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DOI:
10.1137/0731020

Document status and date:
Published: 01/01/1994

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

SIAM J. NUMER. ANAL.
Vol. 31, No. 2, pp. 378–392, April 1994

ON MAXIMUM NORM CONVERGENCE OF MULTIGRID METHODS FOR ELLIPTIC BOUNDARY VALUE PROBLEMS*

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Abstract. Multigrid methods applied to standard linear finite element discretizations of linear elliptic boundary value problems in two dimensions are considered. In the multigrid method, damped Jacobi or damped Gauss–Seidel is used as a smoother. It is proven that the two-grid method with ν pre-smoothing iterations has a contraction number with respect to the *maximum norm* that is (asymptotically) bounded by $C\nu^{-1/2}\ |\ln h_k|^2$, with h_k a suitable mesh size parameter. Moreover, it is shown that this bound is sharp in the sense that a factor $|\ln h_k|$ is necessary.

Key words. multigrid, convergence analysis, maximum norm, elliptic boundary value problems

AMS subject classification. 65N20

1. Introduction.

If one considers elliptic boundary value problems in $\mathbb{R}^N$ $(N = 2, 3)$, then multigrid methods can be used to efficiently solve the large sparse linear systems that arise after discretization. In recent years there has been intensive research into the theoretical understanding of the convergence properties of these methods. We refer to Hackbusch [10], McCormick [14], and the references therein. The main feature of multigrid is that for a broad class of problems the contraction number has an upper bound that is smaller than one and independent of the mesh size. In theoretical analyses this has been shown for several variants of multigrid. Usually in these analyses the energy norm is used; sometimes one uses the Euclidean norm. Some first results about multigrid convergence in the maximum norm are presented in [19]. In that paper, however, only two-point boundary value problems are treated. In this paper we present convergence results in the *maximum norm* for multigrid applied to a class of elliptic *two-dimensional* boundary value problems. We consider a regular linear (nearly) symmetric elliptic boundary value problem on a domain $\Omega \subset \mathbb{R}^2$, and we use linear finite elements on quasi-uniform triangulations.

Two main results of this paper are the following. Firstly, we prove that for a two-grid method with ν damped Jacobi or damped Gauss–Seidel smoothing iterations the contraction number with respect to the maximum norm is (asymptotically) bounded by $C\nu^{-1/2}\ |\ln h_k|^2$ (with h_k a suitable mesh size parameter). Secondly, it is shown that this bound is sharp in the sense that a factor $|\ln h_k|$ is necessary: For a concrete (very regular) example we prove that the contraction number with respect to the maximum norm of a standard two-grid method with a fixed number of smoothing iterations is bounded from below by $C\ |\ln h_k|$.

So instead of an "optimal" bound $C\nu^{-1}$ for the contraction number in the energy norm (or the Euclidean norm) we obtain a "nearly optimal" bound $C\nu^{-1/2}\ |\ln h_k|^2$ if we use the maximum norm and this bound is sharp in some way.

We now outline the remainder of this paper. In §2 we introduce a class of elliptic boundary value problems. Some properties of the usual linear finite element discretization on a sequence of quasi-uniform triangulations are derived in §3. An important property, due to Descloux [7], is that the mass matrix has a condition number which is uniformly bounded (for $h_k \downarrow 0$) with respect to the maximum norm. In §§4 and 5

* Received by the editors March 11, 1992; accepted for publication (in revised form) March 9, 1993.

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we prove the approximation property and smoothing property (cf. Hackbusch [10]), respectively. In the proof of the approximation property we use a regularity result due to Campanato [5] and an L^∞-finite element error estimate due to Rannacher and Frehse [18], [8]. From the latter the factor $|\ln h_k|^2$ originates. The smoothing property is proved using a new technique introduced in [20]. In §6 we derive convergence results for the two-grid method and we discuss multigrid convergence. Finally, in §7, we analyze a specific example. We consider the Poisson equation on the unit square and use a linear finite element discretization on a uniform triangulation. We prove a $C|\ln h_k|$ lower bound for the contraction number of a standard two-grid method with a fixed number of smoothing iterations. Our analysis in §7 is based on the approach used by Haverkamp in [11].

2. Continuous problem. Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain with $\partial\Omega$ sufficiently smooth. We consider the following variational boundary value problem:

$$(2.1)\qquad \begin{cases} \text{find } \varphi^* \in H_0^1(\Omega) \text{ such that for all } \psi \in H_0^1(\Omega), \\ \displaystyle\sum_{i,j=1}^{2} \int_\Omega a_{ij} \frac{\partial \varphi^*}{\partial x_i} \frac{\partial \psi}{\partial x_j}\, dx = \int_\Omega f\, \psi\, dx\, . \end{cases}$$

We use the notation

$$(2.2)\qquad a(\varphi,\psi) = \sum_{i,j=1}^{2} \int_\Omega a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \psi}{\partial x_j}\, dx \qquad (\varphi,\psi \in H_0^1(\Omega))\ ,$$

and we make the following assumptions about Ω, f, $a(\cdot,\cdot)$:

(2.3) $f \in L^2(\Omega)$,

(2.4a) $\partial\Omega \in C^{2,\alpha} \quad (\alpha \in]0,1])$,

(2.4b) $a_{ij} = a_{ji}\ ;\ a_{ij} \in C^{1,\alpha}(\bar\Omega) \quad (\alpha \in]0,1])$,

(2.4c) there are constants $\lambda_1, \lambda_2 > 0$ such that for all $(\xi_1,\xi_2) \in \mathbb{R}^2$ and all $x \in \Omega$

$$\lambda_2\, |\xi|^2 \le \sum_{i,j=1}^{2} a_{ij}(x)\, \xi_i\, \xi_j \le \lambda_1\, |\xi|^2\ .$$

By $H^{m,p}(\Omega)$, $H_0^{m,p}(\Omega)$ $(1 \le p \le \infty,\ m \in \mathbb{N})$ we denote the usual Sobolev spaces with norm $\|\cdot\|_{H^{m,p}}$ and $\|\cdot\|_{H_0^{m,p}}$, respectively. If $p = 2$ we use the notation $H^m(\Omega)$, $H_0^m(\Omega)$.

Remark 2.1. In multigrid convergence theory an important role is played by the regularity of the differential operator. It is well known that under the assumptions (2.3), (2.4a)–(2.4c) we have H^2-regularity: φ^* is an element of $H^2(\Omega) \cap H_0^1(\Omega)$ and $\|\varphi^*\|_{H^2} \le c\,\|f\|_{L^2}$ with c independent of f. If instead of $p = 2$ we take another $p \in]1,\infty[$, then a similar $H^{2,p}$-regularity result holds (cf. [9]). However, even if $\partial\Omega$ and the coefficients a_{ij} are very smooth, a similar $H^{2,\infty}$-regularity estimate $\|\varphi^*\|_{H^{2,\infty}} \le c\,\|f\|_{L^\infty}$ does not hold, as is shown by the following example. Let Ω be the unit sphere and $a(\varphi,\psi) := \int_\Omega \nabla\varphi \cdot \nabla\psi\, dx$. For $\varepsilon \in [0,1]$ we define $\varphi_\varepsilon^*(x_1,x_2) = x_1 x_2\ \ln(|x| + \varepsilon) - x_1 x_2\ \ln(1+\varepsilon)$ (with $\varphi_0^*(0,0) := 0$). Then $\varphi^*_{\varepsilon|\partial\Omega} = 0$ and $\Delta\varphi_\varepsilon^*$ exists on Ω. Now define $f_\varepsilon := -\Delta\varphi_\varepsilon^*$, then $f_\varepsilon \in L^\infty(\bar\Omega)$ for all $\varepsilon \in [0,1]$ and even $f_\varepsilon \in C(\bar\Omega)$ if $\varepsilon \in]0,1]$. However, if $\varepsilon = 0$ we have $\varphi_\varepsilon^* \notin H^{2,\infty}(\Omega)$ and if $\varepsilon \in]0,1]$ we have $\varphi_\varepsilon^* \in H^{2,\infty}$, but $\|\varphi_\varepsilon^*\|_{H^{2,\infty}}\ \|f_\varepsilon\|_{L^\infty}^{-1}$ is unbounded for $\varepsilon \downarrow 0$.

In the proof of the approximation property (§4) with respect to the L^∞-norm another type of regularity result (due to Campanato [5]) is used, in which the usual

$L^p(\Omega)$ space is replaced by the space of John–Nirenberg $E^0(\Omega)$ (also called the space of functions of bounded mean oscillation; cf. [12]). This space satisfies

$$L^\infty(\Omega) \subsetneq E^0(\Omega) \subsetneq L^p(\Omega) \quad \text{for all } p \in [1,\infty[.$$

3. Discretization and two-grid method. We take a sequence of quasi-uniform triangulations $\{\mathcal{T}_k \mid k \in \mathbb{N}_0\}$ as follows. For every k we define $\Omega_k := \cup_{T\in\mathcal{T}_k} T$ and for every $\mathcal{T}_k$ we use a mesh size parameter h_k with $0 < h_{k+1} \le h_k < 1$. We make the following assumptions (with constants c_i independent of T and k):

(3.1a) $\Omega_k \subset \Omega_{k+1} \subset \Omega$ for all k;
(3.1b) $\operatorname{dist}(\partial\Omega_k, \partial\Omega) \le c_0 h_k^2$;
(3.1c) for any two different triangles in $\mathcal{T}_k$ the intersection is empty or consists of a common vertex or of a common side;
(3.1d) for every $T \in \mathcal{T}_k$ there is a disc with radius $c_1 h_k$ containing T and a disc with radius $c_2 h_k$ contained in T;
(3.1e) $h_k h_{k+1}^{-1} \le c_3$.

Continuous piecewise linear functions on such a triangulation $\mathcal{T}_k$ yield a finite-dimensional function space

$$\Phi_k = \{v \in C(\bar\Omega) \mid v \text{ is linear on every } T \in \mathcal{T}_k \text{ and } v \equiv 0 \text{ on } \partial\Omega_k \cup (\bar\Omega\backslash\Omega_k)\}. \tag{3.2}$$

The collection of interior grid points in $\mathcal{T}_k$ is denoted by $\{x_k^i\}_{i\in J_k}$ for some index set J_k with $\#J_k = n_k$. We use the notation $U_k = \mathbb{R}^{n_k}$. The standard basis of Φ_k is given by the functions $\varphi_k^i \in \Phi_k$ which satisfy $\varphi_k^i(x_k^j) = \delta_{ij}$ $(i,j \in J_k)$. This induces the natural bijection

$$P_k : U_k \to \Phi_k, \qquad P_k(u) = \sum_{i\in J_k} u_i\,\varphi_k^i. \tag{3.3}$$

On U_k we use a scaled Euclidean inner product with corresponding norm

$$\langle u, v\rangle_k = h_k^2 \sum_{i\in J_k} u_i v_i, \qquad \|u\|_k = \langle u, u\rangle_k^{\frac12}. \tag{3.4}$$

The maximum norm on U_k is denoted by $\|\cdot\|_\infty$. Below, adjoints are always defined with respect to the L^2-inner product on Φ_k and $\langle\cdot,\cdot\rangle_k$ on U_k. The norms $\|\cdot\|_\infty$ (on U_k) and $\|\cdot\|_{L^\infty}$ (on Φ_k) induce associated operator norms, which are denoted by $\|\cdot\|_\infty$.

Remark 3.1. Lemma 3.2 below yields that the sequences $(P_k)_{k\ge0}$, $(P_k^{-1})_{k\ge0}$, $(P_k^*)_{k\ge0}$, and $((P_k^*)^{-1})_{k\ge0}$ are uniformly bounded in $\|\cdot\|_\infty$. In the analysis of multigrid convergence in the Euclidean norm (cf. [10]) such a uniform boundedness result is used too, but then with resepct to the norms $\|\cdot\|_k$ and $\|\cdot\|_{L^2}$. The latter uniform boundedness is closely related to the well-known fact that the mass matrix $P_k^*P_k$ has a condition number $O(1)$ $(h_k \downarrow 0)$ in the Euclidean norm. The main argument in the proof of Lemma 3.2 is that the condition number of the mass matrix is $O(1)$ $(h_k \downarrow 0)$ with respect to the maximum norm, too. This result has been proved by Descloux in [7].

LEMMA 3.2. *For $P_k : U_k \to \Phi_k$ as in (3.3) the following holds with constants C_1 and C_2 independent of k:*

$$\|P_k u\|_{L^\infty} = \|u\|_\infty \quad \textit{for all } u \in U_k, \tag{1}$$

$$C_1\,\|\varphi\|_{L^\infty} \le \|P_k^*\varphi\|_\infty \le C_2\,\|\varphi\|_{L^\infty} \quad \textit{for all } \varphi \in \Phi_k. \tag{2}$$

Proof. The result in (1) holds because $P_k u$ is piecewise linear and $(P_k u)(x_k^i)$ equals the ith component of u. Due to (1) the statement in (2) is equivalent with

$$(2') \qquad C_1 \|u\|_\infty \le \|P_k^* P_k u\|_\infty \le C_2 \|u\|_\infty \quad \text{for all } u \in U_k .$$

For $i, j \in J_k$ we have (with e_m the mth unit vector in U_k)

$$(P_k^* P_k)_{i,j} = h_k^{-2} \langle P_k^* P_k e_j, e_i \rangle_k = h_k^{-2} \int_\Omega \varphi_k^i \varphi_k^j \, dx .$$

So, using (3.1.d), we get

$$\|P_k^* P_k\|_\infty = \max_i \; h_k^{-2} \sum_{j \in J_k} \int_\Omega \varphi_k^j \varphi_k^i \le \max_i \; h_k^{-2} \operatorname{supp}(\varphi_k^i) \le C_2;$$

thus the second inequality in $(2')$ holds. For the first inequality in $(2')$ we note that due to the quasi-uniformity of the triangulations the assumptions in [7] hold. Theorem 2 in [7] then yields the desired result. □

Galerkin discretization results in a stiffness matrix $L_k : U_k \to U_k$ defined by

$$(3.5) \qquad \langle L_k u, v \rangle_k = a(P_k u, P_k v) \quad \text{for all } u, v \in U_k.$$

Also we have that

$$(3.6) \qquad a(P_k L_k^{-1} g, P_k v) = \langle g, v \rangle_k \quad \text{for all } g, v \in U_k.$$

For the analysis of a multigrid method it is beneficial to have a hierarchy of finite element spaces (i.e., $\Phi_k \subset \Phi_{k+1}$ for all k). Due to $\partial\Omega \in C^{2,\alpha}$ and the assumptions in (3.1) we cannot have such a hierarchy here. Below, in §6, we will point out why we make the assumption that $\partial\Omega$ is smooth instead of the usual assumption (in multigrid convergence analyses) that Ω is polygonal.

In order to separate the coarse to fine strategy (essential for multigrid) from the modifications close to the boundary we will be more specific about the refinement procedure we use. For every $m \ge 0$ we construct $\mathcal{T}_{2m+1}$ and $\mathcal{T}_{2m+2}$ from $\mathcal{T}_{2m}$ as follows: First we use a refinement in which, for every triangle in $\mathcal{T}_{2m}$, the midpoints of the sides are connected; this yields $\mathcal{T}_{2m+1}$; then, in order to fulfill (3.1b), $\mathcal{T}_{2m+2}$ is constructed from $\mathcal{T}_{2m+1}$ by modifying (if necessary) the triangulation close to the boundary (consistent with the other conditions in (3.1)). As a consequence we have

$$(3.1.\text{f}) \qquad \Phi_{2m} \subset \Phi_{2m+1}.$$

Note, however, that in general $\Phi_{2m+1} \not\subset \Phi_{2m+2}$.

In the remainder of this section we assume that k is odd and thus

$$(3.7) \qquad \Phi_{k-1} \subset \Phi_k.$$

For solving a system of the form $L_k u_k = g_k$ we use a standard two-grid method. The iteration matrix of the smoothing method is denoted by S_k. For the prolongation $p = p_k : U_{k-1} \to U_k$ we use the natural one:

$$(3.8) \qquad p = P_k^{-1} P_{k-1}.$$

For the restriction $r = r_k$: $U_k \to U_{k-1}$ we take

$$(3.9) \qquad r = p^*.$$

The iteration matrix of the two-grid method with ν presmoothing iterations is given by

$$(3.10)\qquad T_k(\nu) = (I - pL_{k-1}^{-1}rL_k)\,S_k^\nu = (L_k^{-1} - pL_{k-1}^{-1}r)\,L_kS_k^\nu.$$

Below, in §§4 and 5 we will prove the approximation property and smoothing property (cf. [10]) with respect to the norm $\|\cdot\|_\infty$.

4. The approximation property. We begin this section with a discussion of a rather special finite element L^∞-error estimate that will be used in the proof of the approximation property below.

In most convergence analyses of multigrid the regularity of the underlying boundary value problem is used. If the analysis is based on the energy norm, then h-independent convergence can be proved under very weak regularity conditions (cf., eg., [1]–[4], [10], [13], and [16]). If, however, one wants to prove the approximation property in the scaled Euclidean norm, then an H^2-regularity estimate

$$(4.1)\qquad \|\varphi^*\|_{H^2} \le c\,\|f\|_{L^2}$$

is necessary, as is shown in [6]. The approximation property then follows from a combination of (4.1) with the following (in which φ_k^* is the Galerkin solution in Φ_k):

$$(4.2a)\qquad \|\varphi_k^* - \varphi^*\|_{L^2} \le ch_k^2\,\|\varphi^*\|_{H^2}\ \ \text{(finite element error estimate)},$$

$$(4.2b)\qquad \|\cdot\|_{L^2}\ \text{and}\ \|\cdot\|_k\ \text{are uniformly equivalent for}\ k\to\infty.$$

In this paper we want to prove the approximation property in the maximum norm. Clearly, the analogue of (4.2b) is given in Lemma 3.2. With respect to (4.2a) we note that L^∞- error estimates of the form

$$(4.3)\qquad \|\varphi_k^* - \varphi^*\|_{L^\infty} \le ch_k^2\,|\ln h_k|\ \|\varphi^*\|_{H^{2,\infty}}$$

can be found in the literature (e.g., in [8], [15], [22], and [23]). However, it is shown in Remark 2.1 that, even for very regular problems, an $H^{2,\infty}$-regularity estimate $\|\varphi^*\|_{H^{2,\infty}} \le c\,\|f\|_{L^\infty}$ does not hold. Therefore it is not clear how (4.3) can be used to prove the approximation property.

In [8] and [18] Rannacher and Frehse prove the following type of (asymptotic) error estimate for $f \in L^\infty$:

$$(4.4)\qquad \|\varphi_k^* - \varphi^*\|_{L^\infty} \le ch_k^2\,|\ln h_k|^2\ \|f\|_{L^\infty}.$$

This result, which is a substitute for the combination of (4.1) and (4.2a), and the result of Lemma 3.2 are the main points in the proof of the approximation property in Theorem 4.2 below.

Remark 4.1. We briefly comment on the combination of regularity and approximation properties of Φ_k used in the proof of (4.4). Instead of $L^p(\Omega)$ the John–Nirenberg space $E^0(\Omega)$ is used with a suitable norm denoted by $\|\cdot\|_{L^{2,0}}$. For $E^0(\Omega)$ one has

$$L^\infty(\Omega) \overset{\subset}{\neq} E^0(\Omega) \overset{\subset}{\neq} L^p(\Omega)\quad \text{for every}\ p\in[1,\infty[.$$

Instead of $H^2(\Omega)$ the subspace $H^{2,0}(\Omega)$ of functions for which all generalized second derivatives are in $E^0(\Omega)$ is used. In [5] Campanato proves the following regularity result:

$$(4.5)\qquad \|\varphi^*\|_{H^{2,0}} \le c\,\|f\|_{L^{2,0}}\quad \text{for all}\ \ f\in E^0(\Omega).$$

In [8] Frehse and Rannacher prove the following asymptotic error estimate (for ease $\Omega_k = \Omega$):

(4.6) $$\|\varphi_k^* - \varphi^*\|_{L^\infty} \le c_1 h_k \,|\ln h_k|\, \inf_{\varphi\in\Phi_k} \|\nabla(\varphi - \varphi^*)\|_{L^\infty} + c_2 \inf_{\varphi\in\Phi_k} \|\varphi - \varphi^*\|_{L^\infty}.$$

Furthermore, in [18] Rannacher proves the following (less standard) approximation property of the space Φ_k if $\varphi^* \in H^{2,0}(\Omega) \cap H_0^1(\Omega)$:

(4.7) $$\inf_{\varphi\in\Phi_k} \{h_k \|\nabla(\varphi - \varphi^*)\|_{L^\infty} + \|\varphi - \varphi^*\|_{L^\infty}\} \le c h_k^2 \,|\ln h_k|\, \|\varphi^*\|_{H^{2,0}}.$$

The combination of (4.5)–(4.7) yields (4.4) (note that $\|f\|_{L^{2,0}} \le c\,\|f\|_{L^\infty}$).

THEOREM 4.2. *Assume* (2.4a)–(2.4c) *and* (3.1a)–(3.1f). *Then there are constants* k_0 *and* C_A *such that for all odd* $k \ge k_0$ *the following holds*:

(4.8) $$\|L_k^{-1} - pL_{k-1}^{-1} r\|_\infty \le C_A\, h_k^2 \,|\ln h_k|^2.$$

Proof. Take $g \in U_k$. In the proof different constants C, all independent of k and g, are used. Let $\varphi \in H_0^1(\Omega)$, $\varphi_k \in \Phi_k$, and $\varphi_{k-1} \in \Phi_{k-1}$ be such that

$$\begin{aligned} a(\varphi, \psi) &= ((P_k^*)^{-1} g, \psi)_{L^2} \quad \text{for all } \psi \in H_0^1(\Omega),\\ a(\varphi_k, \psi) &= ((P_k^*)^{-1} g, \psi)_{L^2} \quad \text{for all } \psi \in \Phi_k,\\ a(\varphi_{k-1}, \psi) &= ((P_k^*)^{-1} g, \psi)_{L^2} \quad \text{for all } \psi \in \Phi_{k-1}. \end{aligned}$$

The asymptotic error estimate (4.4) yields that for k large enough, say $k \ge k_0$, we have

$$\|\varphi_m - \varphi\|_{L^\infty} \le C\, h_m^2 \,|\ln h_m|^2\, \|(P_k^*)^{-1} g\|_{L^\infty} \quad \text{if } m \in \{k, k-1\}.$$

Using (3.1.e) this yields for $k \ge k_0$

(4.9) $$\|\varphi_k - \varphi_{k-1}\|_{L^\infty} \le C\, h_k^2 \,|\ln h_k|^2\, \|(P_k^*)^{-1} g\|_{L^\infty}.$$

From (3.6) it follows that $\varphi_k = P_k L_k^{-1} g$ and for k odd $\varphi_{k-1} = P_{k-1} L_{k-1}^{-1} rg$. Using Lemma 3.2 and (4.9) we have that for $k \ge k_0$,

$$\begin{aligned} \|(L_k^{-1} - pL_{k-1}^{-1} r)\, g\|_\infty &= \|P_k L_k^{-1} g - P_{k-1} L_{k-1}^{-1} rg\|_{L^\infty} = \|\varphi_k - \varphi_{k-1}\|_{L^\infty}\\ &\le C\, h_k^2 \,|\ln h_k|^2\, \|(P_k^*)^{-1} g\|_{L^\infty} \le C\, h_k^2 \,|\ln h_k|^2\, \|g\|_\infty. \end{aligned}$$ □

5. The smoothing property. The usual technique for proving the smoothing property requires symmetry (or a nearly symmetric situation), and yields results in the Euclidean norm or in the energy norm. We refer to Wittum [24], where smoothing and the construction of smoothers are discussed in a general framework. A new approach to the smoothing property has been introduced in [20]. The analysis there does not use symmetry and can also be used for the maximum norm. A disadvantage of this new approach is that we need a damping factor less than or equal to 0.5 (whereas the conditions for the damping factor in [24] are less restrictive). The results of this section can be found in a more general setting in [20]. The analysis here is the same as for the one-dimensional case in [19]. For completeness we give proofs here too.

The smoothing iteration we use is based on a splitting

(5.1) $$L_k = W_k - R_k.$$

We make the following assumptions about this splitting:

(5.2a) W_k is regular and $\|W_k^{-1} R_k\|_\infty \le 1$ for all k,

(5.2b) $\|W_k\|_\infty \le c h_k^{-2}$ for all k, with c independent of k.

Remark 5.1. It is well known that if L_k is weakly diagonally dominant (i.e., $\sum_{j=i} |(L_k)_{ij}| \leq |(L_k)_{ii}|$ for all i), then (5.2a) holds if (5.1) corresponds to the Jacobi or Gauss–Seidel relaxation. It can also be shown (cf. [21]) that the following criterion holds: if L_k is such that $L_k\, e \geq 0$ (with $e = (1, 1, \ldots, 1)^T$ and "$\geq$" entrywise) and if (5.1) is a (weak) regular splitting then (5.2a) holds.

This criterion applies to the ILU splitting of an M-matrix L_k.

The proof of the smoothing property in Theorem 5.3 below is based on the following elementary lemma.

LEMMA 5.2. *Let A be a square matrix with $\|A\|_\infty \leq 1$. Then the following holds*:

$$\|(I-A)\,(I+A)^\nu\|_\infty \leq 2\binom{\nu}{[\frac{1}{2}\nu]} < 2^{\nu+1}\sqrt{\frac{2}{\pi\nu}} \qquad (\nu \geq 1). \tag{5.3}$$

Proof.

$$\begin{aligned}(I-A)\,(I+A)^\nu &= (I-A)\sum_{k=0}^{\nu}\binom{\nu}{k}A^k \\ &= I - A^{\nu+1} + \sum_{k=1}^{\nu}\left(\binom{\nu}{k} - \binom{\nu}{k-1}\right)A^k.\end{aligned}$$

So

$$\|(I-A)\,(I+A)^\nu\|_\infty \leq 2 + \sum_{k=1}^{\nu}\left|\binom{\nu}{k} - \binom{\nu}{k-1}\right|. \tag{5.4}$$

Using $\binom{\nu}{k} \geq \binom{\nu}{k-1} \Leftrightarrow k \leq \frac{1}{2}(\nu+1)$ and $\binom{\nu}{k} = \binom{\nu}{\nu-k}$ we get

$$\begin{aligned}&\sum_{k=1}^{\nu}\left|\binom{\nu}{k} - \binom{\nu}{k-1}\right| \\ &= \sum_{1}^{[\frac{1}{2}(\nu+1)]}\left(\binom{\nu}{k} - \binom{\nu}{k-1}\right) + \sum_{[\frac{1}{2}(\nu+1)]+1}^{\nu}\left(\binom{\nu}{k-1} - \binom{\nu}{k}\right) \\ &= \sum_{1}^{[\frac{1}{2}\nu]}\left(\binom{\nu}{k} - \binom{\nu}{k-1}\right) + \sum_{m=1}^{[\frac{1}{2}\nu]}\left(\binom{\nu}{m} - \binom{\nu}{m-1}\right) \\ &= 2\sum_{k=1}^{[\frac{1}{2}\nu]}\left(\binom{\nu}{k} - \binom{\nu}{k-1}\right) = 2\left(\binom{\nu}{[\frac{1}{2}\nu]} - \binom{\nu}{0}\right).\end{aligned}$$

Combined with (5.4) this yields the first inequality in (5.3). Elementary analysis yields that

$$\binom{\nu}{[\frac{1}{2}\nu]} < 2^\nu\sqrt{\frac{2}{\pi\nu}} \quad \text{for all } \nu \geq 1.$$

For details we refer to [20] and [21]. □

As a smoother we use a *damped* iteration based on the splitting in (5.1). We consider an iteration with iteration matrix

$$S_k = I - \theta W_k^{-1} L_k, \qquad \theta \in]0, \tfrac{1}{2}]. \tag{5.5}$$

THEOREM 5.3. *Assume that* (5.2a) *and* (5.2b) *holds. Then we have the following inequality with a constant C_S independent of k and ν:*

$$\|L_k\, S_k^\nu\|_\infty \le C_S\, \frac{1}{\sqrt{\nu}}\, h_k^{-2}. \tag{5.6}$$

Proof. Let $A := I - 2\theta W_k^{-1} L_k = 2\theta W_k^{-1} R_k + (1-2\theta)\, I$. Then due to (5.2a) and $\theta \in]0, \frac{1}{2}]$ we have $\|A\|_\infty \le 1$. Using Lemma 5.2 we get

$$\begin{aligned}
\|L_k S_k^\nu\|_\infty &= \|L_k (I - \theta W_k^{-1} L_k)^\nu\|_\infty \\
&= \|\frac{1}{2\theta}\, W_k (I - A)\, (\tfrac{1}{2})^\nu\, (I + A)^\nu\|_\infty \\
&\le \frac{1}{2\theta}\, \|W_k\|_\infty\, (\tfrac{1}{2})^\nu\, 2^{\nu+1} \sqrt{\frac{2}{\pi\nu}} \\
&\le \frac{1}{\theta} \sqrt{\frac{2}{\pi}}\, c\, \frac{1}{\sqrt{\nu}}\, h_k^{-2} =: C_S\, \frac{1}{\sqrt{\nu}}\, h_k^{-2}. \qquad \square
\end{aligned}$$

6. Contraction number of the two-grid method. Theorems 4.2 and 5.3 combined immediately yield the following result.

THEOREM 6.1. *Assume* (2.4a)–(2.4c), (3.1a)–(3.1f), (5.2a) *and* (5.2b). *Let $T_k(\nu)$ be as in* (3.10) *with S_k from* (5.5). *Then there are constants k_0 and C_{TG} independent of k and ν such that for all odd $k \ge k_0$, we have*

$$\|T_k(\nu)\|_\infty \le C_{TG}\, \frac{1}{\sqrt{\nu}}\, |\ln h_k|^2. \tag{6.1}$$

Clearly, Theorem 6.1 shows that if we take $\nu = \nu_k$ then for $\|T_k(\nu_k)\|_\infty \le c < 1$ to hold, in our upperbound we need $\nu_k = g(h_k)$ with g a logarithmically growing function for $h_k \downarrow 0$. In §7 we show that, with the assumptions as in Theorem 6.1, at least one factor $|\ln h_k|$ is *necessary* in an upper bound for $\|T_k(\nu)\|_\infty$.

Remark 6.2. With respect to *multigrid* convergence we note the following. The proofs of (4.6) and (4.7) also hold if Ω is a *convex polygonal* domain. If a regularity result as in (4.5) would hold for that situation too (as in the case of H^2-regularity), then we could assume Ω convex and polygonal and thus $\Phi_k \subset \Phi_{k+1}$ for all k. Clearly then the estimate in (6.1) holds for all $k \ge k_0$ and using the technique as given by Hackbusch in [10] it is straightforward to derive an L^∞-convergence result for the multigrid W-cycle. However, a regularity result as in (4.5) for a convex polygonal domain is not known to the author. Unfortunately, the best one can hope for is a result as in (4.5) for a very restricted class of convex polygonal domains, namely, domains with a maximum interior angle $\le \frac{1}{2}\pi$. This is clear from the following. Consider the Poisson equation on a convex polygonal domain with maximum interior angle $\gamma\pi$, $0 < \gamma < 1$. Suppose the solution φ^* is such that $\varphi^* \in H^{2,0}$ holds (cf. Remark 4.1). Then φ^* is an element of the Sobolev space $H^{2,p}$ for all $p \in [1, \infty[$ and thus (cf. [9]) $2 - 2/p < 1/\gamma$ must hold for all $p \in [1, \infty[$. This implies $\gamma \le \frac{1}{2}$. This regularity problem explains our assumption "$\partial\Omega$ smooth" instead of "Ω convex and polygonal."

It is clear how to derive a multigrid L^∞-convergence result if we can prove an approximation property as in (4.8) for k even too. We have not solved this problem (yet). It can be shown that for k even the use of the L^2-projection $\hat{p} : \Phi_{k-1} \to \Phi_k$ yields an approximation property as in (4.8) (with $p = P_k^{-1} \hat{p}\, P_{k-1}$). However, the L^2-projection is computationally too expensive. Modification of this projection might yield acceptable results; e.g., one could replace the L^2-inner product by the trapezoidal quadrature rule on $\mathcal{T}_k$ which then yields a prolongation that is easy to compute (this corresponds to lumping of the mass matrix). Clearly some further analysis is needed here.

Remark 6.3. In this paper we only consider the *symmetric* variational problem (2.1). From §5 we see that in the proof of the smoothing property (Theorem 5.3) this symmetry is not used; the only conditions are (5.2a), (5.2b). The proof of the approximation property (Theorem 4.2) remains valid for a more general (e.g., nonsymmetric) problem, provided that the results in Lemma 3.2 and in (4.4) hold. The results in Lemma 3.2 do not depend on $a(\cdot,\cdot)$ at all, but only on the triangulation. So our analysis yields a result as in Theorem 6.1 for a more general second-order elliptic boundary value problem too, if the finite element error estimate (4.4) from [8] and [18] holds. In [8] and [18] only the symmetric situation as in (2.1) is considered; however, in [17] it is remarked that (4.4) can be carried over to general second-order elliptic problems, provided that $\partial\Omega$ and the coefficients are sufficiently regular, and that the corresponding proofs in [8] and [18] require very little technical changes.

7. A lower bound for the contraction number. In this section we show that the estimate in Theorem 6.1 is sharp in the sense that a factor $|\ln h_k|$ is necessary (the power 2 of the $|\ln h_k|$ term, however, may be due to the method of proof).

We consider the Poisson equation on the unit square and use a linear finite element discretization on a uniform triangulation. We analyze the standard two-grid method as in §3 for solving the resulting system of equations.

We show (cf. Theorems 7.6 and 7.7 below) that for a fixed number of smoothing iterations the maximum norm of the iteration matrix is bounded from below by $C\,|\ln h_k|$. Our approach is based on the analysis given in [11]. Also we use an important result from [11] (Lemma 7.3 below). In view of the assumptions in Theorem 6.1 we want a lower bound corresponding to the Poisson equation on a domain Ω with $\partial\Omega$ smooth. In Remark 7.8 below we indicate how the results for the unit square yield similar results for a disc that contains the unit square.

Let Ω be the unit square and consider the variational boundary value problem as in (2.1) and (2.2) with

$$a(\varphi,\psi) := \int_\Omega \nabla\varphi\cdot\nabla\psi\,dx. \tag{7.1}$$

We use a uniform triangulation with mesh size parameter $h_k = 2^{-(k+1)}$ $(k\in\mathbb{N}_0)$ as indicated in Fig. 1.

Now let $\mathcal{T}_{k-1}$ and $\mathcal{T}_k$ be two successive triangulations. We use the definitions of §3 and for ease of notation we introduce $h := h_k$, $\mathcal{T}_h := \mathcal{T}_k$, $J_h := J_k$, $\Phi_h := \Phi_k$, $\varphi_h^i := \varphi_k^i$, $U_h := U_k$, $P_h := P_k$, $\langle u,v\rangle_h := \langle u,v\rangle_k$, $L_h := L_k$. Likewise we use $H := 2h$, $\mathcal{T}_H$, J_H, Φ_H, φ_H^i, U_H, P_H, $\langle\cdot,\cdot\rangle_H$, L_H. Furthermore, the index in J_h (J_H) corresponding to the grid point $Q := (\frac12,\frac12)$ is denoted by i_h^Q (i_H^Q) and the unit vector in U_h (U_H) corresponding to Q is denoted by e_h^Q (e_H^Q).

We now introduce a function $\omega_h \in \Phi_h$, which plays an important role in the analysis below. This function is defined as the piecewise linear interpolant on $\mathcal{T}_h$ of the (h-dependent) function

$$\omega(x_1,x_2) = \tfrac14\,\sin\left(\frac{\pi}{h}\,|x_1-\tfrac12|\right)\,\sin\left(\frac{\pi}{h}\,|x_2-\tfrac12|\right).$$

We write $w_h := P_h^{-1}\,\omega_h$. For all grid points x_h^i in $\mathcal{T}_h$ we have $\omega_h(x_h^i)\in\{-\frac14,\frac14,0\}$. In Fig. 2 we show the pattern of these function values.

$\mathcal{T}_0$ $(h_0 = \frac{1}{2})$ $\qquad$ $\mathcal{T}_1$ $(h_1 = \frac{1}{4})$

FIG. 1.

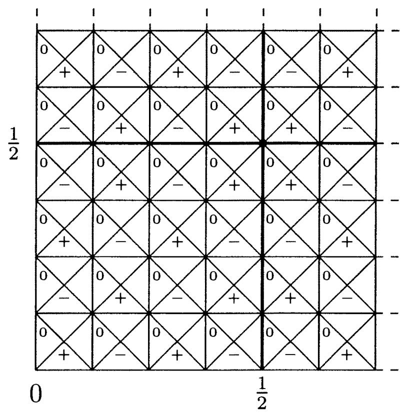


FIG. 2. *Values of the function* ω_h. $+$: *value* $\frac{1}{4}$; $-$: *value* $-\frac{1}{4}$; 0: *value* 0.

It is easy to prove Lemma 7.1 below. A simple proof can be found, e.g., in [11].

LEMMA 7.1. *For* $i, j \in J_h$ *the following holds*

$$a(\varphi_h^i, \varphi_h^j) = \begin{cases} 4 & if \;\; i = j, \\ -1 & if \;\; |x_h^i - x_h^j| = \frac{1}{2}\sqrt{2}\, h, \\ 0 & otherwise. \end{cases} \tag{7.2}$$

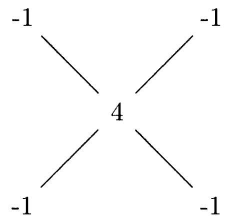
LEMMA 7.2. *The following holds*:

$$L_h w_h = h^{-2}(4w_h - e_h^Q), \tag{7.3}$$

and for $i \in J_H$,

$$a(\omega_h, \varphi_H^i) = \begin{cases} 0 & if \ \ i \neq i_H^Q, \\ 1 & if \ \ i = i_H^Q. \end{cases} \tag{7.4}$$

Proof. We extend the vector $w_h \in U_h$ by taking zero values corresponding to all boundary grid points of $\mathcal{T}_h$. Then Lemma 7.1 yields that $h^2 L_h$ corresponds to the difference star



in all interior points

For $i \in J_h$ the value $(h^2 L_h w_h)_i$ can be found by application of this difference star in the grid point x_h^i to the function $\omega_h = P_h w_h$. Figure 2 then shows that

$$(h^2 L_h w_h)_i = \begin{cases} 4(w_h)_i & \text{if } \ i \neq i_h^Q, \\ -1 & \text{if } \ i = i_h^Q. \end{cases}$$

So $h^2 L_h w_h = 4w_h - e_h^Q$ holds.

The proof of (7.4) runs as follows. For $i \in J_H$ we have

$$a(\omega_h, \varphi_H^i) = \langle L_h w_h, P_h^{-1} \varphi_H^i \rangle_h = 4h^{-2} \langle w_h, P_h^{-1} \varphi_H^i \rangle_h - h^{-2} \langle e_h^Q, P_h^{-1} \varphi_H^i \rangle_h. \tag{7.5}$$

Note that

$$h^{-2} < e_h^Q, P_h^{-1} \varphi_H^i >_h = (P_h^{-1} \varphi_H^i)_{i_h^Q} = \begin{cases} 0 & \text{if } \ i \neq i_H^Q, \\ 1 & \text{if } \ i = i_H^Q. \end{cases} \tag{7.6}$$

With respect to the term $4h^{-2} \langle w_h, P_h^{-1} \varphi_H^i \rangle_h$ in (7.5) we note the following. First we take $i \in J_H$ such that x_H^i is a vertex of four triangles in $\mathcal{T}_H$. Then φ_H^i has nonzero values in five grid points of $\mathcal{T}_h$ as indicated in Fig. 3. Using Fig. 2 we see that in these grid points ω_h has values as indicated in Fig. 3.

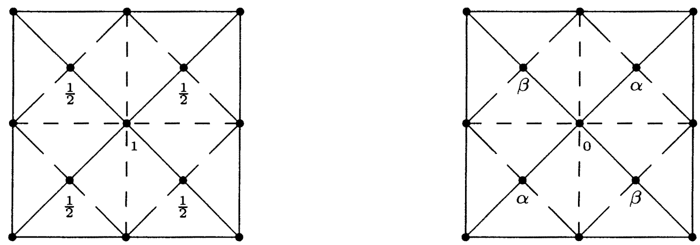
So $4h^{-2} \langle w_h, P_h^{-1} \varphi_H^i \rangle_h = 0$. Secondly, for $i \in J_H \backslash \{i_H^Q\}$ such that x_H^i is a vertex of eight triangles in T_H we have values for φ_H^i and ω_h as illustrated in Fig. 4. So again $4h^{-2} \langle w_h, P_h^{-1} \varphi_H^i \rangle_h = 0$. Finally for $i = i_H^Q$ we have a situation as in Fig. 4 but now with $\alpha = \beta = \gamma = \delta = \frac{1}{4}$, so $4h^{-2} \langle w_h, P_h^{-1} \varphi_H^i \rangle_h = 2$.

Using these results for $4h^{-2} \langle w_h, P_h^{-1} \varphi_H^i \rangle_h$ and the result (7.6) in (7.5) proves (7.4). □

A proof of the following lemma is given by Haverkamp in [11, Lem. 5].

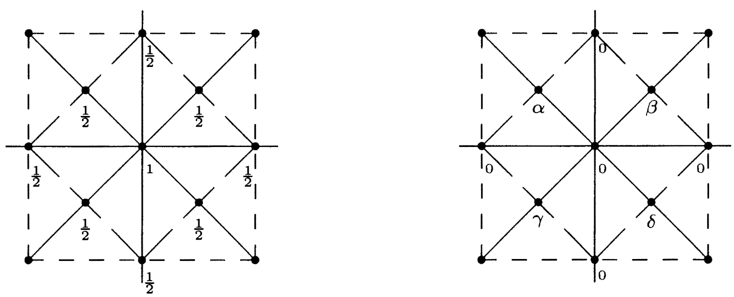
LEMMA 7.3. *If $\psi_H \in \Phi_H$ is such that for $i \in J_H$ we have*

$$a(\psi_H, \varphi_H^i) = \begin{cases} 0 & \text{if } \ i \neq i_H^Q, \\ 1 & \text{if } \ i = i_H^Q \ , \end{cases}$$



φ_H^i ω_h

FIG. 3. $\alpha, \beta \in \{-\frac{1}{4}, \frac{1}{4}\}$, $\alpha + \beta = 0$, —: *triangles in* $\mathcal{T}_H$.



φ_H^i ω_h

FIG. 4. $\alpha, \beta, \gamma, \delta \in \{-\frac{1}{4}, \frac{1}{4}\}$, $\alpha + \beta + \gamma + \delta = 0$, —: *triangles in* $\mathcal{T}_H$.

then $\psi_H(Q) > \pi^{-2}\, |\ln H|$.

LEMMA 7.4. *The following holds:*

$$\|(I - pL_H^{-1} r L_h)\, w_h\|_\infty > \pi^{-2}\, |\ln(2h)|.$$

Proof. We take ψ_H as in Lemma 7.3. Then it follows (cf. Lemma 7.2) that ψ_H is the orthogonal projection of ω_h on Φ_H:

$$a(\psi_H, \varphi_H^i) = a(\omega_h, \varphi_H^i) \quad \text{for all} \quad i \in J_H.$$

So

$$\langle L_H P_H^{-1} \psi_H,\, P_H^{-1} \varphi_H^i \rangle_H = \langle r L_h w_h,\, P_H^{-1} \varphi_H^i \rangle_H \quad \text{for all} \quad i \in J_H,$$

and thus

$$\psi_H = P_H L_H^{-1} r L_h w_h.$$

This yields

$$\begin{aligned} &\|(I - pL_H^{-1} r L_h)\, w_h\|_\infty \\ &\quad = \|P_h (I - pL_H^{-1} r L_h)\, w_h\|_{L^\infty} \\ &\quad = \|\omega_h - \psi_H\|_{L^\infty} \geq |(\omega_h - \psi_H)\ (Q)| = |\psi_H(Q)| > \pi^{-2}\, |\ln(2h)|. \qquad \square \end{aligned}$$

We now prove three theorems in which the main results of this section are given. Theorem 7.5 gives an estimate related to the approximation property in Theorem 4.2. In Theorems 7.6 and 7.7 we give lower bounds for $\|T_k(\nu)\|_\infty$. In Theorem 7.6 we take damped Jacobi and damped Gauss–Seidel as a smoother with a damping parameter $\theta \in]0, \frac{1}{2}[$. In Theorem 7.7 we consider damped Jacobi again, but then with $\theta \in]0, 1[$.

THEOREM 7.5. *The following holds:*

$$\|L_h^{-1} - pL_H^{-1}r\|_\infty > \pi^{-2}\, h^2\, |\ln(2h)|.$$

Proof. From Lemma 7.2 we have $\|L_h w_h\|_\infty = h^{-2}\, \|4w_h - e_h^Q\|_\infty = h^{-2}$. Lemma 7.4 then yields

$$\|L_h^{-1} - pL_H^{-1}r\|_\infty \geq \|(L_h^{-1} - pL_H^{-1}r)\, L_h w_h\|_\infty \Big/ \|L_h w_h\|_\infty > \pi^{-2} h^2\, |\ln(2h)|. \qquad \square$$

We consider a splitting as in (5.1) corresponding to the Jacobi or Gauss–Seidel method. Note that the conditions (5.2a) and (5.2b) are fulfilled. We use a damped version with damping parameter θ ($S_h = I - \theta W_h^{-1} L_h$).

THEOREM 7.6. *For $\theta \in]0, \frac{1}{2}[$ the following holds:*

$$\|T_h(\nu)\|_\infty > 4\pi^{-2}(1 - 2\theta)^\nu\, |\ln(2h)| \qquad (\nu \geq 0).$$

Proof. $\|I - W_h^{-1} L_h\|_\infty \leq 1$ holds and thus $\|W_h^{-1} L_h\|_\infty \leq 2$. So for $\theta \in]0, \frac{1}{2}[$ we have

$$(7.7) \qquad \|(I - \theta W_h^{-1} L_h)^{-1}\|_\infty \leq (1 - \theta\, \|W_h^{-1} L_h\|_\infty)^{-1} \leq (1 - 2\theta)^{-1}.$$

Combining (7.7), Lemma 7.4 and $\|w_h\|_\infty = \frac{1}{4}$ yields

$$\begin{aligned} \|T_h(\nu)\|_\infty &= \|(I - pL_H^{-1} r L_h)\, S_h^\nu\|_\infty \\ &\geq \|(I - pL_H^{-1} r L_h)\, w_h\|_\infty \Big/ \|S_h^{-\nu} w_h\|_\infty > \pi^{-2}\, |\ln(2h)|\, (1 - 2\theta)^\nu\, 4. \qquad \square \end{aligned}$$

THEOREM 7.7. *Let $S_h = I - \theta \frac{1}{4} h^2 L_h$ be the iteration matrix corresponding to the damped Jacobi iteration. Then for $\theta \in]0, 1[$ the following holds with a constant K independent of h:*

$$\|T_h(\nu)\|_\infty > 4\pi^{-2}(1 - \theta)^\nu\, |\ln(2h)| - K \qquad (\nu \geq 0).$$

Proof. The following holds (cf. Lemma 7.2):

$$S_h w_h = \left(I - \theta \frac{1}{4} h^2 L_h\right)\, w_h = (1 - \theta)\, w_h + \frac{1}{4}\theta e_h^Q.$$

Thus

$$(7.8) \qquad S_h^\nu w_h = (1 - \theta)^\nu\, w_h + \tfrac{1}{4}\theta \sum_{k=0}^{\nu-1} (1 - \theta)^k\, S_h^{\nu-1-k}\, e_h^Q.$$

The analysis of the coarse grid correction with respect to the scaled Euclidean norm $\|\cdot\|_h = \langle \cdot, \cdot \rangle_h^{\frac{1}{2}}$ yields (see, e.g., [10]):

$$(7.9) \qquad \|I - pL_H^{-1} r L_h\|_h \leq K, \quad K \text{ independent of } h.$$

Also note that for every $\theta \in]0, 1[$ we have

$$(7.10) \qquad \|S_h^m e_h^Q\|_h \leq \|S_h^m\|_h\, \|e_h^Q\|_h \leq h \qquad (m \geq 0).$$

Combination of Lemma 7.4, (7.8)–(7.10) and $\|u\|_\infty \le h^{-1}\|u\|_h$ yields

$$\begin{aligned}\|(I - pL_h^{-1}rL_h)\,S_h^\nu\|_\infty &\ge 4\,\|(I - pL_H^{-1}rL_h)\,S_h^\nu w_h\|_\infty \\ &\ge 4(1-\theta)^\nu\,\|(I - pL_H^{-1}rL_h)\,w_h\|_\infty \\ &\quad -\theta\left\|(I - pL_H^{-1}rL_h)\sum_{k=0}^{\nu-1}(1-\theta)^k\,S_h^{\nu-1-k}\,e_h^Q\right\|_\infty \\ &> 4(1-\theta)^\nu\,\pi^{-2}\,|\ln(2h)| - \theta h^{-1}\,K\sum_{k=0}^{\nu-1}(1-\theta)^k\,h \\ &> 4(1-\theta)^\nu\,\pi^{-2}\,|\ln(2h)| - K. \qquad \square\end{aligned}$$

Remark 7.8. We now discuss how lower bounds as in Theorems 7.6 or 7.7 can be derived for the Poisson equation on a domain $\tilde\Omega$ with $\partial\tilde\Omega$ smooth. Let $\tilde\Omega$ be the disc with center $(\frac12, \frac12)$ and radius 1. We consider the Poisson equation on $\tilde\Omega$. Let $\{\tilde{\mathcal{T}}_k \mid k \in \mathbb{N}_0\}$ be a sequence of triangulations satisfying (3.1a)–(3.1f) and such that $\tilde{\mathcal{T}}_{k|\Omega} = \mathcal{T}_k$ (with $\mathcal{T}_k$ as in Fig. 1). The notation of this section (for $\Omega = [0,1]^2$) is used for $\tilde\Omega$ too, but then with an upper "$\sim$" (so, e.g., L_h on Ω, $\tilde L_h$ on $\tilde\Omega$). Also for ease we assume that the triangulations $\tilde{\mathcal{T}}_H$ and $\tilde{\mathcal{T}}_h$ are nested (cf. (3.1f)). We take $\tilde\omega_h \in \tilde\Phi_h$ such that $\tilde\omega_h \equiv \omega_h$ on Ω (cf. Fig. 2) and $\tilde\omega_h \equiv 0$ on $\tilde\Omega\backslash\Omega$. Below we use the same arguments as in [11, §4] to derive an estimate as in Lemma 7.4 but now for $\tilde\Omega$. Let $\tilde\psi_H$ be such that $a(\tilde\psi_H - \tilde\omega_h, \tilde\varphi_H^i) = 0$ for all $i \in \tilde J_H$; then $\tilde\psi_H = \tilde P_H \tilde L_H^{-1}\tilde r \tilde L_h \tilde w_h$ with $\tilde w_h = \tilde P_h^{-1}\tilde\omega_h$. Note that $\tilde\psi_H - \psi_H$ (with ψ_H as in Lemma 7.3) is a discrete harmonic on Ω, i.e., $\tilde\psi_H - \psi_H$ is continuous and piecewise linear on $\mathcal{T}_H$ and $a(\tilde\psi_H - \psi_H, \varphi_H^i) = 0$ for all $i \in J_H$. And thus (cf. [11]) for every $u \in C(\bar\Omega)$ with $u_{|\partial\Omega} = 0$ we have $\|u\|_{L^\infty(\Omega)} \le 2\,\|u - (\tilde\psi_H - \psi_H)\|_{L^\infty(\Omega)}$. So we get

$$\begin{aligned}&\|(\tilde I - \tilde p \tilde L_H^{-1}\tilde r \tilde L_h)\,\tilde w_h\|_\infty \\ &= \|\tilde P_h \tilde w_h - \tilde P_H \tilde L_H^{-1}\tilde r \tilde L_h \tilde w_h\|_{L^\infty(\tilde\Omega)} = \|\tilde\omega_h - \tilde\psi_H\|_{L^\infty(\tilde\Omega)} \\ &\ge \|\omega_h - \tilde\psi_H\|_{L^\infty(\Omega)} = \|\omega_h - \psi_H - (\tilde\psi_H - \psi_H)\|_{L^\infty(\Omega)} \ge \tfrac12\,\|\omega_h - \psi_H\|_{L^\infty(\Omega)} \\ &\ge \tfrac12\,\omega_h(Q) > \tfrac12\,\pi^{-2}\,|\ln(2h)|.\end{aligned}$$

Using this result it is straightforward to derive similar results as in Theorems 7.6 and 7.7 for the two-grid method applied to the discretization of the Poisson equation on $\tilde\Omega$.

Acknowledgment. The author would like to thank Professors W. Hackbusch and R. Rannacher and Dr. R. Stevenson for some helpful comments.

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