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Department of Mathematics and Computing Science

RANA 01-24
August 2001

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Reports on Applied and Numerical Analysis
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P.O. Box 513
5600 MB Eindhoven, The Netherlands
ISSN: 0926-4507

MAXIMAL L^p REGULARITY OF FRACTIONAL ORDER EQUATIONS

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Abstract

We study the maximal L^p regularity of the abstract linear problem for the fractional differential equation with Riemann-Liouville fractional derivative of order $\alpha \in (0,2)$. Our analysis combines two different approaches. First we prove maximal L^p regularity of the problem with zero initial conditions using the method of sums of operators. We apply both classical theorems and some very recent results involving the notion of $\mathcal{R}$-boundedness. Concerning the problem with zero forcing function, we use the resolvent approach and prove strict L^p solvability provided the initial data belongs to some real interpolation spaces.

1. Introduction

The notion of maximal L^p regularity plays an important role in the functional analytic approach to parabolic partial differential equations. Many initial and boundary value problems can be reduced to an abstract Cauchy problem of the form

$$u'(t) + Au(t) = f(t), \ t \in I, \quad u(0) = 0, \tag{1}$$

where $I = (0,T)$, $T > 0$, $-A$ generates a bounded analytic semigroup on a Banach space X and f and u are X-valued functions on I. It is well known that (1) has a strong solution for all locally Bochner integrable f, but in many applications we need that u' has the same "smoothness" as f, which is not always the case. In particular, one says that problem (1) has **maximal L^p regularity** on I if for every $f \in L^p(I;X)$ there exists one and only one $u \in L^p(I;D(A)) \cap W^{1,p}(I;X)$ satisfying (1). From the closed graph theorem it follows easily that if there is L^p regularity then there exists $C > 0$ such that

$$\|u\|_{L^p} + \|u'\|_{L^p} + \|Au\|_{L^p} \leq C\|f\|_{L^p}.$$

The theory of strongly continuous semigroups could suggest that it is more natural to study the continuous regularity for (1), i.e. the existence and uniqueness of a solution $u \in C(I;D(A)) \cap C^1(I;X)$ for any continuous f. But Baillon [2] proved that if there is continuous regularity for an unbounded operator A that generates a C_0 semigroup, then the space X must contain a subspace isomorphic to c_0, the space of sequences converging to 0. This fact implies that X cannot be reflexive. On the other hand there are good results of L^p regularity in some reflexive spaces.

There is a rich literature on sufficient conditions for maximal L^p regularity (see for a survey [12]), which implies that for most classical differential operators that may be of interest, there is maximal L^p regularity of problem (1). Quite recently, necessary and sufficient conditions for maximal L^p regularity was obtained in terms of $\mathcal{R}$-boundedness (for a definition see e.g. [9]).

The following theorem is due independently to N. Kalton and L. Weis. For a proof see e.g. [21].

THEOREM 1. *Let $-A$ generates a bounded analytic semigroup on an $\mathcal{HT}$ space X. Then problem (1) has maximal L^p regularity if and only if the set $\{\lambda(\lambda + A)^{-1} | \ \lambda \in i\mathbb{R}, \ \lambda \neq 0\}$ is $\mathcal{R}$-bounded.*

In Hilbert spaces the uniform boundedness of this set already implies maximal L^p regularity, but only in Hilbert space: recently Kalton and Lancien [14] essentially proved that if for every negative generator A of a bounded analytic semigroup on a Banach space X problem (1) has maximal L^p regularity, then X is isomorphic to a Hilbert space. It appears that the additional assumption which we need in more general Banach spaces is namely the $\mathcal{R}$-boundedness of the set.

Maximal L^p regularity is an important tool in treating evolution equations more complex than the basic Cauchy problem (1), such as second order equations, Volterra equations, nonautonomous and quasilinear equations. In this thesis we apply maximal L^p regularity to study fractional order equations: both autonomous and nonautonomous.

2. Preliminaries

2.1. Operators in Banach spaces

Let X be a complex Banach space, and let $A : D(A) \subset X \to X$ be a closed linear densely defined operator in X. In the sequel we suppose that $D(A)$ is equipped with the graph norm of A, i.e. $\|x\|_{D(A)} := \|x\|_X + \|Ax\|_X$; since A is closed, $D(A)$ is a Banach space, continuously and densely embedded into X.

We call an operator $A : D(A) \subset X \to X$ **nonnegative** iff the following two conditions are satisfied:

(i) there exists $K \geq 0$ such that for all $\lambda > 0$ and all $u \in D(A)$,

$$\lambda \|u\|_X \leq K \|\lambda u + Au\|_X \tag{2}$$

holds;

(ii) $R(\lambda I + A) = X$ for all $\lambda > 0$.

Observe that if A satisfies (i) and (ii), it is closed. Moreover, any nonnegative operator in a reflexive Banach space is densely defined [15]. If A is a nonnegative operator on X, then define

$$\phi_A := \sup\{\phi \in [0,\pi] |\ \rho(-A) \supset \overline{\Sigma_\phi},\ \sup_{\lambda\in\overline{\Sigma_\phi}} \|\lambda(\lambda I + A)^{-1}\|_{\mathcal{B}(X)} < \infty\}$$

$$K_A(\phi) := \sup_{\lambda\in\overline{\Sigma_\phi}} \|\lambda(\lambda I + A)^{-1}\|_{\mathcal{B}(X)},\ \phi < \phi_A.$$

The **spectral angle** of A is defined by

$$\omega_A := \pi - \phi_A. \tag{3}$$

An operator A is said to be **positive** if it is nonnegative and $0 \in \rho(A)$.

There are many examples of positive operators. For instance, any positive-definite self-adjoint operator acting in Hilbert space is a positive operator. If A generates a C_0-semigroup of negative type then $-A$ is a positive operator. The reverse statement, however, is untrue, since there exist positive operators which are not generators of suitable semigroups. In many cases, however, condition $0 \in \rho(A)$ is not satisfied, e.g. for the Laplace operator on $L^p(\mathbb{R}^n)$ we have $0 \in \sigma(A)$. Therefore it is desirable to weaken this condition.

An operator A is called **sectorial** if it is nonnegative and $N(A) = \{0\}$ and $\overline{R(A)} = X$.

Obviously, any positive operator is sectorial. Examples of sectorial, but not positive operators are some differential operators on unbounded regions, like the Laplace operator or the Stokes operator on exterior domains.

For the class of sectorial operators one can define complex powers (see e.g. [17]). A sectorial operator A is said to admit **bounded imaginary powers** if the purely imaginary powers A^{is} of A are uniformly bounded for $s \in [-1,1]$. Then it can be shown that A^{is} forms a strongly continuous C_0-group of bounded linear operators. The type θ_A of this group defined by

$$\theta_A := \overline{\lim}_{|s|\to\infty} |s|^{-1} \log \|A^{is}\|_{\mathcal{B}(X)}.$$

is called the **power angle** of A. Then for any $\varphi_A > \theta_A$, there exists constant $M = M(\varphi_A) \geq 1$ such that

$$\|A^{is}\|_{\mathcal{B}(X)} \leq M e^{\varphi_A |s|},\ s \in \mathbb{R}.$$

We denote $A \in \mathcal{BIP}(\mathcal{X}; \mathcal{M}, \varphi_A)$ or $A \in \mathcal{BIP}(\mathcal{X}; \varphi_A)$. The spectral angle ω_A and the power angle θ_A of an operator satisfy the inequality $\theta_A \geq \omega_A$ (see [18]).

Let A be a nonnegative operator in X, $\gamma \in (0,1)$, $p \in (1,\infty)$. Consider the spaces

$$D_A(\gamma,p) := \{x \in X |\ [x]_{D_A(\gamma,p)} < \infty\},$$

where

$$[x]_{D_A(\gamma,p)} := \left\{ \int_0^\infty (t^\gamma \|A(tI+A)^{-1}x\|_X)^p \frac{dt}{t} \right\}^{\frac{1}{p}}, \tag{4}$$

endowed with the norm $\|x\|_{D_A(\gamma,p)} := \|x\|_X + [x]_{D_A(\gamma,p)}$. These spaces coincide up to the equivalence of norms with the **real interpolation spaces** $(X, D(A))_{\gamma,p}$ between X and $D(A)$ ([5], Proposition 3). They are intermediate spaces between $D(A)$ and X in the following sense:

$$D(A) \hookrightarrow D_A(\gamma,p) \hookrightarrow D_A(\gamma',p) \hookrightarrow X, \; 0 < \gamma' < \gamma < 1, \tag{5}$$

where $\hookrightarrow$ denotes continuous embedding. The real interpolation spaces are extensively studied; we refer e.g. to [20] for a more detailed description.

Recall that a Banach space X is said to belong to the **class $\mathcal{HT}$** if the Hilbert transform H defined by

$$(Hf)(t) = \lim_{\varepsilon \to 0+} \int_{|s| \geq \varepsilon} f(t-s) \frac{ds}{\pi s}, \;\; t \in \mathbb{R}, \; f \in C_0^\infty(\mathbb{R}; X),$$

extends to a bounded linear operator on $L^p(\mathbb{R}; X)$ for some $p \in (1, \infty)$. It is well known that Hilbert spaces are of class $\mathcal{HT}$ and if X is of class $\mathcal{HT}$ then $L^p(\mathbb{R}; X)$ is of class $\mathcal{HT}$ for every $p \in (1, \infty)$. Note also that any Banach space of class $\mathcal{HT}$ is reflexive.

2.2. Operators of fractional differentiation in L^p spaces

Let $\alpha > 0$, $m = \lceil \alpha \rceil$, the smallest integer greater than or equal to α, and $I = (0, T)$ for some $T > 0$. For the sake of brevity we use the following notation for $\beta \geq 0$:

$$g_\beta(t) := \begin{cases} \frac{1}{\Gamma(\beta)} t^{\beta-1}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{6}$$

where $\Gamma(\beta)$ is the Gamma function. Note that $g_0(t) = 0$, because $\Gamma(0)^{-1} = 0$. These functions satisfy the semigroup property

$$g_\alpha * g_\beta = g_{\alpha+\beta}. \tag{7}$$

The **Riemann-Liouville fractional integral** of order $\alpha > 0$ is defined as follows:

$$J_t^\alpha f(t) := (g_\alpha * f)(t), \;\; f \in L^1(I), \; t > 0. \tag{8}$$

Set $J_t^0 f(t) := f(t)$. The **Riemann-Liouville fractional derivative** of order α is defined for all f satisfying

$$f \in L^1(I), \; g_{m-\alpha} * f \in W^{m,1}(I) \tag{9}$$

by

$$D_t^\alpha f(t) := D_t^m(g_{m-\alpha} * f)(t) = D_t^m J_t^{m-\alpha} f(t), \tag{10}$$

where $D_t^m := \frac{d^m}{dt^m}$, $m \in \mathbb{N}$. As in the case of differentiation and integration of integer order, D_t^α is a left inverse of J_t^α, but in general it is not a right inverse.

Let X be a complex Banach space. Denote the operators of fractional integration on $L^p(I;X)$ by $\mathcal{J}_\alpha$:

$$D(\mathcal{J}_\alpha) := L^p(I;X), \quad \mathcal{J}_\alpha u := g_\alpha * u, \tag{11}$$

where the integration is in the sense of Bochner. Applying the Young inequality, it follows that $\mathcal{J}_\alpha \in \mathcal{B}(L^p(I;X))$:

$$\|\mathcal{J}_\alpha u\|_{L^p(I;X)} = \|g_\alpha * u\|_{L^p(I;X)} \leq \|g_\alpha\|_{L^1(I)} \|u\|_{L^p(I;X)} = g_{\alpha+1}(T)\|u\|_{L^p(I;X)}.$$

The **Sobolev spaces** can be defined in the following way (see [3], Appendix):

$$W^{m,p}(I;X) := \{f \mid \exists \varphi \in L^p(I;X) : \ f(t) = \sum_{k=0}^{m-1} c_k g_{k+1} + g_m * \varphi(t), \ t \in I\}. \tag{12}$$

Note that $\varphi(t) = f^{(m)}(t)$, $c_k = f^{(k)}(0)$. Let

$$W_0^{m,p}(I;X) := \{f \in W^{m,p}(I;X) \mid f^{(k)}(0) = 0, \ k = 0, 1, \dots, m-1\}.$$

Define the spaces $R^{\alpha,p}(I;X)$ and $R_0^{\alpha,p}(I;X)$ as follows.

If $\alpha \notin \mathbb{N}$, set

$$\begin{aligned} R^{\alpha,p}(I;X) &:= \{u \in L^p(I;X) \mid g_{m-\alpha} * u \in W^{m,p}(I;X)\}, \\ R_0^{\alpha,p}(I;X) &:= \{u \in L^p(I;X) \mid g_{m-\alpha} * u \in W_0^{m,p}(I;X)\}. \end{aligned} \tag{13}$$

If $\alpha \in \mathbb{N}$ we take

$$R^{\alpha,p}(I;X) := W^{\alpha,p}(I;X), \quad R_0^{\alpha,p}(I;X) := W_0^{\alpha,p}(I;X). \tag{14}$$

Denote the extensions of the operators of fractional differentiation in $L^p(I;X)$ by $\mathcal{L}_\alpha \mathcal{L}_\alpha$, i.e.

$$D(\mathcal{L}_\alpha) := R_0^{\alpha,p}(I;X), \quad \mathcal{L}_\alpha u := D_t^\alpha u, \tag{15}$$

where D_t^α is the Riemann-Liouville fractional derivative (10). In the next lemma we study the properties of $\mathcal{L}_\alpha$.

LEMMA 2. *Let $\alpha > 0$, $1 < p < \infty$, X be a complex Banach space, and $\mathcal{L}_\alpha$ be the operators defined by (15). Then*

(a) $\mathcal{L}_\alpha$ are closed, linear, densely defined;

(b) $\mathcal{L}_\alpha = \mathcal{J}_\alpha^{-1}$*;*

(c) $\mathcal{L}_\alpha = \mathcal{L}_1^\alpha$*, the* α*-th power of the operator* $\mathcal{L}_1$*;*

(d) if $\alpha \in (0,2)$ *then* $\mathcal{L}_\alpha$ *are positive operators with spectral angle* $\omega_{\mathcal{L}_\alpha} = \alpha\pi/2$*;*

(e) if X *is of class* $\mathcal{HT}$ *and* $\alpha \in (0,2)$ *then* $\mathcal{L}_\alpha \in \mathcal{BIP}(\mathcal{L}^{\checkmark}(\mathcal{I};\mathcal{X}); \alpha(\pi/\in + \varepsilon))$ *for each* $\varepsilon > 0$*;*

(f) if $\alpha \in (0,1]$ *then* $\mathcal{L}_\alpha$ *are m-accretive operators.*

Proof: The operator $\mathcal{J}_\alpha$ is injective. Indeed, if $\mathcal{J}_\alpha u = 0$, then $\mathcal{J}_1 u = \mathcal{J}_{1-\alpha}\mathcal{J}_\alpha u = 0$, whence $u = 0$. Therefore $\mathcal{J}_\alpha^{-1}$ exists. We shall prove that $\mathcal{L}_\alpha = \mathcal{J}_\alpha^{-1}$. If $u \in R(\mathcal{J}_\alpha)$, then $u = g_\alpha * v$ for some $v \in L^p(I;X)$, and $g_{m-\alpha} * u = g_{m-\alpha} * g_\alpha * v = g_m * v$. Therefore, $g_{m-\alpha} * u \in W_0^{m,p}(I;X)$, that is, $u \in R_0^{\alpha,p} = D(\mathcal{L}_\alpha)$. The identities $\mathcal{L}_\alpha\mathcal{J}_\alpha u = u$, $u \in L^p(I;X)$, $\mathcal{J}_\alpha\mathcal{L}_\alpha v = v$, $v \in D(\mathcal{L}_\alpha)$, can be proven straiughtforwardly. Thus, we proved (b). The representation $\mathcal{L}_\alpha = \mathcal{J}_\alpha^{-1}$ incidentally shows that $\mathcal{L}_\alpha$ is a closed operator as an inverse of a bounded operator and that it is densely defined because $D(\mathcal{L}_\alpha) = R(\mathcal{J}_\alpha)$, which is dense in $L^p(I;X)$. Obviously, it is also linear and (a) is proved.

Let us compute the resolvent of $\mathcal{L}_1$,

$$((sI + \mathcal{L}_1)^{-1}f)(t) = \int_0^t e^{-s(t-\tau)} f(\tau)\, d\tau, \ \mathrm{Re}\, s > 0, \ t \in \mathrm{I}. \tag{16}$$

This representation implies that $\mathcal{L}_1$ is positive with spectral angle $\phi_{\mathcal{L}_1} = \pi/2$. We shall prove that $\mathcal{L}_\alpha = \mathcal{L}_1^\alpha$. Consider first the case $\alpha \in (0,1)$. We have the following representation (see e.g. [1], eq. (4.6.9)):

$$\mathcal{L}_1^{\alpha-1} = \frac{\sin \pi\alpha}{\pi} \int_0^\infty s^{\alpha-1}(sI + \mathcal{L}_1)^{-1}\, ds.$$

Applying (16) and using the definition of the Gamma function and the formula

$$\Gamma(\alpha)\Gamma(1-\alpha) = \pi/\sin \pi\alpha$$

we obtain

$$\mathcal{L}_1^{\alpha-1} f(t) = \frac{\sin \pi\alpha}{\pi} \int_0^\infty s^{\alpha-1} \int_0^t e^{-s(t-\tau)} f(\tau)\, d\tau\, ds = g_{1-\alpha} * f \tag{17}$$

for $t \in I$, $f \in L^p(I;X)$. Since $\mathcal{L}_1$ is an isomorphic mapping from $D(\mathcal{L}_1^\alpha)$ to $D(\mathcal{L}_1^{\alpha-1})$ ([20], Section 1.15.2) then $f \in D(\mathcal{L}_1^\alpha)$ is equivalent to $\mathcal{L}_1^{\alpha-1} f \in D(\mathcal{L}_1)$. This is equivalent to $f \in D(\mathcal{L}_\alpha)$ by (17) and by the definition of $D(\mathcal{L}_\alpha)$. Therefore $D(\mathcal{L}_1^\alpha) = D(\mathcal{L}_\alpha)$. Applying $\mathcal{L}_1$ to (17) we obtain $\mathcal{L}_1^\alpha f = \mathcal{L}_\alpha f$ for $f \in D(\mathcal{L}_1^\alpha) = D(\mathcal{L}_\alpha)$. Let now $\alpha > 1$. Then from the definition of $\mathcal{L}_\alpha$ and the above result one has $\mathcal{L}_\alpha = \mathcal{L}_{m-1}\mathcal{L}_{\alpha-m+1} = \mathcal{L}_1^{m-1}\mathcal{L}_1^{\alpha-m+1} = \mathcal{L}_1^\alpha$.

The facts that $\mathcal{L}_\alpha$, $\alpha \in (0,2)$, are positive and $\phi_{\mathcal{L}_\alpha} = \alpha\pi/2$ follow from the representation $\mathcal{L}_\alpha = \mathcal{L}_1^\alpha$. To see that $\omega_{\mathcal{L}_\alpha} \leq \alpha\pi/2$ one applies [16], Proposition 4. Assume that $\omega_{\mathcal{L}_\alpha} < \alpha\pi/2$. Then $\phi_{\mathcal{L}_\alpha} > \pi(1-\alpha/2)$. But from the representation ([19], Example 42.2)

$$((sI+\mathcal{L}_\alpha)^{-1}f)(t) = \int_0^t (t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-s(t-\tau)^\alpha)f(\tau)\,d\tau$$

and from the asymptotic expansion of the Mittag-Leffler function it follows that if f is a constant then $\|s(sI+\mathcal{L}_\alpha)^{-1}f\|_{L^p(I;X)} \to \infty$ as $|s| \to \infty$ and $|\arg s| > \pi(1-\alpha/2)$. By this contradiction $\omega_{\mathcal{L}_\alpha} = \alpha\pi/2$.

According to [13], Th.3.1, if X belongs to the class $\mathcal{HT}$, then the imaginary powers of $\mathcal{L}_1$ satisfy the estimate

$$\|\mathcal{L}_1^{is}\|_{\mathcal{B}(L^p(I;X))} \leq c(1+s^2)e^{\frac{\pi}{2}|s|},\ s \in \mathbb{R}.$$

Therefore, given $\varepsilon > 0$, there exists $M > 0$ such that

$$\|\mathcal{L}_1^{is}\|_{\mathcal{B}(L^p(I;X))} \leq Me^{(\frac{\pi}{2}+\varepsilon)|s|},\ s \in \mathbb{R}, \tag{18}$$

which means $\mathcal{L}_1 \in \mathcal{BIP}(\mathcal{L}\surd(\mathcal{I};\mathcal{X});\pi/\in+\varepsilon)$. Since $\mathcal{L}_\alpha$, $\alpha \in (0,2)$, are positive, their fractional powers $\mathcal{L}_\alpha^z$, $z \in \mathbb{C}$, are well defined and satisfy ([1], Theorem 4.6.13) $\mathcal{L}_\alpha^{is} = (\mathcal{L}_1^\alpha)^{is} = \mathcal{L}_1^{\alpha is}$. Therefore, by (18), $\mathcal{L}_\alpha$ has bounded imaginary powers and

$$\|\mathcal{L}_\alpha^{is}\|_{\mathcal{B}(L^p(I;X))} \leq Me^{\alpha(\frac{\pi}{2}+\varepsilon)|s|},\ s \in \mathbb{R},$$

that is $\mathcal{L}_\alpha \in \mathcal{BIP}(\mathcal{L}\surd(\mathcal{I};\mathcal{X});\alpha(\pi/\in+\varepsilon))$.

Lastly, (f) follows from [8], Theorem 3.1, because $g_{1-\alpha} \in L^1(I)$ is nonnegative and nonincreasing for $\alpha \in (0,1]$. □

Applying [20], Sections 1.15.4 and 2.10.4, we obtain for $0 \leq \alpha < \beta \leq 1$, $0 < \gamma < 1$, $\alpha(1-\gamma)+\beta\gamma-1/p \notin \mathbb{N}_0$,

$$(R_0^{\alpha,p}(I;X), R_0^{\beta,p}(I;X))_{\gamma,p} = (D(\mathcal{L}_1^\alpha), D(\mathcal{L}_1^\beta))_{\gamma,p} = (L^p(I;X), D(\mathcal{L}_1^\beta))_{\frac{\alpha(1-\gamma)+\beta\gamma}{\beta},p}$$

$$= (L^p(I;X), D(\mathcal{L}_1))_{\alpha(1-\gamma)+\beta\gamma,p} = (L^p(I;X), W_0^{1,p}(I;X))_{\alpha(1-\gamma)+\beta\gamma,p} = W_0^{\alpha(1-\gamma)+\beta\gamma,p}(I;X),$$

and, if $\alpha\gamma - 1/p \notin \mathbb{N}_0$,

$$(L^p(I;X), R_0^{\alpha,p}(I;X))_{\gamma,p} = W_0^{\alpha\gamma,p}(I;X). \tag{19}$$

In the case when X is of class $\mathcal{HT}$, we obtain a precise identification of $R_0^{\alpha,p}$ for $\alpha - 1/p \notin \mathbb{N}_0$. In this case $\mathcal{L}_1$ has bounded imaginary powers and therefore ([20], Theorem 1.15.3)

$$R_0^{\alpha,p} = D(\mathcal{L}_\alpha) = D(\mathcal{L}_1^\alpha) = [L^p, D(\mathcal{L}_1)]_\alpha,\ 0 < \alpha < 1,$$

the **complex interpolation space** between L^p and $D(\mathcal{L}_1) \hookrightarrow L^p$ of order α. Now introduce the **Bessel potential spaces** defined by

$$H^{\alpha,p}(\mathbb{R};X) := \{f \mid \exists f_\alpha \in L^p(\mathbb{R};X): \widetilde{f_\alpha}(\rho) = |\rho|^\alpha \widetilde{f}(\rho),\ \rho \in \mathbb{R}\},$$

$$\|f\|_{H^{\alpha,p}(\mathbb{R};X)} := \|f_\alpha\|_{L^p(\mathbb{R};X)};$$

$$H^{\alpha,p}(I;X) := \{f = g|_I,\ g \in H^{\alpha,p}(\mathbb{R};X)\},\ \|f\|_{H^{\alpha,p}(I;X)} := \inf_{g \in H^{\alpha,p}(\mathbb{R};X)} \|g\|_{H^{\alpha,p}(\mathbb{R};X)};$$

$$H_0^{\alpha,p}(I;X) := \{f \in H^{\alpha,p}(I;X),\ f^{(k)}(0) = 0,\ k = 0,1,\ldots,[\alpha - 1/p]\},\ \alpha - 1/p \notin \mathbb{N}_0.$$

For $k \in \mathbb{N}$ we have $H^{k,p}(I;X) = W^{k,p}(I;X)$. Therefore $D(\mathcal{L}_1) = W_0^{1,p} = H_0^{1,p}$. If $\alpha - 1/p \notin \mathbb{N}_0$, then according to [20], 2.10.4, Theorem 1, $[L^p, H_0^{1,p}]_\alpha = H_0^{\alpha,p}$ and therefore in the case of $\mathcal{HT}$ space we obtain for $\alpha \in (0,1)$

$$R_0^{\alpha,p}(I;X) = H_0^{\alpha,p}(I;X),\quad \alpha - 1/p \notin \mathbb{N}_0. \tag{20}$$

This is also true for all $\alpha > 0$, $\alpha - 1/p \notin \mathbb{N}_0$, because $\mathcal{L}_{m-1}$ is an isomorphism from $H_0^{\alpha,p}$ onto $H_0^{\alpha-m+1,p}$ and from $R_0^{\alpha,p}$ to $R_0^{\alpha-m+1,p} = H_0^{\alpha-m+1,p}$. The identity (20) has been proven in the scalar case in [19], Theorem 18.3 and Remark 18.1, applying another approach.

3. Maximal L^p regularity for fractional order equations

Let A be a linear closed densely defined operator on a Banach space X. Let $I = (0,T)$ for some $T > 0$. Consider the Cauchy problem for the fractional differential equation with Riemann-Liouville fractional derivative of order $\alpha \in (0,2)$

$$D_t^\alpha u(t) + Au(t) = f(t),\quad \text{a.a. } t \in I, \tag{21}$$

with an initial condition $(g_{1-\alpha} * u)(0) = 0$ when $\alpha \in (0,1)$ and two initial conditions $(g_{2-\alpha} * u)(0) = 0$ and $(g_{2-\alpha} * u)'(0) = 0$ when $\alpha \in (1,2)$.

Let $R_0^{\alpha,p}(I;X)$ be the domain of the operator of fractional differentiation, defined by (13). All other notations in this section are also defined in Chapter 1.

Definition 3. We say that there is **maximal L^p regularity** of (21), on I, in X, if for every $f \in L^p(I;X)$ there exists one and only one $u \in L^p(I;D(A)) \cap R_0^{\alpha,p}(I;X)$ satisfying (21).

It follows from the closed graph theorem that if there is L^p regularity of (21) then there exists $C > 0$ such that

$$\|u\|_{L^p} + \|D_t^\alpha u\|_{L^p} + \|Au\|_{L^p} \le C\|f\|_{L^p}. \tag{22}$$

Following Da Prato and Grisvard [11], we rewrite the equation (21) for X-valued functions u and f as an operator equation in $\mathcal{X} = L^p(I;X)$. To this end we define the linear closed operator $\mathcal{A}$ on $\mathcal{X}$ by

$$D(\mathcal{A}) = L^p(I;D(A));\ (\mathcal{A}u)(t) = Au(t),\ u \in D(\mathcal{A}), \tag{23}$$

and take $\mathcal{B} = \mathcal{L}_\alpha$, where $\mathcal{L}_\alpha$ is defined by (15). Then rewrite equation (21) as

$$\mathcal{A}u + \mathcal{B}u = f, \quad f \in \mathcal{X}. \tag{24}$$

More than 20 years ago Da Prato and Grisvard [11] found sufficient conditions for maximal regularity of (24) in real interpolation spaces. Later, Dore and Venni [13] solved this problem in the case of $\mathcal{HT}$ space. Here we present these theorems, reformulated suitably for our application (see [5], Theorem 4, and [1], Theorem 4.9.7 and Corollary 4.9.8).

THEOREM 4. *(Da Prato-Grisvard) Let $\mathcal{X}$ be a complex Banach space and $\mathcal{A}$ and $\mathcal{B}$ be nonnegative operators in $\mathcal{X}$ with spectral angles $\omega_\mathcal{A}$ and $\omega_\mathcal{B}$, respectively, such that*

$$\omega_\mathcal{A} + \omega_\mathcal{B} < \pi.$$

Let moreover $\mathcal{A}$ and $\mathcal{B}$ be resolvent commuting and satisfy $0 \in \rho(A) \cup \rho(B)$. If $\mathcal{Y}$ is one of the spaces $D_\mathcal{A}(\delta, q)$ or $D_\mathcal{B}(\delta, q)$, where $\delta \in (0,1)$ and $q \in [1, \infty]$, then for any $f \in \mathcal{Y}$ there is a unique $u \in D(\mathcal{A}) \cap D(\mathcal{B})$ such that $\mathcal{A}u + \mathcal{B}u = f$. Moreover, $\mathcal{A}u$ and $\mathcal{B}u \in \mathcal{Y}$ and

$$\|u\|_\mathcal{Y} + \|\mathcal{A}u\|_\mathcal{Y} + \|\mathcal{B}u\|_\mathcal{Y} \le C\|f\|_\mathcal{Y},$$

where the constant C depends on $\mathcal{X}$, δ, q, $\omega_\mathcal{A}$, $\omega_\mathcal{B}$, $K_\mathcal{A}(\pi - \theta)$ and $K_\mathcal{B}(\theta)$ for some $\theta \in (\omega_\mathcal{A}, \pi - \omega_\mathcal{B})$, but not on the individual operators $\mathcal{A}$ and $\mathcal{B}$.

THEOREM 5. *(Dore-Venni) Let $\mathcal{X}$ be an $\mathcal{HT}$ space. Assume*

$$\mathcal{A} \in \mathcal{P}_K(\mathcal{X}) \cap \mathcal{BIP}(\mathcal{X}; \mathcal{M}, \varphi_\mathcal{A}), \ \ \mathcal{B} \in \mathcal{P}_K(\mathcal{X}) \cap \mathcal{BIP}(\mathcal{X}; \mathcal{M}, \varphi_\mathcal{B})$$

with

$$\varphi_\mathcal{A} + \varphi_\mathcal{B} < \pi, \tag{25}$$

and let $\mathcal{A}$ and $\mathcal{B}$ be resolvent commuting. Then for any $f \in \mathcal{X}$ there is a unique $u \in D(\mathcal{A}) \cap D(\mathcal{B})$ such that $\mathcal{A}u + \mathcal{B}u = f$. Moreover,

$$\|u\|_\mathcal{X} + \|\mathcal{A}u\|_\mathcal{X} + \|\mathcal{B}u\|_\mathcal{X} \le C\|f\|_\mathcal{X},$$

holds, where the constant C depends on $\mathcal{X}$, K, M, $\varphi_\mathcal{A}$ and $\varphi_\mathcal{B}$, but not on the individual operators $\mathcal{A}$ and $\mathcal{B}$.

Next we apply Theorems 4 and 5 to our concrete problem (21).

COROLLARY 6. *Let $\alpha \in (0,2)$, $1 < p < \infty$, $\delta \in (0,1)$. Let A be a positive operator in a Banach space X with spectral angle satisfying*

$$\omega_A < \pi(1 - \alpha/2). \tag{26}$$

(a) If $Y = (X, D(A))_{\delta,p}$, then (21) has maximal L^p regularity on I in the space Y. More precisely, the following estimate for the solution u holds:

$$\|u\|_{L^p(I;Y)} + \|D_t^\alpha u\|_{L^p(I;Y)} + \|Au\|_{L^p(I;Y)} \leq C_1 \|f\|_{L^p(I;Y)}; \qquad (27)$$

(b) Let $\alpha\delta - 1/p \notin \mathbb{N}_0$. For any $f \in W_0^{\alpha\delta,p}(I;X)$ there exists a unique solution u of (21). Moreover, $D_t^\alpha u,\ Au \in W_0^{\alpha\delta,p}(I;X)$ and

$$\|u\|_{W^{\alpha\delta,p}(I;X)} + \|D_t^\alpha u\|_{W^{\alpha\delta,p}(I;X)} + \|Au\|_{W^{\alpha\delta,p}(I;X)} \leq C_2 \|f\|_{W^{\alpha\delta,p}(I;X)}.$$

The constants C_1 and C_2 depend on α, p, δ, ω_A and $K_A(\theta)$ for some $\theta \in (\alpha\pi/2, \pi - \omega_A)$, but not on T and on the individual operator A.

Proof: We apply first Theorem 4 to the problem on $L^p(\mathbb{R};X)$ in order to obtain a constant, which does not depend on T. Let $f \in L^p(I;Y)$ and define the function

$$f_0 = \begin{cases} f, & t \in [0,T], \\ 0, & t \notin [0,T]. \end{cases} \qquad (28)$$

Then $f_0 \in L^p(\mathbb{R};Y)$. We extend the definition of the fractional derivative to functions on $L^p(\mathbb{R})$ as follows. Let $u \in C_0^\infty(\mathbb{R})$, which is dense in $L^p(\mathbb{R})$. Define

$$L_\alpha u := D_t^m(g_{m-\alpha} * u), \quad m = \lceil \alpha \rceil, \qquad (29)$$

where $*$ is the convolution on $\mathbb{R}$. This operator is nonnegative, therefore closable, and we take its closure in $L^p(\mathbb{R})$ as definition of fractional derivative on $L^p(\mathbb{R})$. We use the same notation L_α. Let $\mathcal{L}_\alpha$, $\mathcal{A}$ denote the extensions of L_α, A to $L^p(\mathbb{R};X)$. For the Fourier transform of $\mathcal{L}_\alpha u$ we have

$$\widetilde{\mathcal{L}_\alpha u}(\rho) = (i\rho)^\alpha \tilde{u}(\rho), \quad \rho \in \mathbb{R} \setminus \{0\}, \ \tilde{f} \in C_0^\infty(\mathbb{R} \setminus \{0\}; X).$$

Therefore, according to [17], Theorem 8.6, $\mathcal{L}_\alpha$ is a sectorial operator in $\mathcal{X} = L^p(\mathbb{R};X)$ with spectral angle $\omega_{\mathcal{L}_\alpha} = \alpha\pi/2$. It is immediate that $\mathcal{A}$ is a positive operator on $\mathcal{X}$ with spectral angle $\omega_{\mathcal{A}} = \omega_A$. Consider the problem on $\mathbb{R}$:

$$\mathcal{L}_\alpha u + \mathcal{A} u = f_0. \qquad (30)$$

By (26) operators $\mathcal{A}$ and $\mathcal{L}_\alpha$ satisfy conditions of Theorem 4. If we take $\mathcal{Y} = (\mathcal{X}, D(\mathcal{A}))_{\delta,p}$, we obtain by [20], Theorem 1.18.4,

$$\mathcal{Y} = (L^p(\mathbb{R};X), L^p(\mathbb{R};D(A)))_{\delta,p} = L^p(\mathbb{R};(X,D(A))_{\delta,p}) = L^p(\mathbb{R};Y),$$

and Theorem 4 implies (a) on $L^p(\mathbb{R};Y)$. Now turn back to our equation (21) on $L^p(I;Y)$. Denote by u_0 the solution of (30). Because of the causality of the equation, $u_0 = 0$ for $t < 0$. This easily implies that u_0 satisfies the initial conditions of problem (21). Therefore, the restriction $u(t)$ of $u_0(t)$ to I will be a solution of (21), satisfying (27), and (a) is proved. The claim concerning the constant follows from the corresponding claim in Theorem 4.

Applying the same argument we prove (b) taking $\mathcal{Y} = (\mathcal{X}, D(\mathcal{L}_\alpha))_{\delta,p}$, which by (19) is equivalent to $\mathcal{Y} = W_0^{\alpha\delta,p}$ for $\alpha\delta - 1/p \notin \mathbb{N}_0$. □

COROLLARY 7. *Let $\alpha \in (0,2)$, A be a positive operator in an $\mathcal{HT}$ space X satisfying $A \in \mathcal{P}_K(X) \cap \mathcal{BIP}(\mathcal{X}; \mathcal{M}, \varphi_{\mathcal{A}})$ with*

$$\varphi_A < \pi(1 - \alpha/2). \tag{31}$$

Then (21) has maximal L^p regularity on I in X. More precisely,

$$\|u\|_{L^p(I;X)} + \|u\|_{H^{\alpha,p}(I;X)} + \|Au\|_{L^p(I;X)} \le C\|f\|_{L^p(I;X)}, \tag{32}$$

where C depends on α, p, K, M, φ_A, T, but not on the individual operator A.

Proof: First note that $\mathcal{X} = L^p(I;X)$ is an $\mathcal{HT}$ space, because X is and $1 < p < \infty$. Since $A \in \mathcal{P}_K(X) \cap \mathcal{BIP}(\mathcal{X}; \mathcal{M}, \varphi_{\mathcal{A}})$, for the extension $\mathcal{A}$ of A to $\mathcal{X}$ we have $\mathcal{A} \in \mathcal{P}_K(\mathcal{X}) \cap \mathcal{BIP}(\mathcal{X}; \mathcal{M}, \varphi_{\mathcal{A}})$. Then Lemma 2, (d), and (31) imply that the conditions of Theorem 5 are satisfied and we obtain the desired result. Because X is of class $\mathcal{HT}$, $R_0^{\alpha,p} = H_0^{\alpha,p}$ for $\alpha - 1/p \neq 0, 1$, and therefore $\|D_t^\alpha u\|_{L^p(I;X)} = \|u\|_{H^{\alpha,p}(I;X)}$. □

If we want to prove that the constant C does not depend on T, we have to apply a generalization of Theorem 5 to sectorial operators [18] and work first on $L^p(\mathbb{R}; X)$ as in the proof of the previous corollary. We skip this argument, because in what follows we present a stronger result.

Next we formulate weaker conditions on A, sufficient for maximal L^p regularity of (21). This is possible applying the following very recent generalization of the Michlin multiplier theorem due to Weis [21], Clément and Prüss [10].

Let $\mathcal{S}(\mathbb{R}; X)$ be the space of Schwartz of smooth rapidly decreasing X-valued functions and $\mathcal{S}'(\mathbb{R}; X)$ be the space of X-valued distributions. Let $m : \mathbb{R} \setminus \{0\} \to \mathcal{B}(X)$ be differentiable and define for $f \in \mathcal{S}(\mathbb{R}; X)$ the function $\mathrm{M}f \in \mathcal{S}'(\mathbb{R}; X)$ by

$$\widetilde{\mathrm{M}f}(\rho) := m(\rho)\tilde{f}(\rho), \ \rho \in \mathbb{R} \setminus \{0\}, \tag{33}$$

where $\tilde{f}$ denotes the Fourier transform of f.

THEOREM 8. *Let X be an $\mathcal{HT}$ space, $1 < p < \infty$, and let $m \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(X))$ be such that the following two conditions are satisfied*

(i) $\mathcal{R}(\{m(\rho) \mid \rho \in \mathbb{R} \setminus \{0\}\}) =: k_0 < \infty$,

(ii) $\mathcal{R}(\{\rho m'(\rho) \mid \rho \in \mathbb{R} \setminus \{0\}\}) =: k_1 < \infty$.

Then the operator M, defined by (33), extends to a bounded operator on $L^p(\mathbb{R}; X)$. Its bound depends only on X, p, k_0 and k_1.

For a proof, we refer to [10], Theorem 1. The statement about the bound is implicitly given in this proof.

It was proven ([10], Proposition 1.) that the $\mathcal{R}$-boundedness condition (i) is also necessary for M to be extended to a bounded operator on $L^p(\mathbb{R}; X)$.

Let just as in the proof of Corollary 4.6 $\mathcal{L}_\alpha$ be the fractional derivative in $L^p(\mathbb{R}; X)$, $\mathcal{A}$ be the extension of A to $L^p(\mathbb{R}; X)$ and define f_0 as in (28). Consider the corresponding problem on $\mathbb{R}$:

$$\mathcal{L}_\alpha u + \mathcal{A}u = f_0. \tag{34}$$

Applying Fourier transform, we obtain

$$\widetilde{u}(\rho) = ((i\rho)^\alpha I + A)^{-1} \widetilde{f}_0(\rho), \quad \rho \in \mathbb{R} \setminus \{0\}.$$

Therefore we have the estimate

$$\|\mathcal{L}_\alpha u\|_{L^p(\mathbb{R};X)} + \|\mathcal{A}u\|_{L^p(\mathbb{R};X)} \leq M \|f_0\|_{L^p(\mathbb{R};X)} \tag{35}$$

iff the operator M defined by

$$\widetilde{\mathrm{M}f}(\rho) := A((i\rho)^\alpha I + A)^{-1} \widetilde{f}(\rho), \ \rho \in \mathbb{R} \setminus \{0\},$$

is a bounded operator on $L^p(\mathbb{R}; X)$. So, set $m(\rho) := A((i\rho)^\alpha I + A)^{-1}$. Suppose that A is nonnegative with spectral angle, satisfying (26).Then $m(\rho)$ and $\rho m'(\rho)$ for $\rho \in \mathbb{R} \setminus \{0\}$ are bounded operator valued functions, that is $m \in C^1(\mathbb{R} \setminus \{0\}; \mathcal{B}(X))$. Hence, to obtain the boundedness of M on $L^p(\mathbb{R}; X)$, $1 < p < \infty$, we have to check conditions (i) and (ii) for $m(\rho)$ of Theorem 4.8. Since we have the representation $\rho m'(\rho) = -\alpha(I - m(\rho))m(\rho)$ and since the product of two $\mathcal{R}$-bounded families is again $\mathcal{R}$-bounded, it follows that condition (i) of Theorem 8 for our concrete function $m(\rho)$ implies condition (ii). That is, the maximal regularity estimate (35) holds iff the family of operators

$$\{\lambda^\alpha (\lambda^\alpha I + A)^{-1} | \ \lambda \in i\mathbb{R}, \ \lambda \neq 0\} \tag{36}$$

is $\mathcal{R}$-bounded.

DEFINITION 9. A sectorial operator A on X is called **$\mathcal{R}$-sectorial** if

$$\mathcal{R}_A(0) := \mathcal{R}\{t(tI + A)^{-1} | \ t > 0\} < \infty.$$

The **$\mathcal{R}$-angle** ω_A^R of A is defined by means of

$$\omega_A^R := \inf\{\theta \in (0, \pi) | \ \mathcal{R}_A(\pi - \theta) < \infty\},$$

where

$$\mathcal{R}_A(\theta) := \mathcal{R}(\{\lambda(\lambda I + A)^{-1} | \ \lambda \in \overline{\Sigma}_\theta \setminus \{0\}\}).$$

It is immediate that $\omega_A^R \geq \omega_A$. It has been shown by Weis [21] that $\mathcal{R}$-sectorial operators behave well under perturbations, like the class of sectorial operators.

We prove now the following result for the problem on I.

Proposition 10. *Let $\alpha \in (0,2)$, $1 < p < \infty$, X be a Banach space of class $\mathcal{HT}$, A be an $\mathcal{R}$-sectorial operator on X with $0 \in \rho(A)$ and $\mathcal{R}$-angle, satisfying*

$$\omega_A^R < \pi(1 - \alpha/2). \tag{37}$$

Then problem (21) has maximal L^p regularity and the following estimate holds

$$\|u\|_{L^p(I;X)} + \|u\|_{H^{\alpha,p}(I;X)} + \|Au\|_{L^p(I;X)} \leq C\|f\|_{L^p(I;X)}, \tag{38}$$

where the constant C depends on X, p, α, $\mathcal{R}_A(\alpha\pi/2)$, but not on T and on the individual operator A.

Proof: Condition (37) implies that the family of operators (36) is $\mathcal{R}$-bounded. Therefore, according to Theorem 4.8, for any $u \in D(\mathcal{L}_\alpha) \cap D(\mathcal{A})$ the estimate (35) holds. Consider equation (34) on $L^p(\mathbb{R};X)$. Since $\mathcal{L}_\alpha$ and $\mathcal{A}$ are resolvent commuting, $0 \in \rho(\mathcal{A})$ and $\omega_{\mathcal{L}_\alpha} + \omega_{\mathcal{A}} < \pi$ (see the proof of Corollary 4.6), the pair of operators $(\mathcal{L}_\alpha, \mathcal{A})$ is an admissible pair in $L^p(\mathbb{R};X)$ in the sense of [4], Definition 3.2, and, according to Theorem 3.3 of the same reference, the equation

$$u + \mathcal{L}_\alpha \mathcal{A}^{-1} u = \mathcal{A}^{-1} f_0, \ f_0 \in L^p(\mathbb{R};X), \tag{39}$$

has a solution u satisfying

$$\|u\|_{L^p(\mathbb{R};X)} \leq C_{\mathcal{A},\theta} K_{\mathcal{L}_\alpha}(\theta) \|f_0\|_{L^p(\mathbb{R};X)} \tag{40}$$

for some $\theta \in (\alpha\pi/2, \pi - \omega_{\mathcal{A}})$, where $C_{\mathcal{A},\theta}$ depends only on $\mathcal{A}$ and θ. This solution is called mild solution of (34) and it becomes its strict solution if $u \in D(\mathcal{L}_\alpha)$ or $u \in D(\mathcal{A})$.

Combining (35) and (40), we obtain the full estimate for the mild solution

$$\|u\|_{L^p(\mathbb{R};X)} + \|\mathcal{A}u\|_{L^p(\mathbb{R};X)} + \|\mathcal{L}_\alpha u\|_{L^p(\mathbb{R};X)} \leq C\|\mathcal{L}_\alpha u + \mathcal{A}u\|_{L^p(\mathbb{R};X)}. \tag{41}$$

Using this estimate, we shall prove that (34) has a strict solution for any $f_0 \in L^p(\mathbb{R};X)$. We know ([4], p.22, Remark), that if $f_0 \in D(\mathcal{L}_\alpha)$ then the solution u of (39) belongs to $D(\mathcal{L}_\alpha)$ and so, it is a strict solution of (34). Take a sequence $f_n \in W^{2,p}(\mathbb{R};X)$ such that $f_n \to f_0$ in L^p. Since $f_n \in W^{2,p} \subset D(\mathcal{L}_\alpha)$, then the equation (34) with right-hand side f_n has a strict solution, denoted by u_n. Applying estimate (41) to the difference of two such equations, we obtain in $L^p(\mathbb{R};X)$:

$$\|u_n - u_m\| + \|\mathcal{A}(u_n - u_m)\| + \|\mathcal{L}_\alpha(u_n - u_m)\| \leq C\|f_n - f_m\|.$$

Hence u_n, $\mathcal{L}_\alpha u_n$, $\mathcal{A}u_n$, are Cauchy sequences. The closedness of the operators $\mathcal{L}_\alpha$ and $\mathcal{A}$ implies that there exists $u \in D(\mathcal{L}_\alpha) \cap D(\mathcal{A})$ such that $u_n \to u$, $\mathcal{L}_\alpha u_n \to \mathcal{L}_\alpha u$, $\mathcal{A}u_n \to \mathcal{A}u$, in L^p. Therefore u is a strict solution of (34).

Turn back to our equation (21) on $L^p(I;X)$. Its solution is obtained as a restriction of the solution of (34) to $[0,T]$. Estimate (38) will follow from (41). □

It is proven in [10], Theorem 4, that if X is of class $\mathcal{HT}$ and $A \in \mathcal{BIP}(\mathcal{X};\theta_{\mathcal{A}})$ then A is $\mathcal{R}$-sectorial and $\omega_A^R \le \theta_A$. Therefore Corollary 7 can be obtained from Proposition 10.

In fact, under the conditions of Proposition 10 we have even more: not only maximal L^p regularity of (21), but also λ-regularity.

DEFINITION 11. Let X be a Banach space. The pair of closed operators $(\mathcal{A},\mathcal{B})$ is called **λ-regular** in X if for any $f \in X$, $\lambda > 0$ the problem

$$\lambda \mathcal{A}u + \mathcal{B}u = f$$

has a unique solution $u \in D(\mathcal{A}) \cap D(\mathcal{B})$ and the following inequality holds

$$\|\lambda \mathcal{A}u\| + \|\mathcal{B}u\| \le M\|\lambda \mathcal{A}u + \mathcal{B}u\|, \quad \lambda > 0$$

for some $M \ge 1$, independent of λ, and for all $u \in D(\mathcal{A}) \cap D(\mathcal{B})$.

Suppose the hypotheses of Proposition 10 are fulfilled. Then the pair of operators $(\mathcal{L}_\alpha,\mathcal{A})$ is λ-regular. Indeed, replacing the operator A by λA, $\lambda > 0$, we obtain the following multiplier function

$$m_\lambda(\rho) := \lambda A((i\rho)^\alpha I + \lambda A)^{-1} = A(\lambda^{-1}(i\rho)^\alpha I + A)^{-1}.$$

Therefore $\mathcal{R}(\{m_\lambda(\rho) | \ \rho \in \mathbb{R}\backslash\{0\}\}) = \mathcal{R}(\{m(\rho) | \ \rho \in \mathbb{R}\backslash\{0\}\}) = k_0$ and so, it does not depend on λ. Applying Theorem 8, the estimate

$$\|\mathcal{L}_\alpha u\|_{L^p(I;X)} + \|\lambda \mathcal{A}u\|_{L^p(I;X)} \le M\|f\|_{L^p(I;X)}$$

follows, where M does not depend on λ.

COROLLARY 12. *Conditions of Proposition 10 are sufficient for $(\mathcal{L}_\alpha,\mathcal{A})$ to be a λ-regular pair in $L^p(I;X)$ and in $L^p(\mathbb{R};X)$.*

4. Strict L^p solutions of fractional order equations

Consider now the fractional evolution equations with nonzero initial conditions

$$\alpha \in (0,1): \quad \begin{aligned} &D_t^\alpha u(t) + Au(t) = f(t), \ \text{a.a. } t > 0,\\ &(g_{1-\alpha} * u)(0) = x_0. \end{aligned} \tag{42}$$

and

$$\alpha \in (1,2): \quad \begin{aligned} &D_t^\alpha u(t) + Au(t) = f(t), \ \text{a.a. } t > 0,\\ &(g_{2-\alpha} * u)(0) = x_0, \ (g_{2-\alpha} * u)'(0) = x_1. \end{aligned} \tag{43}$$

where $x_0, \ x_1 \in X$ and $f \in L^p(I;X)$.

DEFINITION 13. A function $u : I \to X$ is said to be a **strict L^p solution** of (42), resp. (43), on I, in X, if $u \in L^p(I; D(A)) \cap R^{\alpha,p}(I; X)$ and (42), resp. (43), is satisfied.

Obviously, if $x_0 = x_1 = 0$, then (42), resp. (43), has strict L^p solution for any $f \in L^p$ iff it has maximal L^p regularity.

In order to solve (42), we write $u = v + w$, where v satisfies

$$\begin{aligned} &D_t^\alpha v(t) + Av(t) = f(t), \text{ a.a. } t > 0, \\ &(g_{1-\alpha} * v)(0) = 0. \end{aligned} \tag{44}$$

and w satisfies

$$\begin{aligned} &D_t^\alpha w(t) + Aw(t) = 0, \text{ a.a. } t > 0, \\ &(g_{1-\alpha} * w)(0) = x_0. \end{aligned} \tag{45}$$

Similarly, in order to solve (43), we write $u = v + w + z$, where v satisfies

$$\begin{aligned} &D_t^\alpha v(t) + Av(t) = f(t), \text{ a.a. } t > 0, \\ &(g_{2-\alpha} * v)(0) = 0, \ (g_{2-\alpha} * v)'(0) = 0, \end{aligned} \tag{46}$$

w satisfies

$$\begin{aligned} &D_t^\alpha w(t) + Aw(t) = 0, \text{ a.a. } t > 0, \\ &(g_{2-\alpha} * w)(0) = x_0, \ (g_{2-\alpha} * w)'(0) = 0, \end{aligned} \tag{47}$$

and z satisfies

$$\begin{aligned} &D_t^\alpha z(t) + Az(t) = 0, \text{ a.a. } t > 0, \\ &(g_{2-\alpha} * z)(0) = 0, \ (g_{2-\alpha} * z)'(0) = x_1. \end{aligned} \tag{48}$$

We apply different methods to analyse the above problems. For the analysis of (44) and (46) we use the results on maximal L^p regularity given in Proposition 10 and Corollary 6, while for the analysis of (45), (47) and (48) we use the solution operator $P_\alpha(t)$ associated with it, defined as follows.

Let A be a nonnegative operator with spectral angle ω_A satisfying (26). Define the operator-valued function

$$P_\alpha(t)x := \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\lambda t} (\lambda^\alpha I + A)^{-1} x \, d\lambda, \tag{49}$$

where $\theta \in (\frac{\pi}{2}, \min\{\pi, \frac{\pi - \omega_A}{\alpha}\})$ and

$$\Gamma_{r,\theta} := \{re^{i\varphi}; \ |\varphi| \le \theta\} \cup \{\rho e^{i\theta}; \ r \le \rho < \infty\} \cup \{\rho e^{-i\theta} | \ r \le \rho < \infty\}.$$

The orientation of the contour is such that the argument does not decrease along it. Next we summarize some properties of $P_\alpha(t)$.

LEMMA 14. *Assume that $\alpha \in (0,2)$ and A is a nonnegative operator in a complex Banach space X with spectral angle ω_A satisfying (26). Then the following assertions hold*

(a) $P_\alpha(t) \in \mathcal{B}(X)$ *for each* $t > 0$ *and* $\sup_{t>0} t^{1-\alpha}\|P_\alpha(t)\|_{\mathcal{B}(X)} < \infty$;

(b) For any $x \in X,\ t > 0$, $P_\alpha(t)x \in D(A)$ *and* $\sup_{t>0} t\|AP_\alpha(t)\|_{\mathcal{B}(X)} < \infty$;

(c) $P_\alpha(.),\ AP_\alpha^{(k)}(.) \in C^\infty(\mathbb{R}_+;\mathcal{B}(X))$ *and for any integer* $k \geq 0$ *and* $l = 0,1$,

$$\sup_{t>0} t^{1+k+\alpha(l-1)}\|A^l P_\alpha^{(k)}(t)\|_{\mathcal{B}(X)} < \infty;$$

(d) For any fixed $\theta \in \left(0, \min\left\{\pi, \frac{\pi-\omega_A}{\alpha}\right\} - \frac{\pi}{2}\right)$, $k \geq 0$, $l = 0,1$ *there exists an analytic extension of* $A^l P_\alpha^{(k)}(.)$ *to* Σ_θ.

From the definition of $P_\alpha(t)$ it follows

$$(\lambda^\alpha I + A)^{-1}x_0 = \int_0^\infty e^{-\lambda t}P_\alpha(t)x_0\,dt. \tag{50}$$

The maximal L^p regularity of (21) is equivalent to the boundedness in $L^p(I;X)$ of the operator M, defined by

$$Mf(t) := \int_0^t AP_\alpha(t-s)f(s)\,ds,$$

because from the variation of parameters formula for the solution u of (21) we have $Au(t) = Mf(t)$. The solutions of the equations with arbitrary initial conditions and zero forcing function can also be represented in terms of $P_\alpha(t)$. Using this representation, we formulate some results on existence and uniqueness of strict L^p solutions.

4.1. The case $\alpha \in (0,1)$

Our main results concerning the case $\alpha \in (0,1)$ are two theorems on strict L^p solvability in X and in real interpolation spaces $D_A(\delta,p)$, correspondingly. First we prove two lemmas about strict solvability of the equation with zero forcing function.

By (50) and the uniqueness of the Laplace transform it follows that $w(t) := P_\alpha(t)x_0$ satisfies (45). The following lemma gives sufficient conditions for $P_\alpha(t)x_0$ to be a strict solution of (45). In fact we prove a stronger result, which except for our application, is of independent interest, because it gives an equivalent norm in the interpolation spaces $D_A(\frac{p-1}{\alpha p},p)$ in terms of the operator-valued function $AP_\alpha(t)$.

LEMMA 15. *Assume that $\alpha \in (0,1)$ and A is a nonnegative operator in a complex Banach space X with spectral angle ω_A satisfying (26). Then the following assertions hold*

(a) *Let $1 < p < \frac{1}{1-\alpha}$. Then $AP_\alpha(t)x_0 \in L^p(\mathbb{R}_+, X)$ iff $x_0 \in D_A(\frac{p-1}{\alpha p}, p)$. In this case there are constants C_1, C_2, depending only on α, p, ω_A and $K_A(\phi)$ for some $\phi \in (\alpha\pi/2, \pi - \omega_A)$, such that*

$$C_1[x_0]_{D_A(\frac{p-1}{\alpha p},p)} \le \|AP_\alpha(t)x_0\|_{L^p(\mathbb{R}_+;X)} \le C_2[x_0]_{D_A(\frac{p-1}{\alpha p},p)} \tag{51}$$

(b) *Let $p \ge \frac{1}{1-\alpha}$. Then $AP_\alpha(t)x_0 \in L^p(\mathbb{R}_+, X)$ iff $x_0 = 0$.*

Proof: Let $1 < p < \frac{1}{1-\alpha}$ and $x_0 \in D_A(\frac{p-1}{\alpha p}, p)$. According to (49) and using analyticity to change the integration path we get, when we change the integration variable,

$$AP_\alpha(t)x_0 = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\mu} A(\frac{\mu^\alpha}{t^\alpha}I + A)^{-1}x_0\,\frac{d\mu}{t},\ r > 0. \tag{52}$$

By the dominated convergence theorem we can let $r \to 0$ and get

$$AP_\alpha(t)x_0 = \frac{e^{i\theta}}{\pi i}\int_0^\infty e^{\rho e^{i\theta}} A(\frac{\rho^\alpha e^{i\alpha\theta}}{t^\alpha}I + A)^{-1}x_0\,\frac{d\rho}{t}. \tag{53}$$

To estimate the function under the integral sign we use the representation for $s > 0$

$$A(A + se^{\pm i\phi}I)^{-1} - A(A + sI)^{-1} = (e^{\mp i\phi} - 1)se^{\pm i\phi}(A + se^{\pm i\phi}I)^{-1}A(A + sI)^{-1},$$

which implies

$$\|A(A+zI)^{-1}x_0\|_X \le \left(1 + 2\sin\frac{\phi}{2}K_A(\phi)\right)\|A(A+|z|I)^{-1}x_0\|_X,\ |\arg z| = \phi. \tag{54}$$

Therefore

$$\|AP_\alpha(t)x_0\|_X \le c_1\int_0^\infty e^{\rho\cos\theta}\|A(\frac{\rho^\alpha}{t^\alpha}I + A)^{-1}x_0\|_X\,\frac{d\rho}{t},$$

where $c_1 = \pi^{-1}(1 + 2\sin\frac{\alpha\theta}{2}K_A(\alpha\theta))$. Applying the **generalized Minkowski inequality**

$$\left\{\int_{\mathbb{R}_+} d\tau \left|\int_{\mathbb{R}_+} f(\tau,t)\,dt\right|^p\right\}^{1/p} \le \int_{\mathbb{R}_+} dt\left\{\int_{\mathbb{R}_+} |f(\tau,t)|^p\,d\tau\right\}^{1/p}, \tag{55}$$

where $f(\tau,t)$ is a measurable function, defined on $\mathbb{R}_+\times\mathbb{R}_+$ such that the integrals on both sides are well defined, we obtain

$$\begin{aligned} \|AP_\alpha(t)x_0\|_{L^p(\mathbb{R}_+,X)} &\le c_1\int_0^\infty e^{\rho\cos\theta}\left(\int_0^\infty (\|A(\frac{\rho^\alpha}{t^\alpha}I+A)^{-1}x_0\|_X\frac{1}{t})^p\,dt\right)^{\frac{1}{p}}d\rho \quad (56)\\ &= c_1\int_0^\infty e^{\rho\cos\theta}\rho^{\frac{1}{p}-1}\,d\rho\left(\int_0^\infty \|\sigma^{1-\frac{1}{p}}A(\sigma^\alpha I+A)^{-1}x_0\|_X^p\frac{d\sigma}{\sigma}\right)^{\frac{1}{p}}\\ &= c_2[x_0]_{D_A(\frac{p-1}{\alpha p},p)}, \end{aligned}$$

with $c_2=c_1\Gamma(1/p)(-\alpha\cos\theta)^{-1/p}$ and we have used

$$[x]_{D_A(\frac{\gamma}{\alpha},p)}=(\alpha\int_0^\infty (t^\gamma\|A(t^\alpha I+A)^{-1}x\|_X)^p\frac{dt}{t})^{\frac{1}{p}}, \tag{57}$$

easily obtained from (4) for $0<\gamma<\alpha$, $p\in(1,\infty)$.

Suppose now that $AP_\alpha(t)x_0\in L^p(\mathbb{R}_+;X)$. Applying (50) and the generalized Minkowski inequality, we obtain when we change twice the integration variable

$$\begin{aligned} &\left(\int_0^\infty \|\lambda^{1-\frac{1}{p}}A(\lambda^\alpha I+A)^{-1}x_0\|_X^p\frac{d\lambda}{\lambda}\right)^{\frac{1}{p}} \quad (58)\\ =&\left(\int_0^\infty \lambda^{p-2}\|\int_0^\infty e^{-\lambda t}AP_\alpha(t)x_0\,dt\|_X^p\,d\lambda\right)^{\frac{1}{p}}\\ =&\left(\int_0^\infty \lambda^{-2}\|\int_0^\infty e^{-\tau}AP_\alpha(\frac{\tau}{\lambda})x_0\,d\tau\|_X^p\,d\lambda\right)^{\frac{1}{p}}\\ \le&\int_0^\infty e^{-\tau}\left(\int_0^\infty \lambda^{-2}\|AP_\alpha(\frac{\tau}{\lambda})x_0\|_X^p\,d\lambda\right)^{\frac{1}{p}}d\tau\\ =&\int_0^\infty e^{-\tau}\tau^{-\frac{1}{p}}\,d\tau\left(\int_0^\infty \|AP_\alpha(\sigma)x_0\|_X^p\,d\sigma\right)^{\frac{1}{p}}\\ =&\,\Gamma(1-\frac{1}{p})\|AP_\alpha(t)x_0\|_{L^p(\mathbb{R}_+,X)}<\infty. \end{aligned}$$

Therefore, if $1<p<\frac{1}{1-\alpha}$, (57) and (58) imply

$$[x_0]_{D_A(\frac{p-1}{\alpha p},p)}\le c\|AP_\alpha(t)x_0\|_{L^p(\mathbb{R}_+,X)}<\infty$$

and thus $x_0\in D_A(\frac{p-1}{\alpha p},p)$. If $p\ge\frac{1}{1-\alpha}$ then (57) and (58) implies $x_0=0$, because $D_A(1,p)=\{0\}$.□

In case $0\in\rho(A)$, we can take $\|.\|_{D_A(\delta,p)}=[.]_{D_A(\delta,p)}$ as an equivalent norm in $D_A(\delta,p)$ (see [5]) and inequalities (51) imply

$$C_1\|x_0\|_{D_A(\frac{p-1}{\alpha p},p)}\le\|AP_\alpha(t)x_0\|_{L^p(\mathbb{R}_+;X)}\le C_2\|x_0\|_{D_A(\frac{p-1}{\alpha p},p)}. \tag{59}$$

Moreover, Lemma 14 (b) implies

$$\int_0^\infty \|AP_\alpha(t)\|_X^p\,dt =$$

$$\int_0^T \|AP_\alpha(t)\|_X^p\,dt + \int_T^\infty \|AP_\alpha(t)\|_X^p\,dt \le \int_0^T \|AP_\alpha(t)\|_X^p\,dt + \int_T^\infty \frac{C^p}{t^p}\,dt.$$

Since $p > 1$ then $AP_\alpha(t)x_0 \in L^p(I;X)$ is equivalent to $AP_\alpha(t)x_0 \in L^p(\mathbb{R}_+;X)$. These results together with Proposition 10 imply the following theorem.

THEOREM 16. *Suppose that $\alpha \in (0,1)$, $1 < p < \infty$, X is a Banach space of class $\mathcal{HT}$, A is an $\mathcal{R}$-sectorial operator in X with $0 \in \rho(A)$ and with $\mathcal{R}$-angle ω_A^R, satisfying (37), and $f \in L^p(I;X)$. Then the following statements hold:*

(a) if $1 < p < \frac{1}{1-\alpha}$, then there is a unique strict L^p solution u of (42) iff $x_0 \in D_A(\frac{p-1}{\alpha p}, p)$;

(b) if $p \ge \frac{1}{1-\alpha}$ then (42) has a unique strict L^p solution iff $x_0 = 0$.

In both cases the following estimate is satisfied (for (b) we set $x_0 = 0$):

$$\|u\|_{L^p(I;X)} + \|D_t^\alpha u\|_{L^p(I;X)} + \|Au\|_{L^p(I;X)} \le C(\|x_0\|_{D_A(\frac{p-1}{\alpha p},p)} + \|f\|_{L^p(I;X)}), \quad (60)$$

where the constant C depends on X, α, p, ω_A and $K_A(\theta)$ for some $\theta \in (\alpha\pi/2, \pi - \omega_A)$ and on $\mathcal{R}_A(\alpha\pi/2)$, but does not depend on T and on the individual operator A.

To obtain further regularity results we need more detailed estimates on $AP_\alpha(t)x_0$. Next we present conditions under which $AP_\alpha(t)x_0$ belongs to some interpolation spaces.

LEMMA 17. *Assume that $\alpha \in (0,1)$ and A is a nonnegative operator in a complex Banach space X with spectral angle ω_A satisfying (26). If $1 < p < \frac{1}{1-\alpha}$, $0 < \delta < \frac{\alpha p - p + 1}{\alpha p}$ and $x_0 \in D_A(\frac{p-1}{\alpha p} + \delta, p)$ then $AP_\alpha(t)x_0 \in L^p(\mathbb{R}_+; D_A(\delta,p))$. More precisely, there is a constant C depending on α, δ, p, ω_A and $K_A(\phi)$ for some $\phi \in (\alpha\pi/2, \pi - \omega_A)$, such that*

$$\|AP_\alpha(t)x_0\|_{L^p(\mathbb{R}_+;D_A(\delta,p))} \le C\|x_0\|_{D_A(\frac{p-1}{\alpha p}+\delta,p)}. \quad (61)$$

Proof: Set $\gamma = \alpha\delta$. According to (49) we get

$$AP_\alpha(t)x_0 = \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t} A(\lambda^\alpha I + A)^{-1} x_0\,d\lambda, \ r > 0. \quad (62)$$

Take $\mu^\alpha > r$. Since

$$A(\mu^\alpha I + A)^{-1} A(\lambda^\alpha I + A)^{-1} = \frac{\mu^\alpha}{\mu^\alpha - \lambda^\alpha} A(\mu^\alpha I + A)^{-1} - \frac{\lambda^\alpha}{\mu^\alpha - \lambda^\alpha} A(\lambda^\alpha I + A)^{-1},$$

it follows by (62)

$$A(\mu^\alpha I + A)^{-1} A P_\alpha(t) x_0$$

$$= A(\mu^\alpha I + A)^{-1} \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} \frac{\mu^\alpha}{\mu^\alpha - \lambda^\alpha} e^{\lambda t} x_0 \, d\lambda - \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} \frac{\lambda^\alpha}{\mu^\alpha - \lambda^\alpha} e^{\lambda t} A(\lambda^\alpha I + A)^{-1} x_0 \, d\lambda.$$

When we close the path $\Gamma_{r,\theta}$ at infinity by increasing argument, we see that the first integral is 0 by Cauchy's theorem and we get

$$A(\mu^\alpha I + A)^{-1} A P_\alpha(t) x_0 = -\frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} \frac{\lambda^\alpha}{\mu^\alpha - \lambda^\alpha} e^{\lambda t} A(\lambda^\alpha I + A)^{-1} x_0 \, d\lambda. \tag{63}$$

In this integral we may let $r \downarrow 0$ without changing the value of the integral, because the function we integrate is analytic and the integral over a part of the circle with radius r goes to 0 by the assumption that $\theta \in (\frac{\pi}{2}, \min\{\pi, \frac{\pi - \omega_A}{\alpha}\})$, and the definition of $\Gamma_{r,\theta}$.

Thus we have by (63) and (54)

$$\begin{aligned} & \mu^\gamma \| A(\mu^\alpha I + A)^{-1} A P_\alpha(t) x_0 \|_X \\ \leq \; & c_1 \int_0^\infty \frac{\mu^\gamma s^\alpha}{|\mu^\alpha - s^\alpha e^{i\alpha\theta}|} e^{st\cos\theta} \| A(s^\alpha I + A)^{-1} x_0 \|_X \, ds \\ = \; & c_1 \int_0^\infty \frac{\left(\frac{\mu}{s}\right)^\gamma}{\left| \left(\frac{\mu}{s}\right)^\alpha - e^{i\alpha\theta} \right|} s^{\gamma+1} e^{st\cos\theta} \| A(s^\alpha I + A)^{-1} x_0 \|_X \, \frac{ds}{s}, \end{aligned} \tag{64}$$

where $c_1 = \pi^{-1}(1 + 2\sin\frac{\alpha\theta}{2} K_A(\alpha\theta))$. Let for $\tau \in \mathbb{R}$

$$f(\tau) := e^{\tau(\gamma+1)} e^{e^\tau t \cos\theta} \| A(e^{\tau\alpha} I + A)^{-1} x_0 \|_X, \;\; g(\tau) := e^{\tau\gamma} \| A(e^{\tau\alpha} I + A)^{-1} A P_\alpha(t) x_0 \|_X,$$

$$h(\tau) := e^{\tau\gamma} / |e^{\tau\alpha} - e^{i\alpha\theta}|.$$

By changing variables ($s = e^\sigma$) in the integral in (64) we conclude that

$$g(\tau) \leq c_1 \int_{-\infty}^\infty h(\tau - \sigma) f(\sigma) \, d\sigma. \tag{65}$$

Since $h \in L^1(\mathbb{R})$, $f \in L^p(\mathbb{R}; X)$, we can apply the Young inequality to (65) to obtain $\|g\|_{L^p(\mathbb{R};X)} \leq c_1 \|h\|_{L^1(\mathbb{R})} \|f\|_{L^p(\mathbb{R};X)}$. Because a change of variables shows that $\|g\|_{L^p(\mathbb{R})} = [A P_\alpha(t) x_0]_{D_A(\delta,p)}$, we conclude after another change of variables that

$$[A P_\alpha(t) x_0]_{D_A(\delta,p)} \leq$$

$$c_1 \int_0^\infty \frac{s^{\gamma-1}}{|s^\alpha + e^{i\alpha\theta}|} \, ds \left(\int_0^\infty (s^{\gamma+1} e^{st\cos\theta} \| A(s^\alpha I + A)^{-1} x_0 \|_X)^p \, \frac{ds}{s} \right)^{\frac{1}{p}}.$$

Therefore, setting $c_2 = c_1 \int_0^\infty \frac{s^{\gamma-1}}{|s^\alpha + e^{i\alpha\theta}|}\, ds$, we obtain

$$\begin{aligned}
&\left(\int_0^\infty [AP_\alpha(t)x_0]^p_{D_A(\delta,p)}\, dt\right)^{\frac{1}{p}} \qquad (66)\\
\leq\ & c_2 \left(\int_0^\infty \int_0^\infty (s^{\gamma+1} e^{st\cos\theta} \|A(s^\alpha I + A)^{-1}x_0\|_X)^p \frac{ds}{s}\, dt\right)^{\frac{1}{p}}\\
=\ & c_2 \left(\int_0^\infty \int_0^\infty \left(\left(\frac{\sigma}{t}\right)^{\gamma+1} e^{\sigma\cos\theta} \left\|A\left(\left(\frac{\sigma}{t}\right)^\alpha I + A\right)^{-1} x_0\right\|_X\right)^p \frac{d\sigma}{\sigma}\, dt\right)^{\frac{1}{p}}\\
=\ & c_2 \left(\int_0^\infty \left(\sigma^{\gamma+1-\frac{1}{p}} e^{\sigma\cos\theta}\right)^p d\sigma \int_0^\infty \left(t^{-\gamma-1} \left\|A\left(\left(\frac{\sigma}{t}\right)^\alpha I + A\right)^{-1} x_0\right\|_X\right)^p dt\right)^{\frac{1}{p}}\\
=\ & c_2 \left(\int_0^\infty e^{\sigma p\cos\theta}\, d\sigma \int_0^\infty (\tau^{\gamma+1-\frac{1}{p}} \|A(\tau^\alpha I + A)^{-1}x_0\|_X)^p \frac{d\tau}{\tau}\right)^{\frac{1}{p}}\\
=\ & c_3 [x_0]_{D_A(\frac{\gamma+1-\frac{1}{p}}{\alpha},p)} = c_3 [x_0]_{D_A(\frac{p-1}{\alpha p}+\delta,p)} \leq c_3 \|x_0\|_{D_A(\frac{p-1}{\alpha p}+\delta,p)},
\end{aligned}$$

where $c_3 = c_2(-p\cos\theta)^{-\frac{1}{p}}$. By (66), (59) and the embedding

$$D_A(\frac{p-1}{\alpha p} + \delta, p) \hookrightarrow D_A(\frac{p-1}{\alpha p}, p)$$

(i.e. $\|x_0\|_{D_A(\frac{p-1}{\alpha p},p)} \leq c\|x_0\|_{D_A(\frac{p-1}{\alpha p}+\delta,p)}$), we obtain (61).□

This lemma shows that if $x_0 \in D_A(\frac{p-1}{\alpha p} + \delta, p)$ then $P_\alpha(t)x_0$ is a strict L^p solution of (45) in $D_A(\delta, p)$. This result together with Corollary 6,(a), implies the following theorem:

THEOREM 18. *Suppose that $\alpha \in (0,1)$, $1 < p < \infty$, A is a positive operator in a Banach space X with spectral angle ω_A, satisfying (26). If $1 < p < \frac{1}{1-\alpha}$, $0 < \delta < \frac{\alpha p - p + 1}{\alpha p}$, $x_0 \in D_A(\frac{p-1}{\alpha p} + \delta, p)$, $f \in L^p(I; D_A(\delta,p))$, then there is a unique strict L^p solution u of (42) in $D_A(\delta,p)$ satisfying*

$$\|u\|_{L^p(I;D_A(\delta,p))} + \|D_t^\alpha u\|_{L^p(I;D_A(\delta,p))} + \|Au\|_{L^p(I;D_A(\delta,p))} \leq$$

$$C(\|x_0\|_{D_A(\frac{p-1}{\alpha p}+\delta,p)} + \|f\|_{L^p(I;D_A(\delta,p))}).$$

This result holds also if $\frac{\alpha p - p + 1}{\alpha p} \leq \delta < 1$ and $x_0 = 0$. The constant C depends on X, α, p, δ, ω_A and $K_A(\theta)$ for some $\theta \in (\alpha\pi/2, \pi - \omega_A)$, but does not depend on T and on the individual operator A.

4.2. The case $\alpha \in (1,2)$

Applying the Laplace transform to the equations (47) and (48) we obtain formally that the Laplace transforms of their solutions w and z are $\lambda(\lambda^\alpha I+A)^{-1}x_0$ and $(\lambda^\alpha I + A)^{-1}x_1$, respectively. Therefore, besides $P_\alpha(t)$, defined as in (49), we consider also $Q_\alpha(t)$, defined by

$$Q_\alpha(t)x := \frac{1}{2\pi i}\int_{\Gamma_{r,\theta}} e^{\lambda t}\lambda(\lambda^\alpha I + A)^{-1}x\, d\lambda \tag{67}$$

with the same integration path as $P_\alpha(t)$. The properties of $Q_\alpha(t)$ can be derived from Lemma 14 and the fact that $Q_\alpha(t) = P_\alpha'(t)$. It is not difficult to check that $w(t) := Q_\alpha(t)x_0$ satisfies (47) and $z(t) := P_\alpha(t)x_1$ satisfies (48). More information about regularity of these solutions is contained in the following two lemmas, which can be proven in the same way as Lemmas 15 and 17, so we omit their proofs.

LEMMA 19. *Assume that $\alpha \in (1,2)$ and A is a nonnegative operator in a complex Banach space X with spectral angle ω_A satisfying (26). Then the following assertions hold*

(a) *Let $1 < p < \frac{1}{2-\alpha}$. Then $AQ_\alpha(t)x_0 \in L^p(\mathbb{R}_+;X)$ iff $x_0 \in D_A(\frac{2p-1}{\alpha p},p)$. In this case there are constants C_1 and C_2, such that*

$$C_1[x_0]_{D_A(\frac{2p-1}{\alpha p},p)} \le \|AQ_\alpha(t)x_0\|_{L^p(\mathbb{R}_+;X)} \le C_2[x_0]_{D_A(\frac{2p-1}{\alpha p},p)}$$

(b) *Let $p \ge \frac{1}{2-\alpha}$. Then $AQ_\alpha(t)x_0 \in L^p(\mathbb{R}_+;X)$ iff $x_0 = 0$.*

(c) *For all $p > 1$, $AP_\alpha(t)x_1 \in L^p(\mathbb{R}_+;X)$ iff $x_1 \in D_A(\frac{p-1}{\alpha p},p)$. In this case there are constants C_1' and C_2', such that*

$$C_1'[x_1]_{D_A(\frac{p-1}{\alpha p},p)} \le \|AP_\alpha(t)x_1\|_{L^p(\mathbb{R}_+;X)} \le C_2'[x_1]_{D_A(\frac{p-1}{\alpha p},p)}$$

The constants depend on α, p, ω_A and $K_A(\theta)$ for some $\theta \in (\alpha\pi/2, \pi-\omega_A)$.

LEMMA 20. *Assume that $\alpha \in (1,2)$ and A is a nonnegative operator in a complex Banach space X with spectral angle ω_A satisfying (26). Then the following assertions hold*

(a) *If $1 < p < \frac{1}{2-\alpha}$, $0 < \delta < \frac{\alpha p - 2p+1}{\alpha p}$ and $x_0 \in D_A(\frac{2p-1}{\alpha p}+\delta,p)$ then $Q_\alpha(t)x_0$ is a strict solution of (47) in $L^p(\mathbb{R}_+;D_A(\theta,p))$, satisfying*

$$\|AQ_\alpha(t)x_0\|_{L^p(\mathbb{R}_+;D_A(\delta,p))} \le C\|x_0\|_{D_A(\frac{2p-1}{\alpha p}+\delta,p)}.$$

(b) *If $p > 1$, $0 < \delta < \frac{\alpha p - p + 1}{\alpha p}$ and $x_1 \in D_A(\frac{p-1}{\alpha p} + \delta, p)$ then $P_\alpha(t)x_1$ is a strict solution of (48) in $L^p(\mathbb{R}_+; D_A(\theta, p))$ and*

$$\|AP_\alpha(t)x_1\|_{L^p(\mathbb{R}_+;D_A(\delta,p))} \leq C'\|x_1\|_{D_A(\frac{p-1}{\alpha p}+\delta,p)}.$$

The constants depend on α, δ, p, ω_A and $K_A(\theta)$ for some $\theta \in (\alpha\pi/2, \pi - \omega_A)$.

Lemma 19 together with Proposition 10 implies the following theorem:

THEOREM 21. *Let $\alpha \in (1,2)$, $1 < p < \infty$, X be a Banach space of class $\mathcal{HT}$, A be an $\mathcal{R}$-sectorial operator in X with $0 \in \rho(A)$ and with $\mathcal{R}$-angle ω_A^R, satisfying (37), and $f \in L^p(I;X)$. If $1 < p < \frac{1}{2-\alpha}$, $x_0 \in D_A(\frac{2p-1}{\alpha p}, p)$ and $x_1 \in D_A(\frac{p-1}{\alpha p}, p)$, then there is a unique strict L^p solution u of (43) satisfying*

$$\|u\|_{L^p(I;X)} + \|D_t^\alpha u\|_{L^p(I;X)} + \|Au\|_{L^p(I;X)} \leq$$

$$C(\|x_0\|_{D_A(\frac{2p-1}{\alpha p},p)} + \|x_1\|_{D_A(\frac{p-1}{\alpha p},p)} + \|f\|_{L^p(I;X)})$$

This result holds also if $p \geq \frac{1}{2-\alpha}$, $x_0 = 0$ and $x_1 \in D_A(\frac{p-1}{\alpha p}, p)$. The constant C has the same properties as in Theorem 16.

Lemma 20 together with Corollary 6 imply the following theorem.

THEOREM 22. *Suppose that $\alpha \in (1,2)$, $1 < p < \infty$, A is a positive operator in a Banach space X with spectral angle ω_A, satisfying (26), and $f \in L^p(I; D_A(\delta, p))$. If $1 < p < \frac{1}{2-\alpha}$, $0 < \delta < \frac{\alpha p - 2p + 1}{\alpha p}$, $x_0 \in D_A(\frac{2p-1}{\alpha p} + \delta, p)$, $x_1 \in D_A(\frac{p-1}{\alpha p} + \delta, p)$, then there is a unique strict solution u of (43) in $L^p(I; D_A(\delta, p))$ satisfying*

$$\|u\|_{L^p(I;D_A(\delta,p))} + \|D_t^\alpha u\|_{L^p(I;D_A(\delta,p))} + \|Au\|_{L^p(I;D_A(\delta,p))}$$

$$\leq C(\|x_0\|_{D_A(\frac{2p-1}{\alpha p}+\delta,p)} + \|x_1\|_{D_A(\frac{p-1}{\alpha p}+\delta,p)} + \|f\|_{L^p(I;D_A(\delta,p))}).$$

This result holds also if $p \geq \frac{1}{2-\alpha}$, $0 < \delta < \frac{\alpha p - p + 1}{\alpha p}$, $x_0 = 0$, $x_1 \in D_A(\frac{p-1}{\alpha p} + \delta, p)$. The constant C has the same properties as in Theorem 18.

In this way we obtained a complete picture of the strict L^p solvability of fractional autonomous equations. Note that maximal regularity results in the setting of Hölder continuous functions instead of L^p functions are obtained in [6] for $\alpha \in (0,1)$ and [7] for $\alpha \in (1,2)$.

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