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EINDHOVEN UNIVERSITY OF TECHNOLOGY
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**Fitting discrete distributions one the
first two moments**

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FITTING DISCRETE DISTRIBUTIONS ON THE FIRST TWO MOMENTS

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Abstract. In this paper we present a simple method to fit a discrete distribution on the first two moments of a given distribution. With the fitted distribution we solve approximately Lindley's equation for the discrete-time $D/G/1$ queue using a moment-iteration method. Numerical results show excellent performance of the method.

Key words. two-moment fits, discrete distributions, Lindley's equation, moment-iteration method

AMS subject classifications. 60K25, 90B22

1. Introduction. A common way to fit a continuous distribution on the mean, EX , and the coefficient of variation, c_X , of a given distribution is the following (see e.g. Tijms [9]). In case $0 < c_X < 1$ one fits a mixture of Erlangian distributions with the same scale parameter. More specifically, if $1/k \leq c_X^2 \leq 1/(k-1)$, for certain $k = 2, 3, \dots$, one can fit a so-called $E_{k-1,k}$ distribution. A random variable having this distribution is with probability q (respectively $1-q$) distributed as the sum of $k-1$ (respectively k) independent exponentials with common mean. In case $c_X \geq 1$ one fits a hyperexponential with balanced means. A random variable having this distribution is with probability q_1 (respectively q_2) distributed as an exponential variable with parameter μ_1 (respectively μ_2) where $q_1 + q_2 = 1$ and $q_1/\mu_1 = q_2/\mu_2$.

Many applications involve discrete random variables. In such cases it is, of course, more natural to fit discrete distributions instead of continuous ones. The aim of this paper is to construct a discrete analogue of the method described above. It turns out that this problem is not trivial. In fact, in order to fit all possible values of EX and c_X , we need four classes of distributions instead of only two. Ord [8] treats a problem related to the present one. He fits discrete distributions not on the first two moments, but on two parameters depending on the first three moments of a given distribution.

After the construction of the discrete fits on the first two moments, we use these fits for approximating the waiting-time characteristics of the discrete-time $D/G/1$ queue. This queueing model naturally arises, for example, in the analysis of a periodic review (R, S) inventory system and in the analysis of a fixed-cycle traffic light queue. We use a discrete version of the moment-iteration method of de Kok [5], originally introduced for the continuous-time case. Furthermore, we will check the quality of this method by comparing it, for several service-time distributions, with an exact numerical analysis, Fredericks' approximation (see Fredericks [4]), and its continuous-time version.

The rest of the paper is organized as follows. In section 2 we answer the question how to fit a discrete distribution on the first two moments of a given distribution. Next, in section 3 we describe the discrete version of de Kok's iteration method for approximating the waiting-time characteristics of the discrete-time $D/G/1$ queue. Finally, section 4 is devoted to numerical results.

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2. Fitting discrete distributions. Consider an arbitrary discrete distribution on the non-negative integers with first moment EX and coefficient of variation c_X . We use four classes of distributions in order to choose a distribution which has the same mean and coefficient of variation as the given one. These classes of distributions are Poisson, mixtures of binomial, mixtures of negative-binomial and mixtures of geometric distributions with balanced means.

Let us first introduce some notation. A $GEO(p)$ random variable has probability distribution $p_i = (1-p)p^i$, $i = 0, 1, 2, \dots$, and an $NB(k, p)$ variable is the sum of k independent $GEO(p)$ variables. A $POIS(\lambda)$ random variable is Poisson distributed with mean λ , and a $BIN(k, p)$ variable is binomially distributed, where k is the number of trials and p the success probability.

If we define $a := c_X^2 - 1/EX$, then we have that $a \geq -1$, because $EX \leq EX^2$ for random variables on the non-negative integers. Lemma 2.1 shows how to fit a distribution on the first two moments depending on the value of a .

LEMMA 2.1. *Let X be a random variable on the non-negative integers with mean EX and coefficient of variation c_X and let $a = c_X^2 - 1/EX$. Then the random variable Y matches the first two moments of X if Y is chosen as follows:*

1. If $-1/k \leq a \leq -1/(k+1)$ for certain $k = 1, 2, 3, \dots$, then

$$Y = \begin{cases} BIN(k, p) & \text{w.p. } q, \\ BIN(k+1, p) & \text{w.p. } 1-q, \end{cases}$$

where

$$q = \frac{1 + a(1+k) + \sqrt{-ak(1+k) - k}}{1+a}, \quad p = \frac{EX}{k+1-q}.$$

2. If $a = 0$, then $Y = POIS(\lambda)$ with $\lambda = EX$.
3. If $1/(k+1) \leq a \leq 1/k$ for certain $k = 1, 2, 3, \dots$, then

$$Y = \begin{cases} NB(k, p) & \text{w.p. } q, \\ NB(k+1, p) & \text{w.p. } 1-q, \end{cases}$$

where

$$q = \frac{(1+k)a - \sqrt{(1+k)(1-ak)}}{1+a}, \quad p = \frac{EX}{k+1-q+EX}.$$

4. If $a \geq 1$, then

$$Y = \begin{cases} GEO(p_1) & \text{w.p. } q_1, \\ GEO(p_2) & \text{w.p. } q_2, \end{cases}$$

where

$$p_1 = \frac{EX[1+a+\sqrt{a^2-1}]}{2+EX[1+a+\sqrt{a^2-1}]}, \quad q_1 = \frac{1}{1+a+\sqrt{a^2-1}},$$

$$p_2 = \frac{EX[1+a-\sqrt{a^2-1}]}{2+EX[1+a-\sqrt{a^2-1}]}, \quad q_2 = \frac{1}{1+a-\sqrt{a^2-1}}.$$

It is straightforward to check that the given distributions indeed have the same mean and coefficient of variation as X . Therefore, the proof of the lemma is omitted.

The result of Lemma 2.1 is illustrated in Figure 1.

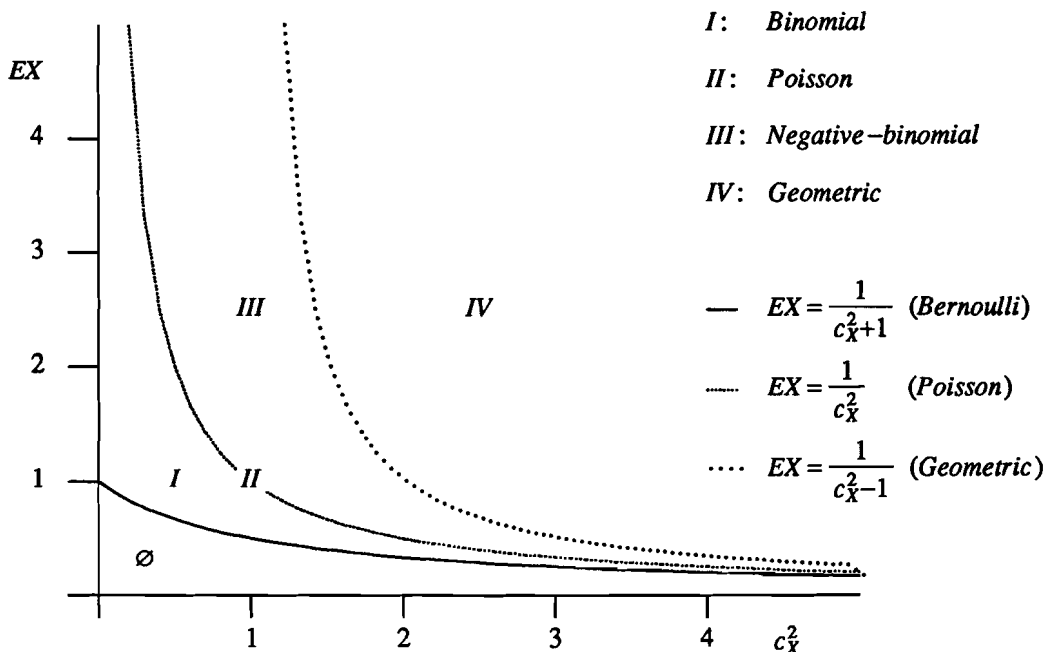


FIG. 1. The four classes of distributions used to match the first two moments of X .

3. Moment-iteration algorithm. In this section, we describe a refined, discrete version of de Kok’s moment-iteration method for approximating waiting-time characteristics of the discrete-time $D/G/1$ queue. Our interest in the discrete-time $D/G/1$ queue arose out of work on the fixed-cycle traffic light (see e.g. Darroch [2] and Newell [7]), and this queue also arises, for example, in the analysis of a periodic review (R, S) inventory system with finite production capacity (see e.g. de Kok [5]). The waiting-time characteristics considered here are the delay probability Π_W , and the first two moments of the waiting time. The moment-iteration method is used in the next section to illustrate the method of fitting discrete distributions as presented in the previous section.

Consider a discrete-time $D/G/1$ queue. Customers arrive at the server with fixed interarrival time A . The service-time probability distribution is given by $\{b_i\}_{i=0}^\infty$ with mean b . The service times of customers are assumed to be mutually independent and customers are served in order of arrival. Furthermore, we assume that the queue is stable, i.e., $\rho := b/A < 1$.

Let, for $n = 0, 1, 2, \dots$, the random variables B_n and W_n denote the service time, respectively, the waiting time of the n -th customer. Further, suppose that the 0-th customer arrives at an empty system, so that $W_0 = 0$. Then, it is easily seen that the following relation holds

$$W_n = \max\{0, W_{n-1} + B_{n-1} - A\}, \quad n = 1, 2, 3, \dots$$

This relation is the starting point for the approximation of the delay probability and the first two moments of the waiting-time distribution. Let the generic random variable B have probability distribution $\{b_i\}_{i=0}^\infty$, then the moment-iteration method (for the discrete-time $D/G/1$ queue) can be described as follows:

Moment-iteration algorithm for the discrete-time $D/G/1$ queue.

1. Initialization.

Set $W_0 := 0$, and compute exactly

$$\begin{aligned} P(W_1 > 0) &= P(B - A > 0), \\ EW_1 &= E(B - A)_+, \\ EW_1^2 &= E(B - A)_+^2, \end{aligned}$$

and set $n := 1$.

2. Iteration.

Set $V_n := (W_n | W_n > 0) - 1$, and compute

$$\begin{aligned} EV_n &= \frac{EW_n}{P(W_n > 0)} - 1, \\ EV_n^2 &= \frac{EW_n^2}{P(W_n > 0)} - \frac{2EW_n}{P(W_n > 0)} + 1. \end{aligned}$$

Fit a tractable probability distribution $\{v_i\}_{i=0}^{\infty}$ to the probability distribution of V_n by matching the first two moments as described in the previous section. Compute

$$\begin{aligned} EW_{n+1} &= E(B - A)_+ P(W_n = 0) + E(1 + V_n + B - A)_+ P(W_n > 0) \\ (1) \quad &= P(W_n = 0) \sum_{i=A+1}^{\infty} b_i(i - A) + P(W_n > 0) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} v_i b_j (1 + i + j - A)_+, \end{aligned}$$

and also

$$\begin{aligned} P(W_{n+1} > 0) &= P(W_n = 0)P(B - A > 0) + P(W_n > 0)P(1 + V_n + B - A > 0), \\ EW_{n+1}^2 &= E(B - A)_+^2 P(W_n = 0) + E(1 + V_n + B - A)_+^2 P(W_n > 0), \end{aligned}$$

where the right-hand sides can be worked out similarly as in (1).

3. Convergence.

If it holds that

$$|EW_{n+1} - EW_n| < \epsilon_1 \quad \text{and} \quad |EW_{n+1}^2 - EW_n^2| < \epsilon_2,$$

then stop, otherwise set $n := n + 1$ and repeat step 2.

4. Stop.

Approximate Π_W by $P(W_{n+1} > 0)$, and the first two moments of the waiting-time distribution by EW_{n+1} and EW_{n+1}^2 , respectively.

In the next section, this algorithm will be applied to the $D/G/1$ queue for several discrete service-time distributions.

4. Numerical results. In this section we present numerical results to illustrate the quality of the approximations obtained with the moment-iteration method. In Table 1 we list the moment approximations (mit) and the exact values (exa) of the delay probability Π_W and the mean waiting time EW for $D/G/1$ queues with binomial, negative-binomial and uniform service times, respectively. The performance characteristics Π_W and EW are evaluated for a range of values of the systems parameters.

We also compared the moment approximation with the well-known approximation of Fredericks (Fred) for the $D/G/1$ queue (see [4]). The results in Table 1 are based on Fredericks' approximation for discrete service-time distributions, which can easily be derived from the continuous version in [4]. The exact results for the $D/BIN(k,p)/1$ queue have been obtained with use of the matrix-geometric approach developed by Neuts [6] (see also [3]). The $D/NB(k,p)/1$ queue has been analyzed exactly by using an embedded Markov chain approach similar as in [1] and the exact results for the $D/U(0,k)/1$ have been obtained by numerically solving the embedded Markov chain using the approach of Tijms and van de Coevering [10], where $U(0,k)$ denotes the uniform distribution on $0, 1, \dots, k$.

TABLE 1
The delay probability Π_w and the mean waiting time EW .

	A	k	p	ρ	Π_w			EW			
					exa	mit	Fred	exa	mit	Fred	
$D/BIN(k,p)/1$	1	2	0.25	0.5	0.1111	0.1111	0.1111	0.1250	0.1250	0.1250	
	1	2	0.45	0.9	0.6694	0.6694	0.6694	2.0250	2.0243	2.0250	
	1	2	0.475	0.95	0.8186	0.8186	0.8186	4.5125	4.5099	4.5125	
	2	10	0.1	0.5	0.1025	0.1025	0.0998	0.1371	0.1371	0.1338	
	2	10	0.18	0.9	0.6885	0.6887	0.6802	3.0759	3.0756	3.0484	
	2	10	0.19	0.95	0.8324	0.8326	0.8270	7.0413	7.0396	7.0080	
	6	10	0.3	0.5	0.0112	0.0112	0.0111	0.0132	0.0132	0.0131	
	6	10	0.54	0.9	0.4701	0.4704	0.4579	1.2498	1.2497	1.2270	
	6	10	0.57	0.95	0.6873	0.6877	0.6762	3.1878	3.1873	3.1533	
	30	40	0.6	0.8	0.0163	0.0163	0.0161	0.0260	0.0260	0.0257	
$D/NB(k,p)/1$	30	40	0.7125	0.95	0.3972	0.3975	0.3806	1.3393	1.3389	1.3028	
	30	40	0.735	0.98	0.6941	0.6948	0.6772	4.9204	4.9188	4.8484	
	10	1	0.8333	0.5	0.2129	0.2129	0.2129	1.6232	1.6232	1.6232	
	10	1	0.9	0.9	0.8075	0.8075	0.8075	41.942	41.941	41.942	
	10	1	0.9048	0.95	0.9018	0.9018	0.9018	96.472	96.467	96.472	
	10	5	0.5	0.5	0.0760	0.0711	0.0690	0.2256	0.2077	0.2028	
	10	5	0.6429	0.9	0.7025	0.7006	0.6872	10.520	10.505	10.383	
	10	5	0.6552	0.95	0.8433	0.8424	0.8337	25.279	25.268	25.114	
	$D/U(0,k)/1$	5	6	-	0.6	0.1727	0.1727	0.1727	0.2087	0.2087	0.2087
		5	8	-	0.8	0.5341	0.5362	0.5100	2.1274	2.1234	2.0814
5		9	-	0.9	0.7535	0.7564	0.7289	6.8230	6.8194	6.7216	
10		15	-	0.75	0.4430	0.4439	0.4180	2.2113	2.2058	2.1545	
10		18	-	0.9	0.7586	0.7615	0.7290	12.312	12.300	12.107	
10		19	-	0.95	0.8765	0.8785	0.8574	30.302	30.326	30.060	

The results in Table 1 show good performance of Fredericks' approximation and excellent performance of the moment approximation. Fredericks' approximation is exact for the $D/GEO(p)/1$ queue and the $D/BIN(k,p)/1$ and $D/U(0,k)/1$ queue with $k = A + 1$, since the waiting-time distribution for these queues is geometric.

Since we fit a tractable distribution to the distribution of $W_n | W_n > 0$, we are also able to compute an approximation for $P(W_{n+1} > i)$ for all $i = 1, 2, \dots$ in the same way as for $P(W_{n+1} > 0)$. From this, we can compute approximations for the α -percentiles of the waiting-time distribution. For each α with $0 \leq \alpha \leq 1$ the α -percentile is defined as the smallest i satisfying $P(W_{n+1} \leq i) \geq \alpha$. In Table 2 we list the moment approximations and the exact values of the α -percentiles for the $D/G/1$ queues in Table 1 with traffic load 0.9 and 0.95, where α is varied from 0.5, 0.9, 0.95 to 0.99. We see in Table 2 that the quality of the moment approximation is good.

TABLE 2
The α -percentiles of the waiting-time distribution.

	A	k	p	α -Percentiles								
				α	0.5		0.9		0.95		0.99	
				exa	mit	exa	mit	exa	mit	exa	mit	
$D/BIN(k, p)/1$	1	2	0.45	1	1	5	5	7	7	11	11	
	1	2	0.475	3	3	11	11	14	14	23	22	
	2	10	0.18	2	2	8	8	11	11	17	17	
	2	10	0.19	5	5	17	17	23	22	35	34	
	6	10	0.54	0	0	4	4	5	5	8	8	
	6	10	0.57	2	2	8	8	11	11	18	17	
	30	40	0.7125	0	0	4	4	6	6	10	10	
$D/NB(k, p)/1$	10	1	0.9	25	25	108	108	144	144	226	226	
	10	1	0.9048	63	63	235	235	308	308	480	480	
	10	5	0.6429	6	6	28	29	38	38	61	59	
	10	5	0.6552	16	16	63	63	83	82	130	125	
$D/U(0, k)/1$	5	9	-	4	4	17	17	23	22	36	33	
	10	18	-	8	8	31	31	42	40	66	60	
	10	19	-	20	22	73	72	96	92	150	136	

Instead of using discrete distributions one may also fit a continuous distribution $W(t)$ on the first two moments of $W_n|W_n > 0$ as described in the introduction. Then equation (1) for EW_{n+1} becomes

$$EW_{n+1} = E(B - A)_+ P(W_n = 0) + P(W_n > 0) \int_{t=0}^{\infty} \sum_{j=0}^{\infty} b_j(t + j - A)_+ dW(t)$$

and the equations for $P(W_{n+1} > 0)$ and EW_{n+1}^2 have to be adapted accordingly. In Table 3 we compare for several examples the quality of discrete (disc) and continuous (cont) fits. The results show that both fits perform excellent for the mean waiting time, but the discrete fits yield a significantly better approximation for the delay probability.

TABLE 3
Comparison of the quality of discrete and continuous fits.

	A	k	p	ρ	Π_w			EW		
					exa	disc	cont	exa	disc	cont
$D/BIN(k, p)/1$	1	2	0.25	0.5	0.1111	0.1111	0.1484	0.1250	0.1250	0.1349
	1	2	0.475	0.95	0.8186	0.8186	0.8704	4.5125	4.5099	4.5595
	2	5	0.3	0.5	0.0815	0.0815	0.0988	0.0990	0.0990	0.1027
	2	5	0.57	0.95	0.8081	0.8083	0.8486	5.2642	5.2630	5.2941
	2	10	0.1	0.5	0.1025	0.1025	0.1233	0.1371	0.1371	0.1413
	2	10	0.19	0.95	0.8324	0.8326	0.8651	7.0413	7.0396	7.0672
$D/NB(k, p)/1$	10	1	0.8333	0.5	0.2129	0.2129	0.2198	1.6232	1.6232	1.6197
	10	1	0.9048	0.95	0.9018	0.9018	0.9049	96.472	96.467	96.464
	5	3	0.4545	0.5	0.1302	0.1284	0.1381	0.3334	0.3283	0.3283
$D/U(0, k)/1$	5	3	0.6129	0.95	0.8689	0.8688	0.8795	23.112	23.108	23.112
	3	4	-	0.67	0.2757	0.2757	0.3076	0.3806	0.3806	0.3872
	3	5	-	0.83	0.5955	0.5979	0.6394	2.0802	2.0788	2.0893
	5	6	-	0.6	0.1727	0.1727	0.1859	0.2087	0.2087	0.2114
	5	9	-	0.9	0.7535	0.7564	0.7764	6.8230	6.8194	6.8211

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