Generalized eigenfunctions with applications to Dirac's formalism

GENERALIZED EIGENFUNCTIONS

with applications to

DIRAC'S FORMALISM

by

S.J.L. van Eijndhoven

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Contents.

Abstract.

In the first part of this paper a theory of generalized eigenfunctions is developed which is based on the theory of generalized functions introduced by De Graaf. For a finite number of commuting self-adjoint operators the existence of a complete set of simultaneous generalized eigenfunctions is proved. A major role in the construction of the proof is played by the commutative multiplicity theory.
The second part is devoted to an Ansatz for a mathematical interpretation of Dirac's formalism. Instead of employing rigged Hilbert space theory Dirac's bracket notion is reinterpreted and extended to the generalized function space $T_{X,A}$. In this way, the concepts of the Fourier expansion of kets, of the orthogonality of complete sets of eigenkets and of matrices of unbounded linear mappings, all in the spirit of Dirac, fit into a mathematical rigorous theory.

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Preliminaries.

The introduction of a theory of generalized eigenfunctions is closely related to a theory of generalized functions, of course. In [GeVi], ch. I, to this end the theory of rigged Hilbert spaces is introduced. Here we employ De Graaf's theory of generalized functions, see [G]. In these preliminaries the main features of this theory will be given.

In a Hilbert space X consider the evolution equation

(p.1) $$\frac{du}{dt} = -Au$$

where A is a positive, unbounded self-adjoint operator. A solution u of (p.1) is called a trajectory if u satisfies

(p.2.i) $\forall_{t>0} : u(t) \in X$

(p.2.ii) $\forall_{t>0}\forall_{\tau>0} : e^{-\tau A}u(t) = u(t+\tau)$.

We emphasize that $\lim_{t\downarrow 0} u(t)$ does not necessarily exist in X-sense. The complex vector space of all trajectories is denoted by $T_{X,A}$. The space $T_{X,A}$ is considered as a space of generalized functions in [G].

The analyticity space $S_{X,A}$ is defined to be the dense linear subspace of X consisting of smooth elements of the form $e^{-tA}h$ where $h \in X$ and $t > 0$. Hence $S_{X,A} = \bigcup_{t>0} e^{-tA}(X)$. For each $f \in S_{X,A}$, there exists $\tau > 0$ such that $e^{\tau A} f \in S_{X,A}$. Further, for each $F \in T_{X,A}$ we have $F(t) \in S_{X,A}$ for all $t > 0$. $S_{X,A}$ is the test function space in De Graaf's theory. In $T_{X,A}$ we take the topology induced by the seminorms

(p.3) $$F \mapsto \|F(t)\| \quad , \; F \in T_{X,A}.$$

Because of the trajectory property (p.2.ii) of elements in $T_{X,A}$, it is a Frêchet space with this topology. In $S_{X,A}$ we take the inductive limit topology. In [G], a set of seminorms on $S_{X,A}$ is produced which generates the inductive limit topology.

The pairing between $S_{X,A}$ and $T_{X,A}$ is defined by

$$\text{(p.4)} \qquad \langle g,F\rangle = (e^{\tau A} g, F(\tau)) \;,\; g \in S_{X,A},\; F \in T_{X,A}.$$

Here $(\cdot,\cdot)$ denotes the inner product in X. Definition (p.4) makes sense for $\tau > 0$ sufficiently small. Due to the trajectory property (p.2.ii) it does not depend on the choice of τ.

The space $S_{X,A}$ is nuclear if and only if A generates a semigroup of Hilbert-Schmidt operators on X. In this case A has an orthonormal basis (v_k) of eigenvectors with respective eigenvalues λ_k, say. Further, for all $t > 0$ the series $\sum_{k=1}^{\infty} e^{-\lambda_k t}$ converges. It can be shown that $f \in S_{X,A}$ if and only if there exists $\tau > 0$ such that

$$\text{(p.5)} \qquad (f,v_k) = O(e^{-\lambda_k \tau})$$

and $F \in T_{X,A}$ if and only if for all $t > 0$

$$\text{(p.6)} \qquad \langle v_k,F\rangle = O(e^{\lambda_k t}) \;.$$

Finally we remark that besides these topics in [G] there can also be found a detailed characterization of continuous linear mappings on these spaces, the introduction of four topological tensor product spaces, and four Kernel theorems.

0. Introduction

First I want to give an illustrative example for the general theory of this paper. Therefore, let $S_{X,A}$ be the test function space with $X = L_2(\mathbb{R})$ and $A = \frac{1}{2}\left(- \frac{d^2}{dx^2} + x^2 + 1 \right)$, the Hamiltonian operator of the harmonic oscillator. This $S_{X,A}$-space is one of the examples discussed in [G].

It is well-known that the Hermite functions ψ_k, $k = 0,1,\ldots$ are the eigenfunctions of A with eigenvalues $k + 1$. So for each $t > 0$, the operator e^{-tA} is Hilbert-Schmidt, and the spaces $S_{X,A}$ and $T_{X,A}$ are nuclear. The self-adjoint operator Q

$$(Qf)(x) = x\, f(x) \quad , \quad x \in \mathbb{R} ,$$

maps $S_{X,A}$ continuously into itself, and can be extended to a continuous linear mapping on $T_{X,A}$, denoted by Q, also.

The linear functional δ_{x_0}, given by

$$\delta_{x_0} : f \mapsto f(x_0)$$

is an eigenfunctional of Q with eigenvalue x_0. The question arises whether $\delta_{x_0} \in T_{X,A}$. The space $S_{X,A}$ consists of entire analytic functions. So for each $f \in S_{X,A}$, $f(x_0)$ exists, and can be written as

$$f(x_0) = \sum_{k=0}^{\infty} (f,\psi_k)\psi_k(x_0) \ .$$

Hence $\delta_{x_0} \in T_{X,A}$ if and only if the series

$$\delta_{x_0}(t) = \sum_{k=0}^{\infty} e^{-(k+1)t}\psi_k(x_0)\psi_k$$

converges in X for all $t > 0$. Because of the growth properties of $|\psi_k(x_0)|$ for large k, this is true in this special case.

In this paper only nuclear $S_{X,A}$ spaces are considered. This implies that all the operators e^{-tA}, $t > 0$, have to be Hilbert-Schmidt. So A has an orthonormal basis of eigenvectors $v_1, v_2, \ldots$ with respective eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ satisfying $\sum_{i=1}^{\infty} e^{-\lambda_i t} < \infty$ for all $t > 0$.
Let T be a self-adjoint operator in X which is continuous on $S_{X,A}$. Since T is self-adjoint, T can always be represented as a multiplication operator in a countably direct sum of L_2-spaces. For convenience in this introduction, we shall consider the special case that T is unitarily equivalent to multiplication by the identity function in $L_2(\mathbb{R},\mu)$ for some finite Borel measure μ. In other words, a unitary operator $U : X \to L_2(\mathbb{R},\mu)$ exists, such that $Q = UTU^*$ is given by

$$(Q f)(x) = x\, f(x)$$

on its domain $D(Q) = U(D(T))$. U maps $S_{X,A}$ continuously onto $S_{Y,B}$, where

$$Y = L_2(\mathbb{R},\mu) \quad \text{and} \quad B = UAU^* .$$

Put $\varphi_k = Uv_k$, $k = 1,2,\ldots$. Then the φ_k's establish an orthonormal basis in Y and they are the eigenvectors of B with eigenvalues $\lambda_1, \lambda_2, \ldots$.
Let $x_0 \in \sigma(T)$, the spectrum of T. It is obvious that x_0 is a (generalized) eigenvalue of T if and only if the linear functional $\Delta_{x_0} : f \mapsto f(x_0)$ is continuous on $S_{Y,B}$. This continuity condition is equivalent to the condition

$$(0.1) \qquad t \mapsto \sum_{k=1}^{\infty} e^{-\lambda_k t} \varphi_k(x_0) \varphi_k \in T_{Y,B} .$$

Of course, there is a problem here. In general $f(x_0)$ has no meaning for L_2-functions. Formula (0.1) makes sense only, if we can choose a representant from each equivalence class $\langle\varphi_k\rangle$ in a unique way. In case

$S_{Y,\mathcal{B}} \subset L_\infty(\mathbb{R},\mu)$ we could employ the lifting theory of Ionescu Tulcea (see [IT]). But in general $S_{Y,\mathcal{B}}$ is not contained in $L_\infty(\mathbb{R},\mu)$.
We shall prove that a unique choice of representants $\hat{\varphi}_k$ in the classes $\langle\varphi_k\rangle$, k = 1,2,... , implies a unique choice of representants in all classes $\langle f\rangle$ of $S_{Y,\mathcal{B}}$, just by defining

$$(0.2) \qquad \hat{f} := \sum_{k=1}^{\infty} (f,\varphi_k)\hat{\varphi}_k \ .$$

Here we take

$$(0.3) \qquad \hat{\varphi}_k \ : \ x \mapsto \lim_{h \downarrow 0} \left\{ \mu(Q_h(x))^{-1} \int_{Q_h(x)} \varphi_k \, d\mu \right\}$$

where $Q_h(x) = [x-h\, ,\, x+h]$. It is clear that Definition (0.3) does not depend on the choice of $\hat{\varphi}_k \in \langle\varphi_k\rangle$.
The general case that T is equivalent to multiplication by the identity function in a countably direct sum of L_2-spaces can be dealt with similarly.

In section 1 we shall show the existence of generalized eigenfunctions for a continuous self-adjoint operator T on $S_{X,A}$. In section 2 excerpts of the commutative multiplicity theory are given. For this theory we refer to Nelson ([N]) and Brown ([Br]). The main theorem in section 3 states that we can a priori remove a set of measure zero N out of the spectrum $\sigma(T)$ of T such, that for all points in $\sigma(T)\backslash N$ with multiplicity m, $0 \le m \le \infty$, there exist precisely m independent generalized eigenfunctions. Section 4 is devoted to a sketchy proof of the result that in an adapted form the conclusions of section 3 remain valid for an n-tuple of commuting self-adjoint operators. Finally, in section 5 an Ansatz is given for a mathematical interpretation of Dirac's formalism.

1. The existence of generalized eigenfunctions

In the sequel A will denote a positive self-adjoint operator in X which generates a semigroup of Hilbert-Schmidt operators. So A has an orthonormal basis of eigenvectors $v_1, v_2, \dots$ with respective eigenvalues $\lambda_1, \lambda_2, \dots$ satisfying $\sum_{i=1}^{\infty} e^{-\lambda_i t} < \infty$ for all $t > 0$. Further, T will denote a self-adjoint operator in X, which maps $S_{X,A}$ continuously into itself. The spectral resolution of T is denoted by $(H_\lambda)_{\lambda\in\mathbb{R}}$.
For $\mathfrak{f} \in X$, the subspace $X_{\mathfrak{f}}$ of X is defined to be the closure of the linear span of the set $\{H(\Delta)\mathfrak{f} \mid \Delta \subset \mathbb{R} \text{ a Borel set}\}$. Here $H(\Delta)$ denotes the spectral projection $\int_\Delta dH_\lambda$.

(1.1) Lemma

The subspace $X_{\mathfrak{f}}$ of X is unitarily equivalent to $L_2(\mathbb{R}, \rho_{\mathfrak{f}})$, where $\rho_{\mathfrak{f}}$ denotes the positive, finite Borel measure $(H_\lambda \mathfrak{f}, \mathfrak{f})_{\lambda\in\mathbb{R}}$.

Proof

The proof will be sketchy. A detailed proof can be found in [Br].
Let $g \in X_{\mathfrak{f}}$. Then there exist sequences $(\alpha_j^{(n)})_{j\in\mathbb{N}}$ and $(\Delta_j^{(n)})_{j\in\mathbb{N}}$ such that

$$(*) \qquad \lim_{n\to\infty} \| g - \sum_{j=1}^{j_n} \alpha_j^{(n)} H(\Delta_j^{(n)})\mathfrak{f} \| = 0 .$$

So we may conclude that the finite series

$$\sum_{j=1}^{j_n} \alpha_j^{(n)} H(\Delta_j^{(n)}\mathfrak{f} \quad , \quad n \in \mathbb{N},$$

are uniformly bounded. Then $\psi = \lim_{n\to\infty} \sum_{j=1}^{j_n} \alpha_j^{(n)} \Delta_j^{(n)}$ exists and because of the completeness of $L_2(\mathbb{R},\rho_\delta)$,

$$\int_{\mathbb{R}} |\psi|^2 \, d\rho_\delta < \infty .$$

By (*) g can be expressed as $g = \psi(T)\delta$ with $\|g\| = \|\psi\|_{L_2}$. On the other hand, if $\psi \in L_2(\mathbb{R},\rho_\delta)$, then

$$\psi = \lim_{n\to\infty} \sum_{j=1}^{j_n} \alpha_j^{(n)} \Delta_j^{(n)}$$

with the limit taken in L_2-sense. So obviously $g = \psi(T)\delta$.

The following equivalence holds

$$g \in X_\delta \Leftrightarrow \exists_{\psi\in L_2(\mathbb{R},\rho_\delta)} : g = \psi(T)\delta .$$

The operator $U : X_\delta \to L_2(\mathbb{R},\rho_\delta)$,

$$Ug = U(\psi(T)\delta) = \psi$$

is unitary. This completes the proof. □

(1.2) <u>Notation</u>

P denotes the set of $x \in \mathbb{R}$ which satisfy

$$\rho_\delta([x-\varepsilon , x+\varepsilon]) > 0$$

for every $\varepsilon > 0$.

For each $x \in P$, define

(1.3) $$G_{t,h}(x) := \mathrm{emb}\left\{ [\rho_\delta(Q_h(x))]^{-1} \int_{Q_h(x)} dH_\lambda \,\delta \right\}(t) \quad , \quad t > 0.$$

Here emb is the continuous linear mapping from X into $T_{X,A}$,

$$\mathrm{emb}(\omega) : t \mapsto e^{-tA}\omega \quad , \quad \omega \in X,$$

and $Q_h(x)$ the closed interval $[x-h, x+h]$.

Since $(v_k)_{k\in\mathbb{N}}$ is an orthonormal basis of eigenvectors of A the Fourier expansion of $G_{t,h}(x)$ is given by

$$G_{t,h}(x) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \left\{ \frac{\int_{Q_h(x)} d(H_\lambda \delta, v_k)}{\int_{Q_h(x)} d(H_\lambda \delta, \delta)} \right\} v_k \quad , \quad t > 0,\ h > 0 .$$

By Lemma (1.1) for each $k \in \mathbb{N}$ there exists $\varphi_k \in L_2(\mathbb{R}, \rho_\delta)$ such that

$$\int_{Q_h(x)} d(H_\lambda \delta, v_k) = \int_{Q_h(x)} \varphi_k \, d\rho_\delta \quad , \quad h > 0.$$

With the aid of Theorem 10.49 in [WZ] we can prove that the limit

$$\hat{\varphi}_k(x) = \lim_{h\downarrow 0} \rho_\delta(Q_h(x))^{-1} \int_{Q_h(x)} \varphi_k \, d\rho_\delta$$

is well defined for almost every $x \in P$ and every $k \in \mathbb{N}$, and $\hat{\varphi}_k$ can be interpreted as a representant of the L_2-class $\langle\varphi_k\rangle$ in the usual way.

Let $t > 0$. The function $\sum_{k\in\mathbb{N}} e^{-\lambda_k t}|\varphi_k|^2$ belongs to $L_1(\mathbb{R}, \rho_\delta)$. So there exists a null set N_t such that for all $x \in P\backslash N_t$

$$\sum_{k\in\mathbb{N}} e^{-\lambda_k t}|\hat{\varphi}_k(x)|^2 = \lim_{h\downarrow 0} \rho_\delta(Q_h(x))^{-1} \left(\int_{Q_h(x)} \sum_{k\in\mathbb{N}} e^{-\lambda_k t}|\varphi_k|^2 d\rho_\delta \right).$$

Put $N = \bigcup_{k\in\mathbb{N}} N_{1/k}$, and let $x \in P\backslash N$. Then N is a null set with respect to

ρ_{δ}. Since for each $t > 0$ there exists $n \in \mathbb{N}$ with $0 < \frac{1}{n} < t$,

$$\sum_{k\in\mathbb{N}} e^{-\lambda_k t} |\hat{\varphi}_k(x)|^2 \leq \sum_{k\in\mathbb{N}} e^{-\lambda_k \frac{1}{n}} |\hat{\varphi}_k(x)|^2 < \infty .$$

Define $G_{t,x}$ by

$$(1.4) \qquad G_{t,x} = \sum_{k=1}^{\infty} e^{-\lambda_k t} \hat{\varphi}_k(x) v_k \quad , \quad t > 0 .$$

Then $t \mapsto G_{t,x}$ is an element of $\mathcal{T}_{X,A}$.

Let $h \in \mathcal{S}_{X,A}$, and put

$$\hat{h} : x \mapsto \sum_{k\in\mathbb{N}} (h, v_k)\hat{\varphi}_k(x) \in L_2(\mathbb{R}, \rho_{\delta}) .$$

Then $|\hat{h}(x)| < \infty$ for all $x \in P\backslash N$. This can be seen as follows:

$$|\hat{h}(x)| \leq \Big(\sum_{k\in\mathbb{N}} e^{2\lambda_k t} |(h, v_k)|^2 \Big)^{\frac{1}{2}} \Big(\sum_{k\in\mathbb{N}} e^{-2\lambda_k t} |\hat{\varphi}_k(x)|^2 \Big)^{\frac{1}{2}} < \infty$$

for $t > 0$ small enough.

(1.5) <u>Theorem</u>

For each $x \in P$, $h > 0$ and $t > 0$, define

$$G_{t,h}(x) := \mathrm{emb}\Big\{ \rho_{\delta}(Q_h(x))^{-1} \int_{Q_h(x)} d\mathcal{H}_\lambda \, \delta \Big\}(t) .$$

Then there exists a null set N_{δ} with respect to ρ_{δ} such that

(i) $\quad G_{t,x} = \lim_{h\downarrow 0} G_{t,h}(x)$ exists for all $x \in P\backslash N_{\delta}$ and all $t > 0$.

(ii) $\quad G_x : t \mapsto G_{t,x} \in \mathcal{T}_{X,A}$ and $G_x \neq 0$ for all $x \in P\backslash N_{\delta}$.

(iii) $\mathcal{T}G_x = xG_x \quad$ for all $x \in P\backslash N_{\delta}$.

Proof

(1.5.i) Let $t > 0$, $\varepsilon > 0$ and let $x \in P\setminus N$, where N is the null set as defined above. Put $M_{x,t} = \left(\sum_{k=1} e^{-\lambda_k t} |\hat{\varphi}_k(x)|^2\right)^{\frac{1}{2}}$. Fix $k_0 \in \mathbb{N}$ so large that

$$e^{-\lambda_k t/2} < \varepsilon(M_{x,t} + 1)^{-1} \quad , \quad k \geq k_0 \ .$$

Then

$$(*) \qquad \Big\| \sum_{k=k_0+1}^{\infty} e^{-\lambda_k t} \hat{\varphi}_k(x) v_k \Big\|^2 = \sum_{k=k_0+1}^{\infty} e^{-2\lambda_k t} |\hat{\varphi}_k(x)|^2 \leq$$

$$\leq e^{-\lambda_{k_0} t} M_{x,t}^2 < \varepsilon^2 \ .$$

Further choose $h > 0$ so small that both

$$\Big|\rho_{\mathfrak{z}}(Q_h(x))^{-1} \int_{Q_h(x)} \varphi_k d\rho_{\mathfrak{z}} - \hat{\varphi}_k(x)\Big| < \varepsilon \quad , \quad k = 1,\ldots,k_0 \ ,$$

and

$$\rho_{\mathfrak{z}}(Q_h(x))^{-1} \int_{Q_h(x)} \left(\sum_{k=0}^{\infty} e^{-\lambda_k t} |\varphi_k|^2\right) d\rho_{\mathfrak{z}} < (M_{x,t} + 1)^2 \ .$$

Then

$$(**) \qquad \Big\| \sum_{k=1}^{k_0} e^{-\lambda_k t} \Big[\rho_{\mathfrak{z}}(Q_h(x))^{-1} \int_{Q_h(x)} \varphi_k d\rho_{\mathfrak{z}} - \hat{\varphi}_k(x) \Big] v_k \Big\| < \varepsilon \| e^{-tA} \|_{\mathfrak{X} \otimes X}$$

and

$$(***) \qquad \Big\| \sum_{k=k_0+1}^{\infty} e^{-\lambda_k t} \rho_{\mathfrak{z}}(Q_h(x)^{-1} \Big(\int_{Q_h(x)} \varphi_k d\rho_{\mathfrak{z}} \Big) v_k \Big\|^2 =$$

$$= \sum_{k=k_0+1}^{\infty} e^{-2\lambda_k t} \Big|\rho_{\mathfrak{z}}(Q_h(x))^{-1} \int_{Q_h(x)} \varphi_k d\rho_{\mathfrak{z}}\Big|^2 \leq$$

$$\leq e^{-\lambda_{k_0} t} \sum_{k=0}^{\infty} e^{-\lambda_k t} \rho_{\mathfrak{z}}(Q_h(x))^{-1} \int_{Q_h(x)} |\varphi_k|^2 d\rho_{\mathfrak{z}} < \varepsilon^2 \ .$$

A combination of the estimates (*), (**) and (***) gives the result

$$\| \operatorname{emb} \rho_{\delta}(Q_h(x))^{-1}\left(\int_{Q_h(x)} dH_\lambda \delta \right)(t) - G_{t,x} \| < \varepsilon(2 + \|e^{-tA}\|_{X \otimes X})$$

for h small enough, where $G_{t,x}$ is defined by (1.4).

(1.5.ii) If G_x is defined by $G_x : t \to G_{t,x}$, it is obvious that $G_x \in T_{X,A}$. Let Γ_0 be the set of all $x \in P \setminus N$ for which $G_x = 0$. We shall show that Γ_0 is a null set with respect to ρ_δ. Note first that $G_x = 0$ implies $\hat{\varphi}_k(x) = 0$ for all $k \in \mathbb{N}$. Hence Γ_0 is a Borel set. Put $\gamma = \int_{\Gamma_0} dE_\lambda \delta$ and let $k \in \mathbb{N}$. Then

$$(\gamma, v_k) = \int_{\Gamma_0} d(E_\lambda \delta, v_k) = \int_{\Gamma_0} \hat{\varphi}_k d\rho_\delta = 0 .$$

Hence $\gamma = 0$ and Γ_0 is a null set with respect to ρ_δ.

(1.5.iii) We have to show that $TG_x = xG_x$.
Since $T - xI$ is continuous on $T_{X,A}$,

$$(*) \qquad (T - xI) \lim_{h \downarrow 0} \rho_\delta(Q_h(x))^{-1} \int_{Q_h(x)} dH_\lambda \delta =$$

$$= \lim_{h \downarrow 0} (T - xI) \rho_\delta(Q_h(x))^{-1} \int_{Q_h(x)} dH_\lambda \delta .$$

Computing the latter limit, we obtain for every $t > 0$

$$\lim_{h \downarrow 0} \left\{ \sum_{k=0}^{\infty} e^{-\lambda_k t} \rho_\delta(Q_h(x))^{-1} \left(\int_{Q_h(x)} d(H_\lambda \delta, (T - xI) v_k) \right) v_k \right\} =$$

$$\lim_{h \downarrow 0} \left\{ \sum_{k=0}^{\infty} e^{-\lambda_k t} \rho_\delta(Q_h(x))^{-1} \left(\int_{Q_h(x)} (\lambda - x) \varphi_k(\lambda) \, d\rho_\delta \right) v_k \right\} .$$

This expression can be treated as follows.

$$\sum_{k=0}^{\infty} e^{-2\lambda_k t} \left| \rho_\delta(Q_h(x))^{-1} \int_{Q_h(x)} (\lambda - x)\, \varphi_k(\lambda)\, d\rho_\delta \right|^2 \leq$$

$$\leq \sum_{k=0}^{\infty} e^{-2\lambda_k t} \rho_\delta(Q_h(x))^{-1} \left(\int_{Q_h(x)} |\varphi_k(\lambda)|^2 d\rho_\delta \right) \cdot$$

$$\cdot\ \rho_\delta(Q_h(x))^{-1} \left(\int_{Q_h(x)} |\lambda - x|^2 d\rho_\delta \right) \leq$$

$$\leq h^2 (M_{x,t} + 1)^2 \quad \text{for } h \text{ small enough.}$$

So the limit (*) is null, and (1.5.iii) is proved. □

2. Commutative multiplicity theory

The commutative multiplicity theorem enables us to set up a theory, which ensures that the notion 'multiplicity of an eigenvalue' also makes sense for generalized eigenvalues. The so called multiplicity theory which leads to this theorem is mainly measure theoretical. It is very well described by Nelson in [Ne], ch. VI, and by Brown in [Br].

(2.1) Definition

Let ρ be a positive, finite Borel measure on $\mathbb{R}$. Then the support of ρ, $\mathrm{supp}(\rho)$, is defined by

$$\mathrm{supp}(\rho) := \{ r \in \mathbb{R} \mid \forall_{\varepsilon>0} : \rho([r-\varepsilon, r+\varepsilon]) > 0 \} .$$

(2.2) Lemma

Let ρ be a positive, finite Borel measure on $\mathbb{R}$. Then the complement of $\mathrm{supp}(\rho)$, $\mathrm{supp}(\rho)^*$, is a set of measure zero with respect to ρ.

Proof

For each $x \in \mathrm{supp}(\rho)^*$, define the set $Q_{x,\varepsilon} := [x-\varepsilon, x+\varepsilon]$ with $\varepsilon > 0$ taken so that $\rho(Q_{x,\varepsilon}) = 0$. Then

$$\mathrm{supp}(\rho)^* \subset \bigcup_{x\in \mathrm{supp}(\rho)^*} Q_{x,\varepsilon} .$$

Let $k \in \mathbb{N}$. The set $\mathrm{supp}(\rho)^* \cap [-k,k]$ is bounded in $\mathbb{R}$. With Besicovitch covering's Lemma ([WZ], p.185) it follows that there is a countable set $\{x_1, x_2, \ldots\}$ such that

$$\mathrm{supp}(\rho)^* \cap [-k,k] \subset \bigcup_{i=1}^{\infty} Q_{x_i,\varepsilon_i} .$$

Hence

$$\rho(\mathrm{supp}(\rho)^* \cap [-k,k]) = 0 .$$

Since $k \in \mathbb{N}$ is arbitrary, $\mathrm{supp}(\rho)^*$ itself is a set of measure zero. □

There is another charaterization of $\mathrm{supp}(\rho)$.

(2.3) Lemma

$\mathrm{supp}(\rho)$ is the complement of the largest measurable open set $\mathcal{O}$ for which $\rho(\mathcal{O}) = 0$.

Proof

Let $\mathrm{supp}_1(\rho)$ denote the complement of the largest measurable open null set, the set $\mathrm{supp}_1(\rho)$ is well defined (see [Bo], p. 16). Suppose $x \notin \mathrm{supp}_1(\rho)$. Then there exists $\varepsilon > 0$ such that the interval

$[x-\varepsilon, x+\varepsilon] \subset \mathrm{supp}_1(\rho)^*$. So $\rho([x-\varepsilon, x+\varepsilon]) = 0$, and $x \notin \mathrm{supp}(\rho)$. Conversely, suppose $x \notin \mathrm{supp}(\rho)$. Then there exists $\varepsilon > 0$ such that $\rho([x-\varepsilon, x+\varepsilon]) = 0$. This implies that $(x-\varepsilon, x+\varepsilon) \subset \mathrm{supp}_1(\rho)^*$. Hence $x \notin \mathrm{supp}_1(\rho)$, completing the proof. □

(2.4) Definition

The Borel measure ν is absolutely continuous with respect to the Borel measure μ, notation $\nu \ll \mu$, if for every Borel set N with $\mu(N) = 0$, also $\nu(N) = 0$.

The Borel measures ν and μ are equivalent, $\nu \sim \mu$, if $\nu \ll \mu$ and $\mu \ll \nu$.

It is clear that $\nu \sim \mu$ implies $\mathrm{supp}(\nu) = \mathrm{supp}(\mu)$. So it makes sense to write $\mathrm{supp}(\langle\nu\rangle)$ meaning the support of each ν in the equivalence class $\langle\nu\rangle$.

(2.5) Definition

Two equivalent classes $\langle\nu\rangle$ and $\langle\mu\rangle$ are called mutually disjoint if

$$\nu(\mathrm{supp}\langle\nu\rangle \cap \mathrm{supp}\langle\mu\rangle) = \mu(\mathrm{supp}\langle\nu\rangle \cap \mathrm{supp}\langle\mu\rangle) = 0 .$$

If one wants a canonical listing of the eigenvalues of a matrix it is natural to list all eigenvalues of multiplicity one, two, etc. We need a way of saying that an operator is of uniform multiplicity one, two, etc. To this end we introduce

(2.6) Definition

A self-adjoint operator T is said to be of uniform multiplicity m, $1 \le m \le \infty$, if T is unitarily equivalent to multiplication by the identity function in $L_2(\mathbb{R},\mu) \oplus \dots \oplus L_2(\mathbb{R},\mu)$, where there are m terms in the sum and μ is a finite Borel measure.

This definition makes sense because if T is also unitarily equivalent to multiplication by the identity function on $L_2(\mathbb{R},\nu) \oplus \ldots \oplus L_2(\mathbb{R},\nu)$ (n times), then m = n and $\mu \sim \nu$ (see [Br]).

(2.7) Theorem (Commutative multiplicity theorem)

Let T be a self-adjoint operator in a Hilbert space X. Then there exists a decomposition $X = X_\infty \oplus X_1 \oplus \ldots \oplus X_m \oplus \ldots$ so that

(i) T acts invariantly in each X_m

(ii) $T \restriction X_m$ has uniform multiplicity m

(iii) The measure classes $<\mu_m>$ associated with the spectral representation of $T \restriction X_m$ are mutually disjoint.

Further, the subspaces $X_\infty, X_1, X_2, \ldots$ (some of which may be zero) and the measure classes $<\mu_\infty>, <\mu_1>, <\mu_2>, \ldots$ are uniquely determined by (i). (ii) and (iii).

Proof

For a proof see Nelson, [N] ch. VI or Brown, [Br]. □

3. A total set of generalized eigenfunctions for the self-adjoint operator T

(3.1) Definition

A set $\Gamma \subset X$ is called cyclic with respect to T if

$$X = \bigoplus_{\gamma \in \Gamma} X_\gamma .$$

Since X is separable, Γ consists of an at most countable number of elements. If Γ can be choosen such that it consists of one element only, this element is called a cyclic vector and the operator T a cyclic operator. The cyclic set Γ is not uniquely determined. The commutative multiplicity theorem brings in some uniqueness.

(3.2) Lemma

T has uniform multiplicity one if and only if T is cyclic. (see Definition 2.6)

By Theorem (2.7) X can be splitted into a countable direct sum,

$$X = X_\infty \oplus X_1 \oplus X_2 \oplus \dots .$$

The restricted operator $T \upharpoonright X_m$, $1 \le m \le \infty$, is unitarily equivalent to multiplication by the identity function in

$$L_2(\mathbb{R},\mu_m) \oplus \dots \oplus L_2(\mathbb{R},\mu_m) \quad , \quad \text{(m times)}.$$

By X_{mj}, $j = 1,\dots,m$, we denote the orthogonal subspace of X_m, which corresponds to the j-th term in the direct sum. Since $T \upharpoonright X_{mj}$ obviously has uniform multiplicity one, there exists a cyclic vector $\gamma_j^{(m)}$ for $T \upharpoonright X_{mj}$. Thus we obtain a set Γ,

$$\Gamma := \{\gamma_j^{(m)} \mid 1 \le j < m+1 \ , \ 1 \le m \le \infty\} ,$$

which is cyclic for T. Note that $1 \le m \le \infty$ means $m = \infty,1,2,\dots$.

Let m, $1 \le m \le \infty$, be fixed so that $X_m \neq \{0\}$, and let j, $1 \le j < m+1$ be fixed. Further, let $\rho_j^{(m)}$ denote the finite Borel measure $\left(\left(H_\lambda \gamma_j^{(m)}, \gamma_j^{(m)} \right)\right)_{\lambda \in \mathbb{R}}$. The projection from X onto X_{mj} is denoted by $P_j^{(m)}$ and the unitary operator from X_{mj} onto $L_2(\mathbb{R},\rho_{\gamma_j^{(m)}})$ by $U_j^{(m)}$. Finally, put $\hat{v}_{k,j}^{(m)} = U_j^{(m)} P_j^{(m)} v_k$.

From Theorem (1.3) we obtain sets $N_j^{(m)}$ of measure zero with respect to $\rho_j^{(m)}$, $m = \infty, 1, 2, \ldots$, such that for each $\sigma \in \mathrm{supp}(\rho_j^{(m)}) \setminus N_j^{(m)}$

$$G_{\sigma,j}^{(m)} : t \to \sum_{k=1}^{\infty} e^{-\lambda_k t} \overline{v_{k,j}^{(m)}(\sigma)}\, v_k$$

is in $T_{X,A}$, and

$$T\, G_{\sigma,j}^{(m)} = \sigma\, G_{\sigma,j}^{(m)} .$$

Following Theorem (2.7) $\rho_i^{(m)} \sim \rho_j^{(m)}$ for all i, $1 \le i < m+1$, i.e. the set $N_j^{(m)}$ is a null set with respect to each $\rho_i^{(m)}$. Put $N^{(m)} = \bigcup_{j=1}^{m} N_j^{(m)}$.

(3.3) Theorem

Let m, $1 \le m \le \infty$, be taken such that $X_m \neq \{0\}$. Then there exists a null set $N^{(m)}$ with respect to $\langle\mu_m\rangle$ with the property that for every $\sigma \in \mathrm{supp}(\langle\mu_m\rangle) \setminus N^{(m)}$ there are precisely m independent generalized eigenfunctions with eigenvalue σ. Further, the set

$$\{G_{\sigma,j}^{(m)} \mid 1 \le j < m+1,\ 1 \le m \le \infty,\ \sigma \in \mathrm{supp}(\langle\mu_m\rangle) \setminus N^{(m)}\}$$

is total.

Proof

Since the measure classes $\langle\mu_m\rangle$ are mutually disjoint, the first assertion has been shown already.

A set $V \subset T_{X,A}$ is said to be total, if

$$\forall_{F \in V} \langle g, F\rangle = 0 \Rightarrow g = 0 .$$

So suppose

$$\langle g, G_{\sigma,j}^{(m)}\rangle = 0$$

for $1 \le j < m+1$, $1 \le m \le \infty$ and $\sigma \in \mathrm{supp}(\langle\mu_m\rangle)\backslash N^{(m)}$. Then it immediately follows that $(U_j^{(m)} P_j^{(m)} g)(\sigma) = 0$ almost everywhere with respect to $\langle\mu_m\rangle$, with $1 \le j < m+1$ and $1 \le m \le \infty$. So $g = 0$. □

(3.4) Lemma

Let $\sigma(T)$ be the spectrum of T. Then

$$\sigma(T) = \overline{\bigcup_{m \in \mathbb{N} \cup \{\infty\}} \mathrm{supp}(\langle\mu_m\rangle)} .$$

Proof

If $x \notin \sigma(T)$, then there exists $\varepsilon > 0$ such that

$$H([x-\varepsilon, x+\varepsilon]) = 0 .$$

So for all m, $1 \le m \le \infty$,

$$\mu_m([x-\varepsilon, x+\varepsilon]) = 0 .$$

This implies $(x-\varepsilon/2, x+\varepsilon/2) \not\subset \mathrm{supp}(\mu_m)$ and hence

$$x \notin \overline{\bigcup_{1 \le m \le \infty} \mathrm{supp}(\langle\mu_m\rangle)} .$$

Conversely, suppose $x \notin \overline{\bigcup_{1 \le m \le \infty} \mathrm{supp}(\langle\mu_m\rangle)}$. Then there exists $\delta > 0$ such that $(x-\delta, x+\delta) \not\subset \mathrm{supp}(\langle\mu_m\rangle)$, $1 \le m \le \infty$. Hence $H([x-\delta, x+\delta])\gamma_j^{(m)} = 0$ for all $m \in \mathbb{N} \cup \{\infty\}$, $1 \le j < m+1$. This implies $H([x-\delta, x+\delta]) = 0$. So $x \notin \sigma(T)$. □

We finish this section with two examples.

(3.5) Example

Let $\lambda_0 \in \sigma(T)$ be an eigenvalue of multiplicity m_0. Then $H(\{\lambda_0\})$ is a non-zero projection on X, and for j, $1 \le j < m_0 + 1$ fixed, we have

$$G^{(m_0)}_{\lambda_{0,j}} = \lim_{h\downarrow 0} \left\{ \frac{Q_h(\lambda_0)^{\int} dH_\lambda \gamma_j^{(m_0)}}{Q_h(\lambda_0)^{\int} d(H_\lambda \gamma_j^{(m_0)}, \gamma_j^{(m_0)})} \right\} = \frac{H(\{\lambda_0\})\gamma_j^{(m_0)}}{\| H\{\lambda_0\}\gamma_j^{(m_0)} \|^2}$$

Hence $G^{(m_0)}_{\lambda_{0,j}} \in X$.

(3.6) Example

Let C be a self-adjoint compact operator on X. Then the vectors

$$\gamma_j^{(m)} := \sum_{k=1}^{\infty} 2^{-k} e_{j,k}^{(m)} \quad , \quad 1 \le j \le m, \ 1 \le m < \infty ,$$

where the series may be a finite sum, establish a cyclic set for C. Here $(e_{j,k}^{(m)})$ is an orthonormal basis of eigenvectors for C; $e_{j,k}^{(m)}$ is the j-th eigenvector, $1 \le j \le m$, with eigenvalue $\mu_k^{(m)}$ of multiplicity m, $1 \le m < \infty$.

4. The case of n-commuting self-adjoint operators

In this section we shall extend the theory of the first part of this paper to the case of n commuting self-adjoint operators, where n is a natural number. We only discuss the frame work of this extension, because there really is no essential difference with the theory of one self-adjoint operator.

Let $(T_1, T_2, \dots, T_n)$ be an n-set of commuting self-adjoint operators in X, which map $S_{X,A}$ continuously into itself. Let $(H_{\lambda_i})_{\lambda_i \in \mathbb{R}}$, $i = 1,\dots,n$, denote their respective spectral resolutions. For $f \in X$, the Hilbert space X_f is the closure in X of the linear span

$$<\{H_1(\Delta_1) \ldots H_n(\Delta_n) f \mid \Delta_i \subset \mathbb{R} \text{ a Borel set}, i = 1,\ldots,n\}> .$$

The Hilbert space X_f is unitarily equivalent to $L_2(\mathbb{R}^n,\rho_f)$, where ρ_f is the well-defined finite measure

$$\rho_f(\lambda_1,\ldots,\lambda_n) = (H_{\lambda_1} \ldots H_{\lambda_n} f, f)$$

over the Borel subsets of $\mathbb{R}^n$. For every $g \in X_f$ there exists $\hat{g} \in L_2(\mathbb{R}^n,\rho_f)$ with the properties

$$g = \int_{\mathbb{R}^n} \hat{g} \, d H_{\lambda_1} \ldots H_{\lambda_n} f$$

$$\| g \|^2 = \int_{\mathbb{R}^n} |\hat{g}|^2 \, d \rho_f .$$

The n-set restricted to X_f, $(T_1,\ldots,T_n) \restriction X_f$ is unitarily equivalent to the n-set $(Q_1,\ldots,Q_n)$, where Q_i denotes multiplication by λ_i in $L_2(\mathbb{R}^n,\rho_f)$.

For $x \in \mathbb{R}^n$ and $h > 0$, we define the cube $Q_h(x)$ by

$$Q_h(x) := \{\xi \in \mathbb{R}^n \mid |x_i - \xi_i| \le h, \ i = 1,\ldots,n\} .$$

Further we define the set $P \subset \mathbb{R}^n$ by

$$P := \{x \in \mathbb{R}^n \mid \forall_{h>0} : \rho_f(Q_h(x)) > 0\} .$$

Then in case of the n-set $(T_1,\ldots,T_n)$, Theorem (1.3) can be reformulated as follows

(4.1) Theorem

For $x \in P$, define

$$G_{x,h}(t) := \text{emb}(\rho_{\mathfrak{F}}(Q_h(x)))^{-1}\left(\int_{Q_h(x)} dH_{\lambda_1}\ldots H_{\lambda_n}\mathfrak{F}\right)(t)$$

There exists a null set N with respect to $\rho_{\mathfrak{F}}$ such that for all $x \in P\setminus N$

(i) $G_x(t) := \lim_{h \downarrow 0} G_{x,h}(t)$ exists in X for all $t > 0$

(ii) $G_x : t \mapsto G_x(t) \in T_{X,A}$ and $G_x \neq 0$

(iii) $T_i G_x = x_i G_x$.

Proof

cf. the proof of Theorem 1.3. □

The measure theoretical part of section 2 can be adapted in the usual way to measures in $\mathbb{R}^n$, cf. Definition (2.1), (2.4), (2.5) and (2.6) and Lemma (2.2) and (2.3).

For a better understanding of the commutative multiplicity theorem for an n-set of self-adjoint commuting operators, we introduce the notion of (generalized) eigentuple of multiplicity m, $1 \le m \le \infty$.

(4.2) Definition

An n-tuple $\lambda = (\lambda_1,\ldots,\lambda_n) \in \mathbb{R}^n$ is an eigentuple of the n-set $(T_1,\ldots,T_n)$ of multiplicity m if there exist m orthonormal simultaneous eigenvectors $e^{(m)}_{\lambda,j}$ such that

$$T_i\, e^{(m)}_{\lambda,j} = \lambda_i\, e^{(m)}_{\lambda,j} \quad , \quad 1 \le j < m+1, \ 1 \le i \le n\,.$$

Similarly, the notion generalized eigentuple can be introduced.

If one wants a canonical listing of the eigentuples of an n-set of commuting matrices it is natural to list all eigentuples of multiplicity one, two,... . We need a way of saying that an n-set of commuting self-adjoint operators is of uniform multiplicity one, two, etc.

(4.3) Definition

An n-set $(T_1,\ldots,T_n)$ of commuting self-adjoint operators is said to be of uniform multiplicity m if each T_i is unitarily equivalent to multiplication by λ_i in $L_2(\mathbb{R}^n,\mu) \oplus \ldots \oplus L_2(\mathbb{R}^n,\mu)$, where there are m terms in the sum and where μ is a finite Borel measure in $\mathbb{R}^n$.

The formulation of the commutative multiplicity theorem for an n-set of commuting self-adjoint operators is quite evident.

(4.4) Theorem

Let $(T_1,\ldots,T_n)$ be an n-set of commuting self-adjoint operators in X. Then there exists a decomposition

$$X = X_\infty \oplus X_1 \oplus X_2 \oplus \ldots$$

such that

(i) The n-set $(T_1,\ldots,T_n)$ acts invariantly in each X_m, $1 \le m \le \infty$.

(ii) The n-set $(T_1,\ldots,T_n)$ restricted to X_m has uniform multiplicity m.

(iii) The measure classes $\langle\mu_m\rangle$ associated with $(T_1,\ldots,T_n) \restriction X_m$ are mutually disjoint.

Further, the subspaces $X_\infty, X_1, X_2, \ldots$ (some of which may be zero) and the classes $\langle\mu_\infty\rangle, \langle\mu_1\rangle, \ldots$ are uniquely determined by (i), (ii) and (iii).

The proof of this theorem can be derived from the proof in the one dimensional case and is essentially the same (see [N], [Br]).

(4.5) Definition

A set $\Gamma \subset X$ is called cyclic with respect to $(T_1,\dots,T_n)$ if

$$X = \bigoplus_{\gamma\in\Gamma} X_\gamma .$$

Note that Γ is at most countable.

If Γ consists of one element, this element is called cyclic vector. Lemma 3.1 can be replaced by

(4.6) Lemma

The n-set $(T_1,\dots,T_n)$ is of uniform multiplicity one if and only if it has a cyclic vector.

Following Theorem (4.4) X can be splitted into a direct sum $X = X_\infty \oplus X_1 \oplus X_2 \oplus \dots$. Each of the restricted operators $T_i \restriction X_m$, $1 \le i < m+1$ is unitarily equivalent to multiplication by λ_i in

$$L_2(\mathbb{R}^n,\mu_m) \oplus \dots \oplus L_2(\mathbb{R}^n,\mu_m) \quad , \quad \text{m-times} .$$

Let X_{mj}, $1 \le j < m+1$ be the orthogonal subspace of X_m, which corresponds to the j-th term in the sum above. Then $(T_1,\dots,T_n) \restriction X_{mj}$ has a cyclic vector $\gamma_j^{(m)}$, say. In this way a set Γ is obtained

$$\Gamma = \{\gamma_j^{(m)} \mid 1 \le j < m+1,\ 1 \le m \le \infty\}$$

which is cyclic for $(T_1,\dots,T_n)$.

(4.7) Theorem

Take m, $1 \le m \le \infty$, such that $X_m \neq \{0\}$. Then there exists a null set $N^{(m)}$ with respect to $<\mu_m>$, such that for all $\lambda \in \text{supp}(<\mu_m>)\backslash N^{(m)}$, there are precisely m independent simultaneous generalized eigenfunctions of $(T_1,\ldots,T_n)$ with generalized eigentuple $\lambda = (\lambda_1,\ldots,\lambda_n)$.
Further, the set of all generalized eigenfunctions is total.

(4.8) Example

Consider $S_{X,A}$ with $X = L_2(\mathbb{R})$ and $A = \frac{1}{2}\left(-\frac{d^2}{dx^2} + x^2 + 1\right)$ and the 2-set (Φ,Q^2) where Φ denotes the parity operator and Q^2 multiplication by x^2; so

$$(Q^2 f)(x) = x^2 f(x) \quad \text{and} \quad (\Phi f)(x) = f(-x) \ .$$

Then the 2-set (Φ,Q^2) has uniform multiplicity 1 because it has a cyclic vector; for instance take

$$\gamma : x \mapsto (1 + x)e^{-\frac{1}{2}x^2} \ .$$

5. A mathematical interpretation of Dirac's formalism

In the preface to his book on the foundations of quantum mechanics von Neumann says that Dirac's formalism *is scarcely to be surpassed in brevity and elegance* but that it *in no way satisfies the requirements of mathematical rigour*. The improper functions of Dirac, the δ-function and its derivatives, have stimulated the growth of a new branch of mathematics: the theory of distributions. Yet, as far as we know, no paper on Dirac's formalism mathematically foundates the bold way in which Dirac treats the continuous spectrum of a self-adjoint operator. Most papers on this

subject only solve the so called generalized eigenvalue problem by means of the rigged Hilbert space theory of Gelfand and Shilov. But Dirac's formalism has more aspects.
In this section an interpretation of the formalism is studied in terms of our distribution theory. It consists of the definition of ket and bra space, of Parseval's identity, of the Fourier expansion of kets with respect to continuous bases, of the existence and orthogonality of complete sets of eigenkets, of matrices of unbounded linear mappings with respect to continuous bases, and of some matrix computation.

We shall only consider quantum systems at a given time without superselection rules. So we do not need to specify whether we are using the Heisenberg or Schrödinger pictures. A quantum system at a given time is determined by states and observables. The space of all states is mostly supposed to be in 1-1 correspondence with the set of all one dimensional subspaces of an infinite dimensional separable Hilbert space X and the set of observables in 1-1 correspondence with the set of all self-adjoint operators in X. But in general we do not need to consider all self-adjoint operators. To describe a quantum system one can make a choice out of the set of observables, e.g. 'energy', 'momentum' and 'spin', which is sufficiently large to completely determine the quantum system and in particular all relevant observables.
In his formalism Dirac treats all points in the spectrum of a self-adjoint operator similarly. So the formalism assumes for instance that the notion multiplicity of λ for every point λ in the spectrum makes sense, and further that for each λ with multiplicity m there exist precisely m independent eigenstates. Of course, Hilbert space theory can not fulfil these wishes.

Hilbert spaces are too small. Therefore, it is natural to look for spaces, which extend Hilbert space, and with structures comparable to Hilbert space structure. For instance, the trajectory spaces $T_{X,A}$ are acceptable candidates.

In Dirac's formalism the dual space of the ket space, the so called bra space, is in 1-1 correspondence with the ket space. So the latter space ought to be self-dual. To this end distribution theory can't ever be of any help. We try to circumvent this problem by a new interpretation of Dirac's bracket notion.

Let QS be a quantum mechanical system. We assume that QS is completely determined by the set of self-adjoint operators $\{P_1,\dots,P_n\}$ in the Hilbert space X. Further, we suppose that there exists a nuclear space $S_{X,A}$ such that each P_i maps $S_{X,A}$ continuously into itself. So the P_i, i = 1,...,n, can be extended to continuous linear mappings on $T_{X,A}$. For instance, when the set $\{P_1,\dots,P_n\}$ is an n-set of commuting self-adjoint operators it is possible to construct such a nuclear space.

In our interpretation the set of observables of QS corresponds uniquely to the set of self-adjoint operators which are continuous on $S_{X,A}$. We note that the choice of the space $S_{X,A}$ depends on the self-adjoint operators $P_1,\dots,P_n$. For the set of states we take the set of one dimensional subspaces of $T_{X,A}$.

In Dirac's terminology the elements of $T_{X,A}$ are the so called ket vectors. Therefore we introduce Dirac's bracket notation and denote them by $|G\rangle$ in the sequel. The label G in the expression $|G\rangle$ is mostly chosen such that it expresses best the properties of $|G\rangle$ which are relevant in the context. To $|G\rangle$ uniquely corresponds the bra $\langle G|$ defined by

$$\langle G| := \sum_{k=1}^{\infty} \langle v_k, |G\rangle\rangle \, v_k$$

where (v_k) denotes the orthonormal basis of eigenvectors of A, and where the series converges in $T_{X,A}$.

The expression $\langle F \mid G\rangle$, called the bracket of $\langle F|$ and $|G\rangle$, denotes the complex valued function

$$\langle F \mid G\rangle : t \mapsto \overline{\langle |F\rangle(t), |G\rangle\rangle} \quad , \quad t > 0 .$$

The function $\langle F \mid G\rangle$ is well defined because $|F\rangle(t) \in S_{X,A}$ for every $t > 0$. It extends to an analytic function on the open right half plane. Let $f \in S_{X,A}$. Then obviously $\langle f \mid G\rangle(-\tau)$ exists for every $|G\rangle$ and $\tau > 0$ sufficiently small and

$$\langle f \mid G\rangle(-\tau) = \overline{\langle |f\rangle(-\tau) \, , \, |G\rangle\rangle} \, ;$$

similarly $\langle G \mid f\rangle(-\tau)$ exists and

$$\langle G \mid f\rangle(-\tau) = \langle |f\rangle(-\tau) \, , \, |G\rangle\rangle \, .$$

To emphasize this nice property of the elements in $S_{X,A}$ the kets and bras corresponding to elements in $S_{X,A}$ are called test kets and test bras. Finally, we remark that for all $t > 0$ the function $\langle F \mid G\rangle$ satisfies

$$\langle F \mid G\rangle(t) = \langle F(t) \mid G\rangle(0) = \overline{\langle G(t) \mid F\rangle(0)} = \overline{\langle G \mid F\rangle(t)}$$

and

$$\langle F \mid G\rangle(t) = \langle F(t) \mid G\rangle(0) = \langle F \mid G(t)\rangle(0) \, .$$

Let $P : S_{X,A} \to S_{X,A}$ be an observable of QS. For simplicity, suppose that P is a cyclic operator in X, Then all points in $\sigma(P)$, the spectrum of P, have multiplicity one. Further, there exists a cyclic vector γ in X such that P is unitarily equivalent to multiplication by λ in the Hilbert space $L_2(\mathbb{R},d(H_\lambda\gamma,\gamma))$. Here $(H_\lambda)_{\lambda\in\mathbb{R}}$ denotes the spectral resolution of the identity with respect to P. As in section 3, the Borel measure $d(H_\lambda\gamma,\gamma)$ is denoted by $d\rho_\gamma(\lambda)$ in the sequel.

Following the preceding sections there exists a null set N with respect to ρ_γ such that for each $\lambda \in \sigma(P)\backslash N$ there is an eigenket $|\lambda>$. With the notation of section 3, $|\lambda>$ has the following Fourier expansion

$$|\lambda> = \sum_{k=1}^{\infty} \overline{\hat{v}_k(\lambda)}\,|v_k>$$

where the series converges in $T_{X,A}$. (As usual v_k denotes the eigenvector of A with eigenvalue λ_k, $k = 1,2,\ldots$.)

Let $g \in S_{X,A}$. Then $g = e^{-tA} f$ for a well chosen $f \in S_{X,A}$ and $t > 0$. Consider the following formal computation

$$\begin{aligned} g &= \sum_{k=1}^{\infty} e^{-\lambda_k t}\,(f,v_k)\,v_k \\ &= \sum_{k=1}^{\infty} e^{-\lambda_k t}\Big(\int_{\mathbb{R}} \hat{f}(\lambda)\,\overline{\hat{v}_k(\lambda)}\,d\rho_\gamma(\lambda)\Big)v_k \\ &\overset{(*)}{=} \int_{\mathbb{R}} \hat{f}(\lambda)\Big(\sum_{k=0}^{\infty} e^{-\lambda_k t}\,\overline{\hat{v}_k(\lambda)}\,v_k\Big)d\rho_\gamma(\lambda)\ . \end{aligned}$$

Hence

$$|g> = \int_{\mathbb{R}} <\lambda\ |\ f>(0)\ |\lambda>(t)\ d\rho_\gamma(\lambda)\ .$$

The only problem in this computation is the equality (*). We shall therefore prove that summation and integration can be interchanged. The following inequalities hold true

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}} e^{-\lambda_k t} \left|\hat{f}(\lambda)\, \overline{\hat{v}_k(\lambda)}\right| \, d\rho_\gamma(\lambda) \le$$

$$\le \frac{1}{2}\left(\sum_{k=1}^{\infty} e^{-\lambda_k t} \int_{\mathbb{R}} |\hat{f}(\lambda)|^2 \, d\rho_\gamma(\lambda) + \sum_{k=1}^{\infty} e^{-\lambda_k t} \int_{\mathbb{R}} |\hat{v}_k(\lambda)|^2 \, d\rho_\gamma(\lambda) \right) =$$

$$= \frac{1}{2} (\| f \|^2 + 1) \left(\sum_{k=1}^{\infty} e^{-\lambda_k t} \right) .$$

By the Fubini-Tonelli theorem equality (*) is verified.

With the aid of the above derivation, $|g\rangle$ can be written as

$$|g\rangle = \int_{\mathbb{R}} \langle \lambda \mid f\rangle(-t) \, |\lambda\rangle(t) \, d\rho_\gamma(\lambda)$$

where the integral converges absolutely in X, and does not depend on the choice of $t > 0$.

(5.1) Theorem

Let $|f\rangle$ be a test ket. Then

$$|f\rangle = \int_{\mathbb{R}} \langle \lambda \mid f\rangle(0) \, |\lambda\rangle \, d\rho_\gamma(\lambda)$$

where the integral converges strongly in $\mathcal{T}_{X,A}$.

Proof

Let $t > 0$. We have seen that

$$|f\rangle(t) = \int_{\mathbb{R}} \langle \lambda \mid f\rangle(0) \, |\lambda\rangle(t) \, d\rho_\gamma(\lambda)$$

with absolute convergence in X. Since $e^{-\tau A}$, $\tau > 0$, is a bounded operator on X

$$e^{-\tau A} \left(\int_{\mathbb{R}} \langle \lambda \mid f\rangle(0) \, |\lambda\rangle(t) \, d\rho_\gamma(\lambda) \right) = \int_{\mathbb{R}} \langle \lambda \mid f\rangle(0) \, |\lambda\rangle(t+\tau) \, d\rho_\gamma(\lambda) .$$

Therefore $t \mapsto |f\rangle(t)$ is a trajectory. □

Parseval's identity is an immediate consequence of section 3

$$(5.2) \qquad \|f\|^2 = \int_{\mathbb{R}} |\hat{f}(\lambda)|^2 \, d\rho_\gamma(\lambda) = \int_{\mathbb{R}} |<f \mid \lambda>(0)|^2 \, d\rho_\gamma(\lambda) .$$

Further, from Theorem (5.1) it is clear that

$$(5.3) \qquad P|f> = \int_{\mathbb{R}} \lambda <\lambda \mid f>(0) \, |\lambda> d\rho_\gamma(\lambda) .$$

Let $F \in T_{X,A}$. Then for every $\tau > 0$, $F(\tau) \in S_{X,A}$ and hence by Theorem 5.1

$$|F>(\tau) = |F(\tau)>(0) = \int_{\mathbb{R}} <\lambda \mid F(\tau)>(0) \, |\lambda> d\rho_\gamma(\lambda)$$

with convergence in $T_{X,A}$. Further, let $t > 0$. Then for every τ, $0 < \tau < t$

$$(5.4) \qquad |F>(t) = e^{-(t-\tau)A} |F>(\tau) = \int_{\mathbb{R}} <\lambda \mid F>(\tau) \, |\lambda>(t-\tau) \, d\rho_\gamma(\lambda) .$$

The integral in (5.4) does not depend on the choice of τ and converges absolutely in X. The ket $|F>$ can thus be represented by

$$|F> \, : \, t \to \int_{\mathbb{R}} <\lambda \mid F>(\tau) \, |\lambda>(t-\tau) \, d\rho_\gamma(\lambda) .$$

By the expression

$$\int_{\mathbb{R}} <\lambda \mid F> |\lambda> d\rho_\gamma(\lambda)$$

is meant the trajectory

$$t \mapsto \int_{\mathbb{R}} <\lambda \mid F>(\tau) \, |\lambda>(t-\tau) \, d\rho_\gamma(\lambda) .$$

Each of the integrals does not depend on the choice of τ, $0 < \tau < t$, and converges absolutely in X. We can write

$$(5.5) \qquad |F\rangle = \int_{\mathbb{R}} \langle\lambda \mid F\rangle \, |\lambda\rangle \, d\rho_\gamma(\lambda)$$

where the integral has to be understood in the above-mentioned sense. It converges strongly in $T_{X,A}$.

The result of Theorem (5.1) can be sharpened. To this end, let $f \in S_{X,A}$. Then there exists $\tau > 0$ such that $e^{\tau A} f \in S_{X,A}$. We have

$$|f\rangle = \int_{\mathbb{R}} \langle\lambda \mid f\rangle \, |\lambda\rangle \, d\rho_\gamma(\lambda) = \int_{\mathbb{R}} \langle\lambda \mid f\rangle(-\tau) \, |\lambda\rangle(\tau) \, d\rho_\gamma(\lambda)$$

where the latter integral converges in X. Since $e^{\frac{\tau}{2}A}$ is a closed operator in X, and since $\int_{\mathbb{R}} \langle\lambda \mid f\rangle(-\tau) \, |\lambda\rangle(\tau/2) \, d\rho_\gamma(\lambda)$ converges absolutely in X, the integral

$$\int_{\mathbb{R}} \langle\lambda \mid f\rangle(-\tau) \, |\lambda\rangle(\tau) \, d\rho_\gamma(\lambda)$$

converges in $S_{X,A}$. Hence in our interpretation for the test ket $|f\rangle$ we have

$$|f\rangle = \int_{\mathbb{R}} \langle\lambda \mid f\rangle \, |\lambda\rangle \, d\rho_\gamma(\lambda)$$

where the integral converges in $S_{X,A}$.

Consider the following equality

$$\langle\mu \mid \lambda\rangle(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t} \, \overline{\hat{v}_k(\lambda)} \, \hat{v}_k(\mu) \quad , \quad \lambda,\mu \in \sigma(P)\backslash N, \; t > 0 \, .$$

Let δ_λ denote the function

$$\delta_\lambda : (\mu,t) \mapsto \langle\mu \mid \lambda\rangle(t)$$

and let U denote the unitary operator from X onto $Y = L_2(\mathbb{R},\rho_\gamma)$. Put $\mathcal{B} = UAU^*$. Then $\delta_\lambda \in T_{Y,\mathcal{B}}$ and for $\hat{f} \in S_{Y,\mathcal{B}}$

$$<\hat{f},\delta_\lambda> = \sum_{k=1}^{\infty}\left(\int_{\mathbb{R}} \hat{f}(\mu)\,\overline{\hat{v}_k(\mu)}\,d\rho_\gamma(\mu)\right)\hat{v}_k(\lambda) = \hat{f}(\lambda)\ .$$

So δ_λ is Dirac's delta function in $T_{Y,\mathcal{B}}$ and consequently we write

(5.6) $$<\mu\mid\lambda> = \delta_\lambda(\mu)\ .$$

Relation (5.6) expresses the generalization of the orthogonality relations for the eigenvectors of P to the eigenkets of P in agreement with Dirac's notation.

For the sake of completeness we rewrite the result (5.5) for the bras and test bras

(5.7) $$<F| = \int_{\mathbb{R}} <F\mid\lambda><\lambda|\,d\rho_\gamma(\lambda)$$

where the integral converges in $T_{X,A}$. Whenever $<F|$ is a test bra the integral converges in $S_{X,A}$.

Another aspect of Dirac's formalism is the so called property of a complete set of eigenkets.

(5.8) <u>Theorem</u> *(closure property)*

$$P^n = \int_{\mathbb{R}} \lambda^n\,|\lambda><\lambda|\,d\rho_\gamma(\lambda)\quad,\quad n = 0,1,2,\ldots$$

where the integral converges in $T_{X\otimes X,A\boxplus A}$. Here $|\lambda><\lambda|$ denotes the tensor product $|\lambda>\otimes|\lambda>$ ($\in T_{X\otimes X,A\boxplus A}$).

(Here $A\boxplus A$ denotes the self-adjoint operator $I\otimes A + A\otimes I$).

Proof

Let $t > 0$. Consider the following formal derivation

$$e^{-t(A \boxplus A)} P^n = \sum_{k,\ell} e^{-t\lambda_k} e^{-t\lambda_\ell} \overline{\langle v_k \otimes v_\ell , P^n \rangle}\, v_k \otimes v_\ell$$

$$= \sum_{k,\ell} e^{-\lambda_k t} e^{-\lambda_\ell t} \left(\int_{\mathbb{R}} \lambda^n \langle v_k \mid \lambda\rangle(0)\, \overline{\langle v_\ell \mid \lambda\rangle(0)}\, d\rho_\gamma(\lambda) \right) v_k \otimes v_\ell$$

$$\overset{(*)}{=} \int_{\mathbb{R}} \lambda^n \left(\sum_{k,\ell} (e^{-\lambda_k t}\, \overline{\hat{v}_k(\lambda)}\, v_k) \otimes (e^{-\lambda_\ell t}\, \hat{v}_\ell(\lambda)\, v_\ell) \right) d\rho_\gamma(\lambda)$$

$$= \int_{\mathbb{R}} \lambda^n\, |\lambda\rangle(t) \otimes |\lambda\rangle(t)\, d\rho_\gamma(\lambda) .$$

We shall prove that summation and integration can be interchanged. The remaining part of the proof is straight forward.

$$\sum_{k,\ell} \int_{\mathbb{R}} |e^{-\lambda_k t} e^{-\lambda_\ell t} \lambda^n\, \hat{v}_k(\lambda)\, \overline{\hat{v}_\ell(\lambda)}|\, d\rho_\gamma(\lambda) \le$$

$$\le \sum_{k,\ell} \int_{\mathbb{R}} \frac{1}{2}\, e^{-\lambda_k t} e^{-\lambda_\ell t} (\lambda^{2n} |\hat{v}_k(\lambda)|^2 + |\hat{v}_\ell(\lambda)|^2)\, d\rho_\gamma(\lambda) \le$$

$$\le \frac{1}{2} \sum_k e^{-\lambda_k t} (\| P^n v_k \|^2 + 1) \left(\sum_\ell e^{-\lambda_\ell t} \right) \le$$

$$\le \frac{1}{2} (|||P^n e^{-\frac{1}{2} t A}|||^2 + 1)\, |||e^{-\frac{1}{2} t A}|||^2 . \qquad \square$$

Next we discuss the general case that $P : S_{X,A} \to S_{X,A}$ has a countable cyclic set. There will appear no essential difference with the case of a cyclic operator P. The same notation as in section 3 will be employed. Proofs will be omitted.

So let $\{\gamma_j^{(m)} \mid m = \infty, 1, 2, \ldots \quad , \quad 1 \le j < m+1\}$ be the cyclic set for P. Then X can be written as

$$X = \bigoplus_{m=1}^{m=\infty} \bigoplus_{j=1}^{m} X_{\gamma_j^{(m)}}$$

where by absence of better notations $\bigoplus_{m=1}^{m=\infty} \bigoplus_{j=1}^{m} X_{\gamma_j^{(m)}}$ will denote

$$\left(\bigoplus_{m=1}^{\infty} \bigoplus_{j=1}^{m} X_{\gamma_j^{(m)}} \right) \oplus \left(\bigoplus_{j=1}^{\infty} X_{\gamma_j^{(\infty)}} \right).$$

The Hilbert space $X_{\gamma_j^{(m)}}$ is unitarily equivalent to $L_2(\mathbb{R}, \rho_{\gamma_j^{(m)}})$ and $P \restriction X_{\gamma_j^{(m)}}$ is unitarily equivalent to multiplication by λ in $L_2(\mathbb{R}, \rho_{\gamma_j^{(m)}})$. Following section 3 there exist sets $N^{(m)}$, each of which has measure zero with respect to $\langle \rho_{\gamma_j^{(m)}} \rangle$, $m = \infty, 1, 2, \ldots$ such that for all λ in $\operatorname{supp}(\langle \rho_{\gamma_j^{(m)}} \rangle) \backslash N^{(m)}$ there are m independent eigenkets $|\lambda, m, j\rangle$, $1 \le j < m+1$. The eigenkets can be written as

$$|\lambda, m, j\rangle = \sum_{k=1}^{\infty} \overline{\hat{v}_{k,j}^{(m)}(\lambda)} \, |v_k\rangle$$

where the series converges in $T_{X,A}$. Then similar to Theorem (5.1)

(5.9) <u>Theorem</u>

Let $f \in S_{X,A}$. Then

$$|f\rangle = \sum_{m=1}^{m=\infty} \sum_{j=1}^{m} \int_{\mathbb{R}} \langle \lambda, m, j \mid f \rangle(0) \, |\lambda, m, j\rangle \, d\rho_{\gamma_j^{(m)}}(\lambda)$$

with convergence in $T_{X,A}$. Further

$$\| f \|^2 = \sum_{m=1}^{m=\infty} \sum_{j=1}^{\infty} \int_{\mathbb{R}} |\langle \lambda, m, j \mid f \rangle(0)|^2 \, d\rho_{\gamma_j^{(m)}}(\lambda)$$

(Parseval's identity) and

$$P|f\rangle = \sum_{m=1}^{m=\infty} \sum_{j=1}^{\infty} \int_{\mathbb{R}} \lambda \langle\lambda,m,j \mid f\rangle(0)\, |\lambda,m,j\rangle\, d\rho_{\gamma_j^{(m)}}(\lambda)\,.$$

Henceforth we will call the set $\{|\lambda,m,j\rangle \mid \lambda \in \sigma(P)\ ,\ 1 \le m \le \infty,\ 1 \le j < m+1\}$ a Dirac basis.

With the same interpretation as in (5.5) we have

$$(5.10) \qquad |F\rangle = \sum_{m=1}^{m=\infty} \sum_{j=1}^{m} \int_{\mathbb{R}} \langle\lambda,m,j \mid F\rangle\, |\lambda,m,j\rangle\, d\rho_{\gamma_j^{(m)}}(\lambda)$$

with convergence in $T_{X,A}$. In particular if $|F\rangle$ in (5.10) is a test ket the convergence takes place even in $S_{X,A}$-sense.

Consider the following equality

$$\langle\mu,n,i \mid \lambda,m,j\rangle(t) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\, \overline{\hat{v}_{k,j}^{(m)}(\lambda)}\, \hat{v}_{k,i}^{(n)}(\mu)$$

where $\lambda \in \operatorname{supp}(\langle\rho_{\gamma_j^{(m)}}\rangle)\backslash N^{(m)}$, $\mu \in \operatorname{supp}(\langle\rho_{\gamma_i^{(n)}}\rangle)\backslash N^{(n)}$, $1 \le j < m+1$, $1 \le i < n+1$ and $m,n = \infty,1,2,\ldots$.

Let $\delta_{\lambda,j}^{(m)}$ denote the function

$$\delta_{\lambda,j}^{(m)} : (\mu,n,i,t) \to \langle\mu,n,i \mid \lambda,m,j\rangle(t)$$

and U the unitary operator from X onto $Y = \bigoplus_{m=1}^{m=\infty} \bigoplus_{j=1}^{m} L_2(\mathbb{R},\rho_{\gamma_j^{(m)}})$. Put $B = UAU^*$. Then $\delta_{\lambda,j}^{(m)} \in T_{Y,B}$, and for $\hat{f} \in S_{Y,B}$

$$\hat{f} : (\mu,n,i) \to \hat{f}_i^{(n)}(\mu) \quad , \quad \hat{f}_i^{(n)} \in L_2(\mathbb{R},\rho_{\gamma_j^{(n)}})$$

and

$$\langle\hat{f},\delta_{\lambda,j}^{(m)}\rangle = \sum_{n=1}^{n=\infty} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \left(\int_{\mathbb{R}} \hat{f}_i^{(n)}(\mu)\, \overline{\hat{v}_{k,i}^{(n)}(\mu)}\, d\rho_{\gamma_i^{(n)}}(\mu) \right) v_{k,j}^{(m)}(\lambda)$$

$$= f_j^{(m)}(\lambda)\,.$$

Hence

$$<\mu,n,i \mid \lambda,m,j> = \delta_\lambda(\mu)\,\delta_{ji}\,\delta_{mn}\,.$$

Finally we give the adaptation of the closure property (5.8).

(5.11) Theorem

$$P^n = \sum_{m=1}^{m=\infty} \sum_{j=1}^{m} \int_{\mathbb{R}} \lambda^n \,|\lambda,m,j><\lambda,m,j|\, d\rho_{\gamma_j^{(m)}}(\lambda) \quad , \quad n = 0,1,2,\ldots$$

with convergence of the integral in $T_{X\otimes X, A \boxplus A}$.

Here we do not intend to discuss the interpretation of Dirac's formalism for an n-set of commuting observables. The generalization to this case is immediate and rather trivial. All results remain valid in an adapted form. We only notice the nice way in which the definition of a complete set of commuting observables in the sense of Dirac can be expressed in our terminology.

(5.12) Proposition

The n-set $(P_1,\ldots,P_n)$ is a complete set of commuting observables iff it has uniform multiplicity one.

Given an orthonormal basis X. Every bounded linear operator B in X is uniquely represented by its matrix [B] with respect to this basis. The product of two operators B_1B_2 has matrix $[B_1B_2]$ which can be derived by formal matrix multiplication, $[B_1B_2]_{k\ell} = \sum_i [B_1]_{ki}\,[B_2]_{i\ell}$. Dirac assumes that the matrix notion can also be introduced in the case of Dirac bases, and that operating with these matrices runs similarly to

the discrete case. Because of this assumption one can choose a representation so that *the representatives of the more abstract quantities occurring in the problem are as simple as possible.* Examples of such representations are the so called x- and p-representations.
Here we shall give a mathematical interpretation of this hypothesis of Dirac. We shall restrict ourselves to representations of observables with repsect to a complete set of generalized eigenfunctions of a cyclic self-adjoint operator. The general case of a non-cyclic self-adjoint operator or of a commuting n-set can be dealt with similarly.

Let $P : S_{X,A} \to S_{X,A}$ be a cyclic self-adjoint operator, and let $|\lambda\rangle$, $\lambda \in \sigma(P)$, denote the eigenkets of P in $T_{X,A}$. The operator $P \otimes P$ is self-adjoint in $X \otimes X$, and maps $S_{X\otimes X, A \boxplus A}$ continuously into itself. Eigenkets in $T_{X\otimes X, A\boxplus A}$ of $P \otimes P$ are $|\lambda\rangle \otimes |\mu\rangle$, $\lambda, \mu \in \sigma(P)$. Following Dirac we shall denote the tensor product $|\lambda\rangle \otimes |\mu\rangle$ by $|\mu\rangle\langle\lambda|$ in the sequel. Every continuous linear mapping from $T_{X,A}$ into $S_{X,A}$ is derived from an element of $S_{X\otimes X, A\boxplus A}$, because of the Kernel theorem. With the methods we employed in the proof of Theorem (5.1) the following result can be shown.

(5.13) Theorem

Let $B \in S_{X\otimes X, A\boxplus A}$. Then

$$B = \iint_{\mathbb{R}^2} \langle\mu \mid B \mid \lambda\rangle(0) \; |\mu\rangle\langle\lambda| \; d\rho_\gamma(\lambda)\, d\rho_\gamma(\mu)$$

where the integral converges in $T_{X\otimes X, A\boxplus A}$, and where

$$\langle\mu \mid B \mid \lambda\rangle(t) = \langle e^{-t\, A\boxplus A} B \,,\, |\lambda\rangle \otimes |\mu\rangle\rangle \,.$$

We note that

$$e^{-t\,A \boxplus A}\,B = \iint_{\mathbb{R}^2} \langle\mu \mid B \mid \lambda\rangle(0)\ (|\mu\rangle\langle\lambda|)(t)\ d\rho_\gamma(\lambda)\ d\rho_\gamma(\mu)\ , \quad t > 0\ ,$$

where the integral converges absolutely in $X \otimes X$.

Similar to the one variable case $T_{X,A}$ (cf. (5.5)), Theorem (5.13) can be adapted such that it is valid for elements in $T_{X\otimes X, A\boxplus A}$.

(5.14) Theorem

Let $G \in T_{X\otimes X, A\boxplus A}$. Then we have with $\langle\mu \mid G \mid \lambda\rangle : t \mapsto \langle\mu \mid G(t) \mid \lambda\rangle$,

$$G = \iint_{\mathbb{R}^2} \langle\mu \mid G \mid \lambda\rangle\ |\mu\rangle\langle\lambda|\ d\rho_\gamma(\lambda)\ d\rho_\gamma(\mu)$$

where similarly to (5.5) the integral has to be understood in the following sense.

$$G : t \mapsto \iint_{\mathbb{R}^2} \langle\lambda \mid G \mid \mu\rangle(\tau)\ (|\mu\rangle\langle\lambda|)(t-\tau)\ d\rho_\gamma(\lambda)\ d\rho_\gamma(\mu)\ .$$

Here the integrals do not depend on the choice of τ, $0 < \tau < t$, and converge in $X \otimes X$.

With respect to the Dirac basis $(|\lambda\rangle)_{\lambda\in\sigma(P)}$ an element B, $B \in S_{X\otimes X, A\boxplus A}$, can be represented by the matrix $[B]$ given by

(5.15) $$[B]_{\mu\lambda} = \langle\mu \mid B \mid \lambda\rangle(0)\ , \quad \mu,\lambda \in \sigma(P)$$

and following Theorem (5.13)

$$B = \iint_{\mathbb{R}^2} [B]_{\mu\lambda}\ |\mu\rangle\langle\lambda|\ d\rho_\gamma(\lambda)\ d\rho_\gamma(\mu)\ .$$

Further for $|F\rangle \in T_{X,A}$, the ket $B|F\rangle$ is a test ket and

$$(5.16) \qquad B|F\rangle = \iint_{\mathbb{R}^2} \langle\mu \mid B \mid \lambda\rangle(-\tau) \langle\lambda \mid F\rangle(\tau) \, |\mu\rangle \, d\rho_\gamma(\lambda) \, d\rho_\gamma(\mu)$$

where $\tau > 0$ has to be taken so small that $B\, e^{\tau A} \in S_{X\otimes X, A \boxplus A}$, and where the integral converges in $T_{X,A}$ and does not depend on the choice of $\tau > 0$. Even convergence in $S_{X,A}$ can be proved. Further

$$(5.17) \qquad \langle\mu \mid B|F\rangle(0) = \int_{\mathbb{R}} \langle\mu \mid B \mid \lambda\rangle(-\tau) \langle\lambda \mid F\rangle(\tau) \, d\rho_\gamma(\lambda)$$

where the integral converges absolutely. Note that $\langle\mu \mid B \mid \lambda\rangle(-\tau)$ exists because $B|F\rangle$ is a test ket for every ket $|F\rangle$.

The matrix notion can be extended to elements of $T_{X\otimes X, A\boxplus A}$. To this end, let $G \in T_{X\otimes X, A\boxplus A}$. Then with the expression $[G]$ we mean the set of functions

$$(5.18) \qquad [G]_{\mu\lambda} = \langle\mu \mid G \mid \lambda\rangle .$$

We note that $G(t) \in S_{X\otimes X, A\boxplus A}$. The expression $[G]$ will be called the matrix of G. By Theorem (5.14) we have

$$G = \iint_{\mathbb{R}^2} [G]_{\mu\lambda} \, |\mu\rangle\langle\lambda| \, d\rho_\gamma(\lambda) \, d\rho_\gamma(\mu) .$$

Let $|f\rangle$ be a test ket. Then $G|f\rangle$ can be represented by

$$(5.19) \qquad G|f\rangle : t \mapsto \iint_{\mathbb{R}^2} [G]_{\mu\lambda}(\tau) \langle\lambda \mid f\rangle(-\tau) \, |\mu\rangle(t-\tau) \, d\rho_\gamma(\lambda) \, d\rho_\gamma(\mu)$$

where τ, $0 < \tau < t$, has to be taken so small that $|f\rangle(-\tau) \in S_{X,A}$, and where the integrals converge absolutely in X and do not depend on the choice of $\tau > 0$. Further

$$(5.20) \qquad \langle\mu\,|\,G\,|\,f\rangle \;:\; t \to \int_{\mathbb{R}} [e^{-(t-\tau)A} G(\tau)]_{\mu\lambda} \langle\lambda\,|\,f\rangle(-\tau)\, d\rho_\gamma(\lambda)$$

where the integrals converge absolutely and do not depend on the choice of $\tau > 0$.

Similarly a matrix notion will be introduced for continuous linear mappings from $S_{X,A}$ into itself resp. $T_{X,A}$ into itself, or equivalently because of the Kernel theorem for elements in $T(S_{X\otimes X, I\otimes A}, A\otimes I)$ resp. $T(S_{X\otimes X, A\otimes I}, I\otimes A)$, i.e. the spaces $\sum_B'$ and $\sum_A'$ as introduced by De Graaf in [G], ch. IV (cf. $[E_1]$).

For $R \in T(S_{X\otimes X, I\otimes A}, A\otimes I)$ the matrix representation $[R]$ is defined by

$$(5.21) \qquad [R]_{\mu\lambda} \;:\; (s,t) \mapsto \langle\mu\,|\,R(t)\,|\,\lambda\rangle(s)\;.$$

Note that $R(t) \in S_{X\otimes X, A\boxplus A}$ $t > 0$, fixed. So there exists $\sigma > 0$ such that $\langle\mu\,|\,R(t)\,|\,\lambda\rangle(-\sigma)$ is well-defined because $R(t)\,|\lambda\rangle$ is a test ket. It can be shown that

$$(5.22) \qquad R \;:\; t \mapsto R(t) = \iint_{\mathbb{R}^2} [R]_{\mu\lambda}(-\sigma,\tau)\,(|\lambda\rangle(t-\tau) \otimes |\mu\rangle(\sigma))\, d\rho_\gamma(\lambda)\, d\rho_\gamma(\mu)$$

where the integrals converge in $X \otimes X$ and do not depend on the choice of τ, $0 < \tau < t$ and of $\sigma > 0$ sufficiently small. We write

$$(5.23) \qquad R = \iint_{\mathbb{R}^2} [R]_{\mu\lambda}\, |\mu\rangle\langle\lambda|\, d\rho_\gamma(\lambda)\, d\rho_\gamma(\mu)$$

where the integral has to be interpreted in the sense of (5.22) and converges in $T_{X\otimes X, A\boxplus A}$ (even in $T(S_{X\otimes X, I\otimes A}, A\otimes I)$). Let $R' \in T(S_{X\otimes X, I\otimes A}, A\otimes I)$. Then the matrix of the product $R'R$ is given by

(5.24) $$[R'R]_{\mu\lambda} : (s,t) \mapsto \int_{\mathbb{R}} [R']_{\mu\nu}(s,\sigma)\, [R]_{\nu\lambda}(-\sigma,t)\, d\rho_\gamma(\nu)$$

where the integrals converge absolutely and do not depend on the choice of σ, and where $\sigma > 0$ has to be taken such that

$$e^{\sigma A} R(t) \in S_{X\otimes X, A\boxplus A} .$$

We write

(5.25) $$[R'R]_{\mu\lambda} = \int_{\mathbb{R}} [R']_{\mu\nu}\, [R]_{\nu\lambda}\, d\rho_\gamma(\nu)$$

where the integral converges in the indicated distributional sense. Further, let $|f\rangle$ be a test ket. Then $R|f\rangle$ is a test ket, also, and

(5.26) $$R|f\rangle = \iint_{\mathbb{R}^2} \langle\mu \mid R(\tau) \mid \lambda\rangle(0)\, \langle\lambda \mid f\rangle(-\tau)\, |\mu\rangle\, d\rho_\gamma(\lambda)\, d\rho_\gamma(\mu)$$
$$= \iint_{\mathbb{R}^2} [R]_{\mu\lambda}(-\sigma,\tau)\, \langle\lambda \mid f\rangle(-\tau)\, |\mu\rangle(\sigma)\, d\rho_\gamma(\lambda)\, d\rho_\gamma(\mu)$$

where the integral converges in $T_{X,A}$ and does not depend on the choice of $\tau > 0$ and of $\sigma > 0$ chosen sufficiently small as indicated in (5.21). Finally, we have

(5.27) $$\langle\mu \mid R \mid f\rangle : s \mapsto \int_{\mathbb{R}} [R]_{\mu\lambda}(s,\tau)\, \langle\lambda \mid f\rangle(-\tau)\, d\rho_\gamma(\lambda) .$$

For $Q \in T(S_{X\otimes X, A\otimes I}, I\otimes A)$ its matrix $[Q]$ is defined by

(5.28) $$[Q]_{\mu\lambda} : (s,t) \to \langle\mu \mid Q(s) \mid \lambda\rangle(t) .$$

Note that $Q(s) \in S_{X\otimes X, A\boxplus A}$. So there exists $\tau > 0$ such that $\langle\mu \mid Q(s) \mid \lambda\rangle(-\tau)$ is well-defined because $Q(s)\,|\lambda\rangle$ is a test ket. It can be shown that

(5.29) $$Q : s \mapsto Q(s) = \iint_{\mathbb{R}^2} [Q]_{\mu\lambda}(\sigma,-\tau)\,(|\lambda\rangle(\tau) \otimes |\mu\rangle(s-\sigma)\, d\rho_\gamma(\lambda)\, d\rho_\gamma(\mu)$$

where σ, $0 < \sigma < s$, and where the integrals converge in $X \otimes X$ and do not depend on the choice of σ, and of $\tau > 0$ sufficiently small (cf. (5.21).

We write

(5.30) $$Q = \iint_{\mathbb{R}^2} [Q]_{\mu\lambda}\,(|\mu\rangle\langle\lambda|)\,d\rho_\gamma(\lambda)\, d\rho_\gamma(\mu)$$

where the integral has to be interpreted in the sense of (5.29) and converges in $T_{X\otimes X, A\boxplus A}$. Let $Q' \in T(S_{X\otimes X, A\otimes I}, I\otimes A)$. Then the matrix of the product $Q'Q$ is given by

(5.31) $$[Q'Q]_{\mu\lambda} : (s,t) \mapsto \int_{\mathbb{R}} [Q']_{\mu\nu}(s,-\tau)\,[Q]_{\nu\lambda}(\tau,t)\, d\rho_\gamma(\nu)$$

where the integrals converge absolutely and do not depend on the choice of τ, and where $\tau > 0$ has to be taken such that

$$Q'(t)e^{\tau A} \in S_{X\otimes X, A\boxplus A} .$$

We write

$$(5.32)\qquad [Q'Q]_{\mu\lambda} = \int_{\mathbb{R}} [Q'Q]_{\mu\nu}\,[Q]_{\nu\lambda}\, d\rho_\gamma(\nu)\ .$$

Again the integral converges in the above-mentioned distributional sense.

Further, $Q|H>$ can be represented by

$$(5.33)\qquad Q|H> \,:\, s \mapsto \iint_{\mathbb{R}^2} [Q]_{\mu\lambda}(\sigma,-\tau) <\lambda \mid H>(\tau)\, |\mu>(s-\sigma)\, d\rho_\gamma(\lambda)\, d\rho_\gamma(\mu)$$

where the integrals converge absolutely in X for every $s > 0$ and do not depend on the choice of σ, $0 < \sigma < s$, and $\tau > 0$, and where $\tau > 0$ has to be taken such that $Q(\sigma)e^{\tau A} \in S_{X\otimes X, A \boxplus A}$.

Finally, note that

$$(5.34)\qquad <\mu \mid Q \mid H> \,:\, s \mapsto \int_{\mathbb{R}} [Q_{\mu\lambda}](s,-\tau) <\lambda \mid H>(\tau)\, d\rho_\gamma(\lambda)\ .$$

Remark

The proofs of most results we gave in the last part of this section become more transparant by the following relation:

Let $B \in S_{X\otimes X, A\boxplus A}$, and let $t_1 > 0$ and $t_2 > 0$. Then

$$(e^{-t_1 A} \otimes e^{-t_2 A})B = \iint <\mu \mid B \mid \lambda>(0)\, (|\lambda>(t_1) \otimes |\mu>(t_2))\, d\rho_\gamma(\lambda)\, d\rho_\gamma(\mu)\ .$$

The proof of this relation runs analogously to the proof of Theorem (5.1).

References to this section:

[An], [Bö], [Di], [Ja], [GeVi], [Mel], [Ro].

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References

[An] : Antoine, J.P., General Dirac formalism, J. Math. Phys. 10, (1969), p. 53.

[Bö] : Böhm, A., The rigged Hilbert space and quantum mechanics, Lect. Notes in Phys., 78, Springer, 1978.

[Bou] : Bourbaki, N., Element des mathematiques, Livre VI, Intégration, Hermann Paris, 1969.

[Br] : Brown, A., A version of multiplicity theory in 'Topics in operator theory', Math. surveys, nr.13, AMS., 1974.

[GeVi] : Gelfand, I.M. and Vilenkin, N.Ya., Generalized Functions, part IV, Ac. Press, New-York, 1964.

[Di] : Dirac, P.A.M., The principles of quantum mechanics, 1958, Clarendon Press, Oxford.

[G] : Graaf, J. De, A theory of generalized functions based on holomorphic semigroup, TH-report, 79-WSK-02, Eindhoven University of Technology, 1979.

[Ja] : Jauch, J.M., On bras and kets, in 'Aspects of quantum theory' edited by A. Salam and E. Wigner, Cambridge University Press, 1972.

[Ne] : Nelson, E., Topics in dynamics I: Flows, Mathematical notes, Princeton University Press, 1969.

[Neu] : Neumann, J. Von, Mathematical foundations of quantum mechanics, Princeton University Press, 1955.

[Me] : Melsheimer, O., Rigged Hilbert space formalism, J. Math. Phys., 15 (1974), p.902.

[Ro] : Rogers, J.E., The Dirac bra and ket formalism. J. Math. Phys., 7 (1966), p. 1097.

[WZ] : Wheeden, R.L., Zygmund, A., Measure and integral, Marcel Dekker inc., New-York, 1977.

[IT] : Ionescu Tulcea, A. and C., Topics in the theory of lifting, Springer, Berlin, 1969.

[E_1] : Eijndhoven. S.J.L. Van, PhD. Thesis, to appear 1983.