

A property of assignment type mixed integer linear programming problems

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EINDHOVEN UNIVERSITY OF TECHNOLOGY
Department of Mathematics and Computing Science

STATISTICS AND OPERATIONS RESEARCH GROUP

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by

J.F. Benders and J.A.E.E. van Nunen

Eindhoven, October 1982

The Netherlands

A PROPERTY OF ASSIGNMENT TYPE MIXED INTEGER LINEAR
PROGRAMMING PROBLEMS

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J.F. Benders* and J.A.E.E. van Nunen**

October 1982

* Eindhoven University of Technology, Department of Mathematics and
Computing Science

** Graduate School of Management Delft

ABSTRACT

In this paper we will prove that rather tight upper bounds can be given for the number of non-unique assignments that are achieved after solving the linear programming relaxation of some types of mixed integer linear assignment problems. Since in these cases the number of splitted assignments is small a heuristic can be used to reach a practically good and feasible assignment. Moreover the type of proof can be exploited for related problems to investigate whether a linear programming relaxation will yield mainly integer assignments.

Key words mixed integer linear programming, assignment problems, location allocation problems, distribution problems.

1. INTRODUCTION

It is well known that solving the linear programming relaxation of mixed integer linear programs often results in a solution which contains only a relatively small number of variables that should be integer but are not integer in the solution of the relaxed problem.

However, it is not clear when one might expect a linear programming relaxation of a mixed integer problem to yield mainly integer solutions or not. Nor is there, up to now, a complete theory on how one can force integer solutions by adding constraints or variables to the relaxed problem formulation.

In this paper we will prove that for a specific class of assignment type problems, rather tight bounds can be given for the number of non-unique assignments, if one uses the linear programming relaxation of the mixed integer problem. The proof can be exploited for related problems to investigate whether the linear programming relaxation will work. Moreover the type of proof might show directions for adding constraints and variables which force a solution to be mainly integer. Originally we encountered problems with a structure as will be described later on, during a practical study on location allocation problems within a Dutch brewery, see [2] [3].

There we were confronted with assignment type problems with over 50.000 zero-one-variables. Later on we met similar problem structures e.g. for the distribution and storage of LPG-gas within an oil company. Also in literature some models show a similar structure e.g. an allocation model for catalogue space planning (see [6]).

In section 2 we will introduce the idea of the mentioned proof for the generalized linear assignment problem. Section 3 will be used to introduce the location allocation problem we met at the brewery and for the description of the proof in that situation.

In the final part of the paper we discuss some extensions and give some comments.

2. THE NUMBER OF NON-UNIQUE ASSIGNMENTS IN THE RELAXATION OF THE GENERALIZED LINEAR ASSIGNMENT PROBLEM

Generalized linear assignment problems arise for example if n jobs have to be assigned to m machines with restricted capacity, and where it is not allowed to split jobs up over more machines. Let a_{ij} be the required capacity if job j is executed on machine i , and let b_i be the capacity of machine i .

Moreover, let c_{ij} be the cost of executing job j on machine i .

If we define x_{ij} as the 0-1 variable which equals 1 if job j is assigned to machine i and 0 if not, then the generalized assignment problem can be formulated by

$$\text{minimize } \sum_{i,j} c_{ij} x_{ij}$$

subject to

$$(1) \quad \sum_j a_{ij} x_{ij} \leq b_i \quad i = 1, 2, \dots, m$$

$$(2) \quad \sum_i x_{ij} = 1 \quad j = 1, 2, \dots, n$$

$$(3) \quad x_{ij} \in \{0, 1\} \quad i = 1, 2, \dots, m \\ j = 1, 2, \dots, n$$

The linear programming relaxation is obtained if one replaces the conditions (3) by $x_{ij} \geq 0$.

The solution of the relaxed problem will, in general, contain some jobs j for which $x_{ij} \neq 0$ for several i . In other words, job j is splitted up over more than one machine. However, a upper bound on the number of splitted jobs can be given.

Theorem 1

If one solves the linear programming relaxation of the generalized linear assignment problem then the number of non-unique assignments is less than or equal to the number of machines of which the capacity

is used completely.

Proof

Consider any basic feasible solution of the relaxed problem. Such a solution will in general contain a number of non-integer assignments. Let m_1 be the number of non-zero slack activities with respect to the capacity restrictions (1). Clearly the number of fully occupied machines can be given by $m_2 = m - m_1$. We denote by n_1 the number of non-splitted jobs, and by λ the average number of machines to which a splitted job is assigned. Consequently $\lambda \geq 2$. Let $n_2 = n - n_1$ be the number of splitted jobs. Now the total number of non-zero activities equals

$$n_1 + \lambda n_2 + m_1 = n + (\lambda - 1)n_2 + m_1$$

However, since the number of constraints in the relaxed problem is $n + m$, we have that any basic feasible solution contains at most $n + m$ non-zero activities. Consequently

$$n + (\lambda - 1)n_2 + m_1 \leq n + m$$

which implies

$$n_2 \leq \left\lceil \frac{m_2}{\lambda - 1} \right\rceil$$

Hence, $n_2 \leq m_2$ since $\lambda \geq 2$.

Note that if $\lambda > 2$, then the number of non-unique assignments is even smaller. Since the number of machines is in general small compared with the number of jobs, the solutions of the relaxed problem will be a good starting point for a heuristic that assigns the remaining splitted jobs. Moreover, the quality of such an heuristic can easily be judged by comparing the costs of the resulting solution with the

costs of the linear programming problem, since the value of the linear programming relaxation is a lower bound for the value of the mixed integer problem. On the other hand, any feasible integer solution that is found by a heuristic forms an upper-bound for the optimal mixed integer solution. In the practical problems we solved, the heuristics led to solutions that were within .1 percent of the lower bound and consequently within .1% of the optimal solution.

3. THE NUMBER OF NON-UNIQUE ASSIGNMENTS IN A LOCATION ALLOCATION PROBLEM WITHIN A BREWERY

The method of counting non-zero activities in combination with some reasonable assumptions about the behaviour of an optimum solution of the problem under consideration, can be applied for more complicated mixed integer linear programming problems involving assignment restrictions.

In fact, we first used it in a practical study we performed for a Dutch brewery. There mixed integer problems formulations were incorporated in a Decision Support System for supporting decision-making with respect to the required production and distribution structure. A simplified model formulation might be based on a distribution network as described in Figure 1.

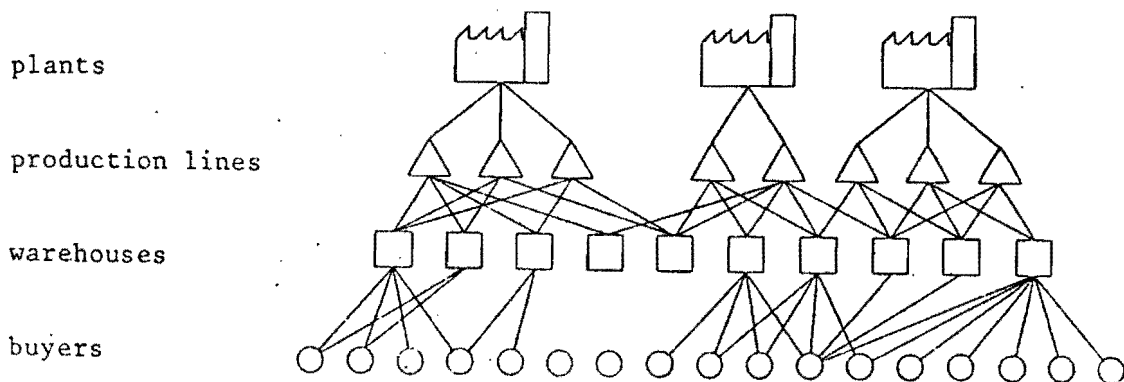


Figure 1 Simplified distribution network for the location allocation problem

In the above network each brewery consists of a number of production lines. Each production line can produce one or several of the n products that are considered. The products have to be transported to and stored in warehouses which are located at different places in the country. The buyers which are themselves wholesalers have to be supplied from the warehouses. It is requested for practical reasons that each buyer is assigned to exactly one warehouse for all his products. The demand of the buyers is supposed to be known. In this actual situation(s) the forecasts for the demand in the main season were rather adequate for mid- and long term planning goals. The maximum time that a product is allowed to be in a warehouse is limited. The problem might be formulated as follows:

which warehouses, out of a given potential set should be used. How should the available buyers be allocated to these warehouses, such that, within the capacity constraints of lines and warehouses, the costs for production, handling and transportation are minimal.

The above problem formulation requires in fact the determination of the amount of each product that has to be transported from each production line to each warehouse. Moreover, it requires for each buyer the determination of the warehouse he is assigned to.

Since the demand is supposed to be known, the determination of the transportation and assignment variables enables us to deduce from them directly, how much each production line should produce of a given product, how much of each product should be stored in each warehouse, etc.

As constraints we encountered capacity restriction for the warehouses and the production lines. In addition, the production time at the different production lines was requested to be more or less equal in order to avoid big differences in the load of older and newer lines. The maximal throughput time of beer in warehouses was equal to 3 weeks for bottles and 1 week for casks. This upper-bound of three weeks is due to internal brewery regulations for the maximal time that beer is allowed to spend in their warehouses. In fact, this throughput time

is a control variable, since reductions from 3 to e.g. 2.7 weeks means that more buyers can be assigned to a warehouse. The required space for a ton of a certain product was warehouse-dependent. Moreover transportations costs were not linear with respect to the distance.

Problems of the above type are described in several articles in literature, see [1], [3], [4]. We reported about the practical applications within the Dutch brewery in [2] and [3].

The model input was based on four lists with names, as given in Figure 2.

Name of the list	Name	Contents
P	Products	the names of the different products
L	Lines	the names of the different production lines
W	Warehouses	the names of the different warehouses
B	Buyers	the names of the different buyers

Figure 2 Basic lists for the location-allocation model

Based on the above list we define the relevant input data as given in Figure 3. The contents of most of the tables of Figure 3 will be clear immediately. An exception might be EGINLP (1,p) which contains the egalization indices for handling differences between the performance of production lines, such differences might be caused by the age of the line as well as by planned or controlled differences. A second exception might be CONVPW (p,w) which is used to indicate the difference in space that is required for storing one ton of product p in each of the warehouses.

Table	Name	Descriptions
DBP (b,p)	Demand (b,p)	The demand of buyer p for product p.
TCBW (b,w)	Transportation costs (b,w)	The transportation costs for one ton of product from buyer b to warehouse w.
HCPW (p,w)	Handling costs (p,w)	The handling costs for one ton of product p at line l.
CAPLP (l,p)	Capacity (l,p)	The production capacity of line l with respect to product p.
EGINLP (l,p)	Egalization index (l,p)	Egalization index of line l for product p.
UPCAPW (w)	Uppercapacity (w)	The maximal capacity of warehouse w.
LOCAPW (w)	Lower capacity (w)	The minimal capacity that should be used in warehouse w.
THRUPW (p,w)	Throughput time (p,w)	The maximal throughput time of product p in warehouse w.
CONVPW (p,w)	Conversion index (p,w)	The conversion index for space occupation of product p in warehouse w.

Figure 3 Notation of the input tables for the location allocation model

Most of the data required for the tables of Figure 3 could be selected from a data base that was available within the brewery. From the above data we deduced the quantities listed in Figure 4.

Table	Description
TPL (p,l)	Time required to produce one ton of product p on line l, after egalization.
CAPBW (b,w)	Capacity required to store the full demand of buyer b in warehouse w.
CPLW (p,l,w)	Costs for production, handling and transportation of one ton of product p from line l to warehouse w.
CBW (b,w)	Cost for handling and transportation of the demand of buyer b from warehouse w to buyer b.

Figure 4 Some additional input tables for the location allocation model

The model formulation can now be given by means of the decision variables. The first group of decision variables contains the transportation variables:

TPLW (p,l,w) The amount of product p that has to be produced on line l for transportation to warehouse w.

The second group contains the assignment variables:

ASBW (b,w) A 0-1 variable, which is 1 when buyer b is assigned to warehouse w and 0 otherwise

The following type of restrictions had to be satisfied:

BALPW (p,w) The balancing restrictions which guarantee that the amount of product p that is transported to w is equal to the demand for product p in warehouse w.

CAPUL (l) The production capacity restriction for lines, which guarantee that (egalized) production capacity for line l is not violated.

CAPUW (w) The capacity restrictions for the warehouses, which

CAPLLW (w) guarantee that the maximum (minimum) capacity of warehouse w is not violated.

ASRB (b) The assignment restriction which guarantees that buyer b is assigned to just one warehouse.

The problem formulation can now be stated as the determination of the decision variables TPLW (p,l,w) and ASBW (b,w) such that the total production, handling and transportation costs are minimal. These costs are represented in the object function COST.

In a practical situation with 10 products, 20 lines, 50 potential warehouses and 900 buyers this yields a MILP with 55.000 columns and 1.520 rows. Solving such large systems is even with the current technology only possible if we use the problem structure in a efficient way. The modularity and efficiency was not only required for finding solutions in an acceptable time but also for creating the possibilities for fast changes in the model as well as in the parameters. The cpu time of a computer run varied from 3 minutes for a 'first run' to less than 30 seconds if a starting basis for the simplex procedure was provided.

column	row	matricelement	relation
TPLW (p,l,w)	BALPW (p,w) CAPUL (l) COST	1 TPL (p,l) CPLW (p,l,w)	
ASBW (b,w)	BALPW (p,w) CAPUW (w) CAPLLW (w) ASRB (b) COST	-DBP (b,p) CBW (b,w) -CBW (b,w) 1 CBW (b,w)	.
RHS	BALPW (p,w) CAPUL (l) CAPUW (w) CAPLLW (w) ASRB (b)	0 1 CAPW (w) -LOCAPW (w) 1	= ≤ ≤ ≤ =

Figure 5 Definition of the non-zero coefficients in the simplex tableau of the location allocation model.

The size of the problem is determined by the number of decision variables and the number of restrictions. The number of decision variables equals

$P * W * L + B * W$ where P, L, W and B stand for the number of products, lines, warehouses and buyers respectively. The number of restrictions is $P * W + 2W + L + B$. Based on these data we can state a similar theorem as for the generalized assignment problem.

Theorem 2

If one solves the linear programming relaxation of the above location allocation problem, then for any feasible basic solution in which every warehouse stores every product we have that the number of non-unique assignments is less than or equal to the number of fully occupied production lines plus the number of fully occupied warehouses plus the number of exactly satisfied lower bounds for the capacity of the warehouses.

Proof

Denote by L_1, UW_1 and LW_1 the number of non-zero slack activities for the capacity restrictions on the production lines, and the warehouses. Let L_2, UW_2 and LW_2 be the number of fully occupied production lines, fully occupied warehouses and minimally occupied warehouses, respectively. Clearly $L = L_1 + L_2$; $W = UW_1 + UW_2$ and $W = LW_1 + LW_2$. Let B_1 and B_2 be the number of unique and non-uniquely assigned buyers, respectively. Let λ be the average number of warehouses over which a non-unique assigned buyer is splitted. Finally, if every warehouse stores every product then at least $P * W$ variables $TPLW (p,l,w)$ are non-zero. In the actual application the customer demand was such that any feasible conditions had to meet this condition. So the number of non-zero activities equals

$$P * W + L_1 + UW_1 + LW_1 + B + (\lambda - 1)B_2$$

which is less than or equal to the maximum number of non-zero activities in a basic feasible solution which equals

$$P * W + L + 2W + B$$

so

$$L_1 + UW_1 + LW_1 + (\lambda-1)B_2 \leq L + 2W$$

or

$$B_2 \leq \frac{L_2 + UW_2 + LW_2}{\lambda - 1}$$

since $\lambda \geq 2$ we have

$$B_2 \leq L_2 + UW_2 + LW_2$$

Since the number of warehouses and production lines is in general small compared with the number of buyers, the linear programming relaxation already gives mainly feasible integer assignments. In the practical examples there remain between 3 and 15 non-unique assignments.

4. EXTENSIONS AND COMMENTS

The above method can be used as said before for related problems. However, the achieved bounds will not always be as tight as in the example given. If it is possible to add constraints of which each can tie at least two variables, then such additions help to reduce the number of non-unique assignments. On the other hand, adding variables of which one can prove that they will be non-zero in the optimal solution of the relaxed linear programming formulation also reduces the number of non-unique assignments.

From a practical point of view one can often directly use the achieved solution. For example, in the brewery problem, if the required storing space for each buyer is relatively small with respect to the available space in each warehouse, then an acceptable solution can be achieved by rounding off the 'integer fractions'.

Such a rounding off was practically feasible since it meant only a small reduction of the throughput time. In the cases where the required

storage space for buyers was not relatively small suitable heuristics could be used. Hence, for these large mixed integer problems with often over 50.000 zero-one variables it was possible to avoid the time consuming mixed integer procedures by using standard linear programming.

As mentioned, the above model formulation was used within a large Decision Support System for analysing distribution and production structures.

For example the opening or closing of warehouses was not explicitly incorporated within the model but could be investigated by changing the input data. For a detailed description and analysis of the brewery problem, see [3].

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