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by

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SOME REMARKS ON THE GAP METRIC

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Abstract

This paper presents the following results on the gap metric defined on the space of closed linear operators: 1) the topology introduced by the gap metric is a diagonal product topology; 2) if the gap of two closed operators is smaller than one, then the two directed gaps are the same; 3) when a closed operator is densely defined, a representation of the orthogonal projection onto its graph is found and using the representation we present an equivalent form of the gap metric.

0 INTRODUCTION   The gap metric was introduced by [Kre.1947] in order to measure perturbations of the closed operators. It was shown [Cor.1963, Kat.1966, Kra.1972, Zam.1980, Zhu.1987[*]] that the gap metric is a valuable tool for perturbation analysis of the closed operators and related problems. The known properties and some of their applications was collected in [Kat.1966]. In 1980, [Zam.1980] used the gap metric in system and control theory for the analysis of robust stability. [Zhu.1987] proved that the topology introduced by the gap metric, restricted to the set of linear time invariant (LTI) systems, which is a subspace of the space of the closed operators, has the diagonal product property as will become clear in the sequel. It was shown in [Zhu1988] that, if the gap of two LTI systems is smaller than one, then the two directed gaps are the same (which is called the coincidence–property in this paper). The diagonal product property of the gap metric on the set of systems can easily be extended to the space of closed operators and the coincidence property of the gap metric on the set of systems needs an alternative procedure to be generalized to the space of the closed operators. The gap metric is defined by the projections

onto the graphs of the related closed operators. For a densely defined closed operator, we present a representation of the orthogonal projection onto its graph and using this representation we give an equivalent form of the gap metric. In section 1, the definition of the gap metric is introduced. In section 2, we present the diagonal product property of the gap metric on the space of closed operators. The equality of the directed gaps on the space of the closed operators will be proved in section 3. Finally, in section 4, we give the representation of the orthogonal projection onto the graph of a closed operator and the equivalent form of the graph metric.

1 PRELIMINARY   Let us first define the gap metric on the space which consists of all closed subspaces in a Hilbert space H. Suppose that φ and ψ are closed subspaces in H and $\Pi(\varphi)$ denotes the orthogonal projection onto φ. Then the gap between φ and ψ is defined as

$$\delta(\varphi,\psi) = \max \{ \overrightarrow{\delta}(\varphi,\psi) , \overrightarrow{\delta}(\psi,\varphi) \}$$

where $\overrightarrow{\delta}(\varphi,\psi)$ is called the directed gap from φ to ψ and is defined as

$$\overrightarrow{\delta}(\varphi,\psi) = \| (I - \Pi(\psi)) \Pi(\varphi) \|$$

It is easy to see that the directed gap has the following equivalent form

$$\overrightarrow{\delta}(\varphi,\psi) = \sup_{\substack{x \in \varphi \\ \|x\|=1}} \inf_{y \in \psi} \| x - y \|$$

One can easily prove that $\delta(.\,,.)$ is a metric on the space consisting of all closed subspaces in H.

Let X and Y be two Hilbert spaces and denote by C(X , Y) the space of all linear closed operators mapping X to Y. Now we define the gap metric on the space C(X , Y). For each element $P \in C(X , Y)$, the graph of P denoted by G(P) is a closed subspace of X×Y. Hence there is an orthogonal projection $\Pi(P)$ mapping X×Y onto G(P). For any two elements P_1 and P_2 in C(X , Y), the gap $\delta(P_1 , P_2)$ between P_1 and P_2 is defined as the gap $\delta(G(P_1) , G(P_2))$ between their graphs i.e.

$$\delta(P_1,P_2) := \delta(G(P_1) , G(P_2))$$

and the directed gap $\overrightarrow{\delta}(P_1,P_2)$ from P_1 to P_2 defined as the directed gap $\overrightarrow{\delta}(G(P_1) , G(P_2))$ from the graph $G(P_1)$ of P_1 to the graph $G(P_2)$ of P_2.

2 THE DIAGONAL PRODUCT PROPERTY

In this section, we introduce, first, two inequalities related to the diagonal form of the closed operators. Their proofs can be found in [Zhu.1987]. The diagonal product property, which is our main result in this section, follows directly from the two inequalities.

Let X^i and Y^i (i=1,2) be Hilbert spaces and define the Hilbert spaces X and Y as

$$X := X^1 \times X^2 \qquad\qquad Y := Y^1 \times Y^2$$

Now suppose that $P_k \in C(X , Y)$ (k=1,2) have the following diagonal form

$$P_k = \begin{bmatrix} P_k^1 & 0 \\ 0 & P_k^2 \end{bmatrix} \qquad (k=1,2)$$

where $P_k^i \in C(X^i , Y^i)$ (i=1,2). Then we have

LEMMA 2.1

(2.1) $\qquad \delta(P_1^1,P_2^1) + \delta(P_1^2,P_2^2) \geq \delta(P_1,P_2) \geq \max \{ \delta(P_1^1,P_2^1) , \delta(P_1^2,P_2^2) \}$

Lemma 2.1 can be proved completely in the same way as the methods presented in [Zhu.1987] for proving the corresponding results for LTI systems.

We introduce a parameter λ which is in a Hausdorff–topology space Γ. Assume that, for each $\lambda \in \Gamma$, $P_\lambda \in C(X , Y)$ has the following diagonal form

$$P_\lambda = \begin{bmatrix} P^1_\lambda & 0 \\ 0 & P^2_\lambda \end{bmatrix}$$

where $P^i_\lambda \in C(X^i , Y^i)$ (i=1,2). The following result follows from (2.1).

THEOREM 2.2

$$\delta(P_\lambda, P_{\lambda_0}) \longrightarrow 0 \qquad (\lambda \longrightarrow \lambda_0)$$

if and only if

$$\delta(P^i_\lambda, P^i_{\lambda_0}) \longrightarrow 0 \qquad (\lambda \longrightarrow \lambda_0)$$

holds for both i=1 and i=2.

It is easy to show that lemma 2.1, and theorem 2.2 are still valid for the case when $P = \text{diag} (P^1 , P^2$ ... $P^1)$. This is the diagonal product property of the gap topology on the space of the linear closed operators. An application of this property can be seen in [Zhu.1987].

3 THE COINCIDENCE PROPERTY   In this section, we directly prove that, if the gap between two closed subspaces is smaller than 1, then the two directed gaps are the same.

THEOREM 3.1   Suppose that φ and ψ are closed subspaces in a Hilbert space H. If $\delta(\varphi , \psi) < 1$, then $\overrightarrow{\delta}(\varphi,\psi) = \overrightarrow{\delta}(\psi,\varphi)$.

To prove this theorem, we need

LEMMA 3.2 [Kra.1972, pp206] $\delta(\varphi , \psi) < 1$ if and only if $\Pi(\varphi)$ maps $G(\psi)$ bijectively onto $G(\varphi)$.

PROOF OF THEOREM 3.1   Suppose $\delta(\varphi , \psi) < 1$. Define $[\Pi(\psi)]_r$ as the restriction of $\Pi(\psi)$ to $G(\varphi)$. By lemma 3.2, $[\Pi(\psi)]_r$ has a bounded inverse. Accordingly, we have

$$\overrightarrow{\delta}(\varphi,\psi)^2 = \| (I - \Pi(\psi))\, \Pi(\varphi) \|^2$$

$$= \sup_{x \in H} \| (I - \Pi(\psi))\, \Pi(\varphi)\, x \|^2$$

$$= \sup_{x \in \varphi} \| (I - \Pi(\psi))\, x \|^2$$

$$= \sup_{x \in \varphi} \| x - [\Pi(\psi)]_r x \|^2$$

$$= 1 - \inf_{x \in \varphi} \| [\Pi(\psi)]_r x \|^2$$

$$= 1 - \| [\Pi(\psi)]_r^{-1} \|^{-2}$$

Similarly

$$\overrightarrow{\delta}(\psi,\varphi)^2 = 1 - \| [\Pi(\varphi)]_r^{-1} \|^{-2}$$

Since

$$< [\Pi(\varphi)]_r x - x \,,\, y > \;=\; < x \,,\, [\Pi(\psi)]_r y - y > \;=\; 0 \qquad \forall x \in \psi \;\; \forall y \in \varphi$$

we have

$$< [\Pi(\varphi)]_r x \,,\, y > \;=\; < x \,,\, y > \;=\; < x \,,\, [\Pi(\psi)]_r y > \qquad \forall x \in \psi \;\; \forall y \in \varphi$$

Hence $[\Pi(\varphi)]_r^* = [\Pi(\psi)]_r$. Consequently, $([\Pi(\varphi)]_r^{-1})^* = ([\Pi(\psi)]_r^{-1})$ and $\|([\Pi(\varphi)]_r^{-1})\| = \|([\Pi(\psi)]_r^{-1})\|$.
This implies that $\overrightarrow{\delta}(\varphi,\psi) = \overrightarrow{\delta}(\psi,\varphi)$.
Q.E.D..

Because the gap of closed operators is defined as the gap of their graphs, it is a trivial consequence that, if the gap of two closed operators is smaller than 1, then the two directed gaps are the same.

4 REPRESENTATION OF Π(P) AND AN EQUIVALENT FORM OF δ(. , .)

Let X and Y be two Hilbert spaces and in this section denote by DC(X , Y) the subspace of C(X , Y) consisting of all elements in C(X , Y) which are densely defined on X. Assume P_i (i=1,2) ∈ DC(X , Y). From section 1 we know that the gap metric is defined as

$$(4.1) \qquad \delta(P_1,P_2) = \max \{ \ \overrightarrow{\delta}(P_1,P_2) \ , \ \overrightarrow{\delta}(P_2,P_1) \ \}$$

where

$$\overrightarrow{\delta}(P_1,P_2) = \| \ (I - \Pi(P_2)) \ \Pi(P_1) \ \|$$

In this section we are mainly going to find a representation of $\Pi(P_1)$

THEOREM 4.1   Suppose P ∈ DC(X , Y), then there exists an operator A mapping X onto G(P) which satisfies

$$(4.2) \qquad A^*A = I$$

$$(4.3) \qquad \Pi(P) = AA^*$$

The proof consists of the following lemmas.

LEMMA 4.2   [Rie.1953, pp. 307–]   If P ∈ DC(X , Y), then $(I + A^*A)^{-1}$ (which will be denoted by R_P) exists as a bounded self–adjoint positive operator mapping X to Dom(A) bijectively. Moreover, PR_P is also bounded and

$$\| R_P \| \ \leq \ 1 \qquad\qquad \| PR_P \| \ \leq \ 1.$$

LEMMA 4.3 [Cor.1963]   If $P \in DC(X , Y)$, then the bounded self—adjoint positive operator $R_p := (I + P^*P)^{-1}$ has a unique bounded self—adjoint positive square root, which we denote by S_p, i.e.

$$R_P = S_p \cdot S_P$$

Moreover, S_p maps X to Dom(P) bijectively and PS_p is bounded and

$$\| S_p \| \leq 1 \qquad \qquad \| PS_p \| \leq 1.$$

PROOF OF THEOREM 4.1   Suppose $P \in DC(X , Y)$ and define

$$(4.4) \qquad A := \begin{bmatrix} I \\ P \end{bmatrix} S_p$$

One can easily check that A satisfies the required conditions. (4.2) follows directly from (4.4) and (4.3) follows from the facts that $(AA^*)^2 = AA^*$, AA^* is self—adjoint, and Range $AA^* = G(P)$. QED

In the rest of this section, we present an equivalent form of the gap metric using the operator A.

THEOREM 4.4   If P_i (i=1,2) $\in DC(X , Y)$ and $\delta(P_1,P_2) < 1$, then

$$\overset{\rightarrow}{\delta}(P_2,P_1)^2 = 1 - \| M^{-1} \|^{-2}$$

where $M = A_1^* A_2$ and A_i is defined by (4.4) with respect to P_i.

To prove this theorem we need

LEMMA 4.5   Let $P \in DC(X , Y)$ and B be a linear bounded operator mapping X to X×Y. Then B maps X bijectively to G(P) if and only if then there exists a unique linear bounded operator U mapping X to X bijectively such that $B = A U$, where A is defined by (4.4).

PROOF ⟸ This is obvious, because A maps X bijectively to G(P).

⟹ Since both B and A map X to G(P) bijectively, for each x ∈ X there is a unique y ∈ X such that

(*) $B x = A y$

and vice versa.

By (*) we can define a linear mapping U

$$U x = y$$

It is obvious that U maps X to X bijectively and $B = A U$. The uniqueness comes from the bijection of B and A. This completes the proof.

LEMMA 4.6 Assume P_i (i=1,2) ∈ DC(X , Y) and define the operator B_1 as

(4.5) $B_1 := \Pi(P_1) A_2$

where A_2 is defined by (4.4) corresponding to P_2. Then $\delta(P_1,P_2) < 1$ if and only if B_1 maps X bijectively to $G(P_1)$.

PROOF This result follows from lemma 3.2 and the fact, that $\Pi(P_1)$ maps $G(P_2)$ bijectively to $G(P_1)$ if and only if $\Pi(P_1) A_2$ maps X bijectively to $G(P_2)$. Q.E.D..

LEMMA 4.7 If P_i (i=1,2) ∈ DC(X , Y) then $\delta(P_1,P_2) < 1$ if and only if $A_1^* A_2$ maps X bijectively to X, where A_i is defined by (4.4) with respect to P_i.

PROOF According to (4.3) and (4.5), $B_1 = A_1 A_1^* A_2$. By lemma 4.6, $\delta(P_1,P_2) < 1$ if and only if B_1 maps X bijectively to $G(P_1)$. By lemma 4.5, B_1 maps X to $G(P_1)$ bijectively if and only if $A_1^* A_2$ maps X to X bijectively. This completes the proof.

PROOF OF THEOREM 4.4   First, we notice that, by lemma 4.7, $\delta(P_1,P_2) < 1$ if and only if M is bijective. Hence M^{-1} exists as a bounded operator.

$$\overrightarrow{\delta}(P_2,P_1)^2 = \| (I - \Pi(P_1)) \, \Pi(P_2) \|^2$$

$$= \sup_{\substack{x\varepsilon X \times Y \\ \|x\|=1}} \| (I - \Pi(P_1)) \, \Pi(P_2) \, x \|^2$$

$$= \sup_{\substack{x\varepsilon G(P_2) \\ \|x\|=1}} \| (I - \Pi(P_1)) \, x \|^2$$

$$= \sup_{\substack{x\varepsilon A_2 X \\ \|x\|=1}} \| (I - \Pi(P_1)) \, x \|^2 \qquad (\, G(P_2) = A_2 X \,)$$

$$= \sup_{\substack{x\varepsilon X \\ \|x\|=1}} \| (I - \Pi(P_1)) \, A_2 x \|^2 \qquad (\, \|A_2 x\| \equiv \|x\| \,)$$

$$= \sup_{\substack{x\varepsilon X \\ \|x\|=1}} (\, \| A_2 x \|^2 - \| \Pi(P_1) \, A_2 x \|^2) \qquad (\, \Pi(P_1) \text{ is orthogonal} \,)$$

$$= 1 - \inf_{\substack{x\varepsilon X \\ \|x\|=1}} \| \Pi(P_1) \, A_2 x \|^2$$

$$= 1 - \inf_{\substack{x\varepsilon X \\ \|x\|=1}} \| A_1 A_1^* \, A_2 x \|^2$$

$$= 1 - \inf_{\substack{x\varepsilon X \\ \|x\|=1}} \| A_1^* \, A_2 x \|^2 \qquad (\, \|A_1 x\| \equiv \|x\| \,)\|$$

$$= 1 - \| (A_1^* \, A_2)^{-1} \|^{-2}$$

This completes the proof.

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