H_2 optimal controllers with observer based architecture for continuous-time systems : separation principle

EINDHOVEN UNIVERSITY OF TECHNOLOGY
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Ali Saberi
Peddapullaiah Sannuti
A.A. Stoorvogel

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Ali Saberi
School of Electrical Engineering
and Computer Science
Washington State University
Pullman, WA 99164-2752
U.S.A.

Peddapullaiah Sannuti
Department of Electrical and
Computer Engineering
Rutgers University, P.O. Box 909
Piscataway, NJ 08855-0909
U.S.A.

Anton A. Stoorvogel
Department of Mathematics
and Computing Science
Eindhoven Univ. of Technology
P.O. Box 513, 5600 MB Eindhoven
The Netherlands

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Abstract

For a general H_2 optimal control problem, at first all H_2 optimal measurement feedback controllers are characterized and parameterized, and then attention is focused on controllers with observer based architecture. Both full order as well as reduced order observer based H_2 optimal controllers are characterized and parameterized. Also, systematic methods of designing them are presented. An important problem that can be coined as an H_2 optimal control problem with simultaneous pole placement, is formulated and solved. That is, since in general there exist many H_2 optimal measurement feedback controllers, utilizing such flexibility and freedom, we can solve the problem of simultaneously placing the closed-loop poles at desirable locations whenever possible while still preserving H_2 optimality. All the design algorithms developed here are easily computer implementable.

1. Introduction

A general H_2 optimal control problem which utilizes measurement feedback is considered. The problem is to find an internally stabilizing controller which attains the infimum of the H_2 norm of a transfer function from an exogenous disturbance to a controlled output of a given linear time invariant system, while utilizing the measured output. For such a problem, two main aspects are addressed in this paper. The first one deals with the characterization and parameterization of all H_2 optimal measurement feedback controllers. The second aspect focuses attention on controllers with observer based architecture, and for such controllers, it characterizes and develops methods of constructing all H_2 optimal controllers. Also, it investigates the freedom and constraints that arise in closed-loop pole placement while preserving H_2 optimality; and in so doing, it solves what can be coined as an H_2 optimal control problem with simultaneous pole placement. Note that this problem studies among optimal H_2 controllers, the available flexibility in the location of the closed loop poles. It does not compromise H_2 performance in favour of better pole locations.

In recent years, there has been a renewed interest in H_2 optimal control utilizing state or measurement feedback. In [12] the necessary and sufficient conditions under which the infimum of the H_2 norm of the concerned transfer function can be attained while utilizing the measured output were developed, i.e. they developed the necessary and sufficient conditions under which an H_2 optimal measurement feedback controller exists. Moreover, they showed that whenever an H_2 optimal measurement feedback controller exists, there exists as well an H_2 optimal controller with observer based architecture. Furthermore, they made an attempt to characterize a subset of all H_2 optimal measurement feedback controllers, and investigated the flexibility such a class of H_2 optimal controllers offer regarding the closed-loop pole placement.

Subsequent to [12], in [2] a complete treatment of the H_2 optimal control problem was provided for the case that the state is available for feedback. More specifically, it completely characterizes all H_2 optimal state feedback controllers including static as well as dynamic ones. Moreover, it solves the H_2 optimal control problem with simultaneous pole placement for the case that the state is available for feedback. In order to do so, for the set of all H_2 optimal state feedback controllers, it constructed an associated set of complex numbers that point out explicitly the freedom and constraints one has in closed-loop pole placement. This set is called the set of H_2 *optimal fixed modes*. Its elements must be included among the closed-loop poles whatever is the H_2 optimal state feedback controller used. A significant aspect of this work is the development of a computationally feasible step by step algorithm called *'Optimal Gains and Fixed Modes'*, abbreviated as *(OGFM)*. Given a matrix quintuple that specifies the given H_2 optimal state feedback control problem, *(OGFM)* algorithm computes among other things, the set of all H_2 optimal static state feedback gains, and the associated set of H_2 optimal fixed modes. A software package implementing the *(OGFM)* algorithm in Matlab is given in [7] and [8].

Although considerable work has been done in H_2 optimal control by various researchers, there still remains a gap regarding the complete characterization of all H_2 optimal controllers with observer based architecture, and the investigation of the freedom and constraints they offer in closed-loop pole placement. The intention of this paper is to fill this gap. In fact, the spirit of this paper is to capture, while using measurement feedback controllers rather than state feedback controllers, all the aspects of H_2 optimal control that were developed in [2]. More specifically, our goals in this paper are, to completely characterize all the H_2 optimal measurement feedback controllers with observer based architecture, and for such controllers to solve the H_2 optimal control problem with simultaneous pole placement. To do so, we construct explicitly the set of all H_2 optimal measurement feedback controllers with a chosen observer based architecture, and *some* associated sets of H_2 *optimal fixed modes*. All the theoretical aspects of these sets are developed in such a way that the explicit construction of these sets can be computationally accomplished by merely using the *(OGFM)* algorithm.

The above task of investigating all the aspects of H_2 optimal control while utilizing observer based controllers, turns out to be complex and involved. The basic reason for complexity arises from the fact that the traditional separation principle does not hold in general. To expand on this, let us note that in the literature on control, the notion of a controller with observer based architecture is very much tied with the notion of separation principle. Two implications arise from the traditional separation principle. The first one relates to the existence of an H_2 optimal measurement feedback controller. It says that whenever an H_2 optimal static state feedback controller and an H_2 optimal state estimator or otherwise called an observer † exist, there exists as well an H_2 optimal observer based measurement feedback controller. This first implication of the traditional separation principle is in general *false* as pointed out in [11]. The second implication of the separation principle relates to the actual construction of an H_2 optimal measurement feedback controller. Suppose there exists an H_2 optimal measurement feedback controller. Then, the traditional separation principle implies that an H_2 optimal measurement feedback controller can be obtained by cascading together *any* H_2 optimal observer and *any* H_2 optimal static state feedback controller. It is shown here that this second implication of the separation principle is in general *not true* either.

This paper is organized as follows. In the next section, we recall some preliminary results needed for our development. Section 3 contains problem statement and our main results regarding controllers with full order observer based architecture, while Section 4 contains the results for controllers with reduced order observer based architecture. Finally, Section 5 draws the conclusions.

Throughout the paper, A' denotes the transpose of A, I denotes an identity matrix, while I_k denotes the identity matrix of dimension $k \times k$. $\mathbb{C}$, $\mathbb{C}^-$, $\mathbb{C}^0$ and $\mathbb{C}^+$ respectively denote the whole complex plane, the open left half complex plane, the imaginary axis, and the open right half complex plane. $\lambda(A)$ denotes the set of eigenvalues of A. A matrix is said to be stable if all its eigenvalues are in $\mathbb{C}^-$. Similarly, a transfer function $G(s)$ is said to be stable if all its poles are in $\mathbb{C}^-$. $\mathrm{Ker}\,[V]$ and $\mathrm{Im}\,[V]$ denote respectively the kernel and the image of V. Given $\mathcal{X}$ a subspace of $\mathbb{R}^n$ or $\mathbb{C}^n$ and a matrix $N \in \mathbb{R}^{n \times m}$, we define

$$N^{-1}\mathcal{X} := \{z \in \mathbb{R}^m \mid Nz \in \mathcal{X}\}.$$

Given a stable transfer function $G(s)$, as usual, its H_2 norm is defined by

$$\|G\|_2 = \left(\frac{1}{2\pi}\mathrm{tr}\left[\int_{-\infty}^{\infty} G(j\omega)G'(-j\omega)d\omega\right]\right)^{1/2}.$$

† The precise notion of H_2 optimal observer is discussed later on in the text.

2. Preliminaries

We consider the following system Σ characterized by,

$$\Sigma : \begin{cases} \dot{x} = Ax + Bu + Ew \\ y = C_1 x + D_1 w \\ z = C_2 x + D_2 u, \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is a state, $u \in \mathbb{R}^m$ is a control input, $w \in \mathbb{R}^l$ is an exogenous disturbance input, $y \in \mathbb{R}^p$ is a measured and $z \in \mathbb{R}^q$ is a controlled output. Without loss of generality, we assume that the matrices $[C_2,\ D_2]$, $[C_1,\ D_1]$, $[B',\ D_2']'$ and $[E',\ D_1']'$ have full rank. Next, we describe a proper controller Σ_C described by

$$\Sigma_C : \begin{cases} \dot{v} = J\,v + L\,y \\ u = Mv + Ny. \end{cases} \tag{2.2}$$

We note that Σ_C, as given in (2.2), is strictly proper when $N = 0$.

We use the following notations. The closed-loop system consisting of the plant Σ and a controller Σ_C is denoted by $\Sigma \times \Sigma_C$. A controller Σ_C is said to be internally stabilizing the system Σ, if the closed-loop system $\Sigma \times \Sigma_C$ is internally stable, i.e., if $\Sigma \times \Sigma_C$ has all its poles in $\mathbb{C}^-$. Also, a controller Σ_C is said to be admissible if it provides internal stability for the closed-loop system $\Sigma \times \Sigma_C$. The transfer matrix from w to z of $\Sigma \times \Sigma_C$ is denoted by $T_{zw}(\Sigma \times \Sigma_C)$.

Next, whenever we say that a system or a subsystem Σ_* is characterized by a quadruple (A, B, C, D), we mean by it that the dynamic equations of it are given by,

$$\Sigma_* : \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du, \end{cases} \tag{2.3}$$

where u and y are respectively some input (control input or disturbance) and output (measured or controlled output) of Σ_*.

Often in our development, we use two subsystems of the given system Σ. These subsystems are, Σ_1 which is characterized by the matrix quadruple $(A,\ E,\ C_1,\ D_1)$, and Σ_2 which is characterized by the matrix quadruple $(A,\ B,\ C_2,\ D_2)$. Also, often in our development, we use two geometric subspaces which are defined below:

Definition 2.1. *Consider a linear system Σ_* characterized by the matrix quadruple (A, B, C, D). Then,*

1. *The $\mathbb{C}_g$-stabilizable weakly unobservable subspace $\mathcal{V}_g(\Sigma_*)$ is defined as the maximal subspace of $\mathbb{R}^n$ which is $(A + BF)$–invariant and contained in* $\operatorname{Ker}[C + DF]$ *such that the eigenvalues of $(A + BF)|\mathcal{V}_g$ are contained in $\mathbb{C}_g \subseteq \mathbb{C}$ for some F.*

2. *The $\mathbb{C}_g$-detectable strongly controllable subspace $\mathcal{S}_g(\Sigma_*)$ is defined as the minimal $(A + KC)$– invariant subspace of $\mathbb{R}^n$ containing in* $\operatorname{Im}[B + KD]$ *such that the eigenvalues of the map which is induced by $(A + KC)$ on the factor space $\mathbb{R}^n/\mathcal{S}_g$ are contained in $\mathbb{C}_g \subseteq \mathbb{C}$ for some K.*

For the case when $\mathbb{C}_g = \mathbb{C}$, $\mathcal{V}_g$ *and* $\mathcal{S}_g$ *are denoted by* $\mathcal{V}^*$ *and* $\mathcal{S}^*$*, respectively. Similarly, for the case when* $\mathbb{C}_g = \mathbb{C}^-$, $\mathcal{V}_g$ *and* $\mathcal{S}_g$ *are denoted by* $\mathcal{V}^-$ *and* $\mathcal{S}^-$*, respectively.*

Next, we have the following definitions regarding H_2 optimal control.

Definition 2.2. *Let a system* Σ *of the form (2.1) be given.* **The** H_2 **optimal control problem** *is to find an internally stabilizing proper controller* Σ_c *which minimizes the* H_2 *norm of the closed loop transfer matrix. The infimum of the performance index is denoted by* γ^**, that is*

$$\gamma^* := \inf\{\|T_{zw}(\Sigma \times \Sigma_c)\|_2 \mid \Sigma_c \text{ is proper and internally stabilizes } \Sigma\}. \tag{2.4}$$

An internally stabilizing proper controller Σ_c *is said to be an* H_2*-optimal controller if it achieves a closed loop* H_2 *norm* γ^**.*

The above definitions correspond to the case when the class of controllers considered are *proper* and are of the form (2.2). One can also consider only *strictly proper* controllers which are again of the form (2.2) but with the additional condition that $N = 0$. Although the conditions for the existence of a *strictly proper* H_2 optimal controller are different from those of *a non-strictly proper* H_2 optimal controller, it turns out that in the case of continuous-time systems (but not in discrete-time systems) the value of the infimum γ^* is the same whether proper or strictly proper controllers are considered (see for details [10]).

Next, as discussed in detail in [10] and in [11], the H_2 optimal control problem for a given system Σ can be reformulated as a disturbance decoupling problem via measurement feedback with internal stability (DDPMS) for an auxiliary system denoted here by Σ_{PQ}. In what follows, we first state the dynamic equations of Σ_{PQ}; recall the definition of a DDPMS; and then recall a lemma that connects the H_2 optimal control problem for Σ to the DDPMS for Σ_{PQ}.

The auxiliary system Σ_{PQ} is described by

$$\Sigma_{PQ} : \begin{cases} \dot{x}_{PQ} = A\,x_{PQ} + Bu_{PQ} + E_Q w_{PQ} \\ y_{PQ} = C_1 x_{PQ} + D_Q w_{PQ} \\ z_{PQ} = C_P x_{PQ} + D_P u_{PQ}. \end{cases} \tag{2.5}$$

Here C_P, D_P, E_Q and D_Q are such that $[C_P,\ D_P]$ and $[E_Q',\ D_Q']'$ have full rank, and

$$F(P) = \begin{pmatrix} C_P' \\ D_P' \end{pmatrix} (C_P \quad D_P) \quad \text{and} \quad G(Q) = \begin{pmatrix} E_Q \\ D_Q \end{pmatrix} (E_Q' \quad D_Q'). \tag{2.6}$$

Moreover,

$$F(P) := \begin{pmatrix} A'P + PA + C_2'C_2 & PB + C_2'D_2 \\ B'P + D_2'C_2 & D_2'D_2 \end{pmatrix}, \tag{2.7}$$

$$G(Q) := \begin{pmatrix} AQ + QA' + EE' & QC_1' + ED_1' \\ C_1Q + D_1E' & D_1D_1' \end{pmatrix}, \tag{2.8}$$

and furthermore, P and Q are positive semi-definite, rank minimizing (see [10]), and are the largest among all symmetric solutions of the respective linear matrix inequalities $F(P) \geq 0$ and $G(Q) \geq 0$.

The following is the definition of the DDPMS for Σ_{PQ}.

Definition 2.3. *Consider a system Σ_{PQ} as in (2.5). The disturbance decoupling problem with measurement feedback and internal stability (DDPMS) for Σ_{PQ} is the problem of finding a proper controller Σ_C of the form (2.2) such that the closed-loop system $\Sigma_{PQ} \times \Sigma_C$ is internally stable, while the resulting closed-loop transfer function is identical to 0.*

The following lemma recalled from [10] connects the H_2 optimal control problem for Σ with the DDPMS for Σ_{PQ}. Such a reformulation plays a significant role in the development of next two sections.

Lemma 2.1. *Consider an H_2 optimal control problem as defined by Definition 2.2 for a system Σ as in (2.1). Assume that (A, B) is stabilizable and (A, C_1) is detectable. Also, consider the auxiliary system Σ_{PQ} as given in (2.5), and a proper controller Σ_C as in (2.2). Then, the following two statements are equivalent.*

1. *Σ_C is an H_2 optimal controller for Σ, i.e., the closed-loop system $\Sigma \times \Sigma_C$ is internally stable, and the H_2 norm of the closed-loop transfer function from w to z is equal to the infimum γ^*.*
2. *Σ_C solves the DDPMS for Σ_{PQ}, i.e., the closed-loop system $\Sigma_{PQ} \times \Sigma_C$ is internally stable, and the resulting transfer function from w_{PQ} to z_{PQ} is equal to zero.*

Moreover, the above equivalence holds even if one considers a strictly proper controller, i.e. a controller Σ_C as in (2.2) with $N = 0$.

To proceed further, let Σ_{1PQ} and Σ_{2PQ} be subsystems of Σ_{PQ} which are respectively characterized by the matrix quadruples (A, E_Q, C_1, D_Q) and (A, B, C_P, D_P). Then, the following theorems recalled from [10] develop the necessary and sufficient conditions under which an H_2 optimal *proper* controller Σ_C of the form (2.2) or an H_2 optimal *strictly proper* controller Σ_C of the form (2.2) with $N = 0$, exists for the given system Σ.

Theorem 2.1. *Consider an H_2 optimal control problem as defined by Definition 2.2 for a system Σ as in (2.1). Then, the following two statements are equivalent:*

1. *There exists a proper controller Σ_C of the form (2.2) such that*
   - (*a*) *the closed-loop system $\Sigma \times \Sigma_C$ is internally stable, and*
   - (*b*) *the closed-loop transfer function $T_{zw}(\Sigma \times \Sigma_C)$ has the H_2 norm γ^*.*
2. *(A, B) is stabilizable, (A, C_1) is detectable and*
   - (*c*) $\operatorname{Im}[E_Q] \subseteq \mathcal{V}^-(\Sigma_{2PQ}) + B \operatorname{Ker}[D_P]$,
   - (*d*) $\operatorname{Ker}[C_P] \supseteq \mathcal{S}^-(\Sigma_{1PQ}) \cap C_1^{-1}\{\operatorname{Im}[D_Q]\}$,
   - (*e*) $\mathcal{S}^-(\Sigma_{1PQ}) \subseteq \mathcal{V}^-(\Sigma_{2PQ})$.

For the class of strictly proper controllers we already noted that we can achieve the same closed-loop H_2 norm. The following theorem is the equivalent of theorem 2.1 for the class of strictly proper controllers.

Theorem 2.2. *Consider an H_2 optimal control problem as defined by Definition 2.2 for a system Σ as in (2.1). The following two statements are equivalent:*

1. *There exists a strictly proper H_2 optimal controller, namely, there exists a controller Σ_C of the form (2.2) with $N = 0$ such that*
   - (*a*) *the closed-loop system $\Sigma \times \Sigma_C$ is internally stable, and*
   - (*b*) *the closed-loop transfer function $T_{zw}(\Sigma \times \Sigma_C)$ has the H_2 norm γ^*.*
2. *(A, B) is stabilizable, (A, C_1) is detectable and*
   - (*c*) $\mathrm{Im}\,[E_Q] \subseteq \mathcal{V}^-(\Sigma_{2PQ})$,
   - (*d*) $\mathrm{Ker}\,[C_P] \supseteq \mathcal{S}^-(\Sigma_{1PQ})$,
   - (*e*) $\mathcal{S}^-(\Sigma_{1PQ}) \subseteq \mathcal{V}^-(\Sigma_{2PQ})$,
   - (*f*) $A\mathcal{S}^-(\Sigma_{1PQ}) \subseteq \mathcal{V}^-(\Sigma_{2PQ})$.

Remark 2.1. *In view of Theorem 2.2, it can be seen easily that the first implication of the traditional separation principle does not hold in general for an H_2 optimal control problem. An H_2 optimal state feedback is a matrix $\bar{F}$ such that $A + B\bar{F}$ is stable and:*

$$\|(C_2 + D_2\bar{F})(sI - A - B\bar{F})^{-1}E\|_2 = \inf_F \left\{ \|(C_2 + D_2F)(sI - A - BF)^{-1}E\|_2 \;\middle|\; A + BF \text{ is stable} \right\}.$$

Similarly an H_2 optimal observer gain is a matrix $\bar{K}$ such that $A + \bar{K}C_1$ is stable and:

$$\|C_2(sI - A - \bar{K}C_1)^{-1}(E + \bar{K}D_1)\|_2 = \inf_K \left\{ \|C_2(sI - A - KC_1)^{-1}(E + KD_1)\|_2 \;\middle|\; A + KC_1 \text{ is stable} \right\}.$$

We can show that the conditions of Theorem 2.2 guarantee the existence of H_2 optimal state feedbacks and observers. However, the converse is not true. There might exist an optimal H_2 state feedback and an H_2 optimal observer and yet there does not exists an H_2 optimal measurement feedback controller.

The following system illustrates that property:

$$\Sigma : \begin{cases} \dot{x} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix} x + \begin{pmatrix} 2 \\ 3 \end{pmatrix} u + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w \\ y = \begin{pmatrix} 2 & 2 \end{pmatrix} x + w \\ z = \begin{pmatrix} 1 & 0 \end{pmatrix} x + u. \end{cases}$$

For this system $P = 0$ and $Q = 0$ and hence Σ_{PQ} is equal to Σ. On the other hand it is easy to check that:

$$\mathcal{V}^-(\Sigma_{2PQ}) = Im \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \mathcal{S}^-(\Sigma_{1PQ}) = Im \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Clearly, the condition (e) of theorem 2.2 is not satisfied. Hence there does not exist an optimal measurement feedback controller. On the other hand, an H_2 optimal state feedback gain is given by

$$\bar{F} = \begin{pmatrix} -1 & -1 \end{pmatrix},$$

while an H_2 optimal observer gain is given by

$$\bar{K} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Traditionally, an H_2 optimal observer based measurement feedback controller whenever it exists, is designed by designing separately an H_2 optimal state estimator or observer and an H_2 optimal state feedback controller, and then cascading them to form a measurement feedback controller.
Let $\mathbf{F}_s^*(A, B, E, C_P, D_P)$ denote the set of all H_2 optimal static state feedback controllers (or gains), i.e. the set of matrices F such that $A + BF$ is asymptotically stable and

$$(C_P + D_P F)(sI - A - BF)^{-1} E \equiv 0.$$

Equivalently this is the set of H_2 optimal static state feedback controllers (or gains) for the state feedback problem associated with the quintuple (A, B, E, C_2, D_2).
Similarly, we define the set of optimal observer gains $\mathbf{K}_s^*(A, E_Q, C_1, C_2, D_Q)$ as the set of matrices K such that $A + KC_1$ is asymptotically stable and

$$C_2(sI - A - KC_1)^{-1}(E_Q + KD_Q) \equiv 0.$$

Clearly K is in $\mathbf{K}_s^*(A, E_Q, C_1, C_2, D_Q)$ if and only if K' is in the set $\mathbf{F}_s^*(A', C_1', C_2', E_Q', D_Q')$.
We have the following additional definitions:

Definition 2.4. *A scalar $\lambda \in \mathbb{C}^-$ is said to be an H_2* **optimal fixed mode** *if λ is an eigenvalue of $A + BF$ for every state feedback which is in $\mathbf{F}_s^*(A, B, E, C_2, D_2)$. Obviously, we can also define fixed modes for the set of H_2 optimal observer gains as the scalars $\lambda \in \mathbb{C}^-$ which are eigenvalues of $A + KC_1$ for every observer gain which is in $\mathbf{K}_s^*(A, E_Q, C_1, C_2, D_Q)$. We will use the following notation:*

$$\Omega^*(A, B, E, C_P, D_P) := \text{the set of } H_2 \text{ optimal fixed modes w.r.t. } \mathbf{F}_s^*(A, B, E, C_P, D_P)$$
$$\Psi^*(A, E_Q, C_1, C_2, D_Q) := \text{the set of } H_2 \text{ optimal fixed modes w.r.t. } \mathbf{K}_s^*(A, E_Q, C_1, C_2, D_Q)$$

Utilization of the sets $\mathbf{F}_s^*$ and $\mathbf{K}_s^*$ to form an appropriate H_2 optimal measurement feedback controller is discussed in the next section. However, at this time, we like to emphasize that an algorithm called *(OGFM)* is developed in [2] to construct explicitly the set of state feedbacks $\mathbf{F}_s^*$, and its associated fixed modes Ω^*. By duality this algorithm can also be used to construct the set of optimal observers $\mathbf{K}_s^*$, and its associated fixed modes Ψ^*.

3. The H_2 control problem with measurement feedback

We have a characterization and parameterization of all H_2 optimal *proper* dynamic measurement feedback controllers which involves the following steps:

1. Find a matrix $F \in \mathbb{R}^{m\times n}$ and a matrix $K \in \mathbb{R}^{n\times p}$ such that the following equations hold,

$$\lambda(A+BF) \subseteq \mathbb{C}^-, \quad \mathrm{Ker}\,[(C_P + D_P F)(sI - A - BF)^{-1}] = \mathcal{V}^-(\Sigma_{2PQ}), \tag{3.1}$$

$$\lambda(A+KC_1) \subseteq \mathbb{C}^-, \quad \mathrm{Im}\,[(sI - A - KC_1)^{-1}(E_Q + KD_Q)] = \mathcal{S}^-(\Sigma_{1PQ}). \tag{3.2}$$

2. Define a set $\mathbf{N}^*$ as,

$$\mathbf{N}^* := \left\{ N \in \mathbb{R}^{m\times p} \mid N \text{ satisfies the LME (3.4)} \right\}, \tag{3.3}$$

$$\begin{pmatrix} XB \\ D_P \end{pmatrix} N \begin{pmatrix} C_1 Y & D_Q \end{pmatrix} = -\begin{pmatrix} XAY & XE_Q \\ C_P Y & 0 \end{pmatrix}. \tag{3.4}$$

   In (3.4) X and Y are any constant matrices such that $\mathcal{V}^-(\Sigma_{2PQ}) = \mathrm{Ker}\,[X]$ and $\mathcal{S}^-(\Sigma_{1PQ}) = \mathrm{Im}\,[Y]$. We note that (3.4) can equivalently be written as:

$$\left[\begin{pmatrix} A & E_Q \\ C_P & 0 \end{pmatrix} + \begin{pmatrix} B \\ D_P \end{pmatrix} N \begin{pmatrix} C_1 & D_Q \end{pmatrix}\right] \left(\mathcal{S}^-(\Sigma_{1PQ}) \oplus \mathbb{R}^l\right) \subseteq \left(\mathcal{V}^-(\Sigma_{2PQ}) \oplus \{0\}\right). \tag{3.5}$$

3. Define a set $\mathbf{Q}_s$ as,

$$\mathbf{Q}_s := \left\{ Q_s \in \mathcal{R}H_2 \mid Q_s \text{ satisfies } G_1 Q_s G_2 = 0 \right\}, \tag{3.6}$$

   where $\mathcal{R}H_2$ denotes the set of strictly proper and stable rational matrices and

$$G_1(s) = [(C_P + D_P F)(sI - A - BF)^{-1}B + D_P], \tag{3.7}$$

$$G_2(s) = [C_1(sI - A - KC_1)^{-1}(E_Q + KD_Q) + D_Q]. \tag{3.8}$$

4. Define a set $\mathbf{Q}$ as,

$$\mathbf{Q} := \left\{ Q = Q_s + N \mid Q_s \in \mathbf{Q}_s \text{ and } N \in \mathbf{N}^* \right\}. \tag{3.9}$$

5. One can define now a set of *proper* dynamic measurement feedback controllers parameterized in $Q(s)$ as

$$\Sigma_C \begin{cases} \dot{\xi} = (A + BF + KC_1)\xi - Ky + By_1 \\ u = F\xi + y_1 \\ y_1 = Q(s)(y - C_1\xi), \end{cases} \tag{3.10}$$

   where F and K satisfy (3.1) and (3.2) and $Q(s) \in \mathbf{Q}$ with $\mathbf{Q}$ as defined in (3.9).

We have the following theorem.

Theorem 3.1. *Consider an H_2 optimal control problem as defined by Definition 2.2 for a system Σ as in (2.1). Assume that the given system Σ satisfies the necessary and sufficient conditions for the existence of an H_2 optimal proper measurement feedback controller as given in the second part of Theorem 2.1. Then the set of controllers of the form Σ_C given in (3.10) with $\mathbf{Q}$ as in (3.9), coincides with the set of all H_2 optimal proper dynamic measurement feedback controllers; i.e. Σ_C internally stabilizes Σ and $\|T_{wz}(\Sigma \times \Sigma_C)\|_2 = \gamma^*$. Moreover, any H_2 optimal proper dynamic measurement feedback controller can be written as in (3.10) for some $Q(s) \in \mathbf{Q}$ which is given by (3.9).*

Proof: See [10]. ∎

Based on the above one can easily derive conditions under which the optimal H_2 controller is unique.

Theorem 3.2. *Consider an H_2 optimal control problem as defined by Definition 2.2 for a system Σ as in (2.1). Assume that the given system Σ satisfies the necessary and sufficient conditions for the existence of an H_2 optimal proper measurement feedback controller as given in the second part of Theorem 2.1, Σ_2 is left-invertible, and Σ_1 is right-invertible. Then there exists a unique H_2 optimal controller.*

Proof: See [3]. ∎

A natural question arises as to what happens if the H_2 optimal controller is not unique. In particular we can enquire what freedom is left and how we can use it for our controller design. We have the following theorem:

Theorem 3.3. *Consider an H_2 optimal control problem as defined by Definition 2.2 for a system Σ as in (2.1). Assume that the given system Σ satisfies the necessary and sufficient conditions for the existence of an H_2 optimal proper measurement feedback controller as given in the second part of Theorem 2.1. Then, the closed-loop transfer matrix from w to z is unique, i.e. for each H_2 optimal controller we obtain the same closed loop transfer matrix.*

Proof: This is a direct consequence of the parameterization of all H_2 optimal stabilizing controllers as given in the beginning of this section. It is easy to check that all these controllers when applied to Σ_{PQ} yield a closed loop transfer matrix equal to 0. In the same way as in [13] for the H_∞ control problem, it can be shown that there is a one to one relationship between the closed loop transfer matrix of $\Sigma \times \Sigma_C$ and the closed loop transfer matrix of $\Sigma_{PQ} \times \Sigma_C$. ∎

The above theorem shows that we cannot use the additional freedom to shape the input-output behaviour. However, in general we have quite a bit of freedom left in placing the closed loop poles. It is the latter flexibility we would like to study in this paper. Note that we presented a complete characterization of all H_2 optimal controllers in this section. However, that parameterization is not very transparent in its effect on closed loop poles. Moreover, the structure of the controller is not very clear. In the following we will study full order and reduced order observer based controllers. These two classes of controllers have a desirable and clear structure and we will completely characterize the freedom we have to place the closed loop poles.

The design methodology for H_2 optimal controllers is the following. We have a complete characterization of all optimal H_2 state feedbacks, namely the set F_s^*. We take an element out of this set which has desirable properties (for instance with respect to pole location) and then we look for an observer such that the interconnection of this observer and the optimal state feedback yields an H_2 optimal dynamic controller.

We first study the following basic question. If we have an optimal state feedback F from the set $\mathbf{F}_s^*$, does there exist an observer such that the interconnection is an H_2 optimal dynamic controller? Note that the set of H_2 optimal state feedbacks for the system Σ is given by $\mathbf{F}_s^*(A, B, E, C_P, D_P)$. On the other hand the set of H_2 optimal state feedbacks for the system Σ_{PQ} is given by $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$.

Theorem 3.4. *Assume the system Σ_2 is left-invertible and an optimal strictly proper H_2 controller exists for the system Σ, i.e. the conditions in the second part of theorem 2.2 are satisfied. Then we have,*

$$\mathbf{F}_s^*(A, B, E_Q, C_P, D_P) \subseteq \mathbf{F}_s^*(A, B, E, C_P, D_P) \tag{3.11}$$

and for each element F_1 in $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$ there exists an output injection K_1 such that

$$\Sigma_C \begin{cases} \dot{\xi} = A\xi + Bu + K_1(C_1\xi - y) \\ u = F_1\xi \end{cases} \tag{3.12}$$

is an H_2 optimal dynamic controller for the system Σ.

Proof: Let F and K satisfy (3.1) and (3.2) respectively. We then have:

$$\begin{aligned} 0 &= (C_P + D_P F)(sI - A - BF)^{-1}E_Q, \\ 0 &= C_P(sI - A - KC_1)^{-1}(E_Q + KD_Q), \\ 0 &= (C_P + D_P F)(sI - A - BF)^{-1}(sI - A)(sI - A - KC_1)^{-1}(E_Q + KD_Q). \end{aligned} \tag{3.13}$$

Next, take an arbitrary element F_1 in $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$. Hence we have:

$$0 = (C_P + D_P F_1)(sI - A - BF_1)^{-1}E_Q.$$

After some extensive algebriac manipulations on the equation (3.13) we find:

$$0 = G(s)\left[(Q_1(s)C_1 - F_1)(sI - A - KC_1)^{-1}(E_Q + KD_Q) + Q_1(s)D_Q\right]$$

where

$$\begin{aligned} G(s) &= (C_P + D_P F_1)(sI - A - BF_1)^{-1}B + D_P, \\ Q_1(s) &= (F - F_1)(sI - A - BF)^{-1}K. \end{aligned}$$

Since Σ_2 is left-invertible, it is not hard to show that $G(s)$ has full column rank as a rational matrix and hence we find:

$$0 = Q_1(s)\left[C_1(sI - A - KC_1)^{-1}(E_Q + KD_Q) + D_Q\right] - F_1(sI - A - KC_1)^{-1}(E_Q + KD_Q).$$

This implies that the disturbance decoupling problem with measurement feedback and stability is solvable by a strictly proper controller for the following system:

$$\Sigma : \begin{cases} \dot{x} &= (A + KC_1)x & & + (E_Q + KD_Q)w \\ y &= C_1 x & & + D_Q w \\ z &= -F_1 x & + u, & \end{cases}$$

Using the results from [14] we find:

$$\mathcal{S}^-(\Sigma_{1PQ}) \subseteq \operatorname{Ker}[F_1]. \tag{3.14}$$

This implies $\mathcal{K}_s^*(A, E_Q, C_1, F_1, D_Q)$ is non-empty and hence there exists a matrix K_1 such that $A + K_1C_1$ is stable and:

$$0 = F_1(sI - A - K_1C_1)^{-1}(E_Q + K_1D_Q).$$

But then it is straightforward to check that for this pair (F_1, K_1) the controller (3.12) stabilizes Σ_{PQ} and achieves disturbance decoupling. According to lemma 2.1 this implies that (3.12) is an H_2 optimal controller for the system Σ.

It remains to show the inclusion (3.11). We have (3.14) and

$$\operatorname{Im}[Q] \subset \mathcal{S}^-(\Sigma_{1PQ}) \subset \operatorname{Ker}[C_P + D_PF_1].$$

This implies

$$E_QE_Q' = EE' + (A + BF_1 - sI)Q + Q(A' + F_1'B' + sI).$$

and then it is trivially checked that

$$\begin{aligned} 0 &= (C_P + D_PF_1)(sI - A - BF_1)^{-1}E_QE_Q'(A' + F_1'B' + sI)^{-1}(C_P' + F_1'D_P') \\ &= (C_P + D_PF_1)(sI - A - BF_1)^{-1}EE'(A' + F_1'B' + sI)^{-1}(C_P' + F_1'D_P'). \end{aligned}$$

This shows that $F_1 \in \mathcal{F}_s^*(A, B, E, C_P, D_P)$. ■

The above theorem identifies a class of state feedback controllers which, when combined with a suitable observer, yield H_2 optimal dynamic controllers. Also, the above theorem shows the intuitive fact that these state feedback controllers are a subset of all H_2 optimal state feedbacks.

Also, the above theorem shows us the available flexibility for the state feedback in a full order observer based controller. Before we point this out in detail, we still have to consider the case that the subsystem Σ_2 is not left-invertible. By choosing an appropriate basis for u we can guarantee that B and D_P have the following form:

$$B = (B_1 \quad B_2), \qquad D_P = (D_{P,1} \quad 0) \tag{3.15}$$

such that $\operatorname{Im} B \cap \mathcal{V}^-(\Sigma_{2PQ}) = \operatorname{Im} B_2$ and B_1 has full row rank and satisfies $\operatorname{Im} B_1 \cap \mathcal{V}^-(\Sigma_{2PQ}) = \{0\}$. We define $\bar{E}_Q$ and Γ by:

$$\bar{E}_Q = (E_Q \quad B_2), \qquad \Gamma = (I_l \quad 0_{l-m}) \tag{3.16}$$

where l is the normal rank of Σ_2 and m the number of inputs, in other words I_l is an identity matrix with the same number of rows as B_1 and 0_{l-m} is a zero matrix with the same number of rows as B_2. Note that $\Gamma = I$ if Σ_2 is left-invertible.

We will investigate feedbacks in the set $\mathcal{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$. We have the following (obvious) properties:

Lemma 3.1. *The set $\mathcal{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ satisfies the following properties:*

1. $\mathcal{F}_s^*(A, B, \bar{E}_Q, C_P, D_P) \subseteq \mathcal{F}_s^*(A, B, E_Q, C_P, D_P)$.

2. $\Omega^*(A, B, \bar{E}_Q, C_P, D_P) = \Omega^*(A, B, E_Q, C_P, D_P)$

3. $F_s^*(A, B, \bar{E}_Q, C_P, D_P)$ *equals* $F_s^*(A, B, E_Q, C_P, D_P)$ *if and only if* Σ_2 *is left invertible (or equivalently* Σ_{2PQ} *is left invertible).*

Proof : Part 1. and 2. are straightforward to check. Part 2. is the tricky part. However, looking at the construction of the set Ω^* in the *(OGFM)* algorithm it is straightforward to establish this fact. ∎

We then obtain an equivalent of theorem 3.4 for non-left-invertible systems:

Theorem 3.5. *Assume an optimal strictly proper* H_2 *controller exists for the system* Σ, *i.e. the conditions in the second part of theorem 2.2 are satisfied. Then, we have,*

$$F_s^*(A, B, \bar{E}_Q, C_P, D_P) \subseteq F_s^*(A, B, E, C_P, D_P)$$

and for each element F_1 *in* $F_s^*(A, B, \bar{E}_Q, C_P, D_P)$ *there exists an output injection* K_1 *such that*

$$\Sigma_C \begin{cases} \dot{\xi} = A\xi + Bu + K_1(C_1\xi - y) \\ u = F_1\xi \end{cases}$$

is an H_2 *optimal dynamic controller for the system* Σ.

Proof : F_1 in $F_s^*(A, B, \bar{E}_Q, C_P, D_P)$ implies that ΓF_1 is in $F_s^*(A, B_1, \bar{E}_Q, C_P, D_{P,1})$. Using the proof of theorem 3.4 we obtain that

$$\mathcal{S}^-(A, \bar{E}_Q, C_1, (D_Q \quad 0)) \subset \operatorname{Ker}[\Gamma F] .$$

On the other hand it is easy to check that

$$\mathcal{S}^-(A, \bar{E}_Q, C_1, D_Q) \subset \mathcal{S}^-(A, \bar{E}_Q, C_1, (D_Q \quad 0)) .$$

This implies $K_s^*(A, E_Q, C_1, \Gamma F_1, D_Q)$ is non-empty and hence there exists a matrix K_1 such that $A + K_1C_1$ is stable and:

$$0 = \Gamma F_1(sI - A - K_1C_1)^{-1}(E_Q + K_1D_Q) .$$

Together with

$$0 = (C_P + D_PF)(sI - A - BF)^{-1}E_Q,$$
$$0 = (C_P + D_PF)(sI - A - BF)^{-1}B_2,$$

this implies that the controller (3.12) stabilizes Σ_{PQ} and achieves disturbance decoupling. According to lemma 2.1 this implies that (3.12) is an H_2 optimal controller for the system Σ. ∎

The next theorem gives the flexibility one has in selecting the observer gain for a given H_2 optimal state feedback in the set $F_s^*(A, B, \bar{E}_Q, C_P, D_P)$.

Theorem 3.6. *Assume that an optimal strictly proper H_2 controller exists for the system Σ, i.e. the conditions in the second part of theorem 2.2 are satisfied. Also, let $F_1 \in \mathcal{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ be given. Then, the set $\mathcal{K}_s^*(F_1) := \mathcal{K}_s^*(A, E_Q, C_1, \Gamma F_1, D_Q)$ is equal to the set of output injections K_1 for which*

$$\Sigma_C \begin{cases} \dot{\xi} = A\xi + Bu + K_1(C_1\xi - y) \\ u = F_1\xi \end{cases} \tag{3.17}$$

is an H_2 optimal controller for Σ. Moreover, given F_1, the set of H_2 optimal full order observer fixed modes associated with F_1, is given by $\Psi^*(A, E_Q, C_1, \Gamma F_1, D_Q)$.

Proof: Given F_1 in $\mathcal{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ and an output injection K_1 such that $A + KC_1$ is stable we know that the controller (3.17) stabilizes the system Σ. It is an H_2 optimal controller if it achieves disturbance decoupling when applied to Σ_{PQ}. Using that F_1 is in $\mathcal{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ we find that (3.17) achieves disturbance decoupling if

$$0 = [(C_P + D_P F_1)(sI - A - BF_1)^{-1}B_1 + D_{P,1}]\Gamma F_1(sI - A - K_1C_1)^{-1}(E_Q + K_1D_Q)\,.$$

Since the system characterized by $(A, B_1, C_P, D_{P,1})$ is left invertible, we find the following necessary and sufficient condition:

$$0 = \Gamma F_1(sI - A - K_1C_1)^{-1}(E_Q + K_1D_Q)\,.$$

The rest of the theorem is then a trivial consequence of earlier results. ∎

Step by Step Sequential Design Procedure:

Consider an H_2 optimal control problem for the system (2.1), while using measurement feedback controllers. Also, assume that an H_2 optimal strictly proper measurement feedback controller exists. Then we have the following steps.

Step 1: Determine P and Q and transform the system Σ to Σ_{PQ}.

Step 2: Using the quintuple $(A, B, \bar{E}_Q, C_P, D_P)$ that characterizes the H_2 optimal static state feedback control problem for Σ_{PQ}, as the input to the *(OGFM)* algorithm, construct the set of H_2 optimal fixed modes $\Omega^*(A, B, \bar{E}_Q, C_P, D_P)$. Choose a set Λ of desired poles which is self conjugate and includes $\Omega^*(A, B, \bar{E}_Q, C_P, D_P)$. Then, following the procedure given in *(OGFM)*, determine the static state feedback gain $F \in \mathcal{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ such that $\lambda(A + BF)$ equals Λ. This is always possible.

Step 3: Consider the quintuple $(A, E_Q, C_1, \Gamma F, D_Q)$ where F is as chosen in Step 2. Using this quintuple as the input to the dual *(OGFM)* algorithm, construct the H_2 optimal full order observer fixed modes, namely,

$$\Psi^*(F) := \Psi^*(A, E_Q, C_1, \Gamma F, D_Q)\,.$$

Next, as in Step 2, first selecting a set of n desired poles which is self conjugate and includes $\Psi^*(F)$, choose a gain $K \in \mathcal{K}_s^*(A, E_Q, C_1, \Gamma F, D_Q)$ such that $\lambda(A + KC_1)$ coincides with the n desired poles. This is always possible.

Step 4: Form a full order observer based controller as in (3.12) with $F_1 = F$ and $K_1 = K$ selected as in Steps 2 and 3.

It is obvious in view of Theorem 3.6 that the full order observer based controller formed in Step 4, is indeed H_2 optimal and places the closed-loop poles at the locations of $\lambda(A + BF)$ and $\lambda(A + KC_1)$. Lemma 3.1 shows that restricting ourselves to the set $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ does not restrict our flexibility in placing the poles of $A + BF$. However, we might have had more flexibility in the observer poles (the poles of $A + KC_1$) if we were able to vary F over the larger class $\mathbf{F}_s^*(A, B, E_Q, C_P, D_P)$. Nevertheless, we do have:

$$\cap_{F\in \mathbf{F}_s^*(A,B,\bar{E}_Q,C_P,D_P)} \Psi^*(F) = \cap_{F\in \mathbf{F}_s^*(A,B,E_Q,C_P,D_P)} \Psi^*(F)$$

which leaves us to believe that we do not loose much flexibility by this restriction.

The above development pertains to H_2 optimal full order observers associated with a given H_2 optimal static state feedback gain $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$. An interesting extension we can pursue next would be to identify a set of full order observers each one of which can be considered as an H_2 optimal observer for *any* $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$. More specifically, we would like to identify next a set of full order observer gains, say $\tilde{\mathbf{K}}^*$, such that $\tilde{\mathbf{K}}^* \subseteq \mathbf{K}^*(F)$ for *any* $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$. Then, *any* full order observer with its gain in $\tilde{\mathbf{K}}^*$, can be utilized to implement *any* H_2 optimal static state feedback law such that the resulting control law would be an H_2 optimal observer based measurement feedback law. To identify the set $\tilde{\mathbf{K}}^*$, we first let T be any matrix such that $\operatorname{Ker}(T) = \mathcal{S}^-(\Sigma_{1PQ})$. Then, the set $\tilde{\mathbf{K}}^*$ can be defined as follows,

$$\tilde{\mathbf{K}}^* = \mathbf{K}_s^*(A, E_Q, C_1, T, D_Q). \tag{3.18}$$

We have the following theorem.

Theorem 3.7. *Consider an H_2 optimal control problem for a system Σ as in (2.1). Assume that the given system Σ satisfies the necessary and sufficient conditions for the existence of an H_2 optimal strictly proper measurement feedback controller as given in the second part of Theorem 2.2. Then, for any $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$, we have,*

$$\tilde{\mathbf{K}}^* \subseteq \mathbf{K}^*(F).$$

Proof: It simply follows from the fact that $\operatorname{Ker}(T) = \mathcal{S}^-(\Sigma_{1PQ})$. ∎

We would like to remark that when one is restricted to the set of observers with their gains in $\tilde{\mathbf{K}}^*$, one looses some freedom in assigning the observer poles. That is, the observer poles must include the fixed modes given by $\Psi^*(A, E_Q, C_1, T, D_Q)$. It is easy to see that, for any $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$,

$$\Psi^*(A, E_Q, C_1, T, D_Q) \supseteq \Psi^*(A, E_Q, C_1, \Gamma F, D_Q).$$

It is also interesting to see how the above development carries over to full order proper controllers of the form:

$$\Sigma_C \begin{cases} \dot{\xi} = A\xi + Bu + K_1(C_1\xi - y) \\ u = F_1\xi - N(C_1\xi - y). \end{cases} \tag{3.19}$$

It is easy from the parameterization of all H_2 optimal controllers that N must be an element of the set $\mathbf{N}^*$ defined by (3.3). By applying the preliminary feedback $u = Ny + v_1$ we see that (3.19) is an H_2 optimal controller for Σ if and only if:

$$\tilde{\Sigma}_C \begin{cases} \dot{\xi} = \tilde{A}\xi + Bu + \tilde{K}_1(C_1\xi - y) \\ u = \tilde{F}_1\xi \end{cases} \tag{3.20}$$

is an H_2 optimal controller for

$$\tilde{\Sigma}: \begin{cases} \dot{x} &= \tilde{A}x + Bu + \tilde{E}w \\ y &= C_1 x + D_1 w \\ z &= \tilde{C}_2 x + D_2 u, \end{cases} \tag{3.21}$$

where

$$\begin{aligned} \tilde{F}_1 &= F - NC_1 , & \tilde{K}_1 &= K_1 - BN , \\ \tilde{A} &= A + BNC_1 & \tilde{E} &= E + BND_1 . \\ \tilde{C}_2 &= C_2 + D_2 N C_1 \end{aligned}$$

The set of solutions of the linear matrix inequalities $F(P) \geq 0$ and $G(Q) \geq 0$ do not change by this step. However, the system Σ_{PQ} takes the following form:

$$\tilde{\Sigma}_{PQ} : \begin{cases} \dot{x}_{PQ} &= \tilde{A}\, x_{PQ} + Bu_{PQ} + \tilde{E}_Q w_{PQ} \\ y_{PQ} &= C_1 x_{PQ} + D_Q w_{PQ} \\ z_{PQ} &= \tilde{C}_P x_{PQ} + D_P u_{PQ}. \end{cases} \tag{3.22}$$

where

$$\begin{aligned} \tilde{E}_Q &= E_Q + BND_Q \\ \tilde{C}_P &= C_P + D_P N C_1 . \end{aligned}$$

We assume B and D_P have the form (3.15) and Γ is given by (3.16). We define $\hat{E}_Q$ by:

$$\hat{E}_Q = (\tilde{E}_Q \quad B_2)$$

We can then apply the previous results since for a fixed N we are looking for a strictly proper controller. Nevertheless, we are naturally interested in the question of how $\mathcal{F}_s^*$ and $\mathcal{K}_s^*$ depend on the preliminary output feedback N.
We have:

$$\begin{aligned} \mathcal{F}_s^*(\tilde{A}, B, \hat{E}_Q, \tilde{C}_P, D_P) &= \{ F - NC_1 \mid F \in \mathcal{F}_s^*(A, B, \tilde{E}_Q + BND_Q, C_P, D_P) \} \\ \mathcal{K}_s^*(\tilde{A}, \tilde{E}_Q, C_1, \Gamma \tilde{F}_1, D_Q) &= \{ K - BN \mid K \in \mathcal{K}_s^*(A, E_Q, C_1, \Gamma(F_1 - NC_1), D_Q) \} . \end{aligned} \tag{3.23}$$

Therefore, the additional flexibility by choosing $N \in \mathbf{N}^*$ has an effect on both $\mathcal{F}_s^*$ and $\mathcal{K}_s^*$. We would like to remark that there is no obvious choice for N which is optimal with respect to pole placement.

4. Reduced Order Observer Based Controller

Let a static state feedback gain F be given. Then, in what follows, we develop a reduced order observer based controller of dynamic order $n - \text{rank}[C_1 , D_1] + \text{rank}[D_1]$ where n as usual is the dynamic order of Σ. At first, without loss of generality, we assume that the matrices C_1 and D_Q have already been transformed to the following form,

$$C_1 = \begin{pmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{pmatrix} \quad \text{and} \quad D_Q = \begin{pmatrix} D_0 \\ 0 \end{pmatrix} . \tag{4.1}$$

Thus, the system Σ_{PQ} as in (2.5) can be partitioned as follows,

$$\begin{cases} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u_{PQ} + \begin{pmatrix} E_{1Q} \\ E_{2Q} \end{pmatrix} w_{PQ} \\ \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 & C_{02} \\ I_{p-m_0} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} D_0 \\ 0 \end{pmatrix} w_{PQ} \\ z_{PQ} = \begin{pmatrix} C_{P1} & C_{P2} \end{pmatrix} x_{PQ} + D_P \; u_{PQ}. \end{cases} \tag{4.2}$$

The idea behind the construction of a reduced order observer based controller is that we only need to build an observer for x_2 as x_1 (or equivalently y_1) is available as a measurement. Our techniques to do so are based on the method discussed in Section 7.2 of [1]. The differential equation for x_2 is given by

$$\dot{x}_2 = A_{22}\; x_2 + \begin{bmatrix} A_{21} & B_2 \end{bmatrix} \begin{pmatrix} y_1 \\ u_{PQ} \end{pmatrix} + E_{2Q}\, w_{PQ}$$

where y_1 is known, and u_{PQ} is temporarily assumed known. Observations of x_2 are made via y_1 and $\tilde{y}$, where

$$\tilde{y} := A_{12}x_2 + E_{1Q}w_{PQ} = \dot{y}_1 - A_{11}x_1 - B_1 u_{PQ}. \tag{4.3}$$

If we do not worry about the differentiation for a moment, we note that we have to build an observer for the following system,

$$\Sigma_r : \begin{cases} \dot{x}_2 = A_{22} \; x_2 + E_{2Q} \; w_{PQ} + \begin{bmatrix} A_{21} & B_2 \end{bmatrix} \begin{pmatrix} y_1 \\ u_{PQ} \end{pmatrix} \\ \begin{pmatrix} y_0 \\ \tilde{y} \end{pmatrix} = \begin{bmatrix} C_{02} \\ A_{12} \end{bmatrix} x_2 + \begin{bmatrix} D_0 \\ E_{1Q} \end{bmatrix} w_{PQ}. \end{cases} \tag{4.4}$$

In order to construct an observer for Σ_r, we need to enquire whether Σ_r is detectable whenever the given system Σ is detectable. The following lemma does this among others.

Lemma 4.1. *Let the system Σ_{re} be defined by the quadruple given below,*

$$\left(A_{22},\; E_{2Q},\; \begin{pmatrix} C_{02} \\ A_{12} \end{pmatrix},\; \begin{pmatrix} D_0 \\ E_{1Q} \end{pmatrix} \right). \tag{4.5}$$

Then we have,

1. *Σ_{re} is detectable if and only if Σ_{1Q} is detectable.*

2. *The invariant zeros of Σ_{re} are the same as the invariant zeros of Σ_{1Q}.*

3. *The infinite zeros of Σ_{re} are the infinite zeros of Σ_{1Q} with order larger than 1. Their order is reduced by 1 when compared with the order of zeros of Σ_{1Q}.*

4. *Σ_{re} is left invertible if and only if Σ_{1Q} is left invertible.*

5. $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{V}_g(\Sigma_{re}) \subseteq \mathcal{V}_g(\Sigma_{1Q})$.

6. $\begin{pmatrix} 0 \\ I \end{pmatrix} \mathcal{S}_g(\Sigma_{re}) \subseteq \mathcal{S}_g(\Sigma_{1Q}) \cap C^{-1}\{\mathrm{Im}\,(D_Q)\}$.

Proof: See Proposition 2.2.2 on p.32 of [9]. ■

Next, we build a full order observer for the system Σ_r defined by (4.4). In fact, we find the following observer which utilizes a gain K_r,

$$\dot{\hat{x}}_2 = A_{22}\hat{x}_2 + A_{21}y_1 + B_2u_{PQ} + K_r\left[\begin{pmatrix} C_{02} \\ A_{12} \end{pmatrix}\hat{x}_2 - \begin{pmatrix} y_0 \\ \dot{y}_1 - A_{11}x_1 - B_1u_{PQ} \end{pmatrix}\right].$$

We partition $K_r = [K_{r0} \quad K_{r1}]$ so as to be compatible with the sizes of $(y_0, \tilde{y})$. Then, using the change of variables $v := \hat{x}_2 + K_{r1}y_1$ results in a reduced order observer,

$$\left\{\begin{aligned} \dot{v} &= (A_{22} + K_{r0}C_{02} + K_{r1}A_{12})v + (B_2 + K_{r1}B_1)u_{PQ} \\ &\qquad + [-K_{r0},\ A_{21} + K_{r1}A_{11} - (A_{22} + K_{r0}C_{02} + K_{r1}A_{12})K_{r1}]y_{PQ} \\ \hat{x}_{PQ} &= \begin{pmatrix} 0 \\ I_r \end{pmatrix} v + \begin{pmatrix} 0 & I_{n-r} \\ 0 & -K_{r1} \end{pmatrix} y_{PQ}, \end{aligned}\right. \tag{4.6a}$$

where r is the dimension of x_2 or equivalently the dimension of v. We use this reduced order observer to obtain the control law as,

$$u_{PQ} = F\hat{x}_{PQ}. \tag{4.6b}$$

Equation (4.6) defines the reduced order observer based controller.

As seen in (4.6), two parameters F and K_r characterize a reduced order observer based controller. In other words, prescribing a pair (F, K_r) is tantamount to prescribing a reduced order observer based controller. Suppose a proper H_2 optimal measurement feedback controller exists. Then, our basic question is how to choose F and K_r so that (4.6) is an H_2 optimal measurement feedback controller.

Again the traditional separation principle does not hold and hence we cannot separate the choice of an observer gain K_r from the choice of a state feedback gain F when trying to construct an H_2 optimal measurement feedback controller.

Given $F \in \mathcal{F}_s^*$, we construct the set $\mathcal{K}_r^*(F)$, such that the given F and *any* $K_r \in \mathcal{K}_r^*(F)$ together specify an H_2 optimal reduced order observer based controller. Again we treat the non-left-invertible case by restricting the state feedback F to the set $\mathcal{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$. Remember that this set is equal to $\mathcal{F}_s^*(A, B, E_Q, C_P, D_P)$ if the system Σ_2 is left-invertible.

Theorem 4.1. *Assume an H_2 optimal strictly proper controller exists for the system Σ, i.e. the conditions in the second part of theorem 2.2 are satisfied. Then, for each F in $\mathcal{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ there exists an output injection $K_r = [K_{r0} \quad K_{r1}]$ such that the controller (4.6) is an H_2 optimal dynamic controller for the system Σ.*

Proof: Given F in $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ we first factorize $\Gamma F = (\, F_1 \quad F_2 \,)$ Then, using the fact that the system $(A, B_1, C_P, D_{P,1})$ is left invertible, we find that a reduced order controller of the form (4.6) is optimal if and only if $A_{22} + K_{r0}C_{02} + K_{r1}A_{12}$ is stable and

$$0 = F_2[(sI - A_{22} - K_{r0}C_{02} - K_{r1}A_{12})^{-1}(E_{2Q} + K_{r0}D_0 + K_{r1}E_{1Q}). \tag{4.7}$$

Since $\mathcal{S}^-(\Sigma_{1PQ}) \subseteq \operatorname{Ker}[\Gamma F]$ (as was shown in the proof of theorem 3.4) we find in combination with lemma 4.1 that $\mathcal{S}^-(\Sigma_{re}) \subseteq \operatorname{Ker}[F_2]$. This guarantees that the existence of a matrix K_r such that (4.7) is satisfied. ∎

Theorem 4.2. *Assume an H_2 optimal optimal strictly proper controller exists for the system Σ, i.e. the conditions in the second part of theorem 2.2 are satisfied. Let F be in $\mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$. Then the class of output injections $K_r = [K_{r0} \quad K_{r1}]$ such that the controller (4.6) is an internally stabilizing H_2 optimal dynamic controller for the system Σ is given by*

$$\mathbf{K}_r^*(F) := \mathbf{K}_s^* \left(A_{22}, E_{2Q}, \begin{pmatrix} C_{02} \\ A_{12} \end{pmatrix}, F_2, \begin{pmatrix} D_0 \\ E_1 \end{pmatrix} \right).$$

Also, given F, the set of H_2 optimal reduced order observer fixed modes associated with F is characterized by

$$\Psi^*(F) := \Psi^* \left(A_{22}, E_{2Q}, \begin{pmatrix} C_{02} \\ A_{12} \end{pmatrix}, F_2, \begin{pmatrix} D_0 \\ E_1 \end{pmatrix} \right).$$

Proof: This is an immediate consequence of the fact that suitable observer gains are characterized by the stability of $A_{22} + K_{r0}C_{02} + K_{r1}A_{12}$ and (4.7).

Step by Step Sequential Design Procedure:

Consider an H_2 optimal control problem for the system (2.1), while using measurement feedback controllers. Assume that an H_2 optimal strictly proper measurement feedback controller exists. Then we have the following steps.

Step 1: Determine P and Q and transform the system Σ to Σ_{PQ} and make sure that C_1 and D_Q have the special form as in 4.1.

Step 2: Using the quintuple $(A, B, \bar{E}_Q, C_P, D_P)$ that characterizes the H_2 optimal static state feedback control problem for Σ_{PQ}, as the input to the *(OGFM)* algorithm, construct the set of H_2 optimal fixed modes $\Omega^*(A, B, \bar{E}_Q, C_P, D_P)$. Choose a set Λ of desired poles which is self conjugate and includes $\Omega^*(A, B, \bar{E}_Q, C_P, D_P)$. Then, following the procedure given in *(OGFM)*, determine the static state feedback gain $F \in \mathbf{F}_s^*(A, B, \bar{E}_Q, C_P, D_P)$ such that $\lambda(A + BF)$ equals Λ. This is always possible.

Step 3: Partition F, the one chosen in Step 2, as $F = [F_1\,,\ F_2]$ in conformity with the partitioning of $x = [x_1',\ x_2']'$. Use the dual *(OGFM)* algorithm, at first construct the set of H_2 optimal reduced order observer fixed modes, namely,

$$\Psi_r^*(F) := \Psi^* \left(A_{22}, E_{2Q}, \begin{pmatrix} C_{02} \\ A_{12} \end{pmatrix}, F_2, \begin{pmatrix} D_0 \\ E_1 \end{pmatrix} \right).$$

Next, choose $K_r \in \mathbf{K}_r^*(F)$ such that $A_{22} + K_{r0}C_{02} + K_{r1}A_{12}$ has all its eigenvalues at r desired locations in $\mathbb{C}^-$. This is possible only if the r desired locations in $\mathbb{C}^-$ include the set $\Psi_r^*(F)$.

Step 4: Form a reduced order observer based controller as in (4.6) with F and K_r selected as in Steps 2 and 3.

It is obvious in view of Theorem 4.2 the reduced order observer based controller formed in Step 4, is indeed H_2 optimal and places the closed-loop poles at the locations of $\lambda(A + BF)$ and $\lambda(A_{22} + K_{r0}C_{02} + K_{r1}A_{12})$.

We now proceed to identify a set of reduced order observers or equivalently a set of reduced order observer gains $\tilde{K}_r^*$ such that *any* element of it can be paired with *any* H_2 optimal static state feedback gain F so that the resulting observer based controller is an H_2 optimal measurement feedback controller. Thus, for this set of reduced order observers, the traditional separation principle holds.
Consider the following set of gains,

$$\tilde{K}_r^* := K_s^*(F) := K_s^* \left(A_{22}, E_{2Q}, \begin{pmatrix} C_{02} \\ A_{12} \end{pmatrix}, T_r, \begin{pmatrix} D_0 \\ E_1 \end{pmatrix} \right) \tag{4.8}$$

where T_r is any matrix such that Ker $(T_r) = \mathcal{S}^-(\Sigma_{re})$.
We have the following theorem.

Theorem 4.3. *Consider an H_2 optimal control problem as defined by Definition 2.2 for a system Σ as in (2.1), while using measurement feedback controllers. Assume that an H_2 optimal strictly proper measurement feedback controller exists. Then, the reduced order observer based controller described by (4.6) where F is any element of $F_s^*(A, B, \bar{E}_Q, C_P, D_P)$ and K_r is any element of $\tilde{K}_r^*$, is an H_2 optimal measurement feedback controller.*

Proof: It follows along the same lines as the proof of Theorem 3.7. ∎
In the above development, it is indeed odd to assume that an optimal strictly proper controller exist and then we construct a reduced order controller which is no longer strictly proper. However, this situation can be rectified in the same way as in section 3. First we assume that there exists a proper H_2 optimal controller, and then choose $\bar{N}$ in the set $\mathbf{N}^*$ defined by (3.3). Then, we design a reduced order H_2 optimal observer for the system (3.21). Note that the existence of an optimal proper controller for (2.1) implies the existence of an optimal strictly proper controller for (3.21). Suppose we have an H_2 optimal, reduced order observer based controller for the system (3.21). This controller is basically of the form (2.2). Combined with the preliminary static output feedback given by $\bar{N}$ we then obtain the following H_2 optimal controller for Σ:

$$\Sigma_C : \begin{cases} \dot{v} = J\,v + L\,y \\ u = Mv + (N + \bar{N})y. \end{cases} \tag{4.9}$$

There is a clear relationship between (3.21) and (2.1). Namely, (4.9) is an H_2 optimal controller for (2.1) if and only if (2.2) is an H_2 optimal controller for (3.21). Moreover, a controller (2.2) applied to (3.21) yields the same closed loop poles as (4.9) applied to (2.1). The flexibility in placing the closed loop poles of (3.21) by strictly proper controllers is clearly described in this section. However, it is not very transparent how this flexibility is influenced by our choice of $\bar{N}$ in $\mathbf{N}^*$. The influence on the state feedback gain is clearly depicted by (3.23). On the other hand the influence of our choice for $\bar{N}$ on the reduced order observer gain is much less transparent.

5. Conclusions

At first we characterize and parameterize all H_2 optimal measurement feedback controllers. Then our attention is focused on controllers with observer based architecture. Both full order as well as reduced order observer based H_2 optimal controllers are considered. Our design of an H_2 optimal observer based controller follows a traditional sequential design philosophy. That is, in the first stage of a design, a static H_2 optimal state feedback law is designed. In the second stage, an observer is designed to implement the given H_2 optimal state feedback law so that the resulting measurement feedback controller is H_2 optimal. A complication that arises in such a design philosophy is that the traditional separation principle does not always hold. For a given H_2 optimal state feedback law, one has to isolate a set of observers that can be termed as H_2 optimal observers associated with that particular H_2 optimal state feedback law. Any observer in such a set of observers can be used to implement that particular H_2 optimal state feedback law so that the resulting measurement feedback controller is H_2 optimal. Here, we characterize, parameterize and develop methods of constructing the set of all H_2 optimal observers associated with any given H_2 optimal static state feedback law.

Since there are in general many H_2 optimal observers associated with a given H_2 optimal static state feedback law, one can formulate a design problem of utilizing such a freedom to assign observer poles to desired locations in the left half complex plane whenever such an assignment is possible. We refer to this problem as an *H_2 optimal control problem with simultaneous closed-loop pole placement*. As is known, the poles of a closed-loop system comprising of the given system and an observer based controller, are the union of observer poles and the poles of the closed-loop system under the state feedback control law alone. In view of this, the problem of assigning the poles of the closed-loop system under an H_2 optimal observer based controller translates into two problems which must be treated sequentially. The first problem is to design a "desired" H_2 optimal state feedback control law that yields a closed-loop system with poles in desired locations whenever it is possible. This problem was studied extensively in an earlier paper [2] and an algorithm called *(OGFM)* was developed there to facilitate the construction of an H_2 optimal state feedback control law with simultaneous closed-loop pole placement. The second problem is to design an H_2 optimal observer associated with the H_2 optimal state feedback control law obtained in the first problem such that its poles are in desired locations. However, it turns out, one cannot in general assign all the poles of an H_2 optimal observer associated with a given H_2 optimal state feedback control law arbitrarily. Some of the poles must be located in certain locations in the left half complex plane in order to guarantee the H_2 optimality of an observer. Obviously, such poles can be refered to as H_2 optimal observer fixed modes associated with a given H_2 optimal state feedback control law. We develop here a method of constructing the set of all such H_2 optimal observer fixed modes associated with a given H_2 optimal state feedback control law in order to identify the freedom that exists in assigning the H_2 optimal observer poles. This finally leads us to a procedure of designing an H_2 optimal measurement feedback controller that places the closed-loop poles at desired locations whenever it can be done.

We also construct here a set of full order as well as reduced order H_2 optimal observers such that *any* element of it can be paired with *any* H_2 optimal static state feedback control law so that the resulting observer based controller is H_2 optimal. When one is restricted to such a set of observers, the traditional separation principle is valid. However, obviously, if we are restricted to use only an observer in such a set, we will have only some (but not the entire possible) freedom in assigning the observer poles.

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