Unit-time open-shop scheduling problems with symmetric objective functions

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Unit-time Open-shop Scheduling Problems with Symmetric Objective Functions *

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Abstract

The paper deals with the open-shop problems with unit-time operations and nondecreasing symmetric objective functions depending on job completion times. We construct two schedules, one of them is optimal for any symmetric convex function, the other is optimal for any symmetric concave function. Each of these two schedules is given by an analytically defined function which assigns to each operation the number of a unit-time slot for its processing.

1 Introduction

We consider the following open-shop problem with n jobs $\mathcal{N} = \{J_1, \ldots, J_n\}$ and m machines $\mathcal{M} = \{M_1, \ldots, M_m\}$. Each job should be processed by each machine, the processing time p_{ik} of each operation O_{ik}, $1 \le i \le n$, $1 \le k \le m$, being given. The operations of each job may be processed in any order. Each machine can handle at most one operation at a time and each job can be processed by at most one machine at a time. If preemption is forbidden, then a schedule s may be defined as a feasible combination of job and machine orders. It can be given by starting or completion times of all operations, and the effectiveness of the schedule s is characterized by vector $\mathbf{C}(s) = (C_1(s), \ldots, C_n(s))$ of completion times of jobs in $\mathcal{N}$. We also consider the analogous problem with preemption: the processing of any operation O_{ik} may be interrupted at any time and resumed later, the total length of all parts of the operation being equal to p_{ik}.

The objective is to construct a schedule which minimizes a given function $F(s) = F(C_1(s), \ldots, C_n(s))$ depending on completion times of jobs $\mathcal{N}$. The function F is nondecreasing with respect to each its argument. This means that if for two schedules s' and s'' we have $\mathbf{C}(s') \le \mathbf{C}(s'')$, then $F(s') \le F(s'')$. We write $\mathbf{X} \le \mathbf{Y}$ for $\mathbf{X} = (x_1, \ldots, x_n), \mathbf{Y} = (y_1, \ldots, y_n)$ if $x_i \le y_i$ for each $i, 1 \le i \le n$.

In this paper we consider open-shop problems with unit-time operations (i.e., $p_{ik} = 1$ for $1 \le i \le n$, $1 \le k \le m$) and nondecreasing symmetric objective functions.

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Function F is *symmetric* if it has the same value for any permutation of its arguments.

Function F is *convex* if for any two vectors $\mathbf{C}^1, \mathbf{C}^2 \in \mathcal{R}^n$ and any number λ, $0 \le \lambda \le 1$,

$$F(\lambda \mathbf{C}^1 + (1-\lambda)\mathbf{C}^2) \le \lambda F(\mathbf{C}^1) + (1-\lambda)F(\mathbf{C}^2).$$

For *concave* function the opposite inequality is valid. We denote nondecreasing symmetric convex function as $\breve{F}$ and nondecreasing symmetric concave function as $\widehat{F}$.

In what follows we use an auxiliary scheduling problem with m identical parallel machines. Each job J_i, $1 \le i \le n$, can be processed by any machine M_k, $1 \le k \le m$, which requires time p_i. The objective is to construct a schedule which minimizes function F.

To denote scheduling problems, we follow classification scheme $\alpha|\beta|\gamma$ [3], where α describes the machine environment, β stands for the job characteristics, and γ is the objective function. In the case of the open-shop problem we have $\alpha = O$, and in the case of the parallel machine problem $\alpha = P$. The job characteristic β may include one of the conditions $p_{ik} = 1$, or $p_i = m$. Parameter $Pmtn$ in the second field denotes that preemption is allowed. The third field γ is equal to $\breve{F}$ or $\widehat{F}$. For instance, the problem of minimizing symmetric convex function in the unit-time open-shop is denoted by $O|p_{ik} = 1|\breve{F}$ and the same problem with preemption is denoted by $O|p_{ik} = 1, Pmtn|\breve{F}$.

The paper is organized as follows. In Sections 2,3 we construct optimal schedules s_1^*, s_2^* for the problems with symmetric convex function $\breve{F}$ and symmetric concave function $\widehat{F}$ respectively. Both problems are considered under the assumption that $n > m$. (Otherwise, these problems are trivial.) For schedules s_1^* and s_2^* we propose analytically defined functions $t_1(i,k)$ and $t_2(i,k)$ respectively, each function assigns to an operation O_{ik}, $1 \le i \le n$, $1 \le k \le m$, the number of a unit-time slot for its processing. It should be mentioned that the result of Section 3 is substantially based on the result from [5].

In Section 4 we extend these results to the case of more general objective functions $\breve{F}^e$ and $\widehat{F}^e$ respectively.

2 Optimal schedule for symmetric convex function

In this section we define a schedule s_1^* which is optimal for both problem $O|p_{ik} = 1, Pmtn|\breve{F}$ and problem $O|p_{ik} = 1|\breve{F}$. In this schedule completion times of the first q jobs are equal to m, where $q = n - m\lfloor \frac{n}{m} \rfloor$ is the remainder of dividing n by m. Completion times of the next m jobs are equal to $q + m$. The next m jobs are completed at time $q + 2m$, etc. (see Fig. 1). For simplicity of notation, we introduce $\lfloor \frac{n}{m} \rfloor + 1$ subsets of job indices: $I^0 \doteq \{1, \ldots, q\}$, $I^l \doteq \{q + (l-1)m + 1, \ldots, q + lm\}$, $1 \le l \le \lfloor \frac{n}{m} \rfloor$.

Schedule s_1^* can be specified by a function $t_1(i,k)$ which for each $i = 1, \ldots, n$ and $k = 1, \ldots, m$ defines the number of a unit-time slot for processing the operation O_{ik}. To calculate the value $t_1(i,k)$ for some $1 \le i \le n$, $1 \le k \le m$, one needs to determine which of inequalities (1), (2) is satisfied:

$$i + k - 1 \le \alpha_i, \tag{1}$$
$$i + k - 1 > \alpha_i, \tag{2}$$

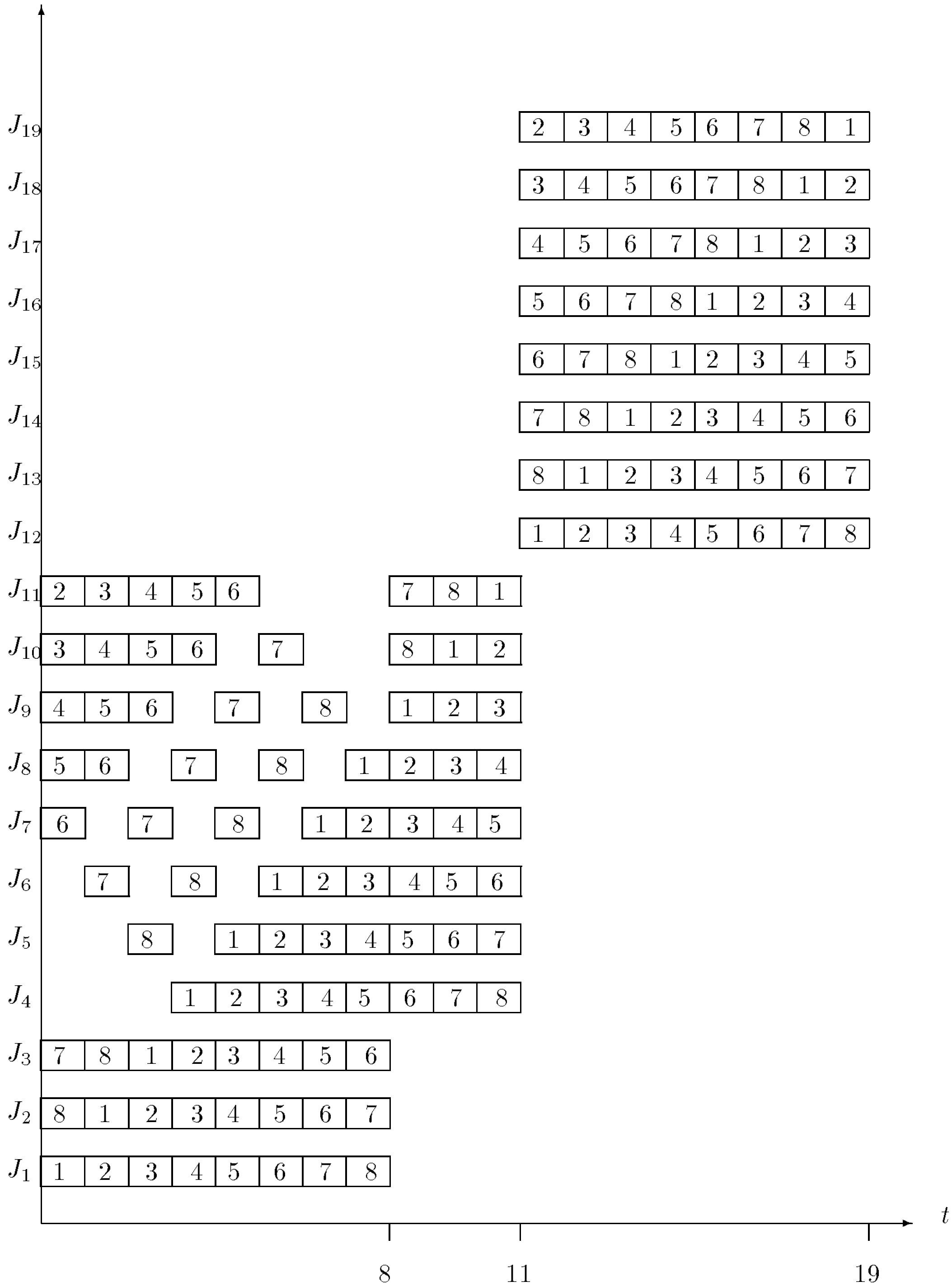

Figure 1: Optimal schedule s_1^* for the problem $O|p_{ik} = 1|\breve{F}(s)$, $\quad n = 19,\ m = 8$

where

$$\alpha_i = \begin{cases} m & \text{for } i \in I^0, \\ q + lm & \text{for } i \in I^l, \quad l = 1, \ldots, \lfloor \frac{n}{m} \rfloor, \end{cases} \tag{3}$$

The formula for $t_1(i,k)$ is given by

$$t_1(i,k) \doteq \begin{cases} i+k-1, & \text{if} \quad i \in I^0 \text{ and } (1), \\ i+k-1-m, & \text{if} \quad i \in I^0 \text{ and } (2), \\ & \\ i+k-1, & \text{if} \quad i \in I^1 \text{ and } (1), \\ i+k-m-q-1, & \text{if} \quad i \in I^1 \text{ and } (2), \;\; k \in K^1, \\ i-2(m-k+1), & \text{if} \quad i \in I^1 \text{ and } (2), \;\; k \in K^2, \\ i-(m-k+1), & \text{if} \quad i \in I^1 \text{ and } (2), \;\; k \in K^3, \\ & \\ i+k-1, & \text{if} \quad i \in I^2 \cup \ldots \cup I^{\lfloor \frac{n}{m} \rfloor} \text{ and } (1), \\ i+k-1-m, & \text{if} \quad i \in I^2 \cup \ldots \cup I^{\lfloor \frac{n}{m} \rfloor} \text{ and } (2), \end{cases} \tag{4}$$

$$\text{where} \quad \begin{array}{rclcccl} K^1 & = & \{ & 1, & \ldots, & m-q+1 & \}, \\ K^2 & = & \{ & m-q+2, & \ldots, & 2m-i+1 & \}, \\ K^3 & = & \{ & 2m-i+2, & \ldots, & m & \}. \end{array}$$

Theorem 1 *The optimal schedule s_1^* for the problems $O|p_{ik}=1|\breve{F}$ and $O|p_{ik}=1, Pmtn|\breve{F}$ can be specified by function $t_1(i,k)$ given by (4).*

In the proof of Theorem 1 we use an auxiliary problem $P|p_i = m, Pmtn|\breve{F}$ and the following result from [2, 6].

Lemma 1 (Gordon, Tanaev, 1973) *A feasible schedule for the problem $P|p_i = m$, $Pmtn|C_i \le D_i$, $D_1 \le D_2 \le \ldots \le D_n$, exists if and only if the following inequalities hold:*

$$\begin{array}{rcll} D_1 & \ge & m, & \\ \sum\limits_{i=\mu-m+1}^{\mu} D_i & \ge & \mu m, & \qquad \mu = m+1, \ldots, n. \end{array} \tag{5}$$

Proof of Theorem 1. It is easy to check that value $t_1(i,k)$ is defined for each pair $(i,k), 1 \le i \le n$, $1 \le k \le m$, and $t_1(i,k) \ge 0$. We prove that formula (4) specifies a feasible open-shop schedule (Subsection 2.1) and that this schedule is optimal for $O|p_{ik}=1|\breve{F}$ and $O|p_{ik} = 1, Pmtn|\breve{F}$ (Subsection 2.2).

2.1 The feasibility of schedule s_1^*. To prove that the schedule defined by formula (4) is feasible, we show that

i) for each job $J_i \in \mathcal{N}$ all values $\{t_1(i,k) | 1 \le k \le m\}$ are different: $t_1(i,k_1) \ne t_1(i,k_2)$ for $k_1 \ne k_2$;

ii) for each machine $M_k \in \mathcal{M}$ all values $\{t_1(i,k) | 1 \le i \le n\}$ are different: $t_1(i_1,k) \ne t_1(i_2,k)$ for $i_1 \ne i_2$.

The proof of conditions i) and ii) for jobs J_i, $i \in I^2 \cup \ldots \cup I^{\lfloor \frac{n}{m} \rfloor}$, is straightforward. We consider jobs J_i, $i \in I^0 \cup I^1$.

i) Observe that condition i) is straightforward for any job J_i, $i \in I^0$.

Consider job $J_i, i \in I^1$, processed by machines M_{k_1}, M_{k_2}.

1. If inequality (1) holds for both machines, M_{k_1} and M_{k_2}, then $t_1(i,k_1) = i + k_1 - 1$ and $t_1(i,k_2) = i + k_2 - 1$, which implies $t_1(i,k_1) \neq t_1(i,k_2)$.

2. If we have (1) for $k = k_1$ and (2) for $k = k_2$, then $t_1(i,k_1) = i + k_1 - 1 \geq i$, whereas $t_1(i,k_2) \leq \max\{i + k - m - q - 1, i - 2(m-k+1), i - (m-k+1)\} < i$.

3. Consider the case that (2) holds for both machine numbers, k_1 and k_2. It is clear that if both numbers, k_1 and k_2, belong to the same subset K^l, then $t_1(i,k_1) \neq t_1(i,k_2)$. Otherwise, for $k_l \in K^l$ we have

$$\begin{array}{rcccl} 1 & \leq & t_1(i,k_1) & < & i - 2q + 1, \\ i - 2q + 2 & \leq & t_1(i,k_2) & < & m, \\ m & < & t_1(i,k_3) & < & q + m. \end{array}$$

   Hence, $t_1(i,k_1) < t_1(i,k_2) < t_1(i,k_3)$, which implies i).

ii) Consider machine $M_k, 1 \leq k \leq m$, processing jobs J_{i_1}, J_{i_2}.

1. Let $k \in K^1$. Consider two subsets of job indices $U^1 \doteq \{1, \ldots, q + m - k + 1\}$ and $U^2 \doteq \{q + m - k + 2, \ldots, q + m\}$. Observe that $U^1 \cup U^2 = I^0 \cup I^1$.

   If $i \in U^1$, then (1) holds and $t_1(i,k) = i + k - 1 \geq k$.
   If $i \in U^2$, then (2) holds and $t_1(i,k) = i + k - m - q - 1 \leq k - 1$.

   This means that if both job indices, i_1 and i_2, belong to the same subset U^l, then condition ii) is valid. Otherwise, if $i_1 \in U^1$, $i_2 \in U^2$, we have $t_1(i_2,k) < t_1(i_1,k)$, which also implies ii).

2. Let $k \in K^2 \cup K^3$. Consider five subsets of job indices:
   $V^1 \doteq \{1, \ldots, m - k + 1\}$,
   $V^2 \doteq \{m - k + 2, \ldots, q\}$,
   $V^3 \doteq \{q + 1, \ldots, q + m - k + 1\}$,
   $V^4 \doteq \{q + m - k + 2, \ldots, 2m - k + 1\}$,
   $V^5 \doteq \{2m - k + 2, \ldots, q + m\}$.

   If $i \in V^1$, then $t_1(i,k) = i + k - 1$.
   If $i \in V^2$, then $t_1(i,k) = i + k - m - 1$.
   If $i \in V^3$, then $t_1(i,k) = i + k - 1$.
   If $i \in V^4$, then $t_1(i,k) = i - 2(m - k + 1)$.
   If $i \in V^5$, then $t_1(i,k) = i - (m - k + 1)$.

   This means that if both job indices belong to the same subset V^l, then condition ii) is valid. Otherwise, for $i_l \in V^l$ we have

$$\begin{array}{rcccl} k & \leq & t_1(i_1,k) & \leq & m, \\ 1 & \leq & t_1(i_2,k) & \leq & q + k - m - 1, \\ q + k - 1 & < & t_1(i_3,k) & \leq & q + m, \\ q + k - m - 1 & < & t_1(i_4,k) & \leq & k - 1, \\ m & < & t_1(i_5,k) & \leq & q + k - 1. \end{array}$$

Hence, $t_1(i_2,k) < t_1(i_4,k) < t_1(i_1,k) < t_1(i_5,k) < t_1(i_3,k)$.

This completes the proof of feasibility of schedule s_1^*.

2.2 The optimality of schedule s_1^*. We consider the parallel-machine problem $P|p_i = m, Pmtn|\breve{F}$ and prove that schedule s_1^* defined by (4) is optimal for it. (Taking into account that s_1^* is a feasible open-shop schedule, this immediately implies that s_1^* is optimal for the problems $O|p_{ik} = 1|\breve{F}$ and $O|p_{ik} = 1, Pmtn|\breve{F}$ as well.)

For this purpose, we consider an arbitrary feasible schedule s^0 for the problem $P|p_i = m$, $Pmtn|\breve{F}$ and prove that $\breve{F}(s^0) \geq \breve{F}(s_1^*)$. To this end, we construct a sequence of n-vectors $\mathbf{C}^0, \mathbf{C}^1, \ldots, \mathbf{C}^r$ for which

$$\breve{F}(s^0) = \breve{F}(\mathbf{C}^0) \geq \breve{F}(\mathbf{C}^1) \geq \ldots \geq \breve{F}(\mathbf{C}^r) \geq \breve{F}(s_1^*). \tag{6}$$

The vector $\mathbf{C}^0 = (C_1^0, \ldots, C_n^0)$ is obtained from the vector $\mathbf{C}(s^0) = (C_1(s^0), \ldots, C_n(s^0))$ by sequencing values $C_i(s^0)$ in nondecreasing order. Since $\breve{F}$ is a symmetric function, we have $\breve{F}(s^0) = \breve{F}(\mathbf{C}^0)$.

The components of the vector $\mathbf{C}^r$ are given by the following formula:

$$C_i^r = \begin{cases} C_i^0 & \text{for } i \in I^0, \\ \overline{C}^k & \text{for } i \in I^k, \quad k = 1, \ldots, \lfloor \frac{n}{m} \rfloor, \end{cases}$$

where $\overline{C}^k$ is the mean value of the components $i \in I^k$ of the vector $\mathbf{C}^0$: $\overline{C}^k = \frac{1}{|I^k|} \sum\limits_{i \in I^k} C_i^0$.

Observe that $\mathbf{C}^r \geq \mathbf{C}(s_1^*)$, which implies $\breve{F}(\mathbf{C}^r) \geq \breve{F}(s_1^*)$. Indeed, since s^0 is a feasible schedule, we have $C_i^0 \geq m$ for any $i \in I^0$, and due to Lemma 1 $\sum_{i \in I^l} C_i^0 \geq (q + lm)m = \alpha_i m$ for any l, $1 \leq l \leq \lfloor \frac{n}{m} \rfloor$. Hence $C_i^r \geq \alpha_i, i = 1, \ldots, n$. From the other hand, due to formula (4) completion times of jobs in schedule s_1^* satisfy the relation $C_i(s_1^*) = \max_{1 \leq k \leq m} t_1(i,k) = \alpha_i$, $1 \leq i \leq n$. Hence, we have $C_i^r \geq C_i(s_1^*)$ for any i, $1 \leq i \leq n$.

Vectors $\mathbf{C}^1, \ldots, \mathbf{C}^r$ are constructed in such a way that the components $\{C_i^j | i \in I^0\}$ are the same for all j, $0 \leq j \leq r$: $C_i^j = C_i^0, \quad i \in I^0$, and for each subset of indices I^k, $k = 1, \ldots, \lfloor \frac{n}{m} \rfloor$, the mean values of the components $\{C_i^j | i \in I^k\}$ are equal for all j, $0 \leq j \leq r$:

$$\frac{1}{|I^k|} \sum_{i \in I^k} C_i^j = \overline{C}^k, \quad 0 \leq j \leq r.$$

The algorithm starts with the last subset $I^{\lfloor \frac{n}{m} \rfloor}$ and modifies only these components. Then it proceeds with $I^{\lfloor \frac{n}{m} \rfloor - 1}, \ldots, I^1$.

Algorithm 1.

> $j := 0$;
> <u>FOR</u> $k := \lfloor \frac{n}{m} \rfloor$ <u>TO</u> 1 <u>BY</u> -1 <u>DO</u>
>   *select*$(\mathbf{C}^j, I^k, u, v)$;
>   <u>WHILE</u> $u \in I^k$ <u>DO</u>
>   *modify*$(C_u^j, C_v^j, \mathbf{C}^{j+1})$;
> $j = j + 1$;
>   *select*$(\mathbf{C}^j, I^k, u, v)$;
>   <u>END</u> ;
> <u>END</u> ;
> <u>STOP</u>.

Procedure *select*$(\mathbf{C}^j, I^k, u, v)$ considers subset of indices I^k of the vector $\mathbf{C}^j$ and selects two components C_u^j and C_v^j in the following way: value C_u^j is the largest component which is less than $\overline{C}^k$, and value C_v^j is the smallest component which is greater than $\overline{C}^k$. If for vector $\mathbf{C}^j$ all components $\{C_i^j | i \in I^k\}$ are equal: $C_i^j = \overline{C}^k$, $i \in I^k$, then we set $u = q + (k-1)m$ to go on with the next subset I^{k-1}.

Procedure *modify*$(C_u^j, C_v^j, \mathbf{C}^{j+1})$ constructs the next vector $\mathbf{C}^{j+1}$ by modifying two selected components C_u^j and C_v^j of vector $\mathbf{C}^j$:

$$C_u^{j+1} = C_u^j + \delta,$$

$$C_v^{j+1} = C_v^j - \delta,$$

where $\delta = \min\{\overline{C}^k - C_u^j, C_v^j - \overline{C}^k\}$. As a result, one of the values C_u^{j+1} or C_v^{j+1} becomes equal to $\overline{C}^k$.

Thus, if we consider the set Q of components that are identical for vectors C^j and C^r, then for initial vector $\mathbf{C}^0$ we have $Q \supset I^0$ and each iteration adds at least one more component to the set Q. Therefore, in at most $n - q$ iterations Algorithm 1 constructs the vector $\mathbf{C}^r$.

To illustrate how Algorithm 1 works, we consider the following instance of the open-shop problem with 11 jobs and 7 machines.

Let s^0 be a feasible schedule with the following vector of job completion times:

$$\mathbf{C}^0 = (7, \quad 7, \quad 7, \quad 7, \quad 8, \quad 8, \quad \overset{\surd}{10}, \quad 11, \quad \overset{\surd}{13}, \quad 13, \quad 14).$$

For this instance we have two subsets of indices: $I^0 = \{1, \ldots, 4\}$ and $I^1 = \{5, \ldots, 11\}$. Applying algorithm 1 to subset I^1, we obtain the following vectors $\mathbf{C}^1, \mathbf{C}^2, \mathbf{C}^3, \mathbf{C}^4 = \mathbf{C}^r$:

$$\begin{aligned}
\mathbf{C}^1 &= (7, \quad 7, \quad 7, \quad 7, \quad 8, \quad \overset{\surd}{8}, \quad 11, \quad 11, \quad \overset{\surd}{12}, \quad 13, \quad 14),\\
\mathbf{C}^2 &= (7, \quad 7, \quad 7, \quad 7, \quad 8, \quad \overset{\surd}{9}, \quad 11, \quad 11, \quad 11, \quad \overset{\surd}{13}, \quad 14),\\
\mathbf{C}^3 &= (7, \quad 7, \quad 7, \quad 7, \quad \overset{\surd}{8}, \quad 11, \quad 11, \quad 11, \quad 11, \quad 11, \quad \overset{\surd}{14}),\\
\mathbf{C}^4 &= (7, \quad 7, \quad 7, \quad 7, \quad 11, \quad 11, \quad 11, \quad 11, \quad 11, \quad 11, \quad 11).
\end{aligned}$$

Let us now prove that $F(\mathbf{C}^j) \geq F(\mathbf{C}^{j+1})$, $0 \leq j \leq r-1$. We introduce a vector $\mathbf{X}^j$ which differs from $\mathbf{C}^j$ by permutation of components C_u^j and C_v^j:

$$\mathbf{C}^j = (C_1^j, \ldots, C_u^j, \ldots, C_v^j, \ldots, C_n^j),$$
$$\mathbf{X}^j = (C_1^j, \ldots, C_v^j, \ldots, C_u^j, \ldots, C_n^j).$$

It is easily checked that vector $\mathbf{C}^{j+1}$ may be represented by $\mathbf{C}^{j+1} = \lambda \mathbf{C}^j + (1-\lambda)\mathbf{X}^j$, where $\lambda = 1 - \delta/(C_v^j - C_u^j)$.

The objective function F is convex: $F(\mathbf{C}^{j+1}) \leq \lambda F(\mathbf{C}^j) + (1-\lambda)F(\mathbf{X}^j)$, and it is symmetric: $F(\mathbf{C}^j) = F(\mathbf{X}^j)$. Thus, $F(\mathbf{C}^{j+1}) \leq F(\mathbf{C}^j)$. This completes the proof of Theorem 1.

∎

Theorem 1 can be applied to $O|p_{ik} = 1|\breve{F}$ with $\breve{F} = C_{\max} = \max_{1 \leq i \leq n}\{C_i\}$ or $\breve{F} = \sum_{i=1}^n C_i^\alpha$ for $\alpha \geq 1$. Since both objective functions are convex symmetric functions, schedule s_1^* is optimal for both criteria: $C_{\max}$ and $\sum C_i^\alpha$.

3 Optimal schedule for symmetric concave function

Theorem 2 *The optimal schedule s_2^* for the problems $O|p_{ik} = 1|\widehat{F}$ and $O|p_{ik} = 1, Pmtn|\widehat{F}$ is specified by function*

$$t_2(i,k) \doteq \begin{cases} i+k-1, & \text{if} \quad i+k-1 \leq \lceil i/m \rceil m; \\ i+k-1-m, & \text{if} \quad i+k-1 > \lceil i/m \rceil m. \end{cases} \tag{7}$$

An instance of schedule s_2^* is represented in Fig. 2.

Proof. It is easy to check that $t_2(i,k)$ is defined for each pair (i,k) and $t_2(i,k) \geq 0$. The feasibility of schedule s_2^* (the validity of conditions i), ii) from Section 2) is straightforward.

To prove the optimality of schedule s_2^*, we show that schedule s_2^* defines by (7) is optimal for the problem $P|p_i = m,\ Pmtn|\widehat{F}$. As it was proved in [5], for the problem $P|Pmtn|\widehat{F}$ there exists an optimal schedule without preemption. This implies that an optimal schedule s_P^* for the problem $P|p_{ik} = m,\ Pmtn|\widehat{F}$ with equal processing times can be constructed by the following simple procedure: schedule m jobs in time interval $[0, m]$, next m jobs in time interval $(m, 2m]$, and so on. It is immaterial which m jobs we choose in each step since the objective function is symmetric. So, job completion times for s_P^* can be given by $C_i(s_P^*) = \lceil i/m \rceil m,\ i = 1, \ldots, n$. From the other hand, job completion times in s_2^* satisfy relations $C_i(s_2^*) = \max_{1 \leq k \leq m}\{t(i,k)\} = \lceil i/m \rceil m$. This means that s_2^* is optimal for $P|p_i = m$, $Pmtn|\widehat{F}$ and hence it is optimal for $O|p_{ik} = 1, Pmtn|\breve{F}$ and $O|p_{ik} = 1|\breve{F}$ as well. Theorem is proved. ∎

Observe that function $\sum C_i$ is convex and concave simultaneously, and hence, both schedules, s_1^* and s_2^*, are optimal for this objective function. It should be mentioned here that optimal schedule for the problem $O|p_{ik} = 1|\sum C_i$ was constructed earlier in [1, 4, 6]. The other useful observation is that function $\sum C_i^\alpha$ is concave for any $\alpha, 0 < \alpha \leq 1$, and therefore, schedule s_2^* is optimal for it.

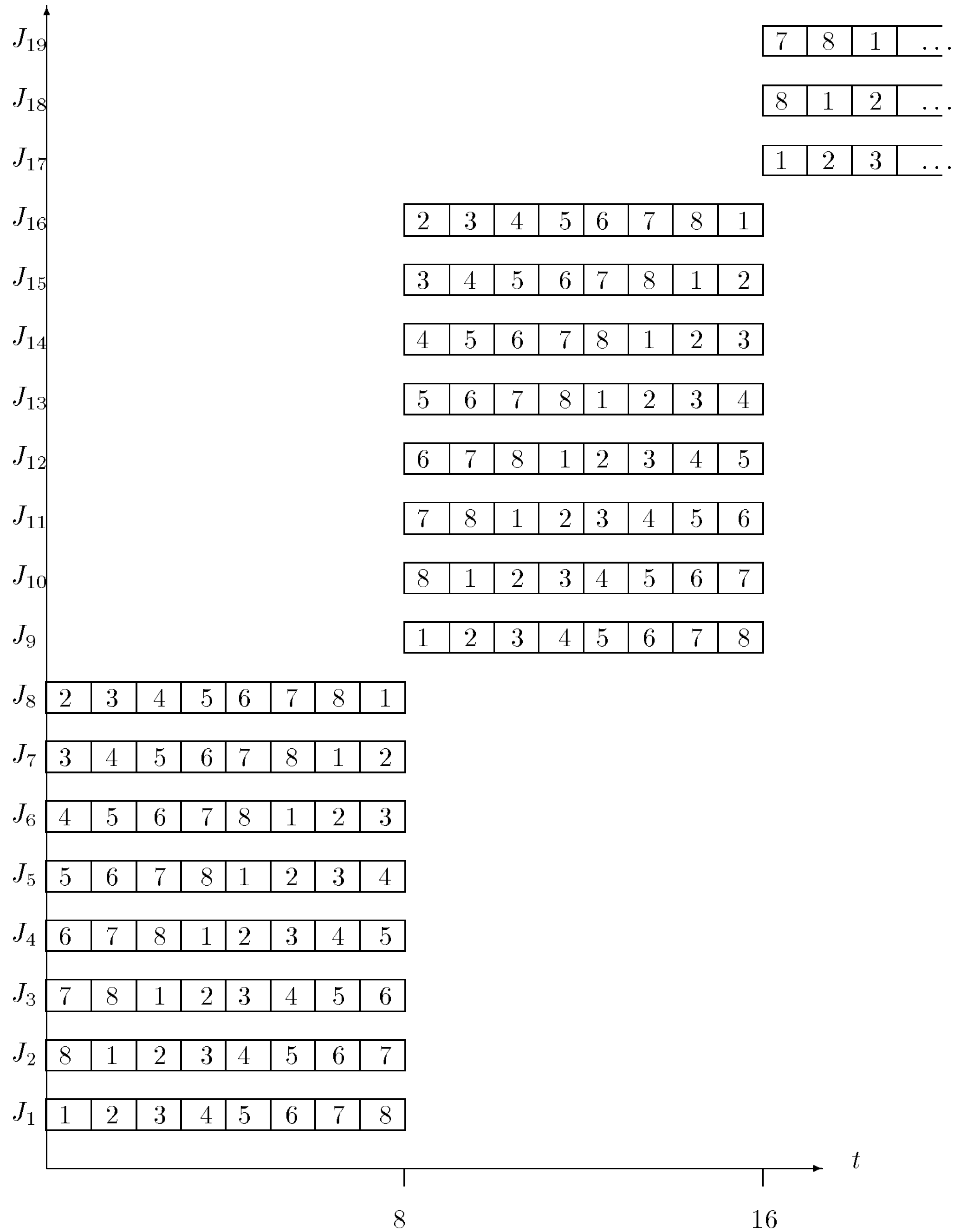

Figure 2: Optimal schedule s_2^* for the problem $O|p_{ik}=1|\widehat{F}(s)$, $n=19$, $m=8$.

4 Some generalizations

In this section we consider more general e-quasiconvex and e-quasiconcave functions. Note that due to [5] the schedule s_2^* is optimal for any e-quasiconcave nondecreasing objective function even if this function is not symmetric.

The definition of convex function is given in Section 1. We remind now the definitions of quasiconvex and e-quasiconvex functions.

Let E_0^n be a set of all vectors in $\mathcal{R}^n$ with components from the set $\{0, 1, -1\}$.

Function $F(\mathbf{C})$ is *quasiconvex* if the inequality

$$F(\lambda \mathbf{C}^1 + (1 - \lambda)\mathbf{C}^2) \leq \max\{F(\mathbf{C}^1), F(\mathbf{C}^2)\} \tag{8}$$

holds for any two vectors $\mathbf{C}^1, \mathbf{C}^2 \in \mathcal{R}^n$ and any number λ, $0 \leq \lambda \leq 1$.

Function F is *e-quasiconvex* if (8) holds for any two vectors $\mathbf{C}^1, \mathbf{C}^2 \in \mathcal{R}^n$ such that $\mathbf{C}^2 = \mathbf{C}^1 + \alpha\mathbf{e}$, $\mathbf{e} \in E_0^n$, and $\alpha \in \mathcal{R}$.

By definition, any convex function is quasiconvex, and any quasiconvex function is e-quasiconvex. It is easy to check that there exist e-quasiconvex functions which are not quasiconvex, and there also exist quasiconvex functions which are not convex.

The definitions of concave, quasiconcave and e-quasiconcave functions are similar: it is just sufficient to replace the sign $\leq$ by $\geq$, and *min* by *max*.

Let us show that the schedule s_1^* constructed in Section 2 is optimal for arbitrary e-quasiconvex nondecreasing symmetric function $\breve{F}^e$.

Indeed, the convexity of function $\breve{F}$ is used in the proof of Theorem 1 to establish that $\breve{F}(\mathbf{C}^{j+1}) \leq \breve{F}(\mathbf{C}^j)$. Let us show that the analogous inequality holds for $\breve{F}^e$. We considered two vectors $\mathbf{C}^j$ and $\mathbf{X}^j$. These vectors satisfy the relation $\mathbf{X}^j = \mathbf{C}^j + \alpha\mathbf{e}$, where $\alpha = C_v^j - C_u^j$ and vector $\mathbf{e}$ has only two nonzero components $e_u = 1$, $e_v = -1$.

Due to the definition of e-quasiconvex function, vector $\mathbf{C}^{j+1} = \lambda\mathbf{C}^j + (1 - \lambda)\mathbf{X}^j$ satisfies the inequality $\breve{F}^e(\mathbf{C}^{j+1}) \leq \max\{\breve{F}^e(\mathbf{C}^j), \breve{F}^e(\mathbf{X}^j)\}$. Taking into account that function $\breve{F}^e$ is symmetric, we obtain $\breve{F}^e(\mathbf{C}^j) = \breve{F}^e(\mathbf{X}^j)$, which implies $\breve{F}^e(\mathbf{C}^{j+1}) \leq \breve{F}^e(\mathbf{C}^j)$.

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