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Abstract: This paper deals with a rigorous derivation (using the singular perturbation technique) of the effective model for the enhanced diffusion through a narrow and long 2D pore. We start with a pore scale model for transport of a reactive solute in a pore space by convection and diffusion. The pore contains initially a soluble substance and the same substance, at different concentration, is injected at $x = 0$. The solute particles undergo a first-order chemical reaction on the pore surfaces. We place ourselves in the conditions of Taylor's study and also in presence of

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chemical reactions. The upscaled behavior for important Peclet's and Damkohler's numbers, using the ratio of the the characteristic transversal and longitudinal times as a measure of smallness, is given. Furthermore, we give a rigorous mathematical justification of the effective behavior, being an approximation of the physical problem. The error estimate is obtained, first in the energy norm, and then in L^∞ and L^1 norms with respect to the space variable. They guarantee the validity of the upscaled model. As a special case, we recover the well-known Taylor dispersion formula. In our knowledge, this is the first time that the Taylor dispersion formula is justified in mathematically rigorous way.

1 Introduction

We study the diffusion of the solute particles transported by the Poiseuille velocity profile in a semi-infinite 2D channel. Solute particles are participants in a first-order chemical reaction with the boundary of the channel. They don't interact between them. The simplest example, borrowed from [8], is described by the following model for the solute concentration c^*

$$\frac{\partial c^*}{\partial t^*}+q(z)\frac{\partial c^*}{\partial x^*}-D^*\frac{\partial^2 c^*}{\partial (x^*)^2}-D^*\frac{\partial^2 c^*}{\partial z^2}=0 \quad \text{in } \Omega^*=(0,+\infty)\times(-H,H), \tag{1}$$

where $q(z)=Q^*(1-(z/H)^2)$ and Q^* (velocity) and D^* (molecular diffusion) are positive constants. At the lateral boundaries $z=\pm H$ the first-order chemical reaction with the solute particles is modeled through the following boundary condition :

$$D^*\partial_z c^*+k^*c^*=0 \quad \text{on } z=\pm H, \tag{2}$$

where k^* is the surface reactivity coefficient.

The natural way of analyzing this problem is to introduce the appropriate scales. They would come from the characteristic concentration $\hat{c}$, the characteristic length L_R, the characteristic velocity Q_R, the characteristic diffusivity D_R and the characteristic time T_c. The characteristic length L_R coincides in fact with the " observation distance ". Set now

$$c=\frac{c^*}{\hat{c}},\ x=\frac{x^*}{L_R},\ y=\frac{z}{H},\ t=\frac{t^*}{T_c},\ Q=\frac{Q^*}{Q_R},\ D=\frac{D^*}{D_R},\ k_0=\frac{k^*}{k_R} \tag{3}$$

Then

$$\Omega=(0,+\infty)\times(-1,1), \quad \Gamma_w=(0,+\infty)\times\{-1,1\}. \tag{4}$$

Then the dimensionless form of (2) reads

$$\frac{\partial c}{\partial t}+\frac{Q_R T_c}{L_R}Q(1-y^2)\frac{\partial c}{\partial x}-\frac{D_R T_c}{L_R^2}D\partial_{xx}c-\frac{D_R T_c}{H^2}D\partial_{yy}c=0 \quad \text{in } \Omega\times(0,T) \quad (5)$$

On Γ_w we impose the condition (2)

$$-\frac{DD_R T_c}{H^2}\frac{\partial c}{\partial y}=k\frac{T_c}{H}c \quad \text{on } \Gamma_w\times(0,T). \quad (6)$$

Problem involves the following time scales:

T_L = characteristic longitudinal time scale = $\dfrac{L_R}{Q_R}$

T_T = characteristic transversal time scale = $\dfrac{H^2}{D_R}$

T_R = superficial chemical reaction time scale = $\dfrac{H}{k_R}$

and the following characteristic non-dimensional numbers

Pe = $\dfrac{L_R Q_R}{D_R}$ (Peclet's number)

Da = $\dfrac{L_R^2 k_R}{H D_R}$ (Damkohler's number)

Further we set $\varepsilon = \dfrac{H}{L_R} << 1$ and choose $T_c = T_L$. Solving the full problem for arbitrary values of coefficients is costly and practically impossible. Consequently, one would like to find the effective (or averaged) values of the dispersion coefficient and the transport velocity and an effective corresponding 1D parabolic equation for the effective concentration.

In the absence of the chemical reaction, in [15] Taylor obtained, for the equation (1) describing only a diffusive transport of a passive scalar, an explicit effective expression for the enhanced diffusion coefficient. It is called in literature *Taylor's dispersion formula.*

The problem studied by Sir Taylor could be easily embedded in our setting. We choose $Q=\mathcal{O}(1)$, and $\dfrac{T_T}{T_L}=\dfrac{HQ_R}{D_R}\varepsilon=\mathcal{O}(\varepsilon^{2-\alpha})=\varepsilon^2\,\mathbf{Pe}$. Then the situation from Taylor's article [15] corresponds to the case when $\alpha = 1$, i.e. Peclet's number is equal to $\frac{1}{\varepsilon}$, and $k_0 = 0$. Our equations in their non-dimensional form are

$$\frac{\partial c}{\partial t} + Q(1-y^2)\frac{\partial c}{\partial x} = D\varepsilon^{\alpha}\partial_{xx}c + D\varepsilon^{\alpha-2}\partial_{yy}c \quad \text{in } I\!R_+ \times (0,1) \times (0,T) \tag{7}$$

$$c(x,y,0) = 1, \quad (x,y) \in I\!R_+ \times (0,1), \tag{8}$$

$$-D\varepsilon^{\alpha-2}\partial_y c|_{y=1} = -D\frac{1}{\varepsilon^2 \mathbf{Pe}}\partial_y c|_{y=1} = k_0\frac{\mathbf{Da}}{\mathbf{Pe}}c|_{y=1} = k_0\varepsilon^{\alpha+\beta}c|_{y=1} \tag{9}$$

$$\partial_y c(x,0,t) = 0, \ (x,t) \in I\!R_+ \times (0,T) \tag{10}$$

$$\text{and} \quad c(0,y,t) = 0, \ (y,t) \in (0,1) \times (0,T), \tag{11}$$

where it was used that c is antisymmetric in y and Damkohler's number was set to ε^{β}. Our domain is now the infinite strip $Z^+ = I\!R_+ \times (0,1)$. We study the behavior of the solution to (7) -(11), with square integrable gradient in x and y, when $\varepsilon \to 0$. Clearly, the most interesting case is $\boxed{\beta = -\alpha}$ and $\boxed{0 \le \alpha < 2}$ and we restrict our considerations to this situation.

In this paper we prove that the correct upscaling of the problem (7)-(11) gives the following 1D parabolic problem

$$\begin{cases} \partial_t c^{Mau} + Q\big(\frac{2}{3} + \frac{4k_0}{45D}\varepsilon^{2-\alpha}\big)\partial_x c^{Mau} + k_0\big(1 - \frac{k_0}{3D}\varepsilon^{2-\alpha}\big)c^{Mau} = \\ (D\varepsilon^{\alpha} + \frac{8}{945}\frac{Q^2}{D}\varepsilon^{2-\alpha})\partial_{xx}c^{Mau} \text{ in } I\!R_+ \times (0,T) \\ c^{Mau}|_{x=0} = 0, \ c^{Mau}|_{t=0} = 1, \ \partial_x c^{Mau} \in L^2(I\!R_+ \times (0,T)). \end{cases} \tag{EFF}$$

We note that for $k_0 = 0$ and $\alpha = 1$ this is exactly the effective model of Sir Taylor.

What is known concerning the derivation of the effective problem (EFF), with or without chemical reactions?

⋄ In the absence of the chemical reactions, there is a formal derivation by R. Aris, using the method of moments. For more details see [1].

⋄ There have been numerous attempts at providing a rigorous justification for the approximation in absence of the chemical reactions. The most convincing is the " *near rigorous* " derivation using the centre manifold theory by G.N. Mercer and A.J. Roberts. For details see [9] , where the initial value problem is studied and the Fourier transform with respect to x is applied. The resulting PDE is written in the form $\dot{u} = \mathcal{A}u + F(u)$, with $u = (k, \hat{c})$. Then the centre manifold theory is applied to obtain effective equations at various orders. Since the corresponding centre manifold isn't finite dimensional, the results aren't rigorous.

⋄ When the chemistry is added (e.g. having an irreversible, 1st order, chemical reaction with equilibrium at $y = 1$, as we have), then there is a paper [11] by M.A. Paine, R.G. Carbonell and S. Whitaker. The authors use the "single-point" closure schemes of turbulence modeling by Launder to obtain a closed model for the averaged concentration.

Hence the mentioned analysis don't provide a rigorous mathematical derivation of the Taylor's dispersion formula and in the presence of the chemical reactions it is even not clear how to average the starting problem.

It should be noted that the real interest is in obtaining *dispersion equations* for reactive flows through porous media. If we consider a porous medium comprised of a bundle of capillary tubes, then we come to our problem. The disadvantage is that a bundle of capillary tubes represents a geometrically oversimplified model of a porous medium. Nevertheless, there is considerable insight to be gained from the analysis of our toy problem.

Our technique is strongly motivated by the paper [14] by J. Rubinstein and R. Mauri, where effective dispersion and convection in porous media is studied, using the homogenization technique. Their analysis is based on the hierarchy of time scales and in getting the dimensionless equations we follow their approach. In our knowledge the only rigorous result concerning the effective dispersion in porous media, in the presence of high Peclet's numbers (and no chemistry), is in the recent paper [2] by A. Bourgeat, M. Jurak and A.L. Piatnitski. Nevertheless, their approach is based on the regular solutions for compatible data for the underlying linear transport equation. They suppose a high order compatibility between the initial and boundary data, involving derivatives up to order five, construct a smooth solution to the linear transport equation and then add the appropriate boundary layer, initial layer and the correction due to the perturbation of the mean flow. The effective solution obtained on this way is an H^1-approximation of order ε and an L^2-approximation of order ε^2. Nevertheless, in problems involving chemistry, it is important to have a jump between the initial values of the concentration and the values imposed at the injection boundary $x = 0$. This is the situation from [15] and simply the compatibility of the data isn't acceptable for the reactive transport.

Homogenization of a problem with dissolution/precipitation at the grain boundaries in porous media, for small Peclet's number, ($\alpha = 0$) is in [3].

For the bounds on convection enhanced diffusion in porous media we refer to papers by Fannjiang, Papanicolaou, Zhikov, Kozlov, Piatnitskii ....

Plan of the paper is the following : In the section 2 we study the homogenized problem. It turns out that it has an explicit solution having the form of moving Gaussian as the 1D boundary layers of parabolic equations, when viscosity goes to zero (see [6]). Its behavior with respect to ε and t is very singular.

Then in section 3 we give a justification of a lower order approximation, using a simple energy argument. In fact such approximation doesn't use Taylor's dispersion formula and gives an error of the same order in $L^\infty(L^2)$ as the solution to the linear transport equation. Furthermore, when $\alpha > 4/3$ this approach doesn't give an approximation any more!

In the section 4 we give a formal derivation of the upscaled problem (EFF), using the approach from [14].

Construction of the spatial boundary layer taking care of the injection boundary is in Section 5.

Then in sections 6 and 7 we prove that the effective concentration satisfying the corresponding 1D parabolic problem, with Taylor's diffusion coefficient and the reactive correction, is an approximation in $L^\infty(L^2)$ and in $L^\infty(L^\infty)$ for the physical concentration.

To satisfy the curiosity of the reader not familiar with the singular perturbation techniques, we give here the simplified version of the results stated in Theorems 21, 22 and 23 in Section 7. For simplicity, we compare only the physical concentration c^ε with c^{Mau}. Keeping the correction terms is necessary in order to have the precision reached in Theorems 21, 22 and 23, Section 7. Throughout the paper $H(x)$ is Heaviside's function.

Theorem 1. *Let c^{Mau} be given by (EFF). Then we have*

$$\|t^3(c^\varepsilon - c^{Mau})\|_{L^\infty((0,T)\times Z^+)} \leq \begin{cases} C\varepsilon^{2-3\alpha/2}, & \textit{if } \alpha < 1, \\ C\varepsilon^{3/2-\alpha-\delta}, \ \forall \delta > 0, & \textit{if } 2 > \alpha \geq 1. \end{cases} \tag{12}$$

$$\|t^3\big(c^\varepsilon - c^{Mau}\big)\|_{L^2(0,T;L^1_{loc}(Z^+))} \leq C\varepsilon^{2-\alpha} \tag{13}$$

$$\|t^3\big(c^\varepsilon - c^{Mau}\big)\|_{L^2(0,T;L^2_{loc}(Z^+)} \leq C\big(\varepsilon^{2-5\alpha/4}H(1-\alpha) + \varepsilon^{3/2-3\alpha/4}H(\alpha-1)\big) \tag{14}$$

$$\|t^3\partial_y c^\varepsilon\|_{L^2(0,T;L^2_{loc}(Z^+))} \leq C\big(\varepsilon^{2-5\alpha/4}H(1-\alpha) + \varepsilon^{3/2-3\alpha/4}H(\alpha-1)\big) \tag{15}$$

$$\|t^3\partial_x\big(c^\varepsilon - c^{Mau}\big)\|_{L^2(0,T;L^2_{loc}(Z^+))} \le C\big(\varepsilon^{2-7\alpha/4}H(1-\alpha) + \varepsilon^{3/2-5\alpha/4}H(\alpha-1)\big) \tag{16}$$

Corollary 2. *In the conditions of Taylor's article [15], $\alpha = 1$ and $k_0 = 0$, we have*

$$\|t^3(c^\varepsilon - c^{Mau})\|_{L^\infty((0,T)\times Z^+)} \le C\varepsilon^{1/2-\delta}, \quad \forall\delta > 0, \tag{17}$$

$$\|t^3\big(c^\varepsilon - c^{Mau}\big)\|_{L^2(0,T;L^1_{loc}(Z^+))} \le C\varepsilon \tag{18}$$

Our result could be restated in dimensional form:

Theorem 3. *Let us suppose that $L_R > \max\{D_R/Q_R, Q_RH^2/D_R, H\}$. Then the upscaled dimensional approximation for (1) reads*

$$\frac{\partial c^{*,eff}}{\partial t^*} + \big(\frac{2}{3} + \frac{4}{45}\mathbf{Da}_T\big)Q^*\frac{\partial c^{*,eff}}{\partial x^*} + \frac{k^*}{H}\big(1 - \frac{1}{3}\mathbf{Da}_T\big)c^{*,eff} = D^*\Big(1 + \frac{8}{945}\mathbf{Pe}_T^2\Big)\frac{\partial^2 c^{*,eff}}{\partial (x^*)^2}, \tag{19}$$

where $\mathbf{Pe}_T = \dfrac{Q^*H}{D^*}$ is the transversal Peclet's number and $\mathbf{Da}_T = \dfrac{k^*H}{D^*}$ is the transversal Damkohler's number.

Finally, let us note that in the known literature on boundary layers for parabolic regularization, the transport velocity is supposed to be zero at the injection boundary (see [5]) and our result doesn't enter into existing framework.

One could try to get even higher order approximations. Unfortunately, our procedure from Section 4 then leads to higher order differential operators and it is not clear if they are easy to handle. In the absence of the boundaries, determination of the higher order terms using the computer program REDUCE was undertaken in [9].

2 Study of the the upscaled diffusion-convection equation on the half-line

For $\bar{Q}, \bar{D}, \varepsilon > 0$ and $\bar{k} \ge 0$, we consider the problem

$$\begin{cases} \partial_t u + \bar{Q}\partial_x u + \bar{k}u = \gamma\bar{D}\partial_{xx}u \ \text{ in } I\!R_+ \times (0,T), & \partial_x u \in L^2(I\!R_+ \times (0,T)) \\ u(x,0) = 1 \ \text{ in } \ I\!R_+, & u(0,t) = 0 \ \text{ at } \ x = 0. \end{cases} \tag{20}$$

The unique solution is given by the following explicit formula

$$u(x,t) = e^{-\bar{k}t}\Big(1 - \frac{1}{\sqrt{\pi}}\left[e^{\frac{\bar{Q}x}{\gamma\bar{D}}} \int_{\frac{x+t\bar{Q}}{2\sqrt{\gamma\bar{D}t}}}^{+\infty} e^{-\eta^2}\ d\eta + \int_{\frac{x-t\bar{Q}}{2\sqrt{\gamma\bar{D}t}}}^{+\infty} e^{-\eta^2}\ d\eta \right]\Big) \tag{21}$$

The explicit formula allows us to find the exact behavior of u with respect to γ. We note that for $\alpha \in [0,1]$, we will set $\gamma = \varepsilon^{\alpha}$. For $\alpha \in [1,2)$, we choose $\gamma = \varepsilon^{2-\alpha}$. The derivatives of u are found using the computer program MAPLE and then their norms are estimated. Since the procedure is standard, we don't give the details. In more general situations there are no explicit solutions and these estimates could be obtained using the technique and results from [6].

1. STEP First, by the maximum principle we have

$$0 \le u(x,t;\gamma) = u(x,t) \le 1 \tag{22}$$

2. STEP Next we estimate the difference between $\chi_{x<\bar{Q}t}$ and u. We have

Lemma 1. *Function u satisfies the estimates*

$$\int_0^{\infty} |e^{-\bar{k}t}\chi_{\{x>\bar{Q}t\}} - u(t,x)|\ dx = 3\sqrt{\gamma\bar{D}t}e^{-\bar{k}t} + C\gamma \tag{23}$$

$$\|e^{-\bar{k}t}\chi_{\{x>\bar{Q}t\}} - u\|_{L^\infty(0,T;L^p(I\!R_+))} \le C\gamma^{1/(2p)}, \quad \forall p \in (1,+\infty). \tag{24}$$

3. STEP For the derivatives of u the following estimates hold

Lemma 2. *Let ζ be defined by*

$$\zeta(t) = \begin{cases} \big(\dfrac{t}{\bar{D}\gamma}\big)^r & \textit{for } 0 \le t \le \bar{D}\gamma, \\ 1 & \textit{otherwise,} \end{cases} \tag{25}$$

with $r \ge q \ge 1$. Then we have

$$\|\zeta(t)\partial_t u\|_{L^q((0,T)\times I\!R_+)} + \|\zeta(t)\partial_x u\|_{L^q((0,T)\times I\!R_+)} \le C(\gamma\bar{D})^{\min\{1/(2q)-1/2,2/q-1\}},\ q \neq 3 \tag{26}$$

$$\|\zeta(t)\partial_t u\|_{L^3((0,T)\times I\!R_+)} + \|\zeta(t)\partial_x u\|_{L^3((0,T)\times I\!R_+)} \le C\big((\gamma\bar{D})^{-1}\log(\frac{1}{\gamma\bar{D}})\big)^{1/3} \tag{27}$$

4. STEP

Now we estimate the second derivatives :

Lemma 3. *Let ζ be defined by (25). Then the second derivatives of u satisfy the estimates*

$$\begin{aligned} \|\zeta(t)u_{tt}\|_{L^q((0,T)\times I\!\!R_+)} + \|\zeta(t)u_{tx}\|_{L^q((0,T)\times I\!\!R_+)} + \|\zeta(t)u_{xx}\|_{L^q((0,T)\times I\!\!R_+)} \\ \leq C_q(\gamma\bar{D})^{\min\{1/(2q)-1,2/q-2\}},\ q\neq 3/2 \qquad (28)\end{aligned}$$

$$\begin{aligned} \|\zeta(t)u_{tt}\|_{L^{3/2}((0,T)\times I\!\!R_+)} + \|\zeta(t)u_{tx}\|_{L^{3/2}((0,T)\times I\!\!R_+)} + \|\zeta(t)u_{xx}\|_{L^{3/2}((0,T)\times I\!\!R_+)} \\ \leq C\big((\gamma\bar{D})^{-1}\log(\frac{1}{\gamma\bar{D}})\big)^{2/3} \qquad (29)\end{aligned}$$

5. STEP

For the 3rd order derivatives we have :

Lemma 4. *Let ζ be defined by (25) . Then*

$$\begin{aligned} \|\partial_{xxx}(\zeta(t)u)\|_{L^q((0,T)\times I\!\!R_+)} + \|\zeta(t)\partial_{xxt}u\|_{L^q((0,T)\times I\!\!R_+)} \\ +\|\zeta(t)\partial_{xtt}u\|_{L^q((0,T)\times I\!\!R_+)} \leq C_q(\gamma\bar{D})^{2/q-3},\ q>1 \qquad (30)\end{aligned}$$

$$\begin{aligned} \|\partial_{xxx}(\zeta(t)u)\|_{L^1((0,T)\times I\!\!R_+)} + \|\zeta(t)\partial_{xxt}u\|_{L^1((0,T)\times I\!\!R_+)} \\ +\|\zeta(t)\partial_{xtt}u\|_{L^1((0,T)\times I\!\!R_+)} \leq C_1(\gamma\bar{D})^{-1}\log\frac{1}{\gamma\bar{D}} \qquad (31)\end{aligned}$$

3 A simple L^2 error estimate

The simplest way to average the problem (7)-(11) is to take the mean value with respect to y. Supposing that the mean of the product is the product of the means, which is in general wrong, we get the following problem for the " averaged " concentration $c_0^{eff}(x,t)$:

$$\left\{\begin{array}{l} \dfrac{\partial c_0^{eff}}{\partial t} + \dfrac{2Q}{3}\dfrac{\partial c_0^{eff}}{\partial x} + k_0 c_0^{eff} = \varepsilon^\alpha D\dfrac{\partial c_0^{eff}}{\partial x^2}\ \text{ in } I\!\!R_+\times(0,T), \\ \\ \partial_x c_0^{eff}\in L^2(I\!\!R_+\times(0,T)),\quad c_0^{eff}|_{t=0}=1,\quad c_0^{eff}|_{x=0}=0. \end{array}\right. \qquad (32)$$

This is the problem (20) with $\tilde{Q} = \dfrac{2}{3}Q$, $\bar{k} = k_0$ and $\bar{D} = D$. The small parameter γ is equal to ε^α. Let the operator $\mathcal{L}^\varepsilon$ be given by

$$\mathcal{L}^\varepsilon\zeta = \frac{\partial\zeta}{\partial t} + Q(1-y^2)\frac{\partial\zeta}{\partial x} - D\varepsilon^\alpha\Big(\partial_{xx}\zeta + \varepsilon^{-2}\partial_{yy}\zeta\Big) \qquad (33)$$

The non-dimensional physical concentration c^ε satisfies (7)-(11), i.e.

$$\mathcal{L}^\varepsilon c^\varepsilon = 0 \quad \text{in } \mathbb{R}_+ \times (0,1) \times (0,T) \tag{34}$$
$$c^\varepsilon(0,y,t) = 0 \quad \text{on } (0,1) \times (0,T) \tag{35}$$
$$\partial_y c^\varepsilon(x,0,t) = 0 \quad \text{on } \mathbb{R}_+ \times (0,T) \tag{36}$$
$$-D\varepsilon^{\alpha-2}\partial_y c^\varepsilon(x,1,t) = k_0 c^\varepsilon(x,1,t) \quad \text{on } \mathbb{R}_+ \times (0,T) \tag{37}$$
$$c^\varepsilon(x,y,0) = 1 \quad \text{on } \mathbb{R}_+ \times (0,1) \tag{38}$$

We want to approximate c^ε by c_0^{eff}. Then

$$\mathcal{L}^\varepsilon(c_0^{eff}) = -k_0 c_0^{eff} + Q\partial_x c_0^{eff}(1/3 - y^2) = R^\varepsilon$$
$$\mathcal{L}^\varepsilon(c^\varepsilon - c_0^{eff}) = -R^\varepsilon \quad \text{in } \mathbb{R}_+ \times (0,1) \times (0,T) \quad \text{and} \tag{39}$$
$$-D\varepsilon^{\alpha-2}\partial_y(c^\varepsilon(x,1,t) - c_0^{eff}) = k_0 c^\varepsilon(x,1,t) \quad \text{on } \mathbb{R}_+ \times (0,T) \tag{40}$$

Let $\Psi(x) = 1/(x+1)$. Then $(\partial_x \Psi^2)^2/\Psi^2 \leq 4\Psi^2$. We have the following proposition, which will be useful in getting the estimates :

Proposition 4. *Let g^ε, ξ_0^ε and R^ε be such that $\Psi g^\varepsilon \in H^1(Z^+ \times (0,T))$, $\Psi \xi_0^\varepsilon \in L^2(Z^+)$ and $\Psi R^\varepsilon \in L^2(Z^+ \times (0,T))$. Let ξ, $\Psi\xi \in C([0,T]; L^2(Z^+))$, $\Psi \nabla_{x,y}\xi \in L^2(Z^+ \times (0,T))$, be a bounded function which satisfies the system*

$$\mathcal{L}^\varepsilon(\xi) = -R^\varepsilon \quad \text{in } Z^+ \times (0,T) \tag{41}$$
$$-D\varepsilon^{\alpha-2}\partial_y \xi|_{y=1} = k_0 \xi|_{y=1} + g^\varepsilon|_{y=1} \text{ and } \partial_y \xi|_{y=0} = 0 \text{ on } \mathbb{R}_+ \times (0,T) \tag{42}$$
$$\xi|_{t=0} = \xi_0^\varepsilon \quad \text{on } Z^+ \quad \text{and } \xi|_{x=0} = 0 \text{ on } (0,1) \times (0,T). \tag{43}$$

Then we have the following energy estimate

$$\mathcal{E}(\xi,t) = \frac{1}{2}\int_{Z^+} \Psi(x)^2 \xi^2(t)\, dxdy + \frac{D}{2}\varepsilon^\alpha \int_0^t \int_{Z^+} \Psi(x)^2 \Big\{ \varepsilon^{-2}|\partial_y \xi|^2 +$$
$$|\partial_x \xi|^2 \Big\}\, dxdyd\tau + k_0 \int_0^t \int_{\mathbb{R}_+} \xi^2|_{y=1} \Psi^2(x)\, dxd\tau \leq -\int_0^t \int_{Z^+} \Psi(x)^2 R^\varepsilon \xi\, dxdyd\tau -$$
$$\int_0^t \int_{\mathbb{R}_+} g^\varepsilon|_{y=1} \xi|_{y=1} \Psi^2(x)\, dxd\tau + 2D\varepsilon^\alpha \int_0^t \int_{Z^+} \Psi(x)^2 \xi^2\, dxdyd\tau. \tag{44}$$

Proof. We test (41)-(43) by $\Psi^2(x)\xi$ and get

$$\frac{1}{2}\int_{Z^+}\xi^2(t)\Psi^2(x)\ dxdy + D\varepsilon^\alpha\int_0^t\int_{Z^+}\Psi^2(x)\Big\{\varepsilon^{-2}|\partial_y\xi|^2+|\partial_x\xi|^2\Big\}\ dxdyd\tau +$$
$$k_0\int_0^t\int_0^{+\infty}\xi^2|_{y=1}\Psi^2\ dxd\tau \le \frac{1}{2}\int_{Z^+}(\xi_0^\varepsilon)^2\Psi^2(x)\ dxdy -$$
$$k_0\int_0^t\int_0^{+\infty}(g^\varepsilon\xi)|_{y=1}\Psi^2\ dxd\tau - D\varepsilon^\alpha\int_0^t\int_{Z^+}\partial_x\xi\xi\partial_x\Psi^2(x)\ dxdyd\tau. \tag{45}$$

Next, we use that

$$D\varepsilon^\alpha\int_0^t\int_{Z^+}\partial_x\xi\xi\partial_x\Psi^2(x)\ dxd\tau \le \frac{D}{2}\varepsilon^\alpha\int_0^t\int_{Z^+}\Psi^2(x)|\partial_x\xi|^2\ dxdyd\tau$$
$$+2D\varepsilon^\alpha\int_0^t\int_{Z^+}\Psi^2(x)|\xi|^2\ dxdyd\tau \tag{46}$$

and get (44). □

This simple proposition allows us to prove

Proposition 5. *In the setting of this section we have*

$$\|\Psi(x)(c^\varepsilon - c_0^{eff})\|_{L^\infty(0,T;L^2(I\!R_+\times(0,1))}\le \varepsilon^{1-\alpha/2}\frac{F^0}{\sqrt{D}} \tag{47}$$
$$\|\Psi(x)\partial_x(c^\varepsilon - c_0^{eff})\|_{L^2(0,T;L^2(I\!R_+\times(0,1))}\le \varepsilon^{1-\alpha}\frac{F^0}{D} \tag{48}$$
$$\|\Psi(x)\partial_y(c^\varepsilon - c_0^{eff})\|_{L^2(0,T;L^2(I\!R_+\times(0,1))}\le \varepsilon^{2-\alpha}\frac{F^0}{D}, \tag{49}$$
$$\textit{where}\quad F^0 = C_1^F\|\partial_x c_0^{eff}\|_{L^2(O_T)} + C_2^F k_0 \le C_3^F\varepsilon^{-\alpha/4} \tag{50}$$

Proof. We are in the situation of Proposition 4 with $\xi_0^\varepsilon = 0$ and $g^\varepsilon = k_0 c_0^{eff}$. Consequently, for $\xi = c^\varepsilon - c_0^{eff}$ we have

$$\mathcal{E}(\xi,t)\le k_0\int_0^t\int_0^{+\infty}c_0^{eff}\Big(\int_0^1 c^\varepsilon\ dy - c^\varepsilon|_{y=1}\Big)\Psi^2\ dxd\tau + 2D\varepsilon^\alpha\cdot$$
$$\int_0^t\int_{Z^+}|\xi|^2\Psi^2(x)\ dxdyd\tau - \int_0^t\int_{Z^+}Q(1/3-y^2)\xi\partial_x c_0^{eff}\Psi^2\ dxdyd\tau. \tag{51}$$

It remains to estimate the first and the third term at the right hand side of (51). We have

$$
\begin{aligned}
&| \int_0^t \int_{Z^+} Q \partial_x c_0^{eff} (1/3 - y^2) \xi \Psi^2(x) \; dxdyd\tau | = \\
&| \int_0^t \int_{Z^+} Q \partial_x c_0^{eff} (y/3 - y^3/3) \partial_y \xi \Psi^2(x) \; dxdyd\tau | \qquad (52)
\end{aligned}
$$

$$
\begin{aligned}
\text{and } \; k_0 | \int_0^t \int_0^{+\infty} c_0^{eff} (\int_0^1 \xi \; dy - \xi|_{y=1}) \Psi^2 \; dxd\tau | \leq \frac{D}{8} \varepsilon^{\alpha} \int_0^t \int_{Z^+} \\
\Psi^2(x) |\partial_y \xi|^2 \; dxdyd\tau + \frac{k_0^2}{D} \varepsilon^{2-\alpha} \int_0^t \int_0^{+\infty} (c_0^{eff})^2 \Psi^2 \; dxd\tau. \qquad (53)
\end{aligned}
$$

After inserting (52)-(53) into (51) we get

$$
\begin{aligned}
\mathcal{E}(c^{\varepsilon} - c_0^{eff}, t) \leq \varepsilon^{2-\alpha} \int_0^t \int_0^{+\infty} \Big\{ \frac{2k_0^2}{D} (c_0^{eff})^2 + \frac{32}{315} \frac{Q^2}{D} (\partial_x c_0^{eff})^2 \Big\} \Psi^2 \; dxd\tau \\
+ \int_0^t \int_0^1 \int_0^{+\infty} 2D \varepsilon^{\alpha} \Psi^2(x) (c^{\varepsilon} - c_0^{eff})^2 \; dxdyd\tau, \qquad (54)
\end{aligned}
$$

and after applying Gronwall's inequality, we obtain (47)-(49). □

Corollary 6.

$$
\| c^{\varepsilon} - c_0^{eff} \|_{L^{\infty}(0,T; L^2_{loc}(\mathbb{R}_+ \times (0,1)))} \leq C \varepsilon^{1-3\alpha/4} \qquad (55)
$$

Remark 7. *It is reasonable to expect some L^1 estimates with better powers for ε. Unfortunately, testing the equation (39) by the regularized sign ($c^{\varepsilon} - c_0^{eff}$), doesn't lead to anything useful. Hence at this stage claiming a $\sqrt{\varepsilon}$ estimate in L^1 is not justified.*

Remark 8. *There are recent papers by Grenier and Gues on singular perturbation problems. In [5] Grenier supposes that Q is zero as x at $x = 0$, together with its derivatives. Such hypothesis allows better estimates.*

Remark 9. *The estimate (23) implies that $\exp\{-k_0 t\} \chi_{\{x > Qt\}}$ is an approximation for the physical concentration which is of the same order in $L^{\infty}(L^2)$ as c_0^{eff}.*

Remark 10. *For $\alpha > 4/3$ the estimate (55) is without any value.*

4 The formal 2-scale expansion leading to Taylor's dispersion

The estimate obtained in the previous section isn't satisfactory. At the other hand, it is known that the Taylor dispersion model gives a very good 1D approximation. With this motivation we briefly explain how to obtain formally the higher precision effective models and notably the variant of Taylor's dispersion formula, by the 2-scale asymptotic expansion.

We start with the problem (34)-(38) and search for c^ε in the form

$$c^\varepsilon = c^0(x,t;\varepsilon) + \varepsilon c^1(x,y,t) + \varepsilon^2 c^2(x,y,t) + \dots \tag{56}$$

After introducing (56) into the equation (34) we get

$$\varepsilon^0 \Big\{ \partial_t c^0 + Q(1-y^2)\partial_x c^0 - D\varepsilon^{\alpha-1}\partial_{yy} c^1 \Big\} + \varepsilon \Big\{ \partial_t c^1 + Q(1-y^2)\partial_x c^1 - D\varepsilon^{\alpha-1}\partial_{xx} c^0 - D\varepsilon^{\alpha-1}\partial_{yy} c^2 \Big\} = \mathcal{O}(\varepsilon^2) \tag{57}$$

In order to have (57) for every $\varepsilon \in (0, \varepsilon_0)$, all coefficients in front of the powers of ε should be zero.

The problem corresponding to the order ε^0 is

$$\begin{cases} -D\partial_{yy} c^1 = -\varepsilon^{1-\alpha} Q(1/3 - y^2)\partial_x c^0 - \varepsilon^{1-\alpha}\big(\partial_t c^0 + 2Q\partial_x c^0/3\big) \ \text{ on } \ (0,1), \\ \partial_y c^1 = 0 \text{ on } \ y=0 \ \text{ and } \ -D\partial_y c^1 = k_0 \varepsilon^{1-\alpha} c^0 \text{ on } \ y=1 \end{cases} \tag{58}$$

for every $(x,t) \in (0,+\infty) \times (0,T)$. By the Fredholm's alternative, the problem (58) has a solution if and only if

$$\partial_t c^0 + 2Q\partial_x c^0/3 + k_0 c^0 = 0 \quad \text{in } \ (0,L) \times (0,T). \tag{59}$$

Unfortunately our initial and boundary data are incompatible and the hyperbolic equation (59) has a discontinuous solution. Since the asymptotic expansion for c^ε involves derivatives of c^0, the equation (59) doesn't suit our needs. In [2] the difficulty was overcome by supposing compatible initial and boundary data. We proceed by following an idea from [14] and suppose that

$$\partial_t c^0 + 2Q\partial_x c^0/3 + k_0 c^0 = \mathcal{O}(\varepsilon) \quad \text{in } \ (0,+\infty) \times (0,T). \tag{60}$$

The hypothesis (60) will be justified *a posteriori*, after getting an equation for c^0.

Hence (58) reduces to

$$\begin{cases} -D\partial_{yy}c^1 = -\varepsilon^{1-\alpha}Q(1/3-y^2)\partial_x c^0 + \varepsilon^{1-\alpha}k_0 c^0 \text{ on } (0,1), \\ \partial_y c^1 = 0 \text{ on } y=0 \text{ and } -D\partial_y c^1 = k_0\varepsilon^{1-\alpha}c^0 \text{ on } y=1 \end{cases} \tag{61}$$

for every $(x,t) \in (0,+\infty)\times(0,T)$, and we have

$$c^1(x,y,t) = \varepsilon^{1-\alpha}\Big(\frac{Q}{D}(\frac{y^2}{6}-\frac{y^4}{12})\partial_x c^0 + \frac{k_0}{D}(\frac{1}{6}-\frac{y^2}{2})c^0 + C_0(x,t)\Big), \tag{62}$$

where C_0 is an arbitrary function.

Let us go to the next order. Then we have

$$\begin{cases} -D\partial_{yy}c^2 = -\varepsilon^{1-\alpha}Q(1-y^2)\partial_x c^1 + D\partial_{xx}c^0 - \varepsilon^{1-\alpha}\partial_t c^1 + D\varepsilon\partial_{xx}c^1 \\ -\varepsilon^{-\alpha}\big(\partial_t c^0 + 2Q\partial_x c^0/3 + k_0 c^0\big) \quad \text{on } (0,1), \\ \partial_y c^2 = 0 \text{ on } y=0 \text{ and } -D\partial_y c^2 = k_0\varepsilon^{1-\alpha}c^1 \text{ on } y=1 \end{cases} \tag{63}$$

for every $(x,t) \in (0,+\infty)\times(0,T)$. The problem (63) has a solution if and only if

$$\partial_t c^0 + 2Q\partial_x c^0/3 + k_0(c^0 + \varepsilon c^1|_{y=1}) + \varepsilon\partial_t(\int_0^1 c^1\, dy) - \varepsilon^\alpha D\partial_{xx}c^0 + \\ \varepsilon\partial_x(\int_0^1 (1-y^2)c^1\, dy) = 0 \quad \text{in } (0,+\infty)\times(0,T). \tag{64}$$

(64) is the equation for c^0 . In order to get the simplest possible equation for c^0 we choose C_0 giving $\int_0^1 c^1\, dy = 0$. Now c^1 takes the form

$$c^1(x,y,t) = \varepsilon^{1-\alpha}\Big(\frac{Q}{D}(\frac{y^2}{6}-\frac{y^4}{12}-\frac{7}{180})\partial_x c^0 + \frac{k_0}{D}(\frac{1}{6}-\frac{y^2}{2})c^0\Big) \tag{65}$$

and the equation (64) becomes

$$\partial_t c^0 + Q\big(\frac{2}{3}+\frac{4k_0}{45D}\varepsilon^{2-\alpha}\big)\partial_x c^0 + k_0\big(1-\frac{k_0}{3D}\varepsilon^{2-\alpha}\big)c^0 = \varepsilon^\alpha \tilde{D}\partial_{xx}c^0 \quad \text{in } (0,+\infty)\times(0,T). \tag{66}$$

with

$$\tilde{D} = D + \frac{8}{945}\frac{Q^2}{D}\varepsilon^{2(1-\alpha)} \tag{67}$$

Now the problem (63) becomes

$$\begin{cases} -D\partial_{yy}c^2 = \varepsilon^{2-2\alpha}\Big\{ -\dfrac{Q^2}{D}\partial_{xx}c^0\big\{\dfrac{8}{945} + (1-y^2)(\dfrac{y^2}{6} - \dfrac{y^4}{12} \\ -\dfrac{7}{180})\big\} + \partial_x c^0 \dfrac{Qk_0}{D}\big\{\dfrac{2}{45} - (1-y^2)(\dfrac{1}{6} - \dfrac{y^2}{2})\big\} + \dfrac{2k_0Q}{45D}\partial_x c^0 - \\ \dfrac{k_0^2}{3D}c^0 - (\dfrac{y^2}{6} - \dfrac{y^4}{12} - \dfrac{7}{180})(\partial_{xt}c^0\dfrac{Q}{D} - \varepsilon^\alpha Q\partial_{xxx}c^0) - \\ (\dfrac{1}{6} - \dfrac{y^2}{2})(\partial_t c^0\dfrac{k_0}{D} - \varepsilon^\alpha k_0 \partial_{xx}c^0)\Big\} \text{ on } (0,1), \quad \partial_y c^2 = 0 \text{ on } y=0 \\ \text{and } \; -D\partial_y c^2 = \dfrac{Qk_0}{D}\varepsilon^{2-2\alpha}\partial_x c^0 \dfrac{2}{45} - \dfrac{k_0^2}{3D}\varepsilon^{2-2\alpha}c^0 \text{ on } y=1. \end{cases} \tag{68}$$

If we choose c^2 such that $\int_0^1 c^2 \, dy = 0$, then

$$c^2(x,y,t) = \varepsilon^{2-2\alpha}\Big\{ -\frac{Q^2}{D^2}\partial_{xx}c^0\Big(\frac{281}{453600} + \frac{23}{1512}y^2 - \frac{37}{2160}y^4 + \frac{1}{120}y^6 \\ -\frac{1}{672}y^8\Big) + (\frac{Q}{D^2}\partial_{xt}c^0 - \varepsilon^\alpha\frac{Q}{D}\partial_{xxx}c^0)\Big(\frac{31}{7560} - \frac{7}{360}y^2 + \frac{y^4}{72} - \frac{y^6}{360}\Big) + \\ \frac{Qk_0}{D^2}\partial_x c^0(\frac{y^6}{60} - \frac{y^4}{18} + \frac{11y^2}{180} - \frac{11}{810}) + (\frac{k_0}{2D^2}\partial_t c^0 - \\ \frac{k_0}{2D}\varepsilon^\alpha\partial_{xx}c^0)\big(-\frac{y^4}{12} + \frac{y^2}{6} - \frac{7}{180}\big) + \big(\frac{Qk_0}{45D^2}\partial_x c^0 - \frac{k_0^2}{6D^2}c^0\big)(\frac{1}{3} - y^2)\Big\} \tag{69}$$

5 Boundary layer

If we add corrections to c^0, the obtained function doesn't satisfy any more the boundary conditions. We correct the new values using the appropriate boundary layer.

Let $Z^+ = (0,+\infty)\times(0,1)$.

$$\begin{cases} -\Delta_{y,z}\beta = 0 \quad \text{for } (z,y)\in Z^+. \\ -\partial_y\beta = 0 \text{ for } y=1, \quad \text{and for } y=0, \\ \beta = \dfrac{y^2}{6} - \dfrac{y^4}{12} - \dfrac{7}{180} \text{ for } z=0. \end{cases} \tag{70}$$

Using the elementary variational theory for PDEs, we get the existence of a unique solution $\beta \in L^2_{loc}(Z^+)$ such that $\nabla\beta \in L^2_{loc}(Z^+)^2$. Next, we note that the average of the boundary value at $z=0$ is zero. This implies that

$\int_0^1 \beta(z, y)\ dy = 0$, for every $z \in (0, +\infty)$. Now we can apply the Poincaré's inequality in H^1:

$$\int_0^1 \varphi^2\ dy \leq \frac{1}{\pi^2} \int_0^1 |\partial_y \varphi|^2\ dy,\ \forall \varphi \in H^1(0,1),\ \int_0^1 \varphi\ dy = 0, \tag{71}$$

and conclude that in fact $\beta \in H^1(Z^+)$. In order to prove that β represents a boundary layer, one should prove the exponential decay. We apply the theory from [10] and get the following result describing the decay of β as $z \to +\infty$:

Proposition 11. *There exists a constant $\gamma_0 > 0$ such that the solution β for (70) satisfies the estimates*

$$\int_z^{+\infty} \int_0^1 |\nabla_{y,z} \beta|^2\ dydz \leq c_0 e^{-\gamma_0 z}, \quad z > 0 \tag{72}$$

$$|\beta(y, z)| \leq c_0 e^{-\gamma_0 z}, \quad \forall (y, z) \in Z^+ \tag{73}$$

6 First Correction

The estimate (55) isn't satisfactory. In order to get a better approximation we take the correction constructed using the formal 2-scale expansion in Section 4.

Let $0 \leq \alpha < 2$. We start by the $\mathcal{O}(\varepsilon^2)$ approximation and consider the function

$$c_1^{eff}(x, y, t; \varepsilon) = c^{Mau}(x, t; \varepsilon) + \varepsilon^{2-\alpha} \zeta(t) \Big(\frac{Q}{D} (\frac{y^2}{6} - \frac{y^4}{12} - \frac{7}{180}) \cdot \frac{\partial c^{Mau}}{\partial x} + \frac{k_0}{D} (\frac{1}{6} - \frac{y^2}{2}) c^{Mau}(x, t; \varepsilon) \Big) \tag{74}$$

where c^{Mau} is the solution to the effective problem with Taylor's dispersion coefficient and reaction terms:

$$\begin{cases} \partial_t c^{Mau} + Q\big(\frac{2}{3} + \frac{4k_0}{45D} \varepsilon^{2-\alpha}\big) \partial_x c^{Mau} + k_0 \big(1 - \frac{k_0}{3D} \varepsilon^{2-\alpha}\big) c^{Mau} = \\ (D\varepsilon^\alpha + \frac{8}{945} \frac{Q^2}{D} \varepsilon^{2-\alpha}) \partial_{xx} c^{Mau},\ \text{in}\ I\!R_+ \times (0, T) \\ c^{Mau}|_{x=0} = 0,\ c^{Mau}|_{t=0} = 1,\ \partial_x c^{Mau} \in L^2(I\!R_+ \times (0, T)), \end{cases} \tag{75}$$

$\tilde{D} = D\varepsilon^{\alpha} + \frac{8}{945}\frac{Q^2}{D}\varepsilon^{2-\alpha}$ is Taylor's dispersion coefficient. The cut-off in time ζ is given by (25) and we use to eliminate the time-like boundary layer appearing at $t = 0$. These effects are not visible in the formal expansion.

Let $\mathcal{L}^{\varepsilon}$ be the differential operator given by (33). Following the formal expansion from Section 4, we know that $\mathcal{L}^{\varepsilon}$ applied to the correction without boundary layer functions and cut-offs would give $F_1^{\varepsilon} + F_2^{\varepsilon} + F_3^{\varepsilon} + F_4^{\varepsilon} + F_5^{\varepsilon}$, where

$$\begin{cases} F_1^{\varepsilon} = \partial_{xx} c^{Mau} \frac{Q^2}{D}\varepsilon^{2-\alpha}\Big\{\frac{8}{945} + (1-y^2)(\frac{y^2}{6} - \frac{y^4}{12} - \frac{7}{180})\Big\} \\ F_2^{\varepsilon} = \partial_{x} c^{Mau} \frac{Qk_0}{D}\varepsilon^{2-\alpha}\Big\{-\frac{2}{45} + (1-y^2)(\frac{1}{6} - \frac{y^2}{2})\Big\} \\ F_3^{\varepsilon} = \varepsilon^{2-\alpha}(\frac{y^2}{6} - \frac{y^4}{12} - \frac{7}{180})\big\{\partial_{xt} c^{Mau}\frac{Q}{D} - \varepsilon^{\alpha}\partial_{xxx}c^{Mau} Q\big\} \\ F_4^{\varepsilon} = \varepsilon^{2-\alpha}(\frac{1}{6} - \frac{y^2}{2})\big\{\partial_{t} c^{Mau}\frac{k_0}{D} - \varepsilon^{\alpha}\partial_{xx}c^{Mau} k_0\big\} \\ F_5^{\varepsilon} = \varepsilon^{2-\alpha}\{-\frac{2}{45}\partial_x c^{Mau}\frac{Qk_0}{D} + \frac{k_0^2}{3D}c^{Mau}\} \end{cases} \tag{76}$$

These functions aren't integrable up to $t = 0$ and we need a cut off ζ in order to deal with them.

After applying $\mathcal{L}^{\varepsilon}$ to c_1^{eff}, we find out that

$$\mathcal{L}^{\varepsilon}(c_1^{eff}) = \zeta(t)\sum_{j=1}^{5} F_j^{\varepsilon} + \Bigg(\varepsilon^{2-\alpha}\partial_{xx}c^{Mau}\frac{Q^2}{D}\frac{8}{945} + Q(1/3 - y^2)\partial_x c^{Mau} - k_0 c^{Mau}\Bigg)(1-\zeta(t)) + \zeta'(t)\varepsilon^{2-\alpha}\Bigg(\partial_x c^{Mau}\frac{Q}{D}\{\frac{y^2}{6} - \frac{y^4}{12} - \frac{7}{180}\} + \frac{k_0}{2D}(\frac{1}{3} - y^2)c^{Mau}\Bigg) \equiv \Phi_1^{\varepsilon} \text{ and } -\mathcal{L}^{\varepsilon}(c_1^{eff}) = \mathcal{L}^{\varepsilon}(c^{\varepsilon} - c_1^{eff}) = -\Phi_1^{\varepsilon} \tag{77}$$

At the lateral boundary $y = 1$ we have

$$-D\varepsilon^{\alpha-2}\partial_y c_1^{eff}|_{y=1} = \zeta(t)k_0 c^{Mau} \tag{78}$$

$$k_0 c_1^{eff}|_{y=1} = k_0\big(c^{Mau} + \varepsilon^{2-\alpha}\frac{Q}{D}\zeta(t)\frac{2}{45}\partial_x c^{Mau} - \varepsilon^{2-\alpha}\frac{k_0}{3D}c^{Mau}\zeta(t)\big) \tag{79}$$

Now $c^\varepsilon - c_1^{eff}$ satisfies the system

$$\mathcal{L}^\varepsilon(c^\varepsilon - c_1^{eff}) = -\Phi_1^\varepsilon \ \text{ in } \ Z^+ \times (0,T) \tag{80}$$
$$-D\varepsilon^{\alpha-2}\partial_y(c^\varepsilon - c_1^{eff})|_{y=1} = k_0(c^\varepsilon - c_1^{eff})|_{y=1} + g^\varepsilon|_{y=1} \ \text{ on } \ I\!R_+ \times (0,T) \tag{81}$$
$$\partial_y(c^\varepsilon - c_1^{eff})|_{y=0} = 0 \ \text{ on } \ I\!R_+ \times (0,T) \tag{82}$$
$$(c^\varepsilon - c_1^{eff})|_{t=0} = 0 \ \text{ on } \ Z^+ \text{ and } (c^\varepsilon - c_1^{eff})|_{x=0} = \eta_0^\varepsilon \text{ on } \ (0,1)\times(0,T). \tag{83}$$

with

$$g^\varepsilon = k_0\zeta(t)\varepsilon^{2-\alpha}\big(\partial_x c^{Mau}\frac{2Q}{45D} - c^{Mau}\frac{k_0}{3D}\big) + (1-\zeta)k_0 c^{Mau} \tag{84}$$
$$\text{and} \quad \eta_0^\varepsilon = -\varepsilon^{2-\alpha}\zeta(t)\partial_x c^{Mau}|_{x=0}(\frac{y^2}{6} - \frac{y^4}{12} - \frac{7}{180})\frac{Q}{D}. \tag{85}$$

Now we should estimate Φ_1^ε to see if the right hand side is smaller than in Section 3. We have

Proposition 12. *Let $O_T = I\!R_+ \times (0,1) \times (0,T)$. Let $\varphi \in H^1(O_T)$, $\varphi = 0$ at $x = 0$. Then we have*

$$|\int_0^t \int_{Z^+} \zeta F_1^\varepsilon \varphi \ dxdyd\tau| \leq C\varepsilon^{3(2-\alpha)/2}\|\zeta(\tau)\partial_{xx}c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}$$
$$\leq C\big(\varepsilon^{3-5\alpha/2}H(1-\alpha) + \varepsilon^{1-\alpha/2}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{86}$$
$$|\int_0^t \int_{Z^+} \zeta(\tau) F_3^\varepsilon \varphi \ dxdyd\tau| \leq C\varepsilon^{3(2-\alpha)/2}\bigg(\|\zeta(\tau)\partial_{xt}c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))} +$$
$$\|\zeta(\tau)\partial_{xx}c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))}\bigg) \cdot \|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \leq$$
$$C\big(\varepsilon^{3-5\alpha/2}H(1-\alpha) + \varepsilon^{1-\alpha/2}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{87}$$

$$|\int_0^t \int_{Z^+} (1-\zeta)\partial_{xx}c^{Mau}\varepsilon^{2-\alpha}\frac{Q^2}{D}\varphi \ dxdyd\tau| \leq C\varepsilon^{2-3\alpha/2}\|\varepsilon^{\alpha/2}\partial_x\varphi\|_{L^2(O_t)}\cdot$$
$$\|(1-\zeta)\partial_x c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))} \leq C\varepsilon^{2-3\alpha/2}\|\varepsilon^{\alpha/2}\partial_x\varphi\|_{L^2(O_t)} \tag{88}$$
$$|\int_0^t \int_{Z^+} (1-\zeta)Q(1/3-y^2)\partial_x c^{Mau}\varphi \ dxdyd\tau| \leq C\varepsilon^{1-\alpha/2}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}\cdot$$
$$\|(1-\zeta)\partial_x c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))} \leq C\varepsilon^{1-\alpha/2}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{89}$$

$$|\int_0^t\int_{Z^+}\zeta'(\frac{t}{D\varepsilon})\varepsilon^{2-\alpha}\{\partial_x c^{Mau}\frac{Q}{D}\{\frac{y^2}{6}-\frac{y^4}{12}-\frac{7}{180}\}-\frac{k_0}{2D}(\frac{1}{3}-y^2)c^{Mau}\}\cdot$$
$$\varphi\ dxdyd\tau|\le C\varepsilon^{3-3\alpha/2}\|\zeta'\partial_x c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}\le$$
$$C\big(\varepsilon^{3-5\alpha/2}H(1-\alpha)+\varepsilon^{1-\alpha/2}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \qquad (90)$$

Proof. Let us note that in (86)-(87) and (89)-(90) the averages of the polynomials in y are zero. We write them in the form $P(y)=\partial_y P_1(y)$, where P_1 has zero traces at $y=0,1$, and after partial integration and applying the results from Section 2, giving us the precise regularity, obtain the estimates. Since $(1-\zeta)\partial_{xx}c^{Mau}$ isn't square integrable, we use the x-derivative in order to obtain (88). □

Proposition 13. *Let $O_T=I\!R_+\times(0,1)\times(0,T)$. Let $\varphi\in H^1(O_T)$, $\varphi=0$ at $x=0$. Then we have*

$$|\int_0^t\int_{Z^+}\zeta F_2^\varepsilon\varphi\ dxdyd\tau|\le C\varepsilon^{3(1-\alpha/2)}\|\zeta\partial_x c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}$$
$$\le C\big(\varepsilon^{3-7\alpha/4}H(1-\alpha)+\varepsilon^{5/2-5\alpha/4}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \qquad (91)$$

$$|\int_0^t\int_{Z^+}\zeta F_4^\varepsilon\varphi\ dxdyd\tau|\le C\varepsilon^{3-3\alpha/2}\Big(\|\zeta\partial_t c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))}+$$
$$\varepsilon^\alpha\|\zeta\partial_{xx}c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))}\Big)\cdot\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}\le$$
$$C\big(\varepsilon^{3-7\alpha/4)}H(1-\alpha)+\varepsilon^{5(2-\alpha)/4}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \qquad (92)$$

$$|\int_0^t\int_0^{+\infty}\zeta\partial_x c^{Mau}\varepsilon^{2-\alpha}(\int_0^1\varphi\ dy-\varphi|_{y=1})\ dxd\tau|\le$$
$$C\varepsilon^{2-\alpha}\|\partial_x c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))}\|\int_0^1\varphi\ dy-\varphi|_{y=1}\|_{L^2(O_t)}$$
$$\le C\big(\varepsilon^{3-7\alpha/4}H(1-\alpha)+\varepsilon^{5(2-\alpha)/4}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \qquad (93)$$

$$|\int_0^t\int_0^{+\infty}\zeta(t)c^{Mau}\varepsilon^{2-\alpha}(\int_0^1\varphi\ dy-\varphi|_{y=1})\ dxd\tau|\le$$
$$C\varepsilon^{3(1-\alpha/2)}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \qquad (94)$$

$$|\int_0^t\int_0^{+\infty}(1-\zeta(t))c^{Mau}(\int_0^1\varphi\ dy-\varphi|_{y=1})\ dxd\tau|\le$$
$$C\big(\varepsilon H(1-\alpha)+\varepsilon^{2-\alpha}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \qquad (95)$$

Corollary 14. *Let $\varphi \in H^1(O_T)$, $\varphi = 0$ at $x = 0$. Let Φ_1^ε be given by (77) and g^ε by (84). Then we have*

$$|\int_0^t \int_{Z^+} \Phi_1^\varepsilon \varphi \; dxdyd\tau + \int_0^t \int_{I\!R_+} g^\varepsilon|_{y=1} \varphi|_{y=1} \; dxd\tau| \leq C\big(\varepsilon^{1-\alpha/2} H(1-\alpha)$$
$$+\varepsilon^{2-3\alpha/2} H(\alpha-1)\big)\Big\{ \|\varepsilon^{\alpha/2-1} \partial_y \varphi\|_{L^2(O_t)} + \|\varepsilon^{\alpha/2} \partial_x \varphi\|_{L^2(O_t)} \Big\} \tag{96}$$

Next we should correct the values at $x = 0$ and apply Proposition 4. Due to the presence of the term containing the first order derivative in x, the boundary layer corresponding to our problem doesn't enter into the theory from [10] and one should generalize it. The generalization in the case of the periodic boundary conditions at the lateral boundary is in the paper [12]. In our knowledge, the generalization to the case of Neumann's boundary conditions at the lateral boundary, was never published. It seems that the results from [12] apply also to this case ([13]). In order to avoid developing the new theory for the boundary layer, we simply use the boundary layer for the Neumann problem for Laplace operator (70). Then the transport term is ignored and a large error in the forcing term is created. The error is concentrated at small times and by eliminating them we would obtain a good estimate.

In order to use this particular point, we prove the following proposition :

Proposition 15. *Let $\Psi(x) = 1/(1+x)$. Let g^ε and Φ^ε be bounded functions such that $\Psi g^\varepsilon \in H^1(Z^+ \times (0,T))$ and $\Psi \Phi^\varepsilon \in L^2(Z^+ \times (0,T))$. Let ξ, $\Psi\xi \in C^{0,\alpha_0}([0,T]; L^2(Z^+))$, $\Psi \nabla_{x,y} \xi \in L^2(Z^+ \times (0,T))$, be a bounded function which satisfies the system*

$$\mathcal{L}^\varepsilon(\xi) = -\Phi^\varepsilon \quad in \;\; Z^+ \times (0,T) \tag{97}$$
$$-D\varepsilon^{\alpha-2} \partial_y \xi|_{y=1} = k_0 \xi|_{y=1} + g^\varepsilon|_{y=1} \;\; and \;\; \partial_y \xi|_{y=0} = 0 \;\; on \;\; I\!R_+ \times (0,T) \tag{98}$$
$$\xi|_{t=0} = 0 \quad on \;\; Z^+ \quad and \, \xi|_{x=0} = 0 \; on \;\; (0,1) \times (0,T). \tag{99}$$

Then we have the following energy estimate

$$\mathcal{E}(t^k \xi, t) = t^{2k} \int_{Z^+} \Psi(x)^2 \xi^2(t) \; dxdy + D\varepsilon^\alpha \int_0^t \int_{Z^+} \Psi(x)^2 \tau^{2k} \Big\{ \varepsilon^{-2} |\partial_y \xi|^2 +$$
$$|\partial_x \xi|^2 \Big\} \; dxdyd\tau + k_0 \int_0^t \int_{I\!R_+} \tau^{2k} \xi^2|_{y=1} \Psi^2(x) \; dxd\tau \leq C_1 |\int_0^t \int_{Z^+} \tau^{2k} \Psi(x)^2 \Phi^\varepsilon \xi \; dxdyd\tau$$

$$+\int_0^t\int_{I\!R_+}\tau^{2k}g^\varepsilon|_{y=1}\xi|_{y=1}\Psi^2(x)\ dxd\tau|+C_2D\varepsilon^\alpha\int_0^t\int_{Z^+}\tau^{2k}\Psi(x)^2\xi^2\ dxdyd\tau,\ \forall k\geq 1. \tag{100}$$

Remark 16. *Clearly we have in our mind $\xi = c^\varepsilon - c_1^{eff}$. Then $\zeta(t)\partial_x c^{Mau}$ has the required regularity, since the cut-off erases the singularity. With c^{Mau} things are more complicated. By a direct calculation we have $\partial_t c^{Mau} \in L^q(0,T;L^2(I\!R_+))$, $\forall q \in [1, 4/3)$ and we get the required Hölder regularity by the Sobolev imbedding. $\int_0^A\int_0^1 |\xi(x,y,t)|^2\ dxdy$ is Hölder-continuous with some exponent $\alpha_0 > 0$, $\forall A < +\infty$, which is* **independent of** *ε. In complete analogy, c_0^{eff} defined by (32) has also the required regularity. Finally, the difference $c^\varepsilon - c_0^{eff}$ satisfies the equations (39) and (40) and it is zero at $x = 0$ and at $t = 0$. Then the classical parabolic regularity theory (see e.g. [7]) implies the Hölder regularity in time of the L^2-norm with respect to x, y. After putting all these results together, we get the required regularity of ξ.*

Proof. By the supposed Hölder continuity, there is $t_M \in [0,T]$, $t_M > 0$, such that

$$\frac{1}{t_M^{\alpha_0}}\int_0^{+\infty}\int_0^1 |\xi(x,y,t_M)|^2\Psi^2(x)\ dxdy = \max_{t\in[0,T]}\frac{1}{t^{\alpha_0}}\int_0^{+\infty}\int_0^1 |\xi(x,y,t)|^2\Psi^2(x)\ dx \tag{101}$$

Then we have

$$\int_0^{t_M} k\tau^{2k-1}\int_{Z^+}|\xi|^2\Psi^2(x)\ dxdyd\tau \leq \int_{Z^+}\frac{|\xi|^2(t_M)}{t_M^{\alpha_0}}\Psi^2(x)\int_0^{t_M}k\tau^{2k-1+\alpha_0}\ d\tau$$
$$= \frac{k}{2k+\alpha_0}t_M^{2k}\int_{Z^+}|\xi|^2(t_M)\Psi^2(x)\ dxdy \tag{102}$$

and

$$\frac{1}{2}t_M^{2k}\int_{Z^+}|\xi|^2(t_M)\Psi^2(x)\ dxdy + k_0\int_0^t\int_0^{+\infty}\tau^{2k}\xi^2|_{y=1}\Psi^2(x)\ dxd\tau+$$
$$\int_0^{t_M} D\Big(\varepsilon^\alpha\int_{Z^+}\tau^{2k}|\partial_x\xi|^2(\tau)\Psi^2(x)\ dxdy + \varepsilon^{\alpha-2}\int_{Z^+}\tau^{2k}|\partial_y\xi|^2(\tau)\Psi^2(x)\ dxdy\Big)\ d\tau$$
$$\leq -\int_0^{t_M}\int_{Z^+}\tau^{2k}\Phi^\varepsilon\xi\ dxdyd\tau - k_0\int_0^t\int_0^{+\infty}\tau^{2k}\xi|_{y=1}g^\varepsilon|_{y=1}\Psi^2(x)\ dxd\tau+$$
$$D\varepsilon^\alpha\int_0^{t_M}\int_{Z^+}\tau^{2k}\Psi^2(x)\xi^2\ dxdyd\tau + k\int_0^{t_M}\int_{Z^+}\tau^{2k-1}|\xi|^2\Psi^2\ dxdyd\tau \tag{103}$$

Using (102) we get (100) for $t = t_M$ and with $C_2 = 0$. Getting the estimates (100) for general $t \in (0, T)$ is now straightforward. $\Box$

Next, in order to use this estimate we should refine the estimates from Propositions 12 and 13 . First we note that the estimate (28) changes to

$$\|t^k \partial_{tt} c^{Mau}\|_{L^q((0,T)\times I\!R_+)} + \|t^k \partial_{tx} c^{Mau}\|_{L^q((0,T)\times I\!R_+)} + \|t^k \partial_{xx} c^{Mau}\|_{L^q((0,T)\times I\!R_+)} \\ \leq C_q(k)(\gamma \bar{D})^{1/(2q)-1}. \quad (104)$$

Hence one gains $\varepsilon^{\alpha/4}$ (respectively $\varepsilon^{1/2-\alpha/4}$) for the L^2-norm. In analogy with Propositions 12 and 13 we have

Proposition 17. *Let $O_T = I\!R_+ \times (0,1) \times (0,T)$. Let $\varphi \in H^1(O_T)$, $\varphi = 0$ at $x = 0$ and $k > 1$. Then we have*

$$|\int_0^t \int_0^\infty \int_0^1 \tau^k \zeta F_1^\varepsilon \varphi \ dxdyd\tau| \leq C\varepsilon^{3(2-\alpha)/2} \|\tau^k \partial_{xx} c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))} \|\varepsilon^{\alpha/2-1} \partial_y \varphi\|_{L^2(O_t)} \\ \leq C\big(\varepsilon^{3-9\alpha/4} H(1-\alpha) + \varepsilon^{3/2-3\alpha/4} H(\alpha-1)\big) \|\varepsilon^{\alpha/2-1} \partial_y \varphi\|_{L^2(O_t)} \quad (105)$$

$$|\int_0^t \int_0^\infty \int_0^1 \tau^k \zeta F_3^\varepsilon \varphi \ dxdyd\tau| \leq C\varepsilon^{3(2-\alpha)/2} \Bigg(\|\tau^k \partial_{xt} c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))} + \\ \|\tau^k \partial_{xx} c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))} \Bigg) \cdot \|\varepsilon^{\alpha/2-1} \partial_y \varphi\|_{L^2(O_t)} \leq C\big(\varepsilon^{3-9\alpha/4} H(1-\alpha) + \\ \varepsilon^{3/2-3\alpha/4} H(\alpha-1)\big) \|\varepsilon^{\alpha/2-1} \partial_y \varphi\|_{L^2(O_t)} \quad (106)$$

$$|\int_0^t \int_{Z^+} \zeta \tau^k F_2^\varepsilon \varphi \ dxdyd\tau| \leq C\varepsilon^{3(1-\alpha/2)} \|\tau^k \zeta \partial_x c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))} \|\varepsilon^{\alpha/2-1} \partial_y \varphi\|_{L^2(O_t)} \\ \leq C\big(\varepsilon^{3-7\alpha/4} H(1-\alpha) + \varepsilon^{5/2-5\alpha/4} H(\alpha-1)\big) \|\varepsilon^{\alpha/2-1} \partial_y \varphi\|_{L^2(O_t)} \quad (107)$$

$$|\int_0^t \int_{Z^+} \zeta \tau^k F_4^\varepsilon \varphi \ dxdyd\tau| \leq C\varepsilon^{3-3\alpha/2} \Bigg(\|\zeta \tau^k \partial_t c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))} + \\ \varepsilon^\alpha \|\zeta \tau^k \partial_{xx} c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))} \Bigg) \cdot \|\varepsilon^{\alpha/2-1} \partial_y \varphi\|_{L^2(O_t)} \leq \\ C\big(\varepsilon^{3-7\alpha/4)} H(1-\alpha) + \varepsilon^{5(2-\alpha)/4} H(\alpha-1)\big) \|\varepsilon^{\alpha/2-1} \partial_y \varphi\|_{L^2(O_t)} \quad (108)$$

$$|\int_0^t \int_0^{+\infty} \zeta \tau^k \partial_x c^{Mau} \varepsilon^{2-\alpha} (\int_0^1 \varphi \ dy - \varphi|_{y=1}) \ dxd\tau| \leq \\ C\varepsilon^{2-\alpha} \|\tau^k \partial_x c^{Mau}\|_{L^2(0,t;L^2(I\!R_+))} \|\int_0^1 \varphi \ dy - \varphi|_{y=1}\|_{L^2(O_t)} \\ \leq C\big(\varepsilon^{3-7\alpha/4} H(1-\alpha) + \varepsilon^{5(2-\alpha)/4} H(\alpha-1)\big) \|\varepsilon^{\alpha/2-1} \partial_y \varphi\|_{L^2(O_t)} \quad (109)$$

$$|\int_0^t\int_0^{+\infty}\zeta(t)\tau^k c^{Mau}\varepsilon^{2-\alpha}(\int_0^1\varphi\ dy-\varphi|_{y=1})\ dxd\tau|\leq$$
$$C\varepsilon^{3(1-\alpha/2)}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{110}$$

Proof. These estimates are straightforward consequences of Propositions 12 and 13 . □

We gain more with other terms:

Proposition 18. *Let $O_T=\mathbb{R}_+\times(0,1)\times(0,T)$. Let $\varphi\in H^1(O_T)$, $\varphi=0$ at $x=0$. Then we have*

$$|\int_0^t\int_0^\infty\int_0^1(1-\zeta)\tau^k\partial_{xx}c^{Mau}\frac{Q^2}{D}\varepsilon^{2-\alpha}\varphi\ dxdyd\tau|$$
$$\leq C\varepsilon^{2-3\alpha/2}\|(1-\zeta)\tau^k\partial_x c^{Mau}\|_{L^2(0,t;L^2(\mathbb{R}_+))}\|\varepsilon^{\alpha/2}\partial_x\varphi\|_{L^2(O_t)}$$
$$\leq C\big(\varepsilon^{k\alpha+2-3\alpha/2}H(1-\alpha)+\varepsilon^{k(2-\alpha)+2-3\alpha/2}H(\alpha-1)\big)\|\varepsilon^{\alpha/2}\partial_x\varphi\|_{L^2(O_t)} \tag{111}$$
$$|\int_0^t\int_0^\infty\int_0^1(1-\zeta)\tau^k Q(1/3-y^2)\partial_x c^{Mau}\varphi\ dxdyd\tau|\leq$$
$$C\varepsilon^{1-\alpha/2}\|(1-\zeta)\tau^k\partial_x c^{Mau}\|_{L^2(0,t;L^2(\mathbb{R}_+))}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}\leq$$
$$C\big(\varepsilon^{k\alpha+1-\alpha/2}H(1-\alpha)+\varepsilon^{k(2-\alpha)+1-\alpha/2}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{112}$$
$$|\int_0^t\int_{Z^+}\zeta'(\frac{t}{D\varepsilon})\tau^k\varepsilon^{2-\alpha}\big\{\partial_x c^{Mau}\frac{Q}{D}\{\frac{y^2}{6}-\frac{y^4}{12}-\frac{7}{180}\}-\frac{k_0}{2D}(\frac{1}{3}-y^2)c^{Mau}\big\}\cdot$$
$$\varphi\ dxdyd\tau|\leq C\varepsilon^{3-3\alpha/2}\|\zeta'\tau^k\partial_x c^{Mau}\|_{L^2(0,t;L^2(\mathbb{R}_+))}\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)}\leq$$
$$C\big(\varepsilon^{3-3\alpha/2+\alpha(k-1)}H(1-\alpha)+\varepsilon^{3-3\alpha/2+(2-\alpha)(k-1)}H(\alpha-1)\big)\|\varepsilon^{\alpha/2-1}\partial_y\varphi\|_{L^2(O_t)} \tag{113}$$

Before applying Proposition 15 and getting the final estimate, we should correct the trace at $x=0$. It is done by adding

$$\bar{c}_1^{eff}=-\varepsilon^{2-\alpha}\zeta(t)\beta^\varepsilon\partial_x c^{Mau}\frac{Q}{D}, \tag{114}$$

where $\beta^\varepsilon(x,y)=\beta(x/\varepsilon,y)$ is the boundary layer function given by (70). Then for $\xi^\varepsilon=c^\varepsilon-c_1^{eff}-\bar{c}_1^{eff}$ we have

$$\mathcal{L}^\varepsilon(\xi)=-\Phi^\varepsilon=-\Phi_1^\varepsilon+\partial_t\zeta\varepsilon^{2-\alpha}\partial_x c^{Mau}\frac{Q}{D}\beta^\varepsilon+\varepsilon^{2-\alpha}\beta^\varepsilon\zeta(t)\big\{\partial_{xt}c^{Mau}\frac{Q}{D}-$$

$$\varepsilon^\alpha \partial_{xxx} c^{Mau} Q\} + \partial_x \beta^\varepsilon \frac{Q^2}{D}(1-y^2)\zeta \varepsilon^{2-\alpha}\partial_x c^{Mau} - \varepsilon^{2-\alpha} Q \partial_{xx} c^{Mau} \zeta(t)\big(2\varepsilon^\alpha \partial_x \beta^\varepsilon -$$

$$\beta^\varepsilon (1-y^2)\frac{Q}{D}\big) \text{ in } Z^+ \times (0,T) \tag{115}$$

$$-D\varepsilon^{\alpha-2}\partial_y \xi^\varepsilon|_{y=1} = k_0 \xi|_{y=1} + g^\varepsilon|_{y=1} - k_0 \varepsilon^{2-\alpha}\zeta \frac{Q}{D}\partial_x c^{Mau} \beta^\varepsilon|_{y=1} \text{ on } I\!R_+ \times (0,T) \tag{116}$$

$$\text{and } \partial_y \xi^\varepsilon|_{y=0} = 0 \text{ on } I\!R_+ \times (0,T) \tag{117}$$

$$\xi^\varepsilon|_{t=0} = 0 \quad \text{on } Z^+ \quad \text{and } \xi^\varepsilon|_{x=0} = 0 \text{ on } (0,1)\times(0,T). \tag{118}$$

We need an estimate for new terms. The estimates will follow from the following auxiliary result

Lemma 5. *Let β be defined by (70), let $k \geq 1$ and c^{Mau} the solution for (75). Then we have*

$$\|\tau^k \zeta' \beta^\varepsilon \partial_x c^{Mau}\|_{L^2((0,t)\times Z^+)} \leq C\varepsilon^{k-3/4}\Big\{\varepsilon^{-\alpha/4} H(1-\alpha) + \varepsilon^{\alpha/4-1/2} H(\alpha-1)\Big\} \leq C\varepsilon^{k-1} \tag{119}$$

$$\|\tau^k \zeta \beta^\varepsilon|_{y=1} \partial_x c^{Mau}\|_{L^2((0,t)\times Z^+)} \leq C\varepsilon^{k+1/4}\Big\{\varepsilon^{-\alpha/4} H(1-\alpha) + \varepsilon^{\alpha/4-1/2} H(\alpha-1)\Big\} \leq C\varepsilon^{k} \tag{120}$$

$$\|\tau^k \zeta \partial_x \beta^\varepsilon \partial_x c^{Mau}\|_{L^2((0,t)\times Z^+)} \leq C\varepsilon^{k-3/4}\Big\{\varepsilon^{-\alpha/4} H(1-\alpha) + \varepsilon^{\alpha/4-1/2} H(\alpha-1)\Big\} \leq C\varepsilon^{k-1} \tag{121}$$

$$\|\tau^k \zeta \partial_x \beta^\varepsilon \partial_t c^{Mau}\|_{L^2((0,t)\times Z^+)} \leq C\varepsilon^{k-5/4}\Big\{\varepsilon^{\alpha/2} H(1-\alpha) + \varepsilon^{1-\alpha/2} H(\alpha-1)\Big\} \leq C\varepsilon^{k-5/4} \tag{122}$$

$$\|\tau^k \zeta \beta^\varepsilon \partial_{xx} c^{Mau}\|_{L^2((0,t)\times Z^+)} \leq C\varepsilon^{k}\Big\{(\varepsilon^{-1/4-\alpha/2} + \varepsilon^{1/4-3\alpha/4}) H(1-\alpha) + (\varepsilon^{\alpha/2-5/4} + \varepsilon^{-5/2+3\alpha/4}) H(\alpha-1)\Big\} \leq C\varepsilon^{k-7/4} \tag{123}$$

$$\|\tau^k \zeta \partial_x \beta^\varepsilon \partial_{xx} c^{Mau}\|_{L^2((0,t)\times Z^+)} \leq C\varepsilon^{k-1}\Big\{ (\varepsilon^{-1/4-\alpha/2} + \varepsilon^{1/4-3\alpha/4})H(1-\alpha) +$$
$$(\varepsilon^{\alpha/2-5/4} + \varepsilon^{-5/2+3\alpha/4})H(\alpha-1)\Big\} \leq C\varepsilon^{k-7/4} \qquad (124)$$

Proof. We have

$$\int_0^{+\infty} |\partial_x c^{Mau} \beta^\varepsilon|^2 \, dx \leq C \int_0^{+\infty} \exp\{-\frac{2\gamma_0 x}{\varepsilon}\} \exp\{-\frac{(x-\tau\bar{Q})^2}{2\gamma\bar{D}\tau}\} \, \frac{dx}{\gamma\tau\bar{D}}$$
$$\leq C(\varepsilon D\tau)^{-1/2} \exp\{-C_0\tau/\varepsilon\} \, dx d\tau \qquad (125)$$

Now (119) , (120) and (121) follow by integration with respect to τ. Next,

$$\int_0^{+\infty} |\partial_t c^{Mau} \beta^\varepsilon|^2 \, dx \leq C \int_0^{+\infty} x^2 \exp\{-\frac{2\gamma_0 x}{\varepsilon}\} \exp\{-\frac{(x-\tau\bar{Q})^2}{2\gamma\bar{D}\tau^3}\} \, \frac{dx}{\gamma\tau\bar{D}}$$
$$\leq C(\varepsilon D\tau^3)^{-1/2} \exp\{-C_0\tau/\varepsilon\} \, dx d\tau \qquad (126)$$

and (122) follows. Since

$$\|\tau^k \zeta \beta^\varepsilon \partial_{xx} c^{Mau}\|_{L^2((0,t)\times Z^+)} \leq C(\|\tau^k \zeta \beta^\varepsilon \partial_x c^{Mau}\|_{L^2((0,t)\times Z^+)} +$$
$$\|\tau^k \zeta \beta^\varepsilon \partial_t c^{Mau}\|_{L^2((0,t)\times Z^+)})(\varepsilon^{-\alpha} H(1-\alpha) + \varepsilon^{\alpha-2} H(\alpha-1)) \qquad (127)$$

we get (122) and (123). □

Proposition 19. *Let $\varphi \in H^1(O_T)$, $\varphi = 0$ at $x = 0$. Then we have*

$$|\int_0^t \int_{Z^+} \varepsilon^{2-\alpha} \tau^k \zeta(\tau) \beta^\varepsilon \big\{\partial_{xt} c^{Mau} \frac{Q}{D} - \varepsilon^\alpha \partial_{xxx} c^{Mau} Q\big\} \varphi \, dx dy d\tau|$$
$$\leq C\varepsilon^{2-\alpha}\Big(\big\{ \|\zeta \tau^k \partial_t c^{Mau} \partial_x \beta^\varepsilon\|_{L^2((0,t)\times Z^+)} + \varepsilon^\alpha \|\tau^k \zeta \partial_x \beta^\varepsilon \partial_{xx} c^{Mau}\|_{L^2((0,t)\times Z^+)} \big\} \cdot$$
$$\|\varphi\|_{L^2((0,t)\times Z^+)} + \varepsilon^{-\alpha/2} \big\{ \|\zeta \tau^k \partial_t c^{Mau} \beta^\varepsilon\|_{L^2((0,t)\times Z^+)} + \varepsilon^\alpha \|\tau^k \zeta \partial_{xx} c^{Mau}\|_{L^2((0,t)\times Z^+)} \big\} \cdot$$
$$\|\varepsilon^{\alpha/2} \partial_x \varphi\|_{L^2((0,t)\times Z^+)} \Big) \leq C\varepsilon^{k+1/4-\alpha} \big(\|\varphi\|_{L^2((0,t)\times Z^+)} + \|\varepsilon^{\alpha/2} \partial_x \varphi\|_{L^2((0,t)\times Z^+)} \big) \qquad (128)$$

$$|\int_0^t \int_{Z^+} \varepsilon^{2-\alpha} \zeta \tau^k \partial_{xx} c^{Mau} \varphi \Big(-\beta^\varepsilon \frac{Q}{D}(1-y^2) + 2\varepsilon^\alpha \partial_x \beta^\varepsilon \Big) \, dx dy d\tau|$$
$$\leq C\varepsilon^{2-\alpha} \big(\|\tau^k \zeta \partial_x \beta^\varepsilon \partial_{xx} c^{Mau}\|_{L^2((0,t)\times Z^+)} + \|\tau^k \zeta \partial_x \beta^\varepsilon \partial_{xx} c^{Mau}\|_{L^2((0,t)\times Z^+)} \big) \cdot$$
$$\|\varphi\|_{L^2(O_t)} \leq C\varepsilon^{k-\alpha+1/4} \|\varphi\|_{L^2(O_t)} \qquad (129)$$

$$|\int_0^t \int_{Z^+} \varepsilon^{2-\alpha}\zeta\tau^k \partial_x c^{Mau}\partial_x\beta^\varepsilon(1-y^2)\varphi\ dxdyd\tau|$$
$$\leq C\varepsilon^{2-\alpha}\|\tau^k\zeta\partial_x\beta^\varepsilon\partial_x c^{Mau}\|_{L^2((0,t)\times Z^+)}\|\varphi\|_{L^2(O_t)} \leq C\varepsilon^{k-\alpha+1}\|\varphi\|_{L^2(O_t)} \quad (130)$$
$$|\int_0^t \int_0^{+\infty} \varepsilon^{2-\alpha}\zeta\tau^k \partial_x c^{Mau}\varphi|_{y=1}\beta^\varepsilon|_{y=1}\ dxd\tau|$$
$$\leq C\varepsilon^{2-\alpha}\|\tau^k\zeta\partial_x\beta^\varepsilon_{y=1}\partial_x c^{Mau}\|_{L^2((0,t)\times I\!R_+)}\|\varphi\|_{L^2(O_t)} \leq C\varepsilon^{k-\alpha+1}\|\varphi|_{y=1}\|_{L^2((0,t)\times I\!R_+))} \quad (131)$$
$$|\int_0^t \int_{Z^+} \varepsilon^{2-\alpha}\zeta'(\tau)\tau^k \partial_x c^{Mau}\varphi\beta^\varepsilon\ dxdyd\tau|$$
$$\leq C\varepsilon^{2-\alpha}\|\tau^k\zeta'\beta^\varepsilon\partial_x c^{Mau}\|_{L^2((0,t)\times Z^+)}\|\varphi\|_{L^2(O_t)} \leq C\varepsilon^{k-\alpha+3/4}\|\varphi\|_{L^2(O_t)} \quad (132)$$

Now the application of Proposition 15 is straightforward and after considering various powers we get

Theorem 20. *Let c^{Mau} be given by (75), let c_1^{eff} be given by (74) and $\bar{c}_1^{eff}$ by (114). Then we have*

$$\|t^3(c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff})\|_{L^\infty(0,T;L^2_{loc}(I\!R_+\times(0,1))} \leq C\big(\varepsilon^{3-9\alpha/4}H(1-\alpha)+ \varepsilon^{3(1-\alpha/2)/2}H(\alpha-1)\big) \quad (133)$$
$$\|t^3\partial_y\big(c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff}\big)\|_{L^2(0,T;L^2_{loc}(I\!R_+\times(0,1))} \leq C\varepsilon^{1-\alpha/2}\big(\varepsilon^{3-9\alpha/4}H(1-\alpha)+\varepsilon^{3(1-\alpha/2)/2}H(\alpha-1)\big) \quad (134)$$
$$\|t^2\partial_x\big(c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff}\big)\|_{L^2(0,T;L^2_{loc}(I\!R_+\times(0,1))} \leq C\varepsilon^{-\alpha/2}\big(\varepsilon^{3-9\alpha/4}H(1-\alpha)+\varepsilon^{3(1-\alpha/2)/2}H(\alpha-1)\big) \quad (135)$$

7 Error estimate involving the second order in expansion

The most important power of α is $\alpha = 1$, which describes Taylor's scaling. In this case our approximation is of order $\varepsilon^{3/4}$. Nevertheless, it is interesting to reach the order ε at least in this case. Also, it could be of interest to get the higher order estimates, which can be useful for ε which is not very small.

Clearly, the estimate isn't sufficiently good due to the terms ζF_1^ε and ζF_3^ε. When deriving formally the effective equation, we have seen that they could be eliminated by introducing the next order correction. Following the formal expansion we find out that c_1^{eff} should be replaced by $c_1^{eff}+c_2^{eff}$, where

$$c_2^{eff}=-\varepsilon^{4-2\alpha}\frac{Q}{D^2}\zeta(t)\Big\{Q\partial_{xx}c^{Mau}\Big(\frac{281}{453600}+\frac{23}{1512}y^2-\frac{37}{2160}y^4+\frac{1}{120}y^6-\frac{1}{672}y^8-\tilde{\beta}_1\Big)-(\partial_{xt}c^{Mau}-D\varepsilon^\alpha\partial_{xxx}c^{Mau})\Big(-\frac{1}{360}y^6+\frac{1}{72}y^4-\frac{7}{360}y^2-\frac{31}{7560}-\tilde{\beta}_2\Big)\Big\}+\varepsilon^{4-2\alpha}\frac{k_0}{D^2}\zeta(t)\Big\{Q\partial_x c^{Mau}\Big(\frac{1}{60}y^6-\frac{1}{18}y^4+\frac{11}{180}y^2-\frac{11}{810}-\tilde{\beta}_3\Big)+\frac{1}{2}(\partial_t c^{Mau}-D\varepsilon^\alpha\partial_{xx}c^{Mau})\Big(-\frac{1}{12}y^4+\frac{1}{6}y^2-\frac{7}{180}-\tilde{\beta}_5\Big)+\frac{Q}{45}\partial_x c^{Mau}\Big(\frac{1}{3}-y^2-\tilde{\beta}_4\Big)-\frac{k_0}{6}c^{Mau}\Big(\frac{1}{3}-y^2\Big)\Big\}, \tag{136}$$

where $\tilde{\beta}_j$, $j=1,\dots,5$, are solutions to the boundary layers analogous to (70) which correct those new values at $x=0$.

Using this additional correction term we have

Theorem 21. *Let c^{Mau} be given by (75), let c_1^{eff} be given by (74), $\bar{c}_1^{eff}$ by (114) and c_2^{eff} by (136). Then we have*

$$\|t^5(c^\varepsilon-c_1^{eff}(x,t;\varepsilon)-\bar{c}_1^{eff})\|_{L^\infty(0,T;L^2_{loc}(I\!\!R_+\times(0,1))}\le C\big(\varepsilon^{4-13\alpha/4}H(1-\alpha)+\varepsilon^{3(1-\alpha/2)/2}H(\alpha-1)\big) \tag{137}$$

$$\|t^5\partial_y\big(c^\varepsilon-c_1^{eff}(x,t;\varepsilon)-\bar{c}_1^{eff}\big)\|_{L^2(0,T;L^2_{loc}(I\!\!R_+\times(0,1))}\le C\varepsilon^{1-\alpha/2}\big(\varepsilon^{4-13\alpha/4}H(1-\alpha)+\varepsilon^{3(1-\alpha/2)/2}H(\alpha-1)\big) \tag{138}$$

$$\|t^5\partial_x\big(c^\varepsilon-c_1^{eff}(x,t;\varepsilon)-\bar{c}_1^{eff}\big)\|_{L^2(0,T;L^2_{loc}(I\!\!R_+\times(0,1))}\le C\varepsilon^{-\alpha/2}\big(\varepsilon^{4-13\alpha/4}H(1-\alpha)+\varepsilon^{3(1-\alpha/2)/2}H(\alpha-1)\big) \tag{139}$$

Proof. After applying the operator $\mathcal{L}^\varepsilon$, given by (33), to $c^\varepsilon-c_1^{eff}-\bar{c}_1^{eff}-c_2^{eff}$ we obtain a forcing term Φ_2^ε, analogous to (115). Let us study it. In fact it is enough to study what happened with $\zeta\sum_{j=1}^5 F_j$. As we have seen in Proposition 18, Lemma 5 and Proposition 19, other terms are small. We have

- F_1^ε and F_3^ε are replaced by

$$\begin{cases} \tilde{F}_1^\varepsilon = (1-y^2)\dfrac{Q^2\varepsilon^{4-2\alpha}}{D^2}\Big\{ -\partial_{xxx}c^{Mau}P_8(y)Q + (\partial_{xxt}c^{Mau} - \\ D\varepsilon^\alpha \partial_{xxxx}c^{Mau})P_6(y)\Big\} \\ \tilde{F}_3^\varepsilon = -\varepsilon^{4-2\alpha}P_8(y)\dfrac{Q^2}{D^2}\Big\{\partial_{xxt}c^{Mau} - \varepsilon^\alpha \partial_{xxxx}c^{Mau}D\Big\} + \\ \varepsilon^{4-2\alpha}P_6(y)\dfrac{Q}{D^2}\Big\{\partial_{xtt}c^{Mau} - 2D\varepsilon^\alpha\partial_{xxxt}c^{Mau} + \varepsilon^{2\alpha}\partial_{xxxxx}c^{Mau}D^2\Big\} \\ \\ P_8(y) = \dfrac{281}{453600} + \dfrac{23}{1512}y^2 - \dfrac{37}{2160}y^4 + \dfrac{1}{120}y^6 - \dfrac{1}{672}y^8, \\ \\ P_4(y) = \dfrac{y^2}{6} - \dfrac{y^4}{12} - \dfrac{7}{180};\ P_6(y) = -\dfrac{1}{60}y^5 + \dfrac{1}{18}y^3 - \dfrac{7}{180}y - \dfrac{31}{7560}. \end{cases} \tag{140}$$

Using (31) we find out, in analogy with (105)-(106), that

$$\int_0^t \int_0^\infty \int_0^1 \tau^k \zeta(|\tilde{F}_1^\varepsilon| + |\tilde{F}_3^\varepsilon|)|\varphi|\ dxdyd\tau \leq$$
$$C\Big(\varepsilon^{4-13\alpha/4}H(1-\alpha) + \varepsilon^{3/2-3\alpha/4}H(\alpha-1)\Big)\|\varphi\|_{L^2(O_t)}, \tag{141}$$

$\forall \varphi \in H^1(O_T),\ \varphi = 0$ at $x=0$ and $k>2$.

- F_2^ε and F_4^ε are replaced by

$$\begin{cases} \tilde{F}_2^\varepsilon = (1-y^2)\dfrac{Qk_0\varepsilon^{4-2\alpha}}{D}\Big\{\partial_{xx}c^{Mau}\dfrac{Q}{D}\tilde{P}_6(y) + (\partial_{xt}c^{Mau}\dfrac{1}{2D} - \\ \varepsilon^\alpha\partial_{xxx}c^{Mau}\dfrac{1}{2})P_4(y) + (\dfrac{Q}{45D}\partial_x c^{Mau} - \dfrac{k_0}{6D}c^{Mau})P_2(y)\Big\} \\ \tilde{F}_4^\varepsilon = -\varepsilon^{4-2\alpha}\tilde{P}_6(y)\dfrac{Qk_0}{D^2}\Big\{\partial_{xt}c^{Mau} - \varepsilon^\alpha\partial_{xxx}c^{Mau}D\Big\} + \\ \varepsilon^{4-2\alpha}P_4(y)\dfrac{k_0}{2D^2}\Big\{\partial_{tt}c^{Mau} - 2D\varepsilon^\alpha\partial_{xxt}c^{Mau} + D^2\varepsilon^{2\alpha}\partial_{xxxx}c^{Mau}\Big\} \\ +\varepsilon^{4-2\alpha}P_2(y)\dfrac{k_0}{3D^2}\Big\{\dfrac{Q}{15}\partial_{xt}c^{Mau} - \dfrac{k_0}{2}\partial_t c^{Mau} - \\ \dfrac{DQ\varepsilon^\alpha}{15}\partial_{xxx}c^{Mau} + \dfrac{Dk_0\varepsilon^\alpha}{2}\partial_{xx}c^{Mau}\Big\}, \end{cases} \tag{142}$$

where $P_2(y) = 1/3 - y^2$ and $\tilde{P}_6 = \frac{y^6}{60} - \frac{y^4}{18} + \frac{11y^2}{180} - \frac{11}{810}$. Using (31) we find out, in analogy with (107)-(108), that

$$\int_0^t \int_0^\infty \int_0^1 \tau^k \zeta(|\tilde{F}_2^\varepsilon| + |\tilde{F}_4^\varepsilon|)|\varphi| \ dxdyd\tau \leq$$

$$C\Big(\varepsilon^{4-11\alpha/4} H(1-\alpha) + \varepsilon^{5/2-5\alpha/4} H(\alpha-1)\Big) \|\varphi\|_{L^2(O_t)}, \tag{143}$$

$\forall \varphi \in H^1(O_T),\ \varphi = 0$ at $x = 0$ and $k > 2$.

- It should be noted that the means of the polynomials in y, contained in $\tilde{F}_1$ and $\tilde{F}_3$ aren't zero any more. Hence we can't gain some powers of ε using the derivative with respect to y of the test function.
- F_5 and the boundary term $k_0\zeta(t)\varepsilon^{2-\alpha}\big(\partial_x c^{Mau}\frac{2Q}{45D} - c^{Mau}\frac{k_0}{3D}\big)$ are canceled. At the boundary $y = 1$ we have a new non-homogeneous term

$$\hat{g}^\varepsilon = (1-\zeta)k_0 c^{Mau} - \zeta\varepsilon^{4-2\alpha}\big(\frac{2Qk_0^2}{45D^2}\partial_x c^{Mau}\tilde{P}_6|_{y=1} + (\frac{k_0}{2D^2}\partial_t c^{Mau} - \varepsilon^\alpha \frac{k_0}{2D}\partial_{xx} c^{Mau})P_4|_{y=1}\big) \tag{144}$$

and the principal boundary contribution is given by

$$|\int_0^t \int_0^\infty \int_0^1 \tau^k \zeta \varepsilon^{4-2\alpha}\big(\frac{2Qk_0^2}{45D^2}\partial_x c^{Mau}\tilde{P}_6|_{y=1} + (\frac{k_0}{2D^2}\partial_t c^{Mau} - \varepsilon^\alpha\frac{k_0}{2D}\partial_{xx}c^{Mau})P_4|_{y=1}\big)\varphi|_{y=1} \ dxdyd\tau| \leq C\Big(\varepsilon^{4-9\alpha/4}H(1-\alpha) + \varepsilon^{7/2-7\alpha/4}H(\alpha-1)\Big)\|\varphi|_{y=1}\|_{L^2((0,t)\times I\!R_+)}, \tag{145}$$

- Other terms are much smaller and don't have to be discussed.

After collecting the powers of ε and applying Proposition 15 we obtain the estimates (137)-(139). □

Theorem 22. *Let c^{Mau} be given by (75), let c_1^{eff} be given by (74), $\bar{c}_1^{eff}$ by (114) and c_2^{eff} by (136). Then we have*

$$\|t^5\big(c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff} - c_2^{eff}\big)\|_{L^2(0,T;L^1_{loc}(I\!R_+\times(0,1))} \leq \\ C\big(\varepsilon^{4-3\alpha}H(1-\alpha) + \varepsilon^{2-\alpha}H(\alpha-1)\big) \qquad (146)$$

$$\|t^5\big(c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff} - c_2^{eff}\big)\|_{L^2(0,T;L^2_{loc}(I\!R_+\times(0,1))} \leq \\ C\big(\varepsilon^{4-3\alpha}H(1-\alpha) + \varepsilon^{2-\alpha}H(\alpha-1)\big) \qquad (147)$$

Proof. First we prove the $L^\infty(L^1)$-estimates (146). We test the equation for $\xi = c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff} - c_2^{eff}$ with regularized sign of ξ multiplied by Ψ^2 and get

$$t^{2k}\int_{Z^+}\Psi(x)^2|\xi|(t)\ dxdy + k_0\int_0^t\int_{I\!R_+}\tau^{2k}|\xi|_{y=1}|\Psi^2(x)\ dxd\tau \leq \\ C_1\int_0^t\int_{Z^+}\tau^{2k}\Psi(x)^2|\Phi_2^\varepsilon|\ dxdyd\tau + \int_0^t\int_{I\!R_+}\tau^{2k}|\hat{g}^\varepsilon|_{y=1}|\Psi^2(x)\ dxd\tau| + \\ C_2\varepsilon^\alpha\int_0^t\int_{Z^+}\tau^{2k}\Psi(x)^2|\xi|\ dxdyd\tau + k\int_0^t\int_{Z^+}\tau^{2k-1}|\xi|\Psi^2\ dxdyd\tau, \qquad (148)$$

$\forall k \geq 3$. As before, the L^1-norm of $\Psi^2\xi$ is Hölder continuous in time with some exponent $\alpha_0 > 0$. Consequently, arguing as in Proposition 15, we obtain

$$\sup_{0\leq t\leq T}\|t^k\Psi^2\xi(t)\|_{L^1(Z^+)} \leq C(\|\Psi^2\Phi_2^\varepsilon\|_{L^1(Z^+\times(0,T))} + \|\Psi^2\hat{g}^\varepsilon|_{y=1}\|_{L^1(I\!R_+\times(0,T))}\big) \qquad (149)$$

and (146) is proved.

The improved $L^2(L^2)$-estimate (147) follows from (146), (138) and the Poincaré's inequality in H^1 (see e.g. [4]). □

Next we prove the corresponding $L^\infty(L^\infty)$-estimate. We have

Theorem 23. *Let c^{Mau} be given by (75), let c_1^{eff} be given by (74), $\bar{c}_1^{eff}$ by (114) and c_2^{eff} by (136). Then we have*

$$\|t^5(c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff} - c_2^{eff})\|_{L^\infty((0,T)\times(I\!R_+\times(0,1))} \leq C(\delta)\big(\varepsilon^{4-7\alpha/2-\delta}H(1-\alpha) \\ + \varepsilon^{3/2-\alpha-\delta}H(\alpha-1)\big), \quad \forall\delta > 0. \qquad (150)$$

Remark 24. *From the proof we see that $C(\delta)$ has an exponential growth when $\delta \to 0$.*

Proof. Let $M > 0$, $\xi = c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff} - c_2^{eff}$ and $\xi_M = \sup\{t^k\xi - M, 0\}$.We test the equation for ξ with $\Psi^2\xi_M$ and get

$$\frac{1}{2}\int_{Z^+}\Psi(x)^2\xi_M^2(t)\ dxdy + D\varepsilon^\alpha\int_0^t\int_{Z^+}\Psi(x)^2|\partial_x\xi_M(\tau)|^2\ dxdyd\tau+ \\ D\varepsilon^{\alpha-2}\int_0^t\int_{Z^+}\Psi(x)^2|\partial_y\xi_M(\tau)|^2\ dxdyd\tau + k_0\int_0^t\int_{I\!R_+}(\xi_M|_{y=1}+ \\ M\tau^k)\xi_M|_{y=1}\Psi^2(x)\ dxd\tau \leq C_1|\int_0^t\int_{Z^+}\tau^k\Psi(x)^2|\Phi_3^\varepsilon|\xi_M\ dxdyd\tau \\ +\int_0^t\int_{I\!R_+}\tau^k|\hat{g}^\varepsilon|_{y=1}\xi_M|_{y=1}\Psi^2(x)\ dxd\tau| + C_2\varepsilon^\alpha\int_0^t\int_{Z^+}\tau^{2k}\Psi(x)^2\xi_M^2\ dxdyd\tau \tag{151}$$

$\forall k \geq 3$, where $\tau^k\Phi_3^\varepsilon = -\tau^k\Phi_2^\varepsilon + k\tau^{k-1}\xi$. We suppose that

$$k_0M \geq \sup_{0\leq\tau\leq T}\tau^k\|\Psi\hat{g}^\varepsilon(\tau)|_{y=1}\|_{L^\infty(I\!R_+)} = c_0\big(\varepsilon^{4-5\alpha/2}H(1-\alpha)+\varepsilon^{3(1-\alpha/2)}H(\alpha-1)\big) \tag{152}$$

As in the classical derivation of the Nash-Moser estimate (see [7], pages 181-186)) we introduce

$$\mu(M) = \int_0^T\int_{Z^+\cap\{t^k\xi-M>0\}}\Psi^2\ dxdydt \tag{153}$$

Now in exactly the same way as in [7], pages 181-186, on a time interval which could be smaller than $[0,T]$, but suppose equal to it without loosing the generality, we get

$$\|\xi_M\|_{V_2}^2 = \sup_{0\leq t\leq T}\int_{Z^+}\Psi(x)^2\xi_M^2(t)\ dxdy + D\varepsilon^\alpha\int_0^T\int_{Z^+}\Psi(x)^2|\partial_x\xi_M(\tau)|^2\ dxdyd\tau \\ +D\varepsilon^{\alpha-2}\int_0^T\int_{Z^+}\Psi(x)^2|\partial_y\xi_M(\tau)|^2\ dxdyd\tau \leq \beta_0^2\|\tau^k\Phi_3^\varepsilon\Psi\|^2_{L^q(Z^+\times(0,T))}\mu(M)^{1-2/q}, \\ q>2. \tag{154}$$

Next, the estimate (154) is iterated in order to conclude that $\xi_M = 0$. Here we should modify the classical argument from [7], pages 102-103, and adapt it to our situation.

We note that, after making appropriate extensions,

$$\|\Psi\varphi\|_{L^4(Z^+\times(0,T))} \leq c_0 \|\Psi\varphi\|^{1/2}_{L^2(Z^+\times(0,T))} \|\Psi\varphi\|^{1/2}_{H^1(Z^+\times(0,T))} \leq c_0 \varepsilon^{-\alpha/4} \|\varphi\|_{V_2}, \tag{155}$$

$\forall \varphi \in V_2$, $\varphi|_{x=0} = 0$. As in [7], page 102, now we take the sequence of levels $k_h = M(2 - 2^{-h})$, $h = 0, 1, \dots$. Then

$$(k_{h+1} - k_h)\mu^{1/4}(k_{h+1}) \leq \|\Psi \xi_{k_h}\|_{L^4(Z^+\times(0,T))} \leq \frac{\bar{\beta}\varepsilon^{-\alpha/4}}{k_{h+1} - k_h} \|\xi_{k_h}\|_{V_2} \tag{156}$$

and

$$\mu^{1/4}(k_{h+1}) \leq 2^h \frac{2\bar{\beta}\beta_0 \|\tau^k \Phi_3^\varepsilon \Psi\|_{L^q(Z^+\times(0,T))} \varepsilon^{-\alpha/4}}{M} \mu^{(1+\kappa)/4}(k_h), \ \kappa = 1 - 2/q > 0. \tag{157}$$

$\mu^{1/4}(k_{h+1})$ will tend to zero for $h \to \infty$ if $\mu^{1/4}(M)$ satisfies

$$\mu^{1/4}(M) \leq \left(\frac{2\bar{\beta}\beta_0 \|\tau^k \Phi_3^\varepsilon \Psi\|_{L^q(Z^+\times(0,T))} \varepsilon^{-\alpha/4}}{M} \right)^{-1/\kappa} 2^{-1/\kappa^2} \tag{158}$$

(158) is satisfied if M equals the right hand side of the estimate (150). $\square$

Next result concern higher order norms. It it not very satisfactory for large α and we state it without giving a proof, which follows from the demonstrations given above.

Theorem 25. *Let c^{Mau} be given by (75), let c_1^{eff} be given by (74), $\bar{c}_1^{eff}$ by (114) and c_2^{eff} by (136). Then we have*

$$\|t^5 \partial_x (c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff})\|_{L^\infty(0,T;L^2_{loc}(\mathbb{R}_+\times(0,1))} \leq C\big(\varepsilon^{4-15\alpha/4} H(1-\alpha) + \varepsilon^{(1-\alpha/2)/2} H(\alpha - 1)\big) \tag{159}$$

$$\|t^5 \partial_t \big(c^\varepsilon - c_1^{eff}(x,t;\varepsilon) - \bar{c}_1^{eff}\big)\|_{L^2(0,T;L^2_{loc}(\mathbb{R}_+\times(0,1))} \leq C\big(\varepsilon^{4-15\alpha/4} H(1-\alpha) + \varepsilon^{(1-\alpha/2)/2} H(\alpha - 1)\big) \tag{160}$$

Final improvement concerns the $L^\infty(L^2)$-norma for small values of α. As mentioned in the proof of Theorem 21, the reason was that $\tilde{F}_1^\varepsilon$ and $\tilde{F}_3^\varepsilon$ didn't have zero means with respect to y. Nevertheless, when computing the term c^2 in the asymptotic expansion, there was a liberty in adding an arbitrary function C_2 of x and t. This function can be chosen such that the appropriate means are zero and estimates (137)-(139) are multiplied by $\varepsilon^{1-\alpha/2}$. Unfortunately, there is a new contribution of the form $QP_2(y)\partial_x C_2$. Its norm destroys the estimate for $\alpha \geq 4/5$. Since this amelioration isn't of real importance we just give it as a result. Proof is completely analogous to the preceding ones.

Corollary 26. *Let c^{Mau} be given by (75), let c_1^{eff} be given by (74), $\bar{c}_1^{eff}$ by (114) and c_2^{eff} by (136). Let the polynomials $P_j(y)$ be defined by (140) and after (142). Finally, let C_2 be given by*

$$\frac{\partial C_2}{\partial t} + \frac{2Q}{3}\frac{\partial C_2}{\partial x} - \varepsilon^\alpha D \frac{\partial^2 C_2}{\partial x^2} = -\frac{Qk_0}{D}\zeta(t)\Big\{\partial_{xx}c^{Mau}\frac{Q}{D}\int_0^1 (1-y^2)\tilde{P}_6(y)\,dy$$
$$+(\partial_{xt}c^{Mau}\frac{1}{2D} - \varepsilon^\alpha \partial_{xxx}c^{Mau}\frac{1}{2})\int_0^1 (1-y^2)P_4(y)\,dy + (\frac{Q}{45D}\partial_x c^{Mau} -$$

$$\frac{k_0}{6D}c^{Mau})\int_0^1 (1-y^2)P_2(y)\,dy\Big\} - \frac{Q^2}{D^2}\zeta(t)\Big\{-\partial_{xxx}c^{Mau}Q\int_0^1(1-y^2)P_8(y)\,dy$$
$$+(\partial_{xxt}c^{Mau} - D\varepsilon^\alpha\partial_{xxxx}c^{Mau})\int_0^1(1-y^2)P_6(y)\,dy\Big\} \quad \text{in } \mathbb{R}_+\times(0,T), \tag{161}$$
$$\partial_x C_2 \in L^2(\mathbb{R}_+\times(0,T)), \quad C_2|_{t=0}=0, \quad C_2|_{x=0}=0. \tag{162}$$

Then for $\alpha\in[0,4/5]$ we have

$$\|t^5(c^\varepsilon - c_1^{eff} - \bar{c}_1^{eff} - c_2^{eff} - C_2)\|_{L^\infty(0,T;L^2_{loc}(\mathbb{R}_+\times(0,1))} \leq C\varepsilon^{5-17\alpha/4} \tag{163}$$
$$\|t^5\partial_y\big(c^\varepsilon - -c_1^{eff} - \bar{c}_1^{eff} - c_2^{eff}\big)\|_{L^2(0,T;L^2_{loc}(\mathbb{R}_+\times(0,1))} \leq C\varepsilon^{6-19\alpha/4} \tag{164}$$
$$\|t^5\partial_x\big(c^\varepsilon - c_1^{eff} - \bar{c}_1^{eff} - c_2^{eff} - C_2\big)\|_{L^2(0,T;L^2_{loc}(\mathbb{R}_+\times(0,1))} \leq \varepsilon^{5-19\alpha/4} \tag{165}$$

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