On Hankel invariant distribution spaces

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by

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Contents

Abstract

Three Hankel invariant test function spaces and the associated generalized function spaces are introduced. The elements of the respective test function spaces are described both in functional analytic and in classical analytic terms. It is proved that one of the test function spaces equals the space H_μ of Zemanian. Finally, some continuous linear mapping in the introduced spaces are discussed.

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Introduction

Formally the Hankel transform of order ν is defined by

$$(0.1) \qquad (\mathbb{H}_\nu f)(x) = \int_0^\infty f(y)\sqrt{xy}\, J_\nu(xy)dy \quad , \quad x > 0 .$$

Here J_ν is the Bessel function of the first kind and of order ν. In this paper we consider the case $\nu \in \mathbb{R}$, $\nu > -1$.

Hankel transforms find their applications amongst others in the discussion of problems posed in spherical coordinates. The Fourier transform $\mathbb{F}\, f$ of f which is a function of r only,

$$(0.2) \qquad r = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}} ,$$

can be expressed in terms of a Hankel transform. In the two-dimensional case this can be seen as follows: Introduce plane polar coordinates (r,φ) in the (x_1,x_2)-plane and (ρ,θ) in the (ξ_1,ξ_2)-plane. Then $x_1\xi_1 + x_2\xi_2 = r\rho\cos(\varphi - \theta)$, and

$$(0.3) \qquad (\mathbb{F}\, f)(\rho) = \frac{1}{2\pi}\int_0^\infty r\,dr \int_0^{2\pi} f(r)\, e^{i r \rho \cos(\varphi - \theta)} d\varphi$$
$$= \int_0^\infty r\, f\,(r)\, J_0(r\varphi)dr$$

because

$$(0.4) \qquad J_0(r\rho) = \frac{1}{2\pi}\int_0^{2\pi} e^{i r \rho \cos(\varphi - \theta)} d\varphi \; .$$

Similarly in the n-dimensional ($n \geq 2$) case we have

$$(0.5) \qquad \rho^{\frac{1}{2}n-1}(\mathbb{F} f)(\rho) = \int_0^\infty r^{\frac{1}{2}n} f(r)\, J_{\frac{1}{2}n-1}(r\rho)\,dr \ .$$

In the appendix to this paper the notion of the Hankel transform is adapted in such a way that the Fourier transform of a spherically symmetric function is just an adapted Hankel transform.

The starting point of our discussion will be the equality

$$(0.6) \qquad e^{-x/2} x^{\alpha/2} L_n^{(\alpha)}(x) = \frac{(-1)^n}{2} \int_0^\infty J_\alpha(\sqrt{xy})\, e^{-y/2} y^{\alpha/2} L_n^{(\alpha)}(y)\,dy$$

where $x > 0$, $\alpha > -1$, and where $L_n^{(\alpha)}$ is the n-th generalized Laquerre polynomial of order α. (For definitions and properties of special functions which occur in this paper we refer to [MOS]) The Hankel transform $\mathbb{H}_\alpha$ is regarded as a linear operator in the Hilbert space $L_2(\mathbb{R}^+,dr)$. We show that we can extend $\mathbb{H}_\alpha$ to the whole of $L_2(\mathbb{R}^+,dr)$. It becomes a unitary operator in this way. Further we apply the two theories of generalized functions as given in [G] and [E] to construct three test function spaces for each $\mathbb{H}_\alpha$. The Hankel transform $\mathbb{H}_\alpha$ acts continuously and bijectively on these three spaces (in fact, infinitely many Hankel invariant test function spaces can be constructed). As a direct consequence of the theories in [G] and [E], $\mathbb{H}_\alpha$ can be extended to a continuous bijection on the dual spaces, i.e. the spaces of generalized functions, of the mentioned test function spaces.

The distribution theories in [G] and [E] are functional analytic theories. Therefore we show that the Hankel transform can be looked upon as a unitary operator in the Hilbert space $L_2(\mathbb{R}^+,dr)$. In this way some results can be proved easily. The price we pay is L_2-convergence of the integrals. In section 2 we introduce three function spaces and the associated generalized function spaces. We characterize them by functional analytic means. The introduced spaces are Hankel invariant. Sections 3, 4 and 5 are devoted to the development of a classical analytic description of the elements in our three test function spaces. In the last section we discuss some continuous linear mappings in one of these spaces.

Besides the usual aspects of distribution theory: the definition of the test function space, the definition of the generalized function space and the pairing, in [G] and [E] we also find a detailed characterization of continuous linear mappings on these spaces, the introduction of four topological tensor product spaces and four Kernel theorems. Since the Hankel invariant test function space H_μ, given by Zemanian in [Z] equals one of our test function spaces, all results of [G] and [E] carry over to this space.

§1 The Hankel Transform

Throughout the whole paper we take $\alpha \in \mathbb{R}$, $\alpha > -1$, fixed.

The following equality holds

$$(1.1) \qquad e^{-x/2} x^{\alpha/2} L_n^{(\alpha)}(x) = \frac{(-1)^n}{2} \int_0^\infty e^{-y/2} y^{\alpha/2} L_n^{(\alpha)}(y)\, J_\alpha(\sqrt{xy})\,dy \ .$$

(see [MOS], p 244). Here J_α is a Bessel function of the first kind and of order α (see [MOS], p 66) and

$$(1.2) \qquad L_n^{(\alpha)}(x) = x^{-\alpha} \frac{e^x}{n!} \left(\frac{d}{dx}\right)^n (e^{-x} x^{n+\alpha})$$

the n-th generalized Laquerre polynomial of type α. Equality (1.1) can be rewritten into

$$(1.3) \qquad \Lambda_n^{(\alpha)}(x) = (-1)^n \int_0^\infty \Lambda_n^{(\alpha)}(y)\ \sqrt{xy}\, J_\alpha(xy)\,dy \quad , \quad x > 0 \ ,$$

where $\Lambda_n^{(\alpha)}(x) = x^{\alpha+\frac{1}{2}} e^{-x^2/2} L_n^{(\alpha)}(x^2)$.

With the aid of the orthogonality relations of the generalized Laquerre polynomials (see [MOS], p 241), we derive

$$(1.4) \qquad \int_0^\infty \Lambda_n^{(\alpha)}(y)\, \Lambda_m^{(\alpha)}(y)\,dy = \tfrac{1}{2} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\, \delta_{nm} \quad , \quad n,m \in \mathbb{N} \cup \{0\} \ .$$

In the sequel $\mathcal{L}_n^{(\alpha)}$ denotes the normalized function $\Lambda_n^{(\alpha)}$.

The functions $\mathcal{L}_n^{(\alpha)}$ are eigenfunctions of the operator

$$(1.5) \qquad A_\alpha : \frac{-d^2}{dx^2} + x^2 + \frac{\alpha^2 - \frac{1}{4}}{x^2} - 2\alpha \ ,$$

and their respective eigenvalues are $4n + 2$, $n \in \mathbb{N} \cup \{0\}$. The operator A_α is positive and self-adjoint in $L_2((0,\infty))$ and its eigenfunctions $L_n^{(\alpha)}$ establish a complete orthonormal basis in $L_2((0,\infty))$. For brevity we shall denote the Hilbert space $L_2((0,\infty))$ by X, in the sequel.

It is obvious that

$$(1.6) \qquad L_n^{(\alpha)}(x) = (-1)^n \int_0^\infty L_n^{(\alpha)}(y)\, \sqrt{xy}\, J_\alpha(xy)\,dy \quad , \quad x > 0 \ .$$

On X we define the Hankel transform $\mathbb{H}_\alpha$ as follows

(1.7) Definition

$$\mathbb{H}_\alpha f = \sum_{n=0}^\infty (-1)^n (f, L_n^{(\alpha)})\, L_n^{(\alpha)} \quad , \quad f \in X \ .$$

Here $(.,.)$ denotes the inner product of X.

Clearly, $\mathbb{H}_\alpha$ is a unitary and self-adjoint operator on X. Since $\mathbb{H}_\alpha L_n^{(\alpha)} = (-1)^n L_n^{(\alpha)}$, we even can write

$$(1.8) \qquad \mathbb{H}_\alpha = -i \exp(\tfrac{1}{4}\pi i\, A_\alpha) \ .$$

If f is in the dense linear span of the $L_n^{(\alpha)}$'s, so $f \in \langle L_1^{(\alpha)}, L_2^{(\alpha)}, \ldots \rangle$, then

$$(1.9) \qquad (\mathbb{H}_\alpha f)(x) = \int_0^\infty f(y)\, \sqrt{xy}\, J_\alpha(xy)\,dy \quad , \quad x > 0 \ .$$

The latter assertion is a corollary of formula (1.6) and Definition (1.7). Here we want to prove that the classical Hankel integral transform has something to do with the Hankel transform $\mathbb{H}_\alpha$ that we defined on X.

The integral

$$\int_0^\infty f(y)\,\sqrt{xy}\,J_\alpha(xy)dy$$

exists for all $x > 0$ if the function $y \to (y^{\alpha+\frac{1}{2}} + 1)f(y)$ is absolutely integrable over $(0,\infty)$. To see this, observe that $J_\alpha(xy)$ is $O(y^\alpha)$ whenever $y \downarrow 0$, and $O(y^{-\frac{1}{2}})$ whenever $y \to \infty$.

(1.10) Theorem

Let $f \in X$ be such that $y \to (y^{\alpha+\frac{1}{2}} + 1)f(y)$ is absolutely integrable over $\mathbb{R}^+$. Then

$$F(x) =: \int_0^\infty f(y)\,\sqrt{xy}\,J_\alpha(xy)dy = (\mathbb{H}_\alpha f)(x) \quad ,$$

for almost every $x > 0$. (So $\mathbb{H}_\alpha f$ has a continuous representant.)

Proof

Since the span $<L_1^{(\alpha)}, L_2^{(\alpha)}, \ldots>$ is dense in X, there exists a sequence (φ_n) in this span such that $\varphi_n \to f$ in X. Take $\not{f}_n = \varphi_n - f$ and $\psi_n = (\mathbb{H}_\alpha \varphi_n) - F$. So

$$\psi_n(x) = \int_0^\infty \not{f}_n(y)\,\sqrt{xy}\,J_\alpha(xy)dy \ .$$

Let $\delta > 0$. We proceed as follows

$$\int_0^\infty e^{-\delta y^2}\,|\psi_n(y)|^2\,dy =$$

$$= \int_0^\infty e^{-\delta y^2} dy \left(\int_0^\infty \varphi_n(v)\sqrt{yv}\, J_\alpha(yv)dv \right)\left(\int_0^\infty \overline{\varphi_n(u)\sqrt{yu}}\, J_\alpha(yu)du \right) =$$

$$(1.11) \qquad = \int_0^\infty \int_0^\infty \sqrt{uv}\,\overline{\varphi_n(u)}\varphi_n(v) \left(\int_0^\infty y\, e^{-\delta y^2} J_\alpha(yv)\, J_\alpha(yu)dy \right) .$$

In the next part of the proof we use the equalities

$$\int_0^\infty J_\nu(\alpha t)\, J_\nu(\beta t)\, e^{-\gamma t^2}\, tdt = \tfrac{1}{2}\gamma^{-1} e^{-(\alpha^2+\beta^2)/4\gamma} I_\nu(\alpha\beta/2\gamma)$$

$$\int_0^\infty I_\nu(\alpha t)\, I_\nu(\beta t)\, e^{-\gamma t^2}\, tdt = \tfrac{1}{2}\gamma^{-1} e^{-(\alpha^2+\beta^2)/4\gamma} J_\nu(\alpha\beta/2\gamma) .$$

where $\mathrm{Re}\,\nu > -1$ and $\mathrm{Re}\,\gamma > 0$. (see [MOS], p 93)

So (1.11) equals

$$(1.12) \qquad \frac{1}{2\delta} \int_0^\infty \int_0^\infty \sqrt{uv}\,\varphi_n(u)\, \varphi_n(v)\, e^{-(u^2+v^2)/4\delta} I_\alpha(\tfrac{1}{2}uv/\delta)dudv .$$

With the aid of Schwartz' inequality it follows that (1.12) is smaller than

$$\frac{1}{2\delta} \left(\int_0^\infty \int_0^\infty |\varphi_n(u)|^2\, |\varphi_n(v)|^2\, dudv \right)^{\frac{1}{2}} \left(\int_0^\infty \int_0^\infty u\, v\, e^{-(u^2+v^2)/2\delta} (I_\alpha(uv/2\delta))^2 dudv \right)^{\frac{1}{2}}$$

Further

$$\int_0^\infty u\, e^{-u^2/2\delta}\, du \left(\int_0^\infty v\, e^{-v^2/2\delta}\, I_\alpha(\tfrac{1}{2}uv/\delta)\, I_\alpha(\tfrac{1}{2}uv/\delta)dv \right) =$$

$$= \delta \int_0^\infty u\, e^{-u^2/2\delta}\, (e^{u^2/4\delta}\, J_\alpha(u^2/4\delta))du$$

$$= 2\delta^2 \int_0^\infty e^{-t} J_\alpha(t)dt = 2\delta^2\, 2^{-\frac{1}{2}}\, (2^{\frac{1}{2}} + 1)^\alpha .$$

Now we have proved the following inequality

$$\forall_{\delta>0} : \int_0^\infty e^{-\delta y^2} |\psi_n(y)|^2 \, dy \le 2^{-3/4}(2^{\frac{1}{2}} + 1)^{\alpha/2} \int_0^\infty |\not\psi_n(u)|^2 \, du .$$

So $\not\psi_n \to 0$ in X imlies $\psi_n \to 0$ in X and $F = \mathbb{H}_\alpha f$. □

As a corollary of Theorem (1.10) we derive

(1.13) Theorem

Let $f \in X$. Then for all $x > 0$

$$(\mathbb{H}_\alpha f)(x) = \underset{R\to\infty}{\mathrm{l.i.m.}} \int_0^R f(y) \sqrt{xy} \, J_\alpha(xy) dy ,$$

i.e.

$$\int_0^\infty \left| (\mathbb{H}_\alpha f)(x) - \int_0^R f(y) \sqrt{xy} \, J_\alpha(xy) dy \right|^2 dx \to 0 \text{ as } R \to \infty .$$

Proof

We take $f_n \in X$ as follows

$$f_n(x) = \begin{cases} 0 & \text{if } x > n \\ f(x) & \text{if } 0 < x \le n \end{cases} , \quad n \in \mathbb{N}.$$

Then

$$\int_0^\infty (x^{\alpha+\frac{1}{2}} + 1) |f_n(x)| \, dx \le \left(\int_0^n (x^{\alpha+\frac{1}{2}} + 1)^2 dx \right)^{\frac{1}{2}} \left(\int_0^\infty |f(x)|^2 dx \right)^{\frac{1}{2}} .$$

So $x \mapsto (x^{\alpha+\frac{1}{2}} + 1) f_n(x)$ is absolutely integrable ($\alpha > -1$!) for all $n \in \mathbb{N}$.

Further more $f_n \to f$ in X. Following Theorem (1.10), this implies

$$\| \mathbb{H}_\alpha f - \mathbb{H}_\alpha f_n \|^2 =$$

$$= \int_0^\infty \left| (\mathbb{H}_\alpha f)(x) - \int_0^n f(y) \sqrt{xy}\, J_\alpha(xy)\, dy \right|^2 dx \to 0 \quad \text{as} \quad n \to \infty .$$

Note that the sequence (n) can be replaced by any sequence (R_n) with $R_n \to \infty$. $\square$

§2 Hankel invariant test function spaces and generalized function spaces

As already noted, the operator A_α introduced in section 1, is positive and self-adjoint in X. Therefore the space S_{X,A_α} is well-defined by [G] and so are the spaces $\tau(X,\log A_\alpha)$ and $\tau(X,A_\alpha)$ by [E]. Here we shall give a short functional analytic characterization of these spaces. Spaces of this kind are studied in great detail in the cited papers [G] and [E].

(2.1) Characterization

(a) $f \in S_{X,A_\alpha} \Leftrightarrow \exists_{\tau>0}\exists_{g\in X} : f = e^{-\tau A_\alpha} g$

$$\text{or} \Leftrightarrow \exists_{\tau>0} : (f, L_n^{(\alpha)}) = O(e^{-n\tau})$$

(b) $f \in \tau(X,\log A_\alpha) \Leftrightarrow \forall_{k\in\mathbb{N}}\exists_{g\in X} : f = A_\alpha^{-k} g$

$$\text{or} \Leftrightarrow \forall_{k\in\mathbb{N}} : (f, L_n^{(\alpha)}) = O(n^{-k})$$

(c) $f \in \tau(X,A_\alpha) \Leftrightarrow \forall_{t>0}\exists_{g\in X} : f = e^{-tA_\alpha} g$

$$\text{or} \Leftrightarrow \forall_{t>0} : (f, L_n^{(\alpha)}) = O(e^{-nt})$$

These three spaces are our test function spaces. The space S_{X,A_α} is a complete topological vector space, the spaces $\tau(X,A_\alpha)$ and $\tau(X,\log A_\alpha)$ are Frêchet spaces. Since A_α^{-1} is a Hilbert-Schmidt operator each of these spaces is nuclear (for details see [G], ch I, and [E], ch.I). The Hankel transform $\mathbb{H}_\alpha$ is well-defined on these spaces. We have

(2.2) Theorem

$\mathbb{H}_\alpha$ is a continuous bijection of S_{X,A_α} onto itself. The same assertion holds true for the spaces $\tau(X,\log A_\alpha)$ and $\tau(X,A_\alpha)$.

Proof

The proof is almost trivial. If for $f \in X$, $(f, L_n^{(\alpha)})$ satisfies the order estimate (2.1.a), then $(\mathbb{H}_\alpha f, L_n^{(\alpha)}) = (-1)^n (f, L_n^{(\alpha)})$ satisfies the same estimate. Thus $\mathbb{H}_\alpha$ is a continuous injection on S_{X,A_α}. Further, $\mathbb{H}_\alpha$ is surjective because $\mathbb{H}_\alpha(\mathbb{H}_\alpha f) = f$. The proofs for the other spaces run similarly. □

The spaces of generalized functions related to the introduced test function spaces are denoted by T_{X,A_α}, $\sigma(X, \log A_\alpha)$ and $\sigma(X, A_\alpha)$. For an extensive investigation of this kind of spaces see [G], ch II, and [E], ch II. Here we give a short characterization. (With $< \cdot , \cdot >$, we denote the respective pairings between the test function spaces and generalized function spaces)

(2.3) Characterization

a) $F \in T_{X,A_\alpha} \Leftrightarrow \forall_{t>0} : \langle L_n^{(\alpha)}, F \rangle = \mathcal{O}(e^{nt})$

b) $F \in \sigma(X, \log A_\alpha) \Leftrightarrow \exists_{k \in \mathbb{N}} : \langle L_n^{(\alpha)}, F \rangle = \mathcal{O}(n^k)$

c) $F \in \sigma(X, A_\alpha) \Leftrightarrow \exists_{t>0} : \langle L_n^{(\alpha)}, F \rangle = \mathcal{O}(e^{nt})$.

As a corollary of Theorem (2.2) we have

(2.4) Corollary

The Hankel transform $\mathbb{H}_\alpha$ can be extended to the spaces of generalized functions T_{X,A_α}, $\sigma(X, \log A_\alpha)$ and $\sigma(X, A_\alpha)$. The extended Hankel transform, also denoted by $\mathbb{H}_\alpha$ is a continuous bijection on each of these spaces.

Proof

We shall prove the assertion for the space T_{X,A_α}.

If $F \in T_{X,A_\alpha}$, then F can be expanded with respect to the basis $(L_n^{(\alpha)})$,

$$F = \sum_{n=0}^{\infty} \overline{\langle L_n^{(\alpha)}, F \rangle}\, L_n^{(\alpha)}$$

where the series converges in $\mathcal{T}_{X,A_\alpha}$. Now define

$$(2.5) \qquad \widetilde{\mathbb{H}}_\alpha F = \sum_{n=0}^{\infty} (-1)^n \langle \overline{L_n^{(\alpha)}}, F\rangle \, L_n^{(\alpha)} .$$

Then $\widetilde{\mathbb{H}}_\alpha$ extends $\mathbb{H}_\alpha$ to $\mathcal{T}_{X,A_\alpha}$. $\widetilde{\mathbb{H}}_\alpha$ is a linear, injective mapping from $\mathcal{T}_{X,A_\alpha}$ into itself. Since $\widetilde{\mathbb{H}}_\alpha^{\,2} F = F$ for all $F \in \mathcal{T}_{X,A_\alpha}$, it is also surjective. The continuity follows from [G], ch IV. Note that for all $g \in S_{X,A_\alpha}$ and $F \in \mathcal{T}_{X,A_\alpha}$

$$\langle \mathbb{H}_\alpha g, \mathbb{H}_\alpha F\rangle = \langle g, F\rangle \qquad \square$$

§3 Analytic characterization of the elements in $\tau(X,\log A_\alpha)$

Let $f \in \tau(X,\log A_\alpha)$. Then f can be written as

$$(3.1) \qquad f = \sum_{n=0}^{\infty} (f,L_n^{(\alpha)})L_n^{(\alpha)} ,$$

where $(f,L_n^{(\alpha)}) = O(n^{-k})$ for all $k \in \mathbb{N}$. We define the function g on $(0,\infty)$ by

$$(3.2) \qquad g(x) = x^{-(\alpha+\frac{1}{2})} f(x) \quad , \quad x > 0 .$$

Then g satisfies

$$(3.3) \qquad g = \sum_{n=0}^{\infty} (g,\tilde{L}_n^{(\alpha)})_\alpha \, \tilde{L}_n^{(\alpha)}$$

with

$$(3.4) \qquad \tilde{L}_n^{(\alpha)}(x) =: x^{-(\alpha+\frac{1}{2})} L_n^{(\alpha)}(x) = \left(\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(\alpha)}(x^2) ,$$

and $(\cdot , \cdot)_\alpha$ the inner product in the Hilbert space X_α,

$$(3.5) \qquad X_\alpha =: L_2((0,\infty) \, , \, x^{2\alpha+1}dx)$$

So $(g,\tilde{L}_n^{(\alpha)})_\alpha = (f,L_n^{(\alpha)})$. The functions $\tilde{L}_n^{(\alpha)}$ establish an orthonormal basis in X_α and they are the eigenfunctions of the self-adjoint operator $\tilde{A}_\alpha$ in X_α,

$$(3.6) \qquad \tilde{A}_\alpha : \frac{-d^2}{dx^2} - \frac{2\alpha+1}{x}\frac{d}{dx} + x^2 - 2\alpha$$

with respective eigenvalues $4n + 2$, $n \in \mathbb{N}$. We have $g \in \tau(X_\alpha,\log \tilde{A}_\alpha)$. We derive for $\beta > -1$, fixed, (see [MOS] p, 248)

$$(3.7)\qquad \tilde{L}_n^{(\beta)}(x) = \mathcal{O}(n^{\gamma+1})$$

with $\gamma =: \max(\beta, \beta/2 - \frac{1}{4})$ on every finite interval $[0,w]$, $w > 0$. Since $g \in \tau(X_\alpha, \log \tilde{A}_\alpha)$, thus $(g, \tilde{L}_n^{(\alpha)}) = O(n^{-k})$ for all $k \in \mathbb{N}$, we have for all $x \geq 0$,

$$(3.8)\qquad g(x) = \sum_{n=0}^{\infty} (g, \tilde{L}_n^{(\alpha)})_\alpha\, \tilde{L}_n^{(\alpha)}(x) .$$

Thus g can be extended to a function of x^2 on $\mathbb{R}$. So g is even. Taking into account the normalization factors (see (1.4)) we derive the following recurrence relations from [MOS], p 241,

$$(3.9)\qquad x^2\tilde{L}_n^{(\alpha)}(x) = -\sqrt{(n+\alpha+1)(n+1)}\,\tilde{L}_{n+1}^{(\alpha)}(x) + (2n+1+\alpha)\,\tilde{L}_n^{(\alpha)}(x) -$$

$$- \sqrt{n(n+\alpha)}\,\tilde{L}_{n-1}^{(\alpha)}(x)$$

and

$$(3.10)\qquad (x^{-1}D)\tilde{L}_n^{(\alpha)}(x) = -\sqrt{n+\alpha+1}\,\tilde{L}_n^{(\alpha+1)}(x) - \sqrt{n}\,\tilde{L}_{n-1}^{(\alpha+1)}(x)$$

where $D = \frac{d}{dx}$, $L_{-1}^{(\alpha+1)} \equiv 0$, $L_{-1}^{(\alpha)} \equiv 0$ and $n = 0,1,2,\ldots$.
With the aid of [E], ch IV, we observe that the linear mapping $Q^2 : \tau(X_\alpha, \log \tilde{A}_\alpha) \to \tau(X_\alpha, \log \tilde{A}_\alpha)$ given by

$$(3.11)\qquad (Q^2 f)(x) = x^2 f(x) \quad , \quad x \geq 0$$

and the linear mapping $R : \tau(X_\alpha, \log \tilde{A}_\alpha) \to \tau(X_{\alpha+1}, \log \tilde{A}_{\alpha+1})$ by

$$(3.12)\qquad Rf(x) = \frac{1}{x}\, f'(x) \quad , \quad x \geq 0 \ ,$$

are continuous. So for each $r,s \in \mathbb{N} \cup \{0\}$ we have

$$(3.13)\qquad \forall_{k\in\mathbb{N}\cup\{0\}}\ \exists_{\ell\in\mathbb{N}}\ :\ \|\tilde{A}^{k}_{\alpha+s}\, Q^{2r} R^{s}\, \tilde{A}^{-\ell}_{\alpha}\| < \infty \ .$$

Especially for $k = 0$ it follows that there exists $\ell \in \mathbb{N}$ and $c > 0$ such that for all $f \in \tau(X_\alpha, \log \tilde{A}_\alpha)$,

$$(3.14)\qquad \|Q^{2r+s} R^{s} f\|_{\alpha} = \|Q^{2r} R^{s} f\|_{\alpha+s} \leq c\, \|\tilde{A}^{\ell}_{\alpha} f\|_{\alpha}$$

Let $i,j \in \mathbb{N} \cup \{0\}$. With the aid of (3.7) and (3.10) it is obvious that there exists $\beta(j) > 0$ such that

$$(3.15)\qquad \sup_{0\leq x\leq 1} |(R^{j}\tilde{L}^{(\alpha)}_{n})(x)| = O(n^{\beta(j)})$$

and therefore also

$$(3.16)\qquad \sup_{0\leq x\leq 1} |(Q^{i} R^{j}\, \tilde{L}^{(\alpha)}_{n})(x)| = O(n^{\beta(j)})$$

So for $k_1 > \beta(j)$, $k_1 \in \mathbb{N}$, and for all $f \in \tau(X_\alpha, \log \tilde{A}_\alpha)$

$$(3.17)\qquad \sup_{0\leq x\leq 1} |(Q^{i} R^{j} f)(x)| \leq \sum_{n=0}^{\infty} |(f, \tilde{L}^{(\alpha)}_{n})| \sup_{0\leq x\leq 1} |Q^{i} R^{j}\, \tilde{L}^{(\alpha)}_{n}(x)|$$

$$\leq c\, \|\tilde{A}^{k_1}_{\alpha} f\|_{\alpha} \left(\sum_{n=0}^{\infty} (4n+2)^{-2k_1} n^{2\beta(j)} \right)^{\frac{1}{2}}$$

where c depends on i,j, only.

By (3.17) we have

$$(3.18) \qquad \left(\int_0^1 |(Q^i R^j f)(x)|^2 x^{2\alpha+1} dx \right)^{\frac{1}{2}} \leq d \| \tilde{A}_\alpha^{k_1} f \|_\alpha$$

for some $d > 0$ and for every $f \in \tau(X_\alpha, \log \tilde{A}_\alpha)$.

Further more for every $f \in \tau(X_\alpha, \log \tilde{A}_\alpha)$

$$\int_1^\infty |(Q^i R^j f)(x)|^2 x^{2\alpha+1} dx \leq \int_1^\infty |(Q^{2i+j} R^j f)(x)|^2 x^{2\alpha+1} dx$$

$$\leq \| Q^{2i+j} R^j f \|_\alpha^2$$

So by (3.14) there exist $k_2 \in \mathbb{N}$ and $d' > 0$ such that

$$(3.19) \qquad \left(\int_1^\infty |(Q^i R^j f)(x)|^2 x^{2\alpha+1} dx \right)^{\frac{1}{2}} \leq d' \| \tilde{A}_\alpha^{k_2} \|_\alpha$$

for all $f \in \tau(X_\alpha, \log \tilde{A}_\alpha)$.

Combining the results (3.18) and (3.19) we obtain

(3.20) Lemma

For each $i,j \in \mathbb{N} \cup \{0\}$ there exist $k \in \mathbb{N}$ and $d > 0$ such that

$$\| Q^i R^j f \|_\alpha \leq d \| \tilde{A}_\alpha^k f \|_\alpha$$

for all $f \in \tau(X_\alpha, \log \tilde{A}_\alpha)$.

Because of Lemma (3.20) there is no problem in defining the following semi-norms on $\tau(X_\alpha, \log \tilde{A}_\alpha)$

$$(3.21) \qquad q_{ij}^{(\alpha)}(f) =: \| Q^i R^j f \|_\alpha$$

Obviously, the seminorms $q_{ij}^{(\alpha)}$ are continuous in the strong topology of $\tau(X_\alpha, \log \tilde{A}_\alpha)$, i.e. the topology generated by the seminorms $f \to \|\tilde{A}_\alpha^k f\|$, $k \in \mathbb{N} \cup \{0\}$.

The operator $\tilde{A}_\alpha$ can be written as

$$(3.22) \qquad \tilde{A}_\alpha = -RQ^2R - 2\alpha R + Q^2 - 2\alpha$$

Because of formula (3.22) and the commutation relation

$$(3.23) \qquad RQ^2 - Q^2R = 2$$

it can be shown that there exist constants $c_{ij} > 0$ such that

$$(3.24) \qquad \|\tilde{A}_\alpha^k f\|_\alpha \le \sum_{i,j=1,1}^{2k,2k} c_{ij}\, q_{ij}^{(\alpha)}(f)$$

for all $f \in \tau(X_\alpha, \log \tilde{A}_\alpha)$.

With the aid of the inequalities (3.20) and (3.24) we derive that the strong topology of $\tau(X_\alpha, \log \tilde{A}_\alpha)$ is the same as the topology generated by the seminorms $q_{ij}^{(\alpha)}$.

Next we want to prove that we can take the supremum norm in stead of $\|\cdot\|_\alpha$ in the definition of the $q_{ij}^{(\alpha)}$'s. So let $i,j \in \mathbb{N} \cup \{0\}$, and let $f \in \tau(X_\alpha, \log \tilde{A}_\alpha)$. Then following (3.17) there exist $d > 0$ and $k_1 \in \mathbb{N}$ such that

$$(3.25) \qquad \sup_{0 \le x \le 1} |(Q^i R^j f)(x)| \le d\, \|\tilde{A}^{k_1} f\|_\alpha \quad .$$

Furthermore, by Sobolev's embedding theorem there exists L > 0 such that

$$(3.26)\qquad \sup_{x\geq 1} |(Q^i R^j f)(x)| \leq L \left(\int_1^\infty (|(Q^i R^j f)(x)|^2 + |(DQ^i R^j f)(x)|^2) dx \right)^{\frac{1}{2}}$$

$$\leq L \left(\int_1^\infty |(Q^{i+1} R^j f)(x)|^2 + |(Q^2 R Q^i R^j f)(x)|^2 x^{2\alpha+1} dx \right)^{\frac{1}{2}}$$

$$\leq L \,(q_{i+1,j}(f) + i\, q_{i+1,j}(f) + q_{i+1,j+1}(f))$$

Combining (3.25) and (3.26) and inserting Lemma (3.20) we find

(3.27) Lemma

For each $i,j \in \mathbb{N}$ there exist $d > 0$ and $k \in \mathbb{N}$ such that

$$\| Q^i R^j f \|_\infty = \sup_{x\geq 0} |(Q^i R^j f)(x)| \leq d \| \tilde{A}_\alpha^k f \|_\alpha$$

for all $f \in \tau(X_\alpha, \log \tilde{A}_\alpha)$.

Define the seminorms $p_{ij}^{(\alpha)}$ on $\tau(X_\alpha, \log \tilde{A}_\alpha)$ by

$$(3.28)\qquad p_{ij}^{(\alpha)}(f) =: \| Q^i R^j f \|_\infty \quad , \quad f \in \tau(X_\alpha, \log \tilde{A}_\alpha).$$

Then we can prove

(3.29) Lemma

The topology generated by the seminorms $p_{ij}^{(\alpha)}$ is the same as the strong topology of $\tau(X_\alpha, \log \tilde{A}_\alpha)$.

Proof

The seminorms $q_{ij}^{(\alpha)}$, $i,j \in \mathbb{N} \cup \{0\}$ are continuous with respect to the seminorms

$p_{ij}^{(\alpha)}$, $i,j \in \mathbb{N} \cup \{0\}$. This can be seen as follows:
Let $i,j \in \mathbb{N} \cup \{0\}$, and let $f \in \tau(X_\alpha, \log \tilde{A}_\alpha)$. Then

$$(q_{ij}^{(\alpha)}(f))^2 = \| Q^i R^j f \|_\alpha^2 = \int_0^\infty |Q^i R^j f(x)|^2 x^{2\alpha+1}\, dx \le$$

$$\le \left(\int_0^\infty \frac{x^{2(\alpha-h)+1}}{(1+x^2)^2}\, dx \right) (2\| Q^{i+h} R^j f \|_\infty^2 + 2\| Q^{i+h+2} R^j f \|_\infty^2) \le$$

$$\le c\,(p_{i+h,j}(f) + p_{i+h+2,j}(f))^2$$

where $h \in \mathbb{N}$ is taken so large that $-1 < (\alpha - h) \le 0$, and where $c = 2 \int_0^\infty \frac{x^{2(\alpha-h)+1}}{(1+x^2)^2}\, dx$. Now the assertion follows by invoking Lemma 3.27 and the result (3.24). □

Going back to our original space $\tau(X, \log A_\alpha)$ we have

(3.30) Theorem

Define the seminorms $\gamma_{ij}^{(\alpha)}$ on $\tau(X, \log A_\alpha)$ by

$$\gamma_{ij}^{(\alpha)}(f) =: \sup_{x \ge 0} \; x^i (x^{-1} D)^j x^{-(\alpha+\frac{1}{2})} f(x) .$$

Then the topology generated by the seminorms $\gamma_{ij}^{(\alpha)}$, $i,j \in \mathbb{N} \cup \{0\}$ is equivalent to the strong topology of $\tau(X, \log A_\alpha)$.

Proof

We have the equivalence

$$f \in \tau(X, \log A_\alpha) \Leftrightarrow g : x \mapsto x^{-(\alpha+\frac{1}{2})} f(x) \in \tau(X_\alpha, \log \tilde{A}_\alpha) .$$

With the aid of Lemma (3.29) the assertion immediately follows. □

(3.31) Theorem

Each element $f \in \tau(X,\log A_\alpha)$ can be written as

$$f(x) = x^{\alpha+\frac{1}{2}}\,\psi(x^2) \quad , \quad x > 0$$

with $\psi \in S$, Schwartz' space of functions of rapid decrease.

Proof

Let $f \in \tau(X,\log A_\alpha)$. Then g, defined by

$$g(x) = x^{-(\alpha+\frac{1}{2})} f(x) \quad , \quad x \geq 0$$

is in $\tau(X_\alpha,\log \tilde{A}_\alpha)$. Thus g can be extended to a function of x^2 on $\mathbb{R}$. So there exists a function h on $[0,\infty)$ such that

$$g(x) = h(x^2) \quad , \quad x \in \mathbb{R} .$$

For all $i,j \in \mathbb{N}$ we have

$$(3.32) \qquad \sup_{x\in\mathbb{R}} \left|(x^{2i}(x^{-1} D_x)^j h)(x^2)\right| < \infty .$$

With the new variable $\xi = x^2$ we derive from (3.32)

$$\sup_{\xi\geq 0} \left|(\xi^i D_\xi^j h)(\xi)\right| < \infty \quad , \quad i,j = 0,1,2,\ldots .$$

Since in $\xi = 0$ all derivatives on the right of h exist, there can be constructed an infinitely differentiable function of bounded support h_1 with $(D_x^m h_1)(0) = (D_x^m h)(+0)$ (Borel's theorem).

Define φ on $\mathbb{R}$ by

$$\varphi(x) \quad \begin{cases} h(x) & x \geq 0 \\ h_1(x) & x < 0 \end{cases}$$

Then $\varphi \in S$ and $f(x) = x^{\alpha+\frac{1}{2}}\varphi(x^2)$, $x > 0$ □

From Theorem (3.30) it follows that $f \in X$ is in $\tau(X,\log A_\alpha)$ if and only if $\gamma_{ij}^{(\alpha)}(f)$ is finite for all $i,j = 0,1,2,\ldots$. Compairing this result with the definition of the space H_μ in [Z] we get as a corollary

(3.33) Corollary

The test function space H_μ in [Z], p 129, equals the space $\tau(X,\log A_\mu)$. Furthermore, the strong topologies of the spaces H_μ and $\tau(X,\log A_\mu)$ coincide.

We can yet give another characterization of the elements in the space $\tau(X,\log A_\alpha)$.

(3.34) Theorem

$f \in \tau(X,\log A_\alpha)$ if and only if the even extension of $x \mapsto x^{-(\alpha+\frac{1}{2})}f(x)$ belongs to Schwartz' space S.

Proof

$\Rightarrow$) Let $f \in \tau(X,\log A_\alpha)$. Then there exists $\varphi \in S$ such that $f(x) = x^{(\alpha+\frac{1}{2})}\varphi(x^2)$, $x > 0$. It is obvious that $x \to \varphi(x^2) \in S$.

$\Leftarrow$) Let g denote the even extension of $x \to x^{-(\alpha+\frac{1}{2})}f(x)$. Then by assumption $g \in S$, thus $g^{(2k+1)}(0) = 0$ for $k = 0,1,\ldots$.

Define h on $[0,\infty)$ by

$$h(x) = g(\sqrt{x}) \quad , \quad x \geq 0 .$$

Then h is indefinitely differentiable on $(0,\infty)$ and for all $k \in \mathbb{N}$,

$$h(x) = g(0) + g^2(0)\frac{x}{2!} + g^4(0)\frac{x^2}{4!} + \ldots + g^{(2k)}(0)\frac{x^k}{(2k)!} + O(x)^k$$

in a right neighbourhood of 0. Therefore all derivatives on the right exist in $x = 0$ and $h^{(k)}$ is continuous on the right in $x = 0$. Similar to the proof of Theorem (3.31) we can show that there exists $\varphi \in S$ with $h(x) = \varphi(x)$ for $x \geq 0$. We have $f(x) = x^{\alpha+\frac{1}{2}}\varphi(x^2)$. So by Theorem (3.31) the result follows. □

In [L], Lee characterizes the elements in H_μ in the same way as we have done in Theorem (3.34), but he adds the condition:

'The Taylor expansions of f near the origin is of the form

$$x^{-(\alpha+\frac{1}{2})}\{a_0 + a_2x^2 + \ldots + a_{2q}x^{2q} + R_{2q}\}'$$

Clearly this extra condition is not necessary. The counter example

$$x \mapsto x^{\alpha+\frac{1}{2}} e^{-|x|}$$

which Lee gives to show necessity, is wrong, because $x \mapsto e^{-|x|} \notin S$.

For completeness we note that $S = \tau(L_2(\mathbb{R}), \log H)$ with

$$H:\ -\frac{d}{dx^2} + x^2 + 1$$

(see [E]).

§4 Analytic characterization of the elements in S_{X,A_α}

We start with the following equality

$$(4.1)\qquad \sum_{n=0}^{\infty} e^{-(4n+2)t}\, L_n^{(\alpha)}(x)\, L_n^{(\alpha)}(y) =$$

$$= \frac{e^{-2\alpha t}(xy)^{\frac{1}{2}}}{\sinh 2t} \exp\left[-\tfrac{1}{2}\,\frac{\cosh 2t}{\sinh 2t}(x^2+y^2)\right] I_\alpha\,(xy/\sinh 2t)\,.$$

Here I_α is the modified Bessel function of the first kind and of order α. Formula (4.1) can be derived from [MOS], p. 242, by a straight forward computation, and it gives an expression for the Hilbert-Schmidt kernel of e^{-tA_α}, $t > 0$, in $L_2(\mathbb{R}^+ \times \mathbb{R}^+)$.

The function I_α can be written as

$$I_\alpha(z) = \frac{(\frac{1}{2}z)^\alpha}{\Gamma(\alpha+1)}\ {}_0F_1\,(\alpha+1, \tfrac{1}{4}z^2)$$

where ${}_0F_1$ is the hypergeometric function (see [MOS], p. 62)

$${}_0F_1\,(\alpha+1, w) = \sum_{m=0}^{\infty} \frac{\Gamma(\alpha+1)}{m!\,\Gamma(\alpha+m+1)}\ w^m \quad ,\quad w \in \mathbb{C}.$$

So I_α can be considered analytic on the region $-\pi < \arg z < \pi$. In the following lemma the growth properties of $|L_n^{(\alpha)}(z)|$ for fixed z and large n are described.

(4.2) Lemma

$$\forall_{\substack{z\in\mathbb{C},\\ |\arg(z)|<\pi}}\ \exists_{K>0}\exists_{\delta>0}\ :\ |L_n^{(\alpha)}(z)| \le K e^{\delta\sqrt{n}}$$

Proof

We have $L_n^{(\alpha)}(z) = \left(\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} z^{\alpha+\frac{1}{2}}\, e^{-\frac{1}{2}z^2}\, \mathrm{L}_n^{(\alpha)}(z^2)$, with

$L_n^{(\alpha)}(z^2) = \sum_{m=0}^{n} (-1)^m \binom{n+\alpha}{n-m} \frac{z^{2m}}{m!}$. So we are ready if we can estimate $L_n^{(\alpha)}(z^2)$ for fixed z.

Let $z \in \mathbb{C}$. We estimate

$$\binom{n+\alpha}{n-m} = \frac{\Gamma(n+\alpha+1)}{\Gamma(n-m+1)} \cdot \frac{1}{\Gamma(m+\alpha+1)} \le (n + [\alpha] + 1)^{m+[\alpha]+2} \cdot \frac{1}{m!} .$$

So

$$|L_n^{(\alpha)}(z^2)| \le \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{|z|^{2m}}{m!} \le \sum_{m=0}^{n} (n + [\alpha] + 1)^{[\alpha]+2} \frac{((n+[\alpha]+1)\,|z|^2)^m}{m!\,m!}$$

$$\le (n + [\alpha] + 1)^{[\alpha]+2} \sum_{m=0}^{n} \frac{(2\sqrt{n+[\alpha]+1}\,|z|)^{2m}}{(2m)!}$$

$$= (n + [\alpha] + 1)^{[\alpha]+2} \cosh (2\sqrt{n+[\alpha]+1}\,|z|) .$$

So $|L_n^{(\alpha)}(z^2)| \le K e^{\gamma\sqrt{n}}$ for well-chosen $K,\gamma > 0$. From this the assertion follows. □

(4.3) Corollary

For each $t > 0$, the series

$$(4.4) \qquad \sum_{n=0}^{\infty} e^{-(4n+2)t} \mathcal{L}_n^{(\alpha)}(z)\, \mathcal{L}_n^{(\alpha)}(w)$$

converges uniformly on compacta in $\mathbb{C}^2$, and

$$(4.5) \qquad \sum_{n=0}^{\infty} e^{-(4n+2)t} \mathcal{L}_n^{(\alpha)}(z)\, \mathcal{L}_n^{(\alpha)}(w) =$$

$$\frac{e^{-2\alpha t}(zw)^{\frac{1}{2}}}{\sinh 2t} \exp(-\tfrac{1}{2} \frac{\cosh 2t}{\sinh 2t} (z^2 + w^2))\, I_\alpha\!\left(\frac{zw}{\sinh 2t}\right)$$

Proof

Follows from Lemma (4.2) and the analytic properties of I_α and the $L_n^{(\alpha)}$'s. □

Since $L_n^{(\alpha)}(\bar{z}) = \overline{L_n^{(\alpha)}(z)}$ from (4.5) we derive the equality

$$(4.6) \qquad \sum_{n=0}^{\infty} e^{-(4n+2)t} |L_n^{(\alpha)}(x+iy)|^2 =$$

$$\frac{e^{-2\alpha t}(x^2+y^2)^{\frac{1}{2}}}{\sinh 2t} \exp\left[-\frac{\cosh 2t}{\sinh 2t}(x^2-y^2)\right] \cdot I_\alpha\left(\frac{x^2+y^2}{\sinh 2t}\right).$$

Now let $g \in X$. Then for $f = e^{-tA_\alpha} g$ we derive

$$|f(x+iy)| = \left|\sum_{n=0}^{\infty} e^{-(4n+2)t}(g, L_n^{(\alpha)}) L_n^{(\alpha)}(x+iy)\right| \leq$$

$$\leq \left(\sum_{n=0}^{\infty} |(g, L_n^{(\alpha)})|^2\right)^{\frac{1}{2}} \left(\sum_{n=0}^{\infty} e^{-(4n+2)2t} |L_n^{(\alpha)}(x+iy)|^2\right)^{\frac{1}{2}} \leq$$

$$\leq \|g\| \frac{e^{-2\alpha t}}{(\sinh 4t)^{\frac{1}{2}}} \exp\left[-\tfrac{1}{2}\frac{\cosh 4t}{\sinh 4t}(x^2-y^2)\right] \left((x^2+y^2)^{\frac{1}{2}} I_\alpha\left(\frac{x^2+y^2}{\sinh 4t}\right)\right)^{\frac{1}{2}}$$

where $z = x + iy$.

Since there exists a constant $K_t > 0$ such that for all $z \in \mathbb{C}$

$$(4.7) \qquad |z|^{-\alpha-\frac{1}{2}} \left(\frac{|z|}{\sinh 4t} I_\alpha\left(\frac{|z|^2}{\sinh 4t}\right)\right)^{\frac{1}{2}} \leq K_t \exp(\tfrac{1}{2}|z|^2/\sinh 4t)$$

we get for all $z = x + iy$

$$(4.8) \qquad |(x+iy)^{-(\alpha+\frac{1}{2})} f(x+iy)| \leq K_t \|g\| \exp \tfrac{1}{2}\left(\frac{1-\cosh 4t}{\sinh 4t} x^2 + \frac{\cosh 4t-1}{\sinh 4t} y^2\right)$$

$$= K_t^1 \exp\left(-\tfrac{1}{2}\frac{\sinh 2t}{\cosh 2t} x^2 + \tfrac{1}{2}\frac{\cosh 2t}{\sinh 2t} y^2\right).$$

Moreover, we can write

$$f(z) = \int_0^\infty g(y)\left[\frac{e^{-2\alpha t}(zy)^{\frac{1}{2}}}{\sinh 2t} \exp\left[-\tfrac{1}{2}\,\frac{\cosh 2t}{\sinh 2t}\,(z^2+y^2)\right] I_\alpha\left(\frac{zy}{\sinh 2t}\right) dy \right. .$$

It is obvious that $z \mapsto z^{-(\alpha+\frac{1}{2})} f(z)$ is an even, entirely analytic function.

We have proved

(4.9) Lemma

Let $w \in X$ and $t > 0$. Put $f = e^{-tA_\alpha} w$. Then

(i) $z \mapsto z^{-(\alpha+\frac{1}{2})} f(z)$ is an even entirely analytic function.

(ii) There are A, $0 < A < 1$ and B, $B > 1$, only depending on t and there is $C > 0$ such that

$$z^{-(\alpha+\frac{1}{2})} f(z) \le C \exp(-\tfrac{1}{2} Ax^2 + \tfrac{1}{2} By^2)$$

for all $z = x + iy$ in $\mathbb{C}$.

We want to show the converse of the above lemma. So let f be a function satisfying (4.9.i) and (4.9.ii) for some fixed A, B and C. We define the even, entirely analytic function g by

$$(4.10) \qquad g(z) = z^{-(\alpha+\frac{1}{2})} f(z) \quad , \quad z \in \mathbb{C} \; .$$

Then we may write

$$g = \sum_{n=0}^{\infty} (g, \tilde{L}_n^{(\alpha)})_\alpha \, \tilde{L}_n^{(\alpha)}$$

where

$$(4.11) \qquad \tilde{L}_n^{(\alpha)} = \left(\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(\alpha)}(x^2)$$

and $(\cdot\,,\cdot)$ denotes the inner product in the Hilbert space X_α, $X_\alpha = L_2(\mathbb{R}^+, x^{2\alpha+1}dx)$. The functions $\tilde{L}_n^{(\alpha)}$ establish an orthonormal basis in X_α and they are the eigenfunctions of the positive self-adjoint operator $\hat{A}_\alpha$ in X_α,

$$(4.12) \qquad \hat{A}_\alpha : -\frac{d^2}{dx^2} + x^2 - \frac{2\alpha+1}{x}\frac{d}{dx}$$

with respective eigenvalues $4n + 2\alpha + 2$ (cf section 3).
We shall show that $g \in S_{X_\alpha, \hat{A}_\alpha}$. It is obvious that $g \in S_{X_\alpha, \hat{A}_\alpha}$ implies $f \in S_{X, A_\alpha}$.

The function g is even and entirely analytic, and g satisfies the estimate $|g(x+iy)| \le C\exp(-\frac{1}{2}Ax^2 + \frac{1}{2}By^2)$. From [B], Theorem 10.1, we can derive that there is $t > 0$ only depending on A and B, such that $g = \sum_{n=0}^{\infty} a_n \phi_{2n}$ with $a_n = \mathcal{O}(e^{-nt})$. Here ϕ_{2n} are the even Hermite functions; we have

$$\phi_{2n} = (-1)^n 2^{-\frac{1}{2}} \tilde{L}_n^{(-\frac{1}{2})} \quad .$$

So $\quad g = \sum_{n=0}^{\infty} (-1)^n 2^{-\frac{1}{2}} a_n \tilde{L}_n^{(-\frac{1}{2})}$, i.e. $g \in e^{-t\hat{A}_{-\frac{1}{2}}}(X_{-\frac{1}{2}})$. Note that $\hat{A}_{-\frac{1}{2}} = H =: -\frac{d^2}{dx^2} + x^2$ and $X_{-\frac{1}{2}} = X = L_2(\mathbb{R}^+, dx)$.
In section 3 we gave the following recurrence relations

$$(4.13) \qquad (x^{-1}D)\,\tilde{L}_n^{(-\frac{1}{2})} = \sqrt{n+\tfrac{1}{2}}\,\tilde{L}_n^{(\frac{1}{2})} - \sqrt{n}\,\tilde{L}_{n-1}^{(\frac{1}{2})} \quad , \qquad n = 0,1,2,\dots$$

where $L_{-1}^{(\frac{1}{2})} \equiv 0$, and $D = \frac{d}{dx}$. Further, the generalized Laguerre polynomials $L_n^{(-\frac{1}{2})}$ and $L_n^{(\frac{1}{2})}$ satisfy the recurrence relations

$$L_n^{(\frac{1}{2})} = \sum_{m=0}^{n} L_m^{(-\frac{1}{2})} \quad , \quad n = 0,1,2,\dots \; .$$

This implies with the aid of (4.11)

$$\tilde{L}_n^{(\frac{1}{2})} = \sum_{m=0}^{n} \left(\frac{\Gamma(n+1)}{\Gamma(n+3/2)} \frac{\Gamma(m+1/2)}{\Gamma(m+1)} \right)^{\frac{1}{2}} \tilde{L}_m^{(-\frac{1}{2})} .$$

With the result (4.13)

$$(x^{-1}D)\, \tilde{L}_n^{(-\frac{1}{2})} =$$

$$= -\sum_{m=0}^{n} \sqrt{n+\tfrac{1}{2}} \left(\frac{\Gamma(n+1)}{\Gamma(n+3/2)} \frac{\Gamma(m+1/2)}{\Gamma(m+1)} \right)^{\frac{1}{2}} \tilde{L}_m^{(-\frac{1}{2})} - \sum_{m=0}^{n-1} \sqrt{n} \left(\frac{\Gamma(n)}{\Gamma(n+\frac{1}{2})} \quad \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} \right)^{\frac{1}{2}} \tilde{L}_m^{(-\frac{1}{2})} .$$

Or equivalently

$$(4.14) \qquad (x^{-1}D)\, \tilde{L}_n^{(-\frac{1}{2})} = -\tilde{L}_n^{-\frac{1}{2}} - 2 \sum_{m=0}^{n-1} \left(\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \quad \frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+1)} \right)^{\frac{1}{2}} \tilde{L}_m^{(-\frac{1}{2})} .$$

The matrix of $x^{-1}D$ is given by

$$(4.15) \qquad ((x^{-1}D)\, L_k^{(-\frac{1}{2})}, L_\ell^{(-\frac{1}{2})}) = \begin{cases} -2\left(\dfrac{\Gamma(k+1)}{\Gamma(k+\frac{1}{2})} \quad \dfrac{\Gamma(\ell+\frac{1}{2})}{\Gamma(\ell+1)} \right)^{\frac{1}{2}} & 0 \le \ell \le k-1 \\ -1 & \ell = k \end{cases}$$

where $\ell, k = 0,1,2,\dots$.

It is obvious that the operator $\hat{A}_\alpha = H + S_\alpha$, where $H = -D^2 + x^2$ and $S_\alpha = -(2\alpha+1)(x^{-1}D)$ is densely defined in X, because its domain contains the linear span $\langle \tilde{L}_0^{(-\frac{1}{2})}, \tilde{L}_1^{(-\frac{1}{2})}, \dots \rangle$.

The next step is to estimate the norm of the operators

$$e^{\tau H} (\hat{A}_\alpha)^n e^{-tH}$$

for $t > 0$, $0 < \tau < t$ and $n = 0,1,2,\dots$.

We proceed therefore as follows. Let $0 < \tau < t$, and $n \in \mathbb{N}$.

$$e^{\tau H}\hat{A}_\alpha^n e^{-tH} =$$

$$= (H e^{-\frac{1}{n}sH}) e^{(\frac{1}{n}s+\tau)H} (I + H^{-1}S_\alpha) e^{-(\frac{1}{n}s+\tau)H} (H e^{-\frac{1}{n}sH}) e^{(\frac{2}{n}s+\tau)H} \cdots$$

$$\cdots (I + H^{-1}S_\alpha) e^{-(\frac{n-1}{n}s+\tau)H} (H e^{-\frac{1}{n}sH}) e^{tH}(I + H^{-1}S_\alpha) e^{-tH}$$

where we take $s = t - \tau$. So

$$\| e^{\tau H}\hat{A}_\alpha^n e^{-tH} \| \le \| H e^{-\frac{1}{n}sH} \|^n \prod_{j=1}^{n} \| e^{(\frac{j}{n}s+\tau)H} (I + H^{-1}S_\alpha) e^{-(\frac{j}{n}s+\tau)H} \| . \tag{4.16}$$

By easy computation it follows that

$$\| H e^{-\frac{1}{n}sH} \|^n \le n^n e^{-n} s^{-n} \le n! s^{-n} . \tag{4.17.i}$$

Further, we have for $r > 0$

$$\| e^{rH} (I + H^{-1}S_\alpha) e^{-rH} \| \le 1 + ||| e^{rH} (H^{-1}S_\alpha) e^{-rH} ||| ,$$

where $||| \cdot |||$ denotes the Hilbert-Schmidt norm of $X \otimes X$.

We estimate as follows

$$||| e^{rH} (H^{-1}S_\alpha) e^{-rH} |||^2 = \sum_{k,\ell=0}^{\infty} | (e^{rH} (H^{-1}S_\alpha) e^{-rH} \tilde{L}_k^{(-\frac{1}{2})}, \tilde{L}_\ell^{(-\frac{1}{2})}) |^2$$

$$= \sum_{k=0}^{\infty} \sum_{\ell=0}^{k} e^{-8r(k-\ell)} \frac{1}{(4\ell+1)^2} | (S_\alpha \tilde{L}_k^{(-\frac{1}{2})}, \tilde{L}_\ell^{(-\frac{1}{2})}) |^2 .$$

By (4.15) it follows that there exists $C > 0$ such that

$$|(S_\alpha \tilde{L}_k^{(-\frac{1}{2})}, \tilde{L}_\ell^{(-\frac{1}{2})})| \le C\, k^{\frac{1}{4}} \; . \quad \text{(Stirling's formula!)}$$

So the latter expression is smaller than

$$C \sum_{\ell=0}^{\infty} \frac{1}{(4\ell+1)^2} \sum_{k=\ell}^{\infty} \sqrt{k}\, e^{-8r(k-\ell)} =$$

$$= C \sum_{\ell=0}^{\infty} \frac{1}{(4\ell+1)^2} \sum_{k=0}^{\infty} \sqrt{k+\ell}\, e^{-8rk} \le$$

$$\le C(\sum_{\ell=0}^{\infty} \frac{\sqrt{\ell}}{(4\ell+1)^2} \sum_{k=0}^{\infty} e^{-8rk} + \sum_{\ell=0}^{\infty} \frac{1}{(4\ell+1)^2} \sum_{k=0}^{\infty} \sqrt{k}\, e^{-8rk}) \le$$

$$\le C' \left(\frac{2}{r} + \frac{4}{r^2}\right) \le \frac{6C'}{r^2} \qquad \text{as } r \le 1 .$$

We can estimate the other factor in the product (4.16)

$$\text{(4.17.ii)} \quad \prod_{j=1}^{n} (\| e^{(\frac{j}{n}s+\tau) H} (I + H^{-1} S_\alpha)\, e^{-(\frac{j}{n}s+\tau) H} \|) \le$$

$$\le \prod_{j=1}^{n} \left(1 + \left(\frac{6C'}{(\frac{j}{n}s+\tau)^2}\right)^{\frac{1}{2}}\right) \le \prod_{j=1}^{n} \left[\frac{(12C')^{\frac{1}{2}} n}{sj}\right]$$

$$= (12C')^{\frac{1}{2}n} \frac{n^n}{n!} s^{-n} \le K^n s^{-n}$$

where we assume that $0 < t \le 1$.

Combining the results (4.17.i) and (4.17.ii) we derive:

There exists a constant D only depending on $t - \tau$ such that for all $n \in \mathbb{N}$

$$\text{(4.18)} \qquad \| e^{\tau H} (H + S_\alpha)^n e^{-t H} \| \le D^n n! \; .$$

We define the operator $e^{r\hat{A}_\alpha}$ by

$$e^{r\hat{A}_\alpha} =: \sum_{n=0}^{\infty} \hat{A}_\alpha^n \frac{r^n}{n!} \quad , \quad r \in \mathbb{R} .$$

We proved that for all $t > 0$ and all τ, $0 < \tau < t$, there is $r_0 > 0$ so that $e^{\tau H} e^{r\hat{A}_\alpha} e^{-tH}$ is a bounded operator on X for all $r \in \mathbb{R}$ with $|r| \leq r_0$, and the series

$$\sum_{n=0}^{\infty} \frac{r^n}{n!} e^{\tau H} \hat{A}_\alpha^n e^{-tH} \quad , \quad |r| \leq r_0$$

converges absolutely and uniformly.

Going back to the function g, which is an element of the space $e^{-tH}(X)$, we have shown that there exist $\tau > 0$ and $r > 0$ such that

$$e^{r\hat{A}_\alpha}(g) \in e^{-\tau H}(X) .$$

By [B], Theorem 6.3 this implies that the function $e^{r\hat{A}_\alpha} g$ is entire and satisfies the estimate

$$|(e^{r\hat{A}_\alpha} g)(x + iy)| \leq C_1 \exp(-\tfrac{1}{2} A_1 x^2 + \tfrac{1}{2} B_1 y^2)$$

for some A_1, B_1, $C_1 > 0$, and all $x,y \in \mathbb{R}$. In particular this implies that

$$e^{r\hat{A}_\alpha} g \in L_2(\mathbb{R}^+, x^{2\alpha+1} dx) = X_\alpha .$$

Since $g = e^{-r\hat{A}_\alpha}(e^{r\hat{A}_\alpha} g)$, the latter assertion implies that $f \in e^{-r\hat{A}_\alpha}(X)$.

We have proved (cf Lemma (4.9)):

(4.19) Theorem

$f \in S_{X,A_\alpha}$ if and only if

(i) $z \to z^{-(\alpha+\frac{1}{2})} f(z)$ is entirely analytic and even.

(ii) There are positive constants A, B, C such that

$$|z^{-(\alpha+\frac{1}{2})} f(z)| \leq C \exp(-\tfrac{1}{2} Ax^2 + \tfrac{1}{2} By^2)$$

or equivalently:

$f \in S_{X,A_\alpha}$ if and only if

the function $z \to z^{-(\alpha+\frac{1}{2})} f(z)$ belongs to the Gelfand-Shilov space $S_{\frac{1}{2}}^{\frac{1}{2}}$ and is even.

We note that $S_{\frac{1}{2}}^{\frac{1}{2}} = S_{L_2(\mathbb{R}),H}$ (see [G]). The latter space is intensively investigated by De Bruijn in [B].

§5 Analytic characterization of the elements in $\tau(X,A_\alpha)$

For convenience we introduce the function classes $S^{(\alpha)}_{A,B}$

(5.1) Definition

$f \in S^{(\alpha)}_{A,B}$ if and only if

(i) $z \mapsto z^{-(\alpha+\frac{1}{2})} f(z)$ is entirely analytic and even.

(ii) $|z^{-(\alpha+\frac{1}{2})} f(z)| \le C \exp(-\frac{1}{2} Ax^2 + \frac{1}{2} By^2)$, $x,y \in \mathbb{R}$, for some $C > 0$, and $z = x + iy$.

By Lemma 4.9 and careful rereading of the arguments which lead to Theorem 4.19 the following inclusions can be derived

$$(5.2) \qquad e^{-tA_\alpha}(X) \subset S^{(\alpha)}_{A,B} \subset e^{-t'A_\alpha}(X)$$

where $t,t' > 0$ depend on the choice of A, $0 < A < 1$ and $B > 1$.

Since

$$(5.3) \qquad \tau(X,A_\alpha) = \bigcap_{t>0} e^{-tA_\alpha}(X)$$

(see [E]), it follows that

$$(5.4) \qquad \tau(X,A_\alpha) = \bigcap_{\substack{0<A<1,\\ B>1}} S^{(\alpha)}_{A,B} .$$

In other words

(5.5) Theorem

$f \in \tau(X,A_\alpha)$ if and only if

(i) $z \mapsto z^{-(\alpha+\frac{1}{2})} f(z)$ is an even, entirely analytic function.

(ii) For each A, $0 < A < 1$ and each $B > 1$ there exists $C > 0$ such that for all $x,y \in \mathbb{R}$

$$|(x + iy)^{-(\alpha+\frac{1}{2})} f(x + iy)| \le C \exp(-\tfrac{1}{2} Ax^2 + \tfrac{1}{2} By^2) .$$

In [E], ch VIII, the space $\tau(L_2(\mathbb{R}),H)$ is characterized with the positive self-adjoint operator

$$H = -\frac{d}{dx^2} + x^2 .$$

As a corollary of Theorem 5.5 we have

(5.6) <u>Corollary</u>

$f \in \tau(X,A_\alpha) \Leftrightarrow g : z \mapsto z^{-(\alpha+\frac{1}{2})} f(z) \in \tau(L_2(\mathbb{R}),H)$ and g is even.

§6 Some linear operators in S_{X,A_α} and T_{X,A_α}.

In this section we shall consider some linear operators in the spaces S_{X,A_α} and T_{X,A_α}. In a similar way we can discuss this subject for the other two pairs of spaces.

In §3 the following recurrence relations were given

$$(6.1) \qquad x^2 L_n^{(\alpha)} = -\sqrt{(n+1)(n+\alpha+1)}\, L_{n+1}^{(\alpha)} + (2n+1+\alpha)\, L_n^{(\alpha)} - \sqrt{n(n+\alpha)}\, L_{n-1}^{(\alpha)}$$

for $n \in \mathbb{N} \cup \{0\}$, where $L_{-1}^{(\alpha)} \equiv 0$. The operator Q^2 (see (3.8)) is positive and self-adjoint in X. With some easy calculations it can be seen that

$$(6.2) \qquad \forall_{t>0} \exists_{\tau>0} : \| e^{\tau A_\alpha} Q^2 e^{-tA_\alpha} \| < \infty .$$

Following [G], ch IV, Q^2 maps S_{X,A_α} contnuously into itself. Since Q^2 is self-adjoint it can be extended to a continuous linear mapping on T_{X,A_α}. We shall denote the extended mapping by Q^2, as well.

(6.3) Theorem

For every $z \in \mathbb{C}$, $|\arg(z)| < \pi$, the generalized function $\delta_z^{(\alpha)}$,

$$(6.4) \qquad \delta_z^{(\alpha)} =: \sum_{n=0}^{\infty} L_n^{(\alpha)}(z)\, L_n^{(\alpha)}$$

is in T_{X,A_α}.

Proof

Let $z \in \mathbb{C}$, $|\arg(z)| < \pi$. By Lemma (4.2) there is $\gamma > 0$ and $K > 0$ such that

$$|L_n^{(\alpha)}(z)| \le K\, e^{\gamma\sqrt{n}} .$$

Therefore for all $t > 0$

$$L_n^{(\alpha)}(z) = O(e^{nt}) \ ,$$

i.e. by Characterization 2.3 the assertion follows. □

We denote the pairing in $S_{X,A_\alpha} \times T_{X,A_\alpha}$ by $<.,.>$. It is easily seen that for all $f \in S_{X,A_\alpha}$ we have

(6.5) $$<f,\delta_z^{(\alpha)}> = (e^{tA_\alpha} f, \delta_z^{(\alpha)}(t)) = f(z)$$

where $t > 0$ is taken sufficiently small. (For the precise definition of $<.,.>$ see [G], ch III)

(6.6) Corollary

For all $z \in \mathbb{C}$, $|\arg z| < \pi$ we have

$$Q^2\delta_z^{(\alpha)} = z^2\delta_z^{(\alpha)} \ .$$

Proof

Let $z \in C$, $|\arg z| < \pi$. Then by (6.5) for all $f \in S_{X,A_\alpha}$

$$<Q^2 f,\delta_z^{(\alpha)}> = z^2 f(z) = z^2<f,\delta_z^{(\alpha)}> \ .$$

So $Q^2\delta_z^{(\alpha)} = z^2\delta_z^{(\alpha)}$. □

We have the following relation

(6.7) $$\mathbb{H}_\alpha Q^2 \mathbb{H}_\alpha = B_\alpha$$

where $B_\alpha = -\dfrac{d^2}{dx^2} + \dfrac{\alpha^2-\frac{1}{4}}{x^2}$.

So the generalized eigenfunctions $e_z^{(\alpha)}$ of $\mathcal{B}_\alpha$ in $\mathcal{T}_{X,A_\alpha}$ are formally given by

(6.8) $$e_z^{(\alpha)} = \mathbb{H}_\alpha \, \delta_z^{(\alpha)} .$$

It is well-known that

(6.9) $$\left(-D_\xi^2 + \frac{\alpha^2-\frac{1}{4}}{\xi^2}\right)[\sqrt{\xi z}\, J_\alpha(\xi z)] = z^2 \sqrt{\xi z}\, J_\alpha(\xi z) .$$

So we derive from (6.8) and (6.9)

(6.10) $$e_z^{(\alpha)}(x) = c_z \sqrt{xz}\, J_\alpha(xz) \quad , \quad |\arg z| < \pi \quad , \quad x > 0$$

for some $c_z \in \mathbb{C}$. We have $c_z = 1$, because

$$L_n^{(\alpha)}(z) = \langle L_n^{(\alpha)}(z), \delta_z^{(\alpha)}\rangle = \langle \mathbb{H}_\alpha L_n^{(\alpha)}, e_z^{(\alpha)}\rangle = (-1)^n \langle L_n^{(\alpha)}, e_z^{(\alpha)}\rangle$$

and

$$L_n^{(\alpha)}(z) = (-1)^n \int_0^\infty L_n^{(\alpha)}(x) \sqrt{xz}\, J_\alpha(xz)\,dx .$$

Consider the following recurrence relations, satisfied by the Laguerre polynomials. ([MOS], p. 241.)

(6.11.a) $$x L_n^{\alpha+1}(x) = (n + \alpha + 1) L_n^{(\alpha)}(x) - (n+1) L_{n+1}^{(\alpha)}(x) \quad , \quad x > 0 ,$$

(6.11.b) $$L_n^{(\alpha)}(x) = L_n^{\alpha+1}(x) - L_{n-1}^{(\alpha+1)}(x) \quad , \quad x > 0 ,$$

where we take $L_{-1}^{(\alpha+1)} \equiv 0$.

From (6.11.a) we derive

$$x(x^{\alpha+1} e^{-x^2/2} L_n^{(\alpha+1)}(x^2)) = x^\alpha e^{-x^2/2} (x^2 L_n^{(\alpha+1)}(x^2)) =$$

$$= (n + \alpha + 1)\, x^\alpha e^{-x^2/2} L_n^{(\alpha)} (x^2) - (n+1)\, x^\alpha e^{-x^2/2} L_{n-1}^{(\alpha)} (x^2)$$

and taking into account the normalization factors of (1.4).

$$(6.12.a)\quad x\,\mathcal{L}_n^{(\alpha+1)}(x) = \sqrt{n+\alpha+1}\,\mathcal{L}_n^{(\alpha)}(x) - \sqrt{n+1}\,\mathcal{L}_{n+1}^{(\alpha)}(x) \quad , \quad x > 0 .$$

Similarly from (6.11.b)

$$(6.12.b)\quad x\,\mathcal{L}_n^{(\alpha)}(x) = \sqrt{n+\alpha}\,\mathcal{L}_n^{(\alpha+1)}(x) - \sqrt{n}\,\mathcal{L}_{n-1}^{(\alpha+1)}(x) \quad , \quad x > 0 .$$

When Q denotes multiplication by x, it can be shown with the aid of (6.12.a) and (6.12.b) that

$$(6.13.a)\quad \forall_{t>0}\exists_{\tau>0} : \| e^{\tau A_\alpha} Q\, e^{-t A_{\alpha+1}} \| < \infty .$$

$$(6.13.b)\quad \forall_{t>0}\exists_{\tau>0} : \| e^{\tau A_{\alpha+1}} Q\, e^{-t A_\alpha} \| < \infty .$$

Following [G], ch IV, Q maps $S_{X,A_{\alpha+1}}$ continuously into S_{X,A_α} and also S_{X,A_α} continuously into $S_{X,A_{\alpha+1}}$. Since Q is self-adjoint in X, the linear mapping Q can be extended to a continuous linear mapping $\bar{Q}$, say, from $T_{X,A_{\alpha+1}}$ into T_{X,A_α} and from T_{X,A_α} into $T_{X,A_{\alpha+1}}$.

In [S], p. 310, the following relations are given

$$(6.14.a)\quad \mathbb{H}_\alpha\, Q\, \mathbb{H}_{\alpha+1} = P_{\alpha+1} .$$

$$(6.14.b) \qquad \mathbb{H}_{\alpha+1}\, Q\, \mathbb{H}_\alpha = -P_\alpha^* \ .$$

with

$$(6.15) \qquad P_\beta = x^{-\beta}\, \frac{d}{dx}\, x^\beta \quad , \quad \beta \in \mathbb{R}\ .$$

From the results of section 2 it follows that $P_{\alpha+1}$ is a continuous linear mapping from $S_{X,A_{\alpha+1}}$ into S_{X,A_α}, which can be extended to a continuous linear mapping from $T_{X,A_{\alpha+1}}$ into T_{X,A_α}. And also that $-P_\alpha^*$ is a continuous linear mapping from S_{X,A_α} into $S_{X,A_{\alpha+1}}$, which can be extended to a continuous linear mapping from T_{X,A_α} into $T_{X,A_{\alpha+1}}$.

Finally we remark that from the theory in [G], ch IV it follows that the operator Q^δ maps S_{X,A_α} continuously into $S_{X,A_{\alpha+\delta}}$ and the operator $\frac{d}{dx}$ maps S_{X,A_α} continuously into $S_{X,A_{\alpha-1}}$. The operators can be extended to continuous linear mappings from T_{X,A_α} into $T_{X,A_{\alpha+\delta}}$ resp. T_{X,A_α} into $T_{X,A_{\alpha-1}}$.

Appendix

We shall adapt the notion: Hankel transform, in order to make it useful for manipulations with spherical coördinates.

For every $\beta \geq 0$ an operator $\mathbb{H}_{\alpha,\beta}$ is introduced on the Hilbert space $L_2((0,\infty),x^\beta dx)$ in such a way that $\mathbb{H}_{\alpha,0} = \mathbb{H}_\alpha$. We start the discussion with equality (1.1)

$$\text{(a.1)} \qquad x^\alpha e^{-x^2/2} L_n^{(\alpha)}(x^2) = (-1)^n \int_0^\infty y^\alpha e^{-y^2/2} L_n^{(\alpha)}(y^2) J_\alpha(xy) y\, dy$$
$$= (-1)^n \int_0^\infty y^\alpha e^{-y^2/2} L_n^{(\alpha)}(y^2)\, y^{1-\beta} J_\alpha(xy) y^\beta\, dy .$$

Following the orthogonality relations (1.4) we have

$$\text{(a.2)} \qquad \int_0^\infty x^{2\alpha} e^{-x^2} L_n^{(\alpha)}(x^2) L_m^{(\alpha)}(x^2) x\, dx = \tfrac{1}{2} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \delta_{nm}$$

or equivalently

$$\text{(a.3)} \qquad \int_0^\infty [x^{\alpha-\frac{1}{2}\beta+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(\alpha)}(x^2)][x^{\alpha-\frac{1}{2}\beta+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(\alpha)}(x^2)] x^\beta\, dx =$$
$$= \tfrac{1}{2} \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)} \delta_{nm} .$$

So with the aid of (a.1) we derive

$$\text{(a.4)} \qquad x^{\alpha-\frac{1}{2}\beta+\frac{1}{2}} e^{-x^2/2} L_n^{(\alpha)}(x^2) =$$
$$= (-1)^n \int_0^\infty y^{\alpha-\frac{1}{2}\beta+\frac{1}{2}} e^{-y^2/2} (xy)^{-\frac{1}{2}\beta+\frac{1}{2}} J_\alpha(xy) y^\beta\, dy .$$

Now define

$$\text{(a.5)} \qquad L_n^{(\alpha,\beta)}(x) =: \left(\frac{2\Gamma(n+1)}{\Gamma(n+\alpha+1)}\right)^{\frac{1}{2}} x^{\alpha-\frac{1}{2}\beta+\frac{1}{2}} e^{-\frac{1}{2}x^2} L_n^{(\alpha)}(x^2) , \quad x > 0 .$$

The $L_n^{(\alpha,\beta)}$'s establish an orthonormal basis in the Hilbert space $L_2((0,\infty),x^\beta dx)$ and they are the eigenfunctions of the self-adjoint operator

$$(a.6) \qquad A_{\alpha,\beta} = -\frac{d^2}{dx^2} - \frac{\beta}{x}\frac{d}{dx} + \frac{\alpha^2-\frac{1}{4}(\beta-1)^2}{x^2} + x^2 .$$

When we define the operator $\mathbb{H}_{\alpha,\beta}$ in $L_2((0,\infty),x^\beta dx)$ formally by

$$(a.7) \qquad (\mathbb{H}_{\alpha,\beta} f)(x) = \int_0^\infty (xy)^{-\frac{1}{2}\beta+\frac{1}{2}} J_\alpha(xy)\, f(y)\, y^\beta\, dy ,$$

it follows from (a.4) that

$$(a.8) \qquad \mathbb{H}_{\alpha,\beta} L_n^{(\alpha,\beta)} = (-1)^n L_n^{(\alpha,\beta)} .$$

Take $X_\beta =: L_2((0,\infty),x^\beta dx)$. Then the test function spaces $\tau(X_\beta,\log A_{\alpha,\beta})$, $S_{X_\beta,A_{\alpha,\beta}}$ and $\tau(X_\beta,A_{\alpha,\beta})$ are well defined and so are the generalized function spaces $\sigma(X_\beta,\log A_{\alpha,\beta})$, $T_{X_\beta,A_{\alpha,\beta}}$, $\sigma(X_\beta,A_{\alpha,\beta})$. Without proof we assert that all results of the previous sections for the Hankel transform $\mathbb{H}_\alpha$ hold in an adapted form for the adapted Hankel transforms $\mathbb{H}_{\alpha,\beta}$.

If we take $\alpha = \frac{1}{2}n-1$ and $\beta = n-1$ with $n \in \mathbb{N}$, $n \geq 2$, then

$$(a.9) \qquad (\mathbb{H}_{\frac{1}{2}n-1,n-1} f)(\rho) = \int_0^\infty (r\rho)^{-\frac{1}{2}n+1} J_{\frac{1}{2}n-1}(r\rho)\, f(r)\, r^{n-1}\, dr .$$

Thus the adapted Hankel transform $\mathbb{H}_{\frac{1}{2}n-1,n-1}f$ of f is equal to its Fourier transform, where f is a function of r

$$r = (x_1^2 + x_2^2 + \dots + x_n^2)^{\frac{1}{2}}$$

only.

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