, we use the fact that

$$q_0 \leq q_1 \leq \dots \leq q_{g-1}, \quad (3.1)$$

together with  $q_j \in [0, 1]$  and  $\sum_{j=0}^{g-1} q_j = \alpha$  (see (2.6)). These three conditions enable us to derive bounds by choosing specific values of the  $q_j$ .

### 3.1 Crude bounds

A crude lower bound can be obtained in the following way:

$$\sum_{j=0}^{g-1} j q_j = \sum_{j=1}^{g-1} q_j \sum_{i=1}^j 1 = \sum_{i=1}^{g-1} \sum_{j=i}^{g-1} q_j \geq \sum_{i=1}^{g-1} (g-i) \frac{\alpha}{g} = \frac{\alpha}{g} \sum_{i=1}^{g-1} i = \frac{\alpha(g-1)}{2}. \quad (3.2)$$

A crude upper bound is obtained by choosing  $q_0 = \dots = q_{g-1} = 1$  (where we intentionally violate the condition  $\sum_{j=0}^{g-1} q_j = \alpha$ ), leading to

$$\sum_{j=0}^{g-1} j q_j \leq \sum_{j=0}^{g-1} j = \frac{g(g-1)}{2}. \quad (3.3)$$

Substituting these bounds into (2.5), and replacing  $\alpha$  by  $(g - c\mu_Y)/(1 - \mu_Y)$  yields the following bounds on  $\mathbb{E}X_g$ :

$$f(\mu_Y, \sigma_Y^2) + \frac{(1 - \mu_Y)(g-1)}{2} \leq \mathbb{E}X_g \leq f(\mu_Y, \sigma_Y^2) + \frac{(1 - \mu_Y)(g-1)}{2} \left( \frac{g - g\mu_Y}{g - c\mu_Y} \right), \quad (3.4)$$

where

$$f(\mu_Y, \sigma_Y^2) = \frac{c\sigma_Y^2 + r^2\mu_Y^2 - g^2(1 - \mu_Y)^2}{2(g - c\mu_Y)} - \frac{\sigma_Y^2}{2(1 - \mu_Y)} + \frac{1 - \mu_Y}{2}. \quad (3.5)$$

Note that the factor  $(g - g\mu_Y)/(g - c\mu_Y)$  is larger than one and tends to one when the ratio  $g/c \uparrow 1$ , i.e. when the green period is relatively long in comparison with the red period.

### 3.2 More precise bounds

Darroch [7] proposes the following upper bound. Let  $[\alpha]$  denote the integral part of  $\alpha$ , and choose

$$q_0 = \dots = q_{g-[\alpha]-2} = 0, \quad q_{g-[\alpha]-1} = \alpha - [\alpha], \quad q_{g-[\alpha]} = \dots = q_{g-1} = 1. \quad (3.6)$$

These values lead to an upper bound on  $\sum_{j=0}^{g-1} j q_j$  given by

$$\sum_{j=0}^{g-1} j q_j \leq (g - [\alpha] - 1)(\alpha - [\alpha]) + \frac{[\alpha](2g - [\alpha] - 1)}{2}. \quad (3.7)$$

Another upper bound may follow from bounding the mean queue length in the bulk service queue  $\mathbb{E}\bar{X}_g$ , see (2.9). It can be seen that, using (2.10),

$$\frac{g - c\mu_Y}{2} = \frac{(\sum_{j=0}^{g-1} x_j(g-j))^2}{2(g - c\mu_Y)} \leq \sum_{j=0}^{g-1} \frac{x_j(g-j)^2}{2(g - c\mu_Y)} \leq g \sum_{j=0}^{g-1} \frac{x_j(g-j)}{2(g - c\mu_Y)} = \frac{g}{2}, \quad (3.8)$$

which yields with (2.9) that

$$\frac{c\sigma_Y^2}{2(g - c\mu_Y)} - \frac{c\mu_Y}{2} \leq \mathbb{E}\bar{X}_g \leq \frac{c\sigma_Y^2}{2(g - c\mu_Y)}. \quad (3.9)$$

Hence, since  $\mathbb{E}X_g \leq \mathbb{E}\bar{X}_g$ , the upper bound on  $\mathbb{E}\bar{X}_g$  in (3.9) is in fact an upper bound on  $\mathbb{E}X_g$ . Sharper bounds on  $\mathbb{E}\bar{X}_g$  are presented in [8].

Darroch [7] also proposed a method to derive a lower bound on  $\sum_{j=0}^{g-1} j q_j$ . He derived upper bounds on  $q_0$  and  $q_1$ , denoted by  $\hat{q}_0$  and  $\hat{q}_1$ , respectively. Darroch constructed a lower bound by setting  $q_0 = \hat{q}_0$ ,  $q_1 = \hat{q}_1$  and  $q_2 = \dots = q_{g-1} = (\alpha - \hat{q}_0 - \hat{q}_1)/(g-2)$ . We now reformulate and extend this method by providing a probabilistic interpretation and upper bounds for all  $q_j$ .

We introduce  $G$  as the random variable representing the effective green period, defined as the number of slots per green period that are actually used for the departure of delayed vehicles. It holds that  $q_0 = \mathbb{P}(G = 0)$  and for  $j = 1, \dots, g - 1$  that

$$q_j = q_{j-1} + (q_j - q_{j-1}) = q_{j-1} + \mathbb{P}(G = j). \quad (3.10)$$

We can bound the probabilities  $\mathbb{P}(G = j)$  by conditioning on the overflow being zero. Assume that the overflow in a cycle is zero. The probability that  $G$  takes on a certain value in the next cycle can be expressed as an exact specification of the arrivals in the intermediate red period and those during the effective green period. The number of combinations that leads to  $G = j$  is largest for  $X_g = 0$ , so  $\mathbb{P}(G = j) < \mathbb{P}(G = j | X_g = 0)$ . This yields the following upper bounds on the  $q_j$ :

$$q_j = \sum_{n=0}^j \mathbb{P}(G = n) < \sum_{n=0}^j \mathbb{P}(G = n | X_g = 0) = \hat{q}_j. \quad (3.11)$$

The values  $\hat{q}_j$  can be expressed fully in terms of the distribution of  $Y$  (with  $y_j = \mathbb{P}(Y = j)$ ). We have for example that  $\hat{q}_0 = y_0^r$ ,  $\hat{q}_1 = y_0^r(1 + ry_1)$ , and

$$\hat{q}_2 = y_0^r \left( 1 + ry_1(1 + y_1) + ry_0y_2 + \binom{r}{2} y_1^2 \right).$$

Let  $m$  be the largest integer value for which

$$\hat{q}_m \leq \frac{\alpha - \sum_{j=0}^m \hat{q}_j}{g - m - 1}. \quad (3.12)$$

Then, we exploit the upper bounds  $\hat{q}_j$  to the maximum, while still satisfying (3.1), by choosing

$$q_0 = \hat{q}_0, \quad q_1 = \hat{q}_1, \quad \dots, \quad q_m = \hat{q}_m, \quad q_{m+1} = q_{m+2} = \dots = q_{g-1} = \frac{\alpha - \sum_{j=0}^m \hat{q}_j}{g - m - 1}. \quad (3.13)$$

A lower bound on  $\sum_{j=0}^{g-1} jq_j$  then follows from setting the  $q_j$  as in (3.13). Note that  $\sum_{j=0}^{g-1} jq_j$  is always positive and therefore the maximum of zero and the newly found bound, yields the real lower bound.

**Example 3.1.** (i) Consider Poisson arrivals with  $\mathbb{P}(Y = j) = e^{-\lambda} \lambda^j / j!$  and  $\mu_Y = \lambda$ . For  $g = r = 5$ , and  $x = c\mu_Y/g = 0.7$  we have  $\alpha = 2.3077$ ,  $m = 2$ ,  $\hat{q}_0 = 0.174$ ,  $\hat{q}_1 = 0.388$ ,  $\hat{q}_2 = 0.520$  and  $q_3 = q_4 = 0.613$ . The resulting lower bound equals 0.333, where the exact value is given by 0.440.

(ii) Consider geometric arrivals with  $\mathbb{P}(Y = j) = (1 - p)p^j$  and  $\mu_Y = p/(1 - p)$ . For  $g = r = 10$ , and  $x = c\mu_Y/g = 0.9$  we have  $\alpha = 1.8182$ ,  $m = 2$ ,  $\hat{q}_0 = 0.024$ ,  $\hat{q}_1 = 0.076$ ,  $\hat{q}_2 = 0.139$  and  $q_3 = \dots = q_9 = 0.226$ . The resulting lower bound equals 4.180, where the exact value is given by 4.745.

## 4 Approximations

In Sec. 3 we have presented bounds on the mean overflow. We now turn to approximations for the mean overflow. We discuss the best-known existing approximations and present a new approximation. The latter is based on the heavy-traffic limit and a scaling argument.

### 4.1 Existing approximations

Miller [15] obtained an approximating formula for the mean overflow based on the discrete bulk service queue. Miller rewrote (2.1) as

$$\bar{X}_{g,n+1} = \bar{X}_{g,n} + A_n - g + U_n, \quad (4.1)$$

where  $U_n = g - (\bar{X}_{g,n} + A_n)$  if  $\bar{X}_{g,n} + A_n < g$ , and 0 otherwise. Let  $U = \lim_{n \rightarrow \infty} U_n$ . The mean overflow can then be shown to be

$$\mathbb{E}\bar{X}_g = \frac{c\sigma_Y^2 - \sigma_U^2}{2(g - c\mu_Y)}. \quad (4.2)$$

Note that (4.2) is the same expression as (2.9) since

$$\sigma_U^2 = \sum_{j=0}^{g-1} x_j(g-j)^2 - \left[ \sum_{j=0}^{g-1} x_j(g-j) \right]^2. \quad (4.3)$$

Of course, this last expression for  $\sigma_U^2$  requires the numerical determination of  $x_0, x_1, \dots, x_{g-1}$  as outlined in Appendix A. Miller aimed for a simple approximation and assumed that  $\sigma_U^2/\mathbb{E}U \approx I$  with  $I$  the index of dispersion given by  $\sigma_Y^2/\mu_Y$ . This gives

$$\mathbb{E}X_g \approx \frac{2c\mu_Y - g}{2(g - c\mu_Y)} \cdot I, \quad (4.4)$$

when  $2c\mu_Y \geq g$ . When  $2c\mu_Y < g$ ,  $\mathbb{E}X_g$  is taken as zero.

Newell [18] lets the red and green period tend to infinity and applies the central limit theorem to obtain the following approximation for the overflow:

$$\mathbb{E}X_g \approx \frac{g - c\mu_Y}{\pi} \int_0^{\pi/2} \frac{\tan^2 \theta}{\exp[(g - c\mu_Y)^2 / (2gI \cos^2 \theta)] - 1} d\theta. \quad (4.5)$$

The best-known approximation formula was derived by Webster [22], which holds only for Poisson arrivals and is formulated in terms of  $\mathbb{E}D$  (with  $\mu_Y = \sigma_Y^2 = \lambda$ ):

$$\mathbb{E}D = \frac{(c-g)^2}{2c(1-\lambda)} + \frac{\lambda c^2}{2g(g-\lambda c)} - 0.65 \left(\frac{c}{\lambda^2}\right)^{1/3} \left(\frac{\lambda c}{g}\right)^{2+5g/c}. \quad (4.6)$$

The first two terms in (4.6) are based on an  $M/D/1$  queue, whereas the last term is obtained by matching simulation results. Miller also derived an approximation for Poisson arrivals only, formulated in terms of  $\mathbb{E}X_g$  (with  $x = \lambda c/g$ ):

$$\mathbb{E}X_g \approx \frac{\exp[-1.33\sqrt{g}(1-x)/x]}{2(1-x)}. \quad (4.7)$$

It should be mentioned that in Rouphail et al. [20] both formulas of Miller (9.28) and (9.32) are incorrectly cited, while Ohno [19] incorrectly cites the formula of Newell (Expression (25)) in his paper.

## 4.2 New approximation formula

We will now derive a new approximation formula for the mean overflow that holds for a general arrival process, where we aim at incorporating the key characteristic of the FCTL queue, see Sec. 2. We offset the added value of the new formula against the best existing formulas (4.4) and (4.5). In particular, we want our approximation to reflect both the impact of the FCTL assumption (Assumption 2.3) and the impact of the type of arrival process.

In (4.4) the FCTL assumption is neglected, since Miller approximates the FCTL queue by the bulk service queue. In (4.5) the impact of the type of arrival process is neglected to a large extent, since Newell relied on the central limit theorem. Thus, the approximation (4.4) is expected to be less accurate for regimes in which the bulk service queue differs severely from the FCTL queue, see Sec. 2, and the approximation (4.5) is expected to be less accurate for small cycles and more volatile arrival processes. This will indeed be demonstrated in Sec. 5.

Let us now turn to the derivation of the new approximation. When the mean number  $c\mu_Y$  of vehicles that arrive to the intersection per cycle approaches its maximum allowed value  $g$ , the

FCTL queue will almost never be empty. As a result hardly any vehicle might pass the intersection without delay. This implies that the FCTL assumption has little influence on the evolution of the queue length. In Sec. 2.1 we argued that it is exactly this assumption that makes the difference between the FCTL queue and the bulk service queue. Hence, when  $c\mu_Y$  approaches  $g$ , the impact of the FCTL assumption gradually vanishes, and so the FCTL queue and the bulk service queue become more and more alike. Among other things, this implies that  $\mathbb{E}X_g$  tends to  $\mathbb{E}\bar{X}_g$ , as demonstrated next.

From the bound (3.4) we see that

$$\lim_{c\mu_Y \uparrow g} \frac{(1 - \mu_Y)^2}{g - c\mu_Y} \sum_{j=0}^{g-1} j q_j = \frac{(1 - \mu_Y)(g - 1)}{2}, \quad (4.8)$$

and plugging this into (2.5) yields

$$\begin{aligned} \lim_{c\mu_Y \uparrow g} \mathbb{E}X_g &= \lim_{c\mu_Y \uparrow g} \left[ \frac{c\sigma_Y^2}{2(g - c\mu_Y)} - \frac{(c - g)\mu_Y}{2} - \frac{\sigma_Y^2}{2(1 - \mu_Y)} \right] \\ &\approx \frac{c\sigma_Y^2}{2(g - c\mu_Y)}. \end{aligned} \quad (4.9)$$

From the bounds (3.9) it follows that the  $\mathbb{E}\bar{X}_g$  is dominated by the same term in heavy traffic, i.e., for  $c\mu_Y \uparrow g$

$$\mathbb{E}X_g \approx \mathbb{E}\bar{X}_g \approx \frac{c\sigma_Y^2}{2(g - c\mu_Y)}. \quad (4.10)$$

We now reduce the problem of determining an approximation of  $\mathbb{E}X_g$  to the problem of finding a good scaling function  $\xi(g, c, \mu_Y)$  such that

$$\mathbb{E}X_g \approx \xi(g, c, \mu_Y) \cdot \frac{c\sigma_Y^2}{2(g - c\mu_Y)}. \quad (4.11)$$

First note that (4.4) can be rewritten as

$$\mathbb{E}X_g \approx \left( 2 - \frac{g}{c\mu_Y} \right) \cdot \frac{c\sigma_Y^2}{2(g - c\mu_Y)}, \quad (4.12)$$

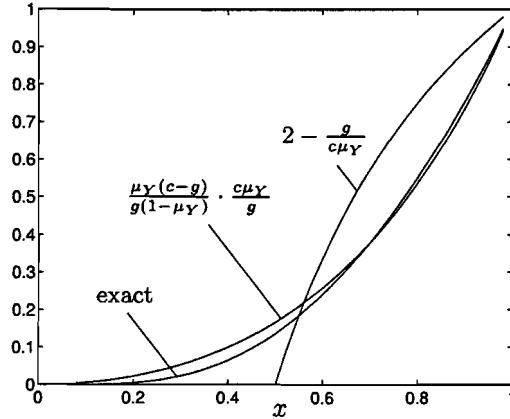
and so Miller's approximation amounts to setting  $\xi(g, c, \mu_Y) = (2 - g/(c\mu_Y))$  if  $2c\mu_Y \geq g$  and  $\xi(g, c, \mu_Y) = 0$  if  $2c\mu_Y < g$ . But what is a proper choice for  $\xi(g, c, \mu_Y)$ ? It should hold that  $\xi(g, c, \mu_Y) \rightarrow 1$  when  $c\mu_Y \uparrow g$ . The scaling function  $\xi(g, c, \mu_Y) = c\mu_Y/g$  obviously would meet this requirement and also reflects the fact that the more vehicles arrive to the intersection, the more dominant the heavy traffic limit will be. However,  $c\mu_Y/g$  equals the mean number of vehicles that should depart during one slot in the green period if all vehicles would be delayed. So,  $\xi(g, c, \mu_Y) = c\mu_Y/g$  does not reflect the FCTL assumption, since a part of the vehicles passes the intersection without delay. Therefore, we multiply  $c\mu_Y/g$  with the probability that an arbitrary slot during the green period is actually used for the departure of delayed vehicles, i.e.

$$\xi(g, c, \mu_Y) = \frac{\mu_Y(c - g)}{g(1 - \mu_Y)} \cdot \frac{c\mu_Y}{g}, \quad (4.13)$$

where (see (2.6) and (2.7))

$$\frac{\mu_Y(c - g)}{g(1 - \mu_Y)} = \frac{1}{g} \left( g - \sum_{k=1}^{g-1} q_k \right) = \frac{1}{g} \left( c - \sum_{k=1}^{g-1} q_k \right) \mu_Y. \quad (4.14)$$

Fig. 6 displays for  $g = r = 5$  and Poisson arrivals the exact value of the overflow divided by the heavy traffic limit, along with the above-mentioned scaling functions  $\xi(g, c, \mu_Y)$ .



**Figure 6:** The exact value of the overflow divided by the heavy traffic limit vs the scaling functions for  $g = r = 5$  and Poisson arrivals and  $0 \leq x < 1$  with  $x = \mu_Y c/g$ .

## 5 Numerical assessment

We now present some numerical results of the mean overflow and mean delay in the FCTL queue for various settings. We compare the newly derived bounds and approximation with the existing ones and assess their quality under various settings of the cycle length, the ratio green period over red period and the type of arrival process.

Table 1 and 2 display the bounds and approximations for the mean overflow, for Poisson arrivals and geometric arrivals, respectively. Observe that upper bound (3.7) and lower bound (3.13) yield very precise bounds, especially for higher values of  $x$ . Also note that the lower bound (3.13) sharpens (3.4), except in cases where  $\alpha < g\hat{q}_0$ .

Miller's approximation (4.7) and the new approximation (4.13) are very accurate for Poisson arrivals,  $g = (1/5)c$  and  $g = (1/2)c$ . In the situation where  $g = (4/5)c$  all approximations perform less well, but again (4.13) produces the best results for both a Poisson as well as a geometric arrival process.

We have seen in Sec. 2, that the impact of the mean overflow on the mean delay is increasing with  $\mu_Y$ . So indeed the difference between the exact value and the approximations of the mean delay is more substantial in these cases as can be seen in Table 3 and 4. Here we should remark that in almost all Poisson cases, Webster's formula is less accurate than our new approximative expression. In case of geometric arrivals, the mean overflow and the mean delay are approximated more precisely with the newly derived formula, than with the existing formulas of Miller (4.4) and Newell (4.5).

In Figs. 7-8 the relative difference of the mean delay is plotted as a function of  $x$  for Poisson and geometric arrivals respectively. With the graphical representation of this difference one can conclude at a single glance which approximation can be best used for what scenario. For these two scenarios the relative difference between new approximation (4.13) and the exact value is within five percent, where this difference is in general larger for most other approximations.

**Table 1: Mean overflow for Poisson arrivals (with  $x = \mu\gamma c/g$ ).**

$g = \frac{1}{2}c$		lb <sup>(3.4)</sup>	lb <sup>(3.13)</sup>	$EX_g$	ub <sup>(3.7)</sup>	ub <sup>(3.9)</sup>	ub <sup>(3.4)</sup>	(4.7)	(4.4)	(4.5)	(4.13)	
c	10	$x=0.5$	0.000	0.053	0.068	0.308	0.500	0.458	0.051	0.000	0.208	0.083
		0.7	0.022	0.333	0.440	0.867	1.167	1.539	0.466	0.667	0.697	0.440
		0.9	2.966	3.082	3.400	4.066	4.500	7.916	3.593	4.000	3.818	3.314
		0.98	22.795	22.795	23.225	23.815	24.500	47.785	23.528	24.000	23.722	23.068
	20	$x=0.5$	0.000	0.131	0.022	0.321	0.500	0.771	0.015	0.000	0.089	0.083
		0.7	0.000	0.000	0.276	0.881	1.167	2.560	0.275	0.667	0.457	0.440
		0.9	1.841	2.437	3.037	4.069	4.500	12.978	3.133	4.000	3.395	3.314
		0.98	21.570	21.852	22.761	23.865	24.500	77.797	22.944	24.000	23.207	23.068

$g = \frac{1}{5}c$		lb <sup>(3.4)</sup>	lb <sup>(3.13)</sup>	$EX_g$	ub <sup>(3.7)</sup>	ub <sup>(3.9)</sup>	ub <sup>(3.4)</sup>	(4.7)	(4.4)	(4.5)	(4.13)	
c	10	$x=0.5$	0.044	0.131	0.174	0.404	0.500	0.404	0.152	0.000	0.397	0.111
		0.7	0.525	0.553	0.684	1.769	1.167	1.328	0.744	0.667	0.982	0.532
		0.9	3.670	3.670	3.850	4.080	4.500	6.622	4.057	4.000	4.225	3.556
		0.98	23.594	23.594	23.781	23.996	24.500	39.353	24.059	24.000	24.186	23.413
	20	$x=0.5$	0.000	0.063	0.098	0.409	0.500	0.724	0.070	0.000	0.253	0.111
		0.7	0.000	0.371	0.527	1.012	1.167	2.373	0.533	0.667	0.772	0.532
		0.9	2.950	3.172	3.590	4.1802	4.500	11.806	3.721	4.000	3.931	3.556
		0.98	22.810	22.810	23.474	24.016	24.500	70.085	23.679	24.000	23.855	23.413

$g = \frac{4}{5}c$		lb <sup>(3.4)</sup>	lb <sup>(3.13)</sup>	$EX_g$	ub <sup>(3.7)</sup>	ub <sup>(3.9)</sup>	ub <sup>(3.4)</sup>	(4.7)	(4.4)	(4.5)	(4.13)	
c	10	$x=0.5$	0.000	0.012	0.022	0.157	0.500	0.187	0.023	0.000	0.123	0.042
		0.7	0.000	0.180	0.238	0.520	1.167	0.689	0.332	0.667	0.535	0.260
		0.9	2.494	2.541	2.714	3.208	4.500	4.258	3.292	4.000	3.543	2.604
		0.98	21.901	21.901	22.085	22.657	24.500	29.310	23.153	24.000	23.392	21.787
	20	$x=0.5$	0.000	0.000	0.004	0.163	0.500	0.267	0.005	0.000	0.036	0.042
		0.7	0.000	0.000	0.129	0.529	1.167	0.950	0.170	0.667	0.301	0.260
		0.9	1.774	2.052	2.396	3.209	4.500	5.554	2.769	4.000	3.543	2.604
		0.98	21.117	21.175	21.647	22.667	24.500	36.993	22.428	24.000	22.749	21.787

**Table 2: Mean overflow for geometric arrivals (with  $x = \mu\gamma c/g$ ).**

$g = \frac{1}{2}c$		lb <sup>(3.4)</sup>	lb <sup>(3.13)</sup>	$EX_g$	ub <sup>(3.7)</sup>	ub <sup>(3.9)</sup>	ub <sup>(3.4)</sup>	(4.4)	(4.5)	(4.13)	
c	10	$x=0.5$	0.000	0.085	0.120	0.392	0.625	0.542	0.000	0.316	0.104
		0.7	0.337	0.552	0.706	1.182	1.575	1.853	0.900	1.076	0.594
		0.9	4.807	4.846	5.181	5.907	6.525	9.757	5.800	5.802	4.805
		0.98	34.564	34.564	34.932	35.584	36.505	59.554	35.760	35.685	34.372
	20	$x=0.5$	0.000	0.000	0.048	0.404	0.625	0.854	0.000	0.153	0.104
		0.7	0.000	0.154	0.490	1.195	1.575	2.874	0.900	0.759	0.594
		0.9	3.682	4.180	4.745	5.909	6.525	14.819	5.800	5.271	4.805
		0.98	33.339	33.474	34.390	35.634	36.505	89.567	35.760	35.051	34.372

$g = \frac{1}{5}c$		lb <sup>(3.4)</sup>	lb <sup>(3.13)</sup>	$EX_g$	ub <sup>(3.7)</sup>	ub <sup>(3.9)</sup>	ub <sup>(3.4)</sup>	(4.4)	(4.5)	(4.13)	
c	10	$x=0.5$	0.089	0.161	0.211	0.449	0.550	0.449	0.000	0.458	0.122
		0.7	0.677	0.677	0.826	1.107	1.330	1.480	0.760	1.160	0.606
		0.9	4.460	4.460	4.627	4.871	5.310	7.412	4.720	5.054	4.196
		0.98	28.372	28.372	28.545	28.774	29.302	44.131	28.704	29.009	28.002
	20	$x=0.5$	0.000	0.079	0.125	0.454	0.550	0.769	0.000	0.300	0.122
		0.7	0.117	0.478	0.652	1.164	1.330	2.525	0.760	0.928	0.606
		0.9	3.740	3.913	4.344	4.971	5.310	12.596	4.720	4.731	4.196
		0.98	27.588	27.588	28.213	28.794	29.302	74.863	28.704	28.646	28.002

$g = \frac{4}{5}c$		lb <sup>(3.4)</sup>	lb <sup>(3.13)</sup>	$EX_g$	ub <sup>(3.7)</sup>	ub <sup>(3.9)</sup>	ub <sup>(3.4)</sup>	(4.4)	(4.5)	(4.13)	
c	10	$x=0.5$	0.000	0.034	0.057	0.223	0.700	0.253	0.000	0.255	0.058
		0.7	0.267	0.407	0.488	0.817	1.820	0.986	1.040	1.075	0.405
		0.9	4.809	4.816	4.981	5.523	7.740	6.573	6.880	6.632	4.478
		0.98	39.686	39.686	39.828	40.442	43.708	47.095	42.816	42.439	38.868
	20	$x=0.5$	0.000	0.000	0.018	0.229	0.700	0.333	0.000	0.100	0.058
		0.7	0.000	0.085	0.325	0.826	1.820	1.247	1.040	0.700	0.405
		0.9	4.089	4.269	4.592	5.524	7.740	7.869	6.880	6.632	4.478
		0.98	38.902	38.902	39.323	40.452	43.708	54.778	42.816	41.567	38.868



**Table 3: Mean delay for Poisson arrivals (with  $x = \mu\gamma c/g$ ).**

$g = \frac{1}{2}c$		ED	(4.6)	err(%)	(4.7)	err(%)	(4.4)	err(%)	(4.5)	err(%)	(4.13)	err(%)	
c	10	$x=0.5$	2.626	2.511	-4.4	2.581	-1.7	2.444	-6.9	2.999	14.2	2.667	1.5
		0.7	3.866	3.690	-4.6	3.923	1.5	4.365	12.9	4.432	14.6	3.866	0.0
		0.9	10.422	9.788	-6.1	10.812	3.7	11.635	11.6	11.266	8.1	10.248	-1.7
		0.98	50.371	49.394	-1.9	50.977	1.2	51.922	3.1	51.366	2.0	50.058	-0.6
	20	$x=0.5$	4.170	4.137	-0.8	4.151	-0.5	4.111	-1.4	4.348	4.3	4.333	3.9
		0.7	5.429	5.466	0.7	5.426	0.0	6.288	15.8	5.828	7.3	5.789	6.6
		0.9	11.962	11.675	-2.4	12.157	1.6	13.907	16.3	12.684	6.0	12.521	4.7
		0.98	51.893	51.310	-1.1	52.259	0.7	54.373	4.8	52.786	1.7	52.509	1.2

$g = \frac{1}{5}c$		ED	(4.6)	err(%)	(4.7)	err(%)	(4.4)	err(%)	(4.5)	err(%)	(4.13)	err(%)	
c	10	$x=0.5$	6.038	5.243	-13.2	5.849	-3.1	4.494	-25.6	8.022	32.9	5.481	-9.2
		0.7	9.270	7.773	-16.1	9.673	4.3	9.157	-1.2	11.253	21.4	8.260	-10.9
		0.9	25.853	23.200	-10.3	26.974	4.3	26.665	3.1	27.884	7.9	24.259	-6.2
		0.98	125.824	122.574	-2.6	127.233	1.1	126.936	0.9	127.880	1.6	123.955	-1.5
	20	$x=0.5$	8.920	8.587	-3.7	8.671	-2.8	8.049	-9.8	10.296	15.4	9.037	1.3
		0.7	11.949	11.031	-7.7	11.990	0.3	12.877	7.8	13.576	13.6	11.981	0.3
		0.9	28.346	26.270	-7.3	29.053	2.5	30.568	7.8	30.196	6.5	28.162	-0.6
		0.98	128.247	125.539	-2.1	129.287	0.8	130.916	2.1	130.179	1.5	127.935	-0.2

$g = \frac{4}{5}c$		ED	(4.6)	err(%)	(4.7)	err(%)	(4.4)	err(%)	(4.5)	err(%)	(4.13)	err(%)	
c	10	$x=0.5$	0.796	0.918	15.3	0.797	0.1	0.778	-2.3	0.880	10.6	0.813	2.1
		0.7	1.391	1.670	20.1	1.468	5.5	1.739	25.0	1.633	17.4	1.409	1.3
		0.9	5.039	5.413	7.4	5.613	11.4	6.315	25.3	5.862	16.3	4.930	-2.2
		0.98	29.616	30.092	1.6	30.876	4.3	31.877	7.6	31.159	5.2	29.263	-1.2
	20	$x=0.5$	1.114	1.241	11.3	1.115	0.1	1.111	-0.3	1.141	2.4	1.146	2.8
		0.7	1.758	2.062	17.3	1.791	1.9	2.194	24.8	1.897	7.9	1.864	6.0
		0.9	5.438	5.886	8.2	5.808	6.8	7.029	29.3	6.576	20.9	5.644	3.8
		0.98	30.024	30.639	2.0	30.946	3.1	32.803	9.3	31.325	4.3	30.189	0.6

**Table 4: Mean delay for geometric arrivals (with  $x = \mu\gamma c/g$ ).**

$g = \frac{1}{2}c$		ED	(4.4)	err(%)	(4.5)	err(%)	(4.13)	err(%)	
c	10	$x=0.5$	2.541	2.222	-12.6	3.065	20.6	2.500	-1.6
		0.7	4.272	4.700	10.0	5.086	19.0	4.027	-5.8
		0.9	13.937	15.188	9.0	15.193	9.0	13.178	-5.4
		0.98	73.775	75.432	2.2	75.281	2.0	72.655	-1.5
	20	$x=0.5$	4.017	3.889	-3.2	4.298	7.0	4.167	3.7
		0.7	5.722	6.623	15.8	6.312	10.3	5.950	4.0
		0.9	15.330	17.461	13.9	16.392	6.9	15.450	0.8
		0.98	75.142	77.883	3.6	76.464	1.8	75.106	0.0

$g = \frac{1}{5}c$		ED	(4.4)	err(%)	(4.5)	err(%)	(4.13)	err(%)	
c	10	$x=0.5$	5.977	4.099	-31.4	8.170	36.7	5.185	-13.2
		0.7	9.823	9.387	-4.4	12.048	22.7	8.366	-14.8
		0.9	29.685	30.187	1.7	31.997	7.8	27.348	-7.9
		0.98	149.633	150.441	0.5	151.988	1.6	146.875	-1.8
	20	$x=0.5$	8.767	7.654	-12.7	10.318	17.7	8.741	-0.3
		0.7	12.389	13.108	5.8	14.225	14.8	12.087	-2.4
		0.9	32.053	34.089	6.4	34.150	6.5	31.250	-2.5
		0.98	151.928	154.421	1.6	154.124	1.4	150.855	-0.7

$g = \frac{4}{5}c$		ED	(4.4)	err(%)	(4.5)	err(%)	(4.13)	err(%)	
c	10	$x=0.5$	0.770	0.722	-6.2	0.935	21.5	0.771	0.2
		0.7	1.656	2.104	27.1	2.133	28.8	1.589	-4.0
		0.9	7.849	9.734	24.0	9.487	20.9	7.351	-6.4
		0.98	51.788	55.317	6.8	54.871	6.0	50.654	-2.2
	20	$x=0.5$	1.070	1.056	-1.4	1.139	6.4	1.104	3.1
		0.7	1.978	2.559	29.4	2.283	15.4	2.044	3.3
		0.9	8.178	10.448	27.8	10.202	24.7	8.065	-1.4
		0.98	52.117	56.242	7.9	54.767	5.1	51.579	-1.0

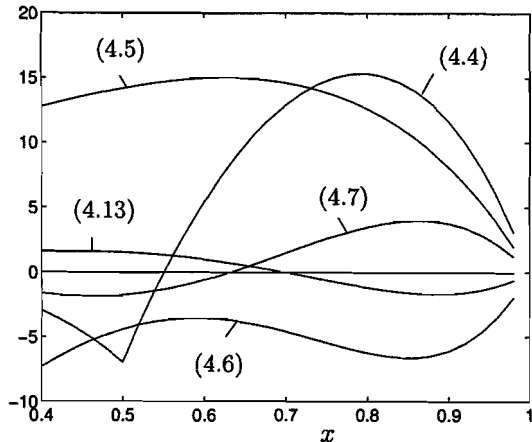


Figure 7:  $100\%(\mathbb{E}\bar{D} - \mathbb{E}D)/\mathbb{E}D$  for  $0.4 \leq x < 0.98$  (with  $x = \mu_Y c/g$ ) and  $g = r = 5$  and Poisson arrivals.

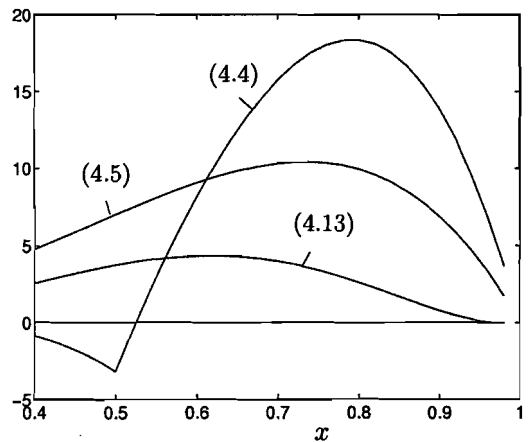


Figure 8:  $100\%(\mathbb{E}\bar{D} - \mathbb{E}D)/\mathbb{E}D$  for  $0.4 \leq x < 0.98$  (with  $x = \mu_Y c/g$ ) and  $g = r = 10$  and geometric arrivals.

## 6 Concluding remarks

Up to this day, Darroch's formula (2.5) for the mean overflow is often considered to be not of practical importance because of the required numerical determination of the roots of  $z^g = Y(z)^c$  on and within the unit circle. However, due to improved numerical algorithms and the increase in computational power, root-finding can be dealt with adequately nowadays, as discussed in Appendix A. Still, though, there is a strong incentive to derive bounds and approximations for the mean overflow that do not involve root-finding. This is because approximations might give more intuitive insight in the behavior of the performance characteristics and can be used for back-of-the-envelope computations. Furthermore, when one has only knowledge of the first two moments of  $Y$ , the exact approaches cannot be applied (or one should fit a distribution on these moments) while most bounds and approximations retain their value.

Many approximations and bounds in the literature have been derived for the mean overflow in the FCTL queue. In deriving these, most authors approximate the FCTL queue by the bulk service queue. In Sec. 2 we have compared both models and concluded that, due to the FCTL assumption (Assumption 2.3), the difference between the models is the largest for intermediate values of the traffic intensity. For low traffic intensities the overflow is negligible, while for high traffic intensities the models tend to each other. The latter is because, for high traffic intensities, the impact of the FCTL queue is marginal and the heavy traffic limit (which is the same in both models) is dominant.

We have presented several bounds and a new approximation for the mean overflow. In doing this, we have not relied on the bulk service queue approximation. The bounds have been derived from Darroch's exact solution. The approximation is based on the heavy traffic limit and a scaling argument that takes into account the FCTL queue assumption. The bounds and approximation hold for a general discrete arrival process and allow for a quick evaluation.

We compare the new approximation with the existing approximations and the newly derived bounds. The existing approximations are either obtained by neglecting the FCTL queue assumption or by assuming that the mean overflow and mean delay are not very sensitive to the detailed stochastic properties (see [15, 18]), so that the FCTL assumption can be ignored. Both considerations might cause problems. In particular, the results presented in Sec. 5 show that the differences in terms of performance characteristics between Poisson and geometric arrivals are considerable. The existing approximations perform less well for geometric arrivals and smaller cycle lengths,

both factors yielding a *more stochastic* system. Our new approximation remains sharp and we conclude that it is of importance to take the stochastic properties of the system into account. The results as presented in this paper allow one to consider distributions with a larger coefficient of variation, distributions with a finite support, or distributions fitted to empirical data.

## A Some analytical results

### A.1 Transform solutions

Darroch [7] derives the following expression for the probability generating function of the overflow  $X_g$ :

$$X_g(z) = \frac{Y(z)^g (\zeta(z) - 1) \sum_{j=0}^{g-1} q_j \zeta(z)^j}{z^g - Y(z)^c}, \quad (\text{A.1})$$

where  $q_j = \mathbb{P}(X_j = 0)$  and  $\zeta(z) = z/Y(z)$ . In this expression there are still  $g$  unknowns  $q_0, \dots, q_{g-1}$ , which can be found using the following classical approach (see e.g. Bailey [3], Darroch [7]). With Rouché's theorem, it can be shown that the denominator of (A.1) has  $g$  zeros on or within the unit circle  $|z| \leq 1$ . Since a pgf is analytic and well-defined in  $|z| \leq 1$ , the numerator of  $X_g(z)$  should vanish at each of the zeros. This gives  $g$  equations. One of the zeros equals 1, and leads to a trivial equation. However, the normalization condition  $X_g(1) = 1$  provides an additional equation. Using l'Hôpital's rule, this condition is found to be (2.6).

Denote the  $g$  roots of  $z^g = Y(z)^c$  on and within the unit circle by  $z_0 = 1, z_1, \dots, z_{g-1}$ . The  $g$  unknowns  $q_0, \dots, q_{g-1}$  then follow from solving the set of linear equations

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \zeta(z_1) & \zeta(z_1)^2 & \dots & \zeta(z_1)^{g-1} \\ 1 & \zeta(z_2) & \zeta(z_2)^2 & \dots & \zeta(z_2)^{g-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \zeta(z_{g-1}) & \zeta(z_{g-1})^2 & \dots & \zeta(z_{g-1})^{g-1} \end{pmatrix} \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ \vdots \\ q_{g-1} \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (\text{A.2})$$

The above system can be solved by applying Cramer's rule. The system can then be transformed into a Vandermonde system, leading to the following explicit solution for  $q_0, \dots, q_{g-1}$  (with  $\tau_k = \zeta(z_k)$ ):

$$q_j = \alpha (-1)^{j+2} \frac{1}{\prod_{k=1}^{g-1} (\tau_k - 1)} \sum_{1 \leq i_1 < i_2 < \dots < i_{g-1-j} \leq g-1} \tau_{i_1} \tau_{i_2} \dots \tau_{i_{g-1-j}}. \quad (\text{A.3})$$

The expression for  $\mathbb{E}X_g$  given by (2.5) follows from taking the first derivative to  $z$  of  $X_g(z)$  and evaluating this in  $z = 1$ .

For the bulk service queue, the pgf of  $\bar{X}_g$  is given by (see e.g. Bailey [3])

$$\bar{X}_g(z) = \sum_{j=0}^{g-1} \frac{x_j (z^g - z^j)}{z^g - Y(z)^c}, \quad (\text{A.4})$$

where  $x_j = \mathbb{P}(\bar{X}_g + A = j)$ . Using the identity (2.10) and a similar reasoning as for the FCTL queue, the  $g$  unknowns  $x_0, \dots, x_{g-1}$  follow from solving the set of linear equations

$$\begin{pmatrix} g & g-1 & \dots & 1 \\ z_1^g - 1 & z_1^g - z_1 & \dots & z_1^g - z_1^{g-1} \\ z_2^g - 1 & z_2^g - z_2 & \dots & z_2^g - z_2^{g-1} \\ \vdots & \vdots & \vdots & \vdots \\ z_{g-1}^g - 1 & z_{g-1}^g - z_{g-1} & \dots & z_{g-1}^g - z_{g-1}^{g-1} \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{g-1} \end{pmatrix} = \begin{pmatrix} \alpha_2 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \quad (\text{A.5})$$

As in case of the FCTL queue, the above system can be solved by applying Cramer's rule and leads to the following explicit solution for  $x_0, \dots, x_{g-1}$ :

$$x_j = \alpha_2 (-1)^{j+2} \frac{1}{\prod_{k=1}^{g-1} (z_k - 1)} \sum_{0 \leq i_1 < i_2 < \dots < i_{g-j} \leq g-1} z_{i_1} z_{i_2} \dots z_{i_{g-j}}. \quad (\text{A.6})$$

## A.2 Numerical determination of the roots

For the above calculations we need to determine the roots  $z_0 = 1, z_1, \dots, z_{g-1}$ . Since this issue of root-finding goes a long way back in queueing theory, it has often been addressed, both from an analytical and numerical perspective. We now give a short overview of this root-finding for the Poisson case  $Y(z) = \exp(\lambda(z-1))$ ,  $\lambda < g/c$ , and point out where extensions can be made to other distributions of  $Y$ .

The easiest way to determine the roots in the Poisson case is to apply successive substitution to a fixed-point equation. We know that the  $g$  roots of  $z^g = Y(z)^c$  in  $|z| \leq 1$  satisfy

$$z = wY(z)^{c/g} = w \exp(c\lambda(z-1)/g), \quad (\text{A.7})$$

$w^g = 1$ . For each feasible  $w$ , (A.7) can be shown to have one unique root in  $|z| \leq 1$ . Moreover, the equations can be solved by successive substitutions as

$$z_k^{(n+1)} = w_k Y(z_k^{(n)})^{c/g}, \quad k = 0, 1, \dots, g-1, \quad (\text{A.8})$$

where  $w_k = \exp(2\pi i k/g)$  and starting values  $z_k^{(0)} = 0$ . It can be shown that the fixed point equations (A.8) lead to convergence to the desired roots. Adan & Zhao [1] distinguish a class of compound Poisson distributions for which the method works. For more general discrete distributions, the method is further investigated in [12].

For the Poisson case, an exact description of the roots can be obtained as well. In [12] it is shown, using the Lagrange inversion theorem, that the roots are given by (with  $\theta = c\lambda/g$ )

$$z_k = \sum_{l=1}^{\infty} e^{-l\theta} \frac{(l\theta)^{l-1}}{l!} w_k^l, \quad k = 0, 1, \dots, g-1. \quad (\text{A.9})$$

One could truncate the infinite series over  $l$  in (A.9) to determine the roots. For a large class of discrete distributions, exact expressions for the roots, similar to (A.9), are derived in [12].

Although the class of distributions of  $Y$  for which one can derive an exact expression like (A.9) is far larger than the class for which the method of successive substitutions (A.8) works, see [12], neither method works for all distributions. Therefore, the most general method relies on numerical techniques. Chaudhry et al. [6] have developed a software program to solve root-finding problems in queueing theory numerically, which works in our experience for almost all distributions.

## B Mean delay for Poisson arrivals

In this appendix we present an elementary derivation of the mean delay expression in (2.3) for Poisson arrivals ( $\mu_Y = \sigma_Y^2 = \lambda$ ), where we rely on the PASTA property and Little's law. We show that the delay of an arbitrary vehicle can be divided into five different parts, each of which is discussed below.

(i) An arriving vehicle finds on average  $\mathbb{E}L^Q$  vehicles in the queue and each of them has a deterministic service time equal to one slot.

(ii) By PASTA we know that the probability that a vehicle arrives during the effective red period is equal to the fraction of time the signal is red during a cycle. This fraction is equal to  $r/c$  and the residual effective red period is given by  $r/2$ .

(iii) When a vehicle arrives, there may be a vehicle in service (driving off at that moment), with a mean residual service time given by  $1/2$ . Again by PASTA, we know that the probability a vehicle finds another vehicle in service, is equal to the fraction of time in a cycle spent on service. For the mean time during a green period that is actually used for the departure of delayed vehicles,  $\mathbb{E}G$ , the following relationship holds

$$\mathbb{E}G = \mu_Y r + \mu_Y \mathbb{E}G. \quad (\text{B.1})$$

(iv) It can happen that a vehicle arrives, but cannot be served in the next green period. Let  $Z$  be the number of times an extra red period of length  $r$  has to be waited. The following version of Little's law can be given

$$\mathbb{E}Z\mu_Y = \frac{1}{c}\mathbb{E}X_g. \quad (\text{B.2})$$

(v) If a vehicle arrives during the green period and there are no vehicles waiting, its service time is equal to zero; otherwise it is equal to 1. By PASTA the probability a vehicle arrives and cannot drive off immediately is equal to the fraction of time the system is not in idle mode, which is during the red period and  $G$ . So with a probability equal to

$$\frac{1}{c}(r + \mathbb{E}G) = \frac{r}{c(1 - \mu_Y)}, \quad (\text{B.3})$$

the service time is equal to 1.

By combining (i) to (v) it follows that the mean delay (without possible service time)  $\mathbb{E}W$  is given by

$$\mathbb{E}W = \mathbb{E}L^Q + \frac{r^2}{2c} + \frac{\mu_Y r}{2c(1 - \mu_Y)} + \frac{r\mathbb{E}X_g}{\mu_Y c}. \quad (\text{B.4})$$

With Little ( $\mathbb{E}L^Q = \lambda\mathbb{E}W$ ) we can now easily determine the mean delay (without possible service time), which is equal to

$$\mathbb{E}W = \frac{r^2}{2c(1 - \mu_Y)} + \frac{\mu_Y r}{2c(1 - \mu_Y)^2} + \frac{r\mathbb{E}X_g}{\mu_Y c(1 - \mu_Y)}. \quad (\text{B.5})$$

Consequently, the mean delay of a vehicle (with possible service time included) is

$$\mathbb{E}D = \frac{r^2}{2c(1 - \mu_Y)} + \frac{\mu_Y r}{2c(1 - \mu_Y)^2} + \frac{r\mathbb{E}X_g}{\mu_Y c(1 - \mu_Y)} + \frac{r}{c(1 - \mu_Y)}, \quad (\text{B.6})$$

which is equal to (2.3).

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