

# Robust point stabilization of underactuated mechanical systems via the extended chained form

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## ROBUST POINT STABILIZATION OF UNDERACTUATED MECHANICAL SYSTEMS VIA THE EXTENDED CHAINED FORM\*

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**Abstract.** This paper addresses point stabilization for the extended chained form (ECF), a control system that may be used to model a number of mechanical underactuated systems. A control law is proposed, based on well-known hybrid open-loop/feedback techniques, which exponentially stabilizes the origin of a dynamic extension of the ECF and ensures a degree of robustness to additive disturbance terms that may represent, for instance, model uncertainties. Numerical simulations are included to illustrate the performance of the presented stabilizers.

**Key words.** extended chained form, second-order chained form, point stabilization, hybrid feedback, underactuated manipulator, surface vessel

**AMS subject classifications.** 93D21, 93C10

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**1. Introduction.** The study of mechanical control systems with fewer actuators than degrees of freedom constitutes a stimulating and active subject of research. Examples of such systems include underactuated manipulators [21], underactuated (surface and underwater) maritime vehicles [30, 5], underactuated spacecraft [18], and mechanical systems with internal degrees of freedom subject to virtual holonomic constraints [15, 24]. Besides the study of properties such as accessibility and controllability, the research efforts have focused mainly on problems such as open-loop steering from one configuration to another, trajectory tracking, and stabilization to an equilibrium point (or configuration). For underactuated mechanical systems, the latter problem is especially challenging since such systems typically do not meet Brockett’s necessary condition for stabilization to a point by continuous, pure-state feedback [3]. As a consequence, solutions usually involve elaborate control techniques, such as time-varying feedback or hybrid control. In this paper we are particularly interested in stabilization to a point.

A valuable tool when addressing control problems is the possibility of transforming the system dynamics, via coordinate change and feedback, into a “canonical” control system with a simpler, more tractable structure. Among such canonical representations, the *extended chained form* (ECF)

$$(1) \quad \begin{aligned} \ddot{x}_1 &= u_1, \\ \ddot{x}_2 &= u_2, \\ \ddot{x}_3 &= u_1 x_2 \end{aligned}$$

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plays, for some underactuated mechanical systems, a role similar to the one played by the *chained form* for driftless nonholonomic systems (cf. [19, 26]). By slight abuse of nomenclature we are calling the particular system (1), with six state variables and two inputs, the ECF, although more general extensions to the chained form can be envisaged and have been considered. System (1) has also been termed *second-order chained form*. However, no definitive unifying notation seems to exist as yet for the family of “chained systems.” In [11], for instance, a two-input control system is introduced which is referred to as an *n-dimensional, high-order generalized chained system*. On the other hand, chained systems having more than two inputs have also been studied under the denomination *multi-input chained systems*, e.g., in [29]. Finally, the reader should be aware that in some references—but not in the present paper—the term *extended chained form* refers to a driftless chained system, as introduced in [19], with integrators added in cascade to each of its inputs, cf., e.g., [31].

The ECF made its appearance in the context of underactuated mechanical systems in [4], where it was shown that the dynamic model of a simplified underwater vehicle is feedback-equivalent to two interconnected ECFs. In [8], the model of a planar PPR̄ manipulator was directly transformed into the ECF (PPR̄ denotes a manipulator with two *prismatic* joints and one *revolute* joint at its most distal end; the bar above “R” designates an *unactuated* or *passive* joint). Among the two-input, three-DOF systems that are feedback-equivalent to the ECF one finds the planar, vertical take-off and landing (VTOL) system in the absence of gravity [25], a simplified underwater vehicle [22], the planar, serial-drive RRR̄ manipulator [32], and the planar, parallel-drive RRR manipulators with any two joints actuated. Two additional examples are multibody systems possessing an unactuated, internal DOF which is required, by design, to satisfy a virtual holonomic constraint, namely the rigid body with internal DOF in [15] and the dynamics of the spring-coupled, third link of a planar PPR manipulator in [24]. It is worth noting that the transformations involved in these examples allow one to map generic equilibrium configurations of the mechanical system to *the origin* of the ECF, thereby reducing stabilization of any such configuration to stabilization of the latter point.

In view of these results, considerable emphasis has been given to the design of controllers for the ECF and some of its generalizations. For instance, a time-varying controller, updated in terms of the state only at isolated time-instants, was developed in [4] to achieve a “discrete-time” version of  $\mathcal{K}$ -exponential stability for the origin of two interconnected ECFs. Tracking controllers were proposed in [8] which, associated with carefully selected state trajectories (cf. also [32]), exponentially drive the state of the ECF towards the origin. In [11], discontinuous controllers were introduced to *almost-exponentially* stabilize the origin of two-input, generalized, *n*-dimensional chained form systems, including the ECF. More recently, the authors of [6] pointed out conditions for two-input systems with drift to be feedback-linearizable by non-smooth (and eventually discontinuous) state and input transformations. Once such a transformation is applied, linear controllers can be used to drive the system state exponentially to the origin, provided the initial conditions belong to a set where the new coordinates are well defined. In [1] a time-varying, continuous, homogeneous control-law was introduced which, to date and to the extent of our knowledge, is the only one capable of ensuring Lyapunov-stability as well as exponential convergence (indeed  $\mathcal{K}$ -exponential stability) for the origin of the ECF.

In this paper we propose controllers that are both *stabilizing* and *robust*—in appropriately defined senses—based on a well-known *hybrid open-loop/feedback* approach (also known as *iterative state steering*). Essentially, this goes along the lines

of *discrete-time* control of *continuous-time* systems: at a given sample instant  $t_k$  the state  $x(t_k)$  is sensed, and an input function  $t \mapsto u_k(t)$  is computed and used to drive the system until the next sample instant  $t_{k+1}$ . Within the interval  $(t_k, t_{k+1})$  the input may change with  $t$ , but it is *independent* of the instantaneous value of the state  $x(t)$ . The input  $u_k$  is designed so that, at the end of the interval, the state  $x(t_{k+1})$  is “closer” to the origin than it was at the beginning. This control algorithm is iterated indefinitely and, under appropriate assumptions, it leads to a robustly stable equilibrium point. Let us remark that the use of iterated control is not new and that important results have been reported in the literature. One example is [4], mentioned above, where iterated controls were developed, but where no robustness study was carried out. A hybrid control combining sampled-time control with continuous-time, linear feedback was proposed in [20] to stabilize chained form systems, with applications to wheeled mobile robots. Among the earliest references addressing the *robustness* of time-varying, iterative control in the framework of nonholonomic systems one finds [2], where control laws are developed for the three- and four-dimensional chained forms. These feedback laws render the origin exponentially stable (in the discrete-time sense) and this stability property is preserved in the presence of additive disturbance vector fields. The authors of [14] consider a large class of systems, possibly with drift, under iterative state steering control. Although no algorithm is presented to construct any such controller—it is assumed that one is known beforehand—conditions are pointed out for discrete-time stability of the origin and robustness to the presence of additive disturbance vector fields. A drawback of the reported conditions for robustness is that some of them are stated in terms of *the flow* of the disturbance vector field(s), thus limiting the class of disturbances for which robustness can be assessed in practice. For driftless systems, a powerful approach was presented in [17], where a constructive algorithm is given to design stabilizers for *any* driftless, analytic, controllable system. The controllers thus obtained guarantee local exponential stability of the origin for a *dynamic extension* of the original system, and the stability is robust to additive disturbance vector fields. Our controller design and methodology share similarities with [4] and [14], although the stability and robustness analysis is inspired by [17]. The presence of a drift term, however, makes the analysis—and the eventual generalization of the present approach to a larger class of systems—more difficult. As a consequence, our result is merely applicable to a class of systems which can be represented as a (perturbed) ECF.

This paper is organized as follows. Section 2 contains definitions of stability and robustness, as used in the present context, as well as a statement of the robust stabilization problem. In section 3, a feedback law is introduced and then shown to be a robust stabilizer in the specific sense considered here. Section 4 contains two simulation examples. Some concluding remarks are given in section 5. Finally, in the appendix, notational conventions are fixed and some technical lemmas are stated or proved.

**2. Preliminaries and definition of the problem.** Prior to stating the problem, let us precisely define the notions of stability and robustness used in this context. To this end consider the ECF, regarded as the *nominal system*, rewritten in the form

$$\dot{x} = b_0(x) + u_1 b_1(x) + u_2 b_2(x),$$

with

$$(2) \quad b_0(x) = x_2 \frac{\partial}{\partial x_1} + x_4 \frac{\partial}{\partial x_3} + x_6 \frac{\partial}{\partial x_5}, \quad b_1(x) = \frac{\partial}{\partial x_2} + x_3 \frac{\partial}{\partial x_6}, \quad b_2(x) = \frac{\partial}{\partial x_4}.$$

As a result of model errors, such as parameter uncertainties, disturbance vector fields may be present in the system to be actually controlled, and one way to model this is by considering the *perturbed system*

$$(3) \quad \dot{x} = b_0(x) + h_0(x, \varepsilon) + \sum_{i=1}^2 u_i(b_i(x) + h_i(x, \varepsilon)),$$

where  $\mathbf{h} = (h_0, h_1, h_2)$  is a 3-tuple of real-analytic mappings  $h_i : U \times E \rightarrow \mathbb{R}^6$ , and  $E \subset \mathbb{R}$  is an interval containing 0.  $\mathbf{h}$ , referred to in what follows as a *disturbance*, is assumed to satisfy  $h_0(0, \varepsilon) = 0$  for every  $\varepsilon \in E$ , so that  $(x, u) = (0, 0)$  is an equilibrium point for the perturbed system. The interpretation of  $\varepsilon$  is that of an additional parameter quantifying the “magnitude” of the perturbation. For ease of reference we denote by  $\mathcal{D}^3$  the set of all disturbances  $\mathbf{h} = (h_0, h_1, h_2)$ , each defined on a set  $U \times E$  ( $E$  may thus depend on the choice of  $\mathbf{h}$ ). In what follows we also write  $h_i^\varepsilon(x) = h_i(x, \varepsilon)$ .

Essentially, these disturbances are intended to represent two kinds of error terms, namely, those that do not depend on  $\varepsilon$ , which may typically encompass “high-order” terms neglected when the model is derived, and those that result from inaccuracies—quantified by  $\varepsilon$ —in the knowledge of the physical dimensions involved in the model (cf. also Remark 2(i) after Proposition 3.1). Obviously, however, not all disturbances may be modeled by additive vector fields as in (3). Phenomena such as neglected modes, nonsmooth effects (e.g., friction) or measurement noise would require different representations. Therefore, the notion of robustness one can aim at by considering such disturbances bears some limitations.

Before we proceed, let us recall the notion of exponential stability for continuous-time systems. Let  $0 \in U \subset \mathbb{R}^n$ , with  $U$  open, and consider the system

$$(4) \quad \dot{z} = f(z, t), \quad f(0, \cdot) = 0, \quad f : U \times \mathbb{R} \rightarrow \mathbb{R}^n.$$

The mapping  $(z, t) \mapsto f(z, t)$  is assumed to be continuous in  $z$  and piecewise continuous in  $t$ . The origin  $z = 0$  is *locally exponentially stable* for (4) if there exist  $K > 0$ ,  $\gamma > 0$  and a neighborhood  $V \subset U$  of 0 such that, for every  $(z_0, t_0) \in V \times \mathbb{R}$ , a solution  $z(\cdot)$  satisfying  $z(t_0) = z_0$  is defined on  $[t_0, \infty)$  and also satisfies

$$(5) \quad \|z(t)\| \leq K \|z_0\| e^{-\gamma(t-t_0)}$$

for all  $t \geq t_0$ .

Now suppose that a continuous, time-varying ( $T$ -periodic) feedback law  $\alpha : U \times \mathbb{R} \rightarrow \mathbb{R}^2$  is given. As mentioned in the introduction, one intends to act on the perturbed system (3) by periodically iterating this control law in the hope that such process stabilizes the system exponentially to a point (the origin, say, without loss of generality). Nevertheless, according to the definition of (local) exponential stability, the iteration of such a control law cannot, in general, achieve that goal since the origin may even fail to be an equilibrium. Indeed, the state of the system may reach the origin at some time  $t_0 \in (kT, (k+1)T)$ , which need not coincide with any of the sampling instants. Since the control operates in “open-loop” between samples, it may continue acting on the system, thus causing the state to leave the origin again. In such a case, inequality (5)—which is required to hold for every choice of “initial data”  $(z_0, t_0) \in V \times \mathbb{R}$ —would not hold for  $(0, t_0)$  and any selection of  $K > 0, \gamma > 0$ . One way to remedy this issue is to consider stability in the *discrete-time*

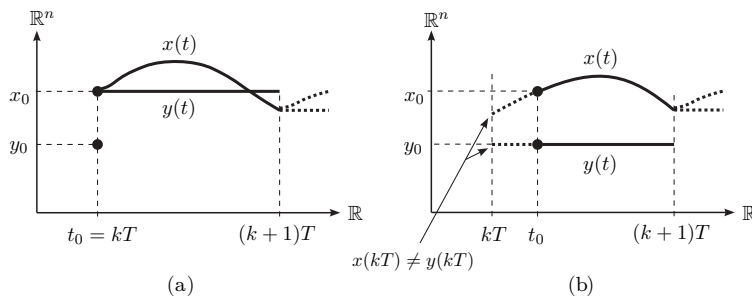


FIG. 1. Initial conditions for system (6): (a) If  $t_0 \bmod T = 0$ , both  $x(\cdot)$  and  $y(\cdot)$  are initialized to  $x_0$ ; (b) If  $t_0 \bmod T \neq 0$ ,  $x(\cdot)$  and  $y(\cdot)$  are initialized to  $x_0$  and  $y_0$ , respectively. Note that in the latter case the solutions are in general not reversible in time, since extending  $x(t)$  and  $y(t)$  for  $t \in [kT, t_0)$ , using the dynamics (6), may lead to the condition  $x(kT) \neq y(kT)$ .

sense and concentrate only on the sequence of state values at the sampling instants,  $(z(kT))_{k \in \mathbb{N}}$ . However, since one is dealing with a continuous-time system (3), we adopt the alternative approach proposed in [17], where local exponential stability is considered for a dynamic extension of the perturbed system (3). More precisely, in order to cope with the case when  $t_0 \bmod T \neq 0$  (so  $t_0$  does not equal any sampling instant) we adjoin a signal  $t \mapsto y(t)$ , which coincides with the state  $x(kT)$  at the update instants indexed by  $k \in \{\lfloor t_0/T \rfloor + 1, \lfloor t_0/T \rfloor + 2, \dots\}$ , and then consider the dynamically extended perturbed system

$$(6) \quad \begin{cases} \dot{x} &= b_0(x) + h_0(x, \varepsilon) + \sum_{i=1}^2 \alpha_i(y, t)(b_i(x) + h_i(x, \varepsilon)), \\ \dot{y} &= \sum_{k=\lfloor t_0/T \rfloor + 1}^{\infty} \delta(t - kT)x(t), \end{cases}$$

under the proviso that its “initial condition” be defined, given any  $(x_0, y_0) \in \mathbb{R}^6 \times \mathbb{R}^6$ , by setting  $(x(t_0), y(t_0))$  equal to  $(x_0, x_0)$  if  $t_0 \bmod T = 0$ , or equal to  $(x_0, y_0)$  otherwise. (The symbol  $\delta(t - kT)$  in (6) represents Dirac’s delta “function” and satisfies  $\int_{-\infty}^{\infty} \delta(t - kT)f(t)dt = f(kT)$  for any mapping  $f : \mathbb{R} \rightarrow \mathbb{R}^n$ .)

The meaning of the initial conditions for system (6) is illustrated in Figure 1. Clearly, the first sample instant after the initial time  $t_0$  occurs at  $t = (\lfloor t_0/T \rfloor + 1)T$  or, using the notation in the figure, at  $t = (k+1)T$ . This explains the initial value for the index  $k$  in the second summation of (6). Note also that the trajectories initialized in this way are defined for forward time ( $t \geq t_0$ ), but they may fail to be reversible in time. In other words, when  $t_0 \bmod T \neq 0$ , the solution  $(x(\cdot), y(\cdot))$  may be prolonged to the interval  $[kT, t_0)$  by using the dynamics (6); however,  $x(kT)$  may differ from  $y(kT)$ .

*Remark 1.* It is worth pointing out that the dynamic extension in (6) is a technical artifice merely used to establish the proofs in a precise setting. In particular, the extension does not have to be “implemented,” nor does it restrain the way the control signals are actually applied to system (1), or the set of allowable initial conditions for the latter.

The problem of robust stabilization may now be formulated as follows.

*Problem 1* (robust stabilization of the extended chained form). Design a control law  $\alpha : U \times \mathbb{R} \rightarrow \mathbb{R}^2$  which ensures that, for every disturbance  $h$  in a given set  $\mathcal{A} \subset \mathcal{D}^3$ , there is a constant  $\varepsilon_0 > 0$  such that the origin  $(x, y) = (0, 0)$  of system (6) is locally

exponentially stable whenever  $\varepsilon \in E$  and  $|\varepsilon| \leq \varepsilon_0$ .

**3. Robust stabilizers for the extended chained form.** In this section we derive a solution to Problem 1 for the ECF system (1). The solution is obtained in two main steps: first the feedback law  $\alpha$  is *designed* to have certain properties; then, in the slightly more involved second step, a stability/robustness *analysis* is carried out to guarantee that  $\alpha$  indeed solves the problem. For more details on the notation used in this and the ensuing sections, the reader may consult section 6.1 in the appendix.

**3.1. Design of the feedback law.** Fix  $T > 0$  and set  $\omega = 2\pi/T$ . Our goal is to design a feedback law  $\alpha \in C^0(\mathbb{R}^6 \times \mathbb{R}; \mathbb{R}^2)$ ,  $T$ -periodic in its second argument, such that the solution  $x(\cdot)$  to the controlled ECF

$$(7) \quad \dot{x} = b_0(x) + \sum_{i=1}^2 \alpha_i(x_0, t)b_i(x), \quad x(0) = x_0 \in \mathbb{R}^6,$$

with  $b_0, b_1, b_2$  given in (2), satisfies

$$(8) \quad x(T) = Ax_0 + o(\|x_0\|),$$

where  $A \in \mathbb{R}^{6 \times 6}$  a *discrete-time-stable* matrix, i.e., a matrix whose spectrum is contained in  $\{z \in \mathbb{C} : |z| < 1\}$ . We propose the following controller structure:

$$(9) \quad \alpha_1(x, t) = a_1x_1 + a_2x_2 + G\rho(x) \cos(\omega t),$$

$$(10) \quad \alpha_2(x, t) = a_3x_3 + a_4x_4 - \frac{2\omega^2}{G} \frac{1}{\rho(x)} (a_5x_5 + a_6x_6) \cos(\omega t),$$

where the vector of control gains  $a \in \mathbb{R}^6$  is determined below,  $G > 0$ , and  $\rho$  is given<sup>1</sup> by  $\rho(x) = (\sum_{i=1}^6 |x_i|^{\frac{2}{r_i}})^{\frac{1}{2}}$ , with  $r = (1, 1, 1, 1, 2, 2)$ . We set  $\alpha(0, \cdot) = 0$ . By virtue of the definition of  $\rho$ , one easily shows that  $\alpha(x, t) \rightarrow 0$  whenever  $x \rightarrow 0$ , uniformly for  $t \in \mathbb{R}$ , so that  $\alpha$  is *continuous on*  $\mathbb{R}^6 \times \mathbb{R}$ .

Now, the closed-loop system can be explicitly integrated thanks to the simple structure of the ECF and the fact that  $u(t) = \alpha(x_0, t)$  is independent of  $x(t)$  on the interval  $(0, T)$ . After some calculations, one verifies the solution  $x(\cdot)$  is of the form

$$(11) \quad x(T) = Ax_0 + w(x_0),$$

where  $A$  is a block-diagonal matrix  $A = \text{diag}(A_1, A_2, A_3)$  with blocks defined by

$$(12) \quad A_i = \begin{pmatrix} 1 + \frac{1}{2}T^2a_{2i-1} & T + \frac{1}{2}T^2a_{2i} \\ Ta_{2i-1} & 1 + Ta_{2i} \end{pmatrix}, \quad i = 1, 2, 3.$$

The spectrum of  $A$  is the union of the spectra of the  $A_i$ , each of which can be made equal to  $\{k_{i1}, k_{i2}\} \subset \{z \in \mathbb{C} : |z| < 1\}$ —thus making  $A$  a discrete-time-stable matrix—by setting

$$(13) \quad a_{2i-1} = \frac{k_{i1} + k_{i2} - k_{i1}k_{i2} - 1}{T^2} \quad \text{and} \quad a_{2i} = \frac{k_{i1} + k_{i2} + k_{i1}k_{i2} - 3}{2T}, \quad i = 1, 2, 3.$$

<sup>1</sup>In the language of homogeneity,  $\rho$  is a homogeneous norm with respect to a dilation of weight  $r$ . In this paper, however, no further use is made of this terminology or the associated results, and the interested reader is referred to, e.g., [7, 9] for more detailed discussions on that subject.

Of course,  $a_{2i-1}$  and  $a_{2i}$  must be real, for which it suffices to choose  $k_{i1}, k_{i2}$  to be complex conjugate. On the other hand, it is readily checked that the function  $w = (w_1, \dots, w_6) : \mathbb{R}^6 \rightarrow \mathbb{R}^6$  in (11) is given by  $w_1 = \dots = w_4 = 0$  and

$$(w_5, w_6)(x_0) = \rho(x_0)L(x_0) + (\rho(x_0))^{-1}P(x_0) + Q(x_0),$$

where  $L : \mathbb{R}^6 \rightarrow \mathbb{R}^2$  is linear and  $P, Q : \mathbb{R}^6 \rightarrow \mathbb{R}^2$  are quadratic. Since  $\rho(x_0) = O(\|x_0\|^{\frac{1}{2}})$ , it follows that  $w(x_0) = O(\|x_0\|^{\frac{3}{2}})$  and hence  $w(x_0) = o(\|x_0\|)$ , so the solution  $x(T)$  has the form (8). Since  $A$  is discrete-time-stable, there exists a symmetric, positive-definite matrix  $P \in \mathbb{R}^{6 \times 6}$  and a real number  $\tau \in [0, 1)$  such that  $\|Ax_0\|_P \leq \tau\|x_0\|_P$  for every  $x_0 \in \mathbb{R}^6$ , with  $\|x\|_P = x^T P x$  denoting the norm of  $x$  induced by  $P$ . This means that, locally around the origin, the mapping which assigns  $x(T)$  to  $x_0$  is a contraction in the norm  $\|\cdot\|_P$ .

### 3.2. Some links between the proposed controller and other approaches.

The remarkably simple structure of the control law (9)–(10) shares common traits with the one in [1]. In particular, both involve terms that are linear in the state components governed by second-order chains of integrators, namely,  $x_1, \dots, x_4$  in the notation of the present paper. In addition, both of them use normalization by  $\rho$ —multiplication of some terms by  $1/\rho$ —in order to adjust the “degree of homogeneity” of the control law  $\alpha$  (see [1] for further details and definitions). The important difference, however, lies in the way the control signals are calculated and applied, to wit, iterative state steering vs. feedback. As a matter of fact, this difference is instrumental in establishing robustness.

Interestingly, the *frequency*  $\omega$  of the time-varying terms in the control law (9)–(10) does not have to be large. In fact, that frequency may be taken arbitrarily small (i.e., the period between samples may be arbitrarily long) without qualitatively altering the nature of the result. This is in opposition with the control laws in [1] or, more generally, with previous results based on averaging of “highly oscillatory” systems, e.g., [28, 16].

Furthermore, in contrast with the control laws in [8], which provide tracking controllers that steer the state asymptotically towards the origin by following an appropriately designed trajectory, the computation of (9)–(10) does not require the use of any such trajectory.

It is also interesting to note that, while our approach and that of [4] exhibit similarities (e.g., both are intended to be implemented as hybrid open-loop/feedback) the control expressions (9)–(10) are less involved than the ones in [4], which make use of time-varying gains determined by the solutions of an exogenous system. Moreover, even though robustness is not explicitly addressed in [4], it seems difficult to assess whether those control laws ensure robustness in the sense considered in this paper or not. In particular, the result in [13], which allows us to ascertain nonrobustness of [1], does not apply in that case.

On the other hand, the work reported in [14], where stability is considered in the discrete-time sense, may be used to ascertain robustness of our controllers with respect to disturbances of a particularly simple nature. It is not clear, however, how a larger class of disturbances (such as the one considered in our main result; cf. Proposition 3.1 below) can be encompassed by the same methodology. In fact, the strongest result in [14] holds when disturbances are simple enough that adding them to the closed-loop system results in a vector field whose flow can be explicitly computed. Since our stability/robustness analysis uses a Chen–Fliess series expansion to scrutinize the



terms that add up to the flow, in a very loose sense it may be regarded as a refinement, for the special case of system (1) controlled by (9)–(10), of the results in [14].

To close this paragraph, let us add that our approach yields control laws that are globally defined on  $\mathbb{R}^6 \times \mathbb{R}$ ; hence they are nonsingular on the whole domain of validity of the coordinate chart containing the point to be stabilized. A slightly different situation occurs for the control laws of [11] and [6], where singularities may appear near the target point due to the nature of the control laws and to the nature of the coordinate transformations, respectively.

**3.3. Stability and robustness analysis.** In this section we present our main result, Proposition 3.1, which characterizes the stability and robustness properties of the feedback law (9)–(10) applied to the ECF.

**PROPOSITION 3.1.** *The control law  $\alpha$  defined in (9)–(10) is a local exponential stabilizer for the origin of system (6), robust to disturbances in  $\mathcal{A} = \{(h_0^\varepsilon, h_1^\varepsilon, h_2^\varepsilon) \in \mathcal{D}^3 : \text{Ord}(h_0^\varepsilon) \geq 1, \text{Ord}(h_0^0) \geq 2 \text{ and } \text{Ord}(h_i^0) \geq 1, i = 1, 2\}$ .*

*Remark 2.* (i) In view of the definition of  $\mathcal{A}$ , for  $\mathbf{h} \in \mathcal{A}$  one can write  $h_i(x, \varepsilon) = w_i^\varepsilon(x) + h_i^0(x)$ , with  $w_i^0(\cdot) = 0$ ,  $h_0^\varepsilon(x) = O(\|x\|^2)$ , and  $h_j^0(x) = O(\|x\|^1)$ , ( $i = 1, 2, 3$ ,  $j = 1, 2$ ). Hence each disturbance vector field can be thought of as consisting of two parts, one containing only “high-order” terms in  $x$  and the other one vanishing identically when  $\varepsilon = 0$ . The terms corresponding to these two parts may have different origins. For instance,  $w_i^\varepsilon(x)$  may arise from uncertainty in the knowledge of the physical parameters; if  $\varepsilon$  is a quantitative measure of the uncertainty, then these terms should vanish when  $\varepsilon$  equals zero. On the other hand,  $h_i^0(x)$  may include high-order terms truncated from a series expansion of the system’s nominal model, and these terms do not necessarily vanish when  $\varepsilon = 0$ .

(ii) A measure of the extent to which robustness is ensured by a feedback law  $\alpha$  lies in the nature of the set  $\mathcal{A}$ . Roughly stated, the larger this set is, the more sources of disturbances  $\alpha$  can tolerate. In this respect, the control law in [1] is *not* robust to disturbances taken from  $\mathcal{A}$ ; thus the origin may be destabilized by the addition of disturbances in  $\mathcal{A}$  regardless of how small their magnitude is (i.e., for arbitrarily small  $|\varepsilon| > 0$ ). This lack of robustness, which can be checked by using the results in [13], is illustrated through numerical simulation in the examples in section 4.

The proof of Proposition 3.1 shares the same basic structure as that of Theorem 1 in [17], and some other technical facts are easy modifications of proofs in [27] and [10]. For the sake of conciseness, we prove only those claims particular to our solution and explicitly refer the reader to the appropriate references when necessary.

*Proof of Proposition 3.1.* Let us fix a disturbance  $\mathbf{h} \in \mathcal{A}$  defined on an open set  $U \times E \subset \mathbb{R}^n \times \mathbb{R}$ . It must be shown that there is  $\varepsilon_0 > 0$  such that the origin of (6) is locally exponentially stable when  $\varepsilon \in [-\varepsilon_0, \varepsilon_0] \cap E$ . The proof is divided into two main steps corresponding to the following two claims.

*Claim 1.* For every compact interval  $E' \subset E$  there is a compact neighborhood  $U' \subset U$  of 0 such that if  $x_0 \in U'$  and  $\varepsilon \in E'$ , the solution  $t \mapsto x(t) = \pi(t, 0, x_0, \varepsilon)$  to

$$(14) \quad \dot{x} = b_0(x) + h_0^\varepsilon(x) + \sum_{i=1}^2 \alpha_i(x_0, t)(b_i(x) + h_i^\varepsilon(x)), \quad x(0) = x_0,$$

satisfies

$$x(T) = Ax_0 + \lambda(\varepsilon, x_0) + \mu(\varepsilon, x_0) + o(\|x_0\|),$$

where the mappings  $\lambda, \mu$  (which need not be uniquely defined) are such that

$$(15) \quad \frac{\|\lambda(\varepsilon, x_0)\|}{\|x_0\|} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0, \quad \text{uniformly for } x_0 \in U' \setminus \{0\},$$

$$(16) \quad \frac{\|\mu(\varepsilon, x_0)\|}{\|x_0\|} \rightarrow 0 \quad \text{as } x_0 \rightarrow 0, \quad \text{uniformly for } \varepsilon \in E'.$$

*Claim 2.* [17, Theorem 1]. There exists a nonempty interval  $E_0 \subset E$  containing 0 such that, for every  $\varepsilon \in E_0$ , the origin of system (6) is locally exponentially stable. The proof that Claim 1 implies Claim 2 can be found in [17, Theorem 1]; here we proceed with the proof of Claim 1. The first step consists in showing that the system’s solution at time  $T$  can be represented by means of a Chen–Fliess series expansion and, to this end, the following lemma is instrumental.<sup>2</sup>

LEMMA 3.2. *Let  $M$  be a real-analytic manifold and let  $\bar{x} \in M$ . Assume that the following hold: (1)  $f_0, \dots, f_m$  are real-analytic vector fields on  $M$ , with  $f_0(\bar{x}) = 0$ ; (2)  $\phi : M \rightarrow \mathbb{R}$  is real-analytic; and (3)  $\alpha \in C^0(M \times \mathbb{R}; \mathbb{R}^m)$  is such that  $\alpha(\bar{x}, \cdot) = 0$  and  $\alpha(x, \cdot)$  is bounded for every  $x \in M$ . Then, given  $T > 0$ , there is a neighborhood  $K$  of  $\bar{x}$  such that, for  $x_0 \in K$  and  $t_0 \in \mathbb{R}$ , the solution  $t \mapsto \pi(t, t_0, x_0)$  to  $\dot{x} = f_0(x) + \sum_{i=1}^m \alpha_i(x_0, t) f_i(x)$ ,  $x(t_0) = x_0$  is defined for  $t \in [t_0, t_0 + T]$ , and the Chen–Fliess series  $\text{Ser}_{\phi, f, \alpha}(t, t_0, x_0) = \sum_I f_I \phi(x_0) \int_{t_0}^t \alpha_I(x_0)$  converges to  $\phi(\pi(t, t_0, x_0))$ , absolutely and uniformly for  $(x_0, t_0) \in K \times \mathbb{R}$  and  $t \in [t_0, t_0 + T]$ .*

*Proof.* The proof of Lemma 3.2 is given in the appendix.  $\square$

Let  $E' \subset E$  be any compact interval containing 0. Define real-analytic vector fields  $g_0, g_1, g_2$  on  $U \times E$  and a feedback law  $\bar{\alpha} \in C^0(U \times E \times \mathbb{R}; \mathbb{R}^m)$  by setting  $g_i(x, \varepsilon) = b_i(x) + h_i^\varepsilon(x)$  and  $\bar{\alpha}_i(x, \varepsilon, t) = \alpha_i(x, t)$ . It is clear that  $g_0(0, \varepsilon) = 0$  for  $\varepsilon \in E$ , and that  $\mathbf{g} = (g_0, g_1, g_2)$  and  $\bar{\alpha}$  satisfy the assumptions of Lemma 3.2. Hence, for every  $\varepsilon \in E'$  there is an open neighborhood  $V_\varepsilon$  of  $(0, \varepsilon) \in U \times E$  for which the conclusion of that lemma holds. But  $(V_\varepsilon)_{\varepsilon \in E'}$  is an open cover for the compact set  $\{0\} \times E'$ ; thus one can extract from it a finite, open subcover. This implies the existence of a neighborhood  $U' \subset U$  of the origin with the property that, for any  $\varepsilon \in E'$ , the solution  $t \mapsto x(t) = \pi(t, 0, x_0, \varepsilon)$  to system (14), issued from any point  $x_0 \in U'$  at  $t = 0$ , is defined on  $[0, T]$ , and the corresponding Chen–Fliess series

$$(17) \quad S(x_0, \varepsilon, t) = \text{Ser}_{\text{id}, b+h^\varepsilon, \alpha}(t, 0, x_0) = \sum_I (b + h^\varepsilon)_I \text{id}(x_0, \varepsilon) \int_0^t \alpha_I(x_0)$$

converges to  $\pi(t, 0, x_0, \varepsilon)$  absolutely, uniformly for  $(x_0, \varepsilon, t) \in U' \times E' \times [0, T]$ . (Here we use the notation  $(b + h^\varepsilon)_I \text{id}(x_0) = (b_{i_1} + h_{i_1}^\varepsilon) \cdots (b_{i_r} + h_{i_r}^\varepsilon) \text{id}(x_0)$ , given a multi-index  $I = (i_1, \dots, i_r)$ .) Note that the terms  $(x_0, \varepsilon) \mapsto (b + h^\varepsilon)_I \text{id}(x_0)$  involved in the series (17) represent real-analytic, first-order differential operators iterated on the function  $\text{id}$ ; hence these terms are real-analytic as well. We may therefore use (17) to express the solution at  $t = T$  in order to prove that it satisfies Claim 1.

<sup>2</sup>In [27, Lemma 4.2], conditions are given for the Chen–Fliess series to converge for every  $t$  in a sufficiently short interval  $[0, \tau]$ . In the present case, however, one requires the value of the solution at the end of the interval  $[0, T]$ , with  $T$  fixed beforehand. When the system is driftless, the interval  $[0, \tau]$  of validity of the series expansion can be made arbitrarily long by imposing small enough bounds on the control inputs  $\|u(\cdot)\|$  (cf. [17, Prop. 1] and the remarks that follow it). Nevertheless, the system here contains a drift term, so the convergence results in [27] cannot be applied without modification. This motivates the role of Lemma 3.2, which states conditions for convergence of the series for arbitrarily large times and initial conditions near an equilibrium point.

Set  $w_i^\varepsilon(x) = h_i^\varepsilon(x) - h_i^0(x)$  so that  $h_i^\varepsilon = w_i^\varepsilon + h_i^0$ ,  $i = 0, 1, 2$ . Obviously, each  $(x, \varepsilon) \mapsto w_i^\varepsilon(x)$  is real-analytic and vanishes when  $\varepsilon = 0$ . For convenience define the sets of vector fields  $\mathcal{B} = \{b_0, b_1, b_2\}$ ,  $\mathcal{W} = \{w_0^\varepsilon, w_1^\varepsilon, w_2^\varepsilon\}$ , and  $\mathcal{H} = \{h_0^0, h_1^0, h_2^0\}$ . Considering that each of the iterated differential operators  $(b + h^\varepsilon)_I$  in (17) can be written as  $(b + w^\varepsilon + h^0)_I$ , it is easy to check that, since  $S(x_0, \varepsilon, T)$  converges absolutely, the series can be rearranged as  $S(x_0, \varepsilon, T) = \sum_{i=1}^5 S_i(x_0, \varepsilon, T)$ , where  $S_1, \dots, S_5$  are absolutely convergent series defined by

$$\begin{aligned} S_1(x_0, \varepsilon, T) &= x_0 + \sum_{1 \leq |I|} b_I \text{id}(x_0) \int_0^T \alpha_I(x_0), \\ S_2(x_0, \varepsilon, T) &= \sum_{1 \leq |I|} X_I \text{id}(x_0, \varepsilon) \int_0^T \alpha_I(x_0), \\ S_3(x_0, \varepsilon, T) &= \sum_{1 \leq |I| \leq 2} Y_I \text{id}(x_0, \varepsilon) \int_0^T \alpha_I(x_0), \\ S_4(x_0, \varepsilon, T) &= \sum_{3 \leq |I|} Y_I \text{id}(x_0, \varepsilon) \int_0^T \alpha_I(x_0), \\ S_5(x_0, \varepsilon, T) &= \sum_{1 \leq |I|} Z_I \text{id}(x_0, \varepsilon) \int_0^T \alpha_I(x_0), \end{aligned}$$

and, for  $I = (i_1, \dots, i_r)$ , the iterated differential operators  $X_I, Y_I, Z_I$  satisfy the following:

1. For  $j = 1, \dots, r$ ,  $X_{i_j}$  and  $Y_{i_j}$  belong to  $\mathcal{B} \cup \mathcal{W} \cup \mathcal{H}$ , whereas  $Z_{i_j}$  belongs to  $\mathcal{B} \cup \mathcal{H}$ .
2. At least one of the  $X_{i_j}$  and at least one of the  $Y_{i_j}$  are contained in  $\mathcal{W}$ .
3. None of the  $X_{i_j}$  belongs to  $\{b_0, w_0^\varepsilon, h_0^0\}$ .
4. At least one of the  $Y_{i_j}$  belongs to  $\{b_0, w_0^\varepsilon, h_0^0\}$ .
5. At least one of the  $Z_{i_j}$  is contained in  $\mathcal{H}$ .

It follows from the first property that all of the  $Z_I$  are independent of  $\varepsilon$  and, from the second, that  $X_I \text{id}(x_0, 0) = Y_I \text{id}(x_0, 0) = 0$  for every  $x_0 \in U$ . In what follows,  $S_1$  through  $S_5$  are analyzed separately in order to show that their sum has the form announced in Claim 1. Let us first present Lemma 3.3, which gathers some simple facts to be used below.

LEMMA 3.3. *Under the assumptions of Proposition 3.1 the following hold:*

- (i) For every compact neighborhood  $U' \subset U$  of the origin there exists  $K > 0$  such that  $\|\alpha(x_0, t)\| \leq K \|x_0\|^{\frac{1}{2}}$  for  $(x_0, t) \in U' \times \mathbb{R}$ .
- (ii) Let  $r \in \{1, 2\}$ . For any nonzero multi-index  $I \in \{0, 1, 2\}^r$ , the iterated integral  $\int_0^T \alpha_I$  satisfies  $\int_0^T \alpha_I(x_0) = O(\|x_0\|)$ , and for any multi-index  $I \in \{1, 2\}^r$  it satisfies  $\int_0^T \alpha_I(x_0) = O(\|x_0\|^2)$ .
- (iii) Say that  $k_0 = 0$  and  $k_1 = k_2 = 1/2$ . Then for any multi-index  $I = (i_1, \dots, i_r) \in \{0, 1, 2\}^r$ ,  $r > 0$ , one has  $\text{Ord}(\int_0^T \alpha_I) \geq \sum_{j=1}^r k_{i_j}$ .
- (iv) For every multi-index  $I$ ,  $x_0 \mapsto \int_0^T \alpha_I(x_0)$  is continuous.
- (v) For  $i = 1, 2$  the following hold:
  - (i)  $\text{Ord}(b_0) = 0$ ,  $\text{Ord}(b_i) = -1$ ,
  - (ii)  $\text{Ord}(h_0^\varepsilon) = \text{Ord}(w_0^\varepsilon) \geq 1$ ,  $\text{Ord}(h_i^\varepsilon) = \text{Ord}(w_i^\varepsilon) \geq -1$ ,
  - (iii)  $\text{Ord}(h_0^0) \geq 2$ ,  $\text{Ord}(h_i^0) \geq 1$ .
- (vi) If  $\phi \in C^\infty(U; \mathbb{R})$  and  $k \geq 1$ , then  $\text{Ord}(b_0^k \phi) \geq k$ . (Here  $b_0^0 \phi = \phi$  and  $b_0^j \phi = b_0(b_0^{j-1} \phi)$ ,  $j \geq 1$ .)

*Proof.* Given in the appendix. □

The first sum  $S_1$  converges to the solution of the nominal system (7) controlled by  $u = \alpha(x_0, t)$ ; thus

$$(18) \quad S_1(x_0, \varepsilon, T) = x_0 + \sum_{1 \leq |I|} b_I \text{id}(x_0) \int_0^T \alpha_I(x_0) = Ax_0 + o(\|x_0\|).$$

Let us now prove that  $S_2$ – $S_4$  can be written in terms of functions satisfying properties analogous to (15)–(16), while  $S_5$  converges to an  $o(\|x\|)$  function. The following lemma is crucial to attaining this goal.

LEMMA 3.4. *Let  $U \times E \subset \mathbb{R}^n \times \mathbb{R}$  be an open neighborhood of  $(0, 0)$  and assume that, for every  $I$  in a countable set  $\mathcal{I}$ ,  $a_I : U \times E \rightarrow \mathbb{R}^n$  is real-analytic and vanishes at  $U \times \{0\}$  and  $b_I : U \rightarrow \mathbb{R}$  is continuous. Assume further that  $\sum_{I \in \mathcal{I}} a_I(x, \varepsilon) b_I(x)$  converges to  $f(x, \varepsilon)$ , absolutely and uniformly for  $(x, \varepsilon) \in U \times E$ . Then there is a compact neighborhood  $U' \subset U$  of 0 such that*

(1)  *$f$  satisfies (15) (with  $\lambda = f$ ) if  $\mathcal{I}$  is finite and any of the following conditions holds:*

- (i)  $a_I(x, \varepsilon) = O(\|x\|)$  for every  $I \in \mathcal{I}$ ,
  - (ii)  $b_I(x) = O(\|x\|)$  for every  $I \in \mathcal{I}$ .
- (2)  *$f$  satisfies (16) (with  $\mu = f$ ) if any of the following conditions holds:*

- (i)  $a_I(x, \varepsilon) = o(\|x\|)$  for every  $I \in \mathcal{I}$ ,
- (ii) *there is  $c > 0$  such that  $b_I(x) = O(\|x\|^{1+c})$  for every  $I \in \mathcal{I}$ ,*
- (iii)  $a_I(x, \varepsilon) = O(\|x\|)$  and *there is  $d > 0$  such that  $b_I(x, \varepsilon) = O(\|x\|^d)$  for every  $I \in \mathcal{I}$ .*

*Proof.* The proof of Lemma 3.4 is given in the appendix. □

Consider the sum  $S_2$ . If  $1 \leq |I| \leq 2$ , Lemma 3.3(ii) yields  $\text{Ord}(\int_0^T \alpha_I) \geq 1$ . On the other hand, since the  $X_I$ 's do not involve any drift term (i.e., none of the indices in  $I$  equals zero), for the terms such that  $|I| \geq 3$  one invokes Lemma 3.3(iii) to conclude that  $\int_0^T \alpha_I(x_0) = O(\|x_0\|^{1+c})$  with  $c = 1/2$ . Thus, by setting  $S_2(x_0, \varepsilon, T) = \lambda_2(x_0, \varepsilon) + \mu_2(x_0, \varepsilon)$ ,

$$\lambda_2(x_0, \varepsilon) = \sum_{1 \leq |I| \leq 2} X_I \text{id}(x_0, \varepsilon) \int_0^T \alpha_I(x_0) \quad \text{and}$$

$$\mu_2(x_0, \varepsilon) = \sum_{|I| \geq 3} X_I \text{id}(x_0, \varepsilon) \int_0^T \alpha_I(x_0);$$

the first is a sum of finitely many terms, and the second is the limit of an absolutely convergent series. By virtue of Lemma 3.4(1)(ii) and Lemma 3.4(2)(ii),  $\lambda_2$  and  $\mu_2$  satisfy properties analogous to (15) and (16), respectively.

Let us turn to  $S_3$ . If  $I \in \{(0), (0, 0)\}$ , then, since  $\text{Ord}(b_0) = 0$ ,  $\text{Ord}(w_0^\varepsilon) \geq 1$ , and  $\text{Ord}(h_0^0) \geq 2$ , one has  $\text{Ord}(Y_I \text{id}) \geq 1$  by virtue of Lemma 6.1(v). If  $I \notin \{(0), (0, 0)\}$ , then Lemma 3.3(ii) implies  $\text{Ord}(\int_0^T \alpha_I) = 2$ . The number of multi-indices  $I$  with  $1 \leq |I| \leq 2$  being finite, one concludes by successive application of points (1)(i) and (1)(ii) of Lemma 3.4 that  $\lambda_3$  defined by

$$(19) \quad S_3(x_0, \varepsilon, T) = \sum_{1 \leq |I| \leq 2} Y_I \text{id}(x_0, \varepsilon) \int_0^T \alpha_I(x_0) = \lambda_3(x_0, \varepsilon)$$

satisfies (15) with  $\lambda = \lambda_3$ .

Now let us turn to  $S_4$  and consider two cases according to the values of the multi-indices  $I$ .

Case (i) ( $|I| \geq 3$  and  $I$  involves three or more *nonzero* indices). Lemma 3.3(iii) implies that  $\int_0^T \alpha_I(x_0) = O(\|x\|^{1+c})$  with  $c = 1/2$ . Thus the sum of the terms for which the multi-index  $I$  involves three or more nonzero indices converges to a function  $(x_0, \varepsilon) \mapsto \mu_{4i}(x_0, \varepsilon)$  which, by virtue of Lemma 3.4(2)(ii), satisfies (16) with  $\mu = \mu_{4i}$ .

Case (ii) ( $|I| \geq 3$  and  $I$  involves two or less *nonzero* indices). Consider the following four subcases:

- *Subcase (a)* ( $|I| \geq 3$  and  $I = (0, \dots, 0)$ ). By the definition of  $Y_I$ ,  $w_0^\varepsilon$  appears at least once in  $Y_I$ ; it follows that  $\text{Ord}(Y_I \text{id}) \geq 2$  as a consequence of Lemma 3.3(v) and Lemma 6.1(v). Thus in this subcase  $Y_I \text{id}(x, \varepsilon) = o(\|x\|)$ , so the sum of these terms converges to a function  $(x, \varepsilon) \mapsto \mu_{4a}(x, \varepsilon)$  which, by Lemma 3.4(2)(i), satisfies (16) with  $\mu = \mu_{4a}$ .
- *Subcase (b)* ( $r = |I| \geq 3$ ,  $I = (0, i_2, \dots, i_r)$  and one or two indices are *nonzero*). Using again Lemma 3.3(v) and Lemma 6.1(v), one deduces that  $\text{Ord}(Y_I \text{id}) \geq 1$ . Also, by virtue of Lemma 3.3(iii),  $\text{Ord}(\int_0^T \alpha_I) \geq \frac{1}{2}$ . Thus the sum of terms in this subcase converges to a function  $(x, \varepsilon) \mapsto \mu_{4b}(x, \varepsilon)$  which, in view of Lemma 3.4(2)(iii), satisfies (16) with  $\mu = \mu_{4b}$ .
- *Subcase (c)* ( $|I| \geq 3$ ,  $I = (i_1, 0, \dots, 0)$ ,  $i_1 \neq 0$ ). It is clear that  $\text{Ord}(\int_0^T \alpha_I) = \frac{1}{2}$  as a consequence of Lemma 3.3(iii). Also, if neither  $w_0^\varepsilon$  nor  $h_0^0$  is involved in  $Y_I$ , then Lemma 3.3(vi) implies  $\text{Ord}(Y_I \text{id}) \geq -1 + 2 = 1$ . If, on the contrary, any of  $w_0^\varepsilon$  or  $h_0^0$  is involved at least once in  $Y_I$ , then  $\text{Ord}(Y_I \text{id}) \geq -1 + \sum_{j=2}^r \text{Ord}(Y_{i_j}) + \text{Ord}(\text{id}) \geq 1$  since, under that condition, one has  $\sum_{j=2}^r \text{Ord}(Y_{i_j}) \geq 1$  in view of Lemma 3.3(v) and Lemma 6.1. Therefore the sum of these terms converges to a function  $(x, \varepsilon) \mapsto \mu_{4c}(x, \varepsilon)$  which, by Lemma 3.4(2)(iii), satisfies (16) with  $\mu = \mu_{4c}$ .
- *Subcase (d)* ( $r = |I| \geq 3$ ,  $I = (i_1, \dots, i_r)$ ,  $i_1 \neq 0$  and *exactly one* of  $i_2, \dots, i_r$  is *nonzero*). Let  $\mathcal{I}$  denote the set of multi-indices corresponding to this subcase. One has  $\text{Ord}(\int_0^T \alpha_I) \geq 1$ , since exactly two indices in  $I$  are nonzero. Assume that the nonzero indices are  $i_1$  and  $i_j$ ,  $2 \leq j \leq r$ , so both  $\text{Ord}(Y_{i_1})$  and  $\text{Ord}(Y_{i_j})$  are  $\geq -1$ . Setting  $\omega_1 = \sum_{k=2}^{j-1} \text{Ord}(Y_{i_k})$  and  $\omega_2 = \sum_{k=j+1}^r \text{Ord}(Y_{i_k})$ , one gets  $\omega_1 \geq 0$  and  $\omega_2 \geq 0$ . For those terms with  $|I| \leq 7$ , Lemma 6.1(v) implies that  $\text{Ord}(Y_I \text{id}) \geq 0$ . For the terms with  $|I| \geq 8$ , on the other hand, either  $b_0$  appears three times consecutively in  $Y_I$ , or it does not. In the former case, if the iterated differential operator  $Y_{i_2} \cdots Y_{i_{j-1}}$  involves the three successive  $b_0$ 's, then Lemma 3.3(vi) yields  $\text{Ord}(Y_I \text{id}) \geq -1 + \max\{1, 3\} = 2$ . If the three successive  $b_0$ 's are involved in  $Y_{i_{j+1}} \cdots Y_{i_r}$ , the same lemma yields  $\text{Ord}(Y_{i_j} \cdots Y_{i_r} \text{id}) \geq -1 + \max\{1, 3\} = 2$ . Thus  $\text{Ord}(Y_{i_2} \cdots Y_{i_r} \text{id}) \geq \omega_1 + 2$  and  $\text{Ord}(Y_I \text{id}) \geq -1 + \max\{1, 2\} = 1$ .

Consider now the case when  $Y_I$  does not involve three consecutive  $b_0$ 's. In this case,

$$\begin{aligned} \text{Ord}(Y_{i_j} \cdots Y_{i_r} \text{id}) &\geq -1 + \max\{1, \omega_2 + 1\} = \omega_2, \\ \text{Ord}(Y_{i_2} \cdots Y_{i_r} \text{id}) &\geq \omega_1 + \max\{1, \omega_2\}; \end{aligned}$$

thus

$$\begin{aligned} \text{Ord}(Y_{i_1} \cdots Y_{i_r} \text{id}) &\geq -1 + \max\{1, \omega_1 + \max\{1, \omega_2\}\} \\ &= \max\{0, \max\{\omega_1, \omega_1 + \omega_2 - 1\}\}. \end{aligned}$$

But since  $|I| \geq 8$ , at least two vector fields from  $\{w_0^\varepsilon, h_0^0\}$  appear in  $Y_I$ , so  $\omega_1 + \omega_2 \geq 2$ ,

$\max\{\omega_1, \omega_1 + \omega_2 - 1\} \geq 1$ , and, consequently,  $\text{Ord}(Y_I \text{id}) \geq 1$ . Therefore, by setting

$$\lambda_4(x_0, \varepsilon) = \sum_{I \in \mathcal{I}, |I| \leq 7} Y_I \text{id}(x_0, \varepsilon) \int_0^T \alpha_I(x_0) \quad \text{and}$$

$$\mu_{4d}(x_0, \varepsilon) = \sum_{I \in \mathcal{I}, |I| \geq 8} Y_I \text{id}(x_0, \varepsilon) \int_0^T \alpha_I(x_0),$$

one sees that these mappings are well defined, the first being the sum of finitely many terms and the second being the limit of an absolutely convergent sequence. But then, with the help of points (1)(ii) and (2)(iii) of Lemma 3.4, one concludes that  $\lambda_4$  and  $\mu_{4d}$  satisfy (15) and (16) with  $\lambda = \lambda_4$  and  $\mu = \mu_{4d}$ , respectively. Summarizing the results from Cases (i) and (ii) for  $S_4$ , one obtains

$$(20) \quad S_4(x_0, \varepsilon, T) = \sum_{3 \leq |I|} Y_I \text{id}(x_0, \varepsilon) \int_0^T \alpha_I(x_0) = \lambda_4(x_0, \varepsilon) + \mu_{4i}(x_0, \varepsilon) + \mu_{4ii}(x_0, \varepsilon)$$

with  $\mu_{4ii} = \mu_{4a} + \mu_{4b} + \mu_{4c} + \mu_{4d}$ .

Finally, let us show that  $S_5$  converges to a function  $f_5$  such that  $f_5(x) = o(\|x\|)$ . Consider three cases according to the value of  $I$ .

*Case (i)* ( $I$  involves three or more *nonzero* indices). From Lemma 3.3(iii), we see that  $\text{Ord}(\int_0^T \alpha) \geq 3/2$ .

*Case (ii)* ( $I$  involves one or two *nonzero* indices). If  $r = |I| \in \{1, 2\}$ , then  $I \in \{1, 2\}^r$  is nonzero, so Lemma 3.3(ii) implies that  $\text{Ord}(\int_0^T \alpha) = 2$ , whereas  $\text{Ord}(Z_I \text{id}) \geq 0$ . Now suppose that  $r = |I| \geq 3$ . From Lemma 3.3(iii),  $\text{Ord}(\int_0^T \alpha_I) \geq 1/2$ . If  $i_1 = 0$ , then  $\text{Ord}(Z_I \text{id}) \geq 0 + \text{Ord}(Z_{i_2} \cdots Z_{i_r} \text{id}) \geq 0 + \max\{1, 0\} = 1$ . Now let us consider the case where  $i_1 \neq 0$ . If  $I = (i_1, 0, \dots, 0)$ , then, by definition of  $Z_I$ , either  $Z_{i_1} \in \mathcal{H}$ , in which case  $\text{Ord}(Z_{i_1} \text{id}) \geq \text{Ord}(Z_{i_1}) + \max\{1, 2\} \geq 3$ , or  $Z_{i_j} \in \mathcal{H}$  for some  $j \in \{2, \dots, r\}$ , in which case  $\text{Ord}(Z_{i_2} \cdots Z_{i_r} \text{id}) \geq \sum_{j=2}^r \text{Ord}(Z_{i_j}) + 1 \geq 2$ , and so,  $\text{Ord}(Z_I \text{id}) \geq -1 + \max\{1, 2\} = 1$ . If  $i_1 \neq 0$  and  $i_j \neq 0$  for some  $j \in \{2, \dots, r\}$ , then  $\text{Ord}(\int_0^T \alpha_I) \geq 1$ . Moreover, either  $Z_{i_1} \in \mathcal{H}$ , in which case  $\text{Ord}(Z_I \text{id}) \geq 1 + \max\{1, 0\} = 2$ , or  $Z_{i_1} \notin \mathcal{H}$ . Suppose the latter is true and set  $\omega_1 = \sum_{k=2}^{j-1} \text{Ord}(Z_{i_k})$  and  $\omega_2 = \sum_{k=j+1}^r \text{Ord}(Z_{i_k})$ , so that  $\omega_1 \geq 0, \omega_2 \geq 0$  and

$$\text{Ord}(Z_{i_j} \cdots Z_{i_r} \text{id}) \geq \text{Ord}(Z_{i_j}) + \max\{1, \omega_2 + 1\} = \text{Ord}(Z_{i_j}) + \omega_2 + 1,$$

$$\text{Ord}(Z_{i_2} \cdots Z_{i_r} \text{id}) \geq \omega_1 + \max\{1, \text{Ord}(Z_{i_j}) + \omega_2 + 1\}.$$

If  $Z_{i_j} \in \mathcal{H}$ , then  $\text{Ord}(Z_{i_j}) \geq 1$  and hence  $\text{Ord}(Z_I \text{id}) \geq -1 + \max\{1, 2 + \omega_2\} \geq 1$ . If  $Z_{i_j} \notin \mathcal{H}$ , then  $Z_{i_k} = h_0^0$  for some  $k \in \{2, \dots, r\} \setminus \{j\}$ . In that case  $\omega_1 + \omega_2 \geq 2 = \text{Ord}(h_0^0)$  and  $\text{Ord}(Z_{i_j}) \geq -1$ ; thus  $\text{Ord}(Z_I \text{id}) \geq -1 + \omega_1 + \max\{1, \omega_2\} \geq 1$ .

Summarizing, every term pertaining to Case (ii) satisfies  $\text{Ord}(Z_I \text{id} \cdot \int_0^T \alpha_I) \geq 3/2$ .

*Case (iii)* ( $I = (0, \dots, 0)$ ,  $|I| = r$ ). Since (a)  $\text{Ord}(Z_{i_j}) \geq 0$  for  $j = 1, \dots, r$ ; (b) at least one of the  $Z_{i_j}$  is equal to  $h_0^0$ ; and (c)  $\text{Ord}(h_0^0) \geq 2$ , one has  $\text{Ord}(Z_I \text{id}) = \sum_{j=1}^r \text{Ord}(Z_{i_j}) + 1 \geq 3$ .

All terms corresponding to Cases (i)–(iii) satisfy  $\text{Ord}(Z_I \text{id} \cdot \int_0^T \alpha_I) \geq 3/2$ ; that is,  $Z_I \text{id}(x_0) \int_0^T \alpha_I(x_0) = o(\|x\|)$ . Thus their sum converges to a function  $f_5$  with the required property. Clearly, the sum of finitely many functions  $f_1, \dots, f_N$  satisfying (15) on compact sets  $U_1, \dots, U_N$  (resp., (16)) also satisfies (15) on  $U' = \bigcap_{i=1}^N U_i$

(resp., (16)). Therefore  $x(T)$  is as in Claim 1, and the proof of Proposition 3.1 is complete.  $\square$

In Proposition 3.1, the condition that the disturbances belong to  $\mathcal{A}$  is *sufficient* but not necessary for stability and robustness. In particular, disturbances in  $\mathcal{A}$  satisfy  $h_0(x, \varepsilon) = O(\|x\|)$  or, stated otherwise, each component of the drift disturbance satisfies  $h_{0,i} = O(\|x\|^2)$ . This is somewhat conservative since in some cases the latter condition is not satisfied and yet the conclusion of the previous proposition seems to hold in simulations. Indeed, a refinement of that result *seems plausible*, although the proof would require surmounting some technical obstacles. We are thus led to formulate the following conjecture which, as we shall see in the examples in section 4, might be of interest when addressing the stabilization of systems whose models can be written as an ECF with additional terms. By viewing these terms as disturbances, one might successfully use the control laws (9)–(10), *without modification*, to stabilize some of those systems to a point. A drawback of the stated condition, however, is that testing it may be difficult in practice.

**CONJECTURE 1.** *Let  $\mathcal{A}'$  be the subset of  $\mathcal{D}^3$  defined by stipulating that  $(h_0^\varepsilon, h_1^\varepsilon, h_2^\varepsilon)$  belongs to  $\mathcal{A}'$  if and only if (1)  $\text{Ord}(h_0^\varepsilon) \geq 0$ ,  $\text{Ord}(h_0^0) \geq 2$ ,  $\text{Ord}(h_i^0) \geq 1$ ,  $i = 1, 2$ , and (2) for every  $k \geq 2$ , every  $X \in \{b_1, b_2, h_1^\varepsilon, h_2^\varepsilon\}$ , and every  $k$ -tuple  $(Y_1, \dots, Y_k) \in \{b_0, h_0^\varepsilon\}^k$  having at least one of the  $Y_i$  equal to  $h_0^\varepsilon$ , one has  $\text{Ord}(XY_1 \cdots Y_k \text{id}) \geq 1$ . Then the control law  $\alpha$  defined in (9)–(10) is a local exponential stabilizer for (6), robust to disturbances in  $\mathcal{A}'$ .*

*Remark 3.* If this conjecture holds true, its proof should essentially coincide with that of Proposition 3.1. The only differences would arise in arguments that explicitly appeal to the assumption  $\text{Ord}(h_0^\varepsilon) \geq 1$  (i.e.,  $\text{Ord}(w_0^\varepsilon) \geq 1$ ), namely, Subcases (a), (c), and (d) of Case (ii) in the sum  $S_4$ . One should show that, by dropping that assumption, the terms pertaining to those subcases satisfy the required properties. For Subcase (c) one has  $\text{Ord}(\int_0^T \alpha_I) \geq 1/2$  and, since in this subcase every multi-index  $I$  is of the form  $I = (i_1, 0, \dots, 0)$  with  $i_1 \neq 0$ , the corresponding terms are of the form  $Y_I \text{id} = XZ_1 \cdots Z_k \text{id}$  with  $X \in \{b_i, h_i^0, w_i^\varepsilon : i = 1, 2\}$  and  $(Z_1, \dots, Z_k) \in \{b_0, h_0^0, w_0^\varepsilon\}^k$ . By a simple induction argument one sees that the definition of  $\mathcal{A}'$ , in particular condition (2) in Conjecture 1, implies that all such terms satisfy  $\text{Ord}(Y_I \text{id}) \geq 1$ , so by virtue of Lemma 3.4(2)(iii) these terms have the required properties. On the other hand, each term of the series involved in Subcase (a) satisfies  $\text{Ord}(Y_I \text{id}) \geq 1$  and  $\text{Ord}(\int_0^T \alpha_I) = 0$  since  $\int_0^T \alpha_I(x_0) = \frac{T^{|I|}}{|I|!}$ , whereas the terms of the series in Subcase (d) satisfy  $\text{Ord}(Y_I \text{id}) \geq 0$  and  $\text{Ord}(\int_0^T \alpha_I) \geq 1$ . To prove the conjecture, then, it would suffice to show that the (infinite) series in these two subcases converge to functions  $\lambda_{4a}$  and  $\lambda_{4d}$  satisfying (15).

#### 4. Examples.

**4.1. Underactuated manipulator.** Consider the example of a PPR̄ manipulator, depicted in Figure 2, with unactuated third joint, constrained to move on a horizontal plane. Considering the links and joints as rigid bodies and neglecting gravitational and frictional forces, this system can be modeled by

$$\begin{aligned}
 (21) \quad & M_1 \ddot{q}_1 - m_3 l \sin(q_3) \ddot{q}_3 - m_3 l \cos(q_3) \dot{q}_3^2 = \tau_1, \\
 & M_2 \ddot{q}_2 + m_3 l \cos(q_3) \ddot{q}_3 - m_3 l \sin(q_3) \dot{q}_3^2 = \tau_2, \\
 & -M_3 l \sin(q_3) \ddot{q}_1 + M_3 l \cos(q_3) \ddot{q}_2 + J \ddot{q}_3 = 0,
 \end{aligned}$$

where  $m_i$ ,  $i = 1, 2, 3$ , is the mass of the  $i$ th link,  $M_i = \sum_{j=i}^3 m_j$ ,  $J$  is the moment of inertia of the third link with respect to the axis of the third joint, and  $l$  is the

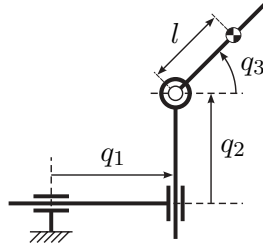


FIG. 2. Schematic representation of the PPR manipulator.

distance from the same axis to the center of mass of the third link. The input vector  $\tau = (\tau_1, \tau_2)$  represents the forces applied in the  $q_1$  and  $q_2$  directions, respectively. The configuration manifold is  $Q = \mathbb{R}^2 \times SS^1$ , for which  $q : Q \rightarrow \mathbb{R}^2 \times (-\pi, \pi)$  is a local coordinate system.

Given a target configuration  $\bar{q} \in Q$ , the dynamics can be transformed into the ECF, locally around  $\bar{q}$ , by using the coordinates of the third link’s “center of percussion.” A detailed description of the corresponding transformation can be found in [8]; for simplicity, however, in what follows we assume without loss of generality that the target configuration—the one that should be stabilized—is given by  $q(\bar{q}) = (0, 0, 0) \in U$ . After simple computations one verifies that, by setting  $K = J/M_3/l$ , the dynamic model (21) can be transformed into the ECF  $\dot{x} = b_0(x) + u_1 b_1(x) + u_2 b_2(x)$  by means of the feedback transformation  $x = \varphi(q, \dot{q})$ ,  $u = A(q, \dot{q}) + B(q)[\tau_1 \ \tau_2]^T$ , where

$$\varphi(q, \dot{q}) = \begin{pmatrix} q_1 + K(\cos(q_3) - 1) \\ \dot{q}_1 - K \sin(q_3)\dot{q}_3 \\ \tan(q_2) \\ (1 + \tan^2(q_3))\dot{q}_2 \\ q_2 + K \sin(q_3) \\ \dot{q}_2 + K \cos(q_3)\dot{q}_3 \end{pmatrix},$$

and

$$A(q, \dot{q}) = \frac{1}{\Delta(q)} \begin{pmatrix} \frac{(JK^2M_2 - J^2 - K^4M_1M_2 + JK^2M_1) \cos(q_3)\dot{q}_3^2}{(2K^2M_1M_2 - 3JM_1 \cos^2(q_3) - 2JM_2 + 3JM_2 \cos^2(q_3)) \sin(q_3)\dot{q}_3^2} \\ \frac{K}{\cos^3(q_3)} \end{pmatrix},$$

$$B(q) = \frac{1}{\Delta(q)} \begin{pmatrix} (K^2M_2 - J) \cos^2(q_3) & (K^2M_1 - J) \cos(q_3) \sin(q_3) \\ \frac{KM_2 \sin(q_3)}{\cos^2(q_3)} & -\frac{KM_1}{\cos(q_3)} \end{pmatrix},$$

$$\Delta(q) = K^2M_1M_2 - JM_1 \cos^2(q_3) - JM_2 \sin^2(q_3).$$

The control laws developed above can be iterated, after the system has been transformed into the ECF, in order to stabilize the origin  $x = 0$ . To this end, at each sample time  $t_k = kT$  one uses the measurements of the state variables to calculate  $x(t_k) = \varphi(q(t_k), \dot{q}(t_k))$ , then the prescribed control law  $u(t) = \alpha(x(t_k), t)$  is computed from (9)–(10). The actual force used to drive the system is obtained by using the inverse transformation  $\tau(t) = [B(q)]^{-1}(u(t) - A(q, \dot{q}))$ .

When the system parameters are not accurately known, which is most often the case, the functions  $\varphi$ ,  $A$ , and  $B$  typically include additional terms. For the sake of illustration let us suppose that uncertainties are present in the values of the (cumulated) masses  $M_i$ , the position of the third link’s center of mass  $l$ , and its inertia



moment  $J$ . This entails that only the erroneous values  $\widetilde{M}_i = M_i + \nu_i$ ,  $\widetilde{l} = l + \nu_4$ , and  $\widetilde{J} = J + \nu_5$ , where  $\nu = (\nu_1, \dots, \nu_5) \in \mathbb{R}^5$  represents the parameter errors, are available to the controller. Note that, if one sets  $\varepsilon = \|\nu\|^2$ , the norm of the error tends to zero as  $\varepsilon \rightarrow 0$ . Ultimately, the effect of the inaccuracies results in disturbance vector fields  $h = (h_0^\varepsilon, h_1^\varepsilon, h_2^\varepsilon)$  being added to the nominal ECF system, yielding a perturbed system in the form of (3). Using a computer algebra package, one readily verifies that for  $i = 1, 2, 3$ , the mappings  $(x, \varepsilon) \mapsto h_i^\varepsilon(x)$  are analytic and have the following structures:

$$\begin{aligned} h_0^\varepsilon(x) &= x_4^2(a_{2,0}^\varepsilon + O(|x_3|^2)) \frac{\partial}{\partial x_2} + x_4^2(a_{4,3}^\varepsilon x_3^3 + O(|x_3|^5)) \frac{\partial}{\partial x_4} + x_4^2(a_{6,1}^\varepsilon x_3 + O(|x_3|^2)) \frac{\partial}{\partial x_6}, \\ h_1^\varepsilon(x) &= (b_{2,0}^\varepsilon + O(|x_3|^2)) \frac{\partial}{\partial x_2} + (b_{4,1}^\varepsilon x_3 + O(|x_3|^3)) \frac{\partial}{\partial x_4} + (b_{6,1}^\varepsilon x_3 + O(|x_3|^3)) \frac{\partial}{\partial x_6}, \\ h_2^\varepsilon(x) &= (c_{2,1}^\varepsilon x_3 + O(|x_3|^3)) \frac{\partial}{\partial x_2} + (c_{4,0}^\varepsilon + O(|x_3|^2)) \frac{\partial}{\partial x_4} + (c_{6,0}^\varepsilon + O(|x_3|^2)) \frac{\partial}{\partial x_6}, \end{aligned}$$

where the symbols  $a_{i,j}^\varepsilon$ ,  $b_{i,j}^\varepsilon$ , and  $c_{i,j}^\varepsilon$  represent real numbers which vanish when  $\varepsilon = 0$  but are nonzero for generic parameter and error values. This implies that  $\text{Ord}(h_0^\varepsilon) = 1$  and  $\text{Ord}(h_1^\varepsilon) = \text{Ord}(h_2^\varepsilon) = -1$ , so the assumptions in Proposition 3.1 are verified. As a result, the iterated application of the control laws (9)–(10) will ensure that the origin of the dynamically extended system (6) is locally exponentially stable, provided  $\varepsilon$  is *small enough*.

Now consider a PPR manipulator whose *nominal*, physical dimensions are as follows. The three masses are equal:  $m_1 = m_2 = m_3 = 10$  kg. The third link is a homogeneous parallelepiped of length  $\ell = 1.5$  m and width  $w = 0.15$  m; its center of mass is located at a distance  $l = \ell/2 = 0.75$  m from the joint axis and its inertia moment is  $J = (\ell^2/3 + w^2/12)m_3 = 7.51875$  kg m<sup>2</sup>. The goal is to stabilize the system to the equilibrium configuration  $(q, \dot{q}) = (0, 0)$  starting at rest ( $\dot{q}_0 = 0$ ) from the initial configuration  $q_0 = (-50$  cm,  $75$  cm,  $\pi/4$ ). A convenient DOF, useful for fine-tuning the transient response, is encompassed by the choice of the controller settings ( $T$ ,  $G$ , and the  $a_i$ 's), which can be made with the aid of some intuitively deduced “rules of thumb.”  $T$  controls the length of the periods during which the system operates in open-loop; smaller values of  $T$  lead to more frequent updates of the feedback terms.  $G$  moderates the control effort exerted on the system due to the oscillatory, time-varying terms; large values of  $G$  lead to shorter settling time (to within a given tolerance) but may require larger control efforts. The values of  $a_i$  set the position of the poles  $\{k_{i1}, k_{i2}\}$ , within the unit circle in  $\mathbb{C}$ , for each of the submatrices  $A_i$  in (12). As can be expected, the closer the poles are to the origin, the shorter the settling time is, but also the larger the control effort becomes. In these simulations the settings are  $\omega = 1$  rad/s, so  $T = 2\pi \approx 6.28$  s;  $G = 0.1$  and  $k_{i,j} = 0.25$  ( $i = 1, 2, 3$ ,  $j = 1, 2$ ); the gain values  $a_1 = a_3 = a_5 \simeq -0.01425$  and  $a_2 = a_4 = a_6 \simeq -0.194$  were determined from (13). In order to perform the numerical simulation in the perturbed case, it is assumed that  $\widetilde{m}_3 = 1.1m_3$  and  $\widetilde{l} = 0.95l$ ; that is, errors of 10% and -5%, respectively, are present in the knowledge of these two parameters. The latter induce an error of -0.7% in the moment of inertia, so that  $\widetilde{J} = 0.993J$ . The response of the perturbed system controlled by (9)–(10) appears in Figure 3, which shows the time history of  $\log(\|(q(t), \dot{q}(t))\|)$ , the configuration variables  $q(t)$ , and velocities  $\dot{q}(t)$ , as well as the input forces  $\tau(t)$ . The differences between the transient responses in the perturbed and nominal cases are barely perceptible, so no simulation for the latter case is included. In order to assess the improved performance of the control law (9)–(10) in the presence of disturbances, let us end this example with a qualitative

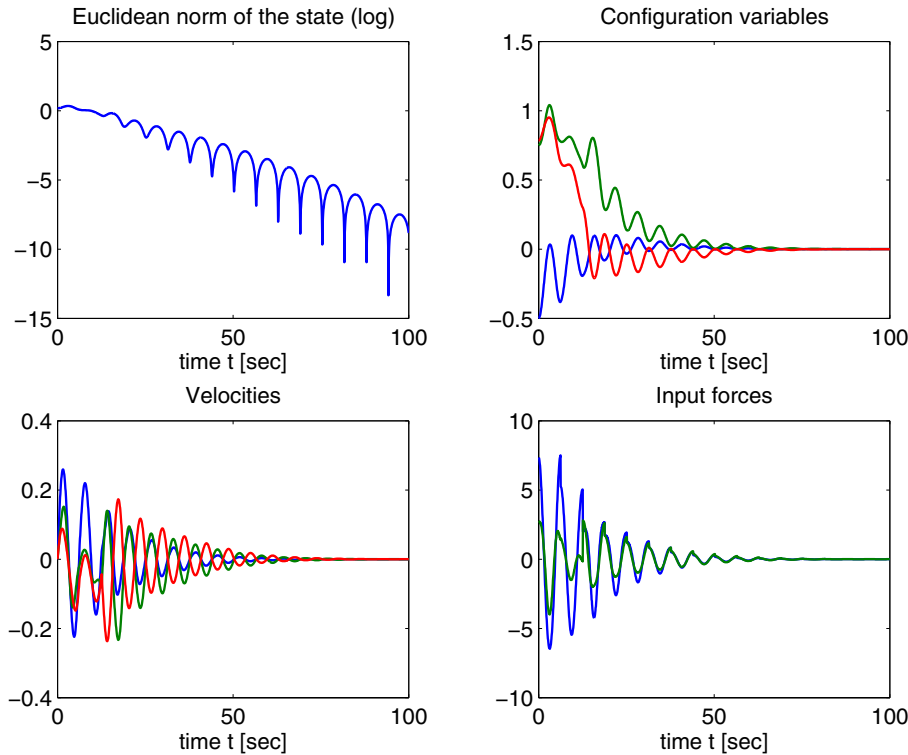


FIG. 3. Transient response of the perturbed  $PP\bar{R}$  manipulator using the hybrid control law (9)–(10).

comparison with another control approach. Recall that in [1], a homogeneous, time-varying feedback law was introduced which  $\rho$ -exponentially stabilizes the ECF to the origin. Nevertheless, by virtue of the main result in [13], these control laws are not robust to disturbances in  $\mathcal{D}^3$  and, in fact, as illustrated in Figure 4, the disturbances considered in this example make the system's solution tend towards what seems to be a limit cycle (in particular the origin is not Lyapunov-stable).

**4.2. Simplified surface vessel.** Consider a simplified surface vessel with configuration variables  $(x, y, \theta)$ , as depicted in Figure 5. Research studies concerning this system are reported in several references, including [23], where more details on the modeling assumptions can be found. In particular, it is shown in that reference that the corresponding dynamic model can be written in the form

$$(22) \quad \begin{aligned} \ddot{x} &= u_1, \\ \ddot{\theta} &= u_2, \\ \ddot{y} &= u_1 \tan(\theta) + \frac{c_y}{m}(-\dot{y} + \tan(\theta)\dot{x}). \end{aligned}$$

Clearly this can be viewed as a perturbed ECF system. More precisely, by setting  $\varepsilon = c_y/m$  and relabeling the state variables  $(x_1, \dots, x_6) = (x, \dot{x}, \theta, \dot{\theta}, y, \dot{y})$  one can also write system (22) as

$$(23) \quad \dot{x} = b_0(x) + h_0^\varepsilon(x) + \sum_{i=1}^2 u_i(b_i(x) + h_i^\varepsilon(x)),$$

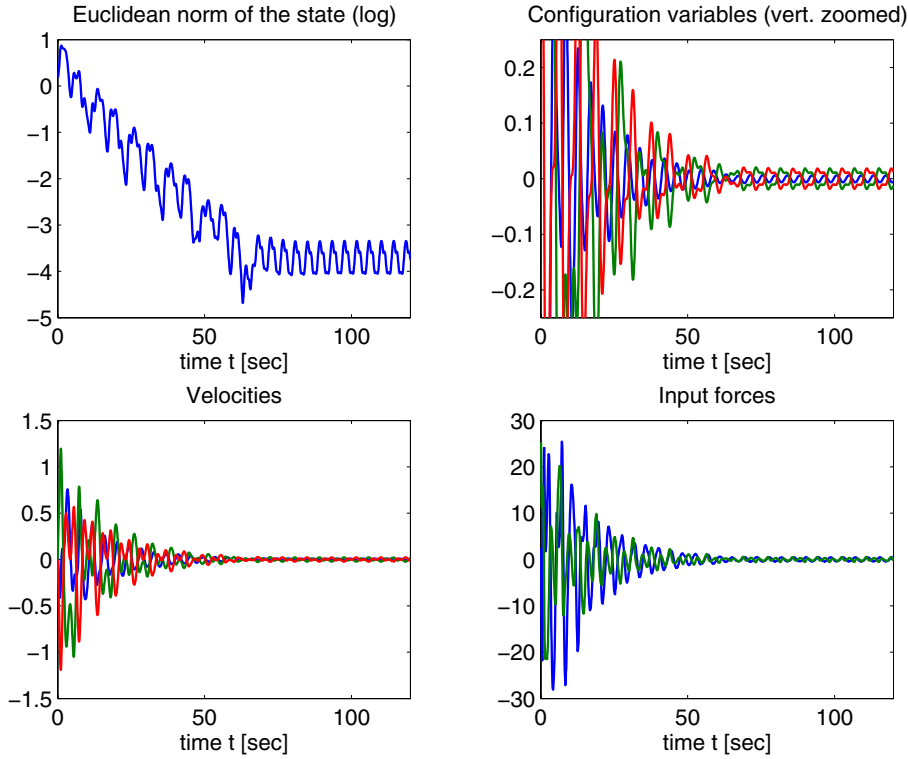


FIG. 4. Transient response of the perturbed  $PP\bar{R}$  manipulator using the continuous, homogeneous, time-varying feedback from [1]. The time histories of the configuration variables are plotted with a different scale to illustrate their ultimately oscillatory nature.

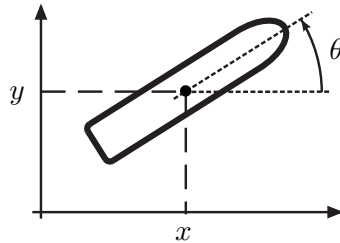


FIG. 5. Configuration variables for the simplified surface vessel model.

with  $b_0, b_1, b_2$  given by (2), and the disturbance vector fields defined by

$$(24) \quad h_0^\varepsilon(x) = \varepsilon(-x_6 + x_2 \tan(x_3)) \frac{\partial}{\partial x_6}, \quad h_1^\varepsilon(x) = (\tan(x_3) - x_3) \frac{\partial}{\partial x_6}, \quad h_2^\varepsilon(x) = 0.$$

Obviously, the family  $\mathbf{h} = (h_0^\varepsilon, h_1^\varepsilon, h_2^\varepsilon)$  is a disturbance in  $\mathcal{D}^3$ , but it is not contained in the set  $\mathcal{A}$  defined in Proposition 3.1 since  $\text{Ord}(h_{0,6}^\varepsilon) = 1$ , i.e.,  $\text{Ord}(h_0^\varepsilon) = 0$ . Let us show, however, that  $\mathbf{h}$  belongs to  $\mathcal{A}'$  and hence that it satisfies the assumptions of Conjecture 1. To this end, let  $g(x) = \varepsilon(-x_6 + x_2 \tan(x_3))$ , so that  $h_0^\varepsilon(x) = g(x) \partial / \partial x_6$ . Note that  $\text{Ord}(h_1^\varepsilon) = 2$  and  $\text{Ord}(h_2^\varepsilon) = +\infty$ ; hence we need only certify that all terms

$XY_1 \cdots Y_k \text{id}$ , with  $X \in \{b_1, b_2\}$ ,  $k \geq 2$  and  $Y_1, \dots, Y_k \in \{b_0, h_0^\varepsilon\}$ , satisfy

$$(25) \quad \text{Ord}(XY_1 \cdots Y_k \text{id}) \geq 1.$$

Since  $b_1\phi(x) = \partial\phi/\partial x_2 + x_3\partial\phi/\partial x_6$  and  $b_2\phi(x) = \partial\phi/\partial x_4$  for any smooth function  $\phi$ , a necessary condition to have  $\text{Ord}(b_1\phi) \geq 1$  and  $\text{Ord}(b_2\phi) \geq 1$  is that  $\text{Ord}(\partial\phi/\partial x_2) \geq 1$  and  $\text{Ord}(\partial\phi/\partial x_4) \geq 1$ . Naturally, this necessary condition holds whenever  $\phi = Y_1 \cdots Y_k \text{id}$  and  $\text{Ord}(\phi) \geq 2$ . In what follows we shall show that it holds even when the latter is not the case. One has

$$b_0 \text{id}_i(x) = \begin{cases} x_{i+1}, & i = 1, 3, 5, \\ 0, & i = 2, 4, 6, \end{cases} \quad \text{and} \quad h_0^\varepsilon \text{id}_i(x) = \begin{cases} 0, & i = 1, \dots, 5, \\ g(x), & i = 6, \end{cases}$$

from which it follows that  $b_0 b_0 \text{id} = 0$ ,  $h_0^\varepsilon b_0 \text{id}_i = 0$  for  $i \neq 5$ , and  $b_0 h_0^\varepsilon \text{id}_i = h_0^\varepsilon h_0^\varepsilon \text{id}_i = 0$  for  $i \neq 6$ . Furthermore,

$$h_0^\varepsilon b_0 \text{id}_5(x) = g(x), \quad b_0 h_0^\varepsilon \text{id}_6(x) = x_4 \frac{\partial g}{\partial x_3}(x) \quad \text{and} \\ h_0^\varepsilon h_0^\varepsilon \text{id}_6(x) = g(x) \frac{\partial g}{\partial x_6}(x) = -\varepsilon g(x).$$

By direct calculation one obtains that

$$(26) \quad \frac{\partial g}{\partial x_2}(x) = -\varepsilon \tan(x_3), \quad \frac{\partial g}{\partial x_4}(x) = 0,$$

$$(27) \quad \frac{\partial}{\partial x_2} \left( x_4 \frac{\partial g}{\partial x_3}(x) \right) = x_4 \frac{\partial^2 g}{\partial x_2 \partial x_3}(x), \\ \frac{\partial}{\partial x_4} \left( x_4 \frac{\partial g}{\partial x_3}(x) \right) = \frac{\partial g}{\partial x_3}(x) = \varepsilon x_2 (1 + \tan^2(x_3)).$$

The orders  $\text{Ord}(\cdot)$  of all of these functions being  $\geq 1$ , the required condition (25) is satisfied for  $k = 2$ . Now consider the case  $k \geq 3$  and note that, since  $\text{Ord}(b_0 b_0 \text{id}) = +\infty$  and  $\text{Ord}(b_0 h_0^\varepsilon \text{id}) = 2$ , all terms  $XY_1 \dots Y_k \text{id}$  which end with  $b_0 b_0 \text{id}$  or with  $b_0 h_0^\varepsilon \text{id}$  satisfy (25) for  $k \geq 3$ . Moreover,  $b_0 h_0^\varepsilon b_0 \text{id}_5 = b_0 h_0^\varepsilon \text{id}_6$  and  $b_0 h_0^\varepsilon h_0^\varepsilon \text{id}_5 = -\varepsilon b_0 h_0^\varepsilon \text{id}_6$ , so those terms that end with  $b_0 h_0^\varepsilon h_0^\varepsilon \text{id}$  and  $b_0 h_0^\varepsilon h_0^\varepsilon \text{id}$  also satisfy (25) for  $k \geq 3$ . It remains only to consider terms ending with  $h_0^\varepsilon h_0^\varepsilon b_0 \text{id}$  and  $h_0^\varepsilon h_0^\varepsilon h_0^\varepsilon \text{id}$ . But  $h_0^\varepsilon h_0^\varepsilon b_0 \text{id}_5 = h_0^\varepsilon h_0^\varepsilon \text{id}_6$ ; thus one needs only to analyze terms of the form  $X(h_0^\varepsilon)^\ell \text{id}_6$  and  $Xb_0(h_0^\varepsilon)^\ell \text{id}_6$ . A routine calculation yields, for  $\ell \geq 1$ ,

$$(28) \quad (h_0^\varepsilon)^\ell \text{id}_6(x) = (-\varepsilon)^{\ell-1} g(x) \quad \text{and} \quad b_0(h_0^\varepsilon)^\ell \text{id}_6(x) = (-\varepsilon)^{\ell-1} x_4 \frac{\partial g}{\partial x_3}(x).$$

Hence, in view of (26)–(28), those terms also satisfy (25) for every  $k \geq 3$ . Consequently  $\mathbf{h} \in \mathcal{A}'$ . A numerical simulation of system (23) with the controller (9)–(10) is shown in Figure 6. For this simulation the size of the error is taken to be  $\varepsilon = c_y/m = 0.1$ , the initial condition is  $x = (1, 0, \pi/4, 0, -1, 0)$ , and the controller settings are  $\omega = 2\pi/T = 1.5$  rad/s,  $G = 1$ , and  $k_{i,j} = 0.1$ ,  $i = 1, 2, 3$ ,  $j = 1, 2$ . The gain values  $a_1 = a_3 = a_5 \simeq -0.0462$  and  $a_2 = a_4 = a_6 \simeq -0.333$  were determined using (13). As depicted in the time-plots, the simulation appears to validate Conjecture 1.

**5. Conclusions.** A controller scheme, based on well-known hybrid open-loop/feedback techniques, has been introduced for the ECF. This controller exponentially

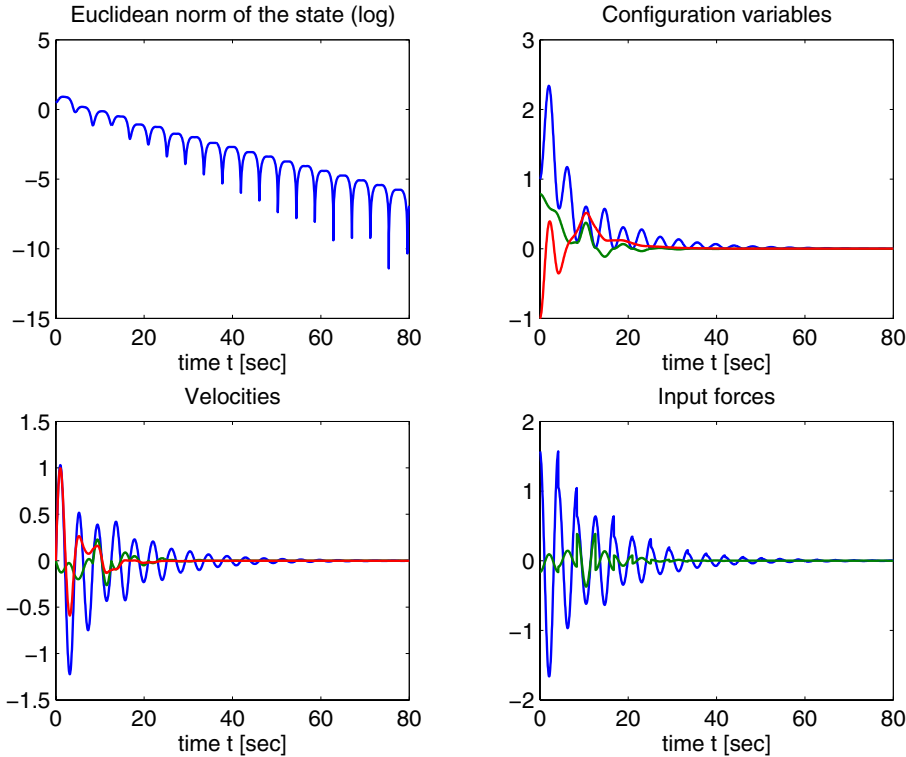


FIG. 6. Transient response of the system (23)—equivalent to the simplified surface vessel model—controlled by (9)–(10).

stabilizes the origin of a dynamic extension of the ECF, with robustness to a class of additive disturbance vector fields. The class of disturbances includes analytic vector fields added to the control vector fields as well as “high-order” drift perturbations. One positive feature of these results is that, for a class of underactuated systems—whose models need not be feedback-equivalent to the ECF—the problem of local point stabilization with exponential convergence can be effectively tackled by using the same control scheme as for the ECF. The typical performance of the proposed control laws seems qualitatively acceptable, as illustrated by the numerical simulations. On the other hand, these controllers clearly have some limitations regarding their robustness, and instability may be induced by disturbances not contained in the class  $\mathcal{A}$  of Proposition 3.1 or by disturbances of a different nature, such as errors in the update time of the control.

A problem that remains open is the extension of the approach in this paper to systems with more inputs and less structure than the ECF. Such an extension would typically involve a design and an analysis stage, the former yielding control laws that stabilize the origin of a dynamically extended, *nominal* system—analogue to (6), but with  $h_i^e = 0$ ,  $i = 1, \dots, m$ . The design stage, of an essentially algebraic nature, might be based on techniques related to the design of oscillatory open-loop controls, such as the ones developed in [12]. By contrast, the analysis can be expected to be significantly involved, all the more so as it would be desirable to guarantee robustness to a large class of admissible disturbances.

**6. Appendix.**

**6.1. Notational conventions.**

**6.1.1. Local order of mappings.** Let us recall some definitions and properties about *local order* of mappings, a notion that simplifies the proofs. In this paragraph,  $n$  and  $m$  represent positive integers,  $\ell$  represents a nonnegative integer, and  $\|\cdot\|$  represents Euclidean norm. Given open sets  $U \subset \mathbb{R}^n$  and  $V \subset \mathbb{R}^m$ , the symbols  $PC(U; V)$ ,  $C^0(U; V)$ ,  $C^\infty(U; V)$ , and  $C^\omega(U; V)$  denote the sets of *piecewise-continuous*, *continuous*, *smooth*, and *(real-)analytic* mappings from  $U$  to  $V$ , respectively. Consider a neighborhood  $U$  of the origin in  $\mathbb{R}^n$ . We deal with mappings defined on  $U \times \Lambda$ , where  $\Lambda \subset \mathbb{R}^\ell$ , and view the elements of  $\Lambda$  as parameters (e.g., “time” or other parameters). Given a mapping  $f : U \times \Lambda \rightarrow \mathbb{R}^m$ , we write  $f(x, \lambda) = o(\|x\|^k)$  if, for every  $\lambda \in \Lambda$ ,

$$(29) \quad \lim_{x \rightarrow 0} \frac{\|f(x, \lambda)\|}{\|x\|^k} = 0.$$

We write  $f(x, \lambda) = O(\|x\|^k)$  if for every  $\lambda \in \Lambda$  there is a constant  $K > 0$  and a neighborhood  $U' \subset U$  of the origin such that, for every  $x \in U' \setminus \{0\}$ ,

$$(30) \quad \frac{\|f(x, \lambda)\|}{\|x\|^k} \leq K.$$

Consider a mapping  $X = (X_1, \dots, X_n) : U \times \Lambda \rightarrow \mathbb{R}^n$  representing a family of *vector fields*  $X(\cdot, \lambda) : U \rightarrow \mathbb{R}^n$ . We write  $X(x, \lambda) = o(\|x\|^k)$  (resp.,  $X(x, \lambda) = O(\|x\|^k)$ ) if  $X_i(x, \lambda) = o(\|x\|^{k+1})$  (resp.,  $X_i(x, \lambda) = O(\|x\|^{k+1})$ ) for  $i = 1, \dots, n$ . We shall also use the function  $\text{Ord} : f \mapsto \text{Ord}(f) \in \mathbb{R} \cup \{+\infty\}$  defined by  $\text{Ord}(f) = \sup\{k \in \mathbb{R} : f(x, \lambda) = O(\|x\|^k)\}$ . Every vector field  $X(\cdot, \lambda)$  is a *differential operator* acting on  $C^\infty(U; \mathbb{R})$ ; thus, for  $\phi \in C^\infty(U; \mathbb{R})$  one has  $X\phi(\cdot, \lambda) \in C^\infty(U; \mathbb{R})$ , where  $X\phi(x, \lambda) = L_X\phi(x, \lambda) = \sum_{i=1}^n \frac{\partial \phi}{\partial x_i}(x, \lambda)$  denotes the Lie derivative of  $\phi$  in the direction of  $X$  evaluated at  $(x, \lambda)$ . We extend this notation to the case when  $\phi \in C^\infty(U; \mathbb{R}^m)$  and use  $X\phi(\cdot, \lambda)$  to denote the  $m$ -tuple  $(X\phi_i)_{i=1, \dots, m}$  of functions  $X\phi_i(\cdot, \lambda) \in C^\infty(U; \mathbb{R})$ . The following properties are easily established:

LEMMA 6.1. *Assume that, for every  $\lambda \in \Lambda$ ,  $f(\cdot, \lambda), g(\cdot, \lambda)$  are  $C^\infty$  mappings  $U \rightarrow \mathbb{R}^m$ , and  $X(\cdot, \lambda), Y(\cdot, \lambda)$  are  $C^\infty$  vector fields  $U \rightarrow \mathbb{R}^n$ . Write  $\mu$  to denote any of these mappings. Then the following hold:*

- (i)  $\text{Ord}(f) \geq 0, \text{Ord}(X) \geq -1$ .
- (ii) *If  $k \in \mathbb{R}$  and  $k \leq \text{Ord}(\mu)$ , then  $\mu(x, \lambda) = O(\|x\|^k)$ .*
- (iii)  $\text{Ord}(f + g) \geq \min\{\text{Ord}(f), \text{Ord}(g)\}$  (where  $(f + g)(x, \lambda) = f(x, \lambda) + g(x, \lambda)$ ).
- (iv)  $\text{Ord}(fg) = \text{Ord}(f) + \text{Ord}(g)$  (where  $fg(x, \lambda) = f(x, \lambda)g(x, \lambda)$ ).
- (v)  $\text{Ord}(Xf) \geq \text{Ord}(X) + \max\{\text{Ord}(f), 1\}$ . In particular  $\text{Ord}(Xf) \geq 0$ .

**6.1.2. Iterated differential operators and iterated integrals.** Assume that  $U \subset \mathbb{R}^n$  is open. Let  $\mathbf{X} = (X_1, \dots, X_m)$  be a family of real-analytic vector fields  $X_i \in C^\omega(U; \mathbb{R}^n)$ , and  $\phi \in C^\omega(U; \mathbb{R})$  be a real-analytic function. Every element of  $\mathcal{I}_{[0, m]} = \bigcup_{k \in \{0, 1, 2, \dots\}} \{0, 1, \dots, m\}^k$  is called a *multi-index*. If  $I = (i_1, \dots, i_r) \in \{0, 1, \dots, m\}^r$ , the multi-index  $I$  is said to have *length*  $r$ , and this is denoted by  $|I| = r$ . By convention,  $I = \emptyset$  is regarded as a multi-index having zero length.

Let  $I = (i_1, \dots, i_r) \in \mathcal{I}_{[0, m]}$  be a multi-index. The *iterated differential operator*  $X_I = X_{i_1} \cdots X_{i_r}$  is defined so that the function  $X_I\phi \in C^\omega(U; \mathbb{R})$  is given by  $X_{i_1} \cdots X_{i_r}\phi$  (each vector field regarded as a first-order differential operator). By convention one sets  $X_\emptyset\phi = \phi$ . We use  $X_I\text{id} : U \rightarrow \mathbb{R}^n$  to denote the  $n$ -tuple

of functions  $(X_I \text{id}_i)_{i=1, \dots, n}$ , where  $\text{id} : U \rightarrow \mathbb{R}^n$  is defined by  $\text{id}_i(x) = x_i$  for  $x = (x_1, \dots, x_n) \in U$ . Given  $\alpha \in C^0(U \times \mathbb{R}; \mathbb{R}^m)$  (e.g., a time-varying feedback law), a multi-index  $I = (i_1, \dots, i_r)$ , and real numbers  $t_0, t$ , one defines the *iterated integral*  $\int_{t_0}^t \alpha_I : U \rightarrow \mathbb{R}$  as follows:

$$\int_{t_0}^t \alpha_I(x) = \int_{t_0}^t \int_{t_0}^{t_r} \cdots \int_{t_0}^{t_2} \alpha_{i_r}(x, t_r) \alpha_{i_{r-1}}(x, t_{r-1}) \cdots \alpha_{i_1}(x, t_1) dt_1 \cdots dt_r.$$

By convention,  $\int_{t_0}^t \alpha_\emptyset(x) = 1$  for every  $x \in U$ .

**6.2. Auxiliary lemmas.**

**6.2.1. Proof of Lemma 3.2.** Since the result is local we may assume, without loss of generality, that  $M$  is an open subset of  $\mathbb{R}^n$  and that  $\bar{x} = 0$ . We shall appeal to the following two technical lemmas; for improved readability, the proof of Lemma 6.2 is relegated to section 6.2.4, whereas Lemma 6.3 follows from a trivial adaptation of the proof of [10, Theorem 2.6].

LEMMA 6.2. *Let  $f_0, \dots, f_m$  be real-analytic vector fields on a real-analytic manifold  $M$ , with  $\bar{x} \in M$ , and let  $\phi : M \rightarrow \mathbb{R}$  be a real-analytic function. Assume that  $f_0(\bar{x}) = 0$ . Then there is a constant  $C > 0$  with the property that, for every  $\eta > 0$ , there exists a neighborhood  $K$  of  $\bar{x}$  such that  $\phi$  and the vector fields  $g_0 = (1/\eta)f_0, g_i = f_i$  ( $i = 1, \dots, m$ ) satisfy the estimate*

$$|(g_{i_1} \cdots g_{i_r} \phi)(x)| \leq C^r r!$$

for every  $x \in K$  and every multi-index  $I = (i_1, \dots, i_r) \in \{0, \dots, m\}^r$  of length  $r \geq 1$ .

LEMMA 6.3. *Let  $f \in C^0(U \times \Lambda \times [t_0, t_1]; \mathbb{R}^n)$ , where  $U \subset \mathbb{R}^n$  is open and connected,  $\Lambda \subset \mathbb{R}^m$  is compact, and  $t_0 < t_1$ . Assume that  $(x_0, \lambda_0) \in U \times \Lambda$  and that (i)  $f(\cdot, \lambda, t)$  is locally Lipschitz on  $U$ , uniformly for  $(\lambda, t) \in \Lambda \times [t_0, t_1]$ , and (ii)  $y : [t_0, t_1] \rightarrow U$  is a solution to  $\dot{y} = f(y, \lambda_0, t)$ , with  $y(t_0) = x_0$ . Then, given  $\varepsilon > 0$ , there are compact neighborhoods  $U' \subset U$  and  $\Lambda' \subset \Lambda$  of  $x_0$  and  $\lambda_0$ , respectively, such that for every  $x \in U'$  and every function  $\varphi \in PC([t_0, t_1]; \Lambda')$ , the system  $\dot{z} = f(z, \varphi(t), t)$  admits a unique solution  $z : [t_0, t_1] \rightarrow U$  which satisfies  $z(t_0) = x$  and  $\|z(t) - y(t)\| \leq \varepsilon$  for all  $t \in [t_0, t_1]$ .*

Fix  $t_0 \in \mathbb{R}$ . Let  $C > 0$  be the constant whose existence is guaranteed by Lemma 6.2 above, and define  $\eta > 0$  such that  $CT(m + 1)^{\frac{3}{2}}\eta < 1$ . Setting  $g_0 = \frac{1}{\eta}f_0$  and  $g_i = f_i, i = 1, \dots, m$ , we apply Lemma 6.2 again to deduce that there is a neighborhood  $K$  of  $0 \in \mathbb{R}^n$  such that  $|g_I \phi(x)| \leq C^r r!$  for every  $x \in K$  and every multi-index  $I$  of length  $r \geq 1$ . Moreover, by defining  $F(x, v, t) = \sum_{i=0}^m g_i(x)v_i$ , with  $v = (v_0, \dots, v_m)$ , we see that  $F$  satisfies the assumptions of Lemma 6.3 if one takes  $\lambda_0 = (\eta, 0, \dots, 0) \in \mathbb{R}^{m+1}$  and  $y : t \mapsto 0 \in \mathbb{R}^n$ . Therefore, there exists a constant  $\delta' \in (0, \eta)$  such that if  $x_0 \in \mathbb{R}^n$ , with  $\|x_0\| < \delta'$ , and if  $v$  is a piecewise-continuous function on  $[t_0, t_0 + T]$  taking values in  $\{u \in \mathbb{R}^m : \|u\| < \delta'\}$ , then the solution to  $\dot{z} = g_0(z)\eta + \sum_{i=1}^m g_i(z)v_i(t)$  with initial value  $z(t_0) = x_0$  satisfies  $z(t) \in K$  for  $t \in [t_0, t_0 + T]$ . But since  $\alpha(x, t) \rightarrow 0$  as  $x \rightarrow 0$ , uniformly for  $t \in \mathbb{R}$ , there exists  $\delta \in (0, \delta')$  such that  $\|\alpha(x, t)\| < \delta'$  whenever  $\|x\| < \delta$  and  $t \in \mathbb{R}$ . It follows that if  $\|x_0\| < \delta$ , then the solution to system  $\dot{x} = f_0(x) + \sum_{i=1}^m \alpha_i(x_0, t)f_i(x), x(t_0) = x_0$ , rewritten as  $\dot{x} = g_0(x)\eta + \sum_{i=1}^m g_i(x)\alpha_i(x_0, t)$ , satisfies  $\pi(t, t_0, x_0) \in K$  for  $t \in [t_0, t_0 + T]$ . Note that, by denoting  $v(x, t) = (\eta, \alpha(x, t))$ , one has  $\|v(x, t)\| < (m + 1)^{\frac{1}{2}}\eta$  for  $(x, t) \in K \times [t_0, t_0 + T]$ . On the other hand, the difference between the

$N$ th partial sum of the Chen–Fliess expansion  $\text{Ser}_{\phi, \mathbf{f}, \alpha}^N(t, t_0, x_0)$  and the actual value of  $\phi$  along that solution is (cf. [27, section 4])

$$|\text{Ser}_{\phi, \mathbf{f}, \alpha}^N(t, t_0, x_0) - \phi(\pi(t, t_0, x_0))| \leq \frac{((m + 1)^{\frac{1}{2}} \eta(t - t_0))^{N+1}}{(N + 1)!} (m + 1)^{N+1} \sup\{|g_I \phi(x)| : x \in K\}.$$

But  $\sup\{|f_I \phi(x)| : x \in K\} < C^{N+1}(N + 1)!$  so

$$|\text{Ser}_{\phi, \mathbf{f}, \alpha}^N(t, t_0, x_0) - \phi(\pi(t, t_0, x_0))| \leq (C(m + 1)^{\frac{3}{2}} \eta(t - t_0))^{N+1}.$$

Since  $t \in [t_0, t_0 + T]$ , one has  $C(m + 1)^{\frac{3}{2}} \eta(t - t_0) < 1$ ; hence the series converges uniformly. It is readily checked that the series is absolutely convergent as well.  $\square$

**6.2.2. Proof of Lemma 3.3.** (i) Let  $U' \subset \mathbb{R}^6$  be a compact set containing 0. From the continuity of  $\alpha$ , the  $T$ -periodicity of  $t \mapsto \alpha(x, t)$ , and the definition of  $\rho$ , which implies that  $\rho(x) = O(\|x\|^{\frac{1}{2}})$ , one deduces that  $\|\alpha_i(x, t)\|/\|x\|^{\frac{1}{2}}$  ( $i = 1, 2$ ) is bounded, say, by  $K' > 0$ , for every  $(x, t) \in U' \times \mathbb{R}$ . Thus the claim holds for any  $K > K'$ .

(ii) Set  $\alpha_0 = 1$  and write  $\alpha_i(x, t) = U_i(x) + V_i(x) \cos(\omega t)$ ,  $i = 0, 1, 2$ , with  $U_0 = 1$ ,  $V_0 = 0$ , and  $U_1, U_2, V_1, V_2$  defined in the obvious way. Note that  $\text{Ord}(U_0) = 0$ ,  $\text{Ord}(U_1) = \text{Ord}(U_2) = 1$ , and  $\text{Ord}(V_1) = \text{Ord}(V_2) = 1/2$ . If  $I = (i) \in \{1, 2\}$ , then  $\int_0^T \alpha_I(x_0) = U_i(x)T + V_i(x) \int_0^T \cos(\omega \tau) d\tau = U_i(x)T$ , so  $\int_0^T \alpha_I(x_0) = O(\|x_0\|)$ . If  $I = (i, j) \in \{0, 1, 2\}^2$ , then

$$\begin{aligned} \int_0^T \alpha_I(x_0) &= \frac{1}{2} U_i \cdot U_j(x_0) T^2 + U_i \cdot V_i(x_0) \int_0^T \int_0^{t_2} \cos(\omega t_1) dt_1 dt_2 \\ &\quad + U_j \cdot V_i(x_0) \int_0^T \tau \cos(\omega \tau) d\tau + V_i \cdot V_j(x_0) \int_0^T \cos(\omega t_2) \int_0^{t_2} \cos(\omega t_1) dt_1 dt_2 \\ &= \frac{1}{2} U_i \cdot U_j(x_0) T^2, \end{aligned}$$

since the three integrals indicated on the right member of this equation vanish. But then, if  $I = (i, j) \neq (0, 0)$ , one gets  $\text{Ord}(\int_0^T \alpha_I) = \text{Ord}(U_i) + \text{Ord}(U_j) \geq 1$ , so  $\int_0^T \alpha_I(x_0) = O(\|x_0\|)$ .

(iii) One easily shows that if a function  $v \in C^0(U \times \mathbb{R}; \mathbb{R})$  satisfies  $v(x_0, t) = O(\|x_0\|^\ell)$ , then  $\int_0^t v(x_0, \tau) d\tau = O(\|x_0\|^\ell)$  for every  $t \in \mathbb{R}$ . By writing  $\text{Ord}(\alpha_j) = k_j$ , with  $k_0 = 0$  and  $k_1 = k_2 = \frac{1}{2}$ , one gets

$$(31) \quad \int_0^t \alpha_j(x_0, t_2) \int_0^{t_2} v(x_0, t_1) dt_1 dt_2 = O(\|x_0\|^{k_j + \ell}), \quad j = 0, 1, 2.$$

Using these facts and an induction argument, one readily deduces that  $\text{Ord}(\int_0^T \alpha_I) \geq \sum_{j=1}^{|I|} k_{i_j}$ .

(iv) This is verified directly by induction on the length of  $I$  using the fact that, for fixed  $T \in \mathbb{R}$ ,  $x_0 \mapsto \int_0^T f(x_0, \tau) d\tau$  is continuous whenever  $f \in C^0(U \times \mathbb{R}; \mathbb{R})$ .

(v) This claim follows immediately by inspecting the components of  $b_0, b_1$ , and  $b_2$  as defined in (2), and from the definition of the set  $\mathcal{A}$ .



(vi) It suffices to show (by induction) that for any  $x \in U$

$$(32) \quad b_0^k \phi(x) = \underbrace{b_0 \cdots b_0}_{k \text{ times}} \phi(x) = \sum_{\mu \in \{1,2,3\}^k} x_{2\mu} \frac{\partial^k \phi}{\partial x_{2\mu-1}}(x),$$

where we write  $x_{2\mu} = x_{2\mu_1} \cdots x_{2\mu_k}$  and  $x_{2\mu-1} = x_{2\mu_1-1} \cdots x_{2\mu_k-1}$  for any multi-index  $\mu \in \{1, 2, 3\}^k$ . Indeed, (v) follows from (32) since every term  $f_\mu(x) = x_{2\mu} \partial^k \phi / \partial x_{2\mu-1}(x)$  in the sum satisfies  $\text{Ord}(f_\mu) \geq k$ . Using (2), one gets  $b_0 \phi(x) = \sum_{i=1}^3 x_{2i} \partial \phi / \partial x_{2i-1}(x)$ , i.e., (32) with  $k = 1$ . If (32) holds for  $k = m \geq 1$ , then

$$\begin{aligned} b_0(b_0^m \phi)(x) &= \sum_{i=1}^3 x_{2i} \left( \sum_{\mu \in \{1,2,3\}^m} \frac{\partial x_{2\mu}}{\partial x_{2i-1}} \frac{\partial^m \phi}{\partial x_{2\mu-1}}(x) + x_{2\mu} \frac{\partial^{m+1} \phi}{\partial x_{2i-1} \partial x_{2\mu-1}}(x) \right) \\ &= \sum_i \sum_\mu x_{2i} x_{2\mu} \frac{\partial^{m+1} \phi}{\partial x_{2i-1} \partial x_{2\mu-1}}(x) = \sum_{\mu \in \{1,2,3\}^{m+1}} x_{2\mu} \frac{\partial^{m+1} \phi}{\partial x_{2\mu-1}}(x), \end{aligned}$$

since each of the terms  $\partial x_{2\mu} / \partial x_{2i-1}$  is zero. Hence (32) holds for all  $k \geq 1$ .  $\square$

**6.2.3. Proof of Lemma 3.4.** (1) Note that, given the finiteness of  $\mathcal{I}$ , if for every  $I \in \mathcal{I}$  the conclusion holds for  $(x, \varepsilon) \mapsto a_I(x, \varepsilon) b_I(x)$  and some compact neighborhood  $U_I \subset U$  of 0, then the conclusion holds for  $f$  by setting  $U' = \bigcap_{I \in \mathcal{I}} U_I$ . Let us then fix  $I \in \mathcal{I}$ .

(1)(i) Since  $a_I$  is real-analytic, we can write  $a_I(x, \varepsilon) b_I(x) = [\frac{\partial a_I}{\partial x}(0, \varepsilon)x + \tilde{a}_I(x, \varepsilon)] b_I(x)$ , where  $\varepsilon \mapsto \frac{\partial a_I}{\partial x}(0, \varepsilon)$  is continuous and vanishes at 0, and  $(x, \varepsilon) \mapsto \tilde{a}_I(x, \varepsilon)$  is a continuous mapping satisfying

$$(\forall \varepsilon \in E) \quad \lim_{x \rightarrow 0} \frac{\|\tilde{a}_I(x, \varepsilon)\|}{\|x\|} = 0 \quad \text{and} \quad \tilde{a}_I(\cdot, 0) = 0.$$

Given any compact neighborhood  $U_I \subset U$  of 0, define  $q(x, \varepsilon) = \frac{\|\tilde{a}_I(x, \varepsilon)\|}{\|x\|}$ , with  $q(0, \cdot) = 0$ , so that  $q$  is continuous—hence bounded—on  $U_I \times E$  and  $q(\cdot, 0) = 0$ . We claim that  $\sup_{x \in U_I} q(x, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . Otherwise there would be  $\eta > 0$  and a sequence  $(\varepsilon_k)_{k \in \mathbb{N}}$ , converging to zero, such that  $\sup_{x \in U_I} q(x, \varepsilon_k) > 2\eta$  for every  $k \in \mathbb{N}$ . By the properties of sup, for every  $k \in \mathbb{N}$  there would exist  $x_k \in U_I$  such that

$$(33) \quad (\forall k \in \mathbb{N}) \quad q(x_k, \varepsilon_k) > \sup_{x \in U_I} q(x, \varepsilon_k) - \eta > \eta.$$

The compactness of  $U_I$  would imply the existence of a subsequence  $(x_{k_j}, \varepsilon_{k_j})_{j \in \mathbb{N}}$ , convergent towards a point  $(\bar{x}, 0) \in U_I \times E$ . But then, since  $q$  is continuous,  $q(x_{k_j}, \varepsilon_{k_j})$  should converge to  $q(\bar{x}, 0) = 0$ , in contradiction to (33). Therefore, in view of the continuity of  $b_I$ , the conclusion follows since

$$\frac{\|a_I(x, \varepsilon) b_I(x)\|}{\|x\|} \leq \left( \left\| \frac{\partial a_I}{\partial x}(0, \varepsilon) \right\| + \sup_{x \in U_I} q(x, \varepsilon) \right) \sup_{x \in U_I} \|b_I(x)\|,$$

the right-hand member of which tends to zero as  $\varepsilon \rightarrow 0$ , uniformly for  $x \in U_I$ .

(1)(ii) The assumption  $b_I(x) = O(\|x\|)$  implies the existence of a constant  $K_I > 0$  and a compact neighborhood  $U_I \subset U$  of 0 such that  $\|b_I(x)\|/\|x\| \leq K_I$  for every  $x \in U_I$ . Furthermore, by the real-analyticity of  $a_I$ , and using  $a_I(\cdot, 0) = 0$ , one can write  $a_I(x, \varepsilon) b_I(x) = [\frac{\partial a_I}{\partial \varepsilon}(x, 0)\varepsilon + \tilde{a}_I(x, \varepsilon)] b_I(x)$ , with  $\frac{\partial a_I}{\partial \varepsilon}(\cdot, 0)$  continuous and hence

bounded on  $U_I$ , say,  $\|\frac{\partial a_I}{\partial \varepsilon}(x, 0)\| \leq K'$ . Moreover, for every  $x \in U_I$ , the function  $q(x, \varepsilon) = \frac{\|\tilde{a}_I(x, \varepsilon)\|}{|\varepsilon|}$  tends to 0 as  $\varepsilon \rightarrow 0$ , so  $q$  is also continuous on  $U_I \times E$ . Proceeding as in the proof of (1)(i) above, one readily shows that  $\sup_{x \in U_I} q(x, \varepsilon) \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . The conclusion then follows directly from the inequality

$$\begin{aligned} \frac{\|a_I(x, \varepsilon)b_I(x)\|}{\|x\|} &\leq \left( \left\| \frac{\partial a_I}{\partial \varepsilon}(x, 0) \right\| + q(x, \varepsilon) \right) |\varepsilon| \frac{\|b_I(x)\|}{\|x\|} \\ &\leq K_I \left( K' + \sup_{x \in U_I} q(x, \varepsilon) \right) |\varepsilon|. \end{aligned}$$

(2) Let  $U' \subset U$  be any compact neighborhood of 0. We claim that if a number  $\eta > 0$  exists such that, for every  $I \in \mathcal{I}$ ,

$$(34) \quad (x, \varepsilon) \mapsto \frac{\|a_I(x, \varepsilon)b_I(x)\|}{\|x\|^{1+\eta}} \quad \text{is continuous on } U' \times E,$$

then the conclusion holds for  $f$  and  $U'$ . Indeed,  $\sum_{I \in \mathcal{I}} a_I(x, \varepsilon)b_I(x)$  converges absolutely and uniformly; therefore

$$(35) \quad \frac{\|f(x, \varepsilon)\|}{\|x\|} \leq \|x\|^\eta \sum_{I \in \mathcal{I}} \frac{\|a_I(x, \varepsilon)b_I(x)\|}{\|x\|^{1+\eta}},$$

and the series on the right side of (35) converges to a function that is bounded on  $U' \times E$ . Consequently, the term on the left side of (35) tends to zero as  $x \rightarrow 0$ , uniformly for  $\varepsilon \in E$ , and this proves the claim. The rest of the proof simply consists of exhibiting such a number  $\eta$ , independent of  $I$ , for each case.

(2)(i) The assumption  $a_I(x, \varepsilon) = o(\|x\|)$  and the real-analyticity of  $a_I$  imply that all terms in  $x$  of degrees  $< 2$  in the Taylor expansion of  $x \mapsto a_I(x, \varepsilon)$  at 0 vanish identically. Thus, for every  $\varepsilon \in E$ ,  $\|a_I(x, \varepsilon)\|/\|x\|^{1+\eta} \rightarrow 0$  as  $x \rightarrow 0$  whenever  $\eta \leq 1$ . By continuity of  $b_I$ , (34) holds with  $\eta = 1/2$ .

(2)(ii) Since  $b_I(x) = O(\|x\|^{1+c})$ , then  $\|b_I(x)\|/\|x\|^{1+c/2} \rightarrow 0$  as  $x \rightarrow 0$ . Taking  $\eta = c/2$  we see that (34) holds.

(2)(iii) The assumptions imply that both  $\|a_I(x, \varepsilon)\|/\|x\|^{1-d/3}$  and  $\|b_I(x)\|/\|x\|^{2d/3}$  tend to zero as  $x \rightarrow 0$ . Thus, (34) holds with  $\eta = d/3$ .  $\square$

**6.2.4. Proof of Lemma 6.2.** This is a straightforward adaptation of the proof of [27, Lemma 4.2]. Since the result is local one may assume, without loss of generality, that  $M$  is an open subset of  $\mathbb{R}^n$  and that  $\bar{x} = 0$ . By real-analyticity, the mappings  $\phi$  and  $f_0, \dots, f_m$  may be extended to complex analytic mappings defined on a polydisc  $D(n, \alpha) = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_i| < \alpha, i = 1, \dots, n\}$  for some  $\alpha > 0$ . Denote the corresponding extensions by  $\tilde{\phi}$  and  $\tilde{f}_0, \dots, \tilde{f}_m$ , respectively.

By Stirling’s formula, there is a constant  $C''$  such that  $r^r \leq C'' e^r r!$  for all  $r \geq 1$ . Let  $C' = \max\{|\tilde{\phi}(q)| : q \in D(n, \frac{2}{3}\alpha)\}$  and define

$$C = e \max\{1, C' C''\}.$$

Select  $\eta > 0$  arbitrarily. Then the vector fields  $\tilde{g}_0 = (1/\eta)\tilde{f}_0, \tilde{g}_i = \tilde{f}_i$  ( $i = 1, \dots, m$ ) are analytic extensions of the vector fields  $g_0, \dots, g_m$ , respectively, to the set  $D(n, \alpha)$ . Consider the complex control system

$$(36) \quad \dot{z} = \sum_{i=0}^m v_i \tilde{g}_i(z), \quad z \in \mathbb{C}^n, \quad v = (v_0, \dots, v_m) \in \mathbb{C}^{m+1}.$$

Clearly, if  $v = (\eta, 0, \dots, 0)$ , then  $(z, v) = (0, v)$  is an equilibrium point; hence the corresponding constant solution  $t \mapsto (0, v)$  is defined for  $t \in [0, 1]$ . Since (36) may be rewritten as a real control system on  $\mathbb{R}^{2n}$ , one can apply Lemma 6.3 to conclude that there is  $\delta \in (0, \frac{2}{3}\alpha)$  with the property that, whenever  $z_0 \in D(n, \delta)$ ,  $v_0 \in A = \{w \in \mathbb{C} : |w - \eta| < \delta\}$ , and  $v_i \in D(1, \delta)$  ( $i = 1, \dots, m$ ), the system (36) has a unique solution  $z : [0, 1] \rightarrow \mathbb{C}^n$  satisfying  $z(t_0) = z_0$  and  $z(t) \in D(n, \frac{2}{3}\alpha)$  for  $t \in [0, 1]$ .

Let us fix a multi-index  $I = (i_1, \dots, i_r) \in \{0, \dots, m\}^r$ ,  $r \geq 1$ , and define the set  $D^I = \prod_{j=1}^r D_j^I$ , where  $D_j^I = A$  if  $i_j = 0$  and  $D_j^I = D(1, \delta)$  otherwise.  $D^I$  is thus an open subset of  $\mathbb{C}^r$ . For any  $z \in D^I$ , define the input function  $v^{I,z} : [0, 1] \rightarrow A \times D(m, \delta)$  by setting, for  $j = 1, \dots, r$  and  $t \in [\frac{j-1}{r}, \frac{j}{r}]$ ,  $v^{I,z}(t) = z_j e_{i_j}$  (here  $\{e_0, \dots, e_m\}$  denotes the canonical basis of the  $\mathbb{C}$ -vector space  $\mathbb{C}^{m+1}$ ). The function  $v^{I,z}$  thus defined is a piecewise-constant function on  $[0, 1]$  taking values in  $A \times D(m, \delta)$ . Therefore, for any  $q \in D(n, \delta)$ , the solution  $t \mapsto \xi_q^{I,z}(t)$  to system (36), with input  $v^{I,z}$  and initial condition  $\xi_q^{I,z}(0) = q$ , is defined and satisfies  $\xi_q^{I,z}(t) \in D(n, \frac{2}{3}\alpha)$  for all  $t \in [0, 1]$ . Moreover, for  $i = 1, \dots, r$ ,  $\xi_q^{I,z}(i/r)$  is analytic in  $q$  and  $z$  for  $(q, z) \in D(n, \delta) \times D^I$  (cf. [27, proof of Lemma 4.2]). Now define a mapping  $\psi_I : D(n, \delta) \times D^I \rightarrow \mathbb{C}$  by setting  $\psi_I(q, z) = \tilde{\phi}(\xi_q^{I,z}(1))$ . Then

$$\frac{\partial^r \psi_I}{\partial z_r \cdots \partial z_1}(q, 0) = \left(\frac{1}{r}\right)^r (\tilde{g}_{i_1} \cdots \tilde{g}_{i_r}, \tilde{\phi})(q).$$

This is readily shown by extending the following basic argument using induction on the length  $r$  of  $I = (i_1, \dots, i_r)$ . For a vector field  $X = (X_1, \dots, X_n)$  defined on some subset  $B$  of  $\mathbb{C}^n$ , denote by  $t \mapsto \Phi_q^X(t)$  the local flow of  $X$  satisfying  $\Phi_q^X(0) = q \in B$ . Hence  $d\Phi_{q,k}^X/dt = X_k(\Phi_q^X(t))$  for  $k = 1, \dots, n$ . It is easy to check that  $\Phi_q^{zX}(t) = \Phi_q^X(zt)$  when  $z \in \mathbb{C}$  and  $|z|$  is small enough. Thus, by setting  $\psi_{(i)}(q, z) = \tilde{\phi}(\Phi_q^{z\tilde{g}_i}(\tau)) = \tilde{\phi}(\Phi_q^{\tilde{g}_i}(z\tau))$ , with  $i \in \{0, \dots, m\}$ , one gets

$$\begin{aligned} \frac{\partial \psi_{(i)}}{\partial z}(q, z) &= \sum_{k=1}^n \frac{\partial \tilde{\phi}}{\partial r_k}(\Phi_q^{\tilde{g}_i}(z\tau)) \frac{d\Phi_{q,k}^{\tilde{g}_i}}{dt}(z\tau) \frac{d}{dz}(z\tau) \\ &= \tau \sum_{k=1}^n \frac{\partial \tilde{\phi}}{\partial r_k}(\Phi_q^{\tilde{g}_i}(z\tau)) \tilde{g}_{i,k}(\Phi_q^{\tilde{g}_i}(z\tau)), \end{aligned}$$

and then, if  $\tau = 1$  and  $z = 0$  (and since  $r=1$  because  $I = (i)$ ),

$$\frac{\partial \psi_I}{\partial z}(q, 0) = \sum_{k=1}^n \frac{\partial \tilde{\phi}}{\partial r_k}(q) \tilde{g}_{i,k}(q) = \left(\frac{1}{r}\right)^r (\tilde{g}_i \tilde{\phi})(q).$$

Since  $q \mapsto \psi_I(q, z)$  is analytic on  $D(n, \delta)$  for  $z \in D^I$ , Cauchy's estimates yield

$$\left| \frac{\partial^r \psi_I}{\partial z_r \cdots \partial z_1}(q, 0) \right| \leq \max \left\{ |\tilde{\phi}(q')| : q' \in D\left(n, \frac{2}{3}\alpha\right) \right\} = C',$$

and this implies in turn that  $|\tilde{g}_{i_1} \cdots \tilde{g}_{i_r} \tilde{\phi}(q)| \leq C' r^r$  for any  $q \in D(n, \frac{2}{3}\alpha)$ . By definition of  $C''$ , one has  $C' r^r \leq C' C'' e^r r!$ . Also,  $C' C'' \leq \max\{1, C' C''\} = C'/e$ . Using the fact that  $1 \leq C'/e$ , and hence  $C'/e \leq (C'/e)^r$ , one gets  $C' C'' e^r r! \leq C' r!$  for  $r \geq 1$ . Therefore, by setting  $K = D(n, \delta) \cap \mathbb{R}^n$ , one concludes that

$$|(g_{i_1} \cdots g_{i_r} \phi)(x)| \leq C' r!$$

for  $x \in K$  and  $r \geq 1$ . Since the constant  $C$  was selected independently of  $\eta$ ,  $r$ , and  $I$ , this finishes the proof.  $\square$

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