

## No current without heat

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# **No current without heat**

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# NO CURRENT WITHOUT HEAT

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**Abstract:** We show for a large class of interacting particle systems that whenever the stationary measure is not reversible for the dynamics, then the mean entropy production in the steady state is strictly positive. This extends to the thermodynamic limit the equivalence between microscopic reversibility and zero mean entropy production: time-reversal invariance cannot be spontaneously broken.

**Keywords:** stochastic interacting particle systems, entropy production, (generalized) detailed balance.

**Dedicated in honor of the 70th birthday of Joel L. Lebowitz**

## 1 Introduction

Reversibility and entropy are words with many meanings. We are very much indebted to Joel Lebowitz who has helped all of us in clarifying the various issues. One class of models that Joel has often considered for learning about nonequilibrium behavior is that of interacting particle systems. These are stochastic dynamics for spatially extended systems in which particles locally interact. They are mostly toy-models remaining far from realistic in their microscopic details. Yet, Joel has convincingly argued that for some good purposes, the details do not matter so much and one should be concerned more with the symmetries, possible conservation laws, locality of the interaction etc. to hope to understand something about real nature.

This paper is about the relation between time-reversal invariance and the positivity of entropy production. We do this in the context of interacting particle systems following work by Joel and Herbert Spohn in [9].

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The physics background will be discussed in Section 3. To understand the mathematical problem, let us look first at a finite Markov chain. Suppose that  $K$  is a finite set on which we have an involution  $\pi : K \rightarrow K, \pi^2 = id$ , called time-reversal. Let  $(X_t, t \in [-T, T])$  be a  $K$ -valued stationary Markov process (steady state) with law  $\mathbb{P}_\rho$ . The subscript refers to the stationary probability measure  $\rho$  on  $K$  for which we assume that  $\rho(a) = \rho\pi(a) > 0, a \in K$ . The rate to go from  $a$  to  $b$  is denoted by  $k(a, b), a, b \in K$  and we assume that  $k(\pi b, \pi a) = 0$  iff  $k(a, b) = 0$  (dynamic reversibility). The generator is

$$Lf(a) = \sum_b k(a, b)[f(b) - f(a)] \quad (1.1)$$

The time-reversed distribution of  $(X_t)$  is the stationary process  $(Y_t, t \in [-T, T])$  with  $Y_t \equiv \pi X_{-t}, t \in [-T, T]$ . We denote its law by  $\tilde{\mathbb{P}}_\rho$  ( $\rho = \rho\pi$  is also stationary for  $(Y_t)$ ).  $(Y_t)$  is again a  $K$ -valued stationary Markov process on  $K$  but now with transition rates

$$\tilde{k}(a, b) \equiv k(\pi b, \pi a) \frac{\rho(b)}{\rho(a)} \quad (1.2)$$

The corresponding generator for this time-reversed process is  $\tilde{L} = \pi L^* \pi$  where the  $*$  refers to the adjoint with respect to the stationary measure  $\rho$ . Of course,  $\tilde{\tilde{L}} = L$ .

We say that the process  $(X_t)_{-T}^T$  is  $\pi$ -reversible if  $\mathbb{P}_\rho = \tilde{\mathbb{P}}_\rho$ . This implies that the stationary measure  $\rho$  satisfies

$$\rho(a)k(a, b) = k(\pi b, \pi a)\rho(b), a, b \in K \quad (1.3)$$

or  $\tilde{k} = k$ , which is generalized (or extended) detailed balance (microscopic reversibility). For the generators, we then have  $L^* = \pi L \pi$  which, in turn, implies the stationarity of  $\rho = \rho\pi$  and the relation (1.3). Note however that (1.3) by itself (without assuming stationarity of  $\rho\pi$ ) does not imply that  $L^* = \pi L \pi$ .

The difference of left and right hand side in (1.3), is called the current

$$J_{ab}(\rho) \equiv k(a, b)\rho(a) - k(\pi b, \pi a)\rho(b)$$

between states  $a$  and  $b$  in the steady state  $\mathbb{P}_\rho$ .

The entropy production is the random variable obtained from taking the relative action on pathspace with respect to time-reversal. Here we are only interested in its expectation value, that is the mean entropy production

$$\text{MEP}_\pi(L, \rho) = \lim_{T \uparrow \infty} \frac{1}{2T} \mathbb{E}_\rho \left[ \log \frac{d\mathbb{P}_\rho}{d\tilde{\mathbb{P}}_\rho} \right] \quad (1.4)$$

The notation  $\text{MEP}_\pi(L, \rho)$  reminds us that this number depends on the transformation  $\pi$ , the dynamics (generated via  $L$ ) and the stationary measure  $\rho$ . The mean entropy production thus measures the degree to which  $\mathbb{P}_\rho$  can be distinguished from

$\tilde{\mathbb{P}}_\rho$ . The main property of the mean entropy production is then:

**Proposition 1:** *Consider the stationary process  $(X_t)$  above with  $\rho = \rho\pi$ .  $\text{MEP}_\pi(L, \rho) = \text{MEP}_\pi(\tilde{L}, \rho) \geq 0$  with equality if and only if the process  $(X_t)$  is  $\pi$ -reversible.*

This says that for finite systems there can be no current without heat, meaning that detailed balance is equivalent with zero mean entropy production. The problem we address here is whether the same remains true in the thermodynamic limit, that is for spatially extended interacting particle systems. In this case we really should be speaking about the mean entropy production *density*, i.e., per unit volume, but we will not use this extension.

We discuss the general physics set-up and further interpretations in Section 3, after stating our mathematical results in Section 2. We start however with three examples illustrating some aspects.

## 1.1 Examples

*Example A:* We consider particles hopping on the one-dimensional lattice with a preferred direction that is itself subject to independent flips. The state space is  $\{-1, +1\} \times \{0, 1\}^{\mathbb{Z}}$  and the process is determined by choosing a constant rate  $c(E, \eta) = 1$  for changes from a configuration  $(E, \eta)$  to  $(-E, \eta)$  and taking rates

$$c(x, E, \eta) = e^E \eta_x (1 - \eta_{x+1}) + e^{-E} \eta_{x+1} (1 - \eta_x)$$

for changes to  $(E, \eta^{x,x+1})$  where  $(\eta^{x,x+1})_y = \eta_y$  if  $x \neq y \neq x+1$ , and  $(\eta^{x,x+1})_y = \eta_x$  when  $y = x+1$  and  $= \eta_{x+1}$  when  $y = x$ . The resulting Markov process has generator

$$\begin{aligned} Lf(E, \eta) &= \sum_x [e^E \eta_x (1 - \eta_{x+1}) + e^{-E} \eta_{x+1} (1 - \eta_x)] [f(E, \eta^{x,x+1}) - f(E, \eta)] \\ &+ f(-E, \eta) - f(E, \eta) \end{aligned} \quad (1.5)$$

For invariant measure  $\rho$  we take

$$\rho(E, d\eta) \equiv \frac{1}{2} (\delta_{E,+1} + \delta_{E,-1}) \times \nu_u(d\eta)$$

where  $\nu_u$  is the Bernoulli measure with specified density  $u \in (0, 1)$ . For time-reversal we take  $\pi(E, \eta) = (-E, \eta)$ .

It is easy to see that the process satisfies generalized detailed balance, like (1.3), in the sense that both

$$c(E, \eta) = c(-E, \eta) \text{ and } c(x, E, \eta) = c(x, -E, \eta^{x,x+1})$$

The last identity depends of course crucially on the fact that  $\pi$  is not the identity and reverses left and right as preferred direction. At the same time, as can be computed

explicitly, the mean entropy production is zero. The same remains true for  $\pi$  a particle-hole transformation,  $(\pi\eta)_x = 1 - \eta_x$ , leaving the field  $E$  unchanged. Then,  $\rho \neq \rho\pi$  for  $u \neq 1/2$  but still generalized detailed balance holds. Finally if, instead, we were to take  $\pi = \text{identity}$  as time-reversal, then we break the detailed balance condition and we obtain a strictly positive mean entropy production.

*Example B:* We take the simplest example of a spinflip dynamics for which the one-dimensional Ising model is stationary but not reversible (for  $\pi = \text{id}$ ). Exactly the same can be done in two dimensions, see [2]. Spinflips are transformations  $U_x : \sigma \rightarrow U_x(\sigma) = \sigma^x, x \in \mathbb{Z}, \sigma \in \{+1, -1\}^{\mathbb{Z}}$  for  $\sigma^x$  equal to  $\sigma$  except at the site  $x$ .

Consider the one-dimensional spinflip dynamics with the following asymmetric rates:

$$c(x, \sigma) = \exp(-2\beta\sigma_x\sigma_{x+1}) \quad (1.6)$$

The invariant measure  $\rho$  is the one-dimensional Ising model at inverse temperature  $\beta$ . The process starting from  $\rho$  is not time-reversal invariant and the entropy production is equal to  $\text{MEP}(L, \rho) = 4\beta \tanh \beta$ . (That is with time-reversal  $\pi = \text{identity}$ .) On the other hand, this time-reversed process is easy to find; it is a spinflip process with generator

$$L^* f(\sigma) = \sum_x e^{-2\beta\sigma_x\sigma_{x-1}} [f(\sigma^x) - f(\sigma)]$$

Let us now take for time-reversal  $\pi$  the reflection:  $(\pi\sigma)_x = \sigma_{-x}$  which leaves  $\rho$  invariant. Since

$$(\pi\sigma)^x = \pi(\sigma^{-x})$$

$L^* = \pi L \pi$  and we have in fact generalized detailed balance (1.3):

$$\frac{c(x, \sigma)}{c(-x, (\pi\sigma)^{-x})} = \frac{d\rho^x}{d\rho} = e^{-2\beta\sigma_x(\sigma_{x-1} + \sigma_{x+1})}$$

The denominator in the left hand side is the rate in the original process by which  $\pi U_x \sigma = \pi(\sigma^x) = (\pi\sigma)^{-x}$  is changed to  $\pi\sigma$ ; to get used to the notation in the next section: here,

$$c(-x, (\pi\sigma)^{-x}) = c(\pi U_x^{-1} \pi, \pi U_x \sigma)$$

As a result,  $\text{MEP}_\pi(L, \rho) = \text{MEP}_\pi(L^*, \rho) = 0$ .

*Example C:* Instead of driving the system in the bulk and breaking detailed balance via some external fields that act on each component of the system, we may also consider boundary driven processes. For this we need to start with finite volume. The simplest interesting case is that of a symmetric exclusion process on a lattice interval that is driven by independent birth and death processes at its boundaries corresponding to different chemical potentials. Take  $\Lambda_n = \{-n, -n+1, \dots, n-1, n\}$

and  $\eta \in \{0, 1\}^{\Lambda_n}$  a particle configuration evolving with generator

$$\begin{aligned}
G_n f(\eta) &= \sum_{x=-n}^{n-1} [f(\eta^{x,x+1}) - f(\eta)] \\
&+ \lambda [e^{h_1 \eta^{-n}} (f(\eta^{-n}) - f(\eta)) + e^{h_2 \eta^n} (f(\eta^n) - f(\eta))] \quad (1.7)
\end{aligned}$$

The first term corresponds to symmetric hopping with exclusion; the two last terms are giving birth and death to particles at the ends of the interval with parameters  $h_1, h_2$ . One can think here of particle reservoirs, to the left of the system with density  $1/(1+e^{h_1})$  and to the right with density  $1/(1+e^{h_2})$ . For  $\lambda = 0$  the system is uncoupled from the reservoirs and it has all uniform product measures as reversible measures with vanishing mean entropy production. For  $\lambda \neq 0, h_1 \neq h_2$  this detailed balance is lost and we have positive mean entropy production. Yet, it remains of order unity, uniformly in the size  $n$  meaning that the mean entropy production density vanishes in the thermodynamic limit. This is an instance of a more general fact for interacting particle systems that will also be treated in the next section: you cannot by driving the system at its boundaries break the time-reversal invariance in the limiting infinite volume process, see Proposition 2 below.

In this paper we show more generally how breaking of detailed balance is strictly equivalent with non-zero mean entropy production. There is no way to get a current and at the same time to have no dissipation (non zero mean entropy production).

In the next section we describe our class of models and we state our main result. In section 3, we discuss this result and we give some more background information concerning entropy production, reversibility and time-reversal. Section 4 is devoted to the proofs.

## 2 Models and main result

### 2.1 Dynamics

This subsection describes the assumptions and introduces the necessary notation. The configuration space is  $\Omega \equiv S^{\mathbb{Z}^d}$  where  $S$  is a finite set and  $\mathbb{Z}^d$  is the regular  $d$ -dimensional lattice. Let  $\pi$  be an involution on  $\Omega$ . A special but important case is when  $\pi = \text{identity}$ . We assume here that  $\pi$  commutes with lattice translations  $\tau_x, x \in \mathbb{Z}^d$ .

Let  $\Lambda_0 \subset \mathbb{Z}^d$  be a finite cube containing the origin and write  $\mathcal{P}_0$  for any specific non-empty set of transformations  $U_0$  satisfying, for every  $U_0 \in \mathcal{P}_0$ , and for every  $\sigma \in \Omega$ :

- i.  $U_0 \sigma \in \Omega$ , and  $(U_0 \sigma)(y) = \sigma(y)$ , for  $y \in \Lambda_0^c$ ,

- ii.  $U_0^{-1} \in \mathcal{P}_0$ ,
- iii.  $\pi \mathcal{P}_0 \pi = \mathcal{P}_0$ ,
- iv.  $U_0 \sigma \neq U'_0 \sigma$  for all  $\sigma \in \Omega$  and  $U_0 \neq U'_0 \in \mathcal{P}_0$  for which  $U_0 \sigma \neq \sigma$  and  $U'_0 \sigma \neq \sigma$  (for convenience only.)

We consider the translations  $\Lambda_x \equiv \{y + x : y \in \Lambda_0\}$  and  $U_x \equiv \tau_x U_0 \tau_{-x}$  to generate a dynamics via local translation invariant rates  $c(U_x, \sigma)$  for the transition  $\sigma \rightarrow U_x \sigma$ . We assume:

- i. Positivity:  $c(U_0, \sigma) = 0$  when  $U_0 \sigma = \sigma$  and if not,  $c(U_0, \sigma) > 0$ ,
- ii. Finite range: there is a finite  $\bar{\Lambda} \subset \mathbb{Z}^d$  such that for all  $\sigma, \eta \in \Omega$ , and  $U_0 \in \mathcal{P}_0$ :  
 $c(U_0, \sigma) = c(U_0, \sigma_{\bar{\Lambda}} \eta_{\bar{\Lambda}^c})$ ,
- iii. Translation invariance: for all  $x \in \mathbb{Z}^d$ ,  $U_x \in \mathcal{P}_x$ ,  $\sigma \in \Omega$ :  $c(U_x, \sigma) = c(U_0, \tau_x \sigma)$

The generator  $L$  corresponding to the given rates is now defined on local functions  $f$  as

$$Lf(\sigma) \equiv \sum_{x \in \mathbb{Z}^d} \sum_{U_x \in \mathcal{P}_x} c(U_x, \sigma) [f(U_x \sigma) - f(\sigma)] \quad (2.8)$$

That is,  $\sigma$  is changed to  $\eta$  at rate  $c(U_x, \sigma)$  if  $\eta = U_x \sigma$ . We will always write  $\rho$  for a translation invariant stationary measure for this dynamics. It can be different from  $\rho\pi$  but we assume that also  $\rho\pi$  is stationary. Finally,  $\rho$  gives positive weight to all cylinders and writing  $\rho^U = \rho U$ , we always assume that  $d\rho^{U_0}/d\rho(\sigma) \geq c > 0$ , which, even in the present rather general set-up, can be expected quite generally.

For  $\Lambda_0 = \{0\}$  and  $S = \{+1, -1\}$ , the choice  $U_x \sigma = \sigma^x$  corresponds to a spinflip process. Taking  $\Lambda_0 = \{0, e_1, e_2, \dots, e_d\}$  with  $e_\alpha$  the lattice unit vectors, we can make a spin exchange process or hopping dynamics. We refer to [8] for further details on constructing the infinite volume process.

## 2.2 Mean Entropy Production

Put  $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$  for large  $n$  and define  $\Lambda_n^\sharp$  as the maximal subset of  $\Lambda_n$ , such that for all  $x \in \Lambda_n^\sharp$  and  $U_x \in \mathcal{P}_x$ ,  $c(U_x, \sigma)$  depends only on coordinates inside  $\Lambda$ , and  $\Lambda_x \subset \Lambda$ . Consider now the Markov chain on  $S^{\Lambda_n}$  with generator

$$L_n f(\sigma) \equiv \sum_{x \in \Lambda_n^\sharp} \sum_{U_x \in \mathcal{P}_x} c(U_x, \sigma) [f(U_x \sigma) - f(\sigma)] \quad (2.9)$$

and started at a fixed configuration  $\omega(-T) \in S^{\Lambda_n}$  at time  $-T$ . Via a Girsanov formula this dynamics gives rise to a Hamiltonian (or action functional) on space-time trajectories  $\omega$  (as in [4]), with corresponding relative energy with respect to

time-reversal equal to

$$\begin{aligned}
R_{T,n,\pi}(\omega) &= \sum_{x \in \Lambda_n^{\dagger}} \sum_{U_x \in \mathcal{P}_x} \int_{-T}^T \log \frac{c(U_x, \omega(s^-))}{c(\pi U_x^{-1} \pi, \pi U_x \omega(s^-))} dN_s^{U_x} \\
&+ \int_{-T}^T [c(U_x, \pi \omega(s)) - c(U_x, \omega(s))] ds
\end{aligned} \tag{2.10}$$

where  $N_t^{U_x} \equiv \sum_{-T \leq s \leq t} I(\omega(s) = U_x(\omega(s^-)) \neq \omega(s^-))$  is the number of times the transformation  $U_x$  appeared in the realization  $\omega$  up to time  $t \in [-T, T]$ .

The mean entropy production for the interacting particle system is defined as

$$\text{MEP}_{\pi}(L, \rho) \equiv \lim_n \lim_{T \uparrow \infty} \frac{1}{2|\Lambda_n|T} \mathbb{E}_{\rho}^{n,T}(R_{T,n,\pi}) \tag{2.11}$$

$\mathbb{E}_{\rho}^{n,T}$  denotes the expectation with respect to the path space measure, in the stationary distribution  $\rho$ , restricted to trajectories within  $S^{\Lambda_n}$ . In other words, the mean entropy production is the expectation of the time-reversal breaking part in the space-time action functional governing the dynamics. We refer to [4] for a mathematical discussion on the existence of the limit (2.11) and for a proof of its non-negativity. We refer to [3, 5] and Section 3 for further background.

### 2.3 Results

The main question is to see whether for a dynamics where the time-reversal symmetry is explicitly broken (in the sense that there is no detailed balance), there still can be zero mean entropy production (dissipationless steady state). Our main result says that this is impossible.

**Main Theorem:** *Under the conditions above,  $\text{MEP}_{\pi}(L, \rho) = \text{MEP}_{\pi}(L, \rho\pi) = 0$  implies that the dynamics satisfies (generalized) detailed balance in the sense that for all  $U_0$*

$$c(\pi U_0^{-1} \pi, \pi U_0 \sigma) \frac{d\rho^{U_0}}{d\rho}(\sigma) = c(U_0, \sigma) \quad \rho - a.s. \tag{2.12}$$

Observe that (2.12) implies that the densities  $d\rho^{U_0}/d\rho$  are  $\pi$ -invariant.

**Corollary 1:** *If  $\text{MEP}_{\pi}(L, \rho) = \text{MEP}_{\pi}(L, \rho\pi) = 0$  and if for all  $\sigma, \sigma' \in \Omega$  for which  $\sigma(x) = \sigma'(x)$ , for all  $x \notin \Lambda_0$ , there exists a transformation  $U_0 \in \mathcal{P}_0$  such that  $\sigma = U_0 \sigma'$ , and  $\pi$  is continuous, then  $\rho$  is a reversible Gibbs measure for the dynamics defined above.*

In [4] the converse to these results was already shown: Suppose that the rates satisfy

$$c(U_x, \sigma) = c(\pi U_x^{-1} \pi, \pi U_x \sigma) \exp(-H(U_x \sigma) + H(\sigma)). \tag{2.13}$$

This is the analogue of (1.3). The energy difference in (2.13) should be interpreted in terms of an absolutely convergent sum of potentials:

$$H(\sigma_\Lambda \eta_{\Lambda^c}) - H(\xi_\Lambda \eta_{\Lambda^c}) = \sum_{A \cap \Lambda \neq \emptyset} (V(A, \sigma_\Lambda \eta_{\Lambda^c}) - V(A, \xi_\Lambda \eta_{\Lambda^c})), \quad (2.14)$$

where  $(V(A, \cdot) : S^A \rightarrow (-\infty, +\infty))$ ,  $A$  finite subsets of  $\mathbb{Z}^d$ , is a translation invariant (uniformly) absolutely summable potential:

$$\sum_{A \ni 0} \max_{\sigma \in S^A} |V(A, \sigma)| < +\infty \quad (2.15)$$

Then,

$$\text{MEP}_\pi(L, \rho) = \text{MEP}_\pi(L, \rho\pi) = 0$$

When we combine the above we obtain a final

**Corollary 2:** *Under the conditions of Corollary 1, if there is one translation invariant stationary measure  $\rho$  for which  $\rho = \rho\pi$  and  $\text{MEP}_\pi(L, \rho) = 0$ , then also  $\text{MEP}_\pi(L, \nu) = 0$  for all translation invariant stationary measures  $\nu$  and they are all Gibbsian for the same potential.*

A caveat in the above main result is to understand better the relation between  $\text{MEP}_\pi(L, \rho)$  and  $\text{MEP}_\pi(L, \rho\pi)$ . To this we can only add that  $\text{MEP}_\pi(\pi L \pi, \rho) = \text{MEP}_\pi(L, \rho\pi)$ , as can be verified from a direct computation starting with (4.22).

The simplest illustration of all this was already obtained in [6] for a spinflip process.

Finally, for completeness we come back to the situation of Example C in Section 1.1. For this we must leave the translation invariant infinite volume context and ask whether boundary driven interacting particle systems can give rise to non-vanishing mean entropy production density in the thermodynamic limit. The question can be formalized as follows. We consider a process on  $S^{\Lambda_n}$  with generator  $G_n$  generalizing (1.7)

$$G_n f(\sigma) \equiv L_n f(\sigma) + \sum_{\substack{A \subset \Lambda_n \setminus \Lambda_n^\# \\ \text{diam} A \leq r}} \sum_{\eta \in S^A} k_A^{(n)}(\sigma, \eta) [f(\sigma^{A, \eta}) - f(\sigma)]$$

where  $\sigma^{A, \eta} \equiv \sigma_{A^c} \eta_A$  equals  $\sigma$  outside the set  $A$  which has a diameter (maximal lattice distance within) less than a given constant  $r$ .

Here the generator  $L_n$  is given by (2.9) but with rates verifying condition (2.13) for a finite range potential, and rates  $k_A^{(n)}(\sigma, \eta)$  as in (1.1) inducing configurational changes at the boundary of  $\Lambda_n$ . We further assume that the  $k_A^{(n)}(\sigma, \eta)$  are uniformly bounded from below and from above. In other words, we have a bulk dynamics generated by  $L_n$  with rates satisfying (generalized) detailed balance, and at the boundary the

configuration can change quite arbitrarily (but in a local and bounded way). We suppose that  $\rho_n$  is the unique stationary measure of this dynamics and for simplicity we only treat the case  $\pi = id$ . We are interested in the mean entropy production  $\text{MEP}(G_n, \rho_n)$  defined in (1.4) (with  $\pi = id$ ).

**Proposition 2:** *There is a constant  $K$  so that  $\text{MEP}_\pi(G_n, \rho_n) \leq Kn^{d-1}$*

The proofs of the above results are postponed to Section 4.

### 3 Discussion

We briefly discuss some concepts that are important for our result.

#### 3.1 Time-reversal

By this we usually mean a transformation on phase space  $\Omega$  which, for a many-particle system, is defined particle-wise or, for spatially extended systems, is sufficiently local. Physically speaking, its precise nature follows from kinematical considerations on the dynamical variables. In classical mechanics, it reverses the momenta of all the particles but in the presence of say an electromagnetic potential, considered part of the system, one can add an extra transformation reversing also the magnetic field and thus making the Lorentz force time-reversal invariant. In our case, we have a configuration space  $\Omega = S^{\mathbb{Z}^d}$  with  $\mathbb{Z}^d$  the  $d$ -dimensional lattice and  $S$  a finite set. Time-reversal is an involution  $\pi$  on  $\Omega$ ,  $\pi^2 = id$ . Time-reversal extends to a transformation on path-space by reversing the trajectories. That is, if we have a trajectory  $(\omega_t, t \in [-T, T])$  then the time-reversed trajectory  $\theta_\pi(\omega)$  is given by  $(\theta_\pi(\omega))_t \equiv \pi\omega_{-t}$ .

#### 3.2 Reversibility

Dynamic reversibility is a property of the dynamics itself under time-reversal. It says that if one trajectory  $\omega$  of the system is possible, so is its time-reversed  $\theta_\pi(\omega)$ . For a deterministic system where  $\omega_t = \phi(t)\omega_0$  with  $\phi(t)$  an invertible flow on phase space, it says that  $\phi(t)^{-1} = \pi\phi(t)\pi$ , that is a symmetry that anticommutes with the time evolution. For a stochastic dynamics this is implied by assuming that if a transition  $\sigma \rightarrow U\sigma$  is possible (positive transition rate), then also the same is true for its time-reversal  $\pi U\sigma \rightarrow \pi\sigma$ .

Microscopic reversibility is a consequence of dynamic reversibility in case of an equilibrium dynamics. For our purposes here we do not make a distinction with the condition of detailed balance. When the dynamics is driven away from equilibrium, the resulting stochastic model will not satisfy detailed balance. Usually this produces a current in the system (but that need not be true in general, see an example in [5]).

On the other hand, a net current signifies the breaking of the detailed balance condition. In general we like to distinguish between two classes of finite volume dynamics where microscopic reversibility is explicitly broken. These are boundary driven versus bulk driven dynamics depending on the extensivity of the perturbation from an equilibrium dynamics. In the bulk driven case, one usually verifies so called *local detailed balance*, i.e., (2.13) is changed into

$$c(U_x, \sigma) = c(\pi U_x^{-1} \pi, \pi U_x \sigma) \exp(-H(U_x \sigma) + H(\sigma)) e^{E\Phi(U_x \sigma, \sigma)}$$

where  $E$  is some amplitude of an external field and  $\Phi$  is antisymmetric,  $\Phi(\pi\eta, \pi\sigma) = -\Phi(\sigma, \eta)$ , see e.g. [9]. Note also that then, necessarily, the relative energies  $H(U_x \sigma) - H(\sigma)$  are  $\pi$ -invariant.

In boundary driven systems, the process becomes non-translation invariant and the rates remain of the form (2.13) in the bulk (that is for  $x$  well inside the considered finite volume) while more or less arbitrary on the boundary. This was the case for Example C in Section 1.1 and was formalized for Proposition 2. Note that there is in fact an example of a boundary driven system where uniformly in the size of the system a bulk current can be maintained. This is the nonequilibrium harmonic crystal treated in [12, 7] where the heat flux is proportional to the boundary temperature difference rather than to the temperature gradient (infinite heat conductivity in the thermodynamic limit). Such ‘superconductors’ do not exist in the context of interacting particle systems as discussed in the present paper.

### 3.3 Entropy production

In phenomenological thermodynamics, entropy production appears in open driven systems as the product of thermodynamic fluxes and forces. The forces are gradients of intensive quantities (like chemical potential) generating the currents. The entropy production is identified from a balance equation for the time-derivative of an entropy density which is defined in systems close to equilibrium. The definition of entropy production as we use it here in statistical mechanics comes from [3, 4, 5, 6, 10, 11, 13]. The mean entropy production appears there as a relative entropy (density) for the process with respect to its time-reversal. That immediately invites the following thought (we are grateful to Senya Shlosman for pointing to this): In equilibrium statistical mechanics, if two translation invariant Gibbs measures have zero relative entropy density, then they must both be Gibbsian for the same interaction potential (but not necessarily equal e.g. because of spontaneous symmetry breaking). Apply this to the space-time measures obtained for the process and the time-reversed process. In some sense, they are Gibbs measures. Thus, if the mean entropy production is zero, then the process itself and its time-reversal have the same (space-time) action functional. Because they also have the same marginals (i.e. the same invariant measure), they must in fact coincide (hence no spontaneous time-reversal breaking). Hence, zero

mean entropy production implies microscopic reversibility. While convincing on a superficial level, unfortunately the details of this argument are technically cumbersome and a direct sufficiently general proof along this line has not been found.

The only more recent paper that we know of concerning time-reversal symmetry and the relation with entropy production is [1]. The set-up there is however quite different from ours. Time-reversal symmetry is there associated with the anticommutation of an involution with the time evolution, what we have called dynamic reversibility in the above. In our discussions here, we deal with spatially extended stochastic dynamics and the breaking of microscopic reversibility.

## 4 Proofs

**Lemma 1:** *Under the conditions of Section 2.1, for a translation invariant stationary measure  $\nu$ ,*

$$\sum_{U_0 \in \mathcal{P}_0} \int d\nu(\sigma) c(U_0, \sigma) \log \frac{d\nu^{U_0}}{d\nu}(\sigma) = 0 \quad (4.16)$$

**Proof:** Let  $\mathcal{F}_\Lambda$  be the  $\sigma$ -field generated by  $\sigma_x, x \in \Lambda$ . Denote by  $\nu_\Lambda$ , respectively  $\nu_\Lambda^{U_0}$  the  $\mathcal{F}_\Lambda$ -restrictions of  $\nu$  and  $\nu^{U_0}$ . Then we have

$$\frac{d\nu_\Lambda^{U_0}}{d\nu_\Lambda} = \mathbb{E}_\nu \left[ \frac{d\nu^{U_0}}{d\nu} \middle| \mathcal{F}_\Lambda \right]$$

Since  $d\nu^{U_0}/d\nu \in L^1(d\nu)$  for all  $U_0$ , we find using the martingale convergence theorem that

$$\lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{d\nu_\Lambda^{U_0}}{d\nu_\Lambda} = \frac{d\nu^{U_0}}{d\nu}, \quad (4.17)$$

in  $L^1(d\nu)$ . Let  $\tilde{\nu}$  be the product measure on  $\Omega$  having as marginals the uniform

measure on  $S$ . From stationarity applied to the local function  $f_\Lambda = d\nu_\Lambda/d\tilde{\nu}_\Lambda$  we find

$$\begin{aligned}
0 &= \sum_{x \in \Lambda'} \sum_{U_x \in \mathcal{P}_x} \int d\nu(\sigma) c(U_x, \sigma) \left[ \log \frac{d\nu_\Lambda^{U_x}}{d\tilde{\nu}_\Lambda} - \log \frac{d\nu_\Lambda}{d\tilde{\nu}_\Lambda} \right] \\
&= \sum_{x \in \Lambda'} \sum_{U_x \in \mathcal{P}_x} \int d\nu(\sigma) c(U_x, \sigma) \log \frac{d\nu_\Lambda^{U_x}}{d\nu_\Lambda} \\
&= \sum_{x \in \Lambda'} \sum_{U_x \in \mathcal{P}_x} \int d\nu(\sigma) c(U_x, \sigma) \log \frac{d\nu^{U_x}}{d\nu} \\
&+ \sum_{x \in \Lambda'} \sum_{U_x \in \mathcal{P}_x} \int d\nu(\sigma) c(U_x, \sigma) \left[ \log \frac{d\nu_\Lambda^{U_x}}{d\nu_\Lambda} - \log \frac{d\nu^{U_x}}{d\nu} \right] \\
&= |\Lambda'| \sum_{U_0 \in \mathcal{P}_0} \int d\nu(\sigma) c(U_0, \sigma) \log \frac{d\nu^{U_0}}{d\nu}(\sigma) \\
&+ \sum_{x \in \Lambda'} \sum_{U_x \in \mathcal{P}_x} \int d\nu(\sigma) c(U_x, \sigma) F_\Lambda^{U_x}(\sigma).
\end{aligned}$$

The last equality uses translation invariance. We have used the notation  $\Lambda' \equiv \{x \in \mathbb{Z}^d | \Lambda_x \cap \Lambda \neq \emptyset\}$  and the expression

$$F_\Lambda^{U_x}(\sigma) \equiv \left( \log \frac{d\nu_\Lambda^{U_x}}{d\nu_\Lambda} - \log \frac{d\nu^{U_x}}{d\nu} \right)$$

We thus have

$$\begin{aligned}
\left| \sum_{U_0 \in \mathcal{P}_0} \int d\nu(\sigma) c(U_0, \sigma) \log \frac{d\nu^{U_0}}{d\nu}(\sigma) \right| &\leq \frac{1}{|\Lambda'|} \sum_{x \in \mathbb{Z}^d} \sum_{U_x \in \mathcal{P}_x} \left| \int d\nu(\sigma) c(U_x, \sigma) F_\Lambda^{U_x}(\sigma) \right| \\
&\leq M \frac{1}{|\Lambda'|} \sum_{x \in \Lambda'} \sum_{U_0 \in \mathcal{P}_0} \int d\nu |F_{\Lambda-x}^{U_0}|, \quad (4.18)
\end{aligned}$$

by the translation invariance of  $\nu$ , and  $M$  bounds the rates. Now we use the general fact that if  $f_n$  converges to  $f$  in  $L^1(d\nu)$  and both  $f_n, f$  are bounded from below by some constant  $c > 0$ , then  $\log f_n$  converges to  $\log f$  in  $L^1(d\nu)$ . This fact implies that for any given  $\varepsilon > 0$ , we can choose  $\Delta \subset \mathbb{Z}^d$  such that for all  $\Delta' \supset \Delta$ :

$$\max_{U_0 \in \mathcal{P}_0} \int d\nu |F_{\Delta'}^{U_0}| \leq \frac{\varepsilon}{2MN}, \quad \text{with } |\mathcal{P}_0| \equiv N$$

Choose now  $\Lambda \subset \mathbb{Z}^d$  so large that

$$\frac{|\{x \in \Lambda' : \Delta + x \cap \Lambda^c \neq \emptyset\}|}{|\Lambda'|} \leq \frac{\varepsilon}{2MN \sup_{V, U_0} \|F_V^{U_0}\|_{L^1(d\nu)}}$$

We then conclude that

$$\left| \sum_{U_0 \in \mathcal{P}_0} \int d\nu(\sigma) c(0, \sigma) \log \frac{d\nu^{U_0}}{d\nu}(\sigma) \right| \leq \frac{1}{|\Lambda'|} \sum_{x \in \Lambda', \Delta+x \subset \Lambda} \frac{\epsilon}{2} + \quad (4.19)$$

$$MN \frac{|\{x \in \Lambda' : \Delta + x \cap \Lambda^c \neq \emptyset\}|}{|\Lambda'|} \sup_{V, U_0} \|F_V^{U_0}\|_{L^1(d\nu)} \quad (4.20)$$

$$\leq \epsilon \quad (4.21)$$

■

**Proof of Main Theorem:** We first recall that the mean entropy production (2.11) equals

$$\begin{aligned} \text{MEP}_\pi(L, \rho) &= \sum_{U_0 \in \mathcal{P}_0} \left( \int \rho(d\sigma) c(U_0, \sigma) \log \frac{c(U_0, \sigma)}{c(\pi U_0^{-1} \pi, \pi U_0 \sigma)} \right. \\ &\quad \left. + \int \rho(d\sigma) [c(U_0, \pi \sigma) - c(U_0, \sigma)] \right) \quad (4.22) \end{aligned}$$

This was derived from (2.10) in [4]. Using Lemma 1 one computes that

$$\text{MEP}_\pi(L, \rho) + \text{MEP}_\pi(L, \rho\pi) = \sum_{U_0 \in \mathcal{P}_0} \int d\rho(\sigma) [c(U_0, \sigma) - \bar{c}(U_0, \sigma)] \log \frac{c(U_0, \sigma)}{\bar{c}(U_0, \sigma)} \quad (4.23)$$

where

$$\bar{c}(U_0, \sigma) \equiv \bar{c}(\pi, \rho; U_0, \sigma) \equiv c(\pi U_0^{-1} \pi, \pi U_0 \sigma) \frac{d\rho^{U_0}}{d\rho}(\sigma)$$

The left hand side is zero by assumption implying the statement of the Theorem. ■

**Proof of Corollary 1** Since the Radon-Nikodym derivative of  $\rho^{U_0}$  with respect to  $\rho$  is a local function for all  $U_0$  and since by assumption, we can generate with the  $U_0$  all local excitations  $\sigma'$  from  $\sigma$ , it means that  $\rho$  has a continuous version for its local conditional distributions. ■

**Proof of Corollary 2** From the main result and Corollary 1 it follows that  $\rho$  is a translation invariant stationary Gibbs measure and (2.13) must be satisfied. All other translation invariant stationary measures must be Gibbsian and for the same potential, see e.g. [2]. From the results in [4] as cited above the statement of Corollary 2, it follows that every other stationary translation invariant measure must have zero mean entropy production. ■

**Proof of Proposition 2** From the definition (1.4) we must first compute the relative action under time reversal, that is

$$R_n \equiv \log \frac{d\mathbb{P}_{\rho_n}}{d\overline{\mathbb{P}}_{\rho_n}}$$

This can be done via a Girsanov formula and we obtain the analogue of (2.10):

$$\begin{aligned} R_n(\omega) &= \sum_{x \in \Lambda_n^\#} \sum_{U_x \in \mathcal{P}_x} \int_{-T}^T \log \frac{c(U_x, \omega(s^-))}{c(U_x^{-1}, U_x \omega(s^-))} dN_s^{U_x}(\omega) \\ &+ \sum_{\substack{A \subset \Lambda_n \setminus \Lambda_n^\# \\ \text{diam}(A) \leq r}} \sum_{\sigma \in S^A} \int_{-T}^T \log \frac{k_A^{(n)}(\omega(s^-), \omega(s^-)^{A, \sigma})}{k_A^{(n)}(\omega(s^-)^{A, \sigma}, \omega(s^-))} dN_s^{A, \sigma, n}(\omega) \end{aligned}$$

The first integral is really a sum over all the times when the trajectory makes a jump from the action of one the  $U_x$ ; the second integral is a sum over all times when a configuration  $\sigma$  is replacing  $\eta$  in a set  $A$  on the boundary. In order to further clarify this formula, let us first look at trajectories where no boundary transitions take place (or, what amounts to the same, take  $k \equiv 0$  for the moment). Then, we only keep the first term, that is just (2.10):

$$\sum_{x \in \Lambda_n^\#} \sum_{U_x \in \mathcal{P}_x} \int_{-T}^T \log \frac{c(U_x, \omega(s^-))}{c(U_x^{-1}, U_x \omega(s^-))} dN_s^{U_x}$$

But if we insert the detailed balance condition (2.13), the above expression telescopes to

$$H(\omega(-T)) - H(\omega(T))$$

and the mean entropy production is zero by stationarity.

Turning to the general case we let  $\{s_i\}_{i=1}^q$  be the set of times at which boundary transitions occur in the sets  $A_i, i = 1, \dots, q$ , for the trajectory  $\omega$ . These are random but we fix them as  $-T \leq s_1 < s_2 < \dots < s_q \leq T$ . The important thing to realize now is that while the perfect telescoping of above is broken at each of these times, it can be restored by adding and subtracting. More precisely, we have

$$\begin{aligned} R_n(\omega) &= H(\omega(-T)) - H(\omega(s_1^-)) + H(\omega(s_1)) - H(\omega(s_2^-)) + \dots \\ &+ H(\omega(s_q)) - H(\omega(T)) + \log \frac{k_{A_1}^{(n)}(\omega(s_1^-), \omega(s_1))}{k_{A_1}^{(n)}(\omega(s_1), \omega(s_1^-))} + \\ &+ \log \frac{k_{A_2}^{(n)}(\omega(s_2^-), \omega(s_2))}{k_{A_2}^{(n)}(\omega(s_2), \omega(s_2^-))} + \dots + \log \frac{k_{A_q}^{(n)}(\omega(s_q^-), \omega(s_q^-))}{k_{A_q}^{(n)}(\omega(s_q), \omega(s_q^-))} \end{aligned}$$

But by the absolute convergence of the interaction potential we have

$$|H(\omega(s_i^-) - H(\omega(s_i)))| \leq rC$$

for some constant  $C$ , since  $\omega(s_i^-)$  and  $\omega(s_i)$  only differ in the set  $A_i$ . Therefore the telescoping of the terms involving energy differences can be restored upon inserting  $q$  terms of order unity.

As for the other terms, we have assumed uniform boundedness so that we get

$$|R_n(\omega)| \leq q(rC + \log \frac{M}{\epsilon})$$

where  $M$  and  $\epsilon$  are constant upper and lower bounds for the transition rates  $k^{(n)}$ . As the expectation of  $q = q(\omega)$  under  $\mathbb{E}_{\rho_n}$  is proportional to  $T|\partial\Lambda_n|$ , the proposition is proved.  $\blacksquare$

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