

A note on dynamic programming with unbounded rewards

Citation for published version (APA):

van Nunen, J. A. E. E., & Wessels, J. (1975). *A note on dynamic programming with unbounded rewards*. (Memorandum COSOR; Vol. 7513). Technische Hogeschool Eindhoven.

Document status and date:

Published: 01/01/1975

Document Version:

Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

Please check the document version of this publication:

- A submitted manuscript is the version of the article upon submission and before peer-review. There can be important differences between the submitted version and the official published version of record. People interested in the research are advised to contact the author for the final version of the publication, or visit the DOI to the publisher's website.
- The final author version and the galley proof are versions of the publication after peer review.
- The final published version features the final layout of the paper including the volume, issue and page numbers.

[Link to publication](#)

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal.

If the publication is distributed under the terms of Article 25fa of the Dutch Copyright Act, indicated by the "Taverne" license above, please follow below link for the End User Agreement:

www.tue.nl/taverne

Take down policy

If you believe that this document breaches copyright please contact us at:

openaccess@tue.nl

providing details and we will investigate your claim.

TECHNOLOGICAL UNIVERSITY EINDHOVEN

Department of Mathematics

STATISTICS AND OPERATIONS RESEARCH GROUP

Memorandum COSOR 75-13

A note on dynamic programming with unbounded rewards

by

J.A.E.E. van Nunen and J. Wessels

Eindhoven, September 1975

A note on dynamic programming with unbounded rewards

by

J.A.E.E. van Nunen and J. Wessels

Summary. In a recent paper, Lippman presents sufficient conditions for Denardo's N-stage contraction in discounted semi-Markov decision processes with unbounded rewards. In this note it is demonstrated that Lippman's conditions may be replaced by weaker conditions which even imply 1-stage contraction. The verification of the conditions of this note is somewhat easier.

Lippman [2] considers a discounted semi-Markov decision process with general state space S and action space A . He presents sufficient conditions for the existence of a normed Banach space of realvalued functions on S in which Denardo's N-stage contraction approach [1] may be used.

In Lippman's notation $q(\cdot|x,a)$, $r(x,a)$ denote the transition probability and one period reward respectively for state $x \in S$ and action $a \in A$; $\alpha > 0$ is the discountfactor; $t(\cdot|x,a)$ is the probability distribution function of the time until the next transition (given state $x \in S$, action $a \in A$).

The conditions in [2] are the following:

A function w on S exists with $w(x) \geq 1$, an integer $m \geq 1$ exists, a number β ($0 \leq \beta < 1$) exists, positive numbers b and M exist, such that for all $x \in S$, $a \in A$:

$$\beta(x,a) := \int_0^{\infty} e^{-\alpha\tau} t(d\tau|x,a) \leq \beta ,$$

$$|r(x,a)|w^{-m}(x) \leq M ,$$

$$\int_S w^n(y)q(dy|x,a) \leq [w(x) + b]^n \quad \text{for } n = 1, \dots, m .$$

Lippman's Banach space consists of realvalued functions u on S with the following norm:

$$\|u\| := \sup_x |u(x)|w^{-m}(x) .$$

Hence Lippman uses weighted supremum norms as introduced more generally for Markov decision processes in [3].

In [2] it is proved that under these conditions there exists an integer $J \geq 1$, such that for any sequence of policies f_1, \dots, f_J the operator T_{f_1, \dots, f_J} is a contraction. Here a policy f maps S into A , and T_f is defined as an operator in the Banach space with

$$(T_f u)(x) := r(x, f(x)) + \beta(x, f(x)) \int_S u(y) q(dy | x, f(x)) .$$

Lemma. Under Lippman's conditions the following holds: For any $\rho > \beta$ there exists a positive function v on S , such that

$$\beta \int_S v(y) q(dy | x, a) \leq \rho v(x) \quad \text{for all } x \in S, a \in A .$$

Proof. Choose a real number c with $c \geq b[(\frac{\rho}{\beta})^{1/m} - 1]^{-1}$ or $b + c \leq (\frac{\rho}{\beta})^{1/m} c$. Define $v(x) := [w(x) + c]^m$. Then

$$\begin{aligned} \int_S v(y) q(dy | x, a) &= \int_S [w(y) + c]^m q(dy | x, a) = \\ &= \sum_{n=0}^m \binom{m}{n} c^{m-n} \int_S w^n(y) q(dy | x, a) \leq \\ &\leq \sum_{n=0}^m \binom{m}{n} c^{m-n} [w(x) + b]^n = [w(x) + b + c]^m \leq \\ &\leq [w(x) + (\frac{\rho}{\beta})^{1/m} c]^m \leq \frac{\rho}{\beta} v(x) . \end{aligned}$$

This lemma enables us to introduce a new weighted supremum norm (and hence a new Banach space, which actually contains the old one if $v = (w + c)^m$) in which T_f itself is already a contraction:

$$\|u\|_v := \sup_x |u(x)| v^{-1}(x) \quad \text{if } v(x) > 0 .$$

Consider the Banach space of realvalued functions u on S with $\|u\|_v < \infty$.

Theorem. Under Lippman's conditions the following holds: For any ρ ($\beta < \rho < 1$) there exists a function v on S with $v(x) > 0$, such that for any policy f

$$\|T_f u_1 - T_f u_2\|_v \leq \rho \|u_1 - u_2\|_v$$

$$\|r_f\|_v \leq M ,$$

where $r_f(x) := r(x, f(x))$.

Proof. Choose c and v as in the lemma. Then

$$\begin{aligned} |(T_f u_1 - T_f u_2)(x)| &\leq \beta \int_S |u_1(y) - u_2(y)| q(dy|x, f(x)) \\ &\leq \beta \|u_1 - u_2\|_v \int_S v(y) q(dy|x, f(x)) \\ &\leq \rho \|u_1 - u_2\|_v v(x) . \end{aligned}$$

Furthermore: $|r(x, a)| v^{-1}(x) \leq |r(x, a)| w^{-m}(x) \leq M$.

Now Lippman's conditions may be replaced by the following weaker and simpler conditions: A function v on S exists with $v(x) > 0$, a number β ($0 \leq \beta < 1$) exists, a number ρ ($\beta < \rho < 1$) exists, a positive number M exists, such that for all $x \in S$, $a \in A$:

$$\beta(x, a) := \int_{0^-}^{\infty} e^{-\alpha \tau} t(d\tau|x, a) \leq \beta ,$$

$$|r(x, a)| v^{-1}(x) \leq M ,$$

$$\beta \int_S v(y) q(dy|x, a) \leq \rho v(x) .$$

Namely, if our conditions are satisfied T_f is a ρ -contraction with respect to the norm $\|\cdot\|_v$ and $\|r_f\|_v \leq M$.

Remarks.

- 1) In order that T_f is contracting it is not necessary that $v(x) \geq 1$; in [2] the condition $w(x) \geq 1$ is essential. Actually we proved that, if Lippman's conditions are satisfied, with $w(x) > 0$ instead of $w(x) \geq 1$, than still a v -norm may be found satisfying our conditions.
- 2) As demonstrated in [3], the discounting requirement is not essential in our analysis: if we replace $\beta(x,a)q(\cdot|x,a)$ by $p(\cdot|x,a)$ then our conditions become:

$$|r(x,a)|v^{-1}(x) \leq M < \infty$$
$$\int_S v(y)p(dy|x,a) \leq \rho v(x) \quad \text{with } \rho < 1 .$$

These conditions allow the situation $\alpha = 0$ in certain cases and give some weakening for $\alpha > 0$.

References

- [1] E.V. Denardo, Contraction mappings in the theory underlying dynamic programming.
SIAM Review 9 (1967), 165-177.
- [2] S.A. Lippman, On dynamic programming with unbounded rewards.
Management Science 21 (1975), 1225-1233.
- [3] J. Wessels, Markov programming by successive approximations with respect to weighted supremum norms.
J. Math. Anal. Appl. (to appear).