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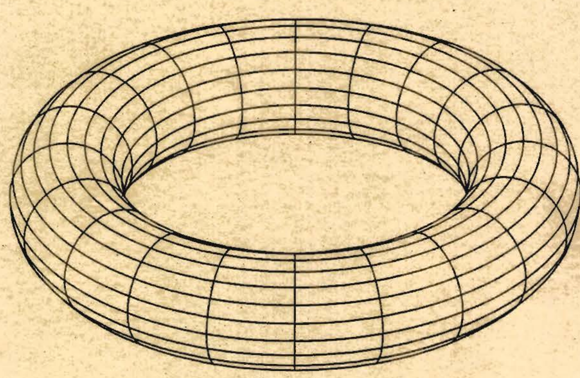

H.M.M. TEN EIKELDER

CANONICAL AND NON-CANONICAL SYMMETRIES FOR HAMILTONIAN SYSTEMS

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PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN DOCTOR IN DE TECHNISCHE WETENSCHAPPEN AAN DE TECHNISCHE HOGESCHOOL EINDHOVEN, OP GEZAG VAN DE RECTOR MAGNIFICUS, PROF. DR. S. T. M. ACKERMANS, VOOR EEN COMMISSIE AANGEWEZEN DOOR HET COLLEGE VAN DEKANEN IN HET OPENBAAR TE VERDEDIGEN OP VRIJDAG 3 FEBRUARI 1984 TE 16.00 UUR

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aan mijn ouders

CONTENTS.

GENERAL INTRODUCTION.

This thesis deals with symmetries of dynamical systems and in particular Hamiltonian systems. Suppose X is a vector field on a manifold M. With this vector field an autonomous dynamical system

$$(0.1) \qquad \dot{u}(t) \equiv \frac{d}{dt} u(t) = X(u(t))$$

on the manifold M is associated. Dynamical systems of this type arise in many places in science, biology, economy and other desciplines. Often, but not always the manifold M is also a linear space. An important special type of dynamical system is the Hamiltonian system. For (autonomous) Hamiltonian systems, as introduced in definition 3.2.14, there always exists a function H on M such that H(u(t)) is constant for every solution u(t) of the system. In physical situations which are described by a Hamiltonian system this function H is often equal to the energy of the system. If the initial state u_o of the system at $t = t_o$ is known, we can try to find the time evolution u(t) of the system by solving (0.1). However, in most cases for a dynamical/Hamiltonian system an explicit form of the solution, corresponding to an initial value $u(t_o) = u_o$, cannot be found, We shall not go into questions concerning existence and uniqueness of the solutions of (0.1) now. By means of numerical methods it is often possible to find a very good approximation for the solution of (0.1) with initial value $u(t_o) = u_o$.

An alternative way to obtain some information about the dynamical system is, instead of looking at a specific solution (as is done in the numerical approach), to find properties which are shared by all solutions or at least classes of solutions. Such properties are for instance the existence of constants of the motion, the existence of symmetries, the stability of the solutions or the behaviour of the solutions for $t \to \infty$. In this thesis we shall only consider symmetries and constants of the motion of dynamical systems and in particular Hamiltonian systems. For a finite-dimensional Hamiltonian system the existence of k constants of the motion in involution (i.e. with vanishing Poisson brackets) allows to reduce the dimension of the phase space by 2k. If the number of constants of the motion in involution equals half the dimension of the manifold (which is always even) the system is called completely integrable. In that case an explicit form for the solutions of (0.1) can be given. This is one of the reasons for the interest in constants of the motion.

For infinite-dimensional Hamiltonian systems the relation between infinite series of constants of the motion and "complete integrability" is not yet clear. During the last years a number of infinite-dimensional Hamiltonian systems have been solved using the so-called "inverse scattering methods". All these equations also have an infinite series of constants of the motion in involution. It is generally assumed that the existence of such a series is strongly related to the possibility of finding general solutions of these equations (for instance by inverse scattering).

In chapter 2 we consider a general dynamical (i.e not necessarily Hamiltonian) system of the form (0.1). A symmetry of a dynamical system is introduced as an infinitesimal transformation of solutions of the dynamical systems into new solutions of the system. We shall consider symmetries which also may depend explicitly on t. So $Y(u,t)$ is a symmetry if for every solution $u(t)$ of (0.1) also $u(t) + \varepsilon\, Y(u(t),t)$ is a solution (up to $o(\varepsilon)$ for $\varepsilon \to 0$). This leads to an interpretation of symmetries of (0.1) as, possibly parameterized, (contravariant) vector fields which satisfy

$$(0.2) \qquad \dot{Y} + [X,Y] = \dot{Y} + L_X Y = 0 \qquad (\dot{Y} = \frac{\partial}{\partial t} Y)$$

where $[X,Y] = L_X Y$ is the Lie bracket of the vector fields X and Y. Sometimes this type of infinitesimal transformation is called a generator of a symmetry; the notion symmetry is then used for a finite (i.e. not infinitesimal) transformation of solutions of (0.1) into new solutions of (0.1). However, we shall use the notion symmetry only for infinitesimal transformations, or more precisely for parameterized vector fields which satisfy (0.2). The relation between symmetries and finite transformations of solutions into (new) solutions is similar to the relation between a Lie algebra and the corresponding Lie group. Therefore it is not surprising that the set of symmetries of a dynamical system has a natural Lie algebra structure.

A second important concept in this thesis is the adjoint symmetry, that is a, possibly parameterized, one-form (covariant vector field) $\sigma(u,t)$ which satisfies

$$(0.3) \qquad \dot{\sigma} + L_X \sigma = 0 .$$

It turns out that every constant of the motion of (0.1) gives rise to an adjoint symmetry. However, the converse is not true in general. The four possible types of linear operators which map (adjoint) symmetries into (adjoint) symmetries are also introduced in chapter 2. These operators are called recursion operators for (adjoint) symmetries, SA- and AS operators. An SA operator maps symmetries into adjoint symmetries, an AS operator acts in the opposite direction. For an arbitrary dynamical system interesting operators of these four types do not exist in general. If there exists a nontrivial recursion operator for symmetries or for adjoint symmetries, it can be shown that under certain conditions its eigenvalues (if they exist) are constants of the motion. This suggests a possible relation between these recursion operators and the eigenvalue problems used in the inverse scattering method. For the Korteweg-de Vries- and Sawada-Kotera equation (sections 5.6 and 5.7) this relation can be given explicitly.

A more interesting situation appears if the dynamical system is a Hamiltonian system. In chapter 2 we introduce Hamiltonian systems using the language of symplectic geometry. The phase space of these Hamiltonian systems is a smooth manifold M . This results in Hamiltonian systems which are more general then the classical Hamiltonian systems written in terms of p_i and q_i. For a classical Hamiltonian system with configuration space Q we have $M = T^*Q$. It turns out that several interesting partial differential equations (Korteweg-de Vries-, sine-Gordon-, Benjamin-Ono equation) can be considered as infinite-dimensional Hamiltonian systems of this type.

In chapter 4 we study symmetries for Hamiltonian systems. The most important consequence of the Hamiltonian character of the system is that there always exists a relation between symmetries and adjoint symmetries, i.e. there always exists an SA- and an AS operator. This implies that every constant of the motion gives rise to a symmetry. This type of symmetry will be called a canonical symmetry. Very often there also exist non-canonical symmetries, i.e. symmetries which are not related in this way to a constant of the motion. For systems which can also be described by a Lagrangian the theorem of Noether gives a relation between special types of symmetries and constants of the motion. It can be shown that Noether's theorem can be applied to symmetries which, in the Hamiltonian setting, are canonical.

A non-canonical symmetry Z (in fact non-semi-canonical; we omit the prefix semi in this introduction) can be used to generate SA- and AS operators out of the already known ones (which are related to the Hamiltonian structure). By combination of these operators we obtain a recursion operator for (adjoint) symmetries $\Lambda(\Gamma)$. Then we can construct an infinite series of symmetries by

$$(0.4) \qquad X_k = \Lambda^{k-1} X .$$

An alternative way to generate infinite series of symmetries is to take the repeated Lie bracket with Z (= Lie derivative in the direction of Z)

$$(0.5) \qquad \tilde{X}_k = L_Z^{\,k-1} X .$$

In section 4.5 we show that if the non-canonical symmetry Z satisfies *additional conditions* the series (0.4) consists of canonical symmetries. So we have constructed an infinite series of constants of the motion (in involution). The series given in (0.5) is considered in section 4.6. It turns out that if $\tilde{X}_2 = bX_2$ then $\tilde{X}_k = b_k X_k$ ($b_k \in \mathbb{R}$). A series which (in general) consists of non-canonical symmetries is given by

$$(0.6) \qquad Z_k = \Lambda^{k-1} Z \qquad (Z_1 = Z).$$

The structure of the Lie algebra of symmetries, generated by the series X_k and Z_k, is also described in section 4.6. Finally we describe a third method for constructing infinite series of constants of the motion. This method is in fact a "combination of the previous two methods". It is clear that the existence of a non-canonical symmetry Z which satisfies the additional conditions mentioned above is a highly nontrivial property, which is in some way related to the "complete integrability" of the system.

Several examples of the preceding theory are considered in chapter 5. The methods described in chapter 4 (sometimes with slight modifications) can be applied to all given examples except the Burgers equation (a non-Hamiltonian system) and the Benjamin-Ono equation. For the Benjamin-Ono equation a non-canonical symmetry which satisfies the additional conditions (and so a nontrivial recursion operator for (adjoint) symmetries) has not been found. However, we can generate a rather complicated algebra of constants of the motion (or canonical symmetries) in another way. Our most extensive example is the Korteweg-de Vries equation. We shall show that there exist an infinite series of canonical symmetries and an infinite series of non-canonical symmetries. So we construct an infinite series of constants of the motion using only "infinitesimal transformations" of solutions (i.e. not by using Bäcklund (finite) transformations).

Some mathematical preliminaries are given in chapter 1. In particular in section 1.1 we shortly describe the differential geometrical methods used and in section 1.2 we show how these methods can be "generalized" to infinite-dimensional systems.

CHAPTER 1: MATHEMATICAL PRELIMINARIES.

1.1 DIFFERENTIAL GEOMETRY.

In this section we shall briefly describe some aspects of differential geometry. For a more comprehensive treatise and also for proofs of the results given here, we refer to the literature, for instance Abraham and Marsden [1,44] or Choquet-Bruhat [3].

<u>Tangent and cotangent spaces.</u>

Let M be a smooth *finite-dimensional manifold* with dimension n. The *tangent space* to M in a point $u \in M$ is denoted by T_uM. This is a linear space with dimension n. The *tangent bundle* TM is the union of all tangent spaces of M, so $TM = \bigcup_{u \in M} T_uM$. The tangent bundle TM is a manifold with dimension 2n. The *tangent bundle projection* $\pi_1 : TM \to M$ is a mapping which sends a tangent vector $A \in TM$ to its point of application. So if $A \in T_uM$ then $\pi_1(A) = u$.

The dual space of T_uM is the *cotangent space* T^*_uM. So an element $\alpha \in T^*_uM$ can be considered as a linear mapping $\alpha : T_uM \to \mathbb{R}$. Since the dimension of T_uM is finite, the dual space of T^*_uM is again T_uM. The *duality map* between T_uM and T^*_uM will be denoted by $\langle\cdot,\cdot\rangle$. So if $A \in T_uM$ and $\alpha \in T^*_uM$ then $\langle\alpha,A\rangle \in \mathbb{R}$.

The *cotangent bundle* T^*M is the union of all cotangent spaces of M, so $T^*M = \bigcup_{u \in M} T^*_uM$. It is again a manifold with dimension 2n. Suppose $\alpha \in T^*M$, so $\alpha \in T^*_uM$ for some $u \in M$. The mapping $\hat{\pi}_1 : T^*M \to M : \alpha \to u$ is called the *cotangent bundle projection*.

<u>Natural bases.</u>

Suppose we choose *local coordinates* $u^i (i=1,\ldots,n)$ on an open subset $U \subset M$ (so U can be described by one chart). By varying the coordinate u^1 and keeping the other coordinates fixed, we obtain a curve in $U \subset M$. The derivative of this curve (with respect to u^1) in a point $u \in M$, is an element of the tangent space T_uM. This tangent vector is denoted by $e_1 = \frac{\partial}{\partial u^1}$.
In a similar way we can construct the tangent vectors $e_i = \frac{\partial}{\partial u^i} \in T_uM$ $(i=2,\ldots,n)$. So in this way we can use the local coordinates u^i to construct a basis $\{ e_i = \frac{\partial}{\partial u^i} \mid i=1,\ldots,n \}$ for T_uM for all $u \in U$. This basis is called a *natural basis*. If $A \in T_uM$ with $u \in U$, it can be written as

(1.1.1) $$A = A^i e_i = A^i \frac{\partial}{\partial u^i} .$$

In this thesis we shall always use the convention that, unless otherwise indicated, summation takes place over all indices which appear twice, once as a subscript and once as a superscript.

A basis $\{ du^i \mid i=1,\dots,n \}$ for $T^*_u M$ is defined by

(1.1.2) $$\langle du^i, e_j \rangle = \delta^i_j \qquad \forall\ i,j = 1,\dots,n.$$

This basis is called the *natural cobasis*. The bases $\{ e_i \mid i=1,\dots,n \}$ for $T_u M$ and $\{ du^i \mid i=1,\dots,n \}$ for $T^*_u M$ are called each others *dual bases*. If $\alpha \in T^*_u M$ with $u \in U$, we can write

(1.1.3) $$\alpha = \alpha_i du^i.$$

Then

(1.1.4) $$\langle \alpha, A \rangle = \langle \alpha_i du^i, A^j e_j \rangle = \alpha_i A^j \delta^i_j = \alpha_i A^i.$$

Tensor fields.

We shall frequently need smooth functions, vector fields, one-forms and (higher order) tensor fields on M. For a formal definition of these objects (using sections of the corresponding vector bundles) see for instance Abraham and Marsden [1,44] or Choquet-Bruhat [3].

1.1.5 Definition.

The set of smooth *functions* on M will be denoted by $F(M)$. The sets of smooth *vector fields* and *(differential) one-forms* on M will be denoted by $X(M)$ respectively $X^*(M)$. Finally the set of smooth *tensor fields* on M with covariant order j and contravariant order i will be denoted by $T^i_j(M)$.

□

So if $A \in X(M)$ then $A(u) \in T_u M$ and if $\alpha \in X^*(M)$ then $\alpha(u) \in T^*_u M$. Of course we can expand vector fields and one-forms in the corresponding natural bases:

(1.1.6) $$A(u) = A^i(u) e_i(u) \text{ and } \alpha(u) = \alpha_i(u) du^i.$$

One-forms are sometimes called *covariant vector fields*, in contrast to vector fields which are called *contravariant vector fields*. Of course functions, vector fields and one-forms on M are special cases of tensor fields, so formally

$$F(M) = T^o_o(M), \; X(M) = T^1_o(M), \; X^*(M) = T^o_1(M).$$

Lie algebra's.

We now make some remarks on the structure of the sets introduced in definition 1.1.5. Of course all these sets are linear spaces (with infinite dimension). The product of two functions on M is again a function on M. This means that $F(M)$ is not only a *linear space* but also a *ring* (with identity). The product of a vector field, one-form or tensor field with a function yields again an object of the same type. This can be expressed by saying that $X(M)$, $X^*(M)$ and $T^j_i(M)$ are *modules over the ring* $F(M)$. The linear space $X(M)$ has additional structure. First we give the following

1.1.7 Definition.

A real linear space E with a bilinear product $[\cdot,\cdot] : E \times E \to E$, which satisfies

i) $[X,X] = 0 \quad \forall\, X \in E$,

ii) $[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \quad \forall\, X,Y,Z \in E$,

is called a *Lie algebra*.

□

Note that i) implies that the product is *antisymmetric*: $[X,Y] = -[Y,X]$. The second condition is called the *Jacobi identity*. It is well-known that the space $X(M)$ of vector fields on M is a Lie algebra. The product $[A,B]$ of two vector fields A and B on M is called the *Lie product* or *Lie bracket* of the vector fields A and B (see section 2.8 for an unusual (and complicated) introduction of the Lie bracket of vector fields). In local coordinates u^i the Lie bracket of the vector fields $A = A^i e_i$ and $B = B^i e_i$ is the vector field

(1.1.8) $$[A,B] = (B^i_{,j} A^j - A^i_{,j} B^j) e_i \, ,$$

where we use the notation $B^i_{,j} = \frac{\partial}{\partial u^j} B_i$, etc.

Tensor products.

In (1.1.6) we showed how vector fields and one-forms can be expanded in the natural bases corresponding to a coordinate system. By taking *tensor products* ($\otimes$) of the elements of these bases, we can construct bases for the various types of tensor fields. Suppose $\Phi \in T^o_2(M)$, $\Lambda \in T^1_1(M)$ and $\Psi \in T^2_o(M)$. Then, in a local coordinate system we can write

$$\Phi = \Phi_{ij} du^i \otimes du^j, \quad \Lambda = \Lambda^i_{\cdot j} e_i \otimes du^j,$$

$$\Psi = \Psi^{ij} e_i \otimes e_j.$$

The *tensor product* of the tensor fields $\Xi \in T^i_j(M)$ and $\Theta \in T^k_\ell(M)$ is a tensor field $\Xi \otimes \Theta \in T^{i+k}_{\ell+j}(M)$. For instance in local coordinates ($A \in X(M)$)

$$\Lambda \otimes \Psi = \Lambda^i_{\cdot j} \Psi^{k\ell} e_i \otimes e_k \otimes e_\ell \otimes du^j,$$

$$\Lambda \otimes A = \Lambda^i_{\cdot j} A^k e_i \otimes e_k \otimes du^j.$$

Contractions.

The tensor product is an operator which yields a tensor field of higher order(s) then the original tensor fields. An operator which lowers both orders of a tensor field is the *contraction*. Suppose $\Xi \in T^j_i(M)$ with $i,j \geq 1$. Then by contraction we obtain a tensor field $\Xi_C \in T^{j-1}_{i-1}(M)$. In fact if $i > 1$ and/or $j > 1$ several types of contraction are possible. As an example consider a tensor field $\Xi \in T^2_1(M)$. So, using a local coordinate system, we can write

$$\Xi = \Xi^{ij}_\ell e_i \otimes e_j \otimes du^\ell.$$

Then by contraction we can obtain the tensor(vector) fields

$$\Xi_{C_1} = \Xi^{ij}_i e_j \text{ and } \Xi_{C_2} = \Xi^{ij}_j e_i.$$

Contracted multiplication.

An operation which will be used very often in this thesis, is *contracted multiplication*, that is a tensor product followed by a contraction. Contracted multiplication of two tensor fields Ξ_1, Ξ_2 will be denoted by

$\Xi_1\Xi_2$. For instance

$$(1.1.9) \qquad \Lambda\Psi = (\Lambda \otimes \Psi)_C = \Lambda^i_k \Psi^{k\ell} e_i \otimes e_\ell,$$

$$(1.1.10) \qquad \Lambda A = (\Lambda \otimes A)_C = \Lambda^i_k A^k e_i.$$

The duality map between a vector field A and a one-form α can also be written as a contracted multiplication

$$\langle \alpha, A \rangle = \alpha A = (\alpha \otimes A)_C.$$

However, it will be convenient to use $\langle \cdot, \cdot \rangle$ for this duality map. It is easily seen from (1.1.10) that by contracted multiplication of a tensor field $\Lambda \in T^1_1(M)$ and a vector field A we obtain again a vector field ΛA on M. This means we can consider Λ also as a linear mapping $\Lambda : X(M) \to X(M)$. Similarly the contracted multiplication of a tensor field $\Gamma \in T^1_1(M)$ and a one-form α yields again a one-form $\Gamma\alpha$ $(= \Gamma^i_j \alpha_i du^j)$. So we can consider Γ also as a linear mapping $\Gamma : X^*(M) \to X^*(M)$. Note that Λ and Γ are tensor fields of the same type. The two different mappings are possible since we can perform different contractions. In general we shall use the symbol Λ for tensor fields which are used as a mapping $\Lambda : X(M) \to X(M)$ and the symbol Γ for tensor fields which are used as a mapping $\Gamma : X^*(M) \to X^*(M)$. Note that this means that in the contracted multiplication $\Lambda\Xi$ we contract "using the lower index of Λ" while in the contracted multiplication $\Gamma\Xi$ we contract "using the upper index of Γ". The contracted multiplication of a tensor field $\Phi \in T^o_2(M)$ and a vector field A yields a one-form $\alpha = \Phi A = \Phi_{ij} A^j du^i$. So we can also consider Φ as a linear mapping $\Phi : X(M) \to X^*(M)$. Finally a tensor field $\Psi \in T^2_o(M)$ can be used to transform a one-form into a vector field. Hence we can consider it as a linear mapping $\Psi : X^*(M) \to X(M)$.

<u>Vector bundle maps.</u>

We have seen that a tensor field $\Lambda \in T^1_1(M)$ can be used as a linear mapping $\Lambda : X(M) \to X(M)$. Of course we can also transform a vector $A \in T_uM$ into a vector $\Lambda A \in T_uM$. So we can also use Λ as a linear mapping $\Lambda : T_uM \to T_uM$. Since $u \in M$ is arbitrary we can also consider the tensor field Λ as a mapping $\Lambda : TM \to TM$. A mapping of this type (with $\Lambda : T_uM \to T_uM$ <u>linear</u>) is called a *vector bundle map*. Similar results hold for the other types of tensor fields.

We summarize the various applications of tensor fields with total order two in the following scheme

(1.1.11)

tensor field	linear map	vector bundle map
$\Lambda \in T^1_1(M)$	$\Lambda : X(M) \to X(M)$	$\Lambda : TM \to TM$,
$\Gamma \in T^1_1(M)$	$\Gamma : X^*(M) \to X^*(M)$	$\Gamma : T^*M \to T^*M$,
$\Phi \in T^o_2(M)$	$\Phi : X(M) \to X^*(M)$	$\Phi : TM \to T^*M$,
$\Psi \in T^2_o(M)$	$\Psi : X^*(M) \to X(M)$	$\Psi : T^*M \to TM$.

The difference between considering Λ as a vector bundle map $\Lambda : TM \to TM$ and as a linear map $\Lambda : X(M) \to X(M)$ is that with the vector bundle map we can transform one vector of TM, while the linear map $\Lambda : X(M) \to X(M)$ transforms a vector field on M.

Lie derivatives.

An extremely important tool in this thesis will be the *Lie derivative*. Suppose Ξ is a tensor field of arbitrary orders and A is a vector field. Then the Lie derivative $L_A\Xi$ is again a tensor field of the same type as Ξ. In the special case that $\Xi = B$ is a vector field, we have

$$L_AB = [A,B] = -L_BA. \tag{1.1.12}$$

In local coordinates the Lie derivatives of $\mathrm{F} \in F(M)$, $B \in X(M)$, $\alpha \in X^*(M)$, $\Phi \in T^o_2(M)$, $\Lambda \in T^1_1(M)$ and $\Psi \in T^2_o(M)$ are given by

$$\left\{\begin{aligned}
L_A\mathrm{F} &= \mathrm{F}_{,k}A^k,\\
L_AB &= [A,B] = (B^i_{,k}A^k - A^i_{,k}B^k)\; e_i,\\
L_A\alpha &= (\alpha_{i,k}A^k + \alpha_kA^k_{,i})\;\mathrm{d}u^i,\\
L_A\Phi &= (\Phi_{ij,k}A^k + \Phi_{ik}A^k_{,j} + \Phi_{kj}A^k_{,i})\;\mathrm{d}u^i \otimes \mathrm{d}u^j,\\
L_A\Lambda &= (\Lambda^i_{j,k}A^k - \Lambda^k_jA^i_{,k} + \Lambda^i_kA^k_{,j})\; e_i \otimes \mathrm{d}u^j,\\
L_A\Psi &= (\Psi^{ij}_{,k}A^k - \Psi^{ik}A^j_{,k} - \Psi^{kj}A^i_{,k})\; e_i \otimes e_j.
\end{aligned}\right. \tag{1.1.13}$$

The Lie derivative satisfies *Leibniz'rule*

$$L_A(\Xi_1 \otimes \Xi_2) = (L_A\Xi_1)\otimes \Xi_2 + \Xi_1 \otimes L_A\Xi_2.$$

Since the Lie derivative "commutes with contraction" this means that the Lie derivative also satisfies Leibniz'rule with respect to contracted multiplication. For instance

$$[A,\Lambda B] = L_A(\Lambda B) = (L_A\Lambda)B + \Lambda L_A B = (L_A\Lambda)B + \Lambda[A,B].$$

Differential forms.

A *(differential) k-form* ξ on M, considered in a point $u \in M$, is a k-linear completely antisymmetric mapping $\xi : T_uM \times T_uM \times \ldots \times T_uM \to \mathbb{R}$. This means we can identify a k-form with a completely antisymmetric tensor field with covariant order k and contravariant order 0. For instance a two-form ϕ can be identified with a tensor field $\Phi \in T_2^0(M)$

(1.1.14) $\qquad \phi(A,B) = \langle\Phi A,B\rangle \qquad \forall\ A,B \in X(M).$

Note that we consider the tensor field Φ as a mapping $\Phi : X(M) \to X^*(M)$. This different way of using a tensor field and the corresponding differential form is the reason for introducing a distinct notation. In general we shall use capital Greek letters for tensor fields. If a tensor field corresponds to a differential form, we denote this form by the corresponding small greek letter (Ξ,ξ; Φ,ϕ; Ω,ω). The *interior product* $i_A\xi$ of a k-form with a vector field yields a (k-1)-form defined by

(1.1.15) $\qquad i_A\xi(B_1, \ldots ,B_{k-1}) = \xi(A,B_1, \ldots ,B_{k-1}).$

It is easily seen that the (k-1)-form $i_A\xi$ corresponds to the tensor field ΞA. The interior product of a two-form with a vector field yields a one-form. From (1.1.14) we obtain

(1.1.16) $\qquad i_A\phi(B_1) \equiv \langle i_A\phi,B_1\rangle = \phi(A,B_1) = \langle\Phi A,B_1\rangle,$

which means $i_A\phi = \Phi A$. For a function $F \in F(M)$ we define $i_AF = 0$.

Exterior differentiation.

The interior product lowers the degree of a differential form. An operation which increases the degree of a differential form is *exterior differentiation*. If ξ is a k-form, the exterior derivative $d\xi$ is a (k+1)-form. In local coordinates the exterior derivative of a function F (= zero-form), one-form α and two-form ϕ are given by

$$(1.1.17)\quad \begin{cases} \langle dF, A\rangle = F_{,i}A^i \ , \\ d\alpha(A,B) = (\alpha_{i,j} - \alpha_{j,i})A^j B^i \ , \\ d\phi(A,B,C) = (\Phi_{ij,k} + \Phi_{jk,i} + \Phi_{ki,j})A^j B^i C^k \ , \end{cases}$$

for all vector fields $A,B,C \in X(M)$.

1.1.18 Definition.

A k-form ξ with $d\xi = 0$ is called a *closed k-form*. A k-form ξ ($k > 0$) which can be written as $\xi = d\zeta$ with ζ a (k-1)-form is called an *exact k-form*.

□

Since $d^2\zeta = dd\zeta = 0$ for all forms ζ, an exact form is always closed. In general the converse is not true.

1.1.19 Lemma (Poincaré).

Suppose ξ is a closed k-form on M. Then for every point $u \in M$ there exists a neighbourhood U such that $\xi|_U$ (ξ restricted to U) is exact.

Proof:

See for instance Abraham and Marsden [1, § 2.4.17].

□

So for every closed k-form ξ and every point $u \in M$ there exists a neighbourhood U of u and a (k-1)-form ζ on U such that $\xi = d\zeta$ on U. Of course this does not imply that $\xi = d\zeta$ on the whole manifold M.

Exterior multiplication.

Suppose $\Xi_1 \in T^o_k(M)$ and $\Xi_2 \in T^o_\ell(M)$ are two completely antisymmetric tensor fields. The corresponding differential forms are denoted by ξ_1 and ξ_2. It is easily seen that the tensor product $\Xi_1 \otimes \Xi_2 \in T^o_{k+\ell}(M)$ is in general not completely antisymmetric. By "antisymmetrization" of this tensor field we obtain a tensor field $\Xi \in T^o_{k+\ell}(M)$ which is again antisymmetric. The corresponding $(k+\ell)$-form ξ is written as

$$\xi = \xi_1 \wedge \xi_2 ,$$

and is called the *exterior product* of the forms ξ_1 and ξ_2. For instance if $k = \ell = 1$ we have

$$\Xi = \Xi_1 \otimes \Xi_2 - \Xi_2 \otimes \Xi_1 ,$$

$$\xi = \xi_1 \wedge \xi_2 .$$

The Lie derivative $L_A\Xi$ of a completely antisymmetric tensor field $\Xi \in T^o_k(M)$ is again an antisymmetric tensor field of the same type. The k-form corresponding to $L_A\Xi$ is denoted as $L_A\xi$, where ξ is the k-form corresponding to the tensor field Ξ. For instance for a two-form ϕ we have (see (1.1.14))

$$(1.1.20) \qquad (L_A\phi)(B_1,B_2) = \langle (L_A\Phi)B_1,B_2\rangle .$$

Note that this formula is only a consequence of the distinct notations we use for a tensor field and the corresponding differential form.

Several formulas.

Now we give a list of various other formulas which will be used in this thesis (see also Choquet-Bruhat [3, chapter IV, § A4]). Suppose Ξ_1 and Ξ_2 are arbitrary tensor fields, A and B are vector fields and α is a one-form on M. Then

$$(1.1.21) \qquad L_A(\Xi_1\Xi_2) = (L_A\Xi_1)\Xi_2 + \Xi_1(L_A\Xi_2),$$

(*Leibniz'rule* for contracted multiplication, same type of contraction in all terms)

(1.1.22) $\quad L_A\langle\alpha,B\rangle = \langle L_A\alpha,B\rangle + \langle\alpha,L_AB\rangle$,

(special case of (1.1.21))

(1.1.23) $\quad L_AB = [A,B] = -L_BA$,

(1.1.24) $\quad L_AL_B - L_BL_A = L_{[A,B]}$.

For the operators L_A, i_A and d on differential forms it can be shown that

(1.1.25) $\quad i_A{}^2 = i_Ai_A = 0$,

(1.1.26) $\quad d^2 = dd = 0$,

(1.1.27) $\quad L_A = di_A + i_Ad$,

(1.1.28) $\quad L_Ai_B - i_BL_A = i_{[A,B]}$,

(1.1.29) $\quad d\alpha(A,B) = L_A\langle\alpha,B\rangle - L_B\langle\alpha,A\rangle - \langle\alpha,[A,B]\rangle \quad$ (α one-form) ,

(1.1.30) $\quad d(\xi_1 \wedge \xi_2) = d\xi_1 \wedge \xi_2 + (-1)^k\xi_1 \wedge d\xi_2 \quad$ (ξ_1 k-form).

It is easily seen from (1.1.27) and (1.1.26) that

(1.1.31) $\quad dL_A = L_Ad.$

Suppose F is a function on M. Then using $i_AF = 0$ we obtain from (1.1.27) that

(1.1.32) $\quad L_AF = i_AdF \equiv \langle dF,A\rangle$.

Transformation properties.

Suppose there exists a diffeomorphism f between M and some other manifold N so f : $M \rightarrow N$. Then using this diffeomorphism all vector fields, differential forms, tensor fields on M can be transformed to objects of the same type on N. All operations described in this section are *natural* with respect to this transformation, i.e. the transformed objects satisfy similar relations as the original objects. For instance suppose A and B are two vector fields on M. The transformed vector fields on N are given by $\tilde{A} = f'A$ and $\tilde{B} = f'B$. Then it can be shown that

$$f'[A,B] = [(f'A),(f'B)],$$

so the transformed Lie bracket of A and B is equal to the Lie bracket of the transformed vector fields.

Parameterized tensor fields.

We shall frequently use functions, vector fields, differential forms and tensor fields on M which also depend on some additional parameter ($t \in \mathbb{R}$).

1.1.33 Definition.

The set of smooth *parameterized functions* on M will be denoted as $F_p(M)$. The sets of smooth *parameterized vector fields* and *one-forms* on M will be denoted as $X_p(M)$ and $X_p^*(M)$. Finally the set of smooth *parameterized tensor fields* on M with covariant order j and contravariant order i will be denoted as $T^i_{jp}(M)$. In all cases the parameter (t) is allowed to take all values in $\mathbb{R}$.

□

So if $Y \in X_p(M)$, then $Y(u,t) \in T_uM$ for all $t \in \mathbb{R}$. Of course $F_p(M) = F(M\times\mathbb{R})$. However, in order to keep a uniform notation, we shall only use $F_p(M)$. Of course $F(M)$, $X(M)$, $X^*(M)$ and $T^i_j(M)$ are (can be identified with) subsets of $F_p(M)$, $X_p(M)$, $X^*_p(M)$ and $T^i_{jp}(M)$.

1.2 "DIFFERENTIAL GEOMETRY" ON A TOPOLOGICAL VECTOR SPACE.

In the preceding section we gave an overview of some aspects of differential geometry on a finite-dimensional manifold M. The notions and relations introduced in that section will extensively be used in chapters 2, 3 and 4. So we can make a straightforward use of the results of those chapters if we consider a dynamical system on a finite-dimensional manifold (for instance the periodic Toda lattice [52]). However, several interesting dynamical systems are described by partial differential equations, i.e. they have "an infinite number of degrees of freedom". So at first sight we need the machinery of differential geometry, as described in section 1.1, also on manifolds of infinite dimension. Fortunately most of the interesting dynamical systems with "an infinite number of degrees of freedom" can be considered in a *topological vector space* instead of on an arbitrary manifold (of infinite dimension). In this way we can avoid the problems associated with differential geometry on manifolds of infinite dimension.

We shall now describe how several differential geometrical objects, introduced in section 1.1, can be "generalized" to the case that the manifold M is an (infinite dimensional) topological vector space W. The (topological) dual of W will be denoted by W^* and the duality map between W and W^* by $\langle\cdot,\cdot\rangle$. We only consider the case $W^{**} = W$, so W is *reflexive*. The space of linear continuous mappings of W into some topological vector space W_1 will be denoted by $L(W,W_1)$. We shall consider $L(W,W_1)$ as a topological vector space with the topology of bounded convergence (see Yosida [45, § IV.7]).

Since $M = W$ is a linear space, we can make the following identifications

$$T_u W = W\ , \quad TW = W \times W\ , \tag{1.2.1}$$

$$T_u^* W = W^*, \quad T^* W = W \times W^*.$$

Using these identifications it is easy to introduce (objects similar to) vector fields, differential forms and tensor fields on W. A vector field A on W is a mapping

$$A : W \to W \times W : u \to (u,\tilde{A}(u)) \tag{1.2.2}$$

where $\tilde{A} : W \to W$ is a , possibly nonlinear, mapping. So we can identify the vector field A with the mapping $\tilde{A}$. Therefore $\tilde{A}$ will also be called a vector field. To simplify notation we shall drop the tilde and write A instead of $\tilde{A}$. In a similar way we can introduce one-forms and tensor fields of higher order. This results in the following list of identifications (c.q. definitions in the infinite-dimensional case) :

	tensor field	"representation"
	$A \in X(W)$	$A : W \to W$,
	$\alpha \in X^*(W)$	$\alpha : W \to W^*$,
	$\Phi \in T^o_2(W)$, considered as vector bundle map $\Phi : TW \to T^*W$	$\Phi : W \to L(W,W^*)$,
(1.2.3)	$\Lambda \in T^1_1(W)$, considered as vector bundle map $\Lambda : TW \to TW$	$\Lambda : W \to L(W,W)$,
	$\Gamma \in T^1_1(W)$, considered as vector bundle map $\Gamma : T^*W \to T^*W$	$\Gamma : W \to L(W^*,W^*)$,
	$\Psi \in T^2_o(W)$, considered as vector bundle map $\Psi : T^*W \to TW$	$\Psi : W \to L(W^*,W)$.

Note that a tensor field in $T^1_1(W)$ can be represented by a linear operator (in fact operator field on W) $\Lambda(u) : W \to W$ and by a linear operator $\Gamma(u) : W^* \to W^*$. If $\Lambda(u)$ and $\Gamma(u)$ correspond to the same tensor field we have $\Lambda(u) = \Gamma^*(u)$ for all $u \in W$. If Φ is antisymmetric (so $\Phi(u)$ is antisymmetric for all $u \in W$) the corresponding differential two-form ϕ on W is given by

(1.2.4) $$\phi(u)(A,B) = \langle\Phi(u)A,B\rangle \qquad \forall\, A,B \in W.$$

In a similar way we can introduce higher order tensor fields and differential forms. However, the tensor fields introduced above will be sufficient for the sequel.

Next we introduce Lie derivatives and (for differential forms) exterior derivatives. First some remarks on differential calculus in topological vector spaces. For a more detailled discussion of this complicated subject we refer to Yamamuro [46]. Suppose W_1 is some topological vector space and f is a (nonlinear) mapping $f : W \to W_1$.

1.2.5 Definition.

We call f *Gateaux differentiable* in $u \in W$ if there exists a mapping $\theta \in L(W,W_1)$ such that for all $A \in W$

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (f(u + \varepsilon A) - f(u) - \theta A) = 0 \tag{1.2.6}$$

in the topology of W_1. The linear mapping $\theta \in L(W,W_1)$ is called the *Gateaux derivative* of f in u and is written as $\theta = f'(u)$.

□

If f is Gateaux differentiable in all points $u \in W$, we can consider the Gateaux derivative as a (in general nonlinear) mapping

$$f' : W \to L(W,W_1).$$

Suppose f' is again Gateaux differentiable in $u \in W$. The second derivative of f in $u \in W$ is a linear mapping $f''(u) \in L(W, L(W,W_1))$. It is easily seen that $f''(u)$ can be considered as a bilinear mapping

$$f''(u) : W \times W \to W_1.$$

Under certain assumptions it can be shown that this mapping is symmetric: $f''(u)(v,w) = f''(u)(w,v)$ for all $w,v \in W$ (see [46]). We shall call a mapping $f : W \to W_1$ *twice differentiable* if its first and second Gateaux derivatives exist and if $f''(u)$ is a symmetric bilinear mapping for all $u \in W$. We assume all mappings in this section are twice differentiable.

1.2.7 Remark.

Note that in the limit given in (1.2.6) a uniformity in w is not required. If this limit is uniform on all sequentially compact subsets of W, the mapping f is called *Hadamard differentiable*. If the limit is uniform on all bounded subsets of W, the mapping f is called *Fréchet differentiable*.

□

Suppose $A : W \to W$ is (represents) a vector field. The Gateaux derivative in $u \in W$ is a linear mapping $A'(u) : W \to W$. The dual of this mapping is denoted by $A'^*(u) : W^* \to W^*$.

1.2.8 Definition.

The *Lie derivatives* in the direction of a vector field A of a function $F : W \to \mathbb{R}$ and of the various tensor fields (vector fields, one-forms) considered in (1.2.3) are defined by

$$
(1.2.9)\quad \begin{cases}
L_A F(u) = F'(u)A \equiv \langle F'(u), A\rangle , \\
L_A B(u) \equiv [A,B](u) = B'(u)A(u) - A'(u)B(u) , \qquad (B \in X(W)), \\
L_A \alpha(u) = \alpha'(u)A(u) + A'^*(u)\alpha(u) , \\
L_A \Phi(u) = (\Phi'(u)A(u)) + \Phi(u)A'(u) + A'^*(u)\Phi(u) , \\
L_A \Lambda(u) = (\Lambda'(u)A(u)) + \Lambda(u)A'(u) - A'(u)\Lambda(u) , \\
L_A \Gamma(u) = (\Gamma'(u)A(u)) - \Gamma(u)A'^*(u) + A'^*(u)\Gamma(u) , \\
L_A \Psi(u) = (\Psi'(u)A(u)) - \Psi(u)A'^*(u) - A'(u)\Psi(u) .
\end{cases}
$$

□

First some remarks on the notation in these expressions. Consider the formula for $L_A\Phi$. Since $\Phi : W \to L(W,W^*)$ we have $\Phi'(u) \in L(W, L(W,W^*))$. So $(\Phi'(u)A) \in L(W,W^*)$ and $(\Phi'(u)A)B \in W^*$. By definition

$$(\Phi'(u)A)B = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} (\Phi(u + \varepsilon A)B - \Phi(u)B).$$

Of course in general this expression is not symmetric in A and B. Therefore we shall always insert brackets in expressions of this type. It is easily seen that the Lie derivative of an object yields again an object of the same type. Note that if $\Gamma^*(u) = \Lambda(u)$ (so Γ and Λ represent the same tensor field) the same holds true for the Lie derivatives: $(L_A\Gamma(u))^* = L_A\Lambda(u)$. Next we define *exterior derivatives* of zero-, one- and two-forms.

1.2.10 Definition.

i) The exterior derivative of a function $F : W \to \mathbb{R}$ is the mapping $dF : W \to W^* : u \to F'(u)$ (so $dF(u) = F'(u)$).

ii) The exterior derivative of a one-form $\alpha : \mathcal{W} \to \mathcal{W}^*$ is the two-form

$$d\alpha(A,B) = \langle\alpha'(u)A,B\rangle - \langle\alpha'(u)B,A\rangle$$

$$= \langle(\alpha'(u) - \alpha'^*(u))A,B\rangle \qquad \forall\ A,B \in \mathcal{W}.$$

iii) The exterior derivative of a two-form ϕ, corresponding to an operator $\Phi(u)$ as in (1.2.4), is given by

$$d\phi(A,B,C) = \langle(\Phi'(u)A)B,C\rangle + \langle(\Phi'(u)B)C,A\rangle + \langle(\Phi'(u)C)A,B\rangle\ ,$$

$$\forall\ A,B,C \in \mathcal{W}.$$

□

Note that the definitions (1.2.8) and (1.2.10) strongly resemble the expressions in local coordinates (1.1.13) and (1.1.17) for the corresponding objects on a finite-dimensional manifold. *Contractions* and *interior products* in the infinite-dimensional case are interpreted via (1.2.3). Also we shall adopt the notions *closed* and *exact* differential forms (see definition 1.1.18).

1.2.11 Theorem.

The relations (1.1.22) up to (1.1.32) included are also valid for Lie derivatives and exterior derivatives given in definitions 1.2.8 and 1.2.10.

Proof:
All proofs are similar to proofs in local coordinates of the corresponding relations on a finite-dimensional manifold. If a second derivative appears, we need its symmetry.

□

Suppose α is a closed one-form with continuous derivative $\alpha'(u) : \mathcal{W} \to \mathcal{W}^*$. Then (definition 1.2.10 ii) $\alpha'(u) = \alpha'^*(u)$ for all $u \in \mathcal{W}$. Since $\mathcal{W}$ is a linear space, a closed differential form is also exact. Define the function $F : \mathcal{W} \to \mathbb{R}$ by

$$(1.2.12) \qquad F(u) = \int_0^1 \langle\alpha(au),u\rangle\ da.$$

Then it is easily verified that $\alpha = dF$, so indeed α is also an exact one-form.

In a somewhat different context an operator $\alpha : \mathcal{W} \to \mathcal{W}^*$ with $\alpha'(u) = \alpha'^*(u)$ is called a *potential operator*. Expression, similar to (1.2.12), can be given for closed higher order differential forms.

Finally we mention that we shall use the same notation as introduced in definition 1.1.33 for *parameterized* functions, vector fields, one-forms and higher order tensor fields on $\mathcal{W}$.

1.3 SOME FUNCTION SPACES.

In chapter 5 we shall consider several nonlinear evolution equations. Some of these equations can be written in the form

$$(1.3.1) \qquad u_t = f(u,u_x,\ldots),$$

where f is a polynomial in u and its derivatives. The Burgers equation (section 5.2), Korteweg-de Vries equation (section 5.6) and the Sawada-Kotera equation (section 5.7) are of this type. In this section we describe function spaces in which we shall consider these equations. For convenience we set $\partial = \frac{d}{dx}$.

1.3.2 Definition.

For $p > 0$ we define the space S_p by

$$S_p = \{ u \in C^\infty(\mathbb{R}) \mid \sqrt{x^2+1}^{\,m+p}\, \partial^m u(x) \in L_1(\mathbb{R}),\ \forall\, m \geq 0 \}.$$

□

The following two theorems describe some properties of the space S_p.

1.3.3 Theorem.

For every function $u \in S_p$ there exists a series of constants C_m such that

$$|\partial^m u(x)| \leq \frac{C_m}{\sqrt{x^2+1}^{\,m+p+1}} \qquad m = 0,1,2,\ldots .$$

Proof:

Set $v_m(x) = \sqrt{x^2+1}^{\,m+p+1}\, \partial^m u(x)$. Then

$$\partial v_m(x) = (m+p+1)\sqrt{x^2+1}^{\,m+p-1} x\partial^m u(x) + \sqrt{x^2+1}^{\,m+p+1}\partial^{m+1}u(x).$$

Hence

$$|\partial v_m(x)| \le (m+p+1)|\sqrt{x^2+1}^{\,m+p}\partial^m u(x)| + |\sqrt{x^2+1}^{\,m+p+1}\partial^{m+1}u(x)|.$$

Since $u \in S_p$ this means that $\partial v_m \in L_1(\mathbb{R})$. Then from

$$v_m(x) = v_m(0) + \int_0^x \partial v_m(x')\, dx'$$

we see that v_m is bounded; there exists a constant C_m such that $|v_m(x)| \le C_m$ $\forall\, x \in \mathbb{R}$.

□

1.3.4 Theorem.

Suppose $u \in S_p$. Then also $xu_x \in S_p$.

Proof:
From $\partial^m(xu_x) = x\partial^{m+1}u + m\partial^m u$ we obtain

$$|\sqrt{x^2+1}^{\,m+p}\partial^m(xu_x)| \le |\sqrt{x^2+1}^{\,m+p+1}\partial^{m+1}u| + |\sqrt{x^2+1}^{\,m+p}\partial^m u|.$$

Both terms of the right hand side are elements of $L_1(\mathbb{R})$, so also the left hand side is an element of $L_1(\mathbb{R})$.

□

We shall also need smooth functions v which satisfy the following conditions

(1.3.5) $\lim_{x\to\infty} v(x) = -\lim_{x\to-\infty} v(x) = a \in \mathbb{R}$, a depends on v,

(1.3.6) $\sqrt{x^2+1}^{\,m+p}\partial^{m+1}v(x) \in L_1(\mathbb{R}) \qquad \forall\, m \ge 0.$

1.3.7 Definition.

For $p > 0$ we define the space $\mathcal{U}_p$ by

$$\mathcal{U}_p = \{\, v \in C^\infty(\mathbb{R}) \mid v \text{ satisfies (1.3.5) and (1.3.6)} \,\}.$$

□

We now consider the relations between the spaces S_p and U_p.

1.3.8 Theorem.

i) $S_p \subset U_p$,
ii) if $v \in U_p$ then $\partial v = v_x \in S_p$,
iii) if $u \in S_p$ and $v \in U_p$ then $uv \in S_p$.

Proof:
The first two parts of this theorem follow immediately from the definitions of S_p and U_p. An elementary calculation yields

$$\sqrt{x^2+1}^{\,m+p}\,\partial^m(uv) = \sum_{i=0}^{m} \binom{m}{i} (\sqrt{x^2+1}^{\,i+p}\,\partial^i u)(\sqrt{x^2+1}^{\,m-i}\,\partial^{m-i} v). \tag{1.3.9}$$

Since $u \in S_p$ we have $\sqrt{x^2+1}^{\,i+p}\,\partial^i u \in L_1(\mathbb{R})$. We now consider the function $\sqrt{x^2+1}^{\,m-i}\,\partial^{m-i} v$. For $i = m$ this is equal to v, which is clearly a bounded function.For $i < m$ we obtain from part ii) of this theorem and theorem 1.3.3 also that this function is bounded. Hence the left hand side of (1.3.9) is an element of $L_1(\mathbb{R})$.

□

1.3.10 Corollary.

If $u \in S_p$ and $v \in S_p$ then also $uv \in S_p$.

□

We have seen that the operator $\partial = \frac{d}{dx}$ maps U_p into S_p. It is possible to define an inverse operator which acts in the opposite direction.

1.3.11 Theorem.

The inverse operator of $\partial : U_p \to S_p$ is the operator $\partial^{-1} : S_p \to U_p$, defined by

$$\partial^{-1}u(x) = \int_{-\infty}^{x} u(y)\,dy - \frac{1}{2}\int_{-\infty}^{\infty} u(y)\,dy. \tag{1.3.12}$$

Proof:
For $u \in S_p$ both integrals exist. We now show that $\partial^{-1}u \in U_p$. It is easily

seen that $\partial^{-1}u$ satisfies (1.3.5) with

$$a = \frac{1}{2} \int_{-\infty}^{\infty} u(y)\, dy.$$

Since $\partial\partial^{-1}u = u$ and $u \in S_p$ it follows from the definition of S_p that $\partial^{-1}u$ also satisfies (1.3.6). The proof is completed by noting that $\partial^{-1}\partial v = v$ for arbitrary $v \in U_p$.

□

Next we introduce a *topology* on S_p and on U_p. For $v \in U_p$ and $u \in S_p$ define

$$\text{(1.3.13)} \qquad \langle v,u\rangle = \int_{-\infty}^{\infty} v(x)u(x)\, dx.$$

This bilinear mapping $U_p \times S_p \to \mathbb{R}$ is called a *duality* or *duality map*. It is easily seen that this duality map is *separating*, i.e. for every nonzero $v \in U_p$ there exists a $u \in S_p$ such that $\langle v,u\rangle \neq 0$ and for every nonzero $u \in S_p$ there exists a $v \in U_p$ with $\langle v,u\rangle \neq 0$. With every $v \in U_p$ corresponds a *seminorm* $p_v(u) = |\langle v,u\rangle|$ on S_p. Also every $u \in S_p$ gives rise to a seminorm $q_u(v) = |\langle v,u\rangle|$ on U_p. Then, using the family of seminorms $\{ p_v \mid v \in U_p \}$, we can supply S_p with a topology. The seminorms $\{ q_u \mid u \in S_p \}$ provide U_p with a topology. Some properties of both topological spaces are described in

1.3.14 Theorem.

The spaces S_p and U_p are *locally convex Hausdorff topological vector spaces*. The *(topological) dual* of S_p is (can be represented by) U_p and the (topological) dual of U_p is S_p, so

$$S_p^* = U_p\ ,\ U_p^* = S_p.$$

Proof:
See Choquet [43; propositions 22.3 and 22.4].

□

Since we now have a topology on S_p and on U_p we can study the continuity of the various mappings between these spaces. Recall that a mapping of a topological space into a topological space is continuous iff the inverse image of an open set is open. Suppose W_1 and W_2 are topological vector spaces with topologies generated by the families of seminorms $\{q_i\}$

respectively $\{p_i\}$. Then a linear mapping $\Theta : \mathcal{W}_1 \to \mathcal{W}_2$ is continuous iff for every seminorm p_i on $\mathcal{W}_2$ there exist a constant C and a seminorm q_j on $\mathcal{W}_1$ such that

$$p_i(\Theta w) \le Cq_j(w) \qquad \forall\ w \in \mathcal{W}_1.$$

If $\mathcal{W}_1 \subset \mathcal{W}_2$ we can consider an element of $\mathcal{W}_1$ also as an element of $\mathcal{W}_2$. This mapping of $\mathcal{W}_1$ into $\mathcal{W}_2$ is called the *embedding operator*.

1.3.15 Theorem.

The mappings $\partial : \mathcal{U}_p \to \mathcal{S}_p$ and $\partial^{-1} : \mathcal{S}_p \to \mathcal{U}_p$ are continuous. Suppose $u \in \mathcal{S}_p$. Then the mapping $m_u : \mathcal{U}_p \to \mathcal{S}_p : v \to uv$ is continuous. The embedding operator of $\mathcal{S}_p$ into $\mathcal{U}_p$ is also continuous.

Proof:
Suppose $v \in \mathcal{U}_p$, then $\partial v = v_x \in \mathcal{S}_p$. For an arbitrary $w \in \mathcal{U}_p$ we have

$$p_w(v_x) = \left|\int_{-\infty}^{\infty} wv_x\, dx\right| = \left|\int_{-\infty}^{\infty} vw_x\, dx\right| = q_{w_x}(v).$$

This means that $\partial : \mathcal{U}_p \to \mathcal{S}_p$ is continuous. The continuity of the other mappings is proved in a similar way.

□

Suppose $u \in \mathcal{S}_p$. To simplify notation we will denote the mapping $m_u : \mathcal{U}_p \to \mathcal{S}_p$ (multiplication by u) by $u : \mathcal{U}_p \to \mathcal{S}_p$. Then, using various parts of this theorem, we see that for instance $u\partial$, ∂u, ∂^3, $u\partial^{-1}u : \mathcal{U}_p \to \mathcal{S}_p$ and $\partial^{-1}u$, $u\partial^{-1}$, $\partial^{-1}u\partial^{-1} : \mathcal{S}_p \to \mathcal{U}_p$ are continuous mappings.

Consider the topological vector spaces $\mathcal{W}_1$ and $\mathcal{W}_2$ with (topological) duals $\mathcal{W}_1^*$ and $\mathcal{W}_2^*$. The *dual operator* of a linear operator $\Theta : \mathcal{W}_1 \to \mathcal{W}_2$ is the linear operator $\Theta^* : \mathcal{W}_2^* \to \mathcal{W}_1^*$ defined by

(1.3.16) $$\langle\Theta^* w_2, w_1\rangle = \langle\Theta w_1, w_2\rangle \qquad \forall\ w_1 \in \mathcal{W}_1,\ w_2 \in \mathcal{W}_2^*.$$

A special situation occurs if $\mathcal{W}_1^* = \mathcal{W}_2$ and $\mathcal{W}_2^* = (\mathcal{W}_1^{**}=)\ \mathcal{W}_1$ (so $\mathcal{W}_1$ is reflexive). Then $\Theta : \mathcal{W}_1 \to \mathcal{W}_2$ and also $\Theta^* : \mathcal{W}_1 \to \mathcal{W}_2$. In this case we call an operator Θ *symmetric* if $\Theta^* = \Theta$ and *antisymmetric* if $\Theta^* = -\Theta$.

1.3.17 Theorem.

The operators $\partial : \mathcal{U}_p \to \mathcal{S}_p$ and $\partial^{-1} : \mathcal{S}_p \to \mathcal{U}_p$ are antisymmetric, so

$$(1.3.18) \qquad \langle v_1, \partial v_2 \rangle = - \langle v_2, \partial v_1 \rangle \qquad \forall\, v_1, v_2 \in \mathcal{U}_p,$$

$$(1.3.19) \qquad \langle \partial^{-1} u_1, u_2 \rangle = - \langle \partial^{-1} u_2, u_1 \rangle \qquad \forall\, u_1, u_2 \in \mathcal{S}_p.$$

Proof:
The first expression follows by partial integration. The proof of (1.3.19) is a straightforward computation using (1.3.12) and (1.3.13).

□

We shall frequently need the dual of an operator which is the *composition* of two other operators. Suppose $\Theta = \Theta_2\Theta_1 : \mathcal{W}_1 \to \mathcal{W}_2$ with $\Theta_1 : \mathcal{W}_1 \to \mathcal{W}_3$ and $\Theta_2 : \mathcal{W}_3 \to \mathcal{W}_2$. Then it is easily seen that $\Theta^* = \Theta_1^*\Theta_2^*$.

Finally we describe some operators which we shall use frequently in chapter 5 (in particular in section 5.6). For $u \in \mathcal{S}_p$ consider the operators $u\partial$, ∂u, $\partial^3 : \mathcal{U}_p \to \mathcal{S}_p$. The dual operators are found to be $(u\partial)^* = -\partial u$, $(\partial u)^* = -u\partial$ and $(\partial^3)^* = -\partial^3$. This means that

$$(1.3.20) \qquad \Phi = u\partial + \partial u - \partial^3 : \mathcal{U}_p \to \mathcal{S}_p$$

is an antisymmetric operator. We shall also meet the operator

$$\Gamma = \partial^{-1}\Phi = \partial^{-1}u\partial + u - \partial^2 : \mathcal{U}_p \to \mathcal{U}_p.$$

The dual operator of Γ is then given by

$$\Lambda = \Gamma^* = \Phi^*(\partial^{-1})^* = \Phi\partial^{-1} = u + \partial u \partial^{-1} - \partial^2 : \mathcal{S}_p \to \mathcal{S}_p.$$

1.4 THE HILBERT TRANSFORM.

In this section we describe some properties of the Hilbert transform, which are used in section 5.8. The *Hilbert transform* of a function $u \in L_2(\mathbb{R})$ is defined by

$$\mathcal{H}u(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{y-x}\, dy \qquad \text{(principal value integral).}$$

1.4.1 Lemma.

Suppose $u \in S_p$ with $0 < p < 1$, then the function

$$(1.4.2) \qquad w(x) = \frac{p}{\pi} \int_{-\infty}^{\infty} \frac{yu(y)}{y-x}\, dy \qquad \text{(principal value integral)}$$

is bounded for all $x \in \mathbb{R}$.

Proof:
It follows from the definition of S_p that $u \in L_1(\mathbb{R})$. Suppose $x > 0$. Then we can write (1.4.2) as

$$(1.4.3) \qquad w(x) = \frac{1}{\pi} \int_{-\infty}^{\frac{1}{2}x} \frac{yu(y)}{y-x}\, dy + \frac{p}{\pi} \int_{\frac{1}{2}x}^{\frac{3}{2}x} \frac{yu(y)}{y-x}\, dy + \frac{1}{\pi} \int_{\frac{3}{2}x}^{\infty} \frac{yu(y)}{y-x}\, dy$$

$$= I_1 + I_2 + I_3.$$

It is easily seen that $|I_1 + I_3| \le \frac{3}{\pi} \int_{-\infty}^{\infty} |u(y)|dy$. Set $v(y) = yu(y)$. Then we obtain from theorem 1.3.3 that

$$(1.4.4) \qquad |v(y)| \le \frac{C_o}{\sqrt{y^2+1}^{\,p}}, \quad |v_y(y)| \le \frac{C_o + C_1}{\sqrt{y^2+1}^{\,p+1}}$$

for all $y \in \mathbb{R}$. Using the mean value theorem we obtain

$$I_2 = \frac{p}{\pi} \int_{\frac{1}{2}x}^{\frac{3}{2}x} \frac{v(x) + (y-x)v_y(a(y))}{y-x}\, dy \qquad (|a(y)-x| < |y-x|).$$

$$= \frac{1}{\pi} \int_{\frac{1}{2}x}^{\frac{3}{2}x} v_y(a(y))\, dy.$$

Then (1.4.4) implies

$$I_2 \le \frac{1}{\pi}\, x\, \frac{C_o + C_1}{\sqrt{(\frac{1}{2}x)^2+1}} < \frac{4(C_o + C_1)}{\pi} \qquad \text{for } x > 0.$$

Hence $w(x)$ is bounded for $x > 0$. A similar estimate can be given for $x < 0$.

□

1.4.5 Lemma.

If $u \in S_p$ with $0 < p < 1$ then $Hu \in C^{\infty}(\mathbb{R})$ and

$$(1.4.6) \qquad \mathcal{H}u(x) \le \frac{C}{\sqrt{x^2+1}} \qquad \forall\, x \in \mathbb{R}.$$

Proof:
Since $u \in S_p$ we have $\partial^m u \in L_2(\mathbb{R})$ for $m = 0,1,2,\ldots$. So $\mathcal{H}\partial^m u = \partial^m \mathcal{H}u \in L_2(\mathbb{R})$, which implies that $\mathcal{H}u \in C^\infty(\mathbb{R})$. Next note that

$$x\mathcal{H}u(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{xu(y)}{y-x}\, dy$$

$$(1.4.7) \qquad = \frac{-1}{\pi} \int_{-\infty}^{\infty} u(y)\, dy + \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{yu(y)}{y-x}\, dy.$$

Then using lemma 1.4.1 and $x\mathcal{H}u(x) \in C^\infty(\mathbb{R})$ we obtain (1.4.6).

□

1.4.8 Corollary.

If $u \in S_p$ and $xu \in S_p$ then

$$(1.4.9) \qquad x\mathcal{H}u(x) = -\frac{1}{\pi} \int_{-\infty}^{\infty} u(y)\, dy + \mathcal{H}(xu(x)).$$

Proof:
This result follows at once from (1.4.7).

□

1.4.10 Theorem.

For $0 < p < 1$ we have $\mathcal{H} : S_p \to \mathcal{U}_p$.

Proof:
It follows from lemma 1.4.5 that $\mathcal{H}u \in C^\infty(\mathbb{R})$ and $\lim_{x\to\pm\infty} \mathcal{H}u(x) = 0$. So we only have to show that $\sqrt{x^2+1}^{\,m+p}\partial^{m+1}\mathcal{H}u(x) \in L_1(\mathbb{R})$.
Note that if $u \in S_p$ then $x^j\partial^m u \in S_p$ for $j \le m$ (see theorem 1.3.4). By using (1.4.9) we obtain

$$x^{m+1}\partial^{m+1}\mathcal{H}u = x^{m+1}\mathcal{H}\partial^{m+1}u$$

$$= x^m\mathcal{H}(x\partial^{m+1}u)$$

$$= \ldots$$

$$= H(x^{m+1}\partial^{m+1}u).$$

Since $x^{m+1}\partial^{m+1}u \in S_p$ we obtain from lemma 1.4.5 and the fact that $Hu \in C^\infty(\mathbb{R})$ that

$$\left|\sqrt{x^2+1}^{\,m+1}\,\partial^{m+1}Hu\right| \leq \frac{C}{\sqrt{x^2+1}}.$$

Since $0 < p < 1$ this implies that $\sqrt{x^2+1}^{\,m+p}\,\partial^{m+1}Hu(x) \in L_1(\mathbb{R})$ for $m = 0,1,2,\ldots$. Thus we proved that $Hu \in U_p$.

□

Finally we mention some other properties of the Hilbert transform:

(1.4.11) $\displaystyle\int_{-\infty}^{\infty} uHv\,dx = -\int_{-\infty}^{\infty} vHu\,dx$ (antisymmetry),

(1.4.12) $HHu(x) = -u(x)$,

(1.4.13) $\partial Hu = H\partial u$,

(1.4.14) $(Hu)(Hv) = uv + H(uHv) + H(vHu)$.

1.5 ANALYTICALLY INDEPENDENT FUNCTIONS.

1.5.1 Definition.

The functions $F_1, \ldots, F_k$ on a possibly infinite-dimensional manifold M are called *analytically independent* if the corresponding one-forms $dF_1(u), \ldots, dF_k(u)$ are linearly independent elements of T_u^*M for all $u \in N$, where N is a dense open subset of M.

□

If the manifold M is finite-dimensional, we can introduce local coordinates u^i $(i=1,\ldots,n)$ on $U \subset M$. Then it is easily seen that the functions $F_1, \ldots, F_k$ are analytically independent iff the Jacobian matrix

$$\begin{pmatrix} \frac{\partial F_1}{\partial u^1} & \cdots & \frac{\partial F_1}{\partial u^n} \\ \vdots & & \vdots \\ \frac{\partial F_k}{\partial u^1} & \cdots & \frac{\partial F_k}{\partial u^n} \end{pmatrix}$$

has rank k. This also implies that on a manifold of dimension n there can exist at most n analytically independent functions. The notion analytically independent is explained in the following

1.5.2 Theorem.

Suppose M is a finite-dimensional manifold. The functions $F_1, \ldots, F_k$ on M are analytically independent iff locally there does not exist a relation

$$g(F_1, \ldots, F_k) = 0$$

where $g : \mathbb{R}^k \to \mathbb{R}$ is a smooth function such that in every point of an open dense subset of $\mathbb{R}^k$ the gradient (one-form dg) does not vanish.

Proof:
See Levi-Civita [54; chapter 1, § 5,6].

□

CHAPTER 2: SYMMETRIES FOR DYNAMICAL SYSTEMS.

2.1 INTRODUCTION.

This chapter deals with some general properties of dynamical systems on manifolds. If the dynamical system is a Hamiltonian system, more specific results can be obtained. Those more specific results will be considered in chapter 4. In section 2.2 we shall introduce two linear equations associated with the dynamical system. Solutions of these equations will be called *symmetries* and *adjoint symmetries*. Since most of the considerations in section 2.2 are of local character, we shall use a local trivialization of the (co)tangent bundle of the manifold. An introduction of symmetries without using a local trivialization of the tangent bundle will be described in the appendix of this chapter. Several properties of symmetries and adjoint symmetries are considered in sections 2.3 and 2.4. The possible relations of symmetries and adjoint symmetries are studied in section 2.5. In section 2.6 we consider a dynamical system for which there exist two infinite series of symmetries. This situation will occur several times in chapters 4 and 5. Finally in section 2.7 we study the transformation properties of (adjoint) symmetries.

A very important tool in this chapter is the Lie derivative of several types of tensor fields in the direction of a vector field. Sometimes we shall also give the more classical formulas, using local coordinates. In that case the manifold is assumed to be finite-dimensional. For an infinite-dimensional manifold our results are formal.

Symmetries (also called invariant variations, infinitesimal transformations or Lie-Bäcklund operators) are also studied by Olver [13], Wadati [14], Fokas [15] , Magri [17] , Fuchssteiner and Fokas [8], etc.. These last mentioned authors also describe adjoint symmetries (which they call conserved covariants). Most authors consider a dynamical system in some (unspecified) topological vector space and write their expressions in terms of Gateaux, Hadamard or Fréchet derivatives. However, the only natural type of derivative for studying symmetries is the (infinite-dimensional version of the) Lie derivative, which replaces complicated combinations of derivatives of one of the previous types. Using this Lie derivative most expressions are considerably simplified and important new relations can be found. Since Lie derivatives are also defined on (in fact invented for) arbitrary smooth manifolds, we can easily describe the theory for dynamical systems on manifolds. In contrast to most authors we also consider (adjoint) symmetries which depend explicitly on the time t. In several applications this type of (adjoint) symmetry turns out to be important.

2.2 DEFINITION OF SYMMETRIES AND ADJOINT SYMMETRIES.

Suppose M is a manifold and X a vector field on M, so $X \in \mathcal{X}(M)$. For a curve u(t) on M we set $\dot{u}(t) = \frac{d}{dt} u(t) \in T_{u(t)}M$.

In this chapter we shall consider the following *autonomous differential equation* on M

$$(2.2.1) \qquad \dot{u}(t) = X(u(t)).$$

The parameter t is called time. This equation can be supplied with an initial condition $u(t_o) = u_o$. Since (2.2.1) is an autonomous system, it is no restriction to take $t_o = 0$. We shall assume that for all $u_o \in M$ and $t_o \in \mathbb{R}$ there exists a unique solution u(t) of (2.2.1), with $u(t_o) = u_o$, defined on some interval $I \in \mathbb{R}$.

Suppose U is an open subset of M which can be described by one chart. This means the tangent bundel TU is a trivial bundle, $TU = U \times W$ for some linear space W. Then we can consider the vector field X as a mapping $X : U \to W$. The derivative of $X(u)$ in a point $u \in U$ is a linear mapping $X'(u) : W \to W$. Suppose u(t) is a solution of (2.2.1) which lies in U. Then we can linearize (2.2.1) around u(t) and obtain

$$(2.2.2) \qquad \dot{v}(t) = X'(u(t))\, v(t) \qquad v(t) \in T_{u(t)}U = W.$$

Since $\frac{d}{dt} X(u(t)) = X'(u(t))X(u(t))$, this equation has always the solution $v(t)= X(u(t))$. Another interesting linear equation, associated with (2.2.1) is the so-called adjoint equation of (2.2.2)

$$(2.2.3) \qquad \dot{w}(t) = -X'^*(u(t))\, w(t) \qquad w(t) \in T^*_{u(t)}U = W^*,$$

where $X'^*(u) : W^* \to W^*$ is the dual operator of $X'(u)$. The equations (2.2.1) and (2.2.3) can be derived from the following *variational principle*

$$(2.2.4) \qquad \mathrm{stat} \int_{t_1}^{t_2} \langle w(t), \dot{u}(t) - X(u(t))\rangle \, dt,$$

over the set of all curves $u(t) \in U$, $w(t) \in W$ for $t \in [t_1, t_2]$ with $u(t_1)$ and $u(t_2)$ fixed. A "variation" of w(t) gives (2.2.1) while a "variation" of u(t) leads to (2.2.3).

With appropriate initial conditions for v and w we could study the Cauchy problems, associated with (2.2.2) and (2.2.3). However, we are only interested in special solutions of (2.2.2) and (2.2.3). Suppose there exists a $Y \in X_p(M)$ (so Y is a vector field on M, depending on an additional parameter t, $Y(u,t) \in T_u M$), such that for all solutions u(t) of (2.2.1) which lie (partly) in U, $v(t) = Y(u(t),t)$ is a solution of (2.2.2). This means

$$\dot{Y}(u(t),t) + Y'(u(t),t)\,\dot{u}(t) = X'(u(t))\,Y(u(t),t).$$

Note that $\dot{Y}$, the partial derivative of the parameterized vector field Y with respect to the parameter (t), is again a vector field on M. Since u(t) is a solution of (2.2.1) we obtain

$$\dot{Y}(u(t),t) + Y'(u(t),t)\,X(u(t)) = X'(u(t))\,Y(u(t),t).$$

This condition has to be satisfied for all solutions u(t) (which lie parly in U) with arbitrary initial condition $u(t_o) = u_o$, hence

$$(2.2.5) \qquad \dot{Y}(u,t) = X'(u)\,Y(u,t) - Y'(u,t)\,X(u) \qquad \forall u \in U,\ t \in \mathbb{R}.$$

The right-hand side can be interpreted as the Lie bracket $[Y,X]$ of the vector fields Y and X. This Lie bracket can also be written in terms of Lie derivatives

$$[Y,X] = -L_X Y = L_Y X.$$

So we can write (2.2.5) as

$$\dot{Y} + [X,Y] = \dot{Y} + L_X Y = 0 \quad \forall u \in U, t \in \mathbb{R} .$$

This condition on the vector field Y does not depend on the local trivialization $TU = U \times W$.
This leads to the following

2.2.6 Definition.

A parameterized vector field Y on M (so $Y \in X_p(M)$), which satisfies

$$\dot{Y} + [X,Y] = 0 \tag{2.2.7}$$

on $M \times \mathbb{R}$ is called a *symmetry of the dynamical system* (2.2.1). The set of symmetries of (2.2.1) will be denoted by $V(X;M)$.

□

In the appendix of this chapter we shall show how (2.2.7) can be derived without using a local trivialization of TM. Since $Y = X \in V(X;M)$ the set $V(X;M)$ contains always a non-zero vector field.

Next we turn to special solutions of (2.2.3). Suppose there exists a $\sigma \in X_p^*(M)$ (so σ is a parameterized one-form or covariant vector field) such that for all solutions u(t) which lie (partly) in U, w(t) = = $\sigma(u(t),t)$ satisfies (2.2.3). This implies

$$\dot{\sigma}(u(t),t) + \sigma'(u(t),t)\, \dot{u}(t) = - X'^*(u(t))\, \sigma(u(t),t).$$

Using (2.2.1) we obtain

$$\dot{\sigma}(u(t),t) + \sigma'(u(t),t)\, X(u(t)) = - X'^*(u(t))\, \sigma(u(t),t).$$

This condition has to be satisfied for all solutions u(t) in U, hence

$$\dot{\sigma}(u,t) + \sigma'(u,t)\, X(u) + X'^*(u)\, \sigma(u,t) = 0 \qquad \forall u \in U, t \in \mathbb{R} .$$

The last two terms in the left-hand side can be written as $L_X \sigma$, the Lie derivative of the one-form σ in direction of the vector field X. This operation results again in a one-form which is independent of the trivialization $TU = U \times W$. Hence the following

2.2.8 Definition.

A parameterized one-form σ (so $\sigma \in X^*_p(M)$) which satisfies

$$(2.2.9) \qquad \dot{\sigma} + L_X \sigma = 0$$

on $M \times \mathbb{R}$ is called an *adjoint symmetry of the dynamical system* (2.2.1). The set of adjoint symmetries of (2.2.1) will be denoted by $V^*(X;M)$.

□

In contrast to $V(X;M)$ the set of adjoint symmetries $V^*(X;M)$ may contain only the trivial one-form $\sigma = 0$. Of course $V(X;M) \subset X_p(M)$ and $V^*(X;M) \subset X^*_p(M)$. Finally we mention that in the remaining part of this chapter (adjoint) symmetries, unless stated otherwise, are meant as (adjoint) symmetries of the dynamical system (2.2.1).

2.3 PROPERTIES OF SYMMETRIES.

First some remarks on the notion of constant of the motion.

2.3.1 Definition.

We call a function $F \in F_p(M)$ a *constant of the motion* or *first integral* of (2.2.1) if, for all solutions u(t) of (2.2.1)

$$\frac{d}{dt} F(u(t),t) = 0.$$

This is equivalent to

$$(2.3.2) \qquad \dot{F} + \langle dF, X \rangle = \dot{F} + L_X F = 0 \quad \text{on } M \times \mathbb{R}.$$

□

Constants of the motion which differ only by a real constant will be identified. The following two lemma's are an immediate consequence of the fact that the evolution equation (2.2.1) is autonomous.

2.3.3 Lemma.

If F is a constant of the motion, then the same holds true for $\dot{F}$.

□

2.3.4 Lemma.

If $Y \in V(X;M)$, then also $\dot{Y} \in V(X;M)$.

□

Some properties of the set of symmetries $V(X;M)$ are described in

2.3.5 Theorem.

$V(X;M)$ is a real linear space. Further if $Y \in V(X;M)$ and F is a constant of the motion, then $FY \in V(X;M)$.

Proof:
Symmetries have to satisfy the linear equation (2.2.7), so the first remark is trivial. Next note that (Leibniz' rule)

$$[X, FY] = L_X(FY) = F[X,Y] + (L_X F)\, Y.$$

Since F is a constant of the motion and Y a symmetry this can be written as

$$[X, FY] = - F\dot{Y} - \dot{F}Y = - \frac{\partial}{\partial t}(FY) \;.$$

So the vector field FY is again a symmetry.

□

Theorem 2.3.5 can be summarized by saying that the set of symmetries $V(X;M)$ is a module over the ring of constants of the motion of (2.2.1).

2.3.6 Theorem.

$V(X;M)$ is a Lie algebra with the same Lie bracket as the algebra $X(M)$ of all vector fields on M. The autonomous symmetries (that is symmetries Y with $\dot{Y} = 0$) form a subalgebra of $V(X;M)$.

Proof:

Suppose Y_1, $Y_2 \in V(X;M)$. Set $Y = [Y_1,Y_2]$. Then

$$\dot{Y} = [\dot{Y}_1,Y_2] + [Y_1,\dot{Y}_2]$$

$$= [[Y_1,X], Y_2] + [Y_1, [Y_2 ,X]] .$$

Using the Jacobi identity for Lie brackets we get

$$\dot{Y} = [[Y_1,Y_2], X] = [Y,X] \quad ,$$

which shows that $V(X;M)$ is a Lie algebra. Finally note that if Y_1 and Y_2 are autonomous, then $Y = [Y_1, Y_2]$ is also autonomous.

□

Next we consider tensor fields which can be used to construct (new) symmetries from (already known) symmetries. Suppose $\Lambda \in T^1_{1p}(M)$, so Λ is a parameterized tensor field of covariant order 1 and contravariant order 1. Then Λ can also be considered as a vector bundle map $\Lambda : TM \to TM$ or as a linear mapping $\Lambda : X_p(M) \to X_p(M)$. We can ask under which conditions Λ maps $V(X;M)$ into $V(X;M)$. This leads to the following

2.3.7 Theorem.

Suppose the tensor field $\Lambda \in T^1_{1p}(M)$ satisfies

(2.3.8) $\quad \dot{\Lambda} + L_X\Lambda = 0 \quad$ on $M \times \mathbb{R}$.

Then if $Y \in V(X;M)$, then also $\Lambda Y \in V(X;M)$.

Proof:

Since the Lie derivative satisfies Leibniz' rule we have

$$\frac{\partial}{\partial t} (\Lambda Y) + [X,\Lambda Y] = \frac{\partial}{\partial t} (\Lambda Y) + L_X(\Lambda Y) = \Lambda(\dot{Y} + [X,Y]) + (\dot{\Lambda} + L_X \Lambda) Y.$$

So if Y is a symmetry and Λ satisfies (2.3.8), we see that ΛY is also a symmetry. □

2.3.9 Definition.

A parameterized linear mapping $\Lambda : X_p(M) \to X_p(M)$, corresponding to a parameterized tensor field (also denoted by) $\Lambda \in T^1_{1p}(M)$ which satisfies (2.3.8), is called a *recursion operator for symmetries*.

□

Recursion operators for symmetries are sometimes called strong symmetries [8,9].

2.3.10 Remark.

Another possibility for constructing (new) symmetries out of already known ones is to compute the Lie bracket with some other symmetry. This method should not be confused with the application of a recursion operator for symmetries. Suppose Y_1 and Z are two symmetries and Λ is a recursion operator for symmetries. Then we can construct the symmetries Y_3 and Y_4 by

$$Y_3 = \Lambda\, Y_1 \;,$$

$$Y_4 = [\; Z,\, Y_1 \;] \;.$$

Then in a point $u \in M$ the vector $Y_3(u,t)$ depends only on $Y_1(u,t)$ and $\Lambda(u,t)$, while $Y_4(u,t)$ depends on $Y_1(u,t)$, $Z(u,t)$ and their derivatives in u.

□

Suppose for a moment M is a finite-dimensional manifold with coordinates u^i $(i = 1,\ldots, n)$. With respect to this coordinate system the tensor field Λ can be represented by a matrix $\Lambda^j_i(u,t)$ (strictly speaking a matrix valued function on $M \times \mathbb{R}$). Then (2.3.8) can be written as

$$\dot{\Lambda}^j_i + \Lambda^j_{i,k}\, X^k - \Lambda^k_i\, X^j_{,k} + \Lambda^j_k\, X^k_{,i} = 0 \;. \tag{2.3.11}$$

For a solution u(t) of (2.2.1) this implies

$$\frac{d}{dt}\, \Lambda^j_i\,(u(t),t) = X^j_{,k}(u(t))\, \Lambda^k_i\,(u(t),t) - \Lambda^j_k\,(u(t),t)\, X^k_{,i}(u(t)).$$

This type of expression is well-known in the theory of isospectral transformations (or " inverse scattering"). In fact it can be shown that under certain assumptions, an eigenvalue of Λ is a constant of the motion. Consider the following eigenvalue problem

$$(2.3.12) \qquad \Lambda\, Y = \lambda\, Y \text{ on } M \times \mathbb{R} \;.$$

Note that the "eigenvalue" λ is a function on $M \times \mathbb{R}$ and that the "eigenvector" Y is a parameterized vector field on M. We assume λ is a smooth function and Y is a smooth vector field. By taking the Lie derivative in the direction of X we obtain

$$(L_X\Lambda)Y + \Lambda(L_X Y) = (L_X\lambda)Y + \lambda\,(L_X Y).$$

Differentiation of (2.3.12) with respect to t gives

$$\dot{\Lambda}Y + \Lambda\dot{Y} = \dot{\lambda}Y + \lambda\dot{Y}\;.$$

After summation of these two expressions we obtain

$$(\Lambda-\lambda)\;(\dot{Y} + L_X Y) = (\dot{\lambda} + L_X\lambda)\;Y.$$

If the recursion operator for symmetries Λ has a complete set of eigenvectors ("eigenvector fields"), this means

$$(2.3.13) \qquad \dot{\lambda} + L_X\lambda = 0,$$

so the function λ is a constant of the motion.

Finally we remark that in most applications the recursion operators for symmetries do not depend explicitly on t (so $\dot{\Lambda} = 0$).

2.4 PROPERTIES OF ADJOINT SYMMETRIES.

The first two results concerning adjoint symmetries correspond to similar results for symmetries.

2.4.1 Lemma.

Suppose $\sigma \in V^*(X;M)$, then also $\dot{\sigma} \in V^*(X;M)$.

□

2.4.2 Theorem.

The set of adjoint symmetries $V^*(X;M)$ is a real linear space. Moreover if F is a constant of the motion and $\sigma \in V^*(X;M)$, then $F\,\sigma \in V^*(X;M)$.

Proof:
Adjoint symmetries have to satisfy the linear equation (2.2.9), so $V^*(X;M)$ is a real linear space. Next assume F is a constant of the motion, $\sigma \in V^*(X;M)$, then

$$\frac{\partial}{\partial t}(F\sigma) + L_X(F\sigma) = F(\dot{\sigma} + L_X\sigma) + (\dot{F} + L_X F)\,\sigma = 0.$$

This means $F\,\sigma \in V^*(X;M)$.

□

This theorem can be summarized by saying that $V^*(X;M)$ is a module over the ring of constants of the motion of (2.2.1). In contrast to $V(X;M)$ the space $V^*(X;M)$ does not have a natural Lie algebra structure.

It turns out that there is a close relation between the space of constants of the motion and a subspace of $V^*(X;M)$. Let F be a function on M (or on $M \times \mathbb{R}$), then its exterior derivative dF is a (parameterized) one-form on M.

2.4.3 Theorem.

Suppose $F \in F_p(M)$ is a constant of the motion. Then the one-form $\sigma = dF$ is an adjoint symmetry.

Proof:
The function F is a constant of the motion, so $\dot{F} + L_X F = 0$.
The exterior derivative d commutes with the Lie derivative and with differentiation with respect to t. Hence

$$\dot{dF} + L_X\, dF = 0.$$

This means that $\sigma = dF$ is an adjoint symmetry. □

2.4.4 Remark.

In fact we proved a little more. Suppose $F \in F_p(M)$ such that for all solutions $u(t)$ of (2.2.1)

$$\frac{d}{dt}\, F(u(t),t) = f(t),$$

where $f: \mathbb{R} \to \mathbb{R}$ is some function. This means $\dot{F} + L_X F = f$. Then the calculation above (with $df = 0$) shows that $\sigma = dF$ is also an adjoint symmetry. In the following theorem we show that σ also can be written as the exterior derivative of a constant of the motion.

□

2.4.5 Theorem.

Let $\sigma \in V^*(X;M)$ be exact, so there exists a function $F \in F_p(M)$ such that $\sigma = dF$. Then there exists a function $g : \mathbb{R} \to \mathbb{R}$ such that $G(\cdot,t) = F(\cdot,t) - g(t)$ is a constant of the motion with $\sigma = dG$.

Proof:
Since σ is an adjoint symmetry, we have $\dot{\sigma} + L_X\sigma = 0$. This can be written as $d(\dot{F} + L_X F) = 0$, which implies that $\dot{F}(u,t) + L_X F(u,t) = f(t)$ on $M \times \mathbb{R}$ for some function $f : \mathbb{R} \to \mathbb{R}$. Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that $\dot{g} = f$. Then $G(\cdot,t) = F(\cdot,t) - g(t)$ is a constant of the motion with $\sigma = dG$.

□

The theorems 2.4.3 and 2.4.5 can be summarized by saying that every constant of the motion gives rise to an (exact) adjoint symmetry and that every exact adjoint symmetry can be written as the exterior derivative of a constant of the motion.

Now we are going to study operators which map $V^*(X;M)$ into itself. Consider a parameterized tensor field $\Gamma \in T^1_{1p}(M)$. Then we can consider Γ also as a linear mapping $\Gamma : X^*_p(M) \to X^*_p(M)$, and we can ask under which conditions Γ maps $V^*(X;M)$ into $V^*(X;M)$. Analogous to theorem 2.3.7

we now have

2.4.6 Theorem.

Suppose the tensor field $\Gamma \in T^1_{1p}(M)$ satisfies

(2.4.7) $\dot{\Gamma} + L_X \Gamma = 0$ on $M \times \mathbb{R}$.

Then for all $\sigma \in V^*(X;M)$ also $\Gamma\sigma \in V^*(X;M)$.

Proof:
Similar to the proof of theorem 2.3.7 we have

$$\frac{\partial}{\partial t}(\Gamma\sigma) + L_X(\Gamma\sigma) = \Gamma(\dot{\sigma} + L_X\sigma) + (\dot{\Gamma} + L_X\Gamma)\sigma \quad .$$

So if $\sigma \in V^*(X;M)$ and Γ satisfies (2.4.7) we see that $\Gamma\,\sigma \in V^*(X;M)$.

□

2.4.8 Definition.

A parameterized linear mapping $\Gamma : X^*_p(M) \to X^*_p(M)$, corresponding to a tensor field (also denoted by) $\Gamma \in T^1_{1p}(M)$ which satisfies (2.4.7), is called a *recursion operator for adjoint symmetries*.

□

2.4.9 Remark.

The conditions (2.3.8) and (2.4.7) for the tensor fields Λ and Γ are identical. This means that a tensor field Λ which satisfies (2.3.8), gives also rise to a recursion operator for adjoint symmetries. In local coordinates on M the tensor field Λ is represented by a matrix Λ^i_j . Suppose Y is a symmetry with coordinates Y^i and σ is an adjoint symmetry with coordinates σ_i. Then the vector field Z with coordinates $Z^i = \Lambda^i_j\, Y^j$ is again a symmetry. But also $\tau_j = \Lambda^i_j\, \sigma_i$ is (represents) an adjoint symmetry. The dual operator of $\Lambda : X_p(M) \to X_p(M)$ is a linear operator $\Lambda^* : X^*_p(M) \to X^*_p(M)$. So, in operator notation, we have $Z = \Lambda Y$ and $\tau = \Lambda^*\sigma$. This leads to

□

2.4.10 Theorem.

Suppose Λ is a recursion operator for symmetries. Then Λ^* is a recursion operator for adjoint symmetries. Also if Γ is a recursion operator for adjoint symmetries then Γ^* is a recursion operator for symmetries.

Proof:
The operators $\Lambda : X_p(M) \to X_p(M)$ and $\Lambda^*: X_p^*(M) \to X_p^*(M)$ correspond both to the tensor field (also denoted by) Λ. If Λ is a recursion operator for symmetries the tensor field satisfies (2.3.8) and so (2.4.7).
Hence Λ^* is a recursion operator for adjoint symmetries. The second part of the theorem is proved in a similar way.

□

In the last part of section 2.3 we have seen that, under certain conditions, the eigenvalues of a recursion operator Λ for symmetries are constants of the motion. In a similar way it can be shown that, under certain conditions, the eigenvalues of a recursion operator Γ for adjoint symmetries are constants of the motion.

2.5 GENERAL RESULTS.

We first consider operators which relate symmetries and adjoint symmetries. Suppose Ψ is a parameterized tensor field of contravariant order 2 and covariant order 0, so $\Psi \in T^2_{o\,p}(M)$. Then we can also consider Ψ as a vector bundle map $\Psi : T^*M \to TM$ or as a linear operator $\Psi : X_p^*(M) \to X_p(M)$. Now we investigate under which conditions Ψ maps adjoint symmetries into symmetries.

2.5.1 Theorem.

Suppose $\Psi \in T^2_{o\,p}(M)$ is a tensor field such that

$$(2.5.2) \qquad \dot{\Psi} + L_X\Psi = 0 \quad \text{on } M \times \mathbb{R} \quad .$$

Then for all $\sigma \in V^*(X;M)$ we have $\Psi\sigma \in V(X;M)$.

Proof:
From

$$\frac{\partial}{\partial t}(\Psi\sigma) + L_X(\Psi\sigma) = \Psi(\dot{\sigma}+L_X\sigma) + (\dot{\Psi} + L_X\Psi)\sigma$$

we see that, if $\sigma \in V^*(X;M)$ and Ψ satisfies (2.5.2), $\Psi\sigma \in V(X;M)$. So Ψ transforms adjoint symmetries into symmetries.

□

2.5.3 Definition.

Suppose the tensor field $\Psi \in T^2_{o\,p}(M)$ satisfies (2.5.2). Then (considered as mapping Ψ: $X^*_p(M) \to X_p(M)$) Ψ is called an *AS operator*.

□

So an AS operator, applied to an adjoint symmetry, yields a symmetry. Next we consider operators acting in the opposite direction.

2.5.4 Theorem.

Suppose $\Phi \in T^o_{2p}(M)$ is a tensor field such that

(2.5.5) $\dot{\Phi} + L_X\Phi = 0$ on $M \times \mathbb{R}$.

Then for all $Y \in V(X;M)$ we have $\Phi\, Y \in V^*(X;M)$.

Proof:
The proof is similar to the proof of theorem (2.5.1).

□

2.5.6 Definition.

Suppose the tensor field $\Phi \in T^o_{2p}(M)$ satisfies (2.5.5). Then (considered as mapping $\Phi : X_p(M) \to X^*_p(M)$) Φ is called an *SA operator*.

□

So an SA operator Φ transforms symmetries into adjoint symmetries.

As expected, if an AS (SA) operator is invertible, the inverse operator is an SA (AS) operator.

2.5.7 Theorem.

Suppose Ψ (Φ) is an invertible AS (SA) operator. Then the inverse operator Ψ^{-1} (Φ^{-1}) is an SA (AS) operator.

Proof:

Since $\Psi\,\Psi^{-1} = I_d : TM \to TM$ we have

$$L_X(\Psi\,\Psi^{-1}) = (L_X\Psi)\,\Psi^{-1} + \Psi\,L_X(\Psi^{-1}) = 0$$

and

$$\frac{\partial}{\partial t}(\Psi\,\Psi^{-1}) = \dot{\Psi}\,\Psi^{-1} + \Psi\,\frac{\partial}{\partial t}(\Psi^{-1}) .$$

This means that if Ψ satisfies (2.5.2), then Ψ^{-1} satisfies (2.5.5).

□

Recall that with a parameterized two-form ϕ always corresponds an (anti-symmetric) tensor field $\Phi \in T^{o}_{2p}(M)$ or equivalently a linear mapping $\Phi : X_p(M) \to X^*_p(M)$, such that

$$\phi(A,B) = \langle \Phi A, B\rangle \qquad \forall\, A,B \in X(M) .$$

This leads to

2.5.8 Theorem.

Let σ be an adjoint symmetry which is not closed, so $d\sigma \neq 0$. Then the operator Φ which corresponds to the two-form $\phi = d\sigma$ is an SA operator.

Proof:

The adjoint symmetry σ satisfies $\dot{\sigma} + L_X\sigma = 0$. After exterior differentiation we obtain $\dot{\phi} + L_X\phi = 0$, which is equivalent to $\dot{\Phi} + L_X\Phi = 0$. Hence Φ is an SA operator.

□

2.5.9 Remark.

Since $d\phi = dd\sigma = 0$ the SA operator Φ corresponds to a closed two-form (ϕ). This means that the SA operator Φ satisfies additional conditions, which are explained in definition 3.2.4 and theorem 3.2.12. Operators of this type will be called *cyclic operators*. If $\phi = d\sigma$ is also nondegenerate, the operator Φ is invertible. In this case the dynamical system (2.2.1) is of a special type, a so called Birkhoffian system (see for instance Santilli [12]). If σ (or ϕ) satisfies one more condition, the system is Hamiltonian. This will be explained in section 3.5 .

□

Of course theorem 2.5.8 is also correct, if σ is closed. However, in that case we obtain the trivial SA operator $\Phi = 0$. In a local coordinate system u^i the adjoint symmetry σ can be written as $\sigma = \sigma_i\, du^i$. The corresponding SA operator is then represented by the matrix $\Phi_{ij} = \sigma_{i,j} - \sigma_{j,i}$.

Recall that with every vector field A and every one-form α corresponds a function on M, defined by their contraction $<\alpha, A> = i_A\alpha$.

2.5.10 Theorem.

Suppose $Y \in V(X;M)$ and $\sigma \in V^*(X;M)$. Then the function $F = <\sigma, Y>$ is a constant of the motion.

Proof:

Using Leibniz' rule we obtain

$$\dot{F} + L_X F = < \sigma \;,\; \dot{Y} + L_X Y > + < \dot{\sigma} + L_X\sigma \;,\; Y > = 0.$$

This means F is a constant of the motion.

□

Starting with two symmetries Y_1 and Y_2 an AS operator Ψ can be defined in the following way. For $\alpha \in X^*_p(M)$ set

$$\Psi\alpha = <\alpha, Y_1> Y_2. \tag{2.5.11}$$

It is easily seen that Ψ is an AS operator. Application of this operator to an adjoint symmetry σ gives $\Psi\sigma = <\sigma, Y_1> Y_2$. By theorem 2.5.10 we see that $<\sigma, Y_1>$ is a constant of the motion. Then, from theorem 2.3.5 we see that $\Psi\sigma$ is a symmetry, so Ψ is indeed an AS operator. Of course we can also verify that Ψ satisfies (2.5.2). This operator Ψ is rather trivial. We obtain always the same vector field Y_2, multiplied by different functions $< \alpha, Y_1 >$. This implies that Ψ is not invertible. It is easily seen that if $\Psi \neq 0$, it is not antisymmetric. This method of constructing an AS operator, starting with two symmetries can be extended. Let $Y_1, \ldots, Y_k \in V(X;M)$ and $c^{ij} \in \mathbb{R}$ for $i,j = 1,\ldots,k$. Then for $\alpha \in X_p^*(M)$ define

$$(2.5.12) \qquad \Psi\alpha = c^{ij} <\alpha, Y_i> Y_j .$$

Then Ψ is an AS operator. This construction yields a symmetric operator if $c^{ij} = c^{ji}$ and an antisymmetric operator if $c^{ij} = -c^{ji}$. Using similar methods we can also construct SA operators and recursion operators for (adjoint) symmetries. For instance, let $\sigma \in V^*(X;M)$ and $Y \in V(X;M)$. Then for $A \in X_p(M)$ define

$$(2.5.13) \qquad \Lambda A = < \sigma, A> Y.$$

Then Ψ is a (rather trivial) example of a recursion operator for symmetries.

There are four different types of operators relating symmetries and (adjoint) symmetries. They were described in the definitions 2.3.9 (Λ, recursion operator for symmetries), 2.4.8 (Γ, recursion operator for adjoint symmetries), 2.5.3 (Ψ, AS operator) and 2.5.6 (Φ, SA operator). If one or more of these operators exist, we can construct new operators by using products and powers of already known operators. For instance, suppose there exists an AS operator Ψ and an SA operator Φ. Then $\Psi\Phi$ is a recursion operator for symmetries and $\Phi\Psi$ is a recursion operator for adjoint symmetries. Also other combinations are possible. Let Λ be a recursion operator for symmetries and Ψ an AS operator. Then $\Lambda\Psi$ is again an AS operator. Of course all these results have a straightforward proof.

We end this section by giving a more general approach of the theory described in this section and in the sections 2.3 and 2.4. Up to now we considered constants of the motion, (adjoint) symmetries and several operators between those symmetries. All these objects are (can be considered as) tensor fields Ξ of different types which satisfy

$$(2.5.14) \qquad \dot{\Xi} + L_X \Xi = 0 \quad \text{on } M \times \mathbb{R} .$$

If Ξ is a completely antisymmetric tensor field of covariant order k and contravariant order 0, we can also consider it as a differential k-form ξ.

There are several methods for constructing new tensor fields out of already known ones. Suppose Ξ is a parameterized tensor field of arbitrary orders and Y is a parameterized vector field. Then new parameterized tensor fields can be constructed by the following methods (see also Abraham and Marsden [1, § 3.4]):

i) Compute $L_Y \Xi$, the Lie derivative of Ξ in the direction of Y.

ii) Compute $\Xi \otimes \Xi_1$, the tensor product of Ξ and some tensor field Ξ_1.

iii) If the co-and contravariant orders of Ξ are both positive, we can perform a contraction.

iv) If Ξ is antisymmetric and has covariant order k and contravariant order 0 we can compute the exterior derivative of the corresponding k-form ξ . Then $d\xi$ corresponds again to a tensor field (with orders k+1 and 0).

v) Suppose Ξ and some other tensor field Ξ_1 correspond to k and ℓ-forms ξ and ξ_1. Then we can construct a tensor field Ξ_2 corresponding to the $(k+\ell)$-form $\xi_2 = \xi \wedge \xi_1$.

There are several relations between these methods. A tensor field Ξ_2 constructed by v), can also be obtained by ii). For instance if Ξ and Ξ_1 have both covariant order 1, then

$$\Xi_2 = \Xi \otimes \Xi_1 - \Xi_1 \otimes \Xi .$$

So we need not consider method v). If Ξ corresponds to a differential k-form ξ then

$$L_Y \xi = d\, i_Y \xi + i_Y\, d\, \xi .$$

The interior product i_Y of a vector field with a differential form can be obtained by a tensor product with Y followed by a contraction. So for k-forms i) can be obtained from ii), iii) and iv). Almost all results of sections 2.3 , 2.4 and this section are in fact special cases of the following

2.5.15 Theorem.

Suppose Ξ, Ξ_1 are parameterized tensor fields of arbitrary orders which satisfy (2.5.14). Let Y be a symmetry. Then

i) $L_Y\Xi$ satisfies (2.5.14),

ii) $\Xi \otimes \Xi_1$ satisfies (2.5.14),

iii) any possible contraction of Ξ satisfies (2.5.14),

iv) if Ξ corresponds to a differential form ξ, the tensor field corresponding to $d\xi$ also satisfies (2.5.14).

Proof:

i) Using the commutation rule for Lie derivatives we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(L_Y\Xi) + L_XL_Y\Xi &= L_{\dot{Y}}\Xi + L_Y\dot{\Xi} + L_XL_Y\Xi \\ &= L_{\dot{Y}}\Xi - L_YL_X\Xi + L_XL_Y\Xi \\ &= L_{\dot{Y}}\Xi + L_{[X,Y]}\Xi.\end{aligned}$$

Since the vector field Y is a symmetry, the last two terms cancel, so $L_Y\Xi$ also satisfies (2.5.14).

ii) This part of the theorem is a straightforward consequence of

$$L_Y(\Xi \otimes \Xi_1) = (L_Y\Xi) \otimes \Xi_1 + \Xi \otimes L_Y\Xi_1.$$

iii) Suppose Ξ is a tensor field with both orders positive. Denote the tensor fields obtained by a contraction in Ξ and (the same contraction in) $L_Y\Xi$ by Ξ_C and Ξ_{LC}. Then $L_Y\Xi_C = \Xi_{LC}$, so "contraction commutes with the Lie derivative". Using this property it is easily shown that, if Ξ satisfies (2.5.14), then also Ξ_C satisfies (2.5.14).

iv) Using $L_Y d = dL_Y$ (for differential forms), this result is also easily proved.

□

We mentioned already that most results of sections 2.3, 2.4 and of this section can be obtained from theorem 2.5.15. For instance the theorems

2.3.5, 2.3.7, 2.4.2, 2.4.6, 2.5.1, 2.5.4 and 2.5.10 follow also from ii) and iii) of theorem 2.5.15. As an example consider theorem 2.4.6. In that theorem Γ and σ are both tensor fields which satisfy (2.5.14). Then also the tensor product $\Gamma\otimes\sigma$ satisfies this condition. After contraction we see that the tensor field $\Gamma\sigma$ (= one-form) also satisfies (2.5.14), so it is an adjoint symmetry. Further theorem 2.3.6 (and in fact also the lemma's 2.3.3, 2.3.4 and 2.4.1) follows from part i) of theorem 2.5.15. The theorems 2.4.3 and 2.5.8 are special cases of part iv) of theorem (2.5.15). Finally we mention that the AS operator Ψ and the recursion operator for symmetries Λ, as given in (2.5.12) and (2.5.13), can be written as

$$\Psi = c^{ij}\, Y_i \otimes Y_j,$$

$$\Lambda = \sigma \otimes A\ .$$

Then by theorem 2.5.15 ii) Ψ is an AS operator and Λ a recursion operator for symmetries.

2.6 THE SPECIAL CASE OF TWO SERIES OF SYMMETRIES.

In the examples we shall sometimes meet dynamical systems (and also Hamiltonian systems), for which there exists a recursion operator for symmetries Λ (with $\dot{\Lambda} = 0$) and two series of symmetries generated by the operator

$$(2.6.1) \qquad X_k = \Lambda^{k-1} X_1\ , \quad Z_k = \Lambda^{k-1} Z_1$$

with $X_1 = X$. (We always consider the dynamical system (2.2.1) $\dot{u} = X(u)$).

2.6.2 Remark.

The situation as described above occurs for instance in the case of the Burgers equation (see section 5.2) and the Korteweg-de Vries equation (see section 5.6). For these equations there also exist symmetries X_o and Z_o such that $X_1 = \Lambda X_o$ and $Z_1 = \Lambda Z_o$. The symmetry X_o is then related to the invariance of solutions of the equation for translation along the x axis, while the symmetry Z_o corresponds to invariance of solutions under a scale transformation. Note that for all (autonomous) evolution equations $\dot{u} = X(u)$, the symmetry X corresponds to the invariance of solutions for translations in time.

□

Recall that (theorem 2.5.15 i) if Λ is a recursion operator for symmetries and Y is a symmetry, then $L_Y\Lambda$ is also a recursion operator for symmetries. If certain conditions on the first elements of the series of symmetries X_k and Z_k are satisfied, the various Lie brackets in and between the series X_k and Z_k are easily computed. These conditions are summarized in

2.6.3 Hypothesis.

Suppose there exist real numbers $a \neq 0$, $b \neq 0$ with $ka + b \neq 0$ for $k = 0,1,2,..$, such that

$$(2.6.4) \qquad L_{Z_1}\Lambda = a\,\Lambda^2,$$

$$(2.6.5) \qquad L_{Z_2}\Lambda = a\,\Lambda^3,$$

$$(2.6.6) \qquad L_{Z_1}X_1 = [Z_1, X_1] = b\,X_2.$$

□

In several cases (for instance Burgers equation, Korteweg-de Vries equation, linear Hamiltonian systems) these conditions are satisfied. *In the remaining part of this section we shall assume this hypothesis is satisfied.* The various Lie brackets and Lie derivatives of the recursion operator Λ can now be found. We start with the following

2.6.7 Theorem.

$[Z_1, X_\ell] = ((\ell-1)a+b)\, X_{\ell+1}$ for $\ell = 1,2,3,\ldots$.

Proof:

The proof is a straightforward computation, using Leibniz' rule , the definition of X_k, (2.6.4) and (2.6.6)

$$[Z_1,X_\ell] = L_{Z_1}(\Lambda^{\ell-1}X_1) = (L_{Z_1}\Lambda^{\ell-1})\, X_1 + \Lambda^{\ell-1}[Z_1,X_1]$$

$$= (\ell-1)\ a\ \Lambda^{\ell}\ X_1\ +\ b\ \Lambda^{\ell-1}\ X_2$$

$$= (\ (\ell-1)a+b)\ X_{\ell+1}.$$

□

This means that the series X_k can also be constructed by using the (repeated) Lie bracket with Z_1.

2.6.8 Theorem.

$L_{X_k}\ \Lambda = 0$ for $k = 1,2,3,\ldots$.

Proof:
The proof is done by induction. Since Λ is a time independent recursion operator for symmetries of $\dot{u} = X(u) = X_1(u)$, we have $L_{X_1}\Lambda = 0$.
Next assume $L_{X_k}\Lambda = 0$. By the preceding theorem

$$X_{k+1} = \frac{1}{(k-1)\ a\ +\ b}\ [Z_1,X_k] \quad .$$

This implies

$$L_{X_{k+1}}\ \Lambda = \frac{1}{(k-1)\ a\ +\ b}\ (L_{Z_1}L_{X_k}\Lambda - L_{X_k}L_{Z_1}\Lambda\ ). \quad .$$

By assumption the first term in the right hand side vanishes. Using (2.6.4) and Leibniz' rule we see that the second term in the right hand side is also zero.

□

Note that this theorem implies that Λ is a recursion operator for symmetries of each equation of the form $\dot{u} = X_k(u)$, $k = 1,2,3,\ldots$. The Lie brackets $[X_k, X_\ell]$ and $[Z_k, Z_\ell]$ are now easily obtained.

2.6.9 Corollary.

$[X_k, X_\ell] = 0$ for $k, \ell = 1,2,3,\ldots$.

Proof:
It is no restriction to assume $\ell = k + m$ with m positive.
Then

$$[X_k, X_\ell] = L_{X_k} (\Lambda^m X_k) = (L_{X_k} \Lambda^m) X_k.$$

From theorem 2.6.8 we obtain that $L_{X_k} \Lambda^m = 0$, which concludes the proof.

□

Since $X_1 = X$ and Λ do not depend explicitly on t, we obtain from (2.6.1) that $\dot{X}_k = 0$. The preceding corollary now implies that X_ℓ is a symmetry for each equation $\dot{u} = X_k(u)$.

2.6.10 Corollary.

$[Z_k, X_\ell] = ((\ell-1)a + b) X_{k+\ell}$ for $k, \ell = 1,2,3,\ldots$.

Proof:

$$[Z_k, X_\ell] = - L_{X_\ell} (\Lambda^{k-1} Z_1) = - \Lambda^{k-1} L_{X_\ell} Z_1$$

$$= \Lambda^{k-1} [Z_1, X_\ell] = ((\ell-1)a + b) X_{k+\ell},$$

where we used Leibniz' rule and the theorems 2.6.7 and 2.6.8 .

□

Next we turn to the Lie bracket in the series Z_k. A simple case is described in the following

2.6.11 Theorem.

$[Z_1, Z_\ell] = a(\ell-1) Z_{\ell+1}$ for $\ell = 1,2,3,\ldots$.

Proof:

$$[Z_1, Z_\ell] = L_{Z_1} (\Lambda^{\ell-1} Z_1).$$

Using Leibniz' rule and (2.6.4) this becomes

$$[Z_1,Z_\ell] = a(\ell-1) \Lambda^\ell Z_1 = a(\ell-1) Z_{\ell+1} .$$

□

To compute the Lie bracket $[Z_k,Z_\ell]$ for $k > 1$ we need the following

2.6.12 Theorem.

$L_{Z_k} \Lambda = a \Lambda^{k+1}$ for $k = 1,2,3...$.

Proof:

The proof is done by induction. For $k = 1$ and $k = 2$ the theorem follows from (2.6.4) and (2.6.5). Suppose the theorem is correct for $k = \ell \geq 2$. By the preceding theorem we have

$$Z_{\ell+1} = \frac{1}{a(\ell-1)} [Z_1, Z_\ell] .$$

Hence

$$L_{Z_{\ell+1}} \Lambda = \frac{1}{a(\ell-1)} (L_{Z_1} L_{Z_\ell} \Lambda - L_{Z_\ell} L_{Z_1} \Lambda)$$

$$= \frac{1}{a(\ell-1)} (L_{Z_1} L_{Z_\ell} \Lambda - a L_{Z_\ell} \Lambda^2).$$

Using the induction assumption we obtain

$$L_{Z_{\ell+1}} \Lambda = \frac{1}{a(\ell-1)} (L_{Z_1} a \Lambda^{\ell+1} - 2a^2 \Lambda^{\ell+2})$$

$$= \frac{1}{a(\ell-1)} (a^2 (\ell+1) \Lambda^{\ell+2} - 2a^2 \Lambda^{\ell+2})$$

$$= a \Lambda^{\ell+2} .$$

So we proved the theorem for $k = \ell+1$.

□

The Lie brackets $[Z_k, Z_\ell]$ are now easily found.

2.6.13 Corollary.

$[Z_k, Z_\ell] = a(\ell-k)\ Z_{k+\ell}$.

Proof:

Suppose $\ell = k+m$ with m positive. Then

$$[Z_k, Z_\ell] = L_{Z_k}(\Lambda^m Z_k) = (L_{Z_k}\Lambda^m)\ Z_k$$

$$= m\ \Lambda^{m-1}\ a\ \Lambda^{k+1}\ Z_k = (\ell-k)\ a\ Z_{k+\ell}.$$

□

For future reference we summarize the Lie brackets and Lie derivatives found in this section

$$(2.6.14)\quad \begin{cases} [X_k, X_\ell] = 0\ , \\ [Z_k, X_\ell] = (\ (\ell-1)\ a+b)\ X_{k+\ell}, \\ [Z_k, Z_\ell] = a\ (\ell-k)\ Z_{k+\ell}\ , \\ L_{X_k}\Lambda = 0\ , \\ L_{Z_k}\Lambda = a\ \Lambda^{k+1}\ . \end{cases}$$

Note that all these results were obtained under the assumption that hypothesis 2.6.3 is satisfied.

2.6.15 Remark.

Note that the results given in (2.6.14) are valid for $k,\ell \geq 1$. Suppose there also exist symmetries X_o and Z_o as described in remark 2.6.2. Then, using the same methods as in the precedent, it can be shown that the relations (2.6.14) are also valid for $k,\ell \geq 0$, if the symmetries X_o and Z_o satisfy

$$(2.6.16) \quad L_{X_o} \Lambda = 0, \quad L_{Z_o} \Lambda = a\Lambda, \quad [Z_o, X_o] = (b-a) X_o.$$

□

2.7 TRANSFORMATION PROPERTIES.

Suppose there exists a diffeomorphism f between M and some other manifold N. Denote the inverse mapping by $\overset{\leftarrow}{f}$, so

$$(2.7.1) \quad \begin{cases} f : M \to N \\ \overset{\leftarrow}{f} : N \to M. \end{cases}$$

Then we can use the derivative of f to transform the equation (2.2.1) to a differential equation on N

$$(2.7.2) \quad \dot{v} = f'(\overset{\leftarrow}{f}(v)) \;\; X(\overset{\leftarrow}{f}(v)) =: \tilde{X}(v).$$

Note that $\tilde{X}$ is a vector field on the manifold N.
Symmetries $\tilde{Y}$ of (2.7.2) are vector fields on N which satisfy

$$\dot{\tilde{Y}} + [\tilde{X},\tilde{Y}] = 0 \quad \text{on } N \times \mathbb{R}.$$

Adjoint symmetries of (2.7.2) are one-forms on N which satisfy

$$\dot{\tilde{\sigma}} + L_{\tilde{X}} \tilde{\sigma} = 0 \quad \text{on } N \times \mathbb{R}.$$

The sets of symmetries and adjoint symmetries of (2.7.2) are denoted by $V(\tilde{X};N)$ respectively $V^*(\tilde{X};N)$. Note that all the expressions given in the sections 2.3, 2.4, 2.5 and 2.6 were given in terms of tensor fields (vector fields, k-forms), Lie derivatives and exterior differentiation. The transformation properties of tensor fields are well-known. Suppose Ξ is an arbitrary tensor field, Y a vector field and η a k-form on M.

The transformed tensor fields, vector fields and k-forms on N are denoted by the same symbol, supplied with a tilde . Then

(2.7.3) $\quad L_{\tilde{Y}}\tilde{\Xi} = \widetilde{L_Y\Xi}$,

(2.7.4) $\quad d\tilde{\eta} = \widetilde{d\eta}$.

This means that the operations L and d are "natural with respect to a diffeomorphism". Suppose Y is a symmetry of (2.2.1). The transformed vector field $\tilde{Y} = f'Y$ on N satisfies

$$\dot{\tilde{Y}} = f'\dot{Y} = f'\, L_X Y = \widetilde{L_X Y}\ .$$

Using (2.7.3) we see that

$$\dot{\tilde{Y}} = L_{\tilde{X}}\tilde{Y} = [\tilde{X},\tilde{Y}]\ ,$$

so the vector field $\tilde{Y}$ on N is a symmetry of (2.7.2).
In the same way we can show that if σ is an adjoint symmetry of (2.2.1), then the one-form $\tilde{\sigma} = \overset{\leftarrow}{f'}{}^{*}\sigma$ on N is an adjoint symmetry of (2.7.2).
So we have proved

2.7.5 Theorem.

If $Y \in V(X;M)$ then $\tilde{Y} = f'Y \in V(\tilde{X};N)$.
Also if $\sigma \in V^*(X;M)$ then $\tilde{\sigma} = \overset{\leftarrow}{f'}{}^{*}\sigma \in V^*(\tilde{X};N)$.

□

Suppose Ψ is an AS operator for equation (2.2.1) on M. Then using (2.7.3) we can show that the transformed operator (tensor field) on N is an AS operator for (2.7.2). Similar results hold for the other possible operators. We summarize them in

2.7.6 Theorem.

Consider the operators Λ, Γ, Ψ, Φ as described in the definitions 2.3.9, 2.4.8, 2.5.3 and 2.5.6. Then the corresponding operators for (2.7.2) on the manifold N are given by

$$(2.7.7)\quad \begin{cases} \tilde{\Lambda} = f'\,\Lambda\, f^{\leftarrow\prime} \;, & \tilde{\Lambda}(v,t) = f'(f^{\leftarrow}(v))\;\Lambda(f^{\leftarrow}(v),t)\; f^{\leftarrow\prime}(v), \\ \tilde{\Gamma} = f^{\leftarrow\prime *}\,\Gamma\, f'^{*} \;, & \tilde{\Gamma}(v,t) = f^{\leftarrow\prime *}(v)\;\Gamma(f^{\leftarrow}(v),t)\; f'^{*}(f^{\leftarrow}(v)), \\ \tilde{\Psi} = f'\,\Psi\, f'^{*} \;, & \tilde{\Psi}(v,t) = f'(f^{\leftarrow}(v))\;\Psi(f^{\leftarrow}(v),t)\; f'^{*}(f^{\leftarrow}(v)), \\ \tilde{\Phi} = f^{\leftarrow\prime *}\,\Phi\, f^{\leftarrow\prime} \;, & \tilde{\Phi}(v,t) = f^{\leftarrow\prime *}(v)\;\Phi(f^{\leftarrow}(v),t)\; f^{\leftarrow\prime}(v). \end{cases}$$

□

2.8 APPENDIX.

In this appendix the evolution equation $\dot{u} = X(u)$ on M is extended to an evolution equation for u and its "variation" $\delta u = v$ on TM. Using this evolution equation for $z = (u,v)$, we show how (2.2.7) can be derived without using a local trivialization of the tangent bundle TM. Since $z(t) \in TM$ and so $\dot{z}(t) \in T_{z(t)}(TM)$ we have to construct a vector field on TM(not on M).

First some mathematical preliminaries (see also Abraham and Marsden [1, § 1.6 and exercise 1.6 D]). The set $T(TM)$ can be considered as a vector bundle in two different ways. First $T(TM)$ is the tangent bundle of TM with projection $\pi_2 : T(TM) \to TM$. In this case, the internal structure of TM is unimportant. However, using the fact that TM is itself a tangent bundle, we can supply $T(TM)$ with another vector bundle structure. Denote the projection of the tangent bundle TM by $\pi_1 : TM \to M$. The derivative of this map is $\pi_1' : T(TM) \to TM$. Using this map we can supply $T(TM)$ with an additional vector bundle structure. Note that with the projection π_1' the bundle $T(TM)$ is not a tangent bundle. The two possible projections are illustrated in figure 1 and figure 2.
Note that in these figures tangent vectors to M can be indicated in two ways, see $y \in T_u M$ in figure 2. The situation is summarized in the "dual tangent rhombic", as shown in figure 3. In the sequel we shall need the following

2.8.1 Lemma.

There exists a map $S_M : T(TM) \to T(TM)$ such that

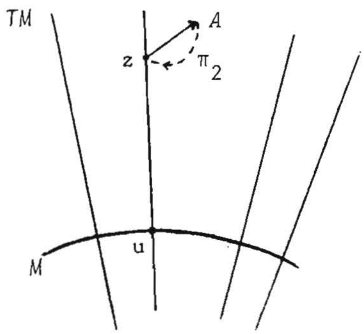
i) $S_M \circ S_M = \mathrm{Id}$ on $T(TM)$,

ii) $\pi_1' \circ S_M = \pi_2,$

$$\pi_2 \circ S_M = \pi_1' \quad .$$

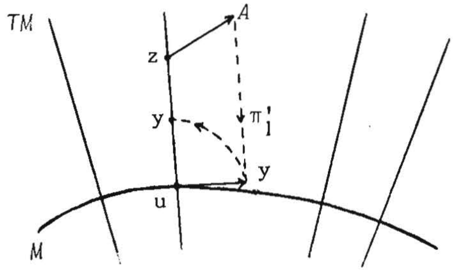
<u>Proof</u>:
See Abraham and Marsden [1, exercise 1.6 D] .

□



$A \in T_z(TM)$
$\pi_2(A) = z$

Figure 1.



$A \in T_z(TM)$
$\pi_1 : TM \to M,\ \pi_1(z) = u$
$\pi_1' : T_z(TM) \to T_uM,\ \pi_1'(A) = y$

Figure 2.

~~Figure 1~~.

$$\begin{tikzcd} & T(TM) \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2"] & \\ TM \arrow[dr, "\pi_1"'] & & TM \arrow[dl, "\pi_1"] \\ & M & \end{tikzcd} \qquad \begin{tikzcd} & A \arrow[dl, "\pi_1'"'] \arrow[dr, "\pi_2"] & \\ y \arrow[dr, "\pi_1"'] & & z \arrow[dl, "\pi_1"] \\ & u & \end{tikzcd}$$

Figure 3.

The map S_M is called the *canonical involution* on M. The lemma may be clearified by looking at figure 2. If we apply the mapping S_M to $A \in T(TM)$ we obtain $\tilde{A} = S_M(A) \in T(TM)$. From $\pi_2(\tilde{A}) = \pi_2(S_M(A)) = \pi_1'(A) = y$ we see that $\tilde{A} \in T_y(TM)$. So we obtain a vector $\tilde{A}$ which is tangent to TM in y. Another application of S_M to $\tilde{A}$ yields again the vector A.

Now we are able to express the Lie bracket of two vector fields on M in terms of the derivatives of the vector fields. Suppose C is a vector field on M. So it is a mapping $C: M \to TM$ such that

$$\pi_1 \circ C = \mathrm{Id} : M \to M \,. \tag{2.8.2}$$

The derivative of the vector field C in a point $u \in M$ is the linear mapping

$$C'(u) : T_u M \to T_{C(u)}(TM).$$

Suppose $E \in T_u M$, then $C'(u)\, E \in T_{C(u)}(TM)$, hence

$$\pi_2(C'(u)E) = C(u) \in TM. \tag{2.8.3}$$

By taking the derivative of (2.8.2) we obtain $\pi_1' \circ C' = \mathrm{Id} : TM \to TM$. This implies

$$\pi_1'(C'(u)E) = E \in TM. \tag{2.8.4}$$

Let B be another vector field on M. Analogous to the expressions for the Lie bracket in local coordinates or in a local trivialization (see (1.1.8) or (2.2.5)), we would like to define $[B,C]$ by computing the difference of $C'(u)\, B(u)$ and $B'(u)\, C(u)$. But since $B'(u)\, C(u) \in T_{B(u)}(TM)$ and $C'(u)\, B(u) \in T_{C(u)}(TM)$ this is not possible.
Now we can use the canonical involution S_M. Using lemma 2.8.1 and (2.8.4) we see that

$$\pi_2(S_M(C'(u)B(u))) = \pi_1'(C'(u)B(u)) = B(u). \tag{2.8.5}$$

This means that $S_M(C'(u)\ B(u)) \in T_{B(u)}\ (TM)$. So we can define

$$(2.8.6) \qquad F(u) := S_M(C'(u)\ B(u)) - B'(u)\ C(u) \in T_{B(u)}\ (TM)\ .$$

We now compute the projection π_1' of $F(u)$. Using lemma 2.8.1, (2.8.3) and (2.8.4) and noting that $\pi_1' : T_{B(u)}(TM) \to T_uM$ is a linear map, we obtain

$$\pi_1'(F(u)) = \pi_2(C'(u)\ B(u)) - \pi_1'\ (B'(u)\ C(u))$$

$$= C(u) - C(u) = 0 \in T_uM.$$

This means that $F(u)$ is not only tangent to TM in the point $B(u)$, but even tangent to T_uM in the point $B(u)$. The situation may be elucidated by the following figure.

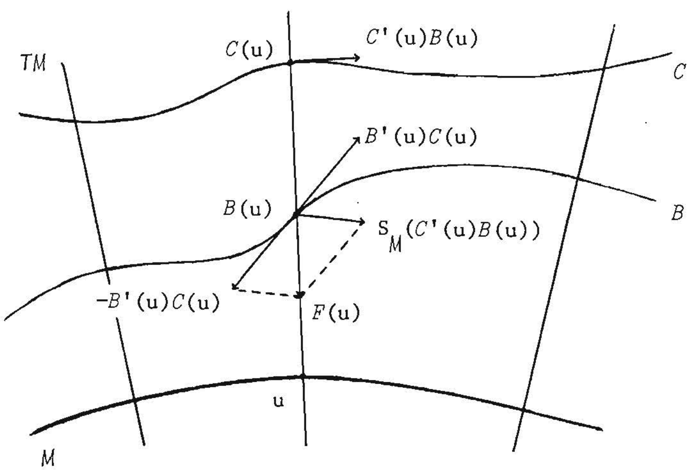


Figure 4.

So $F(u) \in T_{B(u)}(T_uM)$. Finally, using the canonical isomorphism between the linear space T_uM and its tangent space $T_{B(u)}\ (T_uM)$ (see for instance Dieudonné [18], § 16.5.2) we can consider $F(u)$ as an element of T_uM. Since u is arbitrary we constructed a new vector field F on M. By expressing (2.8.6) in local coordinates we see that $F = [B,C]$, the Lie bracket of the vector fields B and C on M. So we have proved the following

2.8.7 Theorem.

The Lie bracket of the vector fields B and C on M is the vector field $[B,C]$ on M, given by

(2.8.8) $[B,C](u) = S_M(C'(u)\, B(u)) - B'(u)\, C(u).$

□

2.8.9 Remark.

In most text-books the Lie bracket of two vector fields is introduced in a much simpler way. However, in the derivation of the condition (2.2.7) for symmetries, both terms of the right hand side of (2.8.8) first appear seperately.

□

2.8.10 Remark.

The preceding construction the Lie bracket is not symmetric. Of course the other possibility (using $S_M(B'(u)\, C(u)) \in T_{C(u)}(TM)$) yields the same result.

□

After these complicated preliminaries the final results are within reach. An evolution equation for u and its "variation" $\delta u = v$ is easily obtained. Suppose $z = (u, \delta u) \in TM$. The expression (2.2.2) suggests to describe the time evolution of z using $X'z$. However, from (2.8.3) we see that $\pi_2(X'z) = X(u)$, which means that (in general) $X'z \notin T_z(TM)$. The correct generalization of (2.2.2) is given by

(2.8.11) $\dot{z} = S_M(X'z)$.

Lemma 2.8.1 and (2.8.4) imply that

$$\pi_2(S_M(X' z)) = \pi_1'(X'z) = z,$$

so $S_M(X'z) \in T_z(TM)$. This means that indeed the right hand side of (2.8.11) is a vector field on TM. From $u = \pi_1(z)$, lemma 2.8.1 and (2.8.3) we obtain

$$\dot{u} = \pi_1'(\dot{z}) = \pi_1' \ (S_M(X'z)) = \pi_2(X'z) = X(u),$$

so we see that (2.2.1) is "contained in " (2.8.11). By using a local trivialization of TM it is also possible to derive (2.2.2) from (2.8.11). *So the evolution equation* (2.8.11) *can be considered as an equation which describes the evolution of* u *(as given in* (2.2.1)) *and the evolution of* $v = \delta u$ *(for a local trivialization given in* (2.2.2)).

Finally we consider again special solutions of (2.8.11). This leads to

2.8.12 Theorem.

Suppose Y is a parameterized vector field on M such that for all solutions $u(t)$ of (2.2.1) $z(t) = Y(u(t),t)$ satisfies (2.8.11). Then

$$(2.8.13) \quad \dot{Y} = [Y,X] .$$

Proof:
Since $z(t) = Y(u(t),t)$ has to be a solution of (2.8.11) for all solutions $u(t)$ of (2.2.1), the vectorfield Y must satisfy

$$(2.8.14) \quad \dot{Y}(u,t) + Y'(u,t)\, X(u) = S_M(X'(u)\, Y(u,t)\,) \quad \forall u \in M,\ t \in \mathbb{R} .$$

Note that $Y'(u,t)\, X(u) \in T_{Y(u,t)}(TM)$ while at first sight $\dot{Y}(u,t) \in T_uM$. However, since T_uM is a linear space, it is canonically isomorphic with its tangent space in an arbitrary point, hence

$$\dot{Y}(u,t) \ \in T_{Y(u,t)}(T_uM) \ \subset T_{Y(u,t)} \ (TM).$$

So (2.8.14) is a correct equation. The theorem now follows from theorem 2.8.7.

□

Thus we have again obtained condition (2.2.7) (which is equivalent to (2.8.13)) for the vector field Y.

CHAPTER 3: HAMILTONIAN SYSTEMS.

3.1 INTRODUCTION.

In this chapter we make some remarks on Hamiltonian systems. Since many results in this chapter are standard, a number of proofs is omitted.
In section 3.2 we introduce Hamiltonian systems using symplectic geometry.
In sections 3.3, 3.4 and 3.6 we describe Poisson brackets, variational principles and completely integrable Hamiltonian systems. The transformation properties of Hamiltonian systems are explained in section 3.7.
In chapter 2 we considered (adjoint) symmetries for general dynamical systems. In section 3.5 we show that, if a certain kind of adjoint symmetry exists, the dynamical system is Hamiltonian. The general theory of symmetries for Hamiltonian systems is described in the next chapter.
To us the formulation of definition 3.2.4 and the results given in lemma 3.2.11, theorem 3.2.12 and in section 3.5 are new. Sometimes we give expressions using local coordinates. In that case the Hamiltonian systems are considered to be finite-dimensional. In this thesis we only consider autonomous (possibly infinite-dimensional) Hamiltonian systems.

Introduce coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ in phase space $\mathbb{R}^{2n}$. Then a *classical Hamiltonian system* can be described by a function $H : \mathbb{R}^{2n} \to \mathbb{R}$, called the Hamiltonian. The system consists of the set of differential equations

$$(3.1.1) \qquad \begin{cases} \dot{q}_i = \dfrac{\partial H}{\partial p_i}(q_1,\ldots,p_n) \\[2ex] \dot{p}_i = -\dfrac{\partial H}{\partial q_i}(q_1,\ldots,p_n) \qquad i = 1,\ldots, n. \end{cases}$$

3.2 DEFINITION OF HAMILTONIAN SYSTEMS.

A very elegant description of Hamiltonian systems is possible in the modern language of symplectic geometry (see for instance Arnold [2], Abraham and Marsden [1] , Souriau [4]). This method will finally result in a system of differential equations, of which (3.1.1) is a special case. Therefore we called (3.1.1) a classical Hamiltonian system. The phase space of these Hamiltonian systems will be a symplectic manifold M.

Consider a two-form ω on M. With this two-form corresponds a vector bundle map $\Omega : TM \to T^*M$, defined by

$$(3.2.1) \qquad < \Omega\, A,B > = \omega(A,B) \qquad \forall A,B\in T_uM,\ \forall u\in M.$$

Of course Ω can also be considered as a tensor field of covariant order 2, $\Omega \in T_2^o(M)$. Mostly we use the last designation. Since a two-form is antisymmetric in its two arguments, the tensor field Ω also is antisymmetric

$$< \Omega(u)\ A,B> = - < \Omega(u)\ B,A > \qquad \forall A,B\in T_uM, \forall u\in M\ .$$

3.2.2 Definition.

We call a two form ω *(strongly) nondegenerate* if the tensor field Ω (considered as vector bundle map $\Omega : TM \to T^*M$) is an isomorphism. The inverse tensor field is then denoted by $\overleftarrow{\Omega}$. If the tensor field (vector bundle map) Ω is one to one, the two-form ω is called *weakly nondegenerate*.

□

A weakly nondegenerate two-form on a finite-dimensional manifold M is (strongly) nondegenerate. A nondegenerate two-form can only exist on a finite-dimensional manifold M if the dimension of M is even. We call Ω and $\overleftarrow{\Omega}$ the tensor fields corresponding to the (nondegenerate) two-form ω. It is easily seen that $\overleftarrow{\Omega}$ is also antisymmetric

$$< \alpha,\ \overleftarrow{\Omega}(u)\ \beta > = - \ < \beta,\overleftarrow{\Omega}(u)\alpha \ > \quad \forall\ \alpha,\ \beta \in T^*_u M,\ \forall\ u \in M.$$

The tensor field Ω can be used to transform a vector field on M into a one-form. So we can consider Ω as a linear mapping $\Omega : X(M) \to X^*(M)$. In the same way we can consider $\overleftarrow{\Omega}$ as a linear mapping $\overleftarrow{\Omega} : X^*(M) \to X(M)$.

3.2.3 Definition.

A *symplectic manifold* is a pair (M,ω) where ω is a closed, nondegenerate two-form on the manifold M. The form ω is called a *symplectic form*.

□

Infinite-dimensional Hamiltonian systems are often described using a closed, weakly nondegenerate two-form ω. Then ω is called a *weak symplectic* *form*.

It is useful to translate the closedness of a two-form ω into properties of the corresponding tensor fields Ω and $\overleftarrow{\Omega}$.

3.2.4 Definition.

Consider the tensor fields $\Phi \in T_2^o(M)$ and $\Psi \in T_o^2(M)$. Define the mappings $f : X(M) \times X(M) \times X(M) \to \mathbb{R}$ and $g: X^*(M) \times X^*(M) \times X^*(M) \to \mathbb{R}$ by

$$f(A,B,C) = < L_A(\Phi B), C > \tag{3.2.5}$$

$$g(\alpha,\beta,\gamma) = < L_{\Psi\alpha} \beta, \Psi\gamma > . \tag{3.2.6}$$

Then the tensor field Φ is called *cyclic* if it is antisymmetric and if for all vector fields A, B, C

$$f(A,B,C) + f(B,C,A) + f(C,A,B) = 0. \tag{3.2.7}$$

The tensor field Ψ is called *canonical* if it is antisymmetric and if for all one-forms α, β, γ

$$g(\alpha,\beta,\gamma) + g(\beta,\gamma,\alpha) + g(\gamma,\alpha,\beta) = 0. \tag{3.2.8}$$

□

In the literature cyclic tensor fields unfortunately are also called symplectic operators (symplectic transformations are explained in remark 3.7.6). For canonical tensor fields various other names are in use, such as inverse symplectic, implectic, co-symplectic. See for instance, Magri [5] , Fuchssteiner and Fokas [8] , Fokas and Anderson [9] . In local coordinates u^i on M the tensor fields Φ and Ψ are represented by matrices $\Phi_{ij}(u)$ and $\Psi^{ij}(u)$. Then (3.2.7) can be written in the following well-known form

$$\Phi_{ij,k}(u) + \Phi_{jk,i}(u) + \Phi_{ki,j}(u) = 0 \quad \forall u \in M. \tag{3.2.9}$$

The condition (3.2.8) for the canonical tensor field Ψ becomes

$$\Psi^{ij}_{,m}(u)\, \Psi^{mk}(u) + \Psi^{jk}_{,m}(u)\, \Psi^{mi}(u) + \Psi^{ki}_{,m}(u)\, \Psi^{mj}(u) = 0 \quad \forall u \in M . \tag{3.2.10}$$

3.2.11 Lemma.

Suppose the tensor field $\Phi \in T^o_2(M)$ is invertible. If Φ is cyclic, the inverse tensor field Φ^{-1} is canonical. Suppose the tensor field $\Psi \in T^2_o(M)$ is invertible. If Ψ is canonical, the inverse tensor field Ψ^{-1} is cyclic.

Proof:
Using $L_A \Phi^{-1} = -\Phi^{-1}(L_A\Phi)\Phi^{-1}$ and the definitions of cyclic and canonical tensor fields the proof is almost trivial.

□

3.2.12 Theorem.

Let ω be a non degenerate two-form with corresponding tensor fields Ω and $\overleftarrow{\Omega}$. Then the following conditions are equivalent

i) ω is closed,
ii) Ω is cyclic,
iii) $\overleftarrow{\Omega}$ is canonical.

Proof:
The equivalence of ii) and iii) follows immediately from lemma 3.2.11. The equivalence of i) and ii) can be shown in the following way. Let A, B, C be vector fields. Define the one-form $\alpha = \Omega A = i_A\omega$. Applying Leibniz rule to the identity

$$d\alpha(B,C) = L_B\langle\Omega A,C\rangle - L_C\langle\Omega A,B\rangle - \langle\Omega A,[B,C]\rangle$$

results in

$$d\alpha(B,C) = \langle L_B(\Omega A),C\rangle + \langle\Omega A,[B,C]\rangle - \langle L_C(\Omega A),B\rangle. \tag{3.2.13}$$

From $L_A\omega = i_A\, d\omega + d\, i_A\omega = i_A\, d\omega + d\alpha$ we obtain

$$\begin{aligned}\langle(L_A\Omega)B,C\rangle &= i_A\, d\omega(B,C) + d\alpha(B,C)\\ &= d\omega(A,B,C) + d\alpha(B,C)\ .\end{aligned}$$

This implies

$$< L_A(\Omega B),\ C > = d\omega(A,B,C) + d\alpha(B,C) + <\Omega[A,B],\ C>.$$

Substitution of (3.2.13) finally results in

$$d\omega(A,B,C) = f(A,B,C) + f(B,C,A) + f(C,A,B)$$

where f is given in (3.2.5) (with $\Phi = \Omega$). So $d\omega = 0$ is equivalent with (3.2.7).

□

Now we are able to define a Hamiltonian vector field on a symplectic manifold (M,ω). Consider a function $H : M \to \mathbb{R}$, then dH is a one-form on M.

3.2.14 <u>Definition.</u>

The vector field $X = \overleftarrow{\Omega}dH$ is called a *Hamiltonian vector field* on the symplectic manifold (M,ω). The function H is called the *Hamiltonian*, the corresponding dynamical system is called a *Hamiltonian system.*

□

Note that $i_X\omega = dH$. Since ω is nondegenerate the vector field X is also uniquely determined by this relation. Let $u : (a,b) \to M$, then we say that u is a solution of this Hamiltonian system if

$$\dot{u}(t) = \overleftarrow{\Omega}(u(t))\, dH(u(t)) \qquad \forall t \in (a,b).$$

In a local coordinate system the tensor field Ω is represented by a matrix $\Omega_{ij}(u)$ and the tensor field $\overleftarrow{\Omega}$ is represented by the inverse matrix $\Omega^{ij}(u)$. Then the coordinates $u^i(t)$ of u(t) satisfy the following system of differential equations

(3.2.15) $$\dot{u}^i(t) = \Omega^{ij}(u(t))\, H_{,j}(u(t)).$$

However, we can always introduce new local coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ such that the system (3.2.15) transformes into the system (3.1.1).

(3.2.16) Theorem (Darboux).

For each $u_o \in M$ there exists a neighbourhood with local coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ such that the symplectic form ω can be written as

(3.2.17) $$\omega = \sum_{i=1}^{n} d\, q_i \wedge d\, p_i \quad .$$

Proof:
See Abraham and Marsden [1] or Choquet-Bruhat [3]

□

The coordinates $q_1, \ldots, p_n$ are called *canonical coordinates*. In this new coordinate system the cyclic tensor field Ω and the canonical tensor field $\overleftarrow{\Omega}$ are represented by

(3.2.18) $$\Omega_{ij} = \delta_{i,j+n} - \delta_{i+n,j} \, ,$$

(3.2.19) $$\Omega^{ij} = \delta^{i+n,j} - \delta^{i,j+n} \qquad \forall\, i,j = 1, \ldots, 2n.$$

With these matrices (3.2.15) reduces to the well-known classical Hamiltonian system (3.1.1).

3.3 POISSON BRACKETS.

Let (M,ω) be a symplectic manifold. With every pair of functions F and G on M corresponds a (new) function on M, called the Poisson bracket of F and G.

3.3.1 Definition.

The *Poisson bracket* of two (possibly explicitly time dependent) functions F and G on M is the function $\{F,G\}$ defined by

(3.3.2) $$\{F,G\} = \langle dF, \overleftarrow{\Omega} dG \rangle .$$

Two functions on M are *in involution* if their Poisson bracket vanishes.

□

In local coordinates (on a finite dimensional manifold) the definition can be written as

$$\{ F,G \} = F,_i \, \Omega^{ij} \, G,_j .$$

3.3.3 Theorem.

The Poisson bracket satisfies the so called *Jacobi identity*

$$\{\{ F,G \}, K\} + \{\{ G,K \}, F\} + \{\{ K,F \}, G\} = 0$$

for any three functions F, G, K $\in F_p(M)$.

Proof:

The proof of this standard result can be found in many text-books, see for instance Arnold [2]. We now give a proof which only uses that $\overleftarrow{\Omega}$ is canonical (and not that $\overleftarrow{\Omega}$ is invertible). Note that

$$\{ F, G \} = \langle dF, \overleftarrow{\Omega} dG \rangle = L_{\overleftarrow{\Omega} dG} F.$$

This implies

$$\{\{ F,G \}, K\} = \langle d\, L_{\overleftarrow{\Omega} dG} F, \overleftarrow{\Omega} dK \rangle$$

$$= \langle L_{\overleftarrow{\Omega} dG} dF, \overleftarrow{\Omega} dK \rangle$$

$$= g(dG, dF, dK)$$

where g is given in (3.2.6) (with $\Psi = \overleftarrow{\Omega}$). The theorem follows now from (3.2.8).

□

Recall (definition 2.3.1) that a function F $\in F_p(M)$ is a constant of the motion or first integral of a dynamical system on M if

$$\frac{d}{dt} F(u(t),t) = 0$$

for all solutions of the dynamical system. For a Hamiltonian system with Hamiltonian H on the symplectic manifold (M,ω) this implies the following

3.3.4 Lemma.

A function $F \in \mathcal{F}_p(M)$ is a constant of the motion iff $\{F,H\} + \dot{F} = 0$ on $M \times \mathbb{R}$. For functions F, which do not depend explicitly on t (so $F \in \mathcal{F}(M)$) this condition is $\{F,H\} = 0$.

Proof:
It is easily seen that

$$\frac{d}{dt} F(u(t),t) = \langle dF, \overleftarrow{\Omega} dH \rangle + \dot{F}$$

$$= \{F,H\} + \dot{F}.$$

□

The following lemma is an immediate consequence of the Jacobi identity.

3.3.5 Lemma.

The set of constants of the motion for a Hamiltonian system is a Lie algebra, if we take the Poisson bracket as Lie product. The set of autonomous constants of the motion is a subalgebra of this Lie algebra.

□

3.4 VARIATIONAL PRINCIPLES.

It is well known that the classical Hamiltonian system (3.1.1) can be derived from the following *variational principle*

$$\underset{\mathcal{U}}{\text{stat}} \int_{t_1}^{t_2} \left(\sum_{i=1}^{n} p_i \dot{q}_i - H(q_1,\ldots,p_n) \right) dt$$

where $\mathcal{U}$ is the set of all curves in phase space $\mathbb{R}^{2n}$ with $q_i(t_1)$ and $q_i(t_2)$ fixed. There also exists a variational principle which yields directly the more general equations (3.2.15):

3.4.1 Theorem.

For every point $u_o \in M$ there exists a neighbourhood $U_o \ni u_o$ and a one-form α defined on U_o, such that a solution $\tilde{u}(t) \in U_o$ for $t \in [t_1, t_2]$ of (3.2.15) is a stationary point of the following functional

$$\int_{t_1}^{t_2} (< \alpha(u(t)), \dot{u}(t) > - H(u(t))) \, dt \tag{3.4.2}$$

over the set of all curves $u(t) \in U_o$ for $t \in [t_1, t_2]$ with $u(t_1) = \tilde{u}(t_1)$, $u(t_2) = \tilde{u}(t_2)$.

Proof:

The two-form ω is closed, so for every point $u_o \in M$ there exists a neighbourhood U'_o and a one-form α defined on U'_o, such that $\omega = - d\alpha$. On a neighbourhood $U_o \subset U'_o$ there exist local coordinates u^i such that $\alpha = \alpha_i \, du^i$. So (3.4.2) can be written as

$$\int_{t_1}^{t_2} (\alpha_i(u(t)) \, \dot{u}^i(t) - H(u(t))) \, dt \tag{3.4.3}$$

Then it is an elementary exercise to show that stationary points of (3.4.3) with $u(t_1) = \tilde{u}(t_1)$, $u(t_2) = \tilde{u}(t_2)$, are solutions of

$$(\alpha_{i,j} - \alpha_{j,i}) \, \dot{u}^i = H_{,j}. \tag{3.4.4}$$

From $\omega = - d\alpha = \frac{1}{2}\Omega_{ji} du^i \wedge du^j$ we obtain $\Omega_{ji} = \alpha_{i,j} - \alpha_{j,i}$. Multiplication of (3.4.4) with Ω^{ij}, the inverse matrix of Ω_{ji}, results in (3.2.15).

□

3.5 HAMILTONIAN SYSTEMS AND ADJOINT SYMMETRIES.

In this section we deal with the question: when is a dynamical system a Hamiltonian system? In general this is a very difficult problem. For a number of equations the Hamiltonian character was only found after a long time. For instance, the Hamiltonian character of the Korteweg - de Vries equation [6,7] was found rather recently by Gardner [11] and Broer [10] .

Consider an autonomous dynamical system on a manifold M

$$(3.5.1) \qquad \dot{u} = X(u).$$

We saw in theorem 2.5.8 that a non-closed adjoint symmetry gives rise to an SA operator.

3.5.2 Theorem.

Suppose the dynamical system (3.5.1) has an adjoint symmetry ρ such that $\omega = d\rho$ is nondegenerate . Then the vector field X can be written as

$$(3.5.3) \qquad X = \overleftarrow{\Omega}\,(dF - \dot{\rho}\,)$$

where $F = - < \rho, X>$, Ω is the SA operator corresponding (theorem 2.5.8) to ρ and $\overleftarrow{\Omega} = \Omega^{-1}$.

Proof:

From $F = - <\rho,X> = - i_X\rho$ we obtain

$$dF = -d\ i_X\ \rho = - L_X\rho + i_X d\rho\ .$$

$\rho \in V^*(X,M)$ implies $L_X\rho = - \dot{\rho}$, so

$$i_X\ d\rho = dF - \dot{\rho}\quad .$$

Then from $<i_X\ d\rho,A> = d\rho\ (X,A) = <\Omega X,A> \quad \forall\ A \in T_uM$, we obtain

$$\Omega X = i_X d\rho = dF - \dot{\rho}\ .$$

Since $d\rho$ is nondegenerate , the inverse tensor field $\overleftarrow{\Omega} \in T^2_0(M)$ of Ω exists. Application of $\overleftarrow{\Omega}$ to the formula above results in (3.5.3).

□

Dynamical systems corresponding to (3.5.3) are sometimes called

Birkhoffian systems, see for instance Santilli [12] . In local coordinates u^i the corresponding differential equations can be written as

$$(3.5.4) \qquad (\rho_{i,j}(u,t) - \rho_{j,i}(u,t))\dot{u}^j = F_{,i}(u,t) - \dot{\rho}_i(u,t)$$

where the matrix $(\rho_{i,j}(u,t) - \rho_{j,i}(u,t))$ is invertible on $M \times \mathbb{R}$.
If the adjoint symmetry ρ satisfies one more condition, the system is Hamiltonian.

3.5.5 Theorem.

Let the dynamical system (3.5.1) have an adjoint symmetry ρ such that
i) $d\rho$ is nondegenerate,
ii) there exists a constant of the motion G such that $\dot{\rho} = dG$.

Then the vector field X is a Hamiltonian vector field with Hamiltonian $H = - < \rho, X> - G$ and symplectic two-form $\omega = d\rho$.

Proof:
Substitution of ii) in (3.5.3) yields

$$X = \overleftarrow{\Omega}\, d(F-G) = \overleftarrow{\Omega}\, dH.$$

In the definition of a Hamiltonian system we required H and ω to be a function and a two-form on M, so they may not depend explicitly on t. This is easily verified

$$\dot{H} = \dot{F} - \dot{G} = - < \dot{\rho}, X> - \dot{G} = - < dG, X> - \dot{G} = 0,$$

since G is a constant of the motion. Also

$$\dot{\omega} = d\dot{\rho} = ddG = 0.$$

□

3.5.6 Remark.

Theorem 3.5.5 gives sufficient conditions for a dynamical system to be Hamiltonian. However, this does not mean that, in trying to find out if a dynamical system is Hamiltonian, one should try to find an adjoint symmetry as described in theorem 3.5.5. There are two reasons for this. First finding an adjoint symmetry as described in theorem 3.5.5 is not simpler then finding the symplectic two-form and Hamiltonian. The second reason is that theorem 3.5.5 gives sufficient but not necessary conditions for a system to be Hamiltonian. A simple example of this is provided by a Hamiltonian system with a symplectic form ω which is closed but not exact. The Hamiltonian form of the Korteweg-de Vries equation (section 5.6) can be found by theorem 3.5.5.

□

3.6 COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS.

For symplicity we now consider the symplectic manifold $M = \mathbb{R}^{2n}$ with canonical coordinates $q_1,\ldots, p_n$. Then $\omega = \sum_i d\, q_i \wedge dp_i$. Suppose we introduce new coordinates $\tilde{q}_1,\ldots, \tilde{p}_n$ on $\mathbb{R}^{2n}$.

3.6.1 Definition.

The transformation from $q_1,\ldots, p_n$ to $\tilde{q}_1,\ldots,\tilde{p}_n$ is called a *canonical coordinate transformation* if, in new coordinates $\omega = \sum_i d\tilde{q}_i \wedge d\tilde{p}_i$.

□

So after a canonical coordinate transformation the differential equations for $\tilde{q}_i$, $\tilde{p}_i$ are also of the form (3.1.1).

Sometimes by means of a canonical coordinate transformation, the system of differential equations is greatly simplified. For instance suppose all the new coordinates $\tilde{q}_i$ are *cyclic*. This means the Hamiltonian, written as function of $\tilde{p}_i$ and $\tilde{q}_i$, depends only on the $\tilde{p}_i$. The solution of the corresponding Hamiltonian system is trivial and the system is called *completely integrable*. Furthermore the functions $\tilde{p}_i$ constitute a set of n constants in involution. In general it turns out that the existence of n constants of the motion in involution, is directly related to the complete integrability of the system.

3.6.2 Theorem (Arnold, Liouville).

Suppose there exist n constants of the motion in involution $F_1 = H, F_2,\ldots,F_n$. Consider the level set of functions F_i

$$M_{\underline{a}} = \{(q_1,\ldots, p_n)\in M = \mathbb{R}^{2n} \mid F_i(q_1,\ldots,p_n) = a_i\} .$$

Assume the one-forms dF_i are linearly independent on $M_{\underline{a}}$ and that $M_{\underline{a}}$ is compact and connected.

Then

i) $M_{\underline{a}}$ is invariant for the Hamilton flow with Hamiltonian H,

ii) $M_{\underline{a}}$ is diffeomorphic to the n-dimensional torus $T^n = \{(\tilde{q}_1,\ldots,\tilde{q}_n) \bmod 2\pi\}$,

iii) there exist n functions $\tilde{p}_i(F_1,\ldots,F_n)$ such that $\tilde{q}_1,\ldots, \tilde{q}_n, \tilde{p}_1,\ldots,\tilde{p}_n$ are coordinates for a neighbourhood of $M_{\underline{a}}$. The transformation $(q_1,\ldots,p_n) \to (\tilde{q}_1,\ldots,\tilde{p}_n)$ is a canonical coordinate transformation and the Hamiltonian H, expressed in the new coordinates, depends only on the $\tilde{p}_i$: $H = H(\tilde{p}_1,\ldots,\tilde{p}_n)$.

Proof:

See Arnold [2].

□

The solution of the corresponding Hamiltonian system

$$(3.6.3)\qquad \begin{cases} \dot{\tilde{p}}_i = 0 \\[2ex] \dot{\tilde{q}}_i = \dfrac{\partial \tilde{H}(\tilde{p}_1,\ldots,\tilde{p}_n)}{\partial \tilde{p}_i} \qquad i = 1,\ldots,n \end{cases}$$

is trivial and the system is completely integrable. The coordinates $\tilde{p}_i$ are called *action variables*, while the $\tilde{q}_i$ are called *angle variables*.

Note that we only discussed complete integrability for finite-dimensional Hamiltonian systems. Some remarks on complete integrability for infinite-dimensional Hamiltonian systems will be made in section 5.5.

3.7 TRANSFORMATION PROPERTIES OF HAMILTONIAN SYSTEMS.

In section 2.7 we discussed the behaviour under transformations of (adjoint) symmetries and the four possible operators between (adjoint) symmetries. The transformation properties of Hamiltonian systems are also easily found. Consider the Hamiltonian vector field $X = \overleftarrow{\Omega}\, dH$, on a manifold M, corresponding to the symplectic form ω and Hamiltonian H.

Suppose there exists a diffeomorphism $f : M \to N$ with inverse $\overleftarrow{f} : N \to M$. Using the derivative map $f' : TM \to TN$ we can transform the vector field X on M to a vector field $\tilde{X} = f'X$ on N.

3.7.1 Theorem.

The transformed vector field $\tilde{X} = f'X$ of the Hamiltonian vector field X is again a Hamiltonian vector field. The corresponding Hamiltonian $\tilde{H}$ and symplectic two-form $\tilde{\omega}$ on N are given by

(3.7.2) $\quad \tilde{H}(v) = H(\overleftarrow{f}(v)) \quad \forall\, v \in N,$

(3.7.3) $\quad \tilde{\omega}(\tilde{A},\tilde{B}) = \omega(\overleftarrow{f}'\tilde{A}, \overleftarrow{f}'\tilde{B}) \quad \forall\, \tilde{A},\tilde{B} \in X(N)$.

The tensor fields $\tilde{\Omega} \in T^{o}_{2}(N)$ and $\overleftarrow{\tilde{\Omega}} \in T^{2}_{o}(N)$ (considered as vector bundle maps we have $\tilde{\Omega}: TN \to T^*N$ and $\overleftarrow{\tilde{\Omega}} : T^*N \to TN$) are given by

(3.7.4) $\quad \tilde{\Omega} = \overleftarrow{f}'^{*}\, \Omega\, \overleftarrow{f}'$,

(3.7.5) $\quad \overleftarrow{\tilde{\Omega}} = f'\overleftarrow{\Omega} f'^{*}$.

Proof:
The relations between functions, differential forms and tensor fields are "natural" with respect to transformations (see also section 2.7). This means that the transformed vector field $\tilde{X} = f'X$ can also be obtained from the transformed Hamiltonian $\tilde{H}$ and the transformed two-form $\tilde{\omega}$. The formulas (3.7.2), (3.7.3), (3.7.4) and (3.7.5) give the usual transformation properties of functions, differential forms and tensor fields.

□

3.7.6 Remark.

By the method used in theorem 3.7.1 we can supply the manifold N with a symplectic two-form $\tilde{\omega}$, the push-forward of ω by f. Suppose there exists already a symplectic form ϕ on N; so (M,ω) and (N,ϕ) are both symplectic manifolds. On N we now have the symplectic forms ϕ and $\tilde{\omega}$. If $\phi = \tilde{\omega}$ the mapping f is called a *symplectic transformation* (symplectic diffeomorphism) or *canonical transformation*. A canonical transformation should not be confused with a canonical coordinate transformation, as described in definition 3.6.1.

□

Other properties of the Hamiltonian system on M are also easily translated to the transformed system on N.

3.7.7 Corollary.

The transformed Poisson bracket of two functions $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ on M is equal to the Poisson bracket of the transformed function F_1, F_2 on N.

Proof:

From $\underset{\sim}{F}_i(v) = F_i(\overleftarrow{f}(v))$ (i = 1,2) we obtain $d\underset{\sim}{F}_i = \overleftarrow{f}'^{*} dF_i$.

The result now follows from the definition of Poisson bracket and of (3.7.5).

□

So if the functions $\underset{\sim}{F}_1$, $\underset{\sim}{F}_2$ on M are in involution, the transformed functions F_1, F_2 on N are also in involution.

CHAPTER 4: SYMMETRIES FOR HAMILTONIAN SYSTEMS.

4.1 INTRODUCTION.

In chapter 2 we considered some general properties of dynamical systems. We studied symmetries, adjoint symmetries and four types of operators between those symmetries. In this chapter we assume the dynamical system is a Hamiltonian system. The most important consequence of this Hamiltonian character is that there always exist at least one SA- and one AS operator. This implies that with a constant of the motion not only corresponds an adjoint symmetry, but also a symmetry. However, there can also exist symmetries which are not related in this way to a constant of the motion. These so called *non-(semi-) canonical symmetries* have interesting properties. In section 4.2 we show how they can be used to construct (new) SA- and AS operators. If the thus constructed (new) SA operator is invertible, the system can be written as a Hamiltonian system in two different ways. These so called *bi-Hamiltonian systems* are considered in section 4.3. Non-(semi-) canonical symmetries can also be used in various ways to construct (new) constants of the motion out of already known ones. In section 4.4 we shall describe three possible ways for doing this. In the sections 4.5 and 4.6 we consider Hamiltonian systems for which there exists a (non-semi-canonical) symmetry which satisfies some additional conditions. Then we show that there exist two infinite series symmetries, one of which corresponds to an infinite series of constants of the motion. The other series consists (in general) of non-semi-canonical symmetries. This method can be applied for several (all?) of the popular completely integrable infinite-dimensional Hamiltonian systems (Korteweg-de Vries equation, sine-Gordon equation). Of course the existence of infinite series of constants of the motion for these equations is well-known. However, several methods we describe for constructing these series of constants of the motion seem to be new. Also the second series of symmetries is generally overlooked.

In this chapter we shall consider an autonomous Hamiltonian system on a symplectic manifold (M,ω) with Hamiltonian H. With the two-form ω correspond the tensor fields $\Omega \in T_2^0(M)$ and $\Omega^{\leftarrow} \in T_0^2(M)$ (see section 3.2). The Hamiltonian vector field on M is then given by

(4.1.1) $\qquad X = \overleftarrow{\Omega}\, dH$

and the corresponding differential equation is

(4.1.2) $\qquad \dot{u}(t) = X(u(t)) = \overleftarrow{\Omega}(u(t))\, dH(u(t))$.

As in chapter 2 we shall assume that for all initial conditions $u(t_o) = u_o$ there exists a smooth unique solution u(t) of (4.1.2), defined on some interval $I \subset \mathbb{R}$.

4.2 SA- AND AS OPERATORS.

In this section we discuss the various possible SA- and AS operators for a Hamiltonian system. The following lemma will be useful in the sequel.

4.2.1 Lemma.

Suppose α is a closed (parameterized) one-form on M and $\Psi \in T^2_o(M)$ is a canonical tensor field. Then $L_{\Psi\alpha}\Psi = 0$.

Proof:
Let β and γ be arbitrary one-forms on M . Then define the vector fields $A = \Psi\alpha$, $B = \Psi\beta$ and $C = \Psi\gamma$. Application of Leibniz'rule to the identity

$$d\alpha(B,C) = L_B\langle\alpha,C\rangle - L_C\langle\alpha,B\rangle - \langle\alpha, [B,C]\rangle$$

results in

(4.2.2) $\qquad d\alpha(B,C) = \langle L_B\alpha,C\rangle + \langle \alpha,[B,C]\rangle - \langle L_C\alpha,B\rangle$.

Using Leibniz' rule and the antisymmetry of Ψ and its Lie derivatives we can write the second term as

$$\langle\alpha,[B,C]\rangle = \langle \alpha, L_B(\Psi\gamma) \rangle$$

$$= \langle\alpha,\Psi\, L_B\gamma\rangle - \langle\gamma,\ (L_B\Psi)\alpha\ \rangle$$

$$= \langle\alpha,\Psi\, L_B\gamma\rangle - \langle\gamma,\ L_B(\Psi\alpha)\rangle + \langle\gamma,\ \Psi L_B\alpha\ \rangle$$

$$= -\ \langle L_B\gamma,\ \Psi\alpha\rangle + \langle\gamma\ ,\ L_A(\Psi\beta)\rangle - \langle L_B\alpha,\ \Psi\gamma\rangle\ .$$

Substitution in (4.2.2) gives

$$\mathrm{d}\alpha(B,C) = -\ \langle L_B\gamma,\Psi\alpha\rangle\ \ + \langle\gamma,\ L_A(\Psi\beta)\rangle\ \ - \langle L_C\alpha\ ,B\ \rangle\ .$$

Since Ψ is canonical (see definition 3.2.4) this becomes

$$\mathrm{d}\alpha(B,C) = \langle L_A\beta,\Psi\gamma\rangle + \langle\gamma,\ L_A(\Psi\beta)\ \rangle$$

$$= -\ \langle\gamma,\Psi L_A\beta\rangle + \langle\gamma,\ L_A(\Psi\beta)$$

$$= \langle\gamma,\ (L_A\Psi)\beta\rangle\ .$$

The one-form α is closed, so the left hand side vanishes. The one-forms β and γ are arbitrary, so $L_A\Psi = L_{\Psi\alpha}\Psi = 0$.

□

The first application of this lemma is described in the following

4.2.3. <u>Lemma</u>.

Let α be a closed (parameterized) one-form on M and let $A = \overleftarrow{\Omega}\alpha$ be the corresponding vector field. Then $L_A\omega = 0$, $L_A\Omega = 0$ and $L_A\overleftarrow{\Omega} = 0$.

<u>Proof</u>:
The tensor field $\overleftarrow{\Omega}$ corresponds to the closed two-form ω, so by theorem 3.2.12 it is a canonical tensor field. Then by the preceding lemma $L_A\overleftarrow{\Omega} = 0$. From $L_A(\Omega\overleftarrow{\Omega}) = 0$ we obtain

$$L_A\Omega = -\ \Omega(L_A\overleftarrow{\Omega})\Omega\ \ = 0\ .$$

Finally $L_A\Omega = 0$ is equivalent with $L_A\omega = 0$.

□

4.2.4 Remark.

The importance of lemma 4.2.1 is that it can be used in cases where a canonical tensor field Ψ is (maybe) not invertible (see for instance section 5.6). In the proof of lemma 4.2.3 we had $\Psi = \overleftarrow{\Omega}$, which is invertible. Using this property the proof that $L_A\overleftarrow{\Omega} = 0$ can be considerably simplified. From $A = \overleftarrow{\Omega}\alpha$ we obtain $\alpha = \Omega A = i_A\omega$. Since α is closed we have $d\, i_A\omega = d\alpha = 0$. Then, because ω is closed, $L_A\omega = i_A d\omega + + d\, i_A\omega = 0$, which is equivalent to $L_A\Omega = 0$. Finally $L_A\overleftarrow{\Omega} = -\overleftarrow{\Omega}(L_A\Omega)\overleftarrow{\Omega} = 0$.

□

Recall that a tensor field $\Psi \in T^2_{op}(M)$ which can be used to map adjoint symmetries into symmetries was called an AS operator (see definition 2.5.3). A tensor field $\Phi \in T^o_{2p}(M)$ which maps symmetries into adjoint symmetries was called an SA operator (see definition 2.5.6). It turns out that for a Hamiltonian system there always exists an SA- and an AS operator .

4.2.5 Theorem.

The tensor field Ω is an SA operator and the tensor field $\overleftarrow{\Omega}$ is an AS operator.

Proof:

The conditions for an SA operator were given in definition 2.5.6. The operator Ω is an SA operator if it satisfies

$$(4.2.6) \qquad \dot{\Omega} + L_X\Omega = 0.$$

It follows form lemma 4.2.3 with $\alpha = dH$ that $L_X\Omega = 0$. Since ω does not depend explicitly on t, the corresponding tensor fields Ω and $\overleftarrow{\Omega}$ also don't. So Ω satisfies (4.2.6). In a similar way we can show that $\overleftarrow{\Omega}$ is an AS operator. This result also follows from theorem 2.5.7.

□

In local coordinates the tensor field Ω is represented by a matrix (matrix valued function) $\Omega_{ij}(u)$ and the tensor field $\overleftarrow{\Omega}$ is represented by the inverse matrix $\Omega^{k\ell}(u)$. A symmetry Y has components $Y^j(u,t)$ and an adjoint symmetry σ has components $\sigma_\ell(u,t)$. Then theorem 4.2.5 says that if Y^j is (represents) a symmetry, $\Omega_{ij}(u)\ Y^j(u,t)$ is an adjoint symmetry. Also if $\sigma_\ell(u,t)$ is an adjoint symmetry, $\Omega^{k\ell}(u)\ \sigma_\ell(u,t)$ is a symmetry.

Theorem 4.2.5 has a very important consequence. Suppose F is a constant of the motion. Then theorem 2.4.3 says that dF is an adjoint symmetry. Next theorem 4.2.5 implies that $\overleftarrow{\Omega}$dF is a symmetry. *So for a Hamiltonian system every constant of the motion* F *gives rise to a symmetry* $\overleftarrow{\Omega}$dF. This leads to the following

4.2.7 Definition.

i) An adjoint symmetry σ which is exact, will be called a *canonical adjoint symmetry*. The corresponding symmetry $Y = \overleftarrow{\Omega}\sigma$ will be called a *canonical symmetry*.

ii) An adjoint symmetry σ which is not exact, will be called a *non-canonical adjoint symmetry*. A symmetry Y such that $\sigma = \Omega Y$ is not exact will be called a *non-canonical symmetry*.

□

4.2.8 Remark.

Suppose $\sigma = \Omega Y$ is a canonical adjoint symmetry. Then there exists a function F on $M \times \mathbb{R}$ such that σ = dF. However, by theorem 2.4.5 there also exists a constant of the motion G such that σ = dG and $Y = \overleftarrow{\Omega}$dG. So the space of canonical adjoint symmetries (a subspace of $V^*(X;M)$) and the space of canonical symmetries (a subspace of $V(X;M)$) are both isomorphic to the space of constants of the motion. (Constants of the motion which differ only by a (numerical) constant are identified).

□

The following notions also turn out to be useful.

4.2.9 Definition.

i) An adjoint symmetry which is closed, will be called a *semi-canonical adjoint symmetry*. The corresponding symmetry $Y = \overleftarrow{\Omega}\sigma$ will be called a *semi-canonical symmetry*.

ii) An adjoint symmetry σ which is not closed, will be called a *non-semi-canonical adjoint symmetry*. Finally a symmetry Y such that $\sigma = \Omega Y$ is not closed will be called a *non-semi-canonical symmetry*.

4.2.10 Remark.

An exact differential form is always closed. In the terminology introduced above, this means that a canonical (adjoint) symmetry is also a semi-canonical (adjoint) symmetry. A differential form which is not closed is also not exact. This implies that a non-semi-canonical (adjoint) symmetry is also a non-canonical (adjoint) symmetry. Since a closed form is not necessarily exact, the converse of these two assertions is not true. By the Poincaré lemma a closed one-form α can *locally* be written as $\alpha = dF$. If this relation holds on $M \times \mathbb{R}$ the form is exact. There is a topological condition which implies that closed k-forms are exact. In our case (one-forms) the condition is that the *first cohomology group* of M vanishes. If the manifold M has this property, (non-) semi-canonical (adjoint) symmetries are identical with (non-) canonical (adjoint) symmetries. This happens for instance if M is also a linear space. □

In local coordinates u^i a canonical adjoint symmetry σ has local coordinates $\sigma_i = G,_i$ for some constant of the motion G. The coordinates σ_i of a semi-canonical adjoint symmetry satisfy $\sigma_{i,j} = \sigma_{j,i}$.

In theorem 4.2.5 we have seen that there always exists an SA- and an AS operator. Non-semi-canonical symmetries also provide us with that type of operators.

4.2.11 Theorem.

Suppose $Z = \overset{\leftarrow}{\Omega}\tau$ is a non-semi-canonical symmetry.
Then

i) $L_Z\Omega$ is an SA operator and $L_Z\overset{\leftarrow}{\Omega}$ is an AS operator,

ii) the operator $L_Z\Omega$ is cyclic and corresponds to the two-form $d\tau$

$$(4.2.12) \qquad < (L_Z\Omega)A,B> = (L_Z\omega)(A,B) = d\tau(A,B) \qquad \forall\, A,B \in X(M).$$

Proof:

In theorem 4.2.5 we have seen that Ω is an SA operator and $\overleftarrow{\Omega}$ is an AS operator. Then by theorem 2.5.15 i the same holds for the Lie derivatives in the direction of a symmetry Z. Using $\tau = \Omega Z = i_Z\omega$ and the closedness of ω we obtain

$$L_Z\omega = d\, i_Z\omega + i_Z d\omega = d\tau$$

which implies (4.2.12). Finally $d\tau$ is closed, so the corresponding operator $L_Z\Omega$ is cyclic (theorem 3.2.12).

□

In local coordinates u^i the operators $L_Z\Omega$ and $L_Z\overleftarrow{\Omega}$ are represented by the following matrices

$$(L_Z\overleftarrow{\Omega})^{ij} = \Omega^{ij}_{,m}\, Z^m - \Omega^{im}\, Z^j_{,m} - \Omega^{mj}\, Z^i_{,m} \quad ,$$

$$(L_Z\Omega)_{ij} = \Omega_{ij,m}\, Z^m + \Omega_{im}\, Z^m_{,j} + \Omega_{mj}\, Z^m_{,i}$$

$$= \tau_{i,j} - \tau_{j,i} \quad .$$

In theorem 2.5.8 we have seen that (also for a non-Hamiltonian system) a non-closed adjoint symmetry $\tau = \Omega Z$ gives rise to an SA operator. Theorem 4.2.11 states that (for a Hamiltonian system) this operator is identical to $L_Z\Omega$. Note that in the proof of theorem 4.2.11 we did not use that the symmetry Z was non-semi-canonical. However, if Z is semi-canonical, the corresponding adjoint symmetry τ is closed. Then by lemma 4.2.3, $L_Z\Omega = 0$ and $L_Z\overleftarrow{\Omega} = 0$. So, if the symmetry Z is semi-canonical, the constructed operators are trivial.

For a symmetry Z which is non-semi-canonical the constructed operators do not vanish. Of course this does not imply that they are invertible. As an example for this consider a system with two analytically independent constants of the motion F and G. Then $Z = \overleftarrow{\Omega}\tau = \overleftarrow{\Omega}(FdG) = F(\overleftarrow{\Omega}dG)$ is a non-semi-canonical adjoint symmetry. The two-form $d\tau$ is given by

$$d\tau(A,B) = \langle dF,A\rangle \langle dG,B\rangle - \langle dG,A\rangle \langle dF,B\rangle \quad .$$

Then (4.2.12) implies that

$$(L_Z\Omega)A = \langle \mathrm{dF}, A\rangle\, \mathrm{dG} - \langle \mathrm{dG}, A\rangle\, \mathrm{dF}.$$

So the SA operator $L_Z\Omega$ maps any vector field A into the module of one-forms spanned by dF and dG. If the manifold M has dimension larger then 2, this means that $L_Z\Omega$ is not invertible. If $L_Z\Omega$ $(L_Z\Omega^{\leftarrow})$ is invertible, then also $L_Z\Omega^{\leftarrow}$ $(L_Z\Omega)$. The two inverse operators are related by

$$\Omega^{\leftarrow}(L_Z\Omega^{\leftarrow})^{-1} + (L_Z\Omega)^{-1}\Omega = 0.$$

Even if $L_Z\Omega$ is not invertible, we can construct several recursion operators for symmetries and for adjoint symmetries. For instance the tensor fields

$$(4.2.13) \qquad \Omega^{\leftarrow} L_Z\Omega \ , \ (L_Z\Omega^{\leftarrow})\Omega \ , \ (L_Z\Omega^{\leftarrow})\ L_Z\Omega$$

are (can be used as) recursion operators for symmetries. However, these recursion operators are not independent. Using the relation $(L_A\Omega^{\leftarrow})\Omega$ + $\Omega^{\leftarrow} L_A\Omega = 0$ for an arbitrary vector field A we can show that

$$(4.2.14) \qquad \begin{aligned} (L_Z\Omega^{\leftarrow})\Omega &= -\ \Omega^{\leftarrow} L_Z\Omega \ , \\ (L_Z\Omega^{\leftarrow}) L_Z\Omega &= -\ (\Omega^{\leftarrow} L_Z\Omega)^2. \end{aligned}$$

Analogously the recursion operators for adjoint symmetries

$$(4.2.15) \qquad (L_Z\Omega)\Omega^{\leftarrow}, \ \ \Omega L_Z\Omega^{\leftarrow}, \ (L_Z\Omega)\ L_Z\Omega^{\leftarrow}$$

are related by

$$(4.2.16) \qquad \begin{aligned} \Omega(L_Z\Omega^{\leftarrow}) &= -\ (L_Z\Omega)\Omega^{\leftarrow} \ , \\ (L_Z\Omega)\ L_Z\Omega^{\leftarrow} &= -\ ((L_Z\Omega)\Omega^{\leftarrow})^2. \end{aligned}$$

So the recursion operators given in (4.2.13) and (4.2.15) can all be expressed as powers of $\Omega^{\leftarrow} L_Z\Omega$ and $(L_Z\Omega)\Omega^{\leftarrow}$. Of course we can also construct SA and AS operators by taking "higher Lie derivatives".

4.2.17 Theorem.

Suppose $Z = \overleftarrow{\Omega}\tau$ is a non-semi-canonical symmetry. Then for all $k \geq 1$

i) $L_Z^k\Omega$ is an SA operator and $L_Z^k\overleftarrow{\Omega}$ is an AS operator,

ii) $L_Z^k\Omega$ is a cyclic operator (cyclic tensor field) and corresponds to the two-form $d\,L_Z^{k-1}\tau$

$$<(L_Z^k\Omega)A,B> = (L_Z^k\omega)(A,B) = (d\ L_Z^{k-1}\tau)(A,B) \qquad \forall\ A,B \in X(M).$$

Proof:

The first part of this theorem follows by induction from theorem 4.2.11 and theorem 2.5.15 i. The SA operator $L_Z^k\Omega$ corresponds to the two-form $L_Z^k\omega$. Since $d\,L_Z^k\omega = L_Z^k\ d\omega = 0$ this SA operator is cyclic. Finally, by theorem 4.2.11

$$L_Z^k\omega = L_Z^{k-1}d\tau = d\,L_Z^{k-1}\tau\ .$$

□

4.2.18 Remark.

By combination of the SA operator $L_Z^k\Omega$ and the AS operator $\overleftarrow{\Omega}$ we see that $(L_Z^k\Omega)\overleftarrow{\Omega}$ is a recursion operator for adjoint symmetries. In this operator the symmetry Z is "used k times". This is also the case for the recursion operator for adjoint symmetries $((L_Z\Omega)\overleftarrow{\Omega})^k$. In general both recursion operators will be different. As an example for this we take again the symmetry $Z = \overleftarrow{\Omega}(FdG) = F(\overleftarrow{\Omega}dG)$ where F and G are constants of the motion. Then it can be shown that, for an arbitrary one-form α,

$$((L_Z\Omega)\overleftarrow{\Omega})^2\alpha = \{F,G\}(<dF,\ \overleftarrow{\Omega}\alpha>dG - <dG,\overleftarrow{\Omega}\alpha>dF)$$

and

$$(L_Z^2\Omega)\overleftarrow{\Omega}\alpha = \{F,G\}(<dF,\overleftarrow{\Omega}\alpha>dG - <dG,\overleftarrow{\Omega}\alpha>dF)$$

$$+F(\ <d\{F,G\},\overleftarrow{\Omega}\alpha>dG - <dG,\overleftarrow{\Omega}\alpha>d\{F,G\}\ ).$$

So in general both operators are not equal. If there exists a symmetry Z, such that the recursion operators for adjoint symmetries $((L_Z\Omega)\overleftarrow{\Omega})^2$ and $(L_Z^2\Omega)\overleftarrow{\Omega}$ are equal up to a multiplicative constant, several interesting properties can be proved. We shall consider that type of symmetry in the sections 4.5 and 4.6.

□

Finally we expand a (non-canonical) symmetry in canonical symmetries. This leads to the following

4.2.19 Theorem.

Suppose there exist constants of the motion G_i, $i = 1,\dots, m$, such that the canonical symmetries $\overleftarrow{\Omega}dG_i$ are linearly independent in every point $u \in M$. Suppose the symmetry Z can be written as

$$(4.2.20) \qquad Z = \sum_{i=1}^{m} F_i \overleftarrow{\Omega} dG_i \quad , \; F_i \in F_p(M) \; .$$

Then the functions F_i $(i=1,\dots,m)$ are constants of the motion.

Proof:

Since $\overleftarrow{\Omega}$ is an AS operator and the G_i are constants of the motion, we have

$$\dot{Z} + L_X Z = \sum_{i=1}^{m} (\dot{F}_i + L_X F)\overleftarrow{\Omega}\, dG_i \, .$$

The vector field Z is a symmetry, so the left hand side vanishes. Since the symmetries $\overleftarrow{\Omega}dG_i$ are linearly independent in every point of M, this implies

$$\dot{F}_i + L_X F_i = 0 \qquad i = 1,\dots,m \quad .$$

This means that the functions F_i are constants of the motion.

□

A completely integrable Hamiltonian system on the finite-dimensional manifold $M = \mathbb{R}^{2n}$ has always 2n constants of the motion with linearly independent corresponding symmetries. In the notation of section 3.6

these constants of the motion are $F_i = \tilde{p}_i$, $F_{i+n} = \tilde{q}_i - t \dfrac{\partial \tilde{H}}{\partial \tilde{p}_i}$ $(i=1,\dots,n)$.

So in this case we can expand any (non-canonical) symmetry as described in theorem 4.2.19.

4.3 BI-HAMILTONIAN SYSTEMS.

In theorem 3.5.5 we have seen that, for a general dynamical system, the existence of a certain adjoint symmetry ρ implies that the system is Hamiltonian. Of course this theorem is also valid if the dynamical system is already a Hamiltonian system. In that case theorem 3.5.5 provides us with a symplectic two-form and a Hamiltonian which may or may not be equal to the original ones. If the two symplectic forms are not equal up to a multiplicative constant, we can write the system as a Hamiltonian system in (at least) two essentially different ways. Systems of this kind are sometimes called *bi-Hamiltonian systems*. Several popular integrable Hamiltonian systems have this property, see for instance Magri [5] . In section 4.5 we shall meet dynamical systems which can be written as a Hamiltonian system in infinitely many ways (see remark 4.5.16). We now reformulate theorem 3.5.5 in case the original system is already a Hamiltonian system.

4.3.1 <u>Theorem</u>.

Let the non-semi-canonical symmetry $Z=\overleftarrow{\Omega}\tau$ satisfy the following conditions:

i) the operator $L_Z\Omega$ is invertible, or equivalently the two-form $d\tau$ is nondegenerate,
ii) the symmetry $\dot{Z}$ is canonical, so there exists a constant of the motion G such that $\dot{Z} = \overleftarrow{\Omega}\dot{\tau} = \overleftarrow{\Omega}dG$.

Then the vector field X is also the Hamiltonian vector field corresponding to the Hamiltonian $\tilde{H} = L_Z H - G$ and the symplectic two-form $\tilde{\omega} = d\tau$.

<u>Proof</u>:
Theorem 3.5.5 yields that X is the Hamiltonian vector field corresponding to the Hamiltonian $\tilde{H} = -\langle\tau,X\rangle - G$ and symplectic two-form $d\tau$. Since

$X = \overleftarrow{\Omega}\, dH$ and $\tau = \overleftarrow{\Omega}\, Z$ we can write

$$\langle\tau, X\rangle = \langle\tau, \overleftarrow{\Omega}\, dH\rangle$$

$$= -\langle dH, Z\rangle$$

$$= -L_Z H.$$

So $\tilde{H} = L_Z H - G$ and this concludes the proof.

□

If a symmetry Z, as described in this theorem exists, the vector field X can be written as

$$X = \overleftarrow{\Omega}\, dH$$

and as

$$X = (L_Z \Omega)^{-1}\, d\tilde{H}.$$

4.4 THE DUALITY MAP.

Suppose $Y_1 = \overleftarrow{\Omega}\,\sigma_1$ and $Y_2 = \overleftarrow{\Omega}\,\sigma_2$ are two symmetries. Then by theorem 2.5.10 the function

(4.4.1) $$G = \langle\sigma_1, Y_2\rangle$$

is a constant of the motion. We shall now compute the canonical symmetry $\overleftarrow{\Omega}\, dG$. First rewrite (4.4.1) as

$$G = i_{Y_2}\sigma_1.$$

The exterior derivative is given by

$$dG = d\, i_{Y_2}\sigma_1$$

$$= L_{Y_2}\sigma_1 - i_{Y_2} d\sigma_1\ .$$

From (4.2.12) (with $\tau = \sigma_1$, $Z = Y_1$) we obtain

$$i_{Y_2} d\sigma_1 = (L_{Y_1}\Omega)\, Y_2$$

$$= L_{Y_1}(\Omega Y_2) - \Omega L_{Y_1} Y_2$$

$$= L_{Y_1}\sigma_2 - \Omega\,[Y_1, Y_2]\ .$$

So

$$dG = L_{Y_2}\sigma_1 - L_{Y_1}\sigma_2 + \Omega[Y_1, Y_2]\ .$$

Application of $\overleftarrow{\Omega}$ results in

$$\overleftarrow{\Omega} dG = L_{Y_2} Y_1 - (L_{Y_2}\overleftarrow{\Omega})\sigma_1 - L_{Y_1} Y_2 + (L_{Y_1}\overleftarrow{\Omega})\sigma_2 + [Y_1, Y_2]$$

(4.4.2)

$$= \overleftarrow{\Omega}(L_{Y_2}\Omega)\, Y_1 - \overleftarrow{\Omega}(L_{Y_1}\Omega)\, Y_2 + [Y_2, Y_1]\quad .$$

By construction this is a canonical symmetry. In the right hand side we recognize the recursion operators for symmetries $\overleftarrow{\Omega} L_{Y_1}\Omega$ and $\overleftarrow{\Omega} L_{Y_2}\Omega$, acting on Y_2 respectively Y_1 and the Lie bracket $[Y_2, Y_1]$. First suppose Y_1 is a canonical symmetry. Then there exists a constant of the motion F_1, such that $Y_1 = \overleftarrow{\Omega}\sigma_1 = \overleftarrow{\Omega} dF_1$. Then by lemma 4.2.3 $L_{Y_1}\Omega = 0$. In this case (4.4.1) and (4.4.2) can be written as

(4.4.3) $\qquad G = \langle dF_1, Y_2\rangle = L_{Y_2} F_1$

and

(4.4.4) $\qquad \overleftarrow{\Omega} dG = \overleftarrow{\Omega}(L_{Y_2}\Omega)\overleftarrow{\Omega} dF_1 + [Y_2, \overleftarrow{\Omega} dF_1]\ .$

Formula (4.4.3) can be considered as a method for constructing a (new) constant of the motion G out of a known constant F_1 and a symmetry Y_2.

Then the canonical symmetry corresponding to G consists of two parts. The first part is $\overleftarrow{\Omega}(L_{Y_2}\Omega)\overleftarrow{\Omega}\, dF_1$, that is the recursion operator $\overleftarrow{\Omega} L_{Y_2}\Omega$ applied to the symmetry $\overleftarrow{\Omega} dF_1$. The second term is the Lie bracket of Y_2 and $\overleftarrow{\Omega} dF_1$. We can also try to use the single terms to construct a (new) constant of the motion. So starting with a constant of the motion F_1 and a (non-semi-canonical) symmetry Y_2 there are three possible ways to construct another constant of the motion.

i) We can compute $G = L_{Y_2} F_1$.

ii) We can apply the recursion operator $\overleftarrow{\Omega} L_{Y_2}\Omega$ to $\overleftarrow{\Omega} dF_1$ and obtain

$$Y_3 = \overleftarrow{\Omega}(L_{Y_2}\Omega)\overleftarrow{\Omega}\, dF_1 .$$

However, the symmetry Y_3 can be canonical or non-canonical. Only in the first case this method yields a constant of the motion.

iii) Compute the Lie bracket

$$Y_4 = [Y_2, \overleftarrow{\Omega} dF_1].$$

Also in this case Y_4 may be canonical or non-canonical.

It follows from (4.4.4) that $\Omega dG = Y_3 + Y_4$. So if method ii) works then also method iii) works and conversely. Method i) seems very attractive because it yields at once a constant of the motion. However, it is easier to describe properties of a constant of the motion which is constructed with one of the other methods. In section 4.5 we consider the problem of constructing an infinite series of constants of the motion using a recursion operator of the type $(L_{Y_2}\Omega)\overleftarrow{\Omega}$. In section 4.6 we investigate under which conditions this infinite series can also be obtained using the (repeated) Lie bracket with Y_2.

We now return to (4.4.2) and assume both symmetries Y_1 and Y_2 are canonical. So there exist constants of the motion F_1 and F_2 such that $Y_1 = \overleftarrow{\Omega} dF_1$ and $Y_2 = \overleftarrow{\Omega} dF_2$. Then (4.4.1) and (4.4.2) can be written as

$$G = \langle dF_1, \overleftarrow{\Omega} dF_2\rangle = \{F_1, F_2\} \quad , \tag{4.4.5}$$

and

(4.4.6) $\qquad \overleftarrow{\Omega} dG = [\overleftarrow{\Omega} dF_2, \overleftarrow{\Omega} dF_1]$.

This means the canonical symmetry corresponding to the Poisson bracket $G = \{F_1, F_2\}$ is equal to the Lie bracket of the canonical symmetries corresponding to F_2 and F_1. So we have proved the following well-known

4.4.7 Theorem.

The canonical symmetries form a subalgebra of the Lie algebra of symmetries $V(X;M)$. This subalgebra is isomorphic to the Lie algebra of constants of the motion, as described in lemma 3.3.5.

□

This theorem has the following consequence. Considerations which only use canonical (adjoint) symmetries can also be held on the level of constants of the motion. It is only useful to work with vector fields (one-forms) if non-canonical (adjoint) symmetries are involved.

4.5 INFINITE SERIES OF CONSTANTS OF THE MOTION I.

A lot of popular integrable Hamiltonian systems have an infinite series of constants of the motion. These constants of the motion F_k do not depend explicitly on t and are in involution

$$\dot{F}_i = 0, \quad \{F_i, F_j\} = 0 .$$

The most obvious way of constructing a new constant of the motion is by taking the Poisson bracket of two already known elements of the series. Since the series F_k is in involution this method will not work. Another possibility is to take the Poisson bracket with some other constant of the motion G. It turns out that several equations have a constant of the motion G, not in the series F_k, such that

$$\{F_k, G\} = c_k F_{k+\ell} \qquad c_k \in \mathbb{R} .$$

However, very often $\ell \leq 0$, which means that in this way we cannot go upwards in the series F_k. For instance, for the Korteweg-de Vries equation there exists a constant of the motion G such that $\ell = -1$ (see for instance Broer and Backerra [25]). In the case of the Sawada Kotera equation there is a constant of the motion G with $\ell = 0$. For both equations this method is not suitable for constructing an infinite series of constants of the motion. For the Benjamin-Ono equation there exists a constant of the motion G such that $\ell = 1$. Then an infinite series of constants of the motion is easily constructed and the following considerations are unnecessary. All these three equations will be used as examples in chapter 5.

In this section we shall consider the problem of constructing an infinite series of constants of the motion using a recursion operator for adjoint symmetries of the type $(L_Z\Omega)\overleftarrow{\Omega}$. Starting with a non-semi-canonical symmetry $Z = Z_1$ we first construct an infinite series of (in general non-semi-canonical) symmetries Z_k. With these symmetries correspond the recursion operators $(L_{Z_k}\Omega)\overleftarrow{\Omega}$. If the symmetry Z satisfies a certain condition (hypothesis 4.5.1) these recursion operators can be expressed in terms of powers of $(L_Z\Omega)\overleftarrow{\Omega}$ (theorem 4.5.5). An infinite series of adjoint symmetries is then constructed by $\rho_{k+1} = \frac{1}{c_k}(L_{Z_k}\Omega)\overleftarrow{\Omega}\, dH (c_k \in \mathbb{R})$. In theorem 4.5.10 we give conditions such that this series consists of semi-canonical adjoint symmetries. Several properties of the (possibly) corresponding constants of the motion are described in theorem 4.5.13.

Suppose there exists a non-semi-canonical symmetry $Z = \overleftarrow{\Omega}\tau$. Then $\Lambda = \overleftarrow{\Omega} L_Z\Omega$ is a recursion operator for symmetries and $\Gamma = (L_Z\Omega)\overleftarrow{\Omega}$ is a recursion operator for adjoint symmetries. If we "use the symmetry Z twice", we can construct the recursion operators for adjoint symmetries

$$((L_Z\Omega)\overleftarrow{\Omega})^2 = \Gamma^2$$

and

$$(L_Z^2\Omega)\overleftarrow{\Omega} \quad .$$

We saw in remark 4.2.18 that in general these recursion operatos will be different. However, there may exist a symmetry Z such that both operators are equal up to a multiplicative constant. This leads to

4.5.1 Hypothesis.

There exists a non-semi-canonical symmetry Z and a real number c with $c \neq (k-1)/k$, $\forall\, k \in \mathbb{N}$, such that

$$L_Z^2\Omega = c(L_Z\Omega)\overset{\leftarrow}{\Omega}\, L_Z\,\Omega \,. \tag{4.5.2}$$

□

Note that since Z is non-semi-canonical the corresponding adjoint symmetry τ is not closed and (theorem 4.2.11) the operator (tensor field) $L_Z\Omega$ does not vanish. For a semi-canonical symmetry the condition 4.5.2 would be trivial. The existence of a symmetry Z as described in this hypothesis, is essential for the theory of this section and section 4.6. For several "completely integrable" Hamiltonian systems (Korteweg-de Vries equation, sine-Gordon equation) this hypothesis is satisfied (and also $L_Z\Omega \neq a\Omega$ for some $a \in \mathbb{R}$). The following lemma will be used several times in the sequel.

4.5.3 Lemma.

Suppose ϕ is a closed parameterized two-form with corresponding tensor field $\Phi \in T^{o}_{2p}(M)$. Let $A \in X_p(M)$ and define the parameterized one-form α by $\alpha = \Phi A$. Then

$$d\alpha(B_1,B_2) = (L_A\phi)(B_1,B_2) = \langle (L_A\Phi)B_1,B_2\rangle \qquad \forall\, B_1,B_2 \in X(M).$$

Proof:
Apart from the dependence on the parameter t, this lemma corresponds to the second part of theorem 4.2.11 with Ω, τ and Z replaced by Φ, α and A. In the proof of that part of theorem 4.2.11 the special role of Ω (we consider the Hamiltonian system $\dot{u} = \overset{\leftarrow}{\Omega} dH$) is not used. So the lemma can be proved in the same way as the second part of theorem 4.2.11.

□

In local coordinates u^i the tensor field Φ (or the two-form ϕ) is represented by a matrix Φ_{ij}. The one-form α and the vector field A have components α_i and A^j such that $\alpha_i = \Phi_{ij}A^j$. Then if ϕ is a closed two-form, lemma 4.5.3 implies that

$$\alpha_{i,j} - \alpha_{j,i} = \Phi_{ij,k} A^k + \Phi_{ik} A^k_{,j} + \Phi_{kj} A^k_{,i} .$$

The operator $\Lambda = \Omega^{\leftarrow} L_Z \Omega$ is a recursion operator for symmetries. So we can construct an infinite series of symmetries Z_k and corresponding adjoint symmetries τ_k by

(4.5.4) $$Z_k = \Lambda^{k-1} Z \quad , \quad \tau_k = \Omega Z_k = \Gamma^{k-1} \tau \quad \text{for } k = 1,2,3,\ldots .$$

We now obtain the important

4.5.5 Theorem.

Suppose hypothesis 4.5.1 is satisfied. Then the symmetries Z_k, defined by (4.5.4), generate SA operators $L_{Z_k} \Omega$ which satisfy

(4.5.6) $$L_{Z_k} \Omega = c_k \Gamma^k \Omega \text{ for } k = 1,2,3,\ldots,$$

with $c_k = c(k-1) + 2-k$.

Proof:
The proof is done by induction. For k=1 (4.5.6) is an identity. Next assume (4.5.6) is correct for $k = j$. Since hypothesis 4.5.1 is satisfied, $c_j \neq 0$. We shall now compute $L_{Z_{j+1}} \Omega$. It follows from lemma 4.5.3 with $\phi = \omega$, $\Phi = \Omega$ and $A = Z_{j+1}$ that

(4.5.7) $$<(L_{Z_{j+1}} \Omega) B_1, B_2> = d\tau_{j+1} (B_1, B_2) \qquad \forall B_1, B_2 \in X(M).$$

By construction $\tau_{j+1} = \Gamma^j \tau$. We assumed (4.5.6) for $k = j$, hence

$$\tau_{j+1} = \frac{1}{c_j} (L_{Z_j} \Omega) \Omega^{\leftarrow} \tau = \frac{1}{c_j} (L_{Z_j} \Omega) Z .$$

The tensor field $L_{Z_j} \Omega$ corresponds to the closed two-form $L_{Z_j} \omega$. So we can again apply lemma 4.5.3 with $\phi = \frac{1}{c_j} L_{Z_j} \omega$, $\Phi = \frac{1}{c_j} L_{Z_j} \Omega$ and $A = Z$. Then

$\alpha = \tau_{j+1}$ and the lemma yields

(4.5.8) $$d\tau_{j+1}(B_1,B_2) = \frac{1}{c_j} <(L_Z L_{Z_{j+1}} \Omega)B_1,B_2> \qquad \forall\ B_1,B_2 \in X(M).$$

Comparing (4.5.7) and (4.5.8) results in

$$L_{Z_{j+1}} \Omega = \frac{1}{c_j} L_Z L_{Z_j} \Omega.$$

By again using the assumption this can be written as

$$L_{Z_{j+1}} \Omega = L_Z(\Gamma^j \Omega) .$$

The proof is now completed by writing out the right hand side with Leibniz'rule

(4.5.8.a) $$L_{Z_{j+1}} \Omega = \sum_{i=0}^{j-1} \Gamma^i (L_Z\Gamma)\ \Gamma^{j-i-1}\Omega + \Gamma^j L_Z\Omega .$$

We first compute $L_Z\Gamma$. Using (4.2.16) and hypothesis 4.5.1 we obtain

$$\begin{aligned} L_Z\Gamma &= L_Z(\ (L_Z\Omega)\Omega^{\leftarrow}) \\ &= (L_Z^2\Omega)\Omega^{\leftarrow} + (L_Z\Omega)(L_Z\Omega^{\leftarrow}) \\ &= (c-1)\ \Gamma^2. \end{aligned}$$

Substitution in (4.5.8.a) finally results in

$$L_{Z_{j+1}} \Omega = (j(c-1)+1)\ \Gamma^{j+1}\ \Omega .$$

So we have proved (4.5.6) for $k = j+1$.

□

The tensor field $L_{Z_k} \Omega$ corresponds to the closed two-form $L_{Z_k} \omega$, so it is a cyclic tensor field. *Theorem* 4.5.5 *now implies that, if hypothesis* 4.5.1 *is satisfied, the SA operators* $\Gamma^k\Omega$ *are also cyclic. If* $\Gamma^k\Omega \neq 0$ *this* *theorem also implies that* Z_k *must be non-semi-canonical.* We now show that, if (besides hypothesis 4.5.1) the non-semi-canonical symmetry Z satisfies an additional condition, it is possible to construct an infinite series of semi-canonical adjoint symmetries. First the following

4.5.9 Lemma.

Suppose Φ is a cyclic SA operator with corresponding (closed) two-form ϕ, which do not depend explicitly on t. Then the adjoint symmetry

$$\rho = \Phi\overleftarrow{\Omega}\,dH = \Phi X$$

is semi-canonical.

Proof:
Since Φ is an autonomous SA operator we have $L_X\Phi = 0$. Since Φ is cyclic we obtain from lemma 4.5.3 (with $A = X$ and $\alpha = \rho$) that $d\rho = 0$, so ρ is semi-canonical.

□

So every autonomous cyclic SA operator gives rise to a semi-canonical adjoint symmetry (and of course a semi-canonical symmetry). If the first cohomology group of M vanishes (so semi-canonical symmetries are canonical symmetries) this means that every autonomous cyclic SA operator gives rise to a constant of the motion.

4.5.10 Theorem.

Suppose there exists a symmetry Z as described in hypothesis 4.5.1. Suppose the time derivative $\dot{Z}$ is a semi-canonical symmetry. Then the adjoint symmetries

$$\rho_{k+1} = \frac{1}{c_k}\,(L_{Z_k}\Omega)\overleftarrow{\Omega}\,dH = \frac{1}{c_k}\,(L_{Z_k}\Omega)X \qquad k = 1,2,3,\ldots \tag{4.5.11}$$

are semi-canonical.

Proof:

The SA operator $L_{Z_k}\Omega$ is cyclic. In view of the preceding lemma we only have to show that $L_{Z_k}\Omega$ does not depend explicitly on t. Since Ω and $\overset{\leftarrow}{\Omega}$ do not depend explicitly on t, we have

$$(4.5.12) \qquad \dot{\Gamma} = \frac{\partial}{\partial t}(L_Z\Omega)\overset{\leftarrow}{\Omega} = (L_{\dot{Z}}\Omega)\overset{\leftarrow}{\Omega}.$$

The symmetry $\dot{Z}$ is canonical, so from lemma 4.2.3 we obtain $L_{\dot{Z}}\Omega = 0$. This implies $\dot{\Gamma} = 0$. Then it follows from theorem 4.5.5 that $\frac{\partial}{\partial t}(L_{Z_k}\Omega) = 0$. This completes the proof.

□

Of course " in practice" we do not have to compute Z_k before we can find ρ_{k+1}. It follows from theorem 4.5.5 that

$$\rho_{k+1} = \frac{1}{c_k}(L_{Z_k}\Omega)\overset{\leftarrow}{\Omega}dH = \Gamma^k\, dH, \qquad k = 0,1,2,\ldots .$$

The corresponding semi-canonical symmetries are

$$X_{k+1} = \overset{\leftarrow}{\Omega}\rho_{k+1} = \Lambda^k X \qquad\qquad k = 0,1,2,\ldots$$

where we have defined $\rho_1 = dH$ and $X_1 = X = \overset{\leftarrow}{\Omega}dH$. The following theorem describes some properties of the constants of the motion which may be associated to the adjoint symmetries ρ_k.

4.5.13 Theorem.

Suppose there exists a symmetry Z which satisfies hypothesis 4.5.1 and suppose $\dot{Z}$ is a canonical symmetry. If the first cohomology group of the manifold M vanishes, there exists an infinite series of constants of the motion $F_1 = H, F_2, F_3, \ldots$ for (4.1.1), defined by

$$(4.5.14) \qquad dF_{k+1} = \rho_{k+1} = \Gamma^k\, dH \qquad\qquad k = 0,1,2,\ldots \quad .$$

These constants of the motion are in involution and do not depend explicitly on t.

Proof:

Theorem 4.5.10 states that the adjoint symmetries ρ_k are semi-canonical. In remark 4.2.10 we explained that, when the first cohomology group of M vanishes, semi-canonical adjoint symmetries are also canonical adjoint symmetries. So (theorem 2.4.5) there exist constants of the motion F_k which satisfy (4.5.14). We now compute the Poisson brackets in these series

$$\{F_{j+1}, F_{k+1}\} = \langle dF_{j+1}, \overleftarrow{\Omega}\, dF_{k+1}\rangle \qquad j,k = 0,1,2,\ldots$$

$$= \langle \Gamma^j dH, \overleftarrow{\Omega}\Gamma^k dH\rangle$$

$$= \langle dH, (\overleftarrow{\Omega} L_Z\Omega)^{j+k}\overleftarrow{\Omega}\, dH\rangle \quad .$$

Since $\overleftarrow{\Omega}$ and $L_Z\Omega$ are both antisymmetric, we obtain

$$\{F_j, F_k\} = 0 \qquad j,k = 1,2,3,\ldots \; .$$

Finally $\dot{F}_k = \{H,F_k\} = \{F_1,F_k\} = 0$ implies that these constants of the motion do not depend explicitly on t.

□

4.5.15 Remark.

Note that we did not prove that the constants of the motion constructed in this way, are (analytically) independent. On a manifold of dimension 2n there can only exist 2n independent functions. So for a finite-dimensional manifold the infinite series F_k cannot be analytically independent. Also it may happen that $F_k = 0$ for $k > k_o$ or that $F_k = f_k H$, $f_k \in \mathbb{R}$. This last situation occurs if $L_Z\Omega = f\Omega$ for some $f \in \mathbb{R}$. However, in the examples in chapter 5 there exist symmetries Z for which it is easy to see that these trivial situations do not occur.

□

4.5.16 Remark.

It follows from (4.5.11) and (4.5.14) that

$$(L_{Z_k} \Omega)X = c_k \, d\, F_{k+1}. \tag{4.5.17}$$

The SA operator $L_{Z_k} \Omega$ corresponds to the closed two-form $L_{Z_k} \omega$. If this two-form is nondegenerate, or equivalently if $L_{Z_k} \Omega$ is invertible, we can consider (4.5.17) as a Hamiltonian system with symplectic two-form $d\tau_k = L_{Z_k} \omega$ and Hamiltonian $c_k F_{k+1}$. Then the vector field X can be written as

$$X = (L_{Z_k} \Omega)^{-1} \; d(c_k F_{k+1}). \tag{4.5.18}$$

The invertability of $L_{Z_k} \Omega$ corresponds (theorem 4.5.5) to the invertability of Γ. So if Γ (or $L_Z\Omega$) is invertible, we can write the vector field X as (4.5.18) for all $k \geq 0$. Systems of this type are called *multi-Hamiltonian*.

□

The methods we describe in this section and also in section 4.6 depend essentially on the use of a series of (in general) non-(semi-)canonical symmetries Z_k. A different method for constructing infinite series of constants of the motion, using a recursion operator for adjoint symmetries, was given by Magri [5] (Nijenhuis operators, compatibility conditions for "symplectic operators") and, using methods similar to those of Magri, by Fuchssteiner and Fokas [8] (heriditary symmetries). These authors do not use non-semi-canonical symmetries (Z_k). This means that several interesting results (various possible methods for constructing new constants of the motion as given in section 4.4; explicitly given symplectic forms for the multi-Hamiltonian description, see remark 4.5.16; all results of section 4.6) cannot be found.

4.6 INFINITE SERIES OF CONSTANTS OF THE MOTION II.

In the preceding section we constructed an infinite series of constants of the motion F_k, using a recursion operator for adjoint symmetries. The corresponding canonical symmetries were $X_k = \Omega^{\leftarrow} dF_k = \Lambda^{k-1} X_1$. We also constructed another series of symmetries $Z_k = \Lambda^{k-1} Z_1$. In the first part of this section we shall consider the various possible Lie brackets between the elements of both series X_k and Z_k. For a general dynamical system (i.e. not necessarily Hamiltonian) a problem of this type was considered in section 2.6. In that section we assumed that hypothesis 2.6.3 was satisfied. We now show that for a Hamiltonian system, this hypothesis follows partly from hypothesis 4.5.1. In this section we use the same notation as in the preceding section (so $(L_{Z_1}\Omega)\Omega^{\leftarrow} = \Gamma$, etc).

4.6.1 Theorem.

Suppose the non-semi-canonical symmetry Z_1 satisfies hypothesis 4.5.1. Then

(4.6.2) $$L_{Z_1}\Lambda = (c-1)\Lambda^2,$$

(4.6.3) $$L_{Z_2}\Lambda = (c-1)\Lambda^3 .$$

Proof:
Using hypothesis 4.5.1 it is easily seen that

$$\begin{aligned} L_{Z_1}\Lambda &= L_{Z_1}(\Omega^{\leftarrow} L_{Z_1}\Omega) \\ &= -\Omega^{\leftarrow}(L_{Z_1}\Omega)\Omega^{\leftarrow} L_{Z_1}\Omega + \Omega^{\leftarrow} L_{Z_1}^2\Omega \\ &= (c-1)\Lambda^2. \end{aligned}$$

Next we prove (4.6.3)

$$\begin{aligned} L_{Z_2}\Lambda &= L_{Z_2}(\Omega^{\leftarrow} L_{Z_1}\Omega) \\ &= -\Omega^{\leftarrow}(L_{Z_2}\Omega)\Omega^{\leftarrow}(L_{Z_1}\Omega) + \Omega^{\leftarrow} L_{Z_2} L_{Z_1}\Omega \end{aligned}$$

(4.6.4) $$= -\Omega^{\leftarrow}(L_{Z_2}\Omega)\Omega^{\leftarrow}(L_{Z_1}\Omega) + \Omega^{\leftarrow}(L_{Z_1} L_{Z_2}\Omega - L_{[Z_1,Z_2]}\Omega).$$

From (4.6.2) we obtain

$$(4.6.5) \qquad [Z_1,Z_2] = L_{Z_1}(\Lambda Z_1) = (L_{Z_1}\Lambda)\, Z_1 = (c-1)\,\Lambda^2 Z_1 = (c-1)\, Z_3.$$

Substitution in the last term of (4.6.4) and using theorem 4.5.5 with $k = 1,2$ and 3 results after some elementary computations in

$$L_{Z_2}\Lambda = (c-1)\Lambda^3.$$

□

It may be surprising that the factor c-1 appears in both Lie derivatives (4.6.2) and (4.6.3). However, if $L_{Z_2}\Lambda = f\,\Lambda^3$ for some $f \in \mathbb{R}$, then necessarily $f = c-1$. For, using the antisymmetry of the Lie bracket

$$0 = [Z_2,Z_2] = L_{Z_2}(\Lambda Z_1) = \Lambda[Z_2,Z_1] + (L_{Z_2}\Lambda)\, Z_1$$

$$= \Lambda[Z_2,Z_1] + f\, Z_4.$$

Then (4.6.5) implies $f = c-1$. Using the results of section 2.6 the following theorems are easily proved.

4.6.6 Theorem.

Suppose the non-semi-canocial symmetry Z_1 satisfies hypothesis 4.5.1. Suppose $[Z_1,X_1] = bX_2$ with $b \neq k(1-c)$ for $k = 0,1,2,\ldots$. Then the Lie brackets between the elements of both series X_k and Z_k are given by

$$(4.6.7) \qquad [X_k,X_\ell] = 0\ ,$$

$$(4.6.8) \qquad [Z_k,X_\ell] = ((\ell-1)(c-1) + b)\, X_{k+\ell},$$

$$(4.6.9) \qquad [Z_k,Z_\ell] = (\ell-k)(c-1)\, Z_{k+\ell} \quad .$$

Proof:

From theorem 4.6.1 and the assumption that $[Z_1,X_1] = b\, X_2$ we see that hypothesis 2.6.3 is satisfied (with $a = c-1$). The results now follow from the corollaries 2.6.9, 2.6.10 and 2.6.13.

□

Note that (4.6.7) corresponds with the fact that the corresponding constants of the motion F_k, as given in theorem (4.5.13), are in involution.

4.6.10 Remark.

From (4.6.8) we see that the series of semi-canonical symmetries X_k can also be constructed using the (repeated) Lie bracket with Z_1. In the preceding section this series was constructed using a recursion operator for (adjoint) symmetries $\Lambda(\Gamma)$. In fact we could also have started with the method of the repeated Lie bracket, that is define a series $\tilde{X}_k = L_{Z_1}^{k-1} X_1$.
Then we can try to prove that this series consists of canonical symmetries.
Suppose that
i) $\tilde{X}_2 = [Z_1, X_1] = \dot{Z}_1$ is semi-canonical,
ii) Z_1 satisfies hypothesis 4.5.1.
Then it can be shown that the series $\tilde{X}_k$ consists of semi-canonical symmetries.
Moreover, if for some $b \neq 0$ $\tilde{X}_2 = \Lambda X_1$, we can show that the series $\tilde{X}_k$ can also be obtained using the recursion operator for symmetries Λ.

□

In section 2.6 we also computed the Lie derivatives of the operator Λ.

4.6.11 Theorem.

Suppose the conditions of theorem 4.6.6 are satisfied.
Then

$$L_{X_k} \Lambda = 0 \quad , \; L_{Z_k} \Lambda = (c-1) \; \Lambda^{k+1} \qquad \text{for } k = 1,2,3,\ldots \, .$$

Proof:
Under the conditions of theorem 4.6.6 hypothesis 2.6.3 is satisfied (with $a = c-1$). The theorem now follows from theorem 2.6.8 and 2.6.12.

□

Note that this theorem implies that Λ is also a recursion operator for symmetries of the Hamiltonian systems $\dot{u} = X_k(u)$, $k = 1,2,3,\ldots$.

In section 4.4 we described three (possible) methods for constructing a (new) constant of the motion, starting with a known constant of the motion and a non-semi-canonical symmetry Z_1. In section 4.5 we constructed an infinite series of constants of the motion using a recursion operator for (adjoint) symmetries (method ii). In the first part of this section we showed that, under certain conditions, the same series can also be obtained using the (repeated) Lie bracket with Z_1 (method iii). We now make some remarks on method i), that is compute the Lie derivative $L_{Z_1} H$.

4.6.12 Theorem.

Suppose the symmetry Z_1 satisfies hypothesis 4.5.1. Let there exist a real number $b \neq k(1-c)$ for $k = 0,1,2,...,$ such that $[Z_1,X_1] = bX_2$. Then

$$d(L_{Z_1}^{k-1} H) = f_k \Omega X_k = f_k \rho_k , \quad k = 1,2,3,... \tag{4.6.13}$$

with $$f_k = \prod_{i=0}^{k-2} (i(c-1)+b+1) .$$

Proof:
The proof is done by induction. Since $f_1 = 1$ the theorem is trivial for $k = 1$. Next assume (4.6.13) holds for $k = j$. Then

$$\begin{aligned} d(L_{Z_1}^{j}H) &= L_{Z_1} d(L_{Z_1}^{j-1}H) \\ &= f_j L_{Z_1} (\Omega X_j) \\ &= f_j (L_{Z_1}\Omega) X_j + f_j \Omega[Z_1,X_j] . \end{aligned}$$

Using theorem 4.6.6 we can write this as

$$\begin{aligned} d(L_{Z_1}^{j}H) &= f_j\Omega \wedge X_j + f_j((j-1)(c-1)+b) \Omega X_{j+1} \\ &= f_j ((j-1)(c-1) + b+1)\Omega X_{j+1} \\ &= f_{j+1}\Omega X_{j+1} = f_{j+1}\rho_{j+1}. \end{aligned}$$

So we proved (4.6.13) for $k = j+1$.

□

4.6.14 Corollary.

Let the conditions of theorem 4.6.12 be satisfied. Then an infinite series of constants of the motion in involution (for the Hamiltonian system $\dot{u} = X(u) = \overset{\leftarrow}{\Omega}(u)\, dH(u)$) is given by

$$(4.6.15) \qquad \hat{F}_k = L_{Z_1}^{k-1} H \quad \text{for } k = 1,2,3,\dots \quad .$$

Proof:
We only have to show that the $\hat{F}_k$ are in involution. From theorem 4.6.12 we obtain $\hat{F}_k = f_k F_k$ where F_k is the series of constants of the motion in involution, as constructed in theorem 4.5.13.

□

4.6.16 Remark.

Note that in theorem 4.6.12 and corollary 4.6.14 we did not have to assume that the first cohomology group of M vanishes. The reason is that in the construction used in corollary 4.6.14 we don't have to reconstruct a constant of the motion from its corresponding canonical (adjoint) symmetry. This also means that "in practice" the method described in the preceding corollary is the most useful of the three considered methods.

□

Note that we did not prove that the constructed constants of the motion are analytically independent. Also it can happen that we obtain only trivial results. See further remark 4.5.15.

CHAPTER 5 EXAMPLES.

5.1 INTRODUCTION.

In this chapter we shall apply the theory, described in the preceding chapters, to several well-known differential equations. Most of these equations have been extensively studied in recent years. However, we obtain some results which, as far as we know, are new and give also different proofs of already known properties. As a first example we consider in section 5.2 the Burgers equation. This equation cannot be written in the form of an autonomous Hamiltonian system. A non-autonomous ("unphysical") Hamiltonian form of the Burgers equation is possible, but the methods described in the preceding chapters apply only to autonomous systems. In the following sections of this chapter we consider various (semi-)Hamiltonian systems. In sections 5.3 and 5.4 we study linear Hamiltonian systems. First, in section 5.3 we study the most general form of a finite-dimensional linear Hamiltonian systems. Then in section 5.4 we consider the one-dimensional wave equation. In section 5.5 we make some remarks on Hamiltonian systems which are related by a (invertible) transformation to a linear Hamiltonian system. As an example we consider a nonlinear system of partial differential equations, which can be transformed into the linear wave equation. The most extensive example of this chapter will be the Korteweg-de Vries equation, discussed in section 5.6. Finally in sections 5.7 and 5.8 we discuss the Sawada-Kotera equation and the Benjamin-Ono equation. The Sawada-Kotera equation is a "higher order Korteweg-de Vries equation", which is not an element of the Lax hierarchy of higher order Korteweg-de Vries equations. For this equation we only describe a "semi-Hamiltonian form". However, methods similar to those in chapter 4, can also be applied to this equation. The Benjamin-Ono equation is an integro-differential equation. Also other properties of this equation are different from the previous ones.

In chapters 2,3 and 4 we discussed some properties of dynamical systems and Hamiltonian systems on manifolds. The differential geometrical methods we used in those chapters, have only a sound foundation if the manifold M is finite-dimensional. So at first sight we can only use the results of the preceding chapters to investigate finite-dimensional systems. An interesting finite-dimensional example is the periodic Toda lattice [52] . However, most of the examples we wish to consider in this chapter are partial differential equations, i.e. dynamical (Hamiltonian) systems on manifolds with infinite dimension. Fortunately these equations can be considered in a topological vector space. In section 1.2 we gave definitions of various differential geometrical objects on a (possibly infinite-dimensional) topological vector space $\mathcal{W}$. Suppose X is a vector field on $\mathcal{W}$ so it is a (possibly nonlinear) mapping $X : \mathcal{W} \to \mathcal{W}$. Then we can consider in $\mathcal{W}$ the dynamical system

(5.1.1) $\qquad \dot{u} = X(u)$.

The following theorem describes (adjoint) symmetries and operators between symmetries for the system (5.1.1).

5.1.2 $\qquad$ Theorem.

Consider the parameterized vector field $Y : \mathcal{W} \times \mathbb{R} \to \mathcal{W}$, the parameterized one-form $Y : \mathcal{W} \times \mathbb{R} \to \mathcal{W}^*$ and parameterized tensor fields Φ, Λ, Γ,Ψ of the same type as in (1.2.3). Then:

i) Y is a symmetry of (5.1.1) iff

(5.1.3) $\qquad \dot{Y}(u,t) + Y'(u,t)X(u) - X'(u)Y(u) = 0,$

ii) σ is an adjoint symmetry of (5.1.1) iff

(5.1.4) $\qquad \dot{\sigma}(u,t) + \sigma'(u,t)X(u) + X'^*(u)\sigma(u,t) = 0,$

iii) Φ is an SA operator for (5.1.1) iff

(5.1.5) $\qquad \dot{\Phi}(u,t) + (\Phi'(u,t)X(u)) + \Phi(u)X'(u) + X'^*(u)\Phi(u) = 0,$

iv) Λ is a recursion operator for symmetries of (5.1.1) iff

(5.1.6) $\qquad \dot{\Lambda}(u,t) + (\Lambda'(u,t)X(u)) + \Lambda(u,t)X'(u) - X'(u)\Lambda(u,t) = 0,$

v) Γ is a recursion operator for adjoint symmetries of (5.1.1) iff

$$(5.1.7) \qquad \dot{\Gamma}(u,t) + (\Gamma'(u,t)\mathcal{X}(u)) - \Gamma(u,t)\mathcal{X}'^*(u) + \mathcal{X}'^*(u)\Gamma(u,t) = 0,$$

vi) Ψ is an AS operator for (5.1.1) iff

$$(5.1.8) \qquad \dot{\Psi}(u,t) + (\Psi'(u,t)\mathcal{X}(u)) - \Psi(u,t)\mathcal{X}'^*(u) - \mathcal{X}'(u)\Psi(u,t) = 0 .$$

All these expressions are assumed to vanish for all $u \in \mathcal{W}$ and $t \in \mathbb{R}$.

Proof:
Using (1.2.9) it is easily seen that all these expressions are equivalent to the corresponding expressions in chapter 2.

□

Suppose u(t) is a solution of (5.1.1). The equation, obtained by linearizing (5.1.1) around u(t) is

$$(5.1.9) \qquad \dot{v}(t) = \mathcal{X}'(u(t))v(t) \qquad\qquad v(t) \in \mathcal{W}.$$

This equation can be considered as an equation for the "variation" $v(t) = \delta u(t)$ of u(t). Similar equations were considered in (2.2.2) (using a local trivialization of the manifold) and in (2.8.11) (differential equation on the tangent bundle). Suppose $\mathcal{Y}(u,t)$ is a symmetry of (5.1.1). Then it is easily seen that $v(t) = \mathcal{Y}(u(t),t)$ is a solution of (5.1.9). So symmetries can be interpreted as solutions of the linearized equation (5.1.9), which can be expressed in u and t. In fact we can even use this property to find symmetries. The adjoint equation of (5.1.9) is given by

$$(5.1.10) \qquad w(t) = - \mathcal{X}'^*(u(t))w(t) \qquad\qquad w(t) \in \mathcal{W}^*.$$

Let $\sigma(u,t)$ be an adjoint symmetry of (5.1.1). Then it is easily verified that $w(t) = \sigma(u(t),t)$ satisfies (5.1.10). So adjoint symmetries σ can be considered as solutions of the "adjoint linearized equation" (5.1.10), which can be written in terms of u and t.

5.1.11 Remark.

Sometimes we shall meet (adjoint) symmetries which do not depend explicitly on t. For symmetries and adjoint symmetries of that type (autonomous symmetries) the first terms in (5.1.3) and (5.1.4) vanish. Almost all recursion operators for (adjoint) symmetries, SA- and AS operators which we shall use in the sequel, do not depend explicitly on t (autonomous operators). So for these operators the first terms in (5.1.5), (5.1.6), (5.1.7) and (5.1.8) also vanish.

□

5.1.12 Remark.

In sections 5.3 and 5.4 we shall meet symmetries of the form $Y(u,t) = \hat{Y}u$ and adjoint symmetries of the form $\sigma(u,t) = \hat{\sigma}u$ where $\hat{Y} : W \to W$ and $\hat{\sigma} : W \to W^*$ are linear operators. In that case the derivatives are easily found: $\dot{Y}(u,t) = 0$, $Y'(u,t) = \hat{Y}$ and $\dot{\sigma}(u,t) = 0$, $\sigma'(u,t) = \hat{\sigma}$. In sections 5.3 and 5.4 we also use recursion operators for (adjoint) symmetries, SA- and AS operators which do not depend explicitly on u and t (i.e. constant operator fields). An SA operator of this type is $\Phi(u,t) = \Xi$ where $\Xi : W \to W^*$ is a linear operator. For operators of this type the derivatives with respect to u and t vanish. This means that in the conditions (5.1.5), (5.1.6), (5.1.7) and (5.1.8) the first two terms are zero.

□

In section 3.2 we considered a closed two-form and the corresponding tensor field(s). In definition 3.2.4 we introduced cyclic tensor fields and canonical tensor fields. The corresponding conditions were written in terms of Lie derivatives. In the case that $M = W$, a topological vector space, these conditions can be simplified somewhat.

5.1.13 Theorem.

An antisymmetric tensor field $\Phi \in T^o_2(W)$ (=antisymmetric operator field $\Phi(u) : W \to W^*$) is cyclic iff

(5.1.14) $\langle(\Phi'(u)A)B,C\rangle + \langle(\Phi'(u)B)C,A\rangle + \langle(\Phi'(u)C)A,B\rangle = 0$

for all A, B, C, $u \in W$.

Proof:

By theorem 3.2.12 an antisymmetric tensor field is cyclic iff the corresponding two-form is closed. Then this theorem follows at once from definition

1.2.10 iii) (and theorem 1.2.11).

□

An alternative proof of this theorem can be given by substitution of $L_A\Phi$, as given in (1.2.9), in definition 3.2.4 (i.e. in (3.2.5)).

5.1.15 Theorem.

An antisymmetric tensor field $\Psi \in T_0^2(W)$ (=antisymmetric operator field $\Psi(u) : W^* \to W$) is canonical iff

(5.1.16) $$\langle\alpha,(\Psi'(u)(\Psi(u)\beta))\gamma\rangle + \langle\beta,(\Psi'(u)(\Psi(u)\gamma))\alpha\rangle$$

$$+ \langle\gamma,(\Psi'(u)(\Psi(u)\alpha))\beta\rangle = 0$$

for all α, β, $\gamma \in W^*$ and $u \in W$.

Proof:
A canonical tensor field Ψ has to satisfy (3.2.8). Substitution of $L_{\Psi\alpha}\beta$, as given in (1.2.9) and some elementary operations yield that (3.2.8) is equivalent to (5.1.16).

□

5.1.17 Remark.
It is easily seen from (5.1.14) and (5.1.16) that antisymmetric operators $\Phi : W \to W^*$ and $\Psi : W^* \to W$, considered as constant operator fields (i.e. $\Phi(u)$ and $\Psi(u)$ do not depend on u) always satisfy (5.1.14) respectively (5.1.16). Hence every antisymmetric operator $\Phi : W \to W^*$ is cyclic (so the corresponding two-form is closed) and every antisymmetric operator $\Psi : W^* \to W$ is canonical.

□

The fact that W is a topological vector space has also consequences for the relation between semi-canonical and canonical symmetries. In section 1.2 we have seen that a closed one-form α on W is also exact. The corresponding function F on W such that $\alpha = dF$ was given in (1.2.12). Of course these results also hold if α (and hence F) depend on a parameter (t). In terms of (adjoint) symmetries this means that *semi-canonical (adjoint) symmetries are canonical (adjoint) symmetries* and that *non-canonical (adjoint) symmetries* are *non-semi canonical* *(adjoint) symmetries*. So we can omit the prefix "semi" in these notions.

Finally we make some remarks on the notation and terminology in this chapter. In the preceding chapters we used the notation and terminology of modern differential geometry. We shall also do this in this chapter, with a few exceptions. If W is infinite-dimensional, the exterior derivative of a function (functional) $F : W \to \mathbb{R}$ is the one-form $dF(u) = F'(u)$, as introduced in definition 1.2.10 i). In cases where the duality map between W and W^* is given by the L_2 innerproduct (all the examples except section 5.3), the derivative of F is frequently denoted as $\frac{\delta F}{\delta u}$ (or $\frac{\delta F(u)}{\delta u}$) instead of $F'(u)$. This expression is called the *variational derivative* of F. In all sections except section 5.3 we shall mostly use this notation, so $dF(u)$ will be replaced by $\frac{\delta F}{\delta u}$.

The derivative of various parameterized objects with respect to the parameter (t) has always been indicated by a dot, for instance $\dot{Y}(u,t) = \frac{\partial}{\partial t} Y(u,t)$ (derivative of a vector field to the parameter). However, when dealing with partial differential equations, derivatives with respect to t (x,y,...) are very often indicated by a subscript t (x,y,...). Apart from section 5.3 we shall also use this notation. So the derivative of a parameterized vector field with respect to the parameter will be written as

$$Y_t(u,t) = \frac{\partial}{\partial t} Y(u,t),$$

a dynamical system (Korteweg-de Vries equation) will be written as

$$u_t = X(u) = 6uu_x - u_{xxx}.$$

5.2 THE BURGERS EQUATION.

This equation was used by Burgers [48,49] in 1939 in a model for turbulent fluid motion. It is the simplest possible equation which describes both non-linear and diffusion effects . The Burgers equation arises in many places in physics, particularly in problems where shock waves are involved (see for instance Whitham [32]). We shall study it in the form

$$(5.2.1) \qquad u_t = X(u) = 2uu_x + u_{xx}, \qquad x \in \mathbb{R} \quad .$$

Various other forms of the Burgers equation can be reduced to (5.2.1), using transformations of the dependent and independent variables. A transformation which relates (5.2.1) to the diffusion equation was found in 1950 by Hopf [50] and in 1951 by Cole [51] . This so-called *Hopf-Cole transformation* is given by

$$(5.2.2) \qquad v = f(u) = e^{\partial^{-1} u} \quad ,$$

$$(5.2.3) \qquad u = f^{\leftarrow}(v) = \frac{v_x}{v} \quad .$$

The corresponding evolution equation for v is given by

$$(5.2.4) \qquad v_t = f'(u)X(u) = \tilde{X}(v) = v_{xx} \quad , x \in \mathbb{R} \qquad (u = f^{\leftarrow}(v)).$$

Various methods are available for solving this linear equation. Suppose we take an initial value $u_o \in S_1$ (see definition 1.3.2) at $t = t_o$. Then, using the relation with (5.2.4), it can be shown that the corresponding solution $u(.,t) \in S_1$ for $t \geq t_o$. Therefore we shall study (5.2.1) in the space S_1. Define the function space $\hat{U}_1 = U_1 \oplus \mathbb{R} = \{u \in C^\infty(\mathbb{R}) \mid u(x) = v(x) + a,\ v \in U_1,\ a \in \mathbb{R}\}$. A duality map between S_1 and $\hat{U}_1$ is given by

$$\langle\alpha, A\rangle = \int_{-\infty}^{\infty} \alpha(x)\, A(x)\,dx \qquad \forall\, \alpha \in \hat{U}_1,\ A \in S_1 \ .$$

Then, similar to theorem 1.3.14, we introduce topologies on S_1 and $\hat{U}_1$ such that $S_1^* = \hat{U}_1$ and $\hat{U}_1^* = S_1$.

We shall now study symmetries and adjoint symmetries for (5.2.1).

Since we consider (5.2.1) on a topological vector space, a symmetry Y is (can be considered as) a mapping $Y : S_1 \times \mathbb{R} \to S_1$ which satisfies (5.1.3). The derivative mapping of X in the point u is given by

(5.2.5) $$X'(u) = 2u\partial + 2u_x + \partial^2 : S_1 \to S_1.$$

Substitution in (5.1.3) yields

$$Y_t(u,t) + Y'(u,t)\,(2uu_x + u_{xx}) - (2u\partial + 2u_x + \partial^2)\,Y(u,t) = 0,$$

$$\forall\, u \in S_1, \forall\, t \in \mathbb{R}.$$

Two simple solutions of this equation are

(5.2.6) $$Y(u,t) = X_o(u) = u_x \text{ and } Y(u,t) = Z_o(u,t) = u + xu_x + 2t(2uu_x+u_{xx}).$$

Note that indeed $X_o : S_1 \to S_1$ and $Z_o : S_1 \times \mathbb{R} \to S_1$. Both symmetries have a simple geometrical interpretation. The equation (5.2.1) is invariant for translations along the x-axis. If u(x,t) is a solution of (5.2.1), then $u(x + \varepsilon,t)$ is also a solution of (5.2.1) for all $\varepsilon \in \mathbb{R}$. The difference between these two solutions is given by

$$u(x+\varepsilon,t) - u(x,t) = \varepsilon\, u_x(x,t) + O(\varepsilon^2) \qquad \text{for } \varepsilon \to 0.$$

This implies that $X_o(u) = u_x$ is a solution of the linearized equation and hence a symmetry (see (5.1.9)). The symmetry Z_o is related to the scaling properties of (5.2.1). It is easily seen that, if u(x,t) satisfies (5.2.1), the function $au(ax,a^2t)$ also satisfies (5.2.1) for all $a \in \mathbb{R}$. By setting $a = 1 + \varepsilon$ and taking the limit $\varepsilon \to 0$ we find that the difference between the two solutions is given by

$$(1+\varepsilon)u((1+\varepsilon)x,(1+\varepsilon)^2t) - u(x,t) = \varepsilon(u(x,t) + xu_x(x,t) + 2tu_t(x,t)) + O(\varepsilon^2).$$

So $Z_o(u) = u + xu_x + 2t(2uu_x + u_{xx})$ is a solution of the linearized equation of (5.2.1) and hence a symmetry (see (5.1.9)).

Recursion operators for symmetries of (5.2.1) can easily be found by using the relation with the linear equation (5.2.4). Suppose we consider the equation (5.2.4) in some linear space $\mathcal{W}$. An autonomous recursion operator for symmetries of (5.2.4) is a linear operator $\tilde{\Lambda}(v) : \mathcal{W} \to \mathcal{W}$, defined for all $v \in \mathcal{W}$, such that (see (5.1.6) and remark 5.1.11)

$$(5.2.7) \qquad (\tilde{\Lambda}'(v)\ \tilde{X}(v)) + \tilde{\Lambda}(v)\tilde{X}'(v) - \tilde{X}'(v)\tilde{\Lambda}(v) = 0 \qquad \forall\ v \in \mathcal{W}\ ,$$

where $\tilde{X}(v) = v_{xx}$ and $\tilde{X}'(v) = \partial^2$. It is easily verified that $\tilde{\Lambda}(v) = \partial$ satisfies this condition.

5.2.8 Remark.

Symmetries of (5.2.4) satisfy the linearized equation of (5.2.4). Since (5.2.4) is a linear equation, symmetries are solutions of (5.2.4). Suppose $w(x,t)$ satisfies (5.2.4), then also $w_x(x,t)$ satisfies (5.2.4). This mapping corresponds to the recursion operator $\tilde{\Lambda} = \partial$.

□

Using the transformations (5.2.2) and (5.2.3) we can formally transform $\tilde{\Lambda}$ to a recursion operator Λ for symmetries of (5.2.1). By theorem 2.7.6 the operator Λ is given by

$$\Lambda(u) = f^{\leftarrow\prime}(v)\tilde{\Lambda}(v)f'(u) \qquad (v=f(u))$$

$$(5.2.9) \qquad = \left(\frac{\partial}{v} - \frac{v_x}{v^2}\right)\partial e^{\partial^{-1}u}\partial^{-1}$$

$$= \partial + \partial u \partial^{-1}.$$

5.2.10 Theorem.

The operator $\Lambda(u) = \partial + \partial u \partial^{-1}$ is a recursion operator for symmetries of (5.2.1).

Proof:

It is easily seen that $\Lambda(u) : S_1 \to S_1$. We have to show that Λ satisfies (5.1.6). Since Λ does not depend on t, this becomes

$$(5.2.11)\qquad (\Lambda'(u)X(u)) + \Lambda(u)X'(u) - X'(u)\Lambda(u) = 0 \qquad \forall\, u \in S_1.$$

Recall that the derivative of $\Lambda(u)$ in $u \in S_1$ is a bilinear operator $\Lambda'(u) : S_1 \times S_1 \to S_1$. Inserting one fixed function $A \in S_1$ this derivative reduces to the linear operator

$$(\Lambda'(u)\, A) = \partial A\, \partial^{-1} : S_1 \to S_1.$$

So the first term of (5.2.11) is the linear operator

$$(\Lambda'(u)X(u)) = \partial(2uu_x + u_{xx})\partial^{-1}.$$

Using (5.2.5) the other terms of (5.2.11) can be found. Then a tedious computation shows that Λ satisfies (5.2.11).

□

This recursion operator for symmetries was already given by Olver [13] . Starting with the symmetries X_o and Z_o given in (5.2.6), we can construct two infinite series of symmetries

$$(5.2.12)\qquad X_k = \Lambda^k X_o,\ Z_k = \Lambda^k Z_o \qquad k = 1,2,3,\ldots\ .$$

As far was we know the series of symmetries Z_k has not been reported before. The first few elements of these series are given by

$$(5.2.13)\qquad \begin{aligned} X_o &= u_x, \\ X_1 &= 2uu_x + u_{xx}, \\ X_2 &= 3u^2u_x + 3uu_{xx} + 3u_x^2 + u_{xxx}, \\ Z_0 &= u + xu_x + 2t(2uu_x + u_{xx}), \\ Z_1 &= u^2 + 2u_x + x(2uu_x + u_{xx}) + 2t(3u^2u_x + 3uu_{xx} + 3u_x^2 + u_{xxx}). \end{aligned}$$

Note that $X_1 = X = u_t$; this symmetry is related to the invariance of (5.2.1) for translations along the t-axis. Some properties of the two series of symmetries are given in the following

5.2.14 <u>Theorem.</u>

The symmetries X_k and Y_k can be written as

$$X_k(u) = \partial r_k(u,u_x,\ldots),$$

$$Z_k(u) = s_k(u,u_x,\ldots) + xX_k(u) + 2tX_{k+1}(u), \qquad k = 0,1,2,\ldots,$$

where $r_k(u,u_x,\ldots)$ and $s_k(u,u_x,\ldots)$ are polynomials in u and its first k derivatives.

<u>Proof:</u>
The recursion operator Λ can be written as $\Lambda = \partial(\partial+u)\partial^{-1}$. Hence $\Lambda^k = \partial(\partial+u)^k\partial^{-1}$. This implies

$$X_k = \partial(\partial+u)^k\partial^{-1}u_x = \partial(\partial+u)^k u.$$

So $r_k(u,u_x,\ldots) = (\partial+u)^k u$. In the same way we obtain

$$Z_k = \partial(\partial+u)^k\partial^{-1}(u+xu_x+2tX_1)$$

$$= \partial(\partial+u)^k xu + 2tX_{k+1}$$

$$= (x\partial(\partial+u)^k + (\partial+u)^k + k\partial(\partial+u)^{k-1})u + 2tX_{k+1}$$

$$= xX_k + s_k(u,u_x,\ldots) + 2tX_{k+1}.$$

with $s_k(u,u_x,\ldots) = (\partial+u)^k u + k\partial(\partial+u)^{k-1}u$.

□

5.2.15 <u>Remark.</u>

The symmetries X_k are mappings $X_k : S_1 \to S_1$ (vector fields on S_1). So we can study the evolution equation

$$u_t = X_k(u) = \Lambda^{k-1}X(u) \qquad k = 1,2,3,\ldots \quad (5.2.16)$$

By formally applying the (derivative of) the transformation (5.2.2) we obtain

$$
\begin{aligned}
v_t &= f'(u)u_t \qquad (u = \overset{\leftarrow}{f}(v)) \\
&= f'(u)\Lambda^{k-1}(u)X(u) \\
&= (f'(u)\Lambda(u)\overset{\leftarrow}{f}{}'(v))^{k-1} f'(u)X(u) \\
&= \tilde{\Lambda}^{k-1}(v)\tilde{X}(v) \\
&= \partial^{k+1}v.
\end{aligned}
$$

So, using the Hopf-Cole transformation, we can transform (5.2.16) into the linear equation

$$(5.2.17) \qquad v_t = \partial^{k+1}v.$$

Note that (with appropriate boundary conditions) (5.2.17) is a Hamiltonian system if k is even ($k = 2\ell$; $\overset{\leftarrow}{\Omega} = \partial$, $H(v) = (-1)^{\ell}\, \frac{1}{2} \int_{-\infty}^{\infty} (\partial^{\ell}v)^2 dx$). If k is odd, say $k = 2\ell+1$, then (5.2.17) is an equation of "diffusion type" if ℓ is even and an equation of "anti-diffusion type" if ℓ is odd. Similar properties hold for the corresponding nonlinear equations (5.2.16). Some properties of the Hamiltonian system

$$u_t = X_2(u) = 3u^2u_x + 3uu_{xx} + 3u_x^2 + u_{xxx}$$

are described by Broer and ten Eikelder [47] .

□

In (5.2.12) we gave two infinite series X_k and Z_k of symmetries for the Burgers equation. We now consider the various Lie brackets between the elements of both series. One possible way for computing these Lie brackets is to transform to the linear equation (5.2.4) and compute the Lie brackets of the corresponding symmetries of (5.2.4). This method is possible because for the Burgers equation a linearizing transformation (Hopf-Cole) is known. However, a straightforward computation using the results of section 2.6 is also possible. We shall follow this second method. Recall that the Lie derivative of the recursion operator Λ is given by (see (1.2.9))

$$L_A\Lambda(u) = (\Lambda'(u)A(u)) + \Lambda(u)A'(u) - A'(u)\Lambda(u) \ .$$

A long computation shows that

(5.2.18) $$L_{Z_1}\Lambda = \Lambda^2 \ , \ L_{Z_2}\Lambda = \Lambda^3$$

and that

(5.2.19) $$[Z_1, X_1] = 2X_2.$$

5.2.20 Theorem.

The Lie brackets between the elements of the series of symmetries X_k and Z_k are given by

$$[X_k, X_\ell] = 0 \ ,$$

$$[Z_k, X_\ell] = (\ell+1)\ X_{k+\ell},$$

$$[Z_k, Z_\ell] = (\ell-k)\ Z_{k+\ell}, \qquad k,\ell = 0,1,2,\dots \ .$$

The Lie derivatives of the recursion operator for symmetries Λ are

$$L_{X_k}\Lambda = 0 \ ,$$

$$L_{Z_k}\Lambda = \Lambda^{k+1} \qquad k = 0,1,2,\dots \ .$$

Proof:
It follows from (5.2.18) and (5.2.19) that hypothesis 2.6.3 is satisfied with a = 1 and b = 2. Then, for $k,\ell \geq 1$, the theorem is a straightforward consequence of the results of section 2.6 (summarized in (2.6.14)).For $k = 0$ and / or $\ell = 0$ a separate proof has to be given. The necessary conditions, given in remark 2.6.15, are easily verified.

□

We conclude this section on the Burgers equation with some remarks on adjoint symmetries for (5.2.1). The function (functional)

$$F(u) = \int_{-\infty}^{\infty} u(x)dx$$

is a constant of the motion of (5.2.1). This function is differentiable, $\frac{\delta F}{\delta u} = 1 \in \hat{U}_1$. So $\sigma(u) = \frac{\delta F}{\delta u} = 1$ is an adjoint symmetry of (5.2.1). A recursion operator for adjoint symmetries is given by

$$\Gamma(u) = \Lambda^*(u) = (\partial + \partial u \partial^{-1})^* = -\partial + \partial^{-1}u\partial : \hat{U}_1 \to \hat{U}_1 .$$

Since $\Gamma(u)\sigma(u) = 0$ we cannot construct a series of adjoint symmetries by using the recursion operator Γ. We did not find adjoint symmetries which were essentially different from σ.

5.3 A FINITE DIMENSIONAL LINEAR HAMILTONIAN SYSTEM.

Suppose $\mathcal{W}$ is a finite-dimensional (real) linear space with dimension 2n; so $\mathcal{W}$ is isomorphic to $\mathbb{R}^{2n}$. The dual space of $\mathcal{W}$ is denoted by $\mathcal{W}^*$. In this section we shall consider a linear Hamiltonian system on the space $\mathcal{W}$. Some general remarks on dynamical systems and Hamiltonian systems on a linear space have been made in section 5.1. Let ω be a symplectic form on $\mathcal{W}$ such that the corresponding operator $\Omega(u) : \mathcal{W} \to \mathcal{W}^*$ does not depend on u. So

$$\omega(A,B) = \langle \Omega A, B \rangle \qquad \forall\, A,B \in \mathcal{W},$$

where $\Omega : \mathcal{W} \to \mathcal{W}^*$ is a linear antisymmetric operator. Since ω is nondegenerate, the operator Ω is invertible. The inverse operator $\overleftarrow{\Omega} : \mathcal{W}^* \to \mathcal{W}$ is also a linear antisymmetric operator. Suppose $H : \mathcal{W} \to \mathbb{R}$ is a homogeneous quadratic function. Then there exists a unique symmetric operator $\hat{H} : \mathcal{W} \to \mathcal{W}^*$ such that

$$H(u) = \tfrac{1}{2}\langle \hat{H}\, u, u \rangle .$$

The corresponding one-form is $dH(u) = \hat{H}\, u$. Then the Hamiltonian system on the symplectic space $(\mathcal{W},\omega)$ with Hamiltonian H is given by

$$\dot{u} = \overleftarrow{\Omega}\, \hat{H}\, u. \tag{5.3.1}$$

With $\hat{X} = \overleftarrow{\Omega}\hat{H} : \mathcal{W} \to \mathcal{W}$, this system can also be written as

$$\dot{u} = X(u) = \hat{X}u. \tag{5.3.2}$$

In theorem 3.4.1 we described a variational principle for a Hamiltonian system. At first sight this theorem provides us with a variational principle on a neighbourhood U_o of some point $u_o \in M = \mathcal{W}$. However, in this case the manifold M is a linear space. This means the second cohomology group of M vanishes, so every closed two-form is exact. Hence the one-form α, such that $\omega = -\,d\alpha$, exists on the whole space $M = \mathcal{W}$. It is easily seen that $\alpha(u) = -\tfrac{1}{2}\Omega\, u$. Then (similar to theorem 3.4.1) a solution $\tilde{u}(t)$ of (5.3.1) is a stationary point of

(5.3.3) $$\int_{t_1}^{t_2} (\tfrac{1}{2}\langle\Omega\dot{u},u\rangle - \tfrac{1}{2}\langle\hat{H}u,u\rangle)\, dt$$

over the set of all curves $u(t)$ in W with $u(t_1) = \tilde{u}(t_1)$ and $u(t_2) = \tilde{u}(t_2)$.

Note that for every initial value $u(t_o) = u_o \in W$ the differential equation (5.3.1) has a unique solution $u(t) \in W$ which exists for all $t \in \mathbb{R}$

$$u(t) = e^{(t-t_o)\overleftarrow{\Omega}\hat{H}} u_o = e^{(t-t_o)\hat{X}} u_o.$$

In the remaining part of this section we shall first consider constants of the motion, (adjoint) symmetries and operators between those symmetries for the Hamiltonian system (5.3.1). The existence of these objects turns out to be related with the existence of operators Ξ which satisfy the condition (5.3.5). Then we shall make some remarks on the space of operators satisfying (5.3.5). Finally we show how the theory described in section 4.5 , can be applied in this example.

Suppose $F : W \to \mathbb{R}$ is a homogeneous quadratic function. Then there exists a symmetric operator $\Xi: W \to W^*$ such that

(5.3.4) $$F(u) = \tfrac{1}{2} \langle\Xi u,u\rangle \quad .$$

The function F is a constant of the motion if $L_X F = \langle dF,X\rangle = 0$ on W. This means

$$\langle\Xi\overleftarrow{\Omega}\hat{H}u,u\rangle = 0 \qquad \forall\, u \in W .$$

This condition is satisfied iff $\Xi\overleftarrow{\Omega}\hat{H}$ is an antisymmetric operator. Since Ξ and $\hat{H}$ are symmetric and $\overleftarrow{\Omega}$ is antisymmetric, this is equivalent to

(5.3.5) $$\Xi\overleftarrow{\Omega}\hat{H} - \hat{H}\overleftarrow{\Omega}\Xi = 0.$$

This condition can also be written in the following two equivalent ways

(5.3.6) $$[\overleftarrow{\Omega}\Xi,\overleftarrow{\Omega}\hat{H}\underline{]} = [\overleftarrow{\Omega}\Xi,\hat{X}\underline{]} = 0$$

and

$$(5.3.7) \qquad [\Xi\overleftarrow{\Omega}, \hat{H}\overleftarrow{\Omega}]_{-} = 0,$$

where $[.,.]_{-}$ is the *commutator of two linear operators*. The linear space of operators $\Xi : \mathcal{W} \to \mathcal{W}^*$ which satisfy (5.3.5) will be denoted by $\mathcal{E}$. The canonical adjoint symmetry and the canonical symmetry, corresponding to the constant of the motion (5.3.4) are given by $\rho(u) = dF(u) = \Xi u$ and $Y(u) = \overleftarrow{\Omega} dF(u) = \overleftarrow{\Omega}\Xi u$. The Poisson bracket of two constants of the motion $F_i(u) = \frac{1}{2} <\Xi_i u,u>$ $(i = 1,2)$ is easily found to be

$$(5.3.8) \qquad \begin{aligned} \{F_1,F_2\}(u) &= <dF_1(u),\overleftarrow{\Omega}\, dF_2(u)> \\ &= \tfrac{1}{2} <(\Xi_1\overleftarrow{\Omega}\Xi_2 - \Xi_2\overleftarrow{\Omega}\Xi_1)u,u> \quad . \end{aligned}$$

Thus we have proved

5.3.9 Theorem.

The function F, defined by (5.3.4), with Ξ a symmetric operator, is a constant of the motion iff Ξ satisfies (5.3.5). The corresponding canonical (adjoint) symmetries are given by $\rho(u) = dF(u) = \Xi u$ and $Y(u) = \overleftarrow{\Omega} dF(u) = \overleftarrow{\Omega}\Xi u$. The Poisson bracket of two homogeneous quadratic functions $F_i(u) = \frac{1}{2} <\Xi_i u,u>$ is given by (5.3.8). It is again a homogeneous quadratic function, corresponding to the symmetric operator $\Xi_1\overleftarrow{\Omega}\Xi_2 - \Xi_2\overleftarrow{\Omega}\Xi_1$.

□

Next we study (adjoint) symmetries for (5.3.1). Note that for all linear operators $\Xi: \mathcal{W} \to \mathcal{W}^*$, $\rho(u) = \Xi u$ is a one-form on $\mathcal{W}$. This one-form is an adjoint symmetry if it satisfies (5.1.4). For a one-form of this type this condition becomes (see also remark 5.1.12)

$$(5.3.10) \qquad \Xi\hat{X} + \hat{X}^*\Xi = 0 .$$

Since $\hat{X}^* = -\hat{H}\overleftarrow{\Omega}$, this condition is equivalent to (5.3.5). Of course, in this case Ξ is not necessarily symmetric. Suppose $\hat{Y} : \mathcal{W} \to \mathcal{W}$ is a linear operator. The vector field $Y(u) = \hat{Y}u$ is a symmetry iff it satisfies (5.1.3). For a vector field of this type this condition becomes

(5.3.11) $\quad [\hat{X}, \hat{Y}]_{-} = 0$.

By setting $\hat{Y}= \overleftarrow{\Omega}\,\Xi$ we obtain again condition (5.3.5) for Ξ. Adjoint symmetries of the form $\rho(u) = \Xi u$ and symmetries of the form $Y(u) = \hat{Y}u = \overleftarrow{\Omega}\,\Xi\, u$ we shall call *linear (adjoint) symmetries*. The manifold W is a linear space, so its first cohomology group vanishes. This implies (see section 5.1) that canonical and semi-canonical (adjoint) symmetries are identical. It is easily seen that the linear (adjoint) symmetries above are canonical iff the operator Ξ is symmetric. The corresponding constant of the motion is then $F(u) = \frac{1}{2}\langle \Xi u,u\rangle$. Also a simple calculation shows that the Lie bracket of two linear symmetries $Y_i(u) = \hat{Y}_i u = \overleftarrow{\Omega}\,\Xi_i u$ (i=1,2) is the linear symmetry

(5.3.12) $\quad Y_3(u) = [Y_1,Y_2](u) = [\hat{Y}_2,\hat{Y}_1]_{-}\, u = \overleftarrow{\Omega}(\Xi_2\overleftarrow{\Omega}\,\Xi_1 - \Xi_1\overleftarrow{\Omega}\,\Xi_2)u$.

Note that the first square bracket is the Lie bracket of two vector fields, while the second square bracket is the commutator of two linear operators. We summarize the results concerning linear symmetries in the following

5.3.13 Theorem.

The following three conditions are equivalent:

i) the linear operator $\Xi : W \to W^*$ satisfies (5.3.5), so $\Xi \in E$,

ii) the one-form $\rho(u) = \Xi\, u$ is a linear adjoint symmetry,

iii) the vector field $Y(u) = \hat{Y}\, u = \overleftarrow{\Omega}\,\Xi\, u$ is a linear symmetry.

These symmetries are canonical iff Ξ is a symmetric operator. The corresponding constant of the motion is given by $F(u) = \frac{1}{2}\langle \Xi\, u,u\rangle$. The Lie bracket of two linear symmetries $Y_i(u) = \hat{Y}_i u$ (i=1,2) is the linear symmetry Y_3 given in (5.3.12).

□

The conditions for the four possible operators between (adjoint) symmetries are also easily derived. Consider the linear operator $\Lambda: W \to W$. This linear operator is a recursion operator for symmetries if it satisfies (5.1.6). Since Λ does not depend on u and t, this implies

(5.3.14) $\quad [\Lambda, \hat{X}]_{-}= 0$.

This relation is also easily obtained from (5.3.11).

Since $\overleftarrow{\Omega}$ is invertible, we can set $\Lambda = \overleftarrow{\Omega}\Xi$. Then the operator $\Xi : \mathcal{W} \to \mathcal{W}^*$ has to satisfy (5.3.5). The conditions for recursion operators between adjoint symmetries and for AS- and SA operators can be derived in a similar way. We summarise them in the following

5.3.15 Theorem.

Suppose $\Xi : \mathcal{W} \to \mathcal{W}^*$ is a linear operator. Then the following conditions are equivalent:

i) $\Lambda = \overleftarrow{\Omega}\Xi : \mathcal{W} \to \mathcal{W}$ is a recursion operator for symmetries,

ii) $\Gamma = \Xi\overleftarrow{\Omega} : \mathcal{W}^* \to \mathcal{W}^*$ is a recursion operator for adjoint symmetries,

iii) Ξ is an SA operator,

iv) $\Psi = \overleftarrow{\Omega}\Xi\overleftarrow{\Omega} : \mathcal{W}^* \to \mathcal{W}$ is an AS operator,

v) Ξ satisfies (5.3.5), so $\Xi \in E$.

If Ξ is antisymmetric, it is a cyclic operator and Ψ is a canonical operator.

Proof:

We showed already that i) and v) are equivalent. In a similar way it can be shown that each of the conditions ii), iii) and iv) is equivalent with v). The fact that antisymmetric operators $\Xi : \mathcal{W} \to \mathcal{W}^*$ are cyclic and antisymmetric operators $\Psi : \mathcal{W}^* \to \mathcal{W}$ are canonical was already explained in remark 5.1.17.

□

In the preceding part of this section we have discussed constants of the motion, (adjoint) symmetries and several operators between those symmetries. It is important to note that these objects not necessarily are of the considered type. For instance there may exist non-quadratic constants of the motion and symmetries which are not linear. The existence of objects of the discussed type was always related to the existence of a linear operator $\Xi : \mathcal{W} \to \mathcal{W}^*$, which satisfies (5.3.5). We shall now make some remarks on the linear space E of operators Ξ satisfying this condition. The following theorem describes some elementary properties of the space E.

5.3.16 Theorem.

i) E is a Lie algebra; if Ξ_1, $\Xi_2 \in E$, then also

$$\Xi_3 = \Xi_1 \overleftarrow{\Omega} \Xi_2 - \Xi_2 \overleftarrow{\Omega} \Xi_1 .$$

The set of symmetric operators $\Xi \in E$ form a subalgebra of E. This subalgebra is isomorphic with the Lie algebra of homogeneous quadratic constants of the motion. Further, if Ξ_1 and Ξ_2 are both antisymmetric, Ξ_3 is symmetric. If one of Ξ_1, Ξ_2 is symmetric and the other is antisymmetric, Ξ_3 is antisymmetric.

ii) If Ξ_1, $\Xi_2 \in E$, then also $\Xi_4 = \Xi_1 \overleftarrow{\Omega} \Xi_2 \in E$.

iii) If $\Xi \in E$, then also $\Xi^* \in E$.

□

It is easily seen that $\hat{H} \in E$ and $\Omega \in E$. So E always contains a symmetric operator and an antisymmetric operator. The fact that $\hat{H} \in E$ gives rise to the following

5.3.17 Corollary.

Suppose $\Xi_1 \in E$. Then $\Xi_2 = \Xi_1 \overleftarrow{\Omega} \hat{H} \in E$. If Ξ_1 is symmetric (antisymmetric), Ξ_2 is antisymmetric (symmetric).

□

Using this corollary we can construct the following series of elements of E

(5.3.18) $\Omega,\ \hat{H},\ \hat{H}\overleftarrow{\Omega}\hat{H},\ \hat{H}(\overleftarrow{\Omega}\hat{H})^2,\ \ldots$.

Note that the operators in this series are alternately antisymmetric and symmetric.

Suppose Ξ_1 is an antisymmetric element of E . If Ξ_1 is invertible, the closed two-form $\tilde{\omega}$, defined by

$$\tilde{\omega}(A,B) = \langle \Xi_1 A, B \rangle$$

is nondegenerate. By corollary 5.3.17 $\Xi_2 = \Xi_1 \overleftarrow{\Omega}\hat{H}$ is a symmetric element of $\tilde{E}$. Hence $\tilde{H}(u) = \frac{1}{2} < \Xi_2 u, u>$ is a quadratic constant of the motion. Then the differential equation (5.3.2) can also be considered as a Hamiltonian system on the symplectic space $(W, \tilde{\omega})$ with Hamiltonian $\tilde{H}$:

$$(5.3.19) \qquad \dot{u} = \Xi_1^{-1} \Xi_2 u.$$

The variational principle corresponding to this Hamiltonian form of (5.3.2) is easily found (see also (5.3.3)). Suppose $\tilde{u}(t)$ is a solution of (5.3.2) (or (5.3.19). The the curve $\tilde{u}(t)$ in W is a stationary point of

$$(5.3.20) \qquad \int_{t_1}^{t_2} (\tfrac{1}{2}<\Xi_1\dot{u},u> - \tfrac{1}{2}< \Xi_2 u,u >)\, dt$$

over the set of all curves $u(t)$ in W with $u(t_1) = \tilde{u}(t_1)$, $u(t_2) = \tilde{u}(t_2)$. If $\Xi_1 \neq a\Omega$ for some $a \in \mathbb{R}$, the two ways (5.3.1) and (5.3.19) of writing the differential equation (5.3.2) as a Hamiltonian system are essentially different and the system is bi-Hamiltonian. If the operator $\hat{X} = \overleftarrow{\Omega}\hat{H}$ is invertible, we can also start with an invertible symmetric operator $\Xi_2 \in E$. Then $\Xi_1 = \Xi_2 \hat{X}^{-1}$ is an antisymmetric element of E and we can write the system again as (5.3.19). So, in the case $\hat{X}$ is invertible, any quadratic constant of the motion $\tilde{H}(u) = \frac{1}{2} <\Xi_2 u,u>$, with Ξ_2 invertible, can be considered as Hamiltonian. The corresponding symplectic form is then

$$\tilde{\omega}(A,B) = < \Xi_2 \hat{X}^{-1} A,B > \qquad \forall\, A,B \in W.$$

Note that if Ξ_1 is an invertible symmetric element of E, we can write (5.3.2) also as

$$(5.3.21) \qquad \dot{u} = \Xi_1^{-1} \Xi_1 \overleftarrow{\Omega}\hat{H}u = \Xi_1^{-1} \Xi_2 u.$$

In this expression Ξ_1 is symmetric and Ξ_2 is antisymmetric.

Next we consider a basis for the Lie algebra E. Recall ((5.3.5) and (5.3.6)) that $\Xi \in E$ iff the operator $\hat{Y} = \overleftarrow{\Omega}\Xi$ is a commutator of $\hat{X} = \overleftarrow{\Omega}\hat{H}$.

5.3.22 Theorem.

Suppose $\hat{X} = \Omega\overleftarrow{\hat{H}}$ is invertible. Then a basis for E consists of the same number (=k) of symmetric and antisymmetric operators. So the dimension of the subalgebra of symmetric operators of E is half the dimension of the Lie algebra E .

Proof:

Suppose the operators $\Phi_1, \ldots, \Phi_\ell$ form a basis for E. Define the symmetric and antisymmetric parts of Φ_i by $\Phi_i^+ = \frac{1}{2}(\Phi_i + \Phi_i^*)$ and $\Phi_i^- = \frac{1}{2}(\Phi_i - \Phi_i^*)$. Then by theorem 5.3.16 iii) these (anti)symmetric operators are also elements of E . Clearly any element Ξ of E can be written as a linear combination of the 2ℓ operators $\Phi_i^{\pm}$ $(i=1,\ldots,\ell)$. We can reduce this set to a new basis $\Xi_1, \ldots, \Xi_\ell$ of E, which consists only of symmetric or antisymmetric operators. Suppose $\Xi_1, \ldots, \Xi_k$ are symmetric and $\Xi_{k+1}, \ldots, \Xi_\ell$ are antisymmetric operators. By corollary 5.3.17 the operators $\Xi_i\Omega\overleftarrow{H}$ $(i=1, \ldots, k)$ are antisymmetric. Since $\hat{X} = \Omega\overleftarrow{\hat{H}}$ is invertible, these operators are linearly independent. Hence $\ell - k \geq k$. In a similar way we can show $\ell - k \leq k$. So $\ell = 2k$ and the basis Ξ_i consists of k symmetric and k antisymmetric operators.

□

The symmetric operators $\Xi_i (i=1,\ldots,k)$ give rise to k quadratic constants of the motion $F_k(u) = \frac{1}{2}\langle\Xi_k u,u\rangle$. Every operator Ξ_i $(i=1,\ldots,2k)$ gives rise to a "*bilinear constant of the motion*". By this we mean a bilinear function $G : W \times W \to \mathbb{R}$ such that for every pair of solutions u(t), v(t) of (5.3.1), the function G(u(t), v(t)) is constant. These "bilinear constants of the motion" are given by

$$G_i(u,v) = \tfrac{1}{2}\langle\Xi_i u, v\rangle \qquad i = 1,2,\ldots, 2k. \tag{5.3.23}$$

Note that $G_i(u,u) = F_i(u)$ for $i = 1, \ldots, k$ and $G_i(u,u) = 0$ for $i = k+1,\ldots,2k$.

If all the eigenvalues of $\hat{X} = \Omega\overleftarrow{\hat{H}}$ are different, a basis for the space of operators which commute with $\hat{X}$, is given by $\{\hat{X}^i \mid i=0,1,\ldots,2n-1\}$. The corresponding basis for E is $\{\Omega\hat{X}^i \mid i = 0,1,\ldots,2n-1\}$. So in the case a basis for E consists of n symmetric operators

$$\hat{H},\ \hat{H}(\Omega\overleftarrow{\hat{H}})^2,\ \ldots,\ \hat{H}(\Omega\overleftarrow{\hat{H}})^{2n-2}, \tag{5.3.24}$$

and of n antisymmetric operators

$$(5.3.25) \qquad \Omega, \hat{H}\Omega^{\leftarrow}\hat{H}, \; \ldots, \; \hat{H}(\Omega^{\leftarrow}\hat{H})^{2n-1} \; .$$

If $\hat{X}$ has eigenvalues which are degenerate, the dimension of the space of operators,which commute with $\hat{X}$, is higher then $2n(2k > 2n)$. A basis for E is then more complicated then the basis given (5.3.24) and (5.3.25).

We shall now show how the theory described in chapter 4, can be applied to the linear Hamiltonian system under consideration. In particular we shall construct an infinite series of constants of the motion, using the methods described in section 4.5. In theorem 4.2.11 we have seen that with a non-semi-canonical symmetry Z corresponds an SA operator $L_Z\Omega$. For a linear symmetry of the form $Z(u) = \Omega^{\leftarrow}\Xi u$ $(\Xi \in E)$, this SA operator is given by

$$(5.3.26) \qquad L_Z\Omega = \Xi - \Xi^* \; .$$

In theorem 4.2.17 we showed that $L_Z^{\,k}$ is also an SA operator. In this case we obtain for $k = 2$

$$(5.3.27) \qquad L_Z^2\,\Omega = (\Xi - \Xi^*)\Omega^{\leftarrow}\,\Xi - \Xi^*\Omega^{\leftarrow}(\Xi - \Xi^*).$$

In section 4.5 we considered the relation between the two SA operators $L_Z^2\Omega$ and $(L_Z\Omega)\Omega^{\leftarrow}L_Z\,\Omega$. In that section we have assumed that hypothesis 4.5.1 is satisfied, i.e. there exists a non-(semi-)canonical symmetry Z such that

$$L_Z^2\Omega = c\,(L_Z\Omega)\Omega^{\leftarrow}L_Z\Omega$$

for some $c \in \mathbb{R}$ with $c \neq (k-1)/k, \qquad \forall\, k \in \mathbb{N}$.

In this case this condition becomes

$$(5.3.28) \qquad (\Xi - \Xi^*)\Omega^{\leftarrow}\Xi - \Xi^*\Omega^{\leftarrow}(\Xi - \Xi^*) = c(\Xi - \Xi^*)\Omega^{\leftarrow}(\Xi - \Xi^*).$$

We shall not try to find the most general solution for Ξ of this equation. However, it is easy to see that every antisymmetric operator Ξ satisfies (5.3.28) with $c = 1$. The theory, described in section 5 of the preceding chapter, leads to the following

5.3.29 Theorem.

Suppose $Z(u) = \overleftarrow{\Omega}\Xi u$ is a non-(semi-)canonical symmetry with Ξ antisymmetric. Then the adjoint symmetries defined by

$$(5.3.30) \qquad \sigma_{k+1}(u) = (\Xi\overleftarrow{\Omega})^k \hat{H}u \qquad k = 0,1,2,\ldots,$$

are canonical. The corresponding constants of the motion are given by

$$(5.3.31 \qquad F_{k+1}(u) = \tfrac{1}{2}< (\Xi\overleftarrow{\Omega})^k \hat{H}u, u > \quad .$$

These constants of the motion are in involution.

Proof:
Since Ξ is antisymmetric, hypothesis 4.5.1 is satisfied with $Z(u) = \overleftarrow{\Omega}\,\Xi\, u$ and $c = 1$. The first cohomology group of $\mathcal{W}$ vanishes (see also section 5.1). So this theorem is a straightforward consequence of theorem 4.5.13.

□

5.3.32 Remark.

A straightforward proof of this theorem can be given in the following way. Define $\Xi_1 = \Xi$ and $\Xi_k = \Xi\overleftarrow{\Omega}\,\Xi_{k-1}$. Then by theorem 5.3.16 ii) the operators $\Xi_k \in E$. Since $\Xi_k = (\Xi\overleftarrow{\Omega})^{k-1}\Xi$ and Ξ is antisymmetric, the operators Ξ_k are also antisymmetric. Then, by corollary 5.3.17, the operator $\Xi_k\overleftarrow{\Omega}\hat{H}$ is a symmetric element of E. Hence σ_{k+1}, defined in (5.3.30), is a canonical adjoint symmetry. It is easily seen that the corresponding constants of the motion, given in (5.3.31), are in involution.

□

It is important to note that the alternative proof of theorem 5.3.29 , given in the preceding remark, depends essentially on the fact that we consider a linear equation in a linear space. *However, the methods described in chapter 4, can also be applied to a nonlinear equation on an arbitrary manifold.*

Theorem 5.3.29 can only be applied if a non-canonical symmetry $Z(u) = \overleftarrow{\Omega}\,\Xi\, u$

(so $\Xi \in E$) with Ξ antisymmetric, is known. A simple example is given by $\Xi = \hat{H}\overleftarrow{\Omega}\hat{H}$. Then the constants of the motion F_k are found to be

$$(5.3.33) \qquad F_{k+1}(u) = \tfrac{1}{2} <(\hat{H}\overleftarrow{\Omega})^{2k}\hat{H}u,u> \qquad k = 0,1,2,\dots .$$

Note that these constants of the motion correspond to the symmetric operators of the series (5.3.18) and that the first n constants correspond to the operators given in (5.3.24). It is a simple exercise to show that $F_{k+1}(u) = (-1)^k H(\hat{X}^k u)$. Note that if u(t) is a solution of (5.3.1), then $v_k(t) = \hat{X}^k u(t)$ is also a solution of (5.3.1). Hence $F_{k+1}(u(t)) = (-1)^k \cdot H(v_k(u(t)))$. So the constants of the motion F_k is, up to the sign, equal to the Hamiltonian, evaluated for a transformed solution.

Finally we remark that, since $M = W$ is a finite-dimensional linear space, the series F_k ($k = 1,2,3,\dots$) given in (5.3.31) or (5.3.33) cannot be analytically independent (see also remark 4.5.15). For instance, if all the eigenvalues of $\hat{X} = \overleftarrow{\Omega}\hat{H}$ are different, only the first n constants of the motion given in (5.3.33), are analytically independent.

5.4 THE WAVE EQUATION.

In this section we shall discuss the Hamiltonian character, constants of the motion, symmetries and operators between symmetries for the linear second order wave equation

$$(5.4.1) \qquad q_{tt} = q_{xx} \quad , \qquad x \in \mathbb{R} .$$

By setting $p = q_t$ we can write this equation as

$$(5.4.2) \qquad \begin{cases} q_t = p \ , \\ \\ p_t = q_{xx} \ , \qquad x \in \mathbb{R} . \end{cases}$$

We shall study this equation on the linear space $\mathcal{W} = \mathcal{U}_1 \times \mathcal{S}_1$ ($q \in \mathcal{U}_1$, $p \in \mathcal{S}_1$, the spaces $\mathcal{U}_1$ and $\mathcal{S}_1$ are introduced in section 1.3). Suppose we take initial values $q(.,0) = q_o(.) \in \mathcal{U}_1$ and $p(.,0) = p_o(.) \in \mathcal{S}_1$. The corresponding solution $q(.,t)$, $p(.,t)$ of (5.4.2) can be obtained with elementary methods. It is easily verified that this solution is an element of $\mathcal{W}$ for all $t \geq 0$. On $\mathcal{S}_1$ we take the topology induced by $\mathcal{U}_1$ and the duality map

$$\langle \alpha_1, A_1 \rangle = \int_{-\infty}^{\infty} \alpha_1(x) A_1(x) \, dx \qquad \text{for } \alpha_1 \in \mathcal{U}_1, \ A_1 \in \mathcal{S}_1 .$$

The dual space of $\mathcal{W}$ is $\mathcal{W}^* = \mathcal{S}_1 \times \mathcal{U}_1$. Note that $\mathcal{W}$ is reflexive $\mathcal{W}^{**} = \mathcal{W}$. The duality map between $\mathcal{W}$ and $\mathcal{W}^*$ is given by

$$(5.4.3) \quad \langle \alpha, A \rangle = \int_{-\infty}^{\infty} (\alpha_1(x) A_1(x) + \alpha_2(x) A_2(x)) \, dx \ , \quad \forall\, A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \in \mathcal{W}, \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \in \mathcal{W}^* .$$

The exterior derivative of a differentiable function (functional) $F : \mathcal{W} \to \mathbb{R}$ is given by

$$dF = \begin{pmatrix} \dfrac{\delta F}{\delta q} \\ \\ \dfrac{\delta F}{\delta p} \end{pmatrix} \in \mathcal{W}^*$$

where $\frac{\delta F}{\delta q} \in \mathcal{S}_1$ and $\frac{\delta F}{\delta p} \in \mathcal{U}_1$ are the variational derivatives (gradients) of F.

We shall now describe the (well-known) Hamiltonian structure of (5.4.2). On W we introduce the standard symplectic form

$$\omega(A,B) = \langle \Omega A, B\rangle = \int_{-\infty}^{\infty} (A_1B_2 - A_2B_1)\, dx \quad \forall\, A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \in W. \tag{5.4.4}$$

So the corresponding operator $\Omega : W \to W^*$ can be represented by

$$\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{5.4.5}$$

It is clear that Ω is invertible, the inverse operator $\overset{\leftarrow}{\Omega} : W^* \to W$ is given by

$$\overset{\leftarrow}{\Omega} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{5.4.6}$$

Define the function H on W by

$$H(q,p) = \tfrac{1}{2}\int_{-\infty}^{\infty} (q_x^2 + p^2)\, dx . \tag{5.4.7}$$

Then the evolution equations (5.4.2) can be written as a Hamiltonian system on W with symplectic form ω and Hamiltonian H

$$\begin{pmatrix} q \\ p \end{pmatrix}_t = \overset{\leftarrow}{\Omega}\, dH = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -q_{xx} \\ p \end{pmatrix}. \tag{5.4.8}$$

Constants of the motion, symmetries and operators between symmetries for the infinite-dimensional linear Hamiltonian system (5.4.8) are easily found. In section 5.3 we studied (quadratic) constants of the motion, (linear) symmetries and operators between symmetries for a finite-dimensional linear Hamiltonian system. Therefore the following considerations will show a strong resemblance with those in section 5.3. We shall also derive two series of symmetries for (5.4.8) which contain t explicitly.

Define the operators $\hat{H}$ and $\hat{X}$ by

$$\hat{H} = \begin{pmatrix} -\partial^2 & 0 \\ 0 & 1 \end{pmatrix} : W \to W^* \qquad , \hat{X} = \begin{pmatrix} 0 & 1 \\ \partial^2 & 0 \end{pmatrix} : W \to W.$$

Denote the vector field, given in the right hand side of (5.4.8) by X, so

$$X = \overset{\leftarrow}{\Omega}\, dH = \overset{\leftarrow}{\Omega}\hat{H}\begin{pmatrix} q \\ p \end{pmatrix} = \hat{X}\begin{pmatrix} q \\ p \end{pmatrix}. \tag{5.4.9}$$

Similar to the theorems 5.3.9, 5.3.13 and 5.3.15 we can prove the following

5.4.10 Theorem.

Suppose $\Xi : W \to W^*$ is a linear operator which satisfies

(5.4.11) $$\Xi \overleftarrow{\Omega} \hat{H} - \hat{H} \overleftarrow{\Omega} \Xi = 0 .$$

Then

i) If Ξ is a symmetric operator the function

(5.4.12) $$F(q,p) = \tfrac{1}{2} < \Xi \begin{vmatrix} q \\ p \end{vmatrix}, \begin{vmatrix} q \\ p \end{vmatrix} >$$

is a constant of the motion of (5.4.8).

ii) An adjoint symmetry σ and a symmetry Y for (5.4.8) are given by

(5.4.13) $$\sigma = \Xi \begin{vmatrix} q \\ p \end{vmatrix} \quad , \; Y = \overleftarrow{\Omega} \sigma = \overleftarrow{\Omega} \Xi \begin{vmatrix} q \\ p \end{vmatrix} \; .$$

These symmetries are canonical if Ξ is symmetric, the corresponding constant of the motion F is given in (5.4.12).

iii) Ξ is an SA operator, $\Gamma = \Xi \overleftarrow{\Omega}$ is a recursion operator for adjoint symmetries and $\Lambda = \overleftarrow{\Omega} \Xi$ is a recursion operator for symmetries of (5.4.8).

□

The following two series of linear operators $\Xi : W \to W^*$ which satisfy (5.4.11) are easily found

(5.4.14) $$\Xi_j = \begin{pmatrix} -\partial^{j+2} & 0 \\ 0 & \partial^j \end{pmatrix}, \qquad \hat{\Xi}_j = \begin{pmatrix} 0 & -\partial^j \\ \partial^j & 0 \end{pmatrix}, \qquad j = 0,1,2,\dots .$$

Note that the operators Ξ_{2k} and $\hat{\Xi}_{2k+1}$ are symmetric while the operators Ξ_{2k+1} and $\hat{\Xi}_{2k}$ are antisymmetric $(k = 0,1,2,\dots)$. To simplify the notation we set $q_{,k} = \partial^k q$ and $p_{,k} = \partial^k p$. The symmetric operators Ξ_{2k} and $\hat{\Xi}_{2k+1}$ give rise to the following two series of constants of the motion in involution

(5.4.15) $$F_k(q,p) = \tfrac{1}{2}(-1)^k < \Xi_{2k} \begin{vmatrix} q \\ p \end{vmatrix}, \begin{vmatrix} q \\ p \end{vmatrix} > = \tfrac{1}{2} \int_{-\infty}^{\infty} (q_{,k+1}^2 + p_{,k}^2)\, dx,$$

$$(5.4.16)\qquad G_k(q,p) = \tfrac{1}{2}(-1)^k < \hat{\Xi}_{2k+1}\begin{pmatrix}q\\p\end{pmatrix}, \begin{pmatrix}q\\p\end{pmatrix} > = \int_{-\infty}^{\infty} q_{,k+1}\,p_{,k}\,dx, \qquad k = 0,1,2,\dots .$$

Note that $F_o = H$. With every operator given in (5.4.14) corresponds an adjoint symmetry and a symmetry

$$(5.4.17)\qquad \rho_j = \Xi_j\begin{pmatrix}q\\p\end{pmatrix} = \begin{pmatrix}-q_{,j+2}\\ p_{,j}\end{pmatrix}, \quad X_j = \Omega^{\leftarrow}\rho_j = \begin{pmatrix}p_{,j}\\ q_{,j+2}\end{pmatrix},$$

$$(5.4.18)\qquad \sigma_j = \hat{\Xi}_j\begin{pmatrix}q\\p\end{pmatrix} = \begin{pmatrix}-p_{,j}\\ q_{,j}\end{pmatrix}, \quad Y_j = \Omega^{\leftarrow}\sigma_j = \begin{pmatrix}q_{,j}\\ p_{,j}\end{pmatrix}, \quad j = 0,1,2,\dots .$$

The (adjoint) symmetries $X_{2j}(\rho_{2j})$ and $Y_{2j+1}(\sigma_{2j+1})$ are canonical:

$$(5.4.19)\qquad X_{2j} = \Omega^{\leftarrow}\rho_{2j} = (-1)^j\Omega^{\leftarrow}\,dF_j, \quad Y_{2j+1} = \Omega^{\leftarrow}\sigma_{2j+1} = (-1)^j\Omega^{\leftarrow}\,dG_j.$$

It is a simple exercise to show that $X_{2j+1}(\rho_{2j+1})$ and $Y_{2j}(\sigma_{2j})$ are non-canonical. The various Lie brackets between the elements of both series of symmetries X_j and Y_j can be found in the same way as in section 5.3 (theorem 5.3.13). It turns out that

$$(5.4.20)\qquad [X_k, X_\ell] = 0,\ [Y_k, Y_\ell] = 0,\ [X_k, Y_\ell] = 0, \qquad k,\ell = 0,1,2,\dots .$$

By theorem 5.4.10 iii) the operators Ξ_j, $\hat{\Xi}_j$, given in (5.4.14), provide us with the following recursion operators for adjoint symmetries

$$(5.4.21)\qquad \Gamma_j = \Xi_j\Omega^{\leftarrow} = \begin{pmatrix}0 & -\partial^{j+2}\\ -\partial^j & 0\end{pmatrix}, \quad \hat{\Gamma}_j = \hat{\Xi}_j\Omega^{\leftarrow} = \begin{pmatrix}\partial^j & 0\\ 0 & \partial^j\end{pmatrix}.$$

Note that every recursion operator of these two series can be written as a product of powers of Γ_o and $\hat{\Gamma}_1$. It is easily seen that

$$(5.4.22)\qquad \begin{aligned} &\hat{\Gamma}_1\rho_j = \rho_{j+1},\ \hat{\Gamma}_1\sigma_j = \sigma_{j+1},\\ &\Gamma_o\rho_j = \sigma_{j+2},\ \Gamma_o\sigma_j = \rho_j . \end{aligned}$$

Together with (5.4.19) this implies

$$(5.4.23)\qquad dF_{j+1} = -\hat{\Gamma}_1^2 dF_j \ , \ dG_{j+1} = -\hat{\Gamma}_1^2 dG_j \ .$$

So the two series of canonical adjoint symmetries can be constructed from dF_o and dG_o using the recursion operator $\hat{\Gamma}_1^2$.

For the system (5.4.8) there also exist symmetries and recursion operators for symmetries which contain t explicitly. Suppose $\Xi, \Phi : \mathcal{W} \to \mathcal{W}^*$ are two linear operators (independent of t). Then we can look for an adjoint symmetry of the form

$$(5.4.24)\qquad \tau = (\Phi + t\Xi)\begin{vmatrix} q \\ p \end{vmatrix} \ .$$

It follows from (5.1.4) that τ is an adjoint symmetry if

$$\Xi + (\Phi + t\Xi)\overleftarrow{\Omega}\hat{H} - \hat{H}\overleftarrow{\Omega}(\Phi + t\Xi) = 0 \ .$$

This implies

$$(5.4.25)\qquad \Xi\overleftarrow{\Omega}\hat{H} - \hat{H}\overleftarrow{\Omega}\Xi = 0$$

and

$$(5.4.26)\qquad \Xi + \Phi\overleftarrow{\Omega}\hat{H} - \hat{H}\overleftarrow{\Omega}\Phi = 0 \ .$$

The condition (5.4.25) is the same as (5.4.11). So operators $\Xi : \mathcal{W} \to \mathcal{W}^*$ which satisfy this condition, are given in (5.4.14). It is a simple exercise to verify that operators Φ_j, $\hat{\Phi}_j$ such that Ξ_j, Φ_j and $\hat{\Xi}_j$, $\hat{\Phi}_j$ satisfy (5.4.26) are given by

$$(5.4.27)\qquad \Phi_j = \begin{pmatrix} 0 & -\partial^j - x\partial^{j+1} \\ x\partial^{j+1} & 0 \end{pmatrix}, \ j \geq 0 \ , \qquad \hat{\Phi}_j = \begin{pmatrix} -\partial^j - x\partial^{j+1} & 0 \\ 0 & x\partial^{j-1} \end{pmatrix}, \ j \geq 1 \ .$$

Thus we constructed two series of (adjoint) symmetries for (5.4.2) which depend explicitly on x and t.

$$(5.4.28)\qquad \begin{aligned} \tau_j &= (\Phi_j + t\Xi_j)\begin{vmatrix} q \\ p \end{vmatrix} = \begin{pmatrix} -p,_j - xp,_{j+1} \\ xq,_{j+1} \end{pmatrix} + t\begin{pmatrix} -q,_{j+2} \\ p,_j \end{pmatrix} \\ Z_j &= \overleftarrow{\Omega}\tau_j = \begin{pmatrix} xq,_{j+1} \\ p,_j + xp,_{j+1} \end{pmatrix} + t\begin{pmatrix} p,_j \\ q,_{j+2} \end{pmatrix} \qquad j \geq 0, \end{aligned}$$

and

$$\mu_j = (\hat{\Phi}_j + t\hat{\Xi}_j)\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -q,_j - xq,_{j+1} \\ xp,_{j-1} \end{pmatrix} + t\begin{pmatrix} -p,_j \\ q,_j \end{pmatrix}$$

(5.4.29)

$$U_j = \Omega^{\leftarrow}\mu_j = \begin{pmatrix} xp,_{j-1} \\ q,_j + xq,_{j+1} \end{pmatrix} + t\begin{pmatrix} q,_j \\ p,_j \end{pmatrix} , \quad j \geq 1 .$$

It can be verified that these (adjoint) symmetries are non-canonical (except Z_o). The symmetry Z_o can be written as

$$Z_o = \begin{pmatrix} xq_x + tp \\ p + xp_x + tq_{xx} \end{pmatrix} = \begin{pmatrix} xq_x + tq_t \\ p + xp_x + tp_t \end{pmatrix} .$$

This symmetry is related to the scaling properties of (5.4.2). Suppose $(q(x,t), p(x,t))$ is a solution of (5.4.2), then $(q(ax,at), a\,p(ax,at))$ is also a solution of (5.4.2) for all $a \in \mathbb{R}$. Set $a = 1 + \varepsilon$ and consider the difference between these two solutions for $\varepsilon \to 0$. We obtain a solution of the (linearized) equation which corresponds to the symmetry Z_o.

The Lie brackets between the elements of the series of symmetries (5.4.17), (5.4.18), (5.4.28) and (5.4.29) are given by

(5.4.30)
$$\begin{cases} [Z_k,Z_\ell] = (k-\ell)Z_{k+\ell}, \quad [U_k,U_\ell] = (\ell-k)Z_{k+\ell-2} , \\ [Z_k,U_\ell] = (\ell-k-1)U_{k+\ell} , \\ [Z_k,X_\ell] = (\ell+1)X_{k+\ell}, \quad [U_k,X_\ell] = (\ell+1)Y_{k+\ell} , \\ [Z_k,Y_\ell] = \ell Y_{k+\ell} , \quad [U_k,Y_\ell] = \ell X_{k+\ell-2} . \end{cases}$$

5.4.31 Remark.

It is also possible to consider (5.4.1) as a Hamiltonian system on W with Hamiltonian $\hat{H} = G_o = \int_{-\infty}^{\infty} pq_x dx$ and symplectic two-form

$$\hat{\omega}(A,B) = \langle\hat{\Omega}A,B\rangle = \int_{-\infty}^{\infty}(-B_1\partial A_1 + B_2\partial^{-1}A_2)dx.$$

Note that indeed ω is nondegenerate on $\mathcal{W} = \mathcal{U}_1 \times \mathcal{S}_1$. The operators $\hat{\Omega} : \mathcal{W} \to \mathcal{W}^*$ and $\hat{\Omega}^{\leftarrow} : \mathcal{W}^* \to \mathcal{W}$ are given by

$$\hat{\Omega} = \begin{pmatrix} -\partial & 0 \\ 0 & \partial^{-1} \end{pmatrix} , \qquad \hat{\Omega}^{\leftarrow} = \begin{pmatrix} -\partial^{-1} & 0 \\ 0 & \partial \end{pmatrix} .$$

Then the differential equations (5.4.2) can also be written as

$$(5.4.32) \qquad \begin{pmatrix} q \\ p \end{pmatrix}_t = \hat{\Omega}^{\leftarrow} d\hat{H} = \begin{pmatrix} -\partial^{-1} & 0 \\ 0 & \partial \end{pmatrix} \begin{pmatrix} -p_x \\ q_x \end{pmatrix} .$$

So the system (5.4.2) can be written as a Hamiltonian system in two ways; it is a bi-Hamiltonian system.

5.4.33 Remark.

It will be clear that constants of the motion, symmetries and operators between symmetries for the system (5.4.8) can relatively easy be found. The reason for this is that (5.4.8) is a very simple linear Hamiltonian system. In fact we can easily derive more properties of the type discussed in this section. For instance, using the solutions for (5.4.25) and (5.4.26) we can also find recursion operators for (adjoint) symmetries which depend explicitly on t. The non-semi-canonical symmetries X_{2j+1} and Z_{2j} can be used if we want to construct infinite series of constants of the motion, using the method described in section 4.5. We then obtain again the series F_k and G_k.

□

5.5 A HAMILTONIAN SYSTEM WITH A LINEARIZING TRANSFORMATION.

The concept "completely integrable Hamiltonian system" is well defined for a Hamiltonian system in a finite-dimensional phase space (see section 3.6). Recall that for a finite-dimensional Hamiltonian system with canonical coordinates $q_1, \dots q_n, p_1, \dots p_n$, the existence of n analytically independent constants of the motion in involution implies complete integrability (theorem 3.6.2). In that case we can perform a transformation to action angle variables and the differential equations can easily be solved.

For an infinite-dimensional Hamiltonian system the situation is much more complicated. There exist infinite-dimensional Hamiltonian systems which posses an infinite series of constants of the motion in involution. In recent years the so called "inverse scattering methods" have become enormously popular for solving certain types of nonlinear evolution equations. The Hamiltonian systems, solvable by this method, turn out to have an infinite series of constants of the motion in involution. Sometimes (always?) an "inverse scattering method" can be considered as a transformation to variables of "action angle type". A famous example is the introduction of action angle variables for the Korteweg-de Vries equation by Zakharov and Faddeev [24] . However, if for an infinite-dimensional Hamiltonian system there exists an infinite series of constants of the motion, the problem of finding the solution for arbitrary initial values (for instance by "inverse scattering") is still unsolved. So a generalization of theorem 3.6.2 to infinite-dimensional Hamiltonian systems is not straightforward.

Next consider an infinite-dimensional (nonlinear) Hamiltonian system for which there exists a (global) invertible transformation to a linear Hamiltonian system. The (formal) solution of the linear Hamiltonian system can always be given. Constants of the motion, (adjoint) symmetries and operators between (adjoint) symmetries can also easily be found. Then, by using the transformation which relates the linear and nonlinear system, we can find the solution, constants of the motion, (adjoint) symmetries and operators between (adjoint) symmetries for the nonlinear Hamiltonian system (see sections 2.7 and 3.7). It will be clear that, independent of the correct way to generalize the concept "completely integrable" for an infinite-dimensional system, an infinite-dimensional Hamiltonian system, for which there exists a global invertible transformation to a linear Hamiltonian system must be "completely integrable". In the remaining part of this section we give an example of such a system.

Consider the following system of partial differential equations

$$(5.5.1) \qquad \begin{cases} u_t = -u_x + 2uv, \\ v_t = v_x - 2uv, \end{cases} \qquad x \in \mathbb{R},$$

with initial values

$$(5.5.2) \qquad \begin{cases} u(x,o) = u_o(x), \\ v(x,o) = v_o(x). \end{cases}$$

We shall consider (5.5.1) on the space $Z = S_1 \times S_1$. On each space S_1 we take the topology induced by the space U_1 and the duality map given in (1.3.13) (see theorem 1.3.14). The dual space of Z is then $Z^* = U_1 \times U_1$ and the duality map between Z and Z^* is given by

$$\langle\alpha,A\rangle = \int_{-\infty}^{\infty} (\alpha_1(x)A_1(x) + \alpha_2(x)A_2(x))dx, \quad \alpha = \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} \in Z^*,\ A = \begin{pmatrix}A_1\\ A_2\end{pmatrix} \in Z .$$

The system (5.5.1) can be used to study some cases of wave-wave interaction in plasma physics. In population dynamics it can be used to describe the growth/decay of two conflicting populations, which meet each other with velocities 1 and -1. In both applications the initial values u_o and v_o satisfy

$$(5.5.3) \qquad \begin{cases} u_o(x) \geq 0, \\ v_o(x) \geq 0, \quad \forall x \in \mathbb{R}. \end{cases}$$

An exact solution of (5.5.1) has been given by Hasimoto [53]. It is obtained by the following linearizing transformation

(5.5.4) $$\begin{pmatrix} u \\ v \end{pmatrix} = f \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} \frac{1}{2q}(p-q_x) \\ \frac{-1}{2q}(p+q_x) \end{pmatrix},$$

(5.5.5) $$\begin{pmatrix} q \\ p \end{pmatrix} = f^{\leftarrow} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} e^{-\partial^{-1}(u+v)} \\ (u-v)e^{-\partial^{-1}(u+v)} \end{pmatrix}.$$

It is a simple calculation to show that q and p satisfy the following linear evolution equations

(5.5.6) $$\begin{cases} q_t = p, \\ p_t = q_{xx} \end{cases}.$$

The initial values q_o and p_o for (5.5.6) are found by transforming u_o and v_o by (5.5.5). This implies $q_o(x) \geq 0$ for all $x \in \mathbb{R}$. So for every pair of initial values u_o and v_o for (5.5.1) there exists a $t_o > 0$ such that the corresponding $q(x,t) > 0$ for $x \in \mathbb{R}$ and $t \in [0,t_o)$. Hence the transformation (5.5.4) is regular for $t \in [0,t_o)$. Thus we obtain a solution $(u(x,t),v(x,t)) \in Z$ for $t \in [0,t_o)$ (local existence). If the initial values u_o and v_o satisfy (5.5.3), it can be shown that $q(x,t) > 0$ for $x \in \mathbb{R}$ and $t \geq 0$. In that case the transformation (5.5.4) is regular for all $t \geq 0$ and we obtain a solution $(u(x,t),v(x,t)) \in Z$ for all $t \geq 0$ (global existence).

5.5.7 Remark.

In section 5.4 we studied the system (5.5.6) on the space $\mathcal{W} = \mathcal{U}_1 \times \mathcal{S}_1$. It will be clear from (5.5.5) that $p \in \mathcal{S}_1$ but $q \notin \mathcal{U}_1$ (for $u,v \in \mathcal{S}_1$). Also the transformation (5.5.4) does not yield regular functions u and v for an arbitrary $q \in \mathcal{U}_1$. So we cannot consider f and $f^{\leftarrow}$ as mappings $f : \mathcal{U}_1 \times \mathcal{S}_1 \to Z$ with inverse $f^{\leftarrow} : Z \to \mathcal{U}_1 \times \mathcal{S}_1$. The most elegant solution of this problem is obtained in the following way. Define the set of functions $\mathcal{V}_1 = e^{\mathcal{U}_1} = \{q \mid q(x) = e^{s(x)} \text{ with } s \in \mathcal{U}_1\}$ and consider (5.5.6) on $\mathcal{V}_1 \times \mathcal{S}_1$. It is easily seen that $f : \mathcal{V}_1 \times \mathcal{S}_1 \to Z$ and $f^{\leftarrow} : Z \to \mathcal{V}_1 \times \mathcal{S}_1$ are correctly defined mappings which are each others inverse. Note that $\mathcal{V}_1$ is not a linear space but an infinite-dimensional manifold. We shall not work out this relative complicated situation further and leave the function spaces for (5.5.6) unspecified.

□

We now indicate how the Hamiltonian structure for (5.5.1) can be obtained from the Hamiltonian structure of (5.5.6), which was explained in the preceding section. A symplectic form ω with corresponding operators Ω and $\overleftarrow{\Omega}$ and a Hamiltonian for (5.5.6) were given in (5.4.4), (5.4.5), (5.4.6) and (5.4.7). In section 3.7 we explained the transformation properties of these objects. We only compute the transformed operator $\tilde{\overleftarrow{\Omega}} = f'\overleftarrow{\Omega}f'^*$. The derivative of f in the point $\begin{pmatrix} q \\ p \end{pmatrix}$ is the linear operator

$$(5.5.8) \qquad f'\begin{pmatrix} q \\ p \end{pmatrix} = e^{\partial^{-1}(u+v)} \begin{pmatrix} -u-\frac{1}{2}\partial & \frac{1}{2} \\ -v-\frac{1}{2}\partial & -\frac{1}{2} \end{pmatrix} .$$

In the right hand side we already replaced q and p by the new variables u and v. The dual operator is given by

$$(5.5.9) \qquad f'^*\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -u+\frac{1}{2}\partial & -v+\frac{1}{2}\partial \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} e^{\partial^{-1}(u+v)} .$$

The transformed operator $\tilde{\overleftarrow{\Omega}}$ is then given by

$$(5.5.10) \qquad \tilde{\overleftarrow{\Omega}} = f'\overleftarrow{\Omega}f'^* = \tfrac{1}{2}e^{2\partial^{-1}(u+v)} \begin{pmatrix} -u-v-\partial & u+v \\ -u-v & u+v+\partial \end{pmatrix} : Z^* \to Z.$$

It can be verified that this is a canonical operator (operator field), as expected from theorem 3.7.1. The Hamiltonian for (5.5.6), given in (5.4.7), is transformed into

$$(5.5.11) \qquad \tilde{H}(u,v) = \int_{-\infty}^{\infty} e^{-2\partial^{-1}(u+v)}\,(u^2+v^2)\,dx$$

Then

$$d\tilde{H} = \begin{pmatrix} \dfrac{\delta\tilde{H}}{\delta u} \\ \dfrac{\delta\tilde{H}}{\delta v} \end{pmatrix} = \begin{pmatrix} 2ue^{-2\partial^{-1}(u+v)} + 2\partial^{-1}((u^2+v^2)e^{-2\partial^{-1}(u+v)}) \\ 2ve^{-2\partial^{-1}(u+v)} + 2\partial^{-1}((u^2+v^2)e^{-2\partial^{-1}(u+v)}) \end{pmatrix} \in Z^*$$

and the system (5.5.1) can be written in the Hamiltonian form

$$(5.5.12) \qquad \begin{pmatrix} u \\ v \end{pmatrix}_t = \tilde{\Omega}^{\leftarrow} d\tilde{H} .$$

Constants of the motion for this Hamiltonian system are found by transforming the series F_k and G_k, given in (5.4.15) and (5.4.16). Since the series F_k and G_k are in involution, the transformed series $\tilde{F}_k$ and $\tilde{G}_k$ are also in involution (see corollary 3.7.7). The first few constants of the motion for (5.5.1) (or (5.5.12)) are given by

$$\tilde{F}_o(u,v) = \tilde{H}(u,v) ,$$

$$(5.5.13) \qquad \tilde{F}_1(u,v) = \int_{-\infty}^{\infty} e^{-2\partial^{-1}(u+v)} ((u^2+v^2)(u^2+v^2+2uv) - 2(u+v)(uu_x+vv_x) + u_x^2 + v_x^2)dx,$$

$$\tilde{G}_o(u,v) = \int_{-\infty}^{\infty} e^{-2\partial^{-1}(u+v)} (v^2 - u^2)dx .$$

Symmetries of (5.5.1) can be obtained by transforming symmetries of (5.5.6) as described in theorem 2.7.5. For instance, the symmetry Y_2, given in (5.4.18), transforms into

$$\tilde{Y}_2 = f'Y_2 = \begin{pmatrix} -2(u+v)u_x + u_{xx} \\ -2(u+v)v_x + v_{xx} \end{pmatrix} ,$$

and the symmetry Z_o, given in (5.4.28), transforms into

$$\tilde{Z}_o = f'Z_o = \begin{pmatrix} u + xu_x + t(2uv - u_x) \\ v + xv_x + t(-2uv + v_x) \end{pmatrix} .$$

The symmetry Z_o has been related to the scale transformation for (5.5.6). It is easily seen that $\tilde{Z}_o$ is related to the scale transformation for (5.5.1).

Finally we given the SA operator $\tilde{\hat{\Xi}}_2$, obtained by transforming the SA operator $\hat{\Xi}_2$ given in (5.4.14)

$$\tilde{\hat{\Xi}}_2 = f^{\leftarrow '*}\hat{\Xi}_2 f^{\leftarrow '} = \begin{pmatrix} \kappa - \kappa^* & \lambda - \kappa^* \\ \kappa - \lambda^* & \lambda - \lambda^* \end{pmatrix} : Z \to Z^* ,$$

where the operators κ and λ are given by

$$\kappa = -\partial^{-1} e^{-\partial^{-1}(u+v)} \partial^2 e^{-\partial^{-1}(u+v)} (1-(u-v)\partial^{-1}) : S_1 \to U_1,$$

$$\lambda = \partial^{-1} e^{-\partial^{-1}(u+v)} \partial^2 e^{-\partial^{-1}(u+v)} (1+(u-v)\partial^{-1}) : S_1 \to U_1 .$$

By combining $\tilde{\hat{\Xi}}_2$ and $\tilde{\Omega}^{\leftarrow}$ we obtain a recursion operator for adjoint symmetries of (5.5.1)

$$\tilde{\Gamma} = \tilde{\hat{\Xi}}_2 \tilde{\Omega}^{\leftarrow} : Z^* \to Z^* .$$

This recursion operator can also be obtained by transforming the corresponding recursion operator for adjoint symmetries of (5.5.6), given in (5.4.21)

$$\hat{\Xi}_2 \Omega^{\leftarrow} = \hat{\Gamma}_2 = \hat{\Gamma}_1^2 .$$

Then, by transforming (5.4.23), we obtain

(5.5.14) $$\tilde{dF}_{k+1} = -\tilde{\Gamma}\tilde{dF}_k, \quad \tilde{dG}_{k+1} = -\tilde{\Gamma}\tilde{dG}_k.$$

5.5.15 Remark.

The justification of (5.5.14) is obtained from the corresponding formula (5.4.23) for the linear Hamiltonian system (5.5.6). However, if we want to investigate whether some nonlinear Hamiltonian system has an infinite series of constants of the motion, looking for an invertible transformation to a linear Hamiltonian system will be an impossible task. If, for instance by trial and error, a non-canonical symmetry Z and / or the corresponding recursion operator $(L_Z\Omega)\Omega^{\leftarrow}$ are found, we can generate an infinite series of adjoint symmetries (starting with "dH"). Then we would like to prove that these adjoint symmetries are canonical. A possible method for doing this was explained in section 4.5. This method can be applied if the non-(semi-) canonical symmetry satisfies hypothesis 4.5.1. It can be shown that the non-canonical symmetry $\tilde{Z}_2 = f'Z_2$ indeed satisfies this hypothesis. □

5.6 THE KORTEWEG-DE VRIES EQUATION.

During the last decennium the Korteweg-de Vries (KdV) equation has become one of the most discussed equations of mathematical physics. The equation was derived by Korteweg and de Vries in 1894 [6,7] for describing long water waves in one direction in a canal. Korteweg and de Vries described periodic solutions (cnoidal waves) and solitary wave solutions of the equation. Solitary waves were already reported by Scott Russell [26] in his famous ride along a channel. His report is quoted in many books on solitons, see for instance Bullough and Caudrey [27] . For a long time the Korteweg-de Vries (KdV) equation gained only limited attention in hydrodynamics. Interest in the equation increased enormously in the sixties. In 1965 Zabusky and Kruskal [28] obtained numerical evidence for the remarkable result that two solitary waves, after their interaction, assume again their original shape. Gardner, Greene, Kruskal and Miura [19] showed in 1967 how the initial value problem for the KdV equation on the real line, with fastly decaying initial value for $|x| \to \infty$, could be solved. The method they used has become known as "inverse scattering". In 1968 Lax [29] found an infinite series of "higher order KdV equations", which all can be solved by this method. These higher order KdV equations are directly related with the infinite series of constants of the motion of the KdV equation, found by Miura, Gardner and Kruskal [30] in the same year. The Hamiltonian character of the KdV equation was pointed out by Gardner [11] and Broer [10] . After this numerous other papers on the KdV and related equations appeared. We mention only the work of Wahlquist and Estabrook on prolongation structures [31] and the paper of Zakharov and Faddeev [24] , in which they show that the KdV equation can be considered as an infinite-dimensional completely integrable Hamiltonian system. The KdV equation has also been derived in several different physical situations, see for instance Whitham [32] or Su and Gardner [33] .

Of course we shall not give many new results on the KdV equation. In this section we consider symmetries of the KdV equation. Besides the well-known series of symmetries which correspond to the higher order KdV equations, we shall describe another infinite series of symmetries. These symmetries depend explicitly on x and t. They are well suited to illustrate the theory described in chapter 4. Using this second series of symmetries we describe several methods for constructing the constants of the motion. One of these methods is a very simple recursion formula for the constants of the motion themselves (i.e. not for their gradients (= adjoint symmetries) or corresponding symmetries). We also show that every constant of the motion of the infinite series can be considered as a Hamiltonian for the KdV equation. The corresponding (weak) symplectic forms are explicitly given. Then we make some remarks on the symmetries which appear in the inverse scattering method. We end this section with some remarks on the higher order KdV equations.

In this section we consider the KdV equation in the form

$$(5.6.1) \qquad u_t = X(u) = 6uu_x - u_{xxx} \qquad x \in \mathbb{R} .$$

Various other forms of the equation can easily be transformed to (5.6.1). We shall study (5.6.1) in the space S_2, provided with the topology induced by U_2 and the duality map (see theorem 1.3.14)

$$<\alpha, A> = \int_{-\infty}^{\infty} \alpha(x)A(x)\, dx \qquad \alpha \in U_2,\ A \in S_2 \quad .$$

We now describe the Hamiltonian form of the KdV equation. Define the two-form ω on S_2 by

$$(5.6.2) \qquad \omega(A,B) = <\partial^{-1}A,B> .$$

Note that $\partial^{-1}: S_2 \to U_2$ is antisymmetric, so ω is correctly defined. The corresponding operators are

$$(5.6.3) \qquad \Omega = \partial^{-1} : S_2 \to U_2,$$

$$(5.6.4) \qquad \Omega^{\leftarrow} = \partial \quad : U_2 \to S_2.$$

It is clear (see remark 5.1.17) that Ω is a cyclic operator, $\Omega^{\leftarrow}$ a canonical operator and ω a symplectic form. Define the function (functional) $H : S_2 \to \mathbb{R}$ by

$$(5.6.5) \qquad H(u) = \int_{-\infty}^{\infty} (u^3 + \tfrac{1}{2} u_x^2)\, dx.$$

The exterior derivative (= variational derivative) is given by

$$dH(u) = \frac{\delta H}{\delta u}(u) = 3u^2 - u_{xx}.$$

Then the KdV equation is a Hamiltonian system on S_2 with Hamiltonian H and symplectic form ω

$$(5.6.6) \qquad u_t = \Omega^{\leftarrow} \frac{\delta H}{\delta u} = \partial(3u^2 - u_{xx}).$$

Clearly the Hamiltonian H is a constant of the motion. Several other constants of the motion are easily found

$$G(u,t) = \int_{-\infty}^{\infty}(xu + 3tu^2)dx,$$

$$(5.6.7) \qquad F_1(u) = \int_{-\infty}^{\infty}udx \ , \ F_2(u) = \int_{-\infty}^{\infty} u^2dx,$$

$$F_3(u) = H(u), \quad F_4(u) = \int_{-\infty}^{\infty} (u^4 + 2uu_x^2 + \frac{1}{5} u_{xx}^2)dx.$$

In 1968 Miura found a relation between the KdV and the so called Modified Korteweg-de Vries (MKdV) equation.

$$(5.6.8) \qquad v_t = 6v^2v_x - v_{xxx} \quad , \quad x \in \mathbb{R}.$$

It is easily verified that for every solution v of (5.6.8) the function

$$(5.6.9) \qquad u = f(v) = v^2 + v_x$$

is a solution of (5.6.1). This transformation has become known as Miura transformation. Using a modified version of the transformation Miura, Gardner and Kruskal [30] proved in 1968 that the KdV equation (and also the MKdV equation) has an infinite series of constants of the motion F_k.

5.6.10 Remark.

The MKdV equation can also formally be written as a Hamiltonian system on some space W of smooth functions, which vanish, together with their derivatives, fast enough for $|x| \to \infty$. Using the canonical operator ∂ and the Hamiltonian $K(v) = \frac{1}{2} \int_{-\infty}^{\infty} (v^4 + \frac{1}{2} v_x^2) dx$ we can write the MKdV equation as

$$(5.6.11) \qquad v_t = \partial \frac{\delta K(v)}{\delta v} = \partial(2v^3 - v_{xx}).$$

□

Symmetries $Y(u,t)$ and adjoint symmetries $\sigma(u,t)$ of the KdV equation have to satisfy the conditions (5.1.3) and (5.1.4). Using

$$X'(u) = 6u\partial + 6u_x - \partial^3 = 6\partial u - \partial^3 : S_2 \to S_2$$

$$X'^*(u) = -6u\partial + \partial^3 : U_2 \to U_2$$

these conditions become

$$(5.6.12) \qquad Y_t(u,t) + Y'(u,t)(6uu_x - u_{xxx}) - (6\partial u - \partial^3)\, Y(u,t) = 0,$$

$$(5.6.13) \qquad \sigma_t(u,t) + \sigma'(u,t)(6uu_x - u_{xxx}) + (-6u\partial + \partial^3)\sigma(u,t) = 0.$$

Define the antisymmetric operator (operator field) $\Psi(u)$ by

$$(5.6.14) \qquad \Psi(u) = 2u\partial + 2\partial u - \partial^3 : U_2 \to S_2.$$

It was observed by Magri [5] that the KdV equation can also be written as

$$(5.6.15) \qquad u_t = X(u) = \Psi(u) \frac{\delta}{\delta u} \tfrac{1}{2} F_2(u) = (2u\partial + 2\partial u - \partial^3)\, u .$$

It is easily verified that $\Psi(u)$ satisfies (5.1.16), so it is a canonical operator. Note that (5.6.15) resembles very much a Hamiltonian system with Hamiltonian $\frac{1}{2}F_2$ and canonical operator Ψ . The fact that we did not prove that Ψ is invertible, prevents us from saying it is a Hamiltonian system. From the two possible ways of writing the KdV equation (5.6.6) and (5.6.15) we can obtain some interesting results.

5.6.16 Theorem.

Consider the operators $\Omega : S_2 \to U_2$, $\overleftarrow{\Omega} : U_2 \to S_2$ and $\Psi : U_2 \to S_2$ as given in (5.6.3), (5.6.4) and (5.6.14). Then Ψ and $\overleftarrow{\Omega}$ are AS operators and Ω is an SA operator (for the KdV equation).

Proof:
The Hamiltonian form (5.6.6) of the KdV equation implies (theorem 4.2.5) that Ω is an SA- and $\overleftarrow{\Omega}$ is an AS operator. The "semi-Hamiltonian form" (5.6.15) suggests that Ψ is also an AS operator. Indeed, even if Ψ is not invertible, we obtain from lemma 4.2.1 (with $\alpha = \frac{1}{2}dF_2$) that $L_X\Psi = 0$. Since Ψ does not depend explicitly on t this means that Ψ is also an AS operator.

□

5.6.17 Corollary.

i) $\Phi = \Omega\Psi\Omega = 2\partial^{-1}u + 2u\partial^{-1} - \partial : S_2 \to U_2$ is an SA operator,
ii) $\Lambda = \Psi\Omega = 2u + 2\partial u\partial^{-1} - \partial^2 : S_2 \to S_2$ is a recursion operator for symmetries,
iii) $\Gamma = \Omega\Psi = \Lambda^* = 2\partial^{-1}u\partial + 2u - \partial^2 : U_2 \to U_2$ is a recursion operator for adjoint symmetries.

□

The recursion operator for symmetries Λ is well-known. It seems first to be found by Lenard. Several other authors use this operator or derive it again, see for instance Olver [13] , Wadati [14] , Magri [5] , Calogero and Degasperis [34] or Gel'fand and Dikii [35]. Using the recursion operators Λ and Γ two infinite series of (adjoint) symmetries are easily constructed. We start with two symmetries, which are related to the invariance of solutions of (5.6.1) for translations along the x-axis and for a scale transformation. Suppose u(x,t) is a solution of (5.6.1). Then it is easily seen that $u(x+\varepsilon,t)$ and $a^2u(ax,a^3t)$ are also solutions of (5.6.1). By taking the limit for $\varepsilon \to 0$ of $u(x+\varepsilon,t) - u(x,t)$ and of $a^2u(ax,a^3t) - u(x,t)$ (with $a = 1+\varepsilon$) we obtain the following two solutions of the linearized KdV equation (linearization around u(x,t))

(5.6.18) $\quad X_o(u) = u_x$,

(5.6.19) $$Z_o(u,t) = \frac{1}{4}(2u + xu_x + 3tu_t) = \frac{1}{2}u + \frac{1}{4}xu_x + \frac{3}{4}t(6uu_x - u_{xxx}).$$

It is easily verified that X_o and Z_o satisfy (5.6.12) and that $X_o(u,t)$, $Z_o(u,t) \in S_2$ for all $u \in S_2$, $t \in \mathbb{R}$. So indeed we have two symmetries; $X_o, Z_o \in V(X; S_2)$. The factor $\frac{1}{4}$ in (5.6.19) may look strange, but turns out to be convenient in the sequel. The corresponding adjoint symmetries are

$$\rho_o = \Omega X_o = u, \ \tau_o = \Omega\ Z_o = \frac{1}{4}\partial^{-1}u + \frac{1}{4}xu + \frac{3}{4}t(3u^2 - u_{xx}).$$

Note that indeed $\rho_o(u,t)$, $\tau_o(u,t) \in U_2$. Using the recursion operators Λ and Γ we now obtain the following

5.6.20 Theorem.

Two infinite series of symmetries for the KdV equation are given by

$$X_k = \Lambda^k X_o, \ Z_k = \Lambda^k Z_o.$$

The corresponding adjoint symmetries are given by

$$\rho_k = \Omega X_k = \Gamma^k \rho_o, \quad \tau_k = \Omega\ Z_k = \Gamma^k \tau_o$$

$\square$

The first few elements of the series X_k and ρ_k are

$$X_1 = X = 6uu_x - u_{xxx},$$

$$\rho_1 = \partial^{-1}X_1 = 3u^2 - u_{xx} = \frac{\delta F_3}{\delta u},$$

(5.6.21) $$X_2 = 30u^2u_x - 20u_xu_{xx} - 10uu_{xxx} + u_{xxxxx},$$

$$\rho_2 = \partial^{-1}X_2 = 10u^3 - 5u_x^2 - 10uu_{xx} + u_{xxxx} = \frac{5}{2}\frac{\delta F_4}{\delta u}.$$

The first elements of the series Z_k and τ_k are

$$Z_1 = 2u^2 + \frac{1}{2} u_x \partial^{-1} u - u_{xx} + \frac{3}{2} xuu_x - \frac{1}{4} xu_{xxx} + \frac{3}{4} tX_2$$

$$= 2u^2 + \frac{1}{2} u_x \partial^{-1} u - u_{xx} + \frac{1}{4} xX_1 + \frac{3}{4} tX_2,$$

(5.6.22)

$$\tau_1 = \partial^{-1} Z_1 = \frac{3}{4} \partial^{-1}(u^2) + \frac{1}{2} u\partial^{-1} u - \frac{3}{4} u_x + \frac{3}{4} xu^2 - \frac{1}{4} xu_{xx} + \frac{3}{4} t\rho_2$$

$$= \frac{3}{4} \partial^{-1}(u^2) + \frac{1}{2} u\partial^{-1} u - \frac{3}{4} u_x + \frac{1}{4} x\rho_1 + \frac{3}{4} t\rho_2.$$

So these series of symmetries and adjoint symmetries depend explicitly on x and t.

5.6.23 Remark.

It is easily shown that the general form of Z_k and τ_k, as suggested by (5.6.19) and (5.6.22), is

$$Z_k(u,t) = f_k(u) + \frac{1}{4} xX_k(u) + \frac{3}{4} tX_{k+1}(u),$$

$$\tau_k(u,t) = g_k(u) + \frac{1}{4} x\rho_k(u) + \frac{3}{4} t\rho_{k+1}(u),$$

where f_k and g_k are functions which can be constructed using u, its derivatives and the operator ∂^{-1}. (So f_k and g_k may not contain x explicitly; a translation of u(x) along the x-axis must correspond to the same translation of $(f_k(u))(x)$ and $(g_k(u))(x)$ along the x-axis).

□

The "variational derivatives" of the constants of the motion F_1 and G are

(5.6.24) $$\frac{\delta F_1}{\delta u} = 1, \quad \frac{\delta G}{\delta u} = x + 6tu.$$

Both derivatives are not elements of U_2, which means that, strictly speaking, F_1 and G are not differentiable (in the choosen topology). The local conservation law corresponding to F_1 is

$$(5.6.25) \qquad u_t = (3u^2 - u_{xx})_x \quad .$$

Because $\int_{-\infty}^{\infty} (3u^2 - u_{xx})dx = 3F_2$, the flux of u in the local conservation law (5.6.25) is again a conserved quantity. Broer [25] has shown that, using this conserved flux property, a new constant of the motion can be constructed. This turns out to be G. In [25] the Poisson brackets between G and the series F_k are also given

$$(5.6.26) \qquad \{F_k, G\} = k\, F_{k-1} \, .$$

If we set $\rho_{-1} = \frac{1}{2}\frac{\delta F_1}{\delta u} = \frac{1}{2}$ and $\tau_{-1} = \frac{1}{8}\frac{\delta G}{\delta u} = \frac{1}{8}x + \frac{3}{4}tu$ then we can verify that ρ_{-1} and τ_{-1} satisfy (5.6.13) and that

$$(5.6.27) \qquad \rho_o = \Gamma\rho_{-1} \, , \; \tau_o = \Gamma\tau_{-1} \, .$$

The series of symmetries X_k is well-known, see for instance Lax [29], Olver [13], Magri [5] or Wadati [14]. The equations $u_t = X_k(u)$ are called *higher order Korteweg-de Vries equations*. The symmetries X_k are canonical and correspond to the constants of the motion F_k by $X_k = a_k\partial \frac{\delta F_{k+2}}{\delta u}$ ($a_k \in \mathbb{R}$). This means that the higher order KdV equations are also Hamiltonian systems. These results were first found by Gardner. In the sequel we shall also prove that the symmetries X_k are canonical. The series of symmetries Z_k, although easily found, has attracted much less attention. As far as we know, it is only reported by Olver [36]. This series is well suited to illustrate the theory, described in the sections 4.5 and 4.6, which we shall do now.

We first study the SA operators which correspond by the theorems 4.2.11 and 4.2.17 to the (adjoint) symmetries Z_o and Z_1 (τ_o and τ_1). Recall that an arbitrary symmetry $Z = \overleftarrow{\Omega}\tau$ gives rise to an SA operator (theorem 4.2.11)

$$L_Z\Omega = (\Omega' Z) + \Omega Z' + Z'^*\Omega$$

$$= \tau' - \tau'^* \quad .$$

Using $\tau_o' = \frac{1}{4}\partial^{-1} + \frac{1}{4}x + \frac{3}{4}t(6u - \partial^2)$ and $\tau_o'^* = -\frac{1}{4}\partial^{-1} + \frac{1}{4}x + \frac{3}{4}t(6u - \partial^2)$ we obtain

$$L_{Z_o}\Omega = \tfrac{1}{2}\partial^{-1} = \tfrac{1}{2}\Omega .$$

So we find again the already known SA operator Ω. This is not surprising since the symmetry Z_o corresponds to the scale properties of KdV. The symmetry Z_1 leads to a more interesting result. The derivative of τ_1 and its dual operator are

$$\tau_1' = \frac{3}{2}\,\partial^{-1}u + \frac{1}{2}(\partial^{-1}u) + \frac{1}{2}\,u\partial^{-1} - \frac{3}{4}\,\partial + \frac{3}{2}\,xu - \frac{1}{4}\,x\partial^2 + \frac{3}{4}\,t\rho_2',$$

$$\tau_1'^{*} = -\frac{3}{2}\,\partial^{-1}u + \frac{1}{2}(\partial^{-1}u) - \frac{1}{2}\,\partial^{-1}u + \frac{3}{4}\,\partial + \frac{3}{2}\,xu - \frac{1}{4}\,\partial^2 x + \frac{3}{4}\,t\rho_2'^{*}.$$

Since ρ_2 is canonical ($\rho_2 = \frac{5}{2}\frac{\delta F_2}{\delta u}$), we have $\rho_2' = \rho_2'^{*}$.

Hence

(5.6.28) $$L_{Z_1}\Omega = 2\partial^{-1}u + 2u\partial^{-1} - \partial = \Phi .$$

So we find the already known SA operator Φ . Because of the normalization factor in (5.6.19) the multiplicative constant in (5.6.28) is equal to 1. We can compute again the Lie derivative (theorem 4.2.17) and obtain the SA operator

$$\begin{aligned}
L^2_{Z_1}\Omega &= L_{Z_1}\Phi \\
&= 9\partial^{-1}u^2 + 9u^2\partial^{-1} + 6u\partial^{-1}u - 3\partial^{-1}u_{xx} - 3u_{xx}\partial^{-1} \\
&\qquad - 6\partial u - 6u\partial + \frac{3}{2}\,\partial^5 \\
&= \frac{3}{2}\,\Phi\,\overleftarrow{\Omega}\,\Phi \\
&= \frac{3}{2}\,(L_{Z_1}\Omega)\overleftarrow{\Omega}\,(L_{Z_1}\Omega).
\end{aligned} \tag{5.6.29}$$

This means that Z_1 satisfies hypothesis 4.5.1 with $c = \frac{3}{2}$. This hypothesis was essential for the theory described in the sections 4.5 and 4.6. As a first result we obtain from theorem 4.5.5 the following

5.6.30 Theorem.

The SA operators corresponding to the symmetries Z_k are given by

$$L_{Z_k}\Omega = \tfrac{1}{2}(k+1)\Gamma^k\Omega = \tfrac{1}{2}(k+1)\Omega\Lambda^k \qquad k = 0,1,2,\ldots \quad .$$

□

5.6.31 Corollary.

The (adjoint) symmetries $Z_k(\tau_k)$ are non-canonical for $k \geq 0$.

Proof:
It is easily seen that $\Gamma^k \neq 0$ for $k \geq 0$. So, by the preceding theorem, $L_{Z_k}\Omega \neq 0$. Then, using lemma 4.2.3 we see that Z_k cannot be canonical.

□

5.6.32 Corollary.

The SA operators $\Gamma^k\Omega = \Omega\Lambda^k$ are cyclic.

Proof:
The SA operator $L_{Z_k}\Omega$ is cyclic (theorem 4.2.11).

□

An infinite series of constants of the motion $\tilde{F}_k$ for the KdV equation is now easily constructed. (We use $\tilde{F}_k$ in stead of F_k since the normalization is different; the coefficient of u^k in F_k is assumed to be 1).

5.6.33 Theorem.

The (adjoint) symmetries $X_k(\rho_k)$ are canonical. The corresponding constants of the motion $\tilde{F}_k$, defined by

$$\frac{\delta\tilde{F}_{k+2}}{\delta u} = \rho_k = \Omega X_k \qquad k = 0,1,2,\ldots$$

are in involution, $\tilde{F}_3 = H$.

Proof:

From (5.6.22) and (5.6.21) we obtain that $Z_{1t} = \frac{3}{4} X_2 = \frac{15}{8} \Omega^{\leftarrow} \frac{\delta F_4}{\delta u}$, so Z_{1t} is a canonical symmetry. For $k \geq 1$ the theorem now follows from theorem 4.5.13. The case $k = 0$ (so $\tilde{F}_2$) has to be considered separately. A simple calculation shows that $\tilde{F}_2 = \frac{1}{2} \int_{-\infty}^{\infty} u^2 dx$ is a constant of the motion. The Poisson bracket

$$\{\tilde{F}_2, \tilde{F}_k\} = < \frac{\delta \tilde{F}_2}{\delta u}, \Omega\Gamma^k \frac{\delta \tilde{F}_2}{\delta u} >$$

vanishes since $\Gamma = \Omega\Psi$, and Ω and Ψ are antisymmetric. So the whole series $\tilde{F}_k$ $(k = 2,3,\ldots)$ is in involution.

□

5.6.34 Remark.

The reason that we have to consider $\tilde{F}_2$ separately is that in theorem 4.5.13 we constructed a series of constants of the motion, starting with the Hamiltonian $H = \tilde{F}_3$. In this case there also exists a constant of the motion $\tilde{F}_2$ "below" the Hamiltonian. We can also consider $\tilde{F}_1 = \frac{1}{2}F_1 = \frac{1}{2}\int_{-\infty}^{\infty} u dx$ as the first element of the series $\tilde{F}_k$. However, formally $\tilde{F}_1$ is not differentiable. If we still compute the corresponding symmetry we obtain

$$X_{-1} = \Omega^{\leftarrow} \frac{\delta \tilde{F}_1}{\delta u} = \partial \tfrac{1}{2} = 0 .$$

This would imply that the Poisson bracket of $\tilde{F}_1 = \frac{1}{2}F_1$ with every other function vanishes.

□

The coefficient of u^k in $\tilde{F}_k$ is found to be $\frac{(2k-3)!}{k!(k-2)!}$.

So if we set

$$F_k = \frac{k!(k-2)!}{(2k-3)!} \tilde{F}_k \qquad \text{for } k > 1$$

we obtain a series of constants of the motion such that the coefficient of u^k in F_k is equal to 1.

Next we consider the various possible Lie brackets.

5.6.35 Theorem.

The Lie brackets in and between the elements of the series X_k and Z_k for $k, \ell \geq 0$ are given by

$$[X_k, X_\ell] = 0,$$

$$[Z_k, X_\ell] = (\tfrac{1}{2}\ell + \tfrac{1}{4}) X_{k+\ell},$$

$$[Z_k, Z_\ell] = \tfrac{1}{2}(\ell - k) Z_{k+\ell}.$$

Proof:
It is easily verified that $[Z_1, X_1] = \frac{3}{4} X_2$. Then for $k, \ell \geq 1$ the theorem follows from theorem 4.6.6 (with $b = \frac{3}{4}$). For $k = 0$ or $\ell = 0$ the proof is also easily given, see remark 2.6.15.

□

Of course the fact that the symmetries of the series X_k commute follows also from the fact that the corresponding constants of the motion $\tilde{F}_{k+2}$ are in involution.

We now have described two methods for constructing the constants of the motion $\tilde{F}_k$ (or F_k). First we used a recursion operator for (adjoint) symmetries $\Lambda(\Gamma)$, viz. the construction described in the theorems 5.6.20 and 5.6.33. The second method consisted in generating the canonical symmetries X_k by using the Lie bracket with Z_1, see theorem 5.6.35. However, the most simple method for constructing the infinite series of constants of the motion is described in

5.6.36 Theorem.

The constant of the motion $F_k (k > 2)$ can be obtained from F_{k-1} by

$$F_k(u) = \frac{2k}{4(k-1)^2-1} L_{Z_1} F_{k-1}(u)$$

$$= \frac{2k}{4(k-1)^2-1} \int_{-\infty}^{\infty} \frac{\delta F_{k-1}}{\delta u} (2u^2 + \frac{1}{2} u_x \partial^{-1} u - u_{xx} + \frac{3}{2} xuu_x - \frac{1}{4} xu_{xxx}) dx.$$

Proof:
For k = 3 this result is easily verified. For k > 3 the first expression follows from corollary 4.6.14 (for the KdV equation $H = F_3 = \tilde{F}_3$). The normalization coefficient is easily found by considering the highest power of u. Using the expression for Z_1, as given in (5.6.22), we obtain

$$L_{Z_1} F_k = \langle \frac{\delta F_k}{\delta u}, Z_1 \rangle$$

$$= \langle \frac{\delta F_k}{\delta u}, 2u^2 + \frac{1}{2} u_x \partial^{-1} u - u_{xx} + \frac{3}{2} xuu_x - \frac{1}{4} xu_{xxx} + \frac{3}{4} tX_2 \rangle .$$

Since $\langle \frac{\delta F_k}{\delta u}, X_2 \rangle = \frac{1}{2} \{F_k, F_2\} = 0$ the term with explicit time dependence vanishes.

□

We shall now show that the KdV equation can also be considered as a Hamiltonian system with Hamiltonian $\frac{1}{2}(k+1)\ \tilde{F}_{k+3} (k=0,1,2,\ldots)$ and an appropriate (weak) symplectic form. Application of $\Gamma^k\Omega$ to (5.6.6) gives

$$\Gamma^k \Omega\ u_t = \Gamma^k \frac{\delta \tilde{F}_3}{\delta u} = \frac{\delta \tilde{F}_{k+3}}{\delta u} .$$

Using theorem 5.6.30 we obtain

$$(5.6.37) \qquad (L_{Z_k} \Omega) u_t = \frac{1}{2}(k+1) \frac{\delta \tilde{F}_{k+3}}{\delta u} .$$

The operator $L_{Z_k} \Omega$ is cyclic, it corresponds to the closed two-form $L_{Z_k} \omega = d\ i_{Z_k} \omega = d\tau_k$. If this two-form is (weakly) nondegenerate we can consider (5.6.37) as a Hamiltonian system with Hamiltonian $\frac{1}{2}(k+1)\tilde{F}_{k+3}$ and (weak) symplectic form $d\tau_k$. This raises the question of the invertability of (theorem 5.6.30)

$$L_{Z_k} \Omega = \frac{1}{2}(k+1)\ \Gamma^k \Omega = \frac{1}{2}(k+1)(\Omega\Psi)^k \Omega .$$

We first consider the operator Ψ . Our attempts to prove that $\Psi(u)$ is invertible were not sucessful. However, we can prove the following

5.6.38 Theorem.

Let $u \in S_2$. Then the linear operator $\Psi(u) : U_2 \to S_2$ is injective.

Proof:
Suppose there exists a function $w \in U_2$, $w \neq 0$ such that

(5.6.39) $$\Psi(u)w = 2u_x w + 4uw_x - w_{xxx} = 0 .$$

We shall show that this leads to a contradiction. After multiplication of (5.6.39) with w we can write this expression as

$$\frac{d}{dx}(2uw^2 - ww_{xx} + \tfrac{1}{2}w_x^2) = 0 .$$

Since $w \in U_2$ and $u \in S_2$ this implies

(5.6.40) $$2uw^2 - ww_{xx} + \tfrac{1}{2}w_x^2 = 0.$$

We shall first show that this implies that w cannot change sign on $\mathbb{R}$. Suppose $w(x_o) = 0$ for some $x_o \in \mathbb{R}$. Then (5.6.40) implies $w_x(x_o) = 0$. Suppose $w_{xx}(x_o) = 0$. Then, by considering (5.6.39) as an initial value problem with initial values $w(x_o) = 0$, $w_x(x_o) = 0$ and $w_{xx}(x_o) = 0$ and using the existence and uniqueness theorems for ordinary differential equations, we obtain $w = 0$ on $\mathbb{R}$, which is a contradiction. So $w_{xx}(x_o) > 0$ or $w_{xx}(x_o) < 0$, which means that w(x) cannot change sign on $\mathbb{R}$. It is no restriction to assume $w(x) \geq 0$ on $\mathbb{R}$. So if $w(x_o) = 0$ then $w_x(x_o) = 0$ and $w_{xx}(x_o) > 0$. Hence $w(x) \sim \tfrac{1}{2}w_{xx}(x_o)(x-x_o)^2$ for $x \to x_o$. This means that $\sqrt{w(x)}$ is continuous but not differentiable in $x = x_o$. Denote the number of zeros of w(x) between x and some point x_1 with $w(x_1) \neq 0$ by n(x). Then it is easily seen that

(5.6.41) $$z(x) = (-1)^{n(x)} \sqrt{w(x)}$$

is again a function with continuous derivatives. Substitution of $w(x) = z^2(x)$ in (5.6.40) results in

(5.6.42) $$-z_{xx} + uz = 0.$$

From $w \in U_2$ and $w(x) \geq 0$ for all $x \in \mathbb{R}$ we obtain $\lim_{x\to\pm\infty} w(x) = 0$. Then (5.6.41) implies

(5.6.43) $$\lim_{x\to\infty} z(x) = 0.$$

The solution z of (5.6.42) and (5.6.43) can be obtained from the following integral equation

(5.6.44) $$z(x) = \int_x^\infty (y-x)u(y)z(y)\, dy.$$

Since $u \in S_2$ the integral exists for every bounded continuous function z. Using a standard contraction argument we show that this equation can only have the trivial solution $z \equiv 0$. Since $u \in S_2$ there exists a real number $A > 0$ such that

(5.6.45) $$B = \int_A^\infty |u(y)|y\, dy < \tfrac{1}{2}.$$

Denote by $C[A,\infty)$ the space of bounded continuous functions on $[A,\infty)$. If we supply $C[A,\infty)$ with the uniform norm it is a Banach space. Define the linear operator $\Theta : C[A,\infty) \to C[A,\infty)$ by

$$(\Theta z)(x) = \int_x^\infty (y-x)u(y)z(y)\, dy.$$

It is easily seen that Θ is a contraction

$$\|(\Theta z)\| \leq \|z\| \int_A^\infty 2y|u(y)|\, dy \leq 2B\|z\| .$$

This means that Θ has only the fixed point $z = 0$. Hence (5.6.42) and (5.6.43) have only the solution $z(x) = 0$ on $[A,\infty)$ and so (uniqueness) $z(x) = 0$ on $\mathbb{R}$. Then (5.6.41) implies that $w(x) = 0$ on $\mathbb{R}$, which is again a contradiction. This completes the proof.

□

5.6.46 Remark.

It is easily seen that a real number A such that (5.6.45) is satisfied also exists for $u \in S_1$. So the theorem also holds if $u \in S_1$ and if we consider

$\Psi(u)$ as an operator $\Psi(u) : U_1 \to S_1$. If $u \notin S_1$ the theorem may be not correct. For instance with the functions

$$u(x) = \frac{2x^2-1}{(x^2+1)^2} \qquad \notin S_1,$$

$$w(x) = \frac{1}{1+x^2} \in U_1$$

we can verify that $\Psi(u)w = 2u_x w + 4uw_x + w_{xxx} = 0$.

□

5.6.47 Remark.

Let $u \in S_2$ be a function which can be obtained by the Miura transformation (5.6.9) from some smooth function v, so $u = v^2 + v_x$. Then it is easily verified that the operator $\Psi(u)$ can be factorized

$$\Psi(u) = 2u\partial + 2\partial u - \partial^3$$

$$= (2v+\partial)\partial(2v-\partial).$$

However, for an arbitrary $u \in S_2$ a function v such that $u = v^2+v_x$ has singularities on the x-axis. So this factorization cannot be used to prove injectivity or even invertability.

□

As a consequence of theorem 5.6.38 we have

5.6.48 Corollary.

The KdV equation can be considered as a Hamiltonian system with Hamiltonian $\frac{1}{2}(k+1)\ \tilde{F}_{k+3}$ and weak symplectic form $d\tau_k$.

Proof:
Since Ψ is injective and Ω is invertible we obtain from theorem 5.6.30 that $L_{Z_k}\Omega : S_2 \to U_2$ is also injective. So the corresponding two-form $L_{Z_k}\omega = d\tau_k$ is a weak symplectic form. The corollary now follows from (5.6.37).

□

Up to now we considered two infinite series of symmetries X_k and $Z_k(k = 0,1,2,\ldots)$ for the KdV equation. A completely different set of symmetries appears in the "inverse scattering method". We shall first describe the scattering and inverse scattering problems for the Schrödinger equation and indicate how the initial value problem for the KdV equation can be solved. Consider the *Schrödinger eigenvalue problem* on $\mathbb{R}$ with a function $u \in S_2$ as potential

$$(5.6.49) \qquad -y_{xx} + uy = \lambda y.$$

For $\lambda = k^2 > 0$ this problem has a continuous spectrum. Define the *Jost functions* $f(x,k)$ and $g(x,k)$ as the solutions of (5.6.49) with $\lambda = k^2$, such that

$$(5.6.50) \qquad \begin{cases} f(x,k) \sim e^{ikx} & \text{for } x \to \infty , \\ g(x,k) \sim e^{-ikx} & \text{for } x \to -\infty . \end{cases}$$

For $k \neq 0$ the pairs $f(x,k)$, $f(x,-k)$ and $g(x,k)$, $g(x,-k)$ form two fundamental systems of solutions. A solution of (5.6.49) which (in quantum mechanics) can be interpreted as a wave, coming from $-\infty$, which is partly reflected and partly transmitted, has the asymptotic behaviour

$$(5.6.51) \qquad \begin{cases} y(x,k) \sim e^{ikx} + R(k)e^{-ikx} & \text{for } x \to -\infty, \\ y(x,k) \sim T(k)\, e^{ikx} & \text{for } x \to \infty. \end{cases}$$

From (5.6.50) we see that this solution can be written as

$$(5.6.51a) \qquad y(x,k) = g(x,-k) + R(k)g(x,k) = T(k)f(x,k).$$

The complex functions R and T are called *reflection* and *transmission coefficient*. The eigenvalue problem (5.6.49) can also have a finite number of discrete (isolated) eigenvalues $\lambda_j = -\mu_j^2 < 0$ for $j = 1,\ldots,n(\mu_j > 0)$. We normalize the corresponding real eigenfunctions y_j by

$$\int_{-\infty}^{\infty} y_j^2(x)\, dx = 1.$$

We fix the sign of $y_j(x)$ by requiring $y_j(x) > 0$ as $x \to -\infty$. For every discrete eigenfunction y_j we define the normalization coefficient by

$$c_j = \lim_{x \to -\infty} e^{-2\mu_j x} y_j^2(x).$$

The set $\{R(k);\ \lambda_j,\ c_j \mid j = 1,\ldots,n\}$ will be called the *scattering data* of the potential u. The problem of reconstructing the potential u from the scattering data is called the *inverse scattering problem.* This problem was solved by Gel'fand and Levitan [21] and Kay and Moses [22]. First define the function $B : \mathbb{R} \to \mathbb{R}$ by

$$B(x) = \sum_{j=1}^{n} c_j e^{\mu_j x} + \frac{1}{2\pi} \int_{-\infty}^{\infty} R(k) e^{-ikx}\, dk. \tag{5.6.52}$$

Then solve the Gel'fand Levitan equation

$$K(x,y) + B(x+y) + \int_{-\infty}^{x} B(y+z) K(x,z)\, dz = 0 \qquad x > y.$$

The potential u can now be obtained from

$$u(x) = 2 \frac{d}{dx} K(x,x).$$

Next suppose the potential u satisfies the KdV equation (5.6.1). Then the scattering data and the (improper) eigenfunctions $f(x,k)$, $g(x,k)$, $y(x,k)$, $y_j(x)$ of (5.6.49) will also depend on t. The remarkable discovery of Gardner, Greene, Kruskal and Miura [19,20] is that, if the potential u of (5.6.49) evolves according to the KdV equation, the evolution of the scattering data is given by

$$\begin{cases} R_t(k,t) = -8\, ik^3 R(k,t), \\ \lambda_{j_t}(t) = 0, \\ c_{j_t}(t) = -8\, \mu_j^3(t) c_j(t) \qquad (\mu_j = \sqrt{-\lambda_j}) \qquad j = 1,\ldots,n. \end{cases} \tag{5.6.53}$$

The solution of these ordinary differential equations is trivial. The initial value problem for the KdV equation can now formally be solved. We first compute the scattering data of the initial value. The time evolution of the scattering data is given by (5.6.53). Then by "inverse scattering" we can find the solution u for arbitrary t. For future reference we also give the time evolution of the solutions of (5.6.49) (see for instance Eckhaus and van Harten [23, § 2.3.1])

$$
(5.6.54) \qquad \begin{cases} f_t = -4ik^3 f - u_x f + 2(u + 2k^2) f_x, \\ g_t = 4ik^3 g - u_x g + 2(u + 2k^2) g_x, \\ y_t = -4ik^3 y - u_x y + 2(u + 2k^2) y_x \end{cases}
$$

and

$$
(5.6.55) \qquad y_{j_t} = -u_x y_j + 2(u+2\lambda_j) y_{j_x} .
$$

5.6.56 Remark.

If u satisfies the KdV equation, the function $B(x,t)$, as given in (5.6.52) satisfies $B_t + 8\, B_{xxx} = 0$. This means that $w(x,t) = B(2x,t)$ satisfies

$$
(5.6.57) \qquad w_t + w_{xxx} = 0 .
$$

So the invertible mapping $u \to w$ is a linearizing transformation for the KdV equation. This means that the KdV equation is also "completely integrable" in the sense discussed in section 5.5. Note that (5.6.57) is also the equation obtained by linearizing the KdV equation around $u = 0$.

□

5.6.58 Remark.

If we want to express the dependence of the scattering data on the potential u, we have to write $\tilde{R}(k,u)$, $\tilde{\lambda}_j(u)$, $\tilde{c}_j(u)$ (and $n(u)$). However, it is usual in inverse scattering theory to consider the reflection coefficient as a function of k and t and the discrete eigenvalues with corresponding normalization coefficients as functions of t (where u is assumed to satisfy the KdV equation). Then

$$R_t(k,t) = \tilde{R}'(k,u)u_t, \quad \lambda_{j_t}(t) = \tilde{\lambda}_j'(u)u_t \text{ and } c_{j_t}(t) = \tilde{c}_j'(u)u_t.$$

If we consider symmetries Y and adjoint symmetries σ also as functions of x and t, they have to satisfy (see (5.6.12) and (5.6.13))

$$Y_t(x,t) - (6\partial u(x,t) - \partial^3)\, Y(x,t) = 0, \tag{5.6.59}$$

$$\sigma_t(x,t) + (-6u(x,t)\partial + \partial^3)\, \sigma(x,t) = 0. \tag{5.6.60}$$

□

It is well-known from first order perturbation theory in quantum mechanics that an infinitesimal change δu in the potential u of the Schrödinger equation (5.6.49) leads to changes in the discrete eigenvalues and reflection coefficient given by

$$\delta\lambda_j = \int_{-\infty}^{\infty} y_j^2(x)\delta u(x)\, dx,$$

$$\delta R(k) = \frac{1}{2ik} \int_{-\infty}^{\infty} y^2(x,k)\delta u(x)\, dx.$$

This implies

$$\frac{\delta\lambda_j}{\delta u} = y_j^2(x) \qquad j = 1,\dots,n, \tag{5.6.61}$$

$$\frac{\delta R(k)}{\delta u} = \frac{1}{2ik}\, y^2(x,k) \qquad k \neq 0\,. \tag{5.6.62}$$

Since y_j and all its x derivatives vanish exponentially for $|x| \to \infty$ we have $y_j \in S_2$. So $\frac{\delta\lambda_j}{\delta u} = y_j^2 \in S_2 \subset U_2$. The asymptotic behaviour of $y(x,k)$ for $|x| \to \infty$, as given in (5.6.51), implies that $\frac{\delta R(k)}{\delta u} \notin U_2$. So formally R(k) is not differentiable (in the topology of S_2). From (5.6.53) we see that a discrete eigenvalue λ_j is a constant of the motion and that

$$\frac{\partial}{\partial t}\left(e^{8ik^3t}\, R(k,t)\right) = 0.$$

This leads to

5.6.63 Theorem.

i) The functions $\sigma_j = y_j^2$ ($j=1,\dots,n$) are canonical adjoint symmetries corresponding to the constants of the motion λ_j; so they satisfy (5.6.60). Further

(5.6.64) $$\Gamma\, y_j^2 = 4\lambda_j y_j^2.$$

ii) For $k \neq 0$ the functions $\zeta_1(x,k,t) = e^{8ik^3t} y^2(x,k,t)$,

$$\zeta_2(x,k,t) = e^{8ik^3t} f^2(x,k,t) \text{ and } \zeta_3(x,k,t) = e^{-8ik^3t} g^2(x,k,t)$$

satisfy (5.6.60) and

(5.6.65) $$\Gamma\, \zeta_m(x,k,t) = 4k^2\, \zeta_m(x,k,t) \qquad m = 1,2,3.$$

Proof:
The discrete eigenvalues λ_j are constants of the motion, so their variational derivatives are adjoint symmetries. Multiplication of (5.6.49) with y_{j_x} and application of $4\partial^{-1}$ yields

$$-2y_{j_x}^2 + 4\,\partial^{-1}(uy_j y_{j_x}) = 2\lambda_j y_j^2 ,$$

while multiplication of (5.6.49) with $2y_j$ gives

$$-2y_j y_{j_{xx}} + 2uy_j^2 = 2\lambda_j y_j^2 .$$

Then (5.6.64) is obtained by adding these two expressions. The fact that the functions ζ_m ($m=1,2,3$) satisfy (5.6.60) follows from a straightforward computation using (5.6.54). The proof of (5.6.65) is similar to the proof of (5.6.64).

□

Although $\zeta_1(.,k,t) = 2ik\,\frac{\delta}{\delta u}(e^{8ik^3t} R(k,t))$ we do not call ζ_1 the canonical adjoint symmetry corresponding to $e^{8ik^3t} R(k,t)$. The reason for this is that $\zeta_1(.,k,t) \notin \mathcal{U}_2$. Also (asymptotic behaviour) $\zeta_2, \zeta_3(.,k,t) \notin \mathcal{U}_2$.

Apart from this difference the two parts of the theorem claim similar results for the squares of the eigenfunctions of the Schrödinger equation (5.6.49). The fact that $\sigma_j (j=1,\ldots,n)$ and ζ_1 satisfy (5.6.60) is already given by Gardner, Greene, Kruskal and Miura [20, theorem 3.6]. However, as far as we know the interpretation of σ_j as canonical adjoint symmetry is new. The relations (5.6.64) and (5.6.65) for the "squared eigenfunctions" are also well-known. Of course $\partial\sigma_j (j=1,\ldots,n)$ and $\partial\zeta_m (m=1,2,3)$ satisfy (5.6.59) and $\partial\sigma_j$ is a canonical symmetry. These functions are also eigenfunctions of the recursion operator for symmetries Λ

$$\text{(5.6.66)} \qquad \Lambda\partial\sigma_j = 4\lambda_j\partial\sigma_j \qquad j = 1,\ldots,n,$$

$$\text{(5.6.67)} \qquad \Lambda\partial\zeta_m = 4k^2\partial\zeta_m \qquad m = 1,2,3 \quad .$$

Recall that at the end of the sections 2.3 and 2.4 we showed that under certain conditions (which we shall not verify here) the eigenvalues of recursion operators for symmetries and for adjoint symmetries are constants of the motion. An example of this situation is given by (5.6.64) and (5.6.66).

We now indicate how a second solution of (5.6.60), corresponding to a discrete eigenvalue λ_j, can be constructed. The Jost functions $f(x,k)$ and $g(x,k)$ can be continued analytically into the upper half of the complex k-plane. In $k=i\mu_j$ we have (for a moment we omit t)

$$g(x,i\mu_j) \sim e^{\mu_j x} \quad \text{for } x \to -\infty \; .$$

A solution $h_j(x)$ of (5.6.49) with $\lambda = -\mu_j^2$ which is independent of $g(x,i\mu_j)$, must have asymptotic behaviour $h(x) \sim e^{-\mu_j x}$ for $x \to -\infty$. Then, by considering the behaviour for $x \to -\infty$ we see that the solution $y_j(x)$ can be written as

$$y_j(x) = \sqrt{c_j}\; g(x,i\mu_j) \; .$$

This means the canonical adjoint symmetry σ_j can be written as

$$\sigma_j(x,t) = y_j^2(x,t) = c_j(t)\; g^2(x,i\mu_j,t).$$

We now consider the derivative of $g(x,k,t)$ with respect to k. The time evolution of this function in $k = i\mu_j$ follows from (5.6.54)

$$g_{kt} = 4\mu_j^3 g_k - u_x g_k + 2(u-2\mu_j^2) g_{kx}$$

(5.6.68)

$$- 12i\mu_j^2 g - 8i\mu_j g_x \quad .$$

Then a long but straightforward computation, using (5.6.53), (5.6.54), (5.6.68) and (derivatives with respect to x and k in $k = i\mu_\ell$ of) the Schrödinger equation (5.6.49) shows that

(5.6.69) $$\hat{\sigma}_j(x,t) = ic_j(t)g(x,i\mu_j,t)g_k(x,i\mu_j,t) - 12\mu_j^2 t\sigma_j(x,t)$$

satisfies (5.6.60). It can be shown that $\hat{\sigma}_j$ is a real function with asymptotic behaviour

$$\hat{\sigma}_j(x,t) \sim c_j(t)xe^{2\mu_j x} \qquad \text{for } x \to -\infty \quad ,$$

$$\hat{\sigma}_j(x,t) \sim 1 \qquad \text{for } x \to \infty.$$

So $\hat{\sigma}_j \notin \mathcal{U}_2$ which means that we cannot call $\hat{\sigma}_j$ an adjoint symmetry. Using derivatives of (5.6.49) with respect to x and k it is a simple exercise to show that

$$\Gamma\hat{\sigma}_j = - 4\mu_j^2\hat{\sigma}_j - 4\mu_j\sigma_j \qquad j = 1,\dots,n.$$

Thus, related with the "inverse scattering method", we constructed the following solutions of (5.6.60):

i) continuous spectrum $\lambda = k^2$, $k \in \mathbb{R}\setminus\{0\}$

$$\zeta_1(x,k,t) = e^{8ik^3 t} y^2(x,k,t),$$

$$\zeta_2(x,k,t) = e^{8ik^3 t} f^2(x,k,t),$$

$$\zeta_3(x,k,t) = e^{-8ik^3t} g^2(x,k,t),$$

with $\Gamma\zeta_m = 4k^2\zeta_m \qquad (m=1,2,3),$

ii) discrete spectrum $\lambda_j = -\mu_j^2$, $j = 1,\dots,n$

$$\sigma_j(x,t) = y_j^2(x,t) = c_j(t)g^2(x,i\mu_j,t),$$

$$\hat{\sigma}_j(x,t) = ic_j(t)g(x,i\mu_j,t)g_k(x,i\mu_j,t) - 12\mu_j^2t\sigma_j(x,t),$$

with $\Gamma\sigma_j = -4\mu_j^2\sigma_j$,

$$\Gamma\hat{\sigma}_j = -4\mu_j^2\hat{\sigma}_j - 4\mu_j\sigma_j .$$

It follows from (5.6.51a) that $\zeta_1(x,k,t) = T^2(k)\,\zeta_2(x,k,t)$. A more profound study of the inverse scattering method shows that a infinitesimal variation δu (smooth, fast decaying as $|x|\to\infty$) can be written in terms of ζ_3, σ_j and $\hat{\sigma}_j$. See for instance Zakharov and Faddeev [24, the first expression in § 2] . This enables us to express the symmetries X_k and Z_k, which we studied in the first part of this section, in terms of ζ_3,σ_j and $\hat{\sigma}_j$. We only give the formal result

$$X_o(x,t) = u_x(x,t)$$

(5.6.70)
$$= \frac{\partial}{\partial x}\Big[\frac{2i}{\pi}\int_{-\infty}^{\infty} kR(k,t)e^{8ik^3t}\,\zeta_3(x,k,t)dk - 4\sum_{j=1}^{n}\mu_j\sigma_j(x,t)\Big],$$

$$Z_o(x,t) = 2u(x,t) + xu_x(x,t) + 3t\,u_t(x,t)$$

$$= \frac{\partial}{\partial x}\Big[\frac{1}{\pi}\int_{-\infty}^{\infty}(kR_k(k,t) + 24ik^3tR(k,t))\,e^{8ik^3t}\,\zeta_3(x,k,t)dk$$

(5.6.71)
$$- \sum_{j=1}^{n}(2\sigma_j(x,t) + 4\mu_j\hat{\sigma}_j(x,t)\Big].$$

The expression (5.6.70) has already been given (in a somewhat different form) by Deift and Trubowitz [37] . By applying the recursion operator Λ (Γ inside the square brackets) we can obtain similar expressions for X_k

and Z_k for $k = 1,2,3,\ldots$.

We end this section by making some remarks on the higher order KdV equations. Denote the "time independent part" of the symmetries Z_k by A_k, so

$$(5.6.72) \qquad A_k = Z_k - \tfrac{3}{4}t\, X_{k+1} = \Lambda^k(\tfrac{1}{2}\, u + \tfrac{1}{4}\, xu_x) \qquad k = 0,1,2,\ldots$$

Then from theorem 5.6.35 we get

$$(5.6.73) \qquad \begin{cases} [A_k, X_\ell] = (\tfrac{1}{2}\ell+\tfrac{1}{4})X_{k+\ell} , \\ \\ [A_k, A_\ell] = \tfrac{1}{2}(\ell-k)A_{k+\ell} . \end{cases}$$

Some properties of higher order KdV equations are described in the following

5.6.74 <u>Theorem.</u>

Consider in S_2 the higher order KdV equation

$$(5.6.75) \qquad u_t = X_m(u) \qquad , \; m = 1,2,3,\ldots .$$

Then

i) this equation is a Hamiltonian system with Hamiltonian $\tilde{F}_{m+2}$ and symplectic form ω

$$u_t = X_m(u) = \Omega^{\leftarrow} \frac{\delta \tilde{F}_{m+2}}{\delta u} ,$$

ii) the functions (functionals) $\tilde{F}_k$ (or F_k) are also constants of the motion for this higher order KdV equation,

iii) the operator $\Lambda(\Gamma)$ is a recursion operator for (adjoint) symmetries of (5.6.75),

iv) two infinite series of symmetries for (5.6.75) are

$$X_k = \Omega^{\leftarrow} \frac{\delta \tilde{F}_{k+2}}{\delta u} \quad \text{(independent of m)},$$

$$U_{m,k} = A_k + (\tfrac{1}{2}m + \tfrac{1}{4})t\, X_{k+m} \qquad k = 0,1,2,\ldots \quad .$$

So X_k, $U_{m,k} \in V(X_m,S_2)$. The symmetries X_k are canonical while the $U_{m,k}$ are non-canonical. The Lie brackets between elements of these series are given by

$$(5.6.76)\qquad \begin{cases} [X_k,X_\ell] = 0, \\ [U_{m,k},X_\ell] = (\frac{1}{2}\ell + \frac{1}{4})X_{k+\ell}, \\ [U_{m,k},U_{m,\ell}] = \frac{1}{2}(\ell-k)U_{m,k+\ell} \qquad k,\ell = 0,1,2,\ldots \quad . \end{cases}$$

Proof:
Part i) and ii) follow at once from theorem 5.6.33. Theorem 4.6.11 yields that $L_{X_m}\Lambda = 0$. Since Λ does not depend explicitly on t this implies that Λ is a recursion operator for symmetries of (5.6.75). Using (5.6.73) we obtain

$$\frac{\partial}{\partial t}(U_{m,k}) + [X_m,U_{m,k}] = 0$$

so $U_{m,k}$ is a symmetry for (5.6.75). The Lie brackets given in (5.6.76) follow immediately from (5.6.73).

□

Note that the structure of the Lie algebra of symmetries $\{X_k, U_{m,k}, k = 0,1,2,\ldots\}$ of (5.6.75) does not depend on m. For the KdV equation itself (m=1) this Lie algebra is already described in theorem 5.6.35.

5.7 THE SAWADA-KOTERA EQUATION.

In this section we consider an equation of "KdV type" found by Sawada and Kotera [38] and also by Caudrey, Dodd and Gibbon [39]. We study this so called Sawada-Kotera (SK) equation in the form

$$(5.7.1) \qquad u_t = X(u) = 180u^2u_x + 30u_xu_{2x} + 30uu_{3x} + u_{5x}, \qquad x \in \mathbb{R}$$

where $u_{nx} = \partial^n u$. The SK equation is essentially different from the higher order KdV equation $u_t = X_2(u)$ in the notation of the preceding section. This equation reads

$$(5.7.2) \qquad u_t = 30u^2u_x - 10uu_{3x} - 20uu_{2x} + u_{5x}.$$

Of course the coefficients of both equations can be changed by scale transformations of x, t and u. However, it is impossible to transform (5.7.1) into (5.7.2) by a scale transformation. It is shown in [39] that (5.7.1) and (5.7.2) are the only equations of this type which have multi-soliton solutions. We shall consider the SK equation in the space S_p ($p = 1,2,\dots$) with the topology induced by U_p and the usual duality map. In this section we study symmetries and constants of the motion of the SK equation. We also make some remarks on the "inverse scattering problem" for (5.7.1). For the SK equation there exists a series of constants of the motion F_k. The first few elements of this series are given by

$$(5.7.3) \qquad \begin{aligned} F_1 &= \int_{-\infty}^{\infty} u\,dx\,, \quad F_3 = \frac{1}{2}\int_{-\infty}^{\infty} (2u^3 - u_x^2)\,dx\,, \\ F_4 &= \frac{1}{12}\int_{-\infty}^{\infty} (12u^4 - 18uu_x^2 + u_{2x}^2)\,dx\,, \\ F_6 &= \frac{1}{576}\int_{-\infty}^{\infty} (576u^6 - 3600u^3u_x^2 - 204u_x^4 + 576u^2u_{2x}^2 + \\ &\qquad + 32u_{2x}^3 - 42uu_{3x}^2 + u_{4x}^2)\,dx\,. \end{aligned}$$

A constant of the motion of a different type is given by

$$G = \int_{-\infty}^{\infty} xu\,dx + 60t\,F_3. \tag{5.7.4}$$

The SK equation (and also (5.7.2)) is invariant for the scale transformation $u(x,t) \to a^2u(ax,a^5t)$. Under this scale transformation the constants of the motion F_k are proportional to a^{2k}. It appears that constants of the motion of the type F_{3k+2} (with densities which are polynomials in u and its derivatives) do not exist. For $k = 0$ this is easily verified. Using a computer program (formula manipulation) it can be shown that also F_5, F_8 and F_{11} do not exist. In the sequel we shall describe several methods to obtain F_{k+3} from F_k. Then, starting with F_1 and F_3 we can construct the series F_{3k+1} and F_{3k+3} for $k = 1,2,3,\ldots$. Of course this does not exclude the possibility that a constant of the motion F_{3k+2} exists for some k ($k > 3$).

Symmetries $Y(u,t)$ and adjoint symmetries $\sigma(u,t)$ of the SK equation have to satisfy (see (5.1.3) and (5.1.4))

$$Y_t(u,t) + Y'(u,t)\,X(u) - X'(u)\,Y(u,t) = 0, \tag{5.7.5}$$

$$\sigma_t(u,t) + \sigma'(u,t)\,X(u) + X'^*(u)\,\sigma(u,t) = 0 \tag{5.7.6}$$

with

$$X'(u) = 180\partial u^2 + 30\partial^3 u - 60\partial u_x\partial + \partial^5 \;:\; S_p \to S_p,$$

$$X'^*(u) = -180u^2\partial - 30u\partial^3 - 60\partial u_x\partial - \partial^5 \;:\; U_p \to U_p.$$

Two "semi-Hamiltonian forms" of the SK equation have been found by Broer and ten Eikelder [40] and also by Fuchssteiner and Oevel [41]. Define the antisymmetric operators (in fact operator fields) $\overleftarrow{\Omega}$ and Φ by

$$\overleftarrow{\Omega}(u) = 12u\partial + 12\partial u + \partial^3 \;:\; U_p \to S_p, \tag{5.7.7}$$

$$\Phi(u) = 6\partial^2 u\partial^{-1} + 6\partial^{-1}u\partial^2 + 18\partial^{-1}u^2 + 18u^2\partial^{-1} + \partial^3 \;:\; S_p \to U_p. \tag{5.7.8}$$

It is easily seen that the SK equation (5.7.1) can be written as

$$(5.7.9) \qquad u_t = \overleftarrow{\Omega}(u)\ \frac{\delta F_3}{\delta u} = (12u\partial + 12\partial u + \partial^3)(3u^2 + u_{xx}).$$

A simple calculation shows that $\overleftarrow{\Omega}$ satisfies (5.1.16), so it is a canonical operator. Note that, up to a scale transformation, $\overleftarrow{\Omega}$ corresponds to the operator Ψ, as given in the preceding section in (5.6.14). So from theorem 5.6.38 we obtain that $\overleftarrow{\Omega}$ is injective. This property is not sufficient to call (5.7.9) a Hamiltonian system. However, using lemma 4.2.1 we obtain from the "semi-Hamiltonian form" (5.7.9) that $L_X\overleftarrow{\Omega} = 0$. Since $\overleftarrow{\Omega}$ does not depend explicitly on t, this means that $\overleftarrow{\Omega}$ is a (canonical) AS operator. Another "semi-Hamiltonian form" of the SK equation is obtained by applying Φ to (5.7.1). This results in

$$(5.7.10) \qquad \Phi(u)\ u_t = 288\ \frac{\delta F_6}{\delta u}.$$

It can be verified that Φ satisfies (5.1.14), so it is a cyclic operator. This means that the two-form ϕ defined by

$$(5.7.11) \qquad \phi(A,B) = \langle\Phi(u)A,B\rangle \qquad \forall\ A,B \in S_p$$

is closed. This two-form is (weakly) nondegenerate iff Φ is invertible (injective). In that case we can consider (5.7.10) as a Hamiltonian system with Hamiltonian 288 F_6 and (weak) symplectic form ϕ. However, we shall not try to prove that $\Phi(u)$ is injective or even invertible. The "semi-Hamiltonian form" (5.7.10) suggests that Φ is an SA operator. Indeed a long but straightforward computation shows that

$$L_X\Phi = (\Phi'X) + \Phi X' + X'^*\Phi = 0.$$

Since Φ does not depend explicitly on t this implies that Φ is an SA operator. Hence we have proved the following

5.7.12 Theorem.

The operator $\overleftarrow{\Omega}$, as given in (5.7.7) is an AS operator. The operator Φ, defined in (5.7.8) is an SA operator. Further $\Lambda = \overleftarrow{\Omega}\Phi : S_p \to S_p$ is a recursion operator for symmetries and $\Gamma = \Phi\overleftarrow{\Omega} : U_p \to U_p$ is a recursion operator for adjoint symmetries.

□

5.7.13 Remark.

Note that, although we have given the "two semi-Hamiltonian forms" (5.7.9) and (5.7.10), we did not prove that the SK equation is a Hamiltonian system. This means that we cannot make straightforward use of the results and definitions of chapter 4 (in particular the sections 4.5 and 4.6). However, a number of results can be obtained by using similar techniques as in chapter 4. We shall adopt the definitions of canonical and non-canonical adjoint symmetries, as given in definition 4.2.7, also for this case, with the restriction that canonical/non-canonical is only defined for symmetries Y which can be written as $Y = \overleftarrow{\Omega}\sigma$. Also we shall use the notion of Poisson bracket (with canonical operator $\overleftarrow{\Omega}$) as explained in section 3.3. Note that we gave a proof of the Jacobi identity (theorem 3.3.3) in which we only used that $\overleftarrow{\Omega}$ is canonical.

□

The "variational derivatives" of F_1 and G are given by

$$\frac{\delta F_1}{\delta u} = 1 \notin U_p, \quad \frac{\delta G}{\delta u} = x + 60t(3u^2 + u_{xx}) \notin U_p.$$

This means that F_1 and G are not differentiable (in the choosen topology). However, if we set

$$\sigma_o = 1 \left(= \frac{\delta F_1}{\delta u}\right), \quad \tau_o = \frac{1}{72}x + \frac{5}{6}t(3u^2 + u_{xx}) \left(= \frac{1}{72}\frac{\delta G}{\delta u}\right),$$

then σ_o and τ_o satisfy (5.7.6). The factor $\frac{1}{72}$ turns out to be convenient in the remaining part of this section. Application of $\overleftarrow{\Omega}$ results in

(5.7.13.a)
$$Y_o = \overleftarrow{\Omega}\sigma_o = 12u_x \in S_p,$$
$$Z_o = \overleftarrow{\Omega}\tau_o = \frac{1}{6}(2u + xu_x + 5tX(u)) \in S_p.$$

It is easily seen that Y_o and Z_o satisfy (5.7.5), so they are symmetries of the SK equation. Note that the symmetry Z_o corresponds to the scale transformation $u(x,t) \to a^2u(ax,a^5t)$ of the SK equation. By applying the SA operator Φ to Y_o and Z_o we obtain the adjoint symmetries

$$\sigma_1 = \Phi Y_o \; (= \Gamma\sigma_o) = 72 \frac{\delta F_4}{\delta u} \quad \text{and} \quad \tau_1 = \Phi Z_o \; (= \Gamma\tau_o).$$

Three infinite series of (adjoint) symmetries are constructed in

5.7.14 Theorem.

The series

(5.7.15) $$\rho_k = \Gamma^{k-1} \frac{\delta F_3}{\delta u}, \quad \sigma_k = \Gamma^{k-1}\sigma_1, \quad \tau_k = \Gamma^{k-1}\tau_1, \quad k = 1,2,3,\ldots$$

consist of adjoint symmetries of the SK equation. The corresponding symmetries are given by

$$X_k = \overleftarrow{\Omega}\rho_k = \Lambda^{k-1}X_1 = \Lambda^k X_o \qquad (X_1 = X),$$

$$Y_k = \overleftarrow{\Omega}\sigma_k = \Lambda^{k-1}Y_1 = \Lambda^k Y_o,$$

$$Z_k = \overleftarrow{\Omega}\tau_k = \Lambda^{k-1}Z_1 = \Lambda^k Z_o.$$

Proof:

This theorem is a straightforward consequence of the fact that $\rho_1 = \frac{\delta F_3}{\delta u}$, σ_1 and τ_1 are adjoint symmetries and that Γ is a recursion operator for adjoint symmetries.

□

We shall show that the series $X_k(\rho_k)$ and $Y_k(\sigma_k)$ consist of canonical (adjoint) symmetries which correspond to the constants of the motion F_{3k} and F_{3k+1}. The (adjoint) symmetries $Z_k(\tau_k)$ turn out to be non-canonical for $k \geq 1$. Since the adjoint symmetry τ_1 and the symmetry $Z_1 = \overleftarrow{\Omega}\tau_1$ are essentially for the following considerations, we give τ_1 explicitly

$$(5.7.16)\qquad \tau_1 = 5\partial^{-1}(u^3) + 3u^2\partial^{-1}u - \frac{5}{2}\partial^{-1}(u_x{}^2) + u_{2x}\partial^{-1}u + 10uu_x +$$

$$+ \frac{5}{6}u_{3x} + x(4u^3 + \frac{3}{2}u_x{}^2 + 3uu_{xx} + \frac{1}{6}u_{4x}) + 240t\,\frac{\delta F_6}{\delta u}$$

$$= \alpha_1 + 240t\,\frac{\delta F_6}{\delta u}\ .$$

It is easily seen that $\tau_1{}' \neq \tau_1{}'^*$, which implies that τ_1 is a non-canonical adjoint symmetry and Z_1 is a non-canonical symmetry. The same property holds for the other elements of the series $Z_k(\tau_k)$.

5.7.17 Theorem.

The (adjoint) symmetries $Z_k(\tau_k)$ are non-canonical for $k \geq 1$.

Proof:
This theorem is proved by considering the terms of τ_k which do not depend explicitly on t and which are proportional to b under the transformation $u \to bu$. For τ_1 these terms are

$$\frac{5}{6}u_{3x} + \frac{1}{6}xu_{4x}.$$

The only term in the recursion operator Γ which generates again terms of this type is the operator ∂^6. So the terms of this type in τ_k are given by

$$\partial^{6(k-1)}(\frac{5}{6}u_{3x} + \frac{1}{6}xu_{4x}) = (k-\frac{1}{6})u_{(6k-3)x} + \frac{1}{6}xu_{(6k-2)x}$$

$$k = 1,2,3,\dots\ .$$

This implies that $\tau_k{}' \neq \tau_k{}'^*$ which means that τ_k and hence $Z_k = \overleftarrow{\Omega}\tau_k$ are non-canonical for $k \geq 1$.

□

Notice that the terms which contain x explicitly in τ_1 can be written as $x\,\frac{\delta F_4}{\delta u}$ (see remark 5.6.23 for a similar property of the non-canonical symmetries of the KdV equation).

By theorem 2.5.15 i the operator $L_Z\overleftarrow{\Omega}$ is again an AS operator and the operator $L_Z\Phi$ is again an SA operator. A very long computation shows that

(5.7.18) $$L_{Z_1}\overleftarrow{\Omega} = -\overleftarrow{\Omega}\Phi\overleftarrow{\Omega} \ ,$$

(5.7.19) $$L_{Z_1}\Phi = 2\Phi\overleftarrow{\Omega}\Phi \ .$$

Suppose for a moment the inverse operator Ω of $\overleftarrow{\Omega}$ exists. Then (5.7.18) implies that $\Phi = L_{Z_1}\Omega$ and (5.7.19) can be written as

$$L_{Z_1}^2\Omega = 2(L_{Z_1}\Omega)\overleftarrow{\Omega}L_{Z_1}\Omega.$$

This would imply that Z_1 satisfies the conditions of hypothesis 4.5.1 with $c = 2$ and so the theory described in the sections 4.5 and 4.6 could be applied. However, since Ω is not known, and even possibly does not exist, a straightforward application of this theory is not possible. Therefore we shall show that the series X_k and Y_k consist of canonical symmetries using methods which differ slightly from those in section 4.5. The following theorem can be compared with theorem 4.5.5.

5.7.20 Theorem.

The SA operators $L_{Z_k}\Phi$ are given by

$$L_{Z_k}\Phi = (k+1)(\Phi\overleftarrow{\Omega})^k\Phi = (k+1)\Gamma^k\Phi \qquad k = 1,2,3,\ldots \ .$$

Proof:
See the proof of theorem 4.5.5 with Ω replaced by Φ.

□

The SA operator $L_{Z_k}\Phi$ corresponds to the two-form $L_{Z_k}\phi$, with ϕ given in (5.7.11). The closedness of ϕ implies that $L_{Z_k}\phi$ is also closed and hence that the SA operator $L_{Z_k}\Phi$ is cyclic. Then theorem (5.7.20) implies the (nontrivial) result that the SA operators $\Gamma^k\Phi$ ($k = 1,2,3,\ldots$) are also cyclic. Next we consider the symmetries X_k and Y_k.

5.7.21 Theorem.

The symmetries $X_k = \overleftarrow{\Omega}\rho_k$, as introduced in theorem 5.7.14 are canonical. The corresponding constants of the motion $\tilde{F}_{3k}$, defined by

$$\frac{\delta\tilde{F}_{3k}}{\delta u} = \rho_k = \Gamma^{k-1}\frac{\delta F_3}{\delta u} \qquad k = 1,2,3,\ldots \tag{5.7.22}$$

are in involution and do not depend explicitly on t.

Proof:

See the proofs of theorem 4.5.10 (with Ω replaced by Φ and H by F_3) and 4.5.13 (with $L_Z\Omega$ replaced by Φ and H by F_3).

□

5.7.23 Theorem.

The symmetries $Y_k = \overleftarrow{\Omega}\sigma_k$, as introduced in theorem 5.7.14 are canonical. The corresponding constants of the motion $\tilde{F}_{3k+1}$, defined by

$$\frac{\delta\tilde{F}_{3k+1}}{\delta u} = \sigma_k = \Gamma^{k-1}\sigma_1 = 72\Gamma^{k-1}\frac{\delta F_4}{\delta u} \qquad k = 1,2,3,\ldots \tag{5.7.24}$$

are in involution and do not depend explicitly on t. The Poisson bracket between the constants of the motion $\tilde{F}_{3k+1}$ and $\tilde{F}_{3\ell}$ also vanish.

Proof:

The proof of this theorem is somewhat different from the proofs of the preceding theorems, therefore we give it completely. We first show that the adjoint symmetries σ_k are canonical. Using $\sigma_1 = \Phi Y_o$ and theorem 5.7.20 we can write

$$\sigma_k = \Gamma^{k-1}\Phi Y_o = \frac{1}{k}\,(L_{Z_{k-1}}\Phi)Y_o \qquad k = 2,3,\ldots\ .$$

The SA operator $L_{Z_{k-1}}\Phi$ is cyclic, it corresponds to the closed two-form $L_{Z_{k-1}}\phi$. So we can apply lemma 4.5.3 (with $A := Y_o$, $\phi := L_{Z_{k-1}}\phi$ and $\Phi := L_{Z_{k-1}}\Phi$). This yields

$$d\sigma_k(B_1,B_2) = \frac{1}{k} < (L_{Y_o}L_{Z_{k-1}}\Phi)B_1\ ,\ B_2 >.$$

By again using theorem 5.7.20 this becomes

$$(5.7.25)\qquad d\sigma_k(B_1,B_2) = \langle\, (L_{Y_0}(\Gamma^{k-1}\Phi))B_1 \,,\, B_2 \,\rangle .$$

A simple calculation shows that $L_{Y_0}\overleftarrow{\Omega}$ and $L_{Y_0}\Phi = 0$. Then by Leibniz'rule we see that the right hand side of (5.7.25) vanishes. So the adjoint symmetries σ_k are canonical. In the same way as in the proof of theorem 4.5.13 we can show that the corresponding constants of the motion $\tilde{F}_{3k+1}$ are in involution. We now consider the Poisson bracket between $\tilde{F}_{3k+1}$ and the "Hamiltonian" $F_3 = \tilde{F}_3$. Since $\tilde{F}_{3k+1}$ is a constant of the motion we have

$$\{\tilde{F}_{3k+1},\tilde{F}_3\} + \frac{\partial}{\partial t}\tilde{F}_{3k+1} = 0.$$

The derivative $\dfrac{\delta\tilde{F}_{3k+1}}{\delta u} = \sigma_k$ does not depend explicitly on t. This means that $\tilde{F}_{3k+1}$ can only depend explicitly on t through an "additive function of t" (see also the proof of theorem 2.4.5). Substitution of $u = 0$ shows that this is impossible, so

$$\{\tilde{F}_{3k+1},\tilde{F}_3\} = 0 \text{ and } \frac{\partial}{\partial t}\tilde{F}_{3k+1} = 0.$$

Finally it follows from

$$\{\tilde{F}_{3k+1},\tilde{F}_{3\ell}\} = \{\tilde{F}_{3k+3\ell-2},\tilde{F}_3\} = 0$$

that the two series $\tilde{F}_{3k+1}$ and $\tilde{F}_{3\ell}$ are also in involution.

□

Thus we have constructed two series of constants of the motion; a series $\tilde{F}_{3k}$ by applying the recursion operator for adjoint symmetries Γ to $\dfrac{\delta F_3}{\delta u}$ and a series $\tilde{F}_{3k+1}$ by applying Γ to $\dfrac{\delta F_4}{\delta u}$. Note that F_3 and F_6 appear in the "semi-Hamiltonian forms" (5.7.9) and (5.7.10). This simplified somewhat the construction of the series $\tilde{F}_{3k}$. By normalizing these constants of the motion so that the coefficient of u^k in F_k is equal to 1, we obtain the series F_{3k} and F_{3k+1}. So there exist real numbers c_k such that

$$(5.7.26)\qquad F_{3k} = c_{3k}\tilde{F}_{3k},\quad F_{3k+1} = c_{3k+1}\tilde{F}_{3k+1}.$$

Next we consider the various possible Lie brackets between the elements of the three series of symmetries X_k, Y_k and Z_k. This type of problem is considered in section 2.6. The results of that section were obtained under the assumption that hypothesis 2.6.3 was satisfied. A careful reading of section 2.6 yields that the second condition of hypothesis 2.6.1 (that is (2.6.5)) is only used in theorem 2.6.12 and corollary 2.6.13.

5.7.27 Theorem.

The Lie brackets between the elements of the series of symmetries X_k, Y_k and Z_k are given by

(5.7.28) $$[X_k,X_\ell] = 0, \quad [X_k,Y_\ell] = 0, \quad [Y_k,Y_\ell] = 0,$$

(5.7.29) $$[Z_k,X_\ell] = (\ell - \tfrac{1}{6})X_{k+\ell},$$

(5.7.30) $$[Z_k,Y_\ell] = (\ell + \tfrac{1}{6})Y_{k+\ell}.$$

Proof:
The symmetries X_k and Y_k are given by

$$X_k = \overleftarrow{\Omega} \frac{\delta \tilde{F}_{3k}}{\delta u}, \quad Y_k = \overleftarrow{\Omega} \frac{\delta \tilde{F}_{3k+1}}{\delta u}.$$

In the theorems 5.7.21 and 5.7.23 we have seen that the various Poisson brackets of the constants of the motion of the series $\tilde{F}_{3k}$ and $\tilde{F}_{3k+1}$ vanish. This implies that the corresponding symmetries commute. The formulas (5.7.29) and (5.7.30) are proved using the methods of section 2.6. We first verify that (2.6.4) and (2.6.6) are satisfied. From (5.7.18) and (5.7.19) we obtain

$$L_{Z_1} \Lambda = L_{Z_1} (\overleftarrow{\Omega} \Phi) = \overleftarrow{\Omega} \Phi \overleftarrow{\Omega} \Phi = \Lambda^2,$$

so (2.6.4) is satisfied with a = 1. Since Z_1 is a symmetry we obtain from (5.7.13.a) that

$$[Z_1,X_1] = \frac{\partial}{\partial t} Z_1 = \Lambda \frac{\partial}{\partial t} Z_o = \frac{5}{6} \Lambda X_1 = \frac{5}{6} X_2,$$

so (2.6.6) is satisfied with $b = \frac{5}{6}$. Then (5.7.29) follows from corollary 2.6.10. In a similar way we can prove (5.7.30).

□

The only Lie bracket which remains is the bracket of two elements of the series Z_k. In section 2.6 this bracket is given in corollary 2.6.13. However, in the proof of the preceding theorem 2.6.12 we used the second condition of hypothesis 2.6.3. So we face the problem of computing

$$(5.7.31) \qquad L_{Z_2} \Lambda = L_{Z_2} (\overset{\leftarrow}{\Omega} \Phi).$$

From theorem 5.7.20 we obtain

$$L_{Z_2} \Phi = 3(\Phi\overset{\leftarrow}{\Omega})^2\Phi$$

which means that we "only" have to compute $L_{Z_2} \overset{\leftarrow}{\Omega}$. Assume for a moment that the inverse operator Ω of $\overset{\leftarrow}{\Omega}$ exists. Then using the theory of section 4.5 it is easily shown that

$$(5.7.32) \qquad L_{Z_2} \overset{\leftarrow}{\Omega} = -2(\overset{\leftarrow}{\Omega}\Phi)^2\overset{\leftarrow}{\Omega}.$$

However, since we do not know whether Ω exists, we have to verify this expression in some other way. The only method we know to verify (5.6.32) is a straightforward computation. We did not carry out this extremely laborous task completely. If (5.7.32) would turn out to be correct, we obtain from corollary (2.6.13) that

$$(5.7.33) \qquad [Z_k, Z_\ell] = (\ell-k)Z_{k+\ell}.$$

We now have discussed two different methods for constructing the series of constants of the motion $\tilde{F}_{3k}$ and $\tilde{F}_{3k+1}$. The first method was to construct the corresponding adjoint symmetries using the recursion operator Γ (see the theorems 5.7.21 and 5.7.23). The second method consisted in generating the corresponding symmetries by using the repated Lie bracket with Z_1 (see theorem 5.7.27). The simplests method for constructing the two series of constants of the motion is described in

5.7.34 Theorem.

The constants of the motion F_{3k} and F_{3k+1} can be found recursively by

$$F_{3k+3} = a_k L_{Z_1} F_{3k} = a_k \int_{-\infty}^{\infty} \frac{\delta F_{3k}}{\delta u} \overleftarrow{\Omega} \alpha_1 \, dx ,$$

$$F_{3k+4} = b_k L_{Z_1} F_{3k+1} = b_k \int_{-\infty}^{\infty} \frac{\delta F_{3k+1}}{\delta u} \overleftarrow{\Omega} \alpha_1 \, dx$$

where α_1 is given in (5.7.16). The normalization constants a_k and b_k have to be choosen such that the coefficients of u^{3k+3} and u^{3k+4} in F_{3k+3} respectively F_{3k+4} are again equal to 1.

Proof:
See the proofs of theorem 4.6.12 (with Ω replaced by Φ), corollary 4.6.14 and theorem 5.6.36.

□

Finally we make some remarks on the "scattering-inverse scattering problem" for the SK equation. A "scattering problem" for the SK equation, given by Satsuma and Kaup [42], reads

$$(5.7.35) \qquad y_{xxx} + 6uy_x = \lambda y.$$

Suppose this equation has a discrete eigenvalue λ with an eigenfunction y such that $\int_{-\infty}^{\infty} y\bar{y}_x \, dx$ exists. Then it can be shown that the eigenvalue λ is purely imaginary and that (formally)

$$\frac{\delta\lambda}{\delta u} = \frac{6y_x\bar{y}_x}{\int_{-\infty}^{\infty} y\bar{y}_x \, dx} .$$

If u evolves according to the SK equation, the discrete eigenvalue λ is a constant of the motion and so $i\,\frac{\delta\lambda}{\delta u}$ is an adjoint symmetry. Indeed, using the time evolution of y given in [42], it can be shown that $i\,\frac{\delta\lambda}{\delta u}$ satisfies (5.7.6). We now can apply the recursion operator Γ to $i\,\frac{\delta\lambda}{\delta u}$. After a long computation, using (x derivatives and complex conjugates of) (5.7.35) we find

$$\Gamma i \frac{\delta\lambda}{\delta u} = 27\lambda\bar{\lambda} i \frac{\delta\lambda}{\delta u} \ . \tag{5.7.36}$$

So the recursion operator Γ has an eigenvalue $27\lambda\bar{\lambda}$ which is again a constant of the motion. Recall that at the end of the sections 2.3 (and 2.4) we showed that, under certain conditions, the eigenvalues of the recursion operators Λ (and Γ) are constants of the motion. The formula (5.7.36) is similar to the relation (5.6.64) in the case of the Korteweg-de Vries equation.

5.8 THE BENJAMIN-ONO EQUATION .

Internal waves in a stratified fluid with infinite depth can be described by the Benjamin-Ono (BO) equation [55,56]. In fact the BO equation can be considered as a limit of a more general equation (this equation is sometimes called the *Whitham* equation, see for instance [57]), which describes internal waves in a stratified fluid with finite depth. In the deep water and shallow water limit this equation reduces to the KdV-respectively the BO equation. We shall consider the BO equation in the form

$$u_t = 2uu_x + \mathcal{H}u_{xx} \qquad x \in \mathbb{R}, \tag{5.8.1}$$

where $\mathcal{H}$ is the *Hilbert transform*

$$\mathcal{H}u(x) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{u(y)}{y-x}\, dy \qquad \text{(principal value integral)}.$$

Multi-soliton solutions of this equation have been found by Matsuno [59] and by Chen, Lie and Pereira [60] . A single soliton solution with velocity $-c$ has the form

$$u(x,t) = \frac{2c}{1+c^2(x+ct)^2} \qquad c > 0. \tag{5.8.2}$$

We shall consider the BO equation in the space $S_p (0 < p < 1)$ with dual space U_p. Clearly the soliton solution given in (5.8.2) is an element of S_p. In theorem 1.4.10 we have proved that the Hilbert transform can be considered as a linear antisymmetric operator $\mathcal{H} : S_p \to U_p$. Several other properties of $\mathcal{H}$ are given in section 1.4. An infinite series of constants of the motion of the BO equation has been constructed by Nakamura [61] and by Bock and Kruskal [62] . The first elements of this series are

$$F_1^o(u) = \int_{-\infty}^{\infty} u\,dx \quad , \quad F_2^o(u) = \frac{1}{2}\int_{-\infty}^{\infty} u^2 dx \ ,$$

$$F_3^o(u) = \frac{1}{3}\int_{-\infty}^{\infty} (u^3 + \tfrac{3}{2}u\mathcal{H}u_x)dx, \tag{5.8.3}$$

$$F_4^o(u) = \frac{1}{4}\int_{-\infty}^{\infty} (u^4+3u^2\mathcal{H}u_x + 2u_x^2)\, dx.$$

It is easily verified that the BO equation can be written in the form

$$(5.8.4) \qquad u_t = \partial \frac{\delta F_3^o}{\delta u} = \partial(u^2 + uHu_x).$$

So we can consider the BO equation as a Hamiltonian system with Hamiltonian F_3^o and canonical operator ∂. A simple calculation shows that the BO equation can also be written in the form

$$(5.8.5) \qquad u_t = \Psi(u) \frac{\delta F_2^o}{\delta u} = (\frac{2}{3} u\partial + \frac{2}{3} \partial u + \partial H\partial)u.$$

However the antisymmetric operator $\Psi(u) = \frac{2}{3} u\partial + \frac{2}{3} \partial u + \partial H\partial : U_p \to S_p$

is not canonical.Hence (5.8.5) is not a Hamiltonian form of the BO equation.

5.8.6 Remark.

The two ways (5.8.4) and (5.8.5) of writing the BO equation strongly resemble the similar expressions (5.6.6) and (5.6.15) for the KdV equation. However, (5.6.6) and (5.6.15) are both ("semi") Hamiltonian forms of the KdV equation. The two corresponding AS operators have been used to construct recursion operators for (adjoint) symmetries of the KdV equation (see theorem 5.6.16 and corollary 5.6.17). Since (5.8.5) is not a (semi-) Hamiltonian form, a similar approach is not possible for the BO equation.

□

It is remarkable that for the BO equation there also exist infinite series of constants of the motion which can only be expressed in terms of densities which depend explicitly on x and t. First define the following functions (functionals) on S_p.

$$C_2^1(u) = \frac{1}{2} \int xu^2 dx, \qquad C_3^1 = \frac{1}{3} \int x(u^3 + \frac{3}{2} uHu_x)dx,$$

$$(5.8.7) \qquad C_4^1(u) = \frac{1}{4} \int x(u^4 + 2u^2 Hu_x - 2uu_x Hu + 2u_x^2)dx$$

and also

$$C_2^2(u) = \frac{1}{2} \int x^2 u^2 dx, \qquad C_3^2 = \frac{1}{3} \int x^2(u^3 + \frac{3}{2} uHu_x)dx,$$

(5.8.8)
$$C_4^2(u) = \tfrac{1}{4}\int x^2(u^4 + 2u^2 Hu_x - 2uu_x Hu + 2u_x^2)dx .$$

Then a long computation shows that

$$F_2^1(u,t) = C_2^1(u) + 2tF_3^o(u) ,$$

(5.8.9)
$$F_3^1(u,t) = C_3^1(u) + 2tF_4^o(u)$$

and also

$$F_2^2(u,t) = C_2^2(u) + 4tC_3^1(u) + 4t^2F_4^o(u),$$

(5.8.10)
$$F_3^2(u,t) = C_3^2(u) + 4tC_4^1(u) + 4t^2F_5^o(u),$$

$$F_4^2(u,t) = C_4^2(u) + 4tC_5^1(u) + 4t^2F_6^o(u)$$

are constants of the motion of the BO equation. In these expressions F_5^o and F_6^o are the following two constants of the motion of the series whose first elements are given (5.8.3). Further $C_5^1(u)$ is an expression of the form given in (5.8.7) ($C_5^1(u) = \frac{1}{5}\int(xu^5 + \ldots)dx$). We do not give the very lengty expressions for F_5^o, F_6^o and C_5^1 explicitly. The symmetry corresponding the constant of the motion F_2^1 is given by

$$X_2^1 = \partial\frac{\delta F_2^1}{\delta u} = xu_x + u + 2t(2uu_x + Hu_{xx}) = xu_x + u + 2tu_t.$$

This symmetry is related to the scale transformation $u(x,t) \to au(ax,a^2t)$ of the BO equation. By taking the repeated Poisson brackets of the constants of the motion given in (5.8.3), (5.8.9) and (5.8.10) (and of already constructed elements) we can generate an infinite dimensional Lie algebra of constants of the motion for the BO equation. However some care is necessary in this construction. The variational derivatives of C_2^2 and C_3^2 are given by

$$\frac{\delta C_2^2}{\delta u} = x^2u , \qquad \frac{\delta C_3^2}{\delta u} = x^2u^2 + \frac{3}{2}x^2Hu_x + \frac{3}{2}H(x^2u)_x.$$

For $u \in S_p$ we have $xu \in \mathcal{U}_p$ but $x^2u = \frac{\delta C_2^2}{\delta u} \notin \mathcal{U}_p$. Also $Hu_x = \partial_x Hu \in S_p$ (see section 1.4), so $xHu_x \in \mathcal{U}_p$ but $x^2Hu_x \notin \mathcal{U}_p$. In a similar way we can show that $\frac{3}{2} H(x^2u)_x \notin \mathcal{U}_p$. Hence $\frac{\delta C_3^2}{\delta u} \notin \mathcal{U}_p$. So formally C_2^2 and C_3^2 are not differentiable in the choosen topology. This means that Poisson brackets between F_2^2, F_3^2 and other (differentiable) constants of the motion may not exist. To avoid these problems we generate a Lie algebra E of constants of the motion of (5.8.1) starting with $\{F_2^o, F_3^o, F_2^1, F_3^1, F_4^2\}$.

Next we make some remarks on the structure of this Lie algebra. The leading terms of the constants of the motion F_ℓ^k given in (5.8.3), (5.8.9) and (5.8.10) are of the form

$$L_\ell^k (u) = \frac{1}{\ell} \int_{-\infty}^{\infty} x^k u^\ell dx.$$

It is easily seen that

$$(5.8.11) \qquad \{L_s^r, L_j^i\} = (i(s-1) - r(j-1)) L_{s+j-2}^{r+i-1} \quad .$$

This means that there can be several methods to construct L_ℓ^k using Poisson bracket of elements L_j^i with "lower orders". Hence it may be possible to generate distinct constants of the motion of the algebra E which have the same leading term L_ℓ^k . For small values of k and ℓ it can be verified that elements of E which have the same leading term L_ℓ^k are identical. We *conjecture* that this also holds true for the other elements of E. In that case a constant of the motion with leading term L_ℓ^k is uniquely determined. We shall denote this constant of the motion by F_ℓ^k. Then, similar to (5.8.11)

$$(5.8.12) \qquad \{F_s^r, F_j^i\} = (i(s-1) - r(j-1)) F_{s+j-2}^{r+i-1} \quad .$$

If the conjecture mentioned above is correct, we can also generate an algebra $\{C_\ell^k\}$, starting with $\{C_2^o = F_2^o, C_3^o = F_3^o, C_3^1, C_4^1, C_4^2\}$. Then it can be shown (see Broer and ten Eikelder [58]) that

$$F_\ell^k = \sum_{i=0}^{k} (2t)^i \binom{k}{i} C_{\ell+i}^{k-i} \quad .$$

In any case we can construct an infinite series of constants of the motion F^o_k by

(5.8.13) $$F^o_{k+1} = \frac{1}{k-1}\{F^o_k, F^1_3\} = \frac{1}{k-1}\{F^o_k, C^1_3\} + \frac{2t}{k-1}\{F^o_k, F^o_4\} .$$

If the Poisson bracket of the F^o_k with F^o_4 vanishes, we obtain

(5.8.14) $$F^o_{k+1} = \frac{1}{k-1}\{F^o_k, C^1_3\} .$$

Then the corresponding symmetries satisfy (see (4.4.5) and (4.4.6))

(5.8.15) $$X^o_{k+1} = \frac{1}{k-1}[A^1_3, X^o_k] \text{ with } A^1_3 = \partial\frac{\delta C^1_3}{\delta u} .$$

This relation has been used by Fokas and Fuchssteiner [63] to generate an infinite series of symmetries and corresponding constants of the motion for the BO equation. However, since all symmetries in this relation are canonical, there is no reason to work with symmetries instead of the corresponding constants of the motion (see also theorem 4.4.7). Moreover a straightforward construction of the constants of the motion using (5.8.14) also avoids the problem of showing that the symmetries constructed in (5.8.15) are canonical. Note that (5.8.14) and (5.8.15) are only correct if $\{F^o_k, F^o_4\} = 0$ for $k \geq 3$. This holds if the series F^o_k is in involution. This last property is often mentioned in the literature, but as far as we know a correct proof has not yet been given. The proof given by Fokas and Fuchssteiner [63] is incomplete. If the conjecture mentioned above turns out to be correct, it follows immediately from (5.8.12) that the series F^o_k is in involution.

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LIST OF SYMBOLS.

A,B,C	: vectors or vector fields
$C^{\infty}(\mathbb{R})$	: infinitly differentiable functions on $\mathbb{R}$
d	: exterior derivative 14,21
$du^1, \ldots, du^n$	: natural cobasis 8
$e_1 = \frac{\partial}{\partial u^1}, \ldots, e_n = \frac{\partial}{\partial u^n}$	: natural basis 7
E	: various Lie algebra's 9
F,G,K	: (parameterized) functions on M (elements of $F(M)$ or $F_p(M)$) or constants of the motion
$F(M)$	: smooth functions on M 8
$F_p(M)$	: smooth parameterized functions on M 17,23
f,g	: various functions or mappings
H	: Hamiltonian 71
H	: Hilbert transform 28,187
i_A	: interior product with a vector field A 13,22
$L_1(\mathbb{R})$	: Lebesgue space of integrable functions
$L_2(\mathbb{R})$	: Lebesgue space of square integrable functions
$L(W,W_1)$	: linear continuous mappings of W into W_1 18
L_A	: Lie derivative in the direction of A 12,21
M,N	: manifolds 7
$p_1, \ldots, p_n, q_1, \ldots, q_n$	: canonical coordinates 72
R(k)	: reflection coefficient 164
T(k)	: transmission coefficient 164
S_p, U_p	: function spaces 23,24
$u^1, \ldots, u^n$	: local coordinates 7
u	: arbitrary point of M
T_uM	: tangent space in $u \in M$ 7
TM	: tangent bundle of M 7
T^*_uM	: cotangent space in $u \in M$ 7
T^*M	: cotangent bundle of M 7
$T^i_j(M)$	: tensor fields on M with covariant order i and contravariant order j 8,19
$T^i_{jp}(M)$	: parameterized tensor fields on M with covariant order i and contravariant order j 17,23
$V(X;M)$	: symmetries of the dynamical system $\dot{u}=X(u)$ on M 36

$V^*(X;M)$	: adjoint symmetries of the dynamical system $\dot{u}=X(u)$ on M 37
$X(M)$	: smooth vector fields on M 8,19
$X_p(M)$	: smooth parameterized vector fields on M 17,23
$X^*(M)$	: smooth one-forms on M 8,19
$X_p^*(M)$	: smooth parameterized one-forms on M 17,23
X,Y,Z	: symmetries (elements of $V(X;M)$)
U,U_o,U_o^1	: open subsets of M
W,Z	: topological vector spaces
W^*,Z^*	: topological duals of W,Z
α,β,γ	: elements of T_u^*M or one-forms on M
Γ	: recursion operator for adjoint symmetries (tensor field)
Λ	: recursion operator for symmetries (tensor field)
Ξ	: various tensor fields or linear mappings
ξ	: differential k-form (corresponding to Ξ)
ρ,σ,τ	: adjoint symmetries (elements of $V^*(X;M)$)
Φ	: SA operator (tensor field)
ϕ	: two-form (corresponding to Φ)
Ψ	: AS operator (tensor field)
Ω	: cyclic (SA) operator (tensor field)
$\overleftarrow{\Omega}$	: canonical (AS) operator (tensor field)
ω	: symplectic two-form (corresponding to Ω)
$\otimes$	: tensor product 10
$\wedge$	: exterior product 15
$\langle\cdot,\cdot\rangle$	: duality map (between T_u^*M and T_uM or between W and W^*) 7,18
$[\cdot,\cdot]$	: Lie bracket of vector fields 9,21
$[\cdot,\cdot\underline{]}$	: commutator of two linear operators 126
$\{\cdot,\cdot\}$	: Poisson bracket of two functions 72
∂	: $\frac{\partial}{\partial x}$ or $\frac{d}{dx}$ 23
∂^{-1}	: inverse of ∂ 25
$\frac{\delta F}{\delta u}$	: variational derivative of F 115

Derivatives with respect to u are indicated by a prime. Derivatives with respect to t are indicated by a dot, except when partial differential equations are considered. In that case derivatives with respect to t are denoted by the subscript t.

INDEX OF TERMS.

SAMENVATTING.

Dit proefschrift behandelt symmetrieën van dynamische systemen en in het bijzonder Hamiltonse systemen van de vorm $\dot{u} = X(u)$, waarbij X een vectorveld op een variëteit M is. Een korte beschrijving van de gebruikte wiskundige methoden is gegeven in hoofdstuk 1. In hoofdstuk 2 bekijken we symmetrieën van dynamische systemen. Symmetrieën worden ingevoerd als infinitesimale transformaties van oplossingen van het systeem in nieuwe oplossingen van het systeem. Dit leidt tot een interpretatie van symmetrieën als vectorvelden Y op M zodat $\dot{Y} + [X,Y] = \dot{Y} + L_X Y = 0$. Naast symmetrieën worden ook geadjungeerde symmetrieën bekeken. Dat zijn een-vormen (covariante vectorvelden) σ die voldoen aan $\dot{\sigma} + L_X\sigma = 0$. Bij elke bewegingsconstante van het dynamisch systeem hoort een geadjungeerde symmetrie; het omgekeerde geldt echter niet.

In hoofdstuk 3 worden (gegeneralizeerde) Hamiltonse systemen ingevoerd. Verschillende partiële differentiaalvergelijkeingen kunnen dan als Hamiltons systeem worden opgevat. Symmetrieën van Hamiltonse systemen worden onderzocht in hoofdstuk 4. Bij een Hamiltons systeem bestaat er altijd (minstens) een operator die symmetrieën overvoert in geadjungeerde symmetrieën en een operator die werkt in de omgekeerde richting. Dit betekent dat bij een Hamiltons systeem elke bewegingsconstante aanleiding geeft tot een symmetrie van het systeem. Dit type symmetrie noemen we een canonieke symmetrie. Vaak bestaan er ook symmetrieën die niet op deze wijze met een bewegingsconstante samenhangen, de zogenaamde niet-canonieke symmetrieën. Het blijkt dat een niet-canonieke symmetrie Z aanleiding geeft tot een recursie operator voor symmetrieën. Uitgaande van X en Z kunnen dan twee oneindige rijen van symmetrieën X_k en Z_k geconstrueerd worden. In paragraaf 4.5 laten we zien dat, als de niet-canonieke symmetrie Z aan een aantal extra voorwaarden voldoet, de rij X_k bestaat uit canonieke symmetrieën. In dat geval bestaat er dus een oneindige rij bewegingsconstanten. De rij Z_k bestaat (in het algemeen) uit niet-canonieke symmetrieën. De Lie algebra voortgebracht door de symmetrieën X_k en Z_k wordt onderzocht in paragraaf 4.6.

In hoofdstuk 5 worden verschillende voorbeelden van de voorafgaande theorie gegeven. Het belangrijkste voorbeeld is de Korteweg-de Vries vergelijking.

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LEVENSLOOP.

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STELLINGEN BEHORENDE BIJ HET PROEFSCHRIFT VAN

H.M.M. TEN EIKELDER

1. Beschouw een Hamiltons systeem op een oneindig dimensionale symplectische variëteit. Een verzameling bewegingsconstanten van dit systeem, die voldoet aan de door Abraham en Marsden gegeven definitie van volledige integreerbaarheid, bestaat niet.

R.Abraham & J.E.Marsden, *Foundations of Mechanics*, Benjamin/Cummings, New York, Second Editon 1978, definition 5.2.20.

2. De behandeling door Mayer van connecties in geassocieerde vectorbundels is zowel voor physici als mathematici onbegrijpelijk.

W.Drechsler & M.E.Mayer, *Fiber Bundle Techniques in Gauge Theories*, Springer Verlag, Berlin 1977, § 4.3.

3. De Bell-ongelijkheden kunnen worden geinterpreteerd als een gevolg van de veronderstelling dat er een gemeenschappelijke meetprocedure bestaat voor de vier betrokken (deels incompatibele) observabelen.

4. Het gebruik van de term volledig integreerbaar bij oneindig dimensionale Hamiltonse systemen zonder verdere toelichting is een bron van verwarring.

5. Beschouw een Hamiltons systeem met een symplectische vorm ω en bijbehorende SA- en AS operatoren Ω en $\overset{\leftarrow}{\Omega}$. Laat Z een niet canonieke symmetrie van dit systeem zijn zoals beschreven in hypothese 4.5.1 van dit proefschrift. Als $\dot{Z}$ een canonieke symmetrie is, is de operator $\Lambda = \overset{\leftarrow}{\Omega} L_Z \Omega$ een erfelijke symmetrie, zoals gedefinieerd door Fuchssteiner.

B.Fuchssteiner, *Application of hereditary symmetries to nonlinear evolution equations*, Nonlinear Anal.Theory Meth.Appl. 3 (1979), 849-862.

6. Beschouw een oneindige Toda-keten met bewegingsvergelijkingen

$$\dot{q}_n = p_n$$

$$\dot{p}_n = e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)}$$

Stel $a_n = \frac{1}{2} e^{-\frac{1}{2}(q_n - q_{n-1})}$ en $b_n = -\frac{1}{2} p_n$. Een oneindige rij bewegingsconstanten, geldig voor oplossingen met $a_n \to \frac{1}{2}$, $b_n \to 0$ voldoend snel als $|n| \to \infty$, kan geconstrueerd worden met de volgende recursie formule

$$F_{k+1} = \sum_{n=-\infty}^{\infty} \Big[\frac{\partial F_k}{\partial a_n} \, a_n((2n+3)\,b_{n+1} - (2n-1)b_n)$$

$$+ \frac{\partial F_k}{\partial b_n} \, (4a_n^2(n+1) - 4a_{n-1}^2(n-1) + 2b_n^2)\Big] \quad ,$$

waarbij $F_1 = \sum_{n=-\infty}^{\infty} b_n$.

7. In de meeste toepassingen van de sine-Gordon vergelijking in de vaste stof fysica speelt de "volledige integreerbaarheid" van deze vergelijking geen rol.

8. De door Aiyer gegeven inverse van de recursie operator voor symmetrieën van de Korteweg-de Vries vergelijking bestaat niet.

   R.N.Aiyer, *Recursion operators for infinitesimal transformations and their inverses for certain nonlinear evolution equations*, J. Phys. A. Math. Gen. 16 (1983) 255-262.

9. De door Sarlet gegeven voorwaarden voor het bestaan van een Lagrangiaan voor een stelsel tweede orde differentiaalvergelijkingen zijn alleen geschikt om het niet bestaan van zo'n Lagrangiaan aan te tonen.

   W.Sarlet, *The Helmholtz conditions revisited. A new approach to the inverse problem of Lagrangian dynamics*, J. Phys. A. Math. Gen. 15 (1982) 1503-1517.

10. Sportief autorijden is niet sportief.

Eindhoven, 3 februari 1984.