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Abstract

The algebraic Riccati equation studied in this paper is related to the suboptimal state feedback H_∞ control problem. It is parameterized by the H_∞ norm bound γ we want to achieve. The objective of this paper is to study the behaviour of the solution to the Riccati equation as a function of γ. It turns out that a stabilizing solution exists for all but finitely many values of γ larger than some a priori determined boundary γ_-. On the other hand for values smaller than γ_- there does not exist a stabilizing solution. The finite number of exception points turn out to be switching points where eigenvalues of the stabilizing solution can switch from negative to positive with increasing γ. After the final switching point the solution will be positive semi-definite. We obtain the following interpretation: the Riccati equation has a stabilizing solution with k negative eigenvalues if and only if there exist a static feedback such that the closed loop transfer matrix has no more than k unstable poles and an L_∞ norm strictly less than γ.

Keywords: The H_∞ control problem, The Algebraic Riccati Equation, J-spectral factorization, Wiener-Hopf factorization.

1 Introduction

The algebraic Riccati equation has a long history. The algebraic Riccati equation with a sign-definite quadratic term has played an important role in control theory. It was used in linear quadratic control, Kalman filtering and the combination of the latter two: the Linear Quadratic Gaussian or H_2 control problem (see e.g. [2, 1, 12, 20, 6]. The specific properties of this Riccati equation have also been studied extensively (see e.g. [17]).
But also a more general form of the algebraic Riccati equation has appeared in the literature. In this case, the quadratic term is not necessarily sign-definite. This more general Riccati equation first appeared in the game theory literature (see e.g. [4, 13, 14]). More recently, it turned out to play an important role in H_∞ control theory (see e.g. [8, 16, 19]). In the

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latter case the Riccati equation is parameterized by a parameter γ. It turns out that there exists a state feedback which makes the H_∞ norm strictly less than γ if and only if there exists a positive semi-definite, stabilizing solution to the algebraic Riccati equation. An iterative search then determines the minimal achievable H_∞ norm, say γ_*. In the process of determining γ_* one also checks for existence of stabilizing solutions for values of γ smaller than γ_*. It turned out that either the solution does not exist or is indefinite. The objective of this paper is to study the behaviour and existence of stabilizing solutions of the algebraic Riccati equation also for values of γ smaller than γ_*.
This is not a purely theoretical exercise. This study might help to find more efficient ways to perform the before-mentioned γ-iteration. We also derive a bounded real lemma for transfer matrices which are unstable. In general this Riccati equation plays such an important role in current day controller design that it is important to study its properties.
On the other hand we also obtain a very nice interpretation which is valid for all but finitely many γ: the Riccati equation has a stabilizing solution with k negative eigenvalues if and only if there exist a static feedback such that the closed loop transfer matrix has no more than k unstable poles and an L_∞ norm strictly less than γ.

2 Problem formulation

This paper studies the following Riccati equation:

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C - (PB + C^{\mathrm{T}}D)(D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + \gamma^{-2}PEE^{\mathrm{T}}P \qquad (2.1)$$

In this paper we only study stabilizing solutions of this equation, i.e. solutions for which the following matrix is asymptotically stable:

$$A - B(D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C) + \gamma^{-2}EE^{\mathrm{T}}P \qquad (2.2)$$

One of the main reasons for studying this Riccati equation is related to the H_∞ control problem. Consider the following system:

$$\begin{cases} \dot{x} = Ax + Bu + Ew \\ z = Cx + Du \end{cases} \qquad (2.3)$$

where $x \in \mathbb{R}^n$. The following theorem stems from e.g. [16, 8, 19]:

Theorem 2.1 : *Consider the system (2.3) and let $\gamma > 0$. Assume that the system (A, B, C, D) has no invariant zeros on the imaginary axis and D is injective. Then the following statements are equivalent:*

(*i*) *There exists a static feedback law $u = Fx$ such that after applying this compensator to the system (2.3) the resulting closed-loop system is internally stable and the closed-loop transfer matrix G_F has H_∞ norm less than γ, i.e. $\|G_F\|_\infty < \gamma$.*

(*ii*) *There exists a positive semi-definite solution P of the Riccati equation (2.1) such that the matrix in (2.2) is asymptotically stable.*

If P satisfies the conditions in part (ii), then a controller satisfying the conditions in part (i) is given by:

$$F := -(D^{\mathrm{T}}D)^{-1}(D^{\mathrm{T}}C + B^{\mathrm{T}}P). \tag{2.4}$$

□

For later use, we define the infimal achievable H_∞ norm via a stabilizing state feedback by γ_*.
It is our objective to extend the above result to show that for $\gamma < \gamma^*$ you still have solutions of the algebraic Riccati equation but the solution being sign-indefinite and the number of negative eigenvalues determining the number of unstable poles we have to admit to guarantee an L_∞ performance bound of γ.More precisely stated, the main result of this paper is the following theorem:

Theorem 2.2 : *Consider the system (2.3) and let $\gamma > 0$. Assume that the system (A, B, C, D) has no invariant zeros on the imaginary axis and D is injective. Then for all but finitely many γ the following statements are equivalent:*

- (*i*) *There exists a static feedback law $u = Fx$ such that after applying this compensator to the system (2.3) the resulting closed-loop system has at most i unstable eigenvalues and the closed-loop transfer matrix G_F has L_∞ norm less than γ, i.e. $\|G_F\|_\infty < \gamma$.*
- (*ii*) *There exists a solution P of the Riccati equation (2.1) such that the matrix in (2.2) is asymptotically stable. Moreover P has no more than i negative eigenvalues.*

If P satisfies the conditions in part (ii), then a controller satisfying the conditions in part (i) is given by (2.4). □

Remark : Note that we do not suggest that people should start designing controllers which do not stabilize the system. The importance of the above theorem lies in the fact that it tells us a great deal about the algebraic Riccati equation and the behaviour of its stabilizing solution as a function of γ. For large γ the equation has a positive semi-definite stabilizing solution. Then after a certain switching point the Riccati equation may still have a solution but it will have at least one negative eigenvalue. There are at most n switching points. These switching points are the only values of γ where the number of positive eigenvalues of the stabilizing solution can change. The finitely many values of γ for which the above theorem might not hold are precisely these switching points. Hence we also know a priori that there are no more than n values of γ for which the theorem might not be true.
We will denote the minimal achievable L_∞ norm via a static state feedback (without any stability requirements) by γ_-. Later we will give an explicit characterization of γ_-.

3 The bounded real lemma

In our derivation of theorem 2.2, the bounded real lemma will play a role. But this is a different version then the classical result (see e.g. [3, 20]). Instead of a test to check whether a stable transfer matrix has H_∞ norm strictly less than γ, we derive a test whether an arbitrary (not necessarily stable) rational matrix has L_∞ norm strictly less than γ.

Theorem 3.1 : *Consider a transfer matrix G with stabilizable and detectable realization $[A, B, C, D]$ such that A has no eigenvalues on the imaginary axis Then the following statements are equivalent:*

(*i*) *We have $\|G\|_\infty < \gamma$.*

(*ii*) *We have $D^{\mathrm{T}} D < \gamma$. Moreover, there exists a solution P of the algebraic Riccati equation:*

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C + (PB + C^{\mathrm{T}}D)\left(\gamma^2 I - D^{\mathrm{T}}D\right)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C)$$

such that the following matrix is asymptotically stable.

$$A + B\left(\gamma^2 I - D^{\mathrm{T}}D\right)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C).$$

(*iii*) *We have $DD^{\mathrm{T}} < \gamma$. Moreover, there exists a solution Q of the algebraic Riccati equation:*

$$0 = AQ + QA^{\mathrm{T}} + BB^{\mathrm{T}} + (QC^{\mathrm{T}} + BD^{\mathrm{T}})\left(\gamma^2 I - DD^{\mathrm{T}}\right)^{-1}(CQ + DB^{\mathrm{T}}) \tag{3.1}$$

such that the following matrix is asymptotically stable.

$$A + (QC^{\mathrm{T}} + BD^{\mathrm{T}})\left(\gamma^2 I - DD^{\mathrm{T}}\right)^{-1} C.$$

If P satisfies condition (ii) or Q satisfies condition (iii) then it has no more negative eigenvalues than the number of unstable eigenvalues of A. □

Proof : Conditions (ii) and (iii) are clearly dual to each other. Hence it suffices to prove equality between conditions (i) and (iii).

Due to detectability there exists a solution Y of

$$AY + YA^{\mathrm{T}} - YC^{\mathrm{T}}CY = 0$$

such that $A - YC^{\mathrm{T}}C$ is asymptotically stable. Define a transfer matrix H with realization:

$$[A - YC^{\mathrm{T}}C, B - YC^{\mathrm{T}}D, C, D]$$

It is then easy to check that $G^\sim G = H^\sim H$ where $G^\sim(s) = G^{\mathrm{T}}(-s)$. Then it is immediate that G and H have the same L_∞ norm. On the other hand since H is stable we know from the classical small gain theorem that H has H_∞ norm less than γ if and only if $DD^{\mathrm{T}} < \gamma$ and if there exists a solution Z of

$$\begin{aligned} 0 &= [A - YC^{\mathrm{T}}C]Z + Z[A - YC^{\mathrm{T}}C]^{\mathrm{T}} + [B - YC^{\mathrm{T}}D][B - YC^{\mathrm{T}}D]^{\mathrm{T}} + \\ &\qquad (ZC^{\mathrm{T}} + [B - YC^{\mathrm{T}}D]D^{\mathrm{T}})\left(\gamma^2 I - DD^{\mathrm{T}}\right)^{-1}(CZ + D[B - YC^{\mathrm{T}}D]^{\mathrm{T}}) \end{aligned}$$

such that the following matrix is stable:

$$[A - YC^{\mathrm{T}}C] + (ZC^{\mathrm{T}} + [B - YC^{\mathrm{T}}D]D^{\mathrm{T}})\left(\gamma^2 I - DD^{\mathrm{T}}\right)^{-1} C$$

The proof is completed by noting via some algebraic manipulations that Z satisfies the above equations if and only if $P := Z - \gamma^2 Y$ satisfies the conditions of theorem 3.1. ■

4 Relation of H_∞ control problems to J-spectral factorization

In this section we will show the relation between the existence of suitable H_∞ control problems and J-spectral factorization. This section is strongly based on the paper [11]. We basically study how the results change if we allow unstable closed loop poles. In this section we study the classical one and two block problems and relate the existence of a controller with at most i unstable poles to the existence of a J-spectral factorization with a specific additional feature. In the next section we relate J spectral factorization to Riccati equations based on a theorem of [5, 11]. Finally, in the section thereafter, we use our results for the two-block problem to prove our theorem 2.2.

4.1 The Nehari problem

Let $R \in L_\infty$ be given with Hankel singular values $\sigma_1^H \geq \cdots \sigma_i^H \geq \cdots \geq \sigma_n^H$. Denote by H_∞^i the set of transfer matrices with at most i unstable poles, i.e. the McMillan degree of the unstable part is less than or equal to i. Moreover $\mathcal{G}H_\infty$ denote those transfer matrices in H_∞ that are invertible and whose inverse is again in H_∞. Then we have the following theorem:

Theorem 4.1 : *For $\gamma \neq \sigma_j^H (j = 1, \ldots, n)$, the following statements are equivalent:*

(*i*) $\gamma > \sigma_i^H$.

(*ii*) *There exists $Q \in H_\infty^i$ such that $\|R + Q\|_\infty < \gamma$.*

(*iii*) *There exists $W \in \mathcal{G}H_\infty$ where W_{11} is invertible with $W_{11}^{-1} \in H_\infty^i$ satisfying:*

$$G^\sim J G = W^\sim J W$$

where

$$G = \begin{pmatrix} I & R \\ 0 & I \end{pmatrix}, \qquad J = \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix}$$

□

Proof : The equivalence of (i) and (ii) has been shown in [10].

(i) $\Longrightarrow$ (iii): We split $R = R_+ + R_-$ where R_+ is stable while R_- is strictly proper and anti-stable and R_- has realization $[A_-, B_-, C_-, 0]$. Let P and Q be the controllability and observability gramians of R_-. Since $\gamma \neq \sigma_j^H (j = 1, \ldots, n)$ we have that $N := (I - PQ)^{-1}$ is well-defined. Define X by

$$X := \left[-A_-^{\mathrm{T}} \left(C^{\mathrm{T}} - QB \right), \gamma^{-2} \begin{pmatrix} CPN \\ B^{\mathrm{T}} N \end{pmatrix}, \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \right]$$

Then it can be easily checked that $G_-^{\sim} J G_- = X^{\sim} J X$ and $X \in \mathcal{G}H_\infty$ where

$$G_- := \begin{pmatrix} I & R_- \\ 0 & I \end{pmatrix}$$

Moreover the $(1,1)$-block of X, denoted by X_{11}, is invertible and $X_{11}^{-1} \in H_\infty^i$. The proof of this implication is completed by noting that

$$W := X \begin{pmatrix} I & R_+ \\ 0 & I \end{pmatrix}$$

satisfies all the requirements of the above theorem.

(iii) $\Longrightarrow$ (ii): Suppose a W exists satisfying the conditions of part (iii). Define $V = W^{-1}$ and partition V and W conformably with G. Define $Q = V_{12} V_{22}^{-1}$. It is easy to check that $Q \in H_\infty^i$. Moreover:

$$(R + Q)^{\sim}(R + Q) - \gamma^2 I = -\gamma^2 (V_{22} V_{22}^{\sim})^{-1} < 0$$

This implies part (ii). ∎

4.2 The two-block problem

Theorem 4.2 : *Let $S, T \in H_\infty$ be given where T has full row rank on the imaginary axis. For all but finitely many γ, the following statements are equivalent:*

(*i*) *There exists $Q \in H_\infty^i$ such that*

$$\|T + SQ\|_\infty < \gamma. \tag{4.1}$$

(*ii*) *There exists $W \in \mathcal{G}H_\infty$ where W_{11} is invertible with $W_{11}^{-1} \in H_\infty^i$ satisfying:*

$$G^{\sim} J G = W^{\sim} J W \tag{4.2}$$

where

$$G = \begin{pmatrix} S & T \\ 0 & I \end{pmatrix}, \qquad J = \begin{pmatrix} I & 0 \\ 0 & -\gamma^2 I \end{pmatrix}$$

□

Proof : Factorize $T = T_i T_o$ where $T_o \in \mathcal{G}H_\infty$ and T_i is inner. Choose $T_\perp \in H_\infty$ such that $[T_i \ \ T_\perp]$ is square and inner. Then we have (4.1) if and only if

$$(R_1 + T_o Q)^\sim (R_1 + T_o Q) < \gamma^2 I - R_2^\sim R_2$$

where $R_1 = T_i^\sim S$ and $R_2 = T_\perp^\sim S$. Therefore there exists a $Q \in H_\infty^i$ such that (4.1) is satisfied if and only if there exist $N \in \mathcal{G}H_\infty$ such that

$$N^\sim N = \gamma^2 I - R_2^\sim R_2 \tag{4.3}$$

and

$$\|R_1 N^{-1} + \hat{Q}\|_\infty < 1 \tag{4.4}$$

(where $\hat{Q} = T_o Q N^{-1}$). If $1 \neq \sigma_j^H(R_1 N^{-1})$ $(j = 1, \ldots, n)$ we can apply theorem 4.1. We get that (4.4) is satisfied if and only if there exist X such that

$$\begin{pmatrix} I & R_1 N^{-1} \\ 0 & I \end{pmatrix}^\sim \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} I & R_1 N^{-1} \\ 0 & I \end{pmatrix} = X^\sim \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} X \tag{4.5}$$

with $X \in \mathcal{G}H_\infty$ with X_{11} invertible and $X_{11}^{-1} \in H_\infty^i$. Finally X satisfies the above properties if and only if

$$W := X \begin{pmatrix} T_o & 0 \\ 0 & \gamma^{-1} N \end{pmatrix}$$

satisfies (4.2 with $W \in \mathcal{G}H_\infty$ with W_{11} invertible and $W_{11}^{-1} \in H_\infty^i$.
The proof is complete if we show that the existence of W satisfying the conditions of part (ii) implies the existence of N satisfying 4.3. The latter follows since

$$G^\sim J G = \begin{pmatrix} T_o & 0 \\ S^\sim T_i & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & R_2^\sim R_2 - \gamma^2 I \end{pmatrix} \begin{pmatrix} T_o & T_i^\sim S \\ 0 & I \end{pmatrix}. \tag{4.6}$$

Because W has full rank on the imaginary axis, (4.2) implies that $G^\sim JG$ evaluated on the imaginary axis has the same inertia as J. According to (4.6) this requires that

$$R_2^\sim R_2 - \gamma^2 I < 0$$

which in turn implies the existence of the required N. ■

Remark: From the above proof we see that γ should not be such that $R_1 N^{-1}$ has a Hankel singular value equal to 1. It is easy to check that the Hankel singular values of $R_1 N^{-1}$ are decreasing functions of γ. Hence the number of exception points is no more than the McMillan degree of $R_1 N^{-1}$.

5 J-spectral factorization

In this section we would like to show the relation between the existence of a J-spectral factorization and the existence of a solution to the algebraic Riccati equation. Also, since the factorization is not unique, we show that the number of unstable poles of the inverse of the $(1,1)$-block of the J-spectral factor is independent of the specific choice for the J-spectral factor. This is needed because that number played an important role in the previous section. We have:

Theorem 5.1 : *Let S and T have realizations $[A, B_1, C, D]$ and $[A, B_2, C, 0]$ respectively with A stable. Then there exists $W \in \mathcal{G}H_\infty$ such that (4.2) is satisfied if and only if there exists a solution P of the following algebraic Riccati equation*

$$0 = A^{\mathrm{T}}P + PA + C^{\mathrm{T}}C - (PB_1 + C^{\mathrm{T}}D)(D^{\mathrm{T}}D)^{-1}(B_1^{\mathrm{T}}P + D^{\mathrm{T}}C) + \gamma^{-2}PB_2B_2^{\mathrm{T}}P \qquad (5.1)$$

such that the following matrix is asymptotically stable:

$$A - B_1(D^{\mathrm{T}}D)^{-1}(B_1^{\mathrm{T}}P + D^{\mathrm{T}}C) + \gamma^{-2}B_2B_2^{\mathrm{T}}P \qquad (5.2)$$

Proof : This is a direct result from [5, 11]. ■

Next, we focus on the question whether the existence of one J spectral vector W of $G^{\sim}JG$ for which W_{11} is invertible with $W_{11}^{-1} \in H_\infty^i$ implies that every spectral factor of $G^{\sim}JG$ has this property. We first need a preliminary lemma:

Lemma 5.2 : *Let $H \in L_\infty$ be a given rational matrix with $\|H\|_\infty < 1$. Then $(I+H)^{-1}$ exists and has the same number of unstable poles as H.* □

Proof : Let $[A, B, C, D]$ be a minimal realization of H. Then $(I+H)^{-1}$ has a realization

$$[A - B(I+D)^{-1}C, B(I+D)^{-1}, -C(I+D)^{-1}, (I+D)^{-1}]$$

Since H has norm less than 1 we can apply theorem 3.1. In other words there exists a matrix Q of the algebraic Riccati equation (3.1). Then, after some algebraic manipulations we get:

$$\left[A - B(I+D)^{-1}C\right]Q + Q\left[A - B(I+D)^{-1}C\right] + S = 0 \qquad (5.3)$$

where

$$S = \left[XC^{\mathrm{T}} + B(I+D)^{-1}(I+D^{\mathrm{T}})\right](I - DD^{\mathrm{T}})^{-1}\left[CX + (I+D)(I+D^{\mathrm{T}})^{-1}B^{\mathrm{T}}\right] \geq 0$$

We know that (A, B) is controllable and that A has no imaginary axis eigenvalues. Hence, if we view (3.1) as a Lyapunov equation, we get that the number of unstable eigenvalues of A is equal to the number of negative eigenvalues of X. Moreover X is not singular.

It is immediate that $A - B(I+D)^{-1}C$ has no eigenvalues on the imaginary axis. Hence, using some classical results for the Lyapunov equation (see e.g. [10]), we find that the Lyapunov equation (5.3) implies that $A - B(I+D)^{-1}C$ has as many unstable poles as A (whether or not $(A - B(I+D)^{-1}C, S)$ is not controllable is immaterial).

■

The above is for SISO systems a direct consequence of the classical theorem by Rouché (see [18]). The above allows us to derive the following theorem establishing that the number of unstable zeros of the (1,1) block of a J-spectral factorization is independent of the specific factorization chosen.

Theorem 5.3 : *Let G be given as in theorem 4.2. Let $V, W \in \mathcal{G}H_\infty$ be two spectral factors of $G^\sim JG$, i.e.*

$$V^\sim JV = G^\sim JG = W^\sim JW.$$

Then V_{11}^{-1} and W_{11}^{-1} both exist and they have the same number of unstable poles. □

Proof : Note that (4.2) together with S full row rank implies that

$$W_{11}^\sim G_{11} - \gamma^2 W_{21}^\sim W_{21} = S^\sim S > 0$$

Hence W_{11} is invertible and $\|W_{21}W_{11}^{-1}\|_\infty < \gamma^{-1}$. Also note that the number of unstable zeros of $W_{21}W_{11}^{-1}$ is equal to the number of unstable poles of W_{11}^{-1}, i.e. no pole-zero cancellations can occur.
It is easy to show (see [11]) that J-spectral factors are unique up to a constant J-unitary matrix, i.e. there exists a constant matrix A such that $V = AW$ where

$$A^\sim JA = J$$

This condition for A implies

$$A_{11}^T A_{11} - \gamma^2 A_{21}^T A_{21} = I$$

and therefore A_{11} is invertible and $\|A_{11}^{-1}A_{12}\| < \gamma$.
We find $\|A_{11}^{-1}A_{12}W_{21}W_{11}^{-1}\| < 1$ and hence, according to lemma 5.2,

$$(I + A_{11}^{-1}A_{12}W_{21}W_{11}^{-1})^{-1}$$

has at most as many poles as W_{11}^{-1}. Hence

$$\begin{aligned} V_{11}^{-1} &= (A_{11}W_{11} + A_{12}W_{21})^{-1} \\ &= W_{11}((I + A_{11}^{-1}A_{12}W_{21}W_{11}^{-1})^{-1})^{-1}A_{11} \end{aligned}$$

has no more unstable poles then W_{11}^{-1}. The above argument is symmetric and hence the theorem is proved. ■

Using the above we can extend theorem 5.1 to include the number of unstable poles of the inverse of the $(1,1)$ block.

Theorem 5.4 : *Let T and S have realizations $[A, B_1, C, D]$ and $[A, B_2, C, 0]$ respectively with A stable. Then there exists $W \in \mathcal{G}H_\infty$ where W_{11} is invertible with $W_{11}^{-1} \in H_\infty^i$ such that (4.2) is satisfied if and only if there exists a solution P of the algebraic Riccati equation (5.1) such that the matrix in (5.2) is asymptotically stable and P has no more than i unstable poles.* □

Proof : It is easily checked that if P satisfying the conditions of theorem 5.1 exists then one particular J-spectral factor $W \in \mathcal{G}H_\infty$ is given by:

$$W = [A, (B_1 \;\; B_2), C_W, D_W]$$

where:

$$C_W := \begin{pmatrix} (D^{\mathrm{T}}D)^{-1/2}(D^{\mathrm{T}}C + B_1^{\mathrm{T}}P) \\ -\gamma^{-2}B_2^{\mathrm{T}}P \end{pmatrix}$$

$$D_W := \begin{pmatrix} (D^{\mathrm{T}}D)^{1/2} & 0 \\ 0 & I \end{pmatrix}$$

Therefore we find the following realization for W_{11}^{-1}:

$$W_{11}^{-1} := [A_W, -B_1(D^{\mathrm{T}}D)^{-1/2}, (D^{\mathrm{T}}D)^{-1}(D^{\mathrm{T}}C + B_1^{\mathrm{T}}P), (D^{\mathrm{T}}D)^{-1/2}]$$

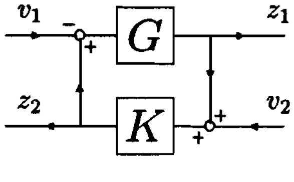
where $A_W := A - B_1(D^{\mathrm{T}}D)^{-1}(D^{\mathrm{T}}C + B_1^{\mathrm{T}}P)$. The algebraic Riccati equation for P can be rewritten as:

$$0 = A_W^{\mathrm{T}}P + PA_W + C_{W,1}^{\mathrm{T}}C_{W,1} + \gamma^{-2}PB_2B_2^{\mathrm{T}}P \tag{5.4}$$

where $C_{W,1} := C - D(D^{\mathrm{T}}D)^{-1}(D^{\mathrm{T}}C + B_1^{\mathrm{T}}P)$. Treating this equation as a Lyapunov equation, [10] tells us that the number of negative eigenvalues of P is equal to the number of unstable eigenvalues of A_W. In other words, the number of unstable poles of W_{11}^{-1} is equal to the number of negative eigenvalues of P. Because of theorem 5.3 it is sufficient to prove the result for one particular J-spectral factorization and hence the proof is complete. ■

6 Youla parameterization

The Youla parameterization is an often used tool in modern control theory (see e.g. [9, 21, 7]). However, since we allow for a fixed number of unstable poles, we need to extend this theory. First of all, we need to define the unstable closed loop poles of the closed loop system. Suppose we have the following interconnection:



(6.1)

The closed loop transfer matrix from v to z is equal to

$$T(G,K) := \begin{pmatrix} -G(I-KG)^{-1} & G(I-KG)^{-1}K \\ -K(I-GK)^{-1}G & (I-KG)^{-1}K \end{pmatrix}$$

Our standing assumption in this paper is that G is stabilizable and detectable. Then we define the unstable closed loop poles as the unstable poles of $T(G,K)$ and the number of unstable poles as the McMillan degree of the unstable part of $T(G,K)$.
We obtain left and right coprime factorizations over H_∞ of K and G:

$$\begin{aligned} K &= \tilde{V}^{-1}\tilde{U} &&= UV^{-1} \\ G &= \tilde{M}^{-1}\tilde{N} &&= NM^{-1} \end{aligned} \tag{6.2}$$

Then it is easy to show that a right coprime factorization of $T(G,K)$ is given by:

$$T(G,K) = \begin{pmatrix} -N & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} M & U \\ N & V \end{pmatrix}^{-1}.$$

Therefore the number of unstable poles of the closed loop system is equal to the number of unstable zeros of

$$\begin{pmatrix} M & U \\ N & V \end{pmatrix}$$

We can now derive the following theorem:

Theorem 6.1 : *The set of all proper controllers K of G such that the closed loop system has no more than i unstable poles is parameterized by*

$$K = (Y - MQ)(X - NQ)^{-1} = (\tilde{X} - Q\tilde{N})^{-1}(\tilde{Y} - Q\tilde{M}), \qquad Q \in H_\infty^i$$

where $N, M, \tilde{M}, \tilde{N}, X, Y, \tilde{X}, \tilde{M}$ form a doubly coprime factorization of G, i.e. (6.2) is satisfied and

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I$$

□

7 Proof of Theorem 2.2

Using the classical technique from [9] we transform the state feedback H_∞ control problem into a model-matching problem.
The following result is a direct consequence of our extended Youla parameterization as given in theorem 6.1 and an explicit expression for the doubly coprime factorization which can be found in e.g. [9, 15]:

Theorem 7.1 : *There exists a (possibly dynamic) feedback $u = Kx$ which, when applied to (2.3), yields a closed loop system with at most i unstable poles if and only if there exists a $Q \in H_\infty^i$ such that*

$$K = (Y - MQ)(X - NQ)^{-1}$$

where M, N, Y, X are defined by:

$$\begin{aligned} M &:= [A + BF, B, F, I] \\ N &:= [A + BF, B, I, 0] \\ Y &:= [A + BF, -H, F, 0] \\ X &:= [A + BF, -H, I, I] \end{aligned}$$

and F and H are such that $A + BF$ and $A + H$ are asymptotically stable. Moreover, the resulting closed loop transfer matrix is equal to:

$$G_{cl} := T_1 - T_2 Q T_3$$

where

$$\begin{aligned} T_1 &= \left[\begin{pmatrix} A + BF & -BF \\ 0 & A + H \end{pmatrix}, \begin{pmatrix} E \\ E \end{pmatrix}, \begin{pmatrix} C + DF & -DF \end{pmatrix}, 0 \right], \\ T_2 &= [A + BF, B, C + DF, D], \\ T_3 &= [A + H, E, I, 0]. \end{aligned}$$

□

It turns out that the parameterization as obtained from [9] can be simplified by replacing Q by $Q + F$. Clearly $Q + F \in H_\infty^i$ if and only if $Q \in H_\infty^i$. In this way we obtain the following corollary:

Corollary 7.2 : *There exists a (possibly dynamic) feedback $u = Kx$ which, when applied to (2.3), yields a closed loop system with at most i unstable poles if and only if there exists a $\hat{Q} \in H_\infty^i$ such that*

$$K = (\hat{Y} - M\hat{Q})(\hat{X} - N\hat{Q})^{-1}$$

where M, N, Y, X are defined by:

$$\begin{aligned} M &:= [A + BF, B, F, I] \\ N &:= [A + BF, B, I, 0] \\ \hat{Y} &:= [A + BF, -H + BF, F, F] \\ \hat{X} &:= [A + BF, -H + BF, I, I] \end{aligned}$$

Moreover, the resulting closed loop transfer matrix is equal to:

$$G_{cl} := \hat{T}_1 - T_2 \hat{Q} T_3$$

where

$$\begin{aligned}\hat{T}_1 &= [A + BF, E, C + DF, 0],\\ T_2 &= [A + BF, B, C + DF, D],\\ T_3 &= [A + H, E, I, 0].\end{aligned}$$ □

The implication (i) ⇒ (ii) in theorem 2.2 is now a direct consequence of the above corollary, theorem 5.4 and theorem 4.2. After all, the existence of a suitable feedback implies according to the above corollary the existence of a matrix $\hat{Q} \in H_\infty^i$ such that $\|\hat{T}_1 - T_2\hat{Q}T_3\|_\infty < \gamma$. Hence $\tilde{Q} := \hat{Q}T_3$ satisfies $\|\hat{T}_1 - T_2\tilde{Q}\|_\infty < \gamma$ and according to theorem 4.2 this implies the existence of a certain J-spectral factorization for all but finitely many γ. By theorem 5.4, this J-spectral factorization exists if and only if there exists a solution to an algebraic Riccati equation. Finally it is easily checked that the solution of this Riccati equation satisfies all the requirements of part (ii) of theorem 2.2.

The implication (ii) ⇒ (i) in theorem 2.2 is almost immediate. The feedback given by (2.4) results in a closed loop system $[A_W, E, C_W, 0]$ where:

$$\begin{aligned}A_W &:= A - B(D^T D)^{-1}(D^T C + B^T P),\\ C_W &:= C - D(D^T D)^{-1}(D^T C + B^T P).\end{aligned}$$

It is easy to check that the algebraic Riccati equation for P can be rewritten as:

$$0 = A_W^T P + P A_W + C_W^T C_W + \gamma^{-2} P E E^T P.$$

Moreover

$$A_W + \gamma^{-2} E E^T P$$

is asymptotically stable. It is then a direct consequence of theorem 3.1 that this feedback satisfies the conditions of part (i) of theorem 2.2.

8 Existence of a stabilizing solution to the Riccati equation

In this section we will determine γ_- which is uniquely defined by the fact that for all but finitely many γ larger than γ_- there exists a stabilizing solution to the algebraic Riccati equation. Moreover, the stabilizing solution does not exists for γ smaller than γ_-. According to theorem 2.2, γ_- is the minimal achievable L_∞ norm of the closed loop system without any stability requirements. According to corollary 7.2, we have

$$\gamma_- = \inf_{\hat{Q} \in L_\infty} \|\hat{T}_1 - T_2\hat{Q}T_3\|_\infty$$

Since T_3 is minimum-phase and $\hat{T}_1$ is strictly proper, it is easy to see that:

$$\gamma_- = \inf_{\hat{Q} \in L_\infty} \|\hat{T}_1 - T_2\hat{Q}\|_\infty$$

We still have the freedom to pick F. We choose F such that T_2 becomes co-inner. In other words, $F = -(D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}R + D^{\mathrm{T}}C)$ where R is a stabilizing solution of:

$$0 = A^{\mathrm{T}}R + RA + C^{\mathrm{T}}C - (RB + C^{\mathrm{T}}D)(D^{\mathrm{T}}D)^{-1}(B^{\mathrm{T}}P + D^{\mathrm{T}}C)$$

Then we directly obtain the following result:

$$\gamma_- = \|T_2^{\sim}\hat{T}_1\|_\infty$$

We obtain the following realization for $T_2^{\sim}\hat{T}_1$:

$$[-(A+BF)^{\mathrm{T}}, -RE, B^{\mathrm{T}}, 0] \tag{8.1}$$

In conclusion, the minimal achievable L_∞ norm is equal to the L_∞ norm of $T_2^{\sim}\hat{T}_1$ whose realization is given by (8.1).

9 Conclusion

In this paper we established a very general result regarding the existence of stabilizing solutions to the algebraic Riccati equation. The stabilizing solution exists for all but finitely many γ larger than γ_-. Moreover, the stabilizing solution does not exists for γ smaller than γ_-. Moreover, we related the number of negative eigenvalues of the stabilizing solution to the number of unstable poles needed to achieve the required L_∞ performance.

Using the techniques of this paper one can also derive conditions for the measurement feedback L_∞ control problem where we look for dynamic controllers which yield no more than i unstable closed loop poles and achieve an a priori given bound on the L_∞ norm of the closed loop. However, this seems to be mainly of theoretical interest.

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