Back-order lead time behaviour in (s,Q)-inventory models with compound renewal demand

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Abstract

In the practice of inventory management customer-oriented performance characteristics as opposed to availability-oriented performance characteristics have received more and more attention. Instead of measuring the probability of a stockout one measures the probability that a customer's demand is satisfied within one week. In this paper we derive approximate expressions for the customer waiting time distribution in care a single stockpoint is replenished according to an (s,Q)-policies. The demand process is compound renewal, i.e. customer has a demand distributed according to some arbitrary distribution funtion. Extensive simulations show that the approximations yield excellent results. The results can be applied to analyze distribution networks where each stockpoint is controlled according to an (s,Q)-model.

1. Introduction

Over the last twenty years most successful companies have recognized the importance of customer focus. This has had its influence on inventory management concepts. Through better inventory management information systems like DRP and MRP it has become possible to react more adequately to customer wishes. Yet proper inventory management is still based on setting appropriate safety stock levels for components and finished goods, so that most customer wishes can be satisfied routinely. In spite of the customer focus most companies still use performance measures which are not truly customer-based, only in an indirect manner because they measure the availability of stock. Examples of such performance measures include stockout probability, fraction of demand satisfied directly from the shelf. In this paper we focus on customer-oriented performance measures, such as the probability that a customer has to wait less than a prespecified maximum allowed waiting time and the average customer waiting time. Furthermore we include as performance measure the fraction of demand satisfied within a maximum allowed waiting time. This measure is especially useful in situations where customers hold inventory themselves, so that customer orders are merely stock replenishments, which need not be served immediately from stock.

In this paper we consider a one-product single echelon model. We assume that the inventory from which customer demand is satisfied is controlled according to a continuous review (s,Q)-policy. I.e. as soon as the inventory position (which equals physical stock plus stock on order minus backorders) drops below s a quantity kQ ($k \in N$) is ordered such that the inventory position is raised to a level between s and s + Q. We assume s > 0. The demand process is a compound renewal process, i.e. customers arrive at the stockpoint according to a renewal process and the demands of customers are i.i.d. random variables independent of the arrival process. We assume such a general demand process since we observed in many practical situations, especially for stockpoints of manufacturers that the demand process is non-Poisson. This is mainly caused by the fact that in those situations the number of customers is limited. Finally we assume that the lead times of orders are identically distributed. Yet the lead times are not independent since we assume that consecutive orders do not overtake. Note that this assumption is valid for almost any practical situation. We derive approximate expressions for the above mentioned performance measures and validate the approximations through extensive computer simulations.

The literature on customer backorder behaviour is limited. The most explicit contribution is by Van der Veen [1981], who first mentions the lead time shift conjecture given in section 3 of this paper without proof and without explicit definition of the demand process. Using the conjecture Van der Veen derives for an (s,Q)-model an approximate expression for the fraction of demand delivered from stock within a given time. This expression only holds for large values of Q. In a later paper Van der Veen [1984] further elaborates on the backorder behaviour. In Van Beek [1981] the backorder behaviour of an (s,Q)-model is used to analyze a two-echelon inventory model. Van Beek claims that the backorder lead time is exponentially distributed. Our analysis reveals that this is not true. In the context of multi-echelon systems various authors have either implicitly or explicitly derived expressions for the backorder lead time (or equivalently customer waiting time). We mention Svoronos and Zipkin [1988], who analyze the backorder behaviour of an (s,Q)-model under the assumption of compound Poisson demand with fixed demand sizes within the context of a two-echelon model with pure Poisson demand at the downstream warehouses. In a later paper Svoronos and Zipkin [1991] analyze the backorder behaviour for arbitrary multi-echelon systems, where each stockpoint is controlled according to a one-for-one replenishment policy. In both papers Svoronos and Zipkin derive the results for backorder behaviour from expressions for the first and second moment of the stationary backlog and application of Little's formula. This approach is only valid for the specific (s,Q)-model with fixed demand sizes and lot-for-lot replenishment models, as studied in their papers. In our model Little's formula linking waiting time and backlog cannot be applied. This has also been argued in Van der Heijden and De Kok[1992]. The major contribution of our paper is that for (s,Q)-models with arbitrary demand processes and lead time distributions, simple and robust approximations are given for performance characteristics related to the backorder behaviour. Because of the fact that the results are derived for general demand processes it is possible to use these results as part of an analysis framework that enables to analyze arbitrary multi-echelon systems where each node in the network is controlled according to some (s,Q)-model.

The paper is organized as follows. In section 2 we give a detailed description of the model and we define the performance measures studied. In section 3 we derive approximate expressions for these performance measures. In section 4 we show the quality of the approximations by comparing the analytically derived results with computer simulation. In section 5 we discuss the application of our results and directions for further research.

2. Description of the model and performance characteristics.

In this section we give a detailed description of the mathematical model that serves as the basis of our analysis of the inventory management problem described in the introduction.

We assume that the demand process is a compound renewal process. I.e. customers arrive according to a renewal process $\{A_n\}$ and the demand of the n^{th} customer equals D_n. Hence

A_n := time between the arrival of the $(n-1)^{th}$ and n^{th} customer.

We assume that the 0^{th} customer arrives at time 0.

Both $\{A_n\}$ and $\{D_n\}$ are series of independent identically distributed random variables and $\{A_n\}$ is independent of $\{D_n\}$. We define the distribution function F_A and F_D as follows

$$F_A(t) \quad \cdot = \quad P\{A_n \leq t\} \ \forall\, n.$$

$$F_D(x) \quad = \quad P\{D_n \leq x\} \ \forall\, n.$$

Let A and D denote a generic interarrival time and demand, respectively. We assume that $F_A(0) = 0$ and $F_D(0) = 0$, i.e. A_n and D_n are positive random variables.

The customers' demand is satisfied from stock. Stock is replenished according to an (s,Q)-policy with $s > 0$. This policy can be described as follows. First define

$Y(t)$:= sum of physical stock at time t and the stock on order at time t minus the backorders at time t.

As soon as $Y(t)$ drops below the reorder level s at the arrival of a customer, a quantity kQ ($k \in \mathbf{N}$) is ordered, where k is determined such that

$$s \leq Y(t^o) + kQ \leq s + Q.$$

Here $Y(t^o)$ denotes the inventory position immediately after the arrival of the customer that causes the undershoot of s, but just before the replenishment decision is taken. In cases where D_n is small compared with Q, k is usually equal to 1.

The lead time of an order is defined as the time that elapses between the initiation of an order upon undershoot of the reorder level s and the arrival of the order at the stock point. Define

Q_m := the m^{th} order quantity after time 0.

L_m := the lead time of Q_m.

We assume that the lead times L_m are identically distributed with distribution function L. So we have that

$$P\{L_m \leq t\} = F_L(t)$$

Furthermore, we assume that consecutive orders Q_m and Q_{m+1} do not overtake, i.e. Q_m alwayjs arrives earlier at the stock point than Q_{m+1}. This implies that L_m and L_{m+1} are dependent. The non-overtaking assumption is the classical assumption in almost all publications on inventory management models (cf. Hadley and Whitin [1963], Silver and Peterson [1985]).

Having defined the inventory management model, we now can define the relevant performance characteristics of the model. In the notation below we explicitly take into account the fact that the performance characteristics depend on the probability distribution of the lead times, F_L. Instead of using an argument F_L, we use the argument L. We can consider L to be the generic lead time with probability distribution F_L.

$f(t;L)$:= fraction of demand delivered from stock within t time units for our model with lead time L.

$p_W(t;L)$:= probability a customer's demand cannot be satisfied within t time units, for our model with lead time L.

$E[W;L]$:= average backorder lead time of a customer for our model with lead time L.

$E[W^2;L]$:= second moment of the backorder lead time of a customerfor our model with lead time L.

In the next section we derive approximations for these performance characteristics. We give only an outline of the analysis. The technical details are given in appendix II.

3. Outline of the analysis.

With the definition of our performance characteristics we can start to find expressions for them. From standard probability theory we find (cf. Tijms [1986])

$$E[W;L] = \int_0^\infty p_W\,(t;L)dt \tag{3.1}$$

$$E[W^2;L] = 2\int_0^\infty t p_W(t;L)dt \tag{3.2}$$

Hence an expression for $p_W(t;L)$ will give us to derive expressions for $E[W;L]$ and $E[W^2;L]$.

Before we derive expressions for $p_W(t;L)$ and $f(t;L)$ we give the following theorem, which yields a major simplification of our analysis.

Lead time shift theorem.

For all t and any lead time L we have for the (b,Q)-model with b > 0.

$f(t;L)$:= $f(0;(L-t)^+)$ (3.3)

$p_W(t;L)$:= $p_W(0;(L-t)^+)$ (3.4)

with

x^+ := $\max(0,x)$.

We conjectured this result based on simulation experiments. This conjecture has been mentioned before by Van der Veen [1981], and has been elaborated on in Van der Veen [1984]. The theorem is recently proven in Van der Wal [1993]. In Van der Wal [1993] it is also shown that the theorem holds for continuous review (s,S)-models, but not for periodic review models like the (R,S)-model studied in Van der Heijden and De Kok [1992] and Zheng and Chen [1992].

The assumption that $s > 0$ is essential, since it can be seen that if $s < 0$ then a customer may wait longer than L, even if L is a constant. This happens when the inventory position drops below 0 but remains greater than s, and the time until the next arrival is greater than L.

The consequence of the theorem is that it suffices to find expressions for $f(0;L)$ and $p_W(0;L)$. Note that

$f(0;L)$	=	fraction of demand satisfied directly from stock on hand, for our model with lead time L.
$p_W(0;L)$	=	probability a customer cannot be satisfied directly from stock on hand, for ouw model with lead time L.

An expression for $f(0;L)$ has been derived by several authors, e.g. Silver and Peterson [1985] and De Kok [1991]. For ease of reference we outline the analysis.

Let us assume an amount Q is ordered at time 0 immediately after an undershoot U_0 of s. At time σ again an amount Q is ordered after an undershoot U_1 of s. The order at time 0 arrives at time L_0 at the stockpoint, the order at time σ arrives at time $L_1 + \sigma$. The physical stock immediately after arrival of the order at time L_0 equals the inventory position at time 0, i.e. $s + Q - U_0$, minus the demand during $(0, L_0]$. The physical stock immediately before arrival of the order at time $\sigma + L$, equals $s + Q - U_0$ minus the demand during $(0, \sigma + L_1]$. Define

$D(s,t]$	:=	demand during $(s,t]$

The amount backordered during the replenishment cycle $(L_0, \sigma + L_1]$, B, equals

$$B = \{D(0, \sigma + L_1] - (s - U_0 + Q)\}^+ - \{D(0, L_0] - (s - U_0 + Q)\}^+ \tag{3.5}$$

This expression is derived by distinguishing between the three possible situations

(i) physical stock immediately before $\sigma + L_1$ positive;

(ii) physical stock immediately before $\sigma + L_1$ negative and physical stock immediately after L_0 positive;

(iii) physical stock immediately after L_0 negative.

Now we note that

$$\begin{aligned} D(0, \sigma + L_1] &= D(0, \sigma] + D(\sigma, \sigma + L_1] \\ &= s + Q - U_0 - (s - U_1) + D(\sigma, \sigma + L_1] \\ &= Q - U_0 + U_1 + D(\sigma, \sigma + L_1] \end{aligned}$$

Substitution into (3.5) yields

$$B = \{D(\sigma, \sigma + L_1] + U_1 - s\}^+ - \{D(0, L_0] + U_0 - (s + Q)\}^+$$

Now we note that $D(\sigma, \sigma + L_1]$ and $D(0, L_0]$ are identically distributed. Furthermore we assume that U_0 and U_1 are distributed according to the stationary residual lifetime associated with the renewal process constituted by $\{D_n\}$, i.e.

$$P\{U \le x\} = \frac{1}{E[D]} \int_0^x (1 - F_D(y))dy$$

Then we find the following approximation for E[B],

$$E[B] \approx E[\{D(0, L] + U - s\}^+] - E[\{D(0, L] + U - (s + Q)\}^+]$$

Finally noting that f(0,L] equals one minus the fraction of demand backordered during a replenishment cycle, i.e. f(0,L] = 1 - E[B]/Q, we find

$$f(0;L] = 1 - \frac{E[(D(0,L]+U-s)^+]-E[(D(0,L]+U-(s+Q))^+]}{Q} \tag{3.6}$$

In our derivation we implicitly assumed that every order Q_m equals Q. Yet discrete simulation experiments have shown that expression (3.6) also applies to situations where $Q_m = kQ$ with $k > 1$ (cf. De Kok [1991]).

We observe that in general the distribution of D(0,L] is intractable. Therefore we calculate f(0;L) from (3.6) by fitting a mixture of Erlang distributions to approximations the first two moments of D(0,L] + U. In Appendix I we give expressions for these first two moments (formulae (I.1) and (I.2)) and the procedure to fit the mixture of Erlang distributions.

It remains to find an expression for $p_W(0;L)$. The derivation is given in Appendix II. This derivation is based on renewal-theoretic arguments and pseudo-memorylessness of renewal processes. Pseudo-memorylessness is defined through the assumption that at an arbitrary point in time t the residual life is identically distributed as the **stationary** residual life. This assumption is only valid for exponentially distributed times between renewals. The derivation in Appendix II yields

$$p_W(0\,;L) = \frac{E[(D(0,L]+D-s)^+]-E[(D(0,L]+D-(s+Q))^+]}{Q} \tag{3.7}$$

where (3.7) is exact for Poisson arrivals and exponential demand per customer. In this case we fit a mixture of Erlang distributions to the first two moments of D(0,L] + D. It is important to note that the dependence of f(0;L) and $p_W(0;L)$ on the arrival process is only through D(0,L]. Note also that (3.6) and (3.7) show that the waiting probability is not equal to the fraction of demand backordered, as is often implicitly presumed (e.g. Van Beek [1981]), especially when L is small. Yet (3.6) and (3.7) are identical for exponentially distributed demand per customer.

Note that (3.7) yields a positive value even if the lead time is zero! This is caused by the fact that customers, whose demand causes a stockout, are counted as waiting customers, even if their waiting time is zero in the case of zero lead time.

Based on our theorem we derive from (3.3), (3.4), (3.6) and (3.7)

$$f(t;L) = 1 - \frac{E[(D(0,(L-t)^{+}]+U-s)^{+}]-E[(D(0,(L-t)^{+}]+U-(s+Q))^{+}]}{Q} \tag{3.8}$$

$$p_W(t;L) = \frac{E[(D(0,(L-t)^{+}]+D-s))^{+}]-E[(D(0,(L-t)^{+}]+D-(s+Q))^{+}]}{Q} \tag{3.9}$$

A complicating factor for calculations is the truncated lead time $(L-t)^{+}$. However, if we fit a mixture of Erlang distributions to L, it is straightforward to calculate $E[(L-t)^{+}]$ and $E[((L-t)^{+})^{2}]$. Then we substitute these expressions in (I.1) and (I.2) to arrive at the first two moments of $D(0,(L-t)^{+}]$. Note that if L is a constant then

$E[(L-t)^{+}] = L-t \;,\; t \leq L.$

$E[((L-t)^{+})^{2}] = (L-t)^{2} \;,\; t \leq L.$

It remains to find expressions for $E[W;L]$ and $E[W^{2};L]$ for a given (s,Q)-policy. Towards this end we use equations (3.1), (3.2) and (3.9). Let us first derive an expression along these lines for $E[W;L]$. Assume the system is stationary. Let L(j) denote the lead time of the order from which an arbitrary customer j is satisfied. Let W(j) denote the waiting time of customer j. Then we obtain

$$E[W;L] = \int_0^\infty P\{W(j) > t\}\, dt$$

$$= \int_0^\infty \int_0^\infty P\{W(j) > t \mid L(j) = y\}\, dF_{L(j)}(y)$$

It follows from the lead time theorem that

P{W(j) > t | L(j) < t} = 0

Applying this result we find

$$E[W;L] \quad = \quad \int_0^\infty \int_0^y P\,\{W > t \,|\, L = y\}\; dt\, dF_{L(j)}(y)$$

Suppose now that customer j has to wait. Next we observe that the waiting time W(j) of customer j, whose associated lead time L(j) equals y is exactly equal to the waiting time of this same customer in an (s,Q)-model, where the lead time is a constant equal to y. This follows from the fact that the lead time distribution does not affect the ordering process, which is based on the inventory position, and from the fact that orders do not overtake. Hence it follows from (3.9) that for t ≤ y

$$P\{W(j) > t \,|\, L(j) = y\} = \frac{1}{Q}\{E[(D(0,y-t] + D - s)^+] - E[(D(0,y-t] + D - (s+Q))^+]$$

and thus

$$E[W;L] \quad = \quad \frac{1}{Q}\int_0^\infty \int_0^y (E[(D(0,y-t]+D-s)^+]-E[(D(0,y-t]+D-(s+Q))^+])dt\; dF_L(y)$$

$$= \quad \frac{1}{Q}\int_0^\infty \int_0^y (E[(D(0,t]+D-s)^+]-E[(D(0,t]+D-(s+Q))^+])dt\; dF_L(y)$$

$$= \quad \frac{1}{Q}\int_0^\infty \int_t^\infty (E[(D(0,t]+D-s)^+]-E[(D(0,t]+D-(s+Q))^+)]dF_L(y)\; dt$$

Changing the order of integration we find

$$E[W;L] \quad = \quad \frac{1}{Q}\int_0^{\infty}(E[(D(0,t]+D-s)^+]-E[(D(0,t]+D-(s+Q))^+)](1-F_L(t))dt \tag{3.10}$$

To compute E[W;L] from (3.10) is quite complicated, through the intractability of the distribution of D(0,t]. Even fitting mixtures of Erlang distributions would imply integration of a complicated function of t. In principle numerical integration could be applied, yet that could also be applied directly to relations (3.1) and (3.2). Expression (3.10) gives us the possibility to arrive at a tractable approximation for E[W;L]. Towards this end we define $\hat{L}$ as the generic random variable with the probability distribution function $F_{\hat{L}}$, with

$$F_{\hat{L}}(t)=\frac{1}{E[L]}\int_{0.}^{t}(1-F_L(t))dt \tag{3.11}$$

Substitution of (3.11) into (3.10) yields

$$E[W;L] \quad = \quad \frac{E[L]}{Q}\int_0^{\infty}(E[(D(0,t]+D-s)^+]-E[(D(0,t]+1)-(s+Q))^+])dF_{\hat{L}}(t)$$

which is by definiton equal to

$$E[W;L] \quad = \quad \frac{E[L]}{Q}(E[(D(0,\hat{L}]+D-s)^+]-E(D(0,\hat{L}]+D-(s+Q))^+]) \tag{3.12}$$

Combining (3.7) and (3.12) we find

$$E[W;L]=E[L]p_W(0;\hat{L}) \tag{3.13}$$

Clearly, approximation (3.13) is tractable, since we already indicated a feasible approach to compute $p_W(o;L)$ for an arbitrary lead time distribution F_L. Analogously one may derive from (3.2) and (3.9) that

$$E[W^2; L] = E[L^2]p_W(0; \tilde{L}) \tag{3.14}$$

where

$$P\{\tilde{L} \le t\} = \int_0^t \int_y^\infty (w-y)\, dF_L(w)\, dy \tag{3.15}$$

We derive only approximations for $p_W(0;L)$ through fitting mixtures of Erlangian distribution to the first two moments of D(0,L] + D. Since these two moments only involve the first two moments of L, it suffices to give the first two moments of $\hat{L}$ and $\tilde{L}$. These follow from

$$E[\hat{L}^k] = \frac{E[L^{k+1}]}{kE[L]} \tag{3.16}$$

$$E[\tilde{L}^k] = \frac{E[L^{k+2}]}{k(k+1)E[L]} \tag{3.17}$$

Having derived approximations for the performance characteristics, we must check our results by simulation.

4. Numerical evaluation

In this section we report on the extensive numerical evaluation of our approximations. We focus on the performance of our approximations for the backorder lead time characteristics. We studied the following models as described by its parameters,

$E[A] = 1$

$c_A^2 = 1/4, 3/4, 2$

$E[D] = 100$

$c_D^2 = 1/2, 2$

$E[L] = 2, 8, 32$

$c_L^2 = 0, 1/5, 1/3$

$Q = 200, 1000$

Here c_A, c_D and c_L denote the squared coefficient of variation of the interarrival time, customer demand and lead time respectively. Note that we did not simulate compound Poisson demand. The main reason for this is that the approximations are most accurate for compound Poisson demand, because in that case the assumption of memorylessness of the arrival process holds.

We calculated the reorder level based on the following service level restriction,

$f(0;L) = \beta,$

with

$\beta = 0.5, 0.75, 0.95$

For 18 combinations of parameters we found $s<0$. These cases were removed, since our approximations only hold for $s \geq 0$. For each model we executed two independent simulation runs of 50,000 customers. The results of these 612 simulation runs are summarized in figures 5.1, 5.2 and 5.3, associated with $P_W(0;L)$, the **conditional** waiting time $E[W^+;L]$ and the coefficient of variation $c(W^+;L)$ of the **conditional** waiting time, respectively.

The results from test runs indicated that our estimates of $E[W;L]$ and $E[W^2;L]$ for L deterministic deteriorate as L increases. A possible explanation is that $\hat{L}$ and $\bar{L}$ have only probability mass on (0,L) in the deterministic lead time case. For long lead times L this appears to impact the assumption that $D(0,\hat{L}]$ and $D(0,\bar{L}]$ have approximately a gamma distribution, as is often assumed (cf. Burgin [1975], De Kok [1991]). We could not find a straightforward solution to this problem other than to use numerical integration. The results for constant lead times are therefore calculated directly from (3.1) and (3.2).

We used the trapezoid integration method (Abramowitz and Stegun [1984]) and split (0,L] in 20 intervals. This turned out to be satisfactory as was shown by our simulation results.

We conclude that the quality of the approximations for $P_W(0;L]$ and $E[W^+;L]$ are excellent. The results for $C[W^+;L]$ are somewhat worse, yet quite acceptable for practical purposes. Furthermore, the less accurate approximations occurred mostly for small values of $P_W(0;L]$, when $E[W;L]$ and $E[W^2;L]$ are small.

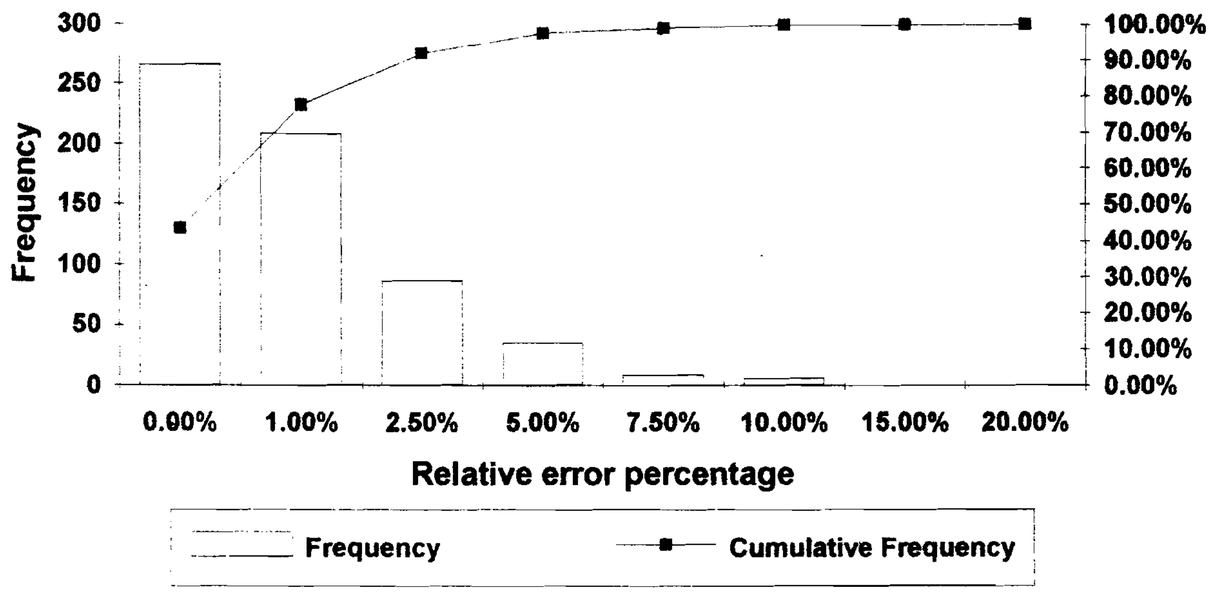

Figure 5.1 Quality of approximation of $P_W(0;L)$

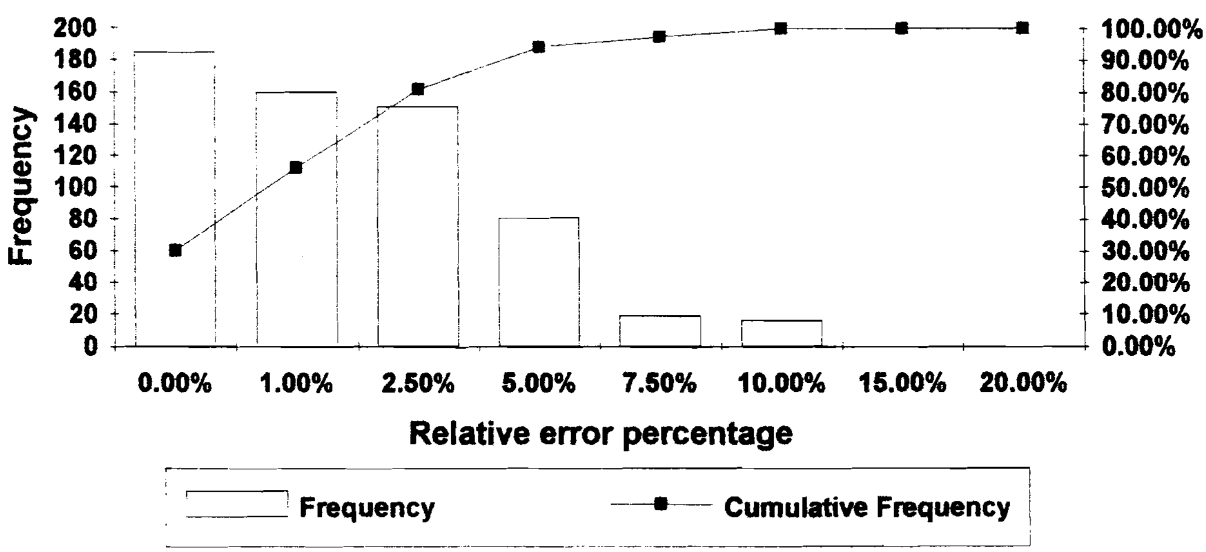

Figure 5.2 Quality of approximation of $E[W^+;L]$

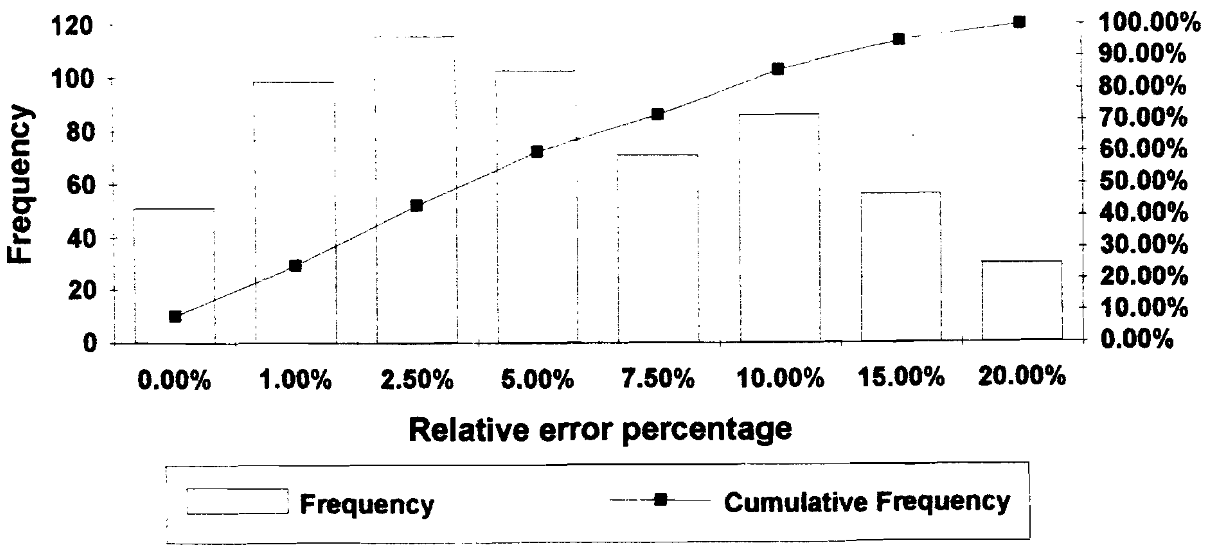

Figure 5.3 Quality of approximation of $c(W^{+};L)$

In figure 5.4 and 5.5 we show the performance of the approximations of f(t;L) and $P_W(t;L)$. We have chosen t equal to 1/2E[L]. We find that the quality of these approximations is again quite good.

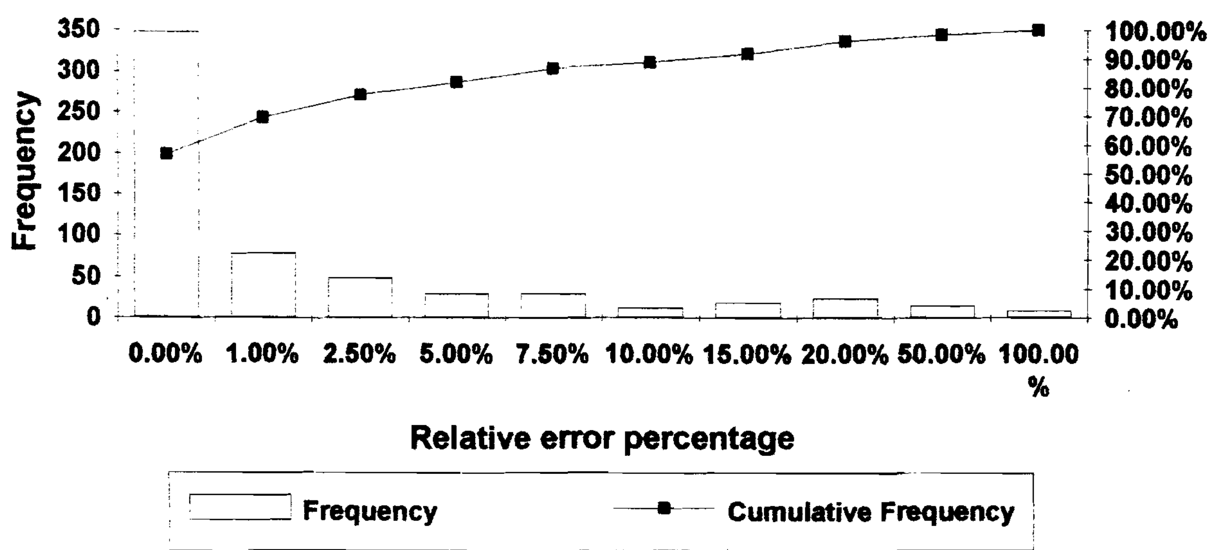

Figure 5.4 Quality of approximation of f(½E[L];L)

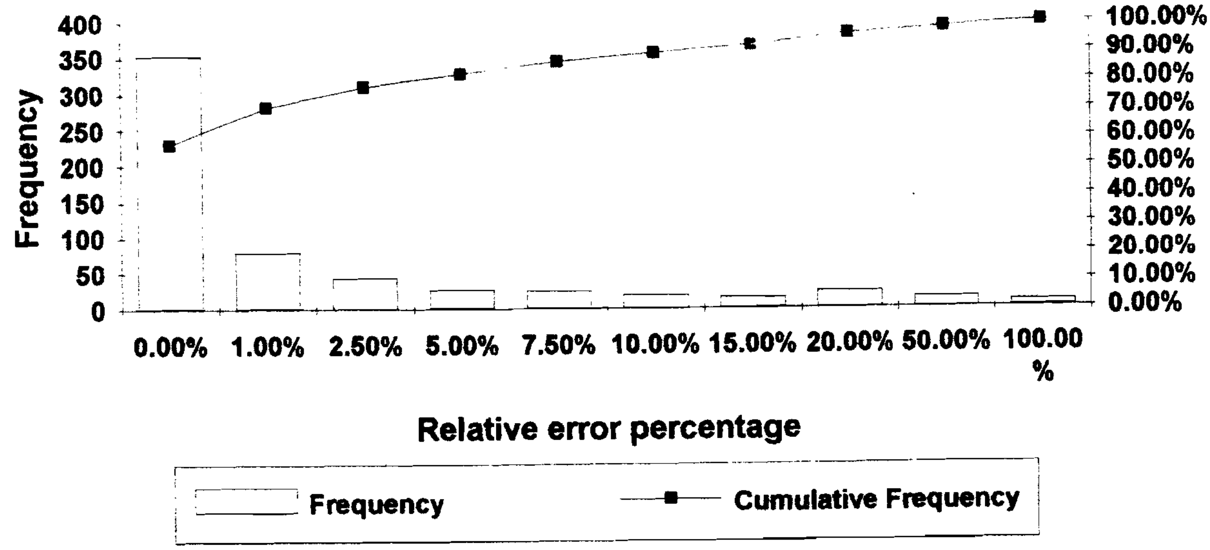

Figure 5.5 Quality of approximation of $1-p_W(\frac{1}{2}E[L],E[L]$

Concluding, the approximations derived in the previous sections are robust and accurate. The computation of the approximations is straightforward as is shown in Appendix I.

5. Conclusions and further research.

The model described in this paper is the standard (s,Q)-model. The major contribution of the paper is the derivation of expressions for model characteristics associated with backorder behaviour, like customer waiting times and fraction of demand satisfied within a given backorder lead time. The expressions yield approximations that are easy to compute and in fact take similar forms as well-known performance characteristics like the fraction of demand satisfied from stock on hand. The results derived can be used for what-if simulations to study the effect of lead time reduction, batch size reduction, but also the effect of market concentration. The latter usually results in a more regular demand process, provided big customers use sound replenishment rules. This can be modelled by a change of the demand process described by the parameters ($E[A]$, c_A^2, $E[D]$, c_D^2).

The model discussed in quite general through the use of a compound renewal demand process. By being able to cope with compound renewal demand processes the model can be used as a building block for analyzing multi-echelon inventory models, where each stockpoint is controlled by an (s,Q)-model. We will report on this application in a subsequent paper.

The analysis given here can be easily extended to (s,S)-models, although some of the expressions change. This extension is possible through the application of the lead time shift theorem.

By studying the backorder behaviour of (s,Q)-models we provide a means of studying the differences and similarities of make-to-stock and make-to-order policies. In fact, by deliberately creating backorders we find a mixed scenario. Since we must distinguish between types of customers, e.g. big customers, whose demand is produced on order and small customers, whose demand is delivered from stock, a really practical model is more complicated. This will be subject of further research.

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Appendix I Moments of lead time demand and undershoot and fitting distributions to these moments.

The expressions for the performance characteristics all involve D(0,L] or D(0,L] + U. The undershoot can be approximated by the stationary residual life time associated with the renewal process {Dn}. This yields

$$P\{U \le x\} = \frac{1}{E[D]} \int_0^x (1-F_D(y))dy$$

$$E[U] = \frac{E[D^2]}{2E[D]}$$

$$E[U^2] = \frac{E[D^3]}{3E[D]}$$

This approximation for U yields good results for all values of Q.

The approximation for the first two moments of D(0,L] is more complicated. A major assumption to be made is that at time L the renewal process {An} is stationary and L can be considered an arbitrary point in time. This assumption is reasonable provided,

$$P\{L < \frac{3}{2} c_A^2 E[A]\} = 0$$

We refer to De Kok [1987] for further motivation of this statement. Yet even if we take E[L] = E[A] we find that our assumption required to derive the first two moments of D(0,L] yields good results, unless $c_A{}^2$ gets large.

The derivation of E[D(0,L]] and E(D^2(0,L]] is based on asymptotic results from renewal theory. Assume at time 0 a customer arrives.

Define

N(t): = number of arrivals in (0,t]

Then

$$D(0,L]=\sum_{n=1}^{N(L)} D_n$$

and thus

$$E[D(0,L]] = E[N(L)]E[D]$$

$$E[D^2(0,L]]=E[N(L)]\sigma^2(D)+E[N^2(L)]E^2(D]$$

It follows from application of the *key renewal theorem* that

$$\lim_{t\to\infty} E(N(t)]-\left(\frac{t}{E[A]}+\frac{(c_A^{\,2}-1)}{2}\right)=0$$

$$\lim_{t\to\infty} E[N^2(t)]-\left(\frac{t^2}{E^2[A]}+(2c_A^2-1)\frac{t}{E[A]}+\frac{3E^2[A^2]}{2E^4[A]}-\frac{2E[A^3]}{3E^3[A]}-\frac{3E[A^2]}{2E^2[A]}+1\right)=0$$

with

$$c_A^2=\sigma^2(A)/E^2[A]$$

From the above equations we find

$$E[D(0,L]]\approx\left(\frac{E[L]}{E[A]}+\frac{(c_A^{\,2}-1)}{2}\right)E[D] \tag{I.1}$$

$$\begin{aligned}\sigma^2(D(0,L])\approx{}&\frac{E[L]}{E[A]}\sigma^2(D)+\frac{E[L]}{E[A]}c_A^{\,2}E^2[D]\\&+\sigma^2(L)\frac{E^2[D]}{E^2[A]}+\frac{(c_A^{\,2}-1)}{2}\sigma^2(D)+\frac{(1-c_A^{\,4})}{12}E^2[D]\end{aligned} \tag{I.2}$$

In the expression for $\sigma^2(D(0,L])$ we assumed that F_A is a gamma distribution function. The above expression is exact in the case of compound Poisson demand.

From the above results we can compute the first two moments of D(0,L] + U, since D(0,L] and U are independent. To compute the expressions for the performance characteristics we need the probability distribution function of D(0,L] and D(0,L] + U. We assume that these distribution functions are mixtures of Erlang distributions, i.e.

$$F(x)=p\left(1-\sum_{j=0}^{k_1-1} e^{-\mu_1 x}\frac{(\mu_1 x)^j}{j!}\right)+(1-p)\left(1-\sum_{j=0}^{k_2-1} e^{\mu_2 x}\frac{(\mu_2 x)^j}{j!}\right)$$

Such distribution functions can be used to explicitly evaluate the various integrals associated with the performance characteristics.

The choice of the parameters p, k_1, k_2, μ_1 and μ_2 is based on the first two moments of F. Let X denote the random variable associated with F and let E[x] and c_x denote its expectation and coefficient of variation, respectively. Then the parameters are computed as follows (cf. Tijms [1986])

(*i*) $c_X^2<1$:

$$\frac{1}{c_X^2}\le k_2\le\frac{1}{c_X^2}+1 \qquad k_1\in\mathbf{N}$$

$$k_1=k_2-1$$

$$p=\frac{k_2c_X^2-\sqrt{k_2(1+c_X^2)-k_2^2c_X^2}}{1+c_X^2}$$

$$\mu_1=\mu_2=\frac{k_2-p}{E[X]}$$

(*ii*) $c_X^2\ge1$

$k_1 = k_2 = 1$

$$p=\frac{1}{2}+\frac{1}{2}\sqrt{\frac{c_X^2-1}{c_X^2+1}}$$

$$\mu_1=\frac{2p}{E[x]}\,,\;\mu_2=\frac{2(1-p)}{E[X]}$$

Appendix II: Derivation of expression for $P_W(0;L]$

Consider the reorder cycle $(0, \sigma_1]$, i.e. at time 0 the reorder level s is undershot and at time σ_1 the reorder level s is undershot the next time. Define

U_0: = undershoot of s at time 0

U_1: = undershoot of s at time σ_1.

L_0: = lead time of order initiated at time 0

L_1: = lead time of order initiated at time σ_1.

To derive an expression for $P_W(0,L]$ we will study the inventory process during the replenishment cycle $(L_0, \sigma_1 + L_1]$. Before doing so we first study the behaviour of the inventory process under special conditions.

Let us assume that at time 0 the arrival process is stationary and the time origin 0 is an arbitrary point in time. Assume that the inventory at time 0 equals $x \geq 0$. Furthermore, assume that no replenishments arrive after time 0. Define

$N^+(x;t)$: = the number of customers that do not wait in $(0,t]$

$N^+(x;\infty)$: = the number of customers that do not wait in $(0,\infty)$

It is easy to see that

$$E[N^+(x;\infty)]=M_D(x)-1$$

where M_D is the renewal function associated with the renewal process $\{D_n\}$. Conditioning on the demand during $(0,t]$ and using the assumption that the arrival process is stationary and the time of the first arrival after time 0 is distributed according to the stationary residual life distribution associated with $\{A_n\}$ we find

$$E[N^+(x;\infty)]=E[N^+(x;t)]+\int_0^x E[N^+(x-y;\infty)]dF_{D(0,t]}(y)$$

and hence

$$E[N^+(x;t)]=M_D(x)-1-\int_0^x (M_D(x-y)-1)dF_{D(0,t]}(y)$$

Now let us consider the replenishment cycle $(L_0, \sigma_1 + L_1]$. Let us define

$N^+(b,Q)$: = number of customers that do not wait during $(L_0, \sigma_1 + L_1]$

By conditioning on $\sigma_1 + L_1 - L_0$ and $(D(0,L_0]^+ U_0)$ we find

$$E[N^+(s,Q)]=\int_0^\infty \int_0^{s+Q} E[N^+(b+Q-y,t)]dF_{U_0+D(0,L_0]|\sigma_1+L_1-L_0=t}(y)dF_{\sigma_1+L_1-\sigma_o}(t)$$

$$=\int_0^\infty \int_0^{s+Q} \left\{ (M_D(s+Q-y)-1) - \int_0^{s+Q-y} (M_D(s+Q-y-z)-1)dF_{D(0,t]}(z) \right\} dF_{U_0+D(0,L_0]|\sigma_1+L_1-L_0=t}(y)dF_{\sigma_1+L_1-L_0}(t)$$

Now note that $D(0,t] + (U_0 + D(0,L_0])/\sigma_1 + L_1 - L_0 = t$ equals $U_0 + D(0, \sigma_1 + L_1]/\sigma_1 + L_1 - L_0 = t$ for all t. By taking the unconditioned expectations we find

$$E[N^+(b,Q)]= \int_0^{b+Q} (M_D(b+Q-y)-1)dF_{U_0+D(0,L_0]}(y)$$
$$- \int_0^{b+Q} (M_D(b+Q-y)-1)dF_{U_0+D(0,\sigma_1+L_1]}(y)$$

Now we use the following equation

$$\begin{aligned} U_0+D(0,\sigma_1+L_1] &= U_0+D(0,\sigma_1]+D(0,\sigma_1+L_1] \\ &= U_0+(b+Q-U_0-(b-U_1))+D(0,\sigma_1+L_1] \\ &= Q+U_1+D(0,\sigma_1+L_1] \end{aligned}$$

Hence $P\{U_0 + D(0, \sigma_1 + L_1] > Q\} = 1$

Since $U_1 + D(0, \sigma_1 + L_1]$ is identically distributed as $U_0 + D(0, L_0\}$ we find

$$E[N^+(b,Q)] = \int_0^{b+Q} (M_D(b+Q-y)-1)\,dF_{U_0+D(0,L_0]}(y) - \int_0^{b} (M_D(b-y)-1)\,dF_{U_0+D(0,L_0]}(y) \tag{II.1}$$

Let us consider the integrals at the right-hand side of (II.1). They are both of the following form,

$$\int_0^x (M_D(x-y)-1)\,dF_{U+X}(y)$$

with X some random variable. We have that

$$\int_0^x (M_D(x-y)-1)\,dF_{U+X}(y) = \int_0^x \int_0^{x-y} (M_D(x-y-z)-1)\,dF_U(z)\,dF_X(y) \tag{II.2}$$

Let us consider the following equation

$$\int_0^x (M_D(x-y)-1)\,dF_U(y) = M_D \cdot F_U(x) - F_U(x)$$

We can simplify this equation by taking Laplace-Stieltjes transforms. Define the Laplace-Stieltjes transform $\tilde{G}(s)$ by

$$\tilde{G}(s) = \int_0^\infty e^{-sx}\,dG(x).$$

Then it follows that from the definition of $M_D(x)$ and the assumption that $F_U(x)$ is the stationary residual life distribution associated with the renewal process $\{D_u\}$ (cf. Appendix I).

$$\tilde{M}_D(s) = \frac{1}{1-\tilde{F}(s)}$$

$$\tilde{F}_U(s) = \frac{(1-\tilde{F}(s))}{E[D]}$$

Then we find that

$$\tilde{M}_D(s)\tilde{F}_U(s) - \tilde{F}_U(s) = \frac{\tilde{F}_D(s)}{sE[D]} = \frac{\tilde{F}_D(s)}{E[D]} \cdot \frac{1}{s}$$

Since $\frac{1}{s}$ is the Laplace transform of the identity function f(x) = x, we find by inversion of the Laplace transform

$$\int_0^x (M_D(x-y)-1)dF_U(y) = \int_0^x (x-y)dF_D(y)$$

Substitution of this result into (II.2) we find

$$\int_0^x (M_D(x-y)-1)dF_{U+X}(y) = \int_0^x (x-y)dF_{D+X}(y)$$

for any random variable X. Substitution of this result into (II.1) we obtain
Hence we find

$$E[N^+(b,Q)] = \int_0^{b+Q} \frac{(b+Q-y)}{E[D]} dF_{D(0,L_0)+D}(y) - \int_0^b \frac{(b-y)}{E(D)} dF_{D(0,L_0]+D}(y)$$

The waiting probability P_W (0;L] equals the average number of customers waiting during a replenishment cycle divided by the average number of customers arriving during a replenishment cycle, i.e.

$$P_W(0;L] = 1 - \frac{E[N^+(b,Q)]}{E[\sigma_1]}$$

Since

$$\sigma_1 = \sum_{n=1}^{N(\sigma_1)} A_n$$

and

$$Q = \sum_{n=1}^{N(\sigma_1)} D_n$$

and the fact that $N(\sigma_1)$ is a stopping time for $\{D_n\}$ and independent of $\{A_n\}$ we find

$$\begin{aligned} E[\sigma_1] &= E[N(\sigma_1)]E[A] \\ &= \frac{Q}{E[D]} \cdot E[A] \end{aligned}$$

and thus

$$P_W(0;L) = 1 - \frac{E[N^+(b,Q)]}{\left(\dfrac{Q}{E[D]}\right)}.$$

This finally yields

$$\begin{aligned} P_W(0;L) &= 1 - \frac{1}{Q}\left\{ \int_0^{b+Q} (b+Q-y)\,dF_{D(0,L_0]+D}(y) - \int_0^{b} (b+Q-y)\,dF_{D(0,L_0]+D}(y) \right\} \\ &= \frac{1}{Q}\left\{ \int_b^{\infty} (y-b)\,dF_{D(0,L_0]+D}(y) - \int_{b+Q}^{\infty} (y-(b+Q))\,dF_{D(0,L_0]+D}(y) \right\} \end{aligned}$$

which is identical to (3.7).

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