On finite simple subgroups of the complex Lie group of type E_8

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On Finite Simple Subgroups of the Complex Lie Group of Type E_8

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Abstract. The object of this paper is to produce a list of finite quasisimple groups which may be embedded in the Lie group $G = E_8(\mathbf{C})$. We provide additional information about the ways these finite groups are embedded and have strong restrictions for the cases which are questionable. We hope that the results of this paper can be used to give a classification of all finite subgroups of G.

1. Introduction and statement of results. We can regard the embedding of a finite group L in a Lie group as a variation of the theme of representation theory. If the Lie group is of classical type B_n, C_n, or D_n, the embedding of L is, up to central extensions, just a faithful representation of L of degree $2n+1$, $2n$, or $2n$ leaving invariant a nondegenerate form which is symmetric, alternating and symmetric, in the respective cases; and so the embeddings into these Lie groups form part of the classical theory. This leaves the five exceptional types of simple Lie groups. The finite subgroups of $G_2(\mathbf{C})$ have been determined (up to conjugacy) in Cohen and Wales [**10**], and those of $E_6(\mathbf{C})$ and $F_4(\mathbf{C})$ are described in Cohen and Wales [**11**]. In this paper, we begin the case of $E_8(\mathbf{C})$ and obtain some results for $E_7(\mathbf{C})$ and other Lie subgroups as a byproduct.

Our main result is Theorem 1.1 below. To prove it, we go through the list of finite simple groups. Thus, we take as an axiom the completeness of the list of finite simple groups. A second axiom here is the correctness of the relevant information in the Atlas [**12**]. We examine conjugacy classes, centralizers, and character tables extensively, even though we make reasonable efforts to use elementary arguments and general theory whenever possible. To reduce the chance of having to adapt our arguments, or (worse) our conclusions in case of a corrected mistake in the Atlas, we have tried to check nonexistence arguments in at least two ways (although we usually present one in the paper).

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Finite subgroups of a Lie group are compact (0-dimensional) Lie subgroups. For complex simple Lie groups G, maximal closed Lie subgroups of strictly positive dimension are essentially determined by Dynkin [**15**] (see also Borel and De Siebenthal [**4**] for the case of maximal Lie rank). In this context, it is natural to ask for the list of all finite subgroups (up to conjugacy, or coarser, up to isomorphism) of a given Lie group G, which are not contained in a closed Lie subgroup of G of dimension > 0. In the next section, we describe how our main theorem fits into such a classification for $G = E_8(\mathbf{C})$. (See §1.5 for notation used in the theorem below.)

1.1. THEOREM. *Suppose L is a finite nonabelian simple group and G is the complex Lie group of type E_8.*

(i) *If L can be embedded in G, then L is isomorphic to one of the following groups*: Alt_n $(5 \le n \le 10)$, $L(2,q)$ $(q = 7,8,11,13,16,17,19,25,27,29,31,32,61)$, $L(3,q)$ $(q = 3,5)$, $PSU(3,3)$, $PSU(3,8)$, $PSU(4,2)$, $PS\Omega(7,2)$, $PS\Omega^+(8,2)$, $G_2(3)$, ${}^3D_4(2)$, ${}^2F_4(2)'$, M_{11}.

(ii) *If a perfect nonsplit central extension $\hat{L}$ of L can be embedded in G, then $\hat{L}$ is isomorphic to one of*: $2 \cdot \mathrm{Alt}_n$ $(5 \le n \le 17)$, $3 \cdot \mathrm{Alt}_n$ $(6 \le n \le 7)$, $6 \cdot \mathrm{Alt}_n$ $(6 \le n \le 7)$, $SL(2,q)(q = 7,8,11,13,17,19,25,29,31,32,37)$, $SL(3,q)$ $(q = 3,4)$, $4 \cdot L(3,4)$, $6 \cdot L(3,4)$, $2 \cdot PSU(4,2)$, $6 \cdot PSU(4,3)$, $\mathrm{Sp}(4,5)$, $2 \cdot \Omega(7,2)$, $2 \cdot \Omega^+(8,2)$, $2^2 \cdot \Omega^+(8,2)$, $2 \cdot M_{12}$, $2 \cdot J_2$.

As of this moment, we have no full answer to the converse, but in the open cases we give feasible characters of L on the adjoint module $\mathbf{g}$ for G. We do not deal with the question of conjugacy. In Table 1, the symbols preceding L give the status of embeddings of L into G. A + indicates that an embedding exists, and a ? indicates that the existence is plausible but not established. In some cases, the shape of an interesting character of L is given in the column following L (an irreducible character is given by its dimension and an index frequently related to the order of appearance in the Atlas [**12**]; the expression $12 \times 3_*$ stands for a sum of twelve (nonspecified) irreducible characters of degree 3). The character given is (or, if there is a question mark in the first column: is conjectured to be) the restriction to a subgroup isomorphic to L of the adjoint character χ for G. If relevant, a statement about uniqueness (notation !) of $\chi|_L$ (sometimes as a fixed point free character, notation fpf) is added. The last column gives information on a known connected Lie subgroup of G containing a subgroup isomorphic to L (of course, in the case $PSU(3,8)$, where an embedding has not been established, we mean that an embedding in G would imply an embedding of L in $E_7(\mathbf{C})$). It should be stressed that there may be other embeddings of L via (possibly distinct) connected Lie subgroups. In the cases $L(2,q)$ $(q \in \{31,32,61\})$, there is no proper closed Lie subgroup of strictly positive dimension containing a subgroup of G isomorphic to $L(2,q)$; this is expressed by the abbreviation max for maximal in the last column.

TABLE 1

Finite nonabelian simple groups L and their embeddability in $E_8(\mathbf{C})$

status	L	$\chi\|_L$	via, e.g.,
+	Alt_5	! fpf: $14 \times (3_a + 3_b) + 16 \times 4_a + 20 \times 5_a$	$SL(3,\mathbf{C})$
+	Alt_6		$SL(5,\mathbf{C})$
+	Alt_7		$SL(6,\mathbf{C})$
+	Alt_8	no fpf	$SL(7,\mathbf{C})$
+	Alt_9	no fpf	$SL(8,\mathbf{C})$
+	Alt_{10}	!: $9_a + 35_a + 36_a + 2 \times 84_a$	$3 \cdot PSL(9,\mathbf{C})$
+	$L(2,7)$	!fpf: $5 \times 3_a + 5 \times 3_b + 6 \times 6_a + 10 \times 7_a + 14 \times 8_a$	$SL(3,\mathbf{C})$
+	$L(2,8)$		$G_2(\mathbf{C})$
+	$L(2,11)$		$SL(5,\mathbf{C})$
+	$L(2,13)$		$G_2(\mathbf{C})$
+	$L(2,16)$	$12 \times 15_* + 17_d + 17_e + 17_f + 17_g$	$\mathrm{HSpin}(16,\mathbf{C})$
+	$L(2,17)$		$3 \cdot PSL(9,\mathbf{C})$
+	$L(2,19)$		$3 \cdot PSL(9,\mathbf{C})$
+	$L(2,25)$		$F_4(\mathbf{C})$
+	$L(2,27)$	$14 \times 1_a + 26_a + 26_b + 7 \times 26_d$	$F_4(\mathbf{C})$
+	$L(2,29)$	$(2\varepsilon + 1) \times 15_* + 28_a + 28_b + 28_c + 29_a + 30_d$ $+30_e + 30_f + (1-\varepsilon) \times 30*, \varepsilon = 0, 1$	$\mathrm{Spin}(15,\mathbf{C})$
?	$L(2,31)$	$2 \times 30_b + 2 \times 30_c + 32_d + 32_e + 32_f + 32_g$	max
?	$L(2,32)$	$8 \times 31_*$	max
?	$L(2,61)$	$62_a + 62_b + 62_c + 62_d$	max
+	$L(3,3)$		via $F_4(\mathbf{C})$
+	$L(3,5)$	!: $124_a + 124_b$	$3^5 : SL(3,4)$
+	$PSU(3,3)$		$G_2(\mathbf{C})$
?	$PSU(3,8)$	$3 \times 1_a + 133_* + 2 \times 56_a$	$2 \cdot E_7(\mathbf{C})$
+	$PSU(4,2)$		$SL(5,\mathbf{C})$
+	$P S\Omega(7,2)$	$4 \times 1_a + 6 \times 7_a + 5 \times 21_a + 27_a + 2 \times 35_a$	$SL(7,\mathbf{C})$
+	$P S\Omega^+(8,2)$	!: $3 \times 1_a + 5 \times 28_a + 35_a + 35_b + 35_c$	$4 \cdot PSL(8,\mathbf{C})$
+	${}^3D_4(2)$	!: $14 \times 1_a + 7 \times 26_a + 52_a$	$F_4(\mathbf{C})$
+	${}^2F_4(2)'$	!: $8 \times 1_a + 3 \times 27_c + 3 \times 27_d + 78_a$	$3 \cdot E_6(\mathbf{C})$
+	$G_2(3)$	!: $1_a + 2 \times 14_a + 64_a + 64_b + 91_a$	$\mathrm{Spin}(14,\mathbf{C})$
+	M_{11}		$\mathrm{Spin}(10,\mathbf{C})$

In Table 2 we give information about the embedding in G of nonsimple quasisimple groups $\hat{L}$ with simple quotients L (i.e., of nontrivial perfect central extensions $\hat{L}$ of nonabelian simple groups L). The meaning of + and ? in the first column is as for Table 1. In the third column, we give a (possible) type of Lie subgroup $C_G(Z(\hat{L}))$ and, in the fourth column, a connected Lie subgroup of $C_G(Z(\hat{L}))$ containing $\hat{L}$ (with the understanding that the embedding of $\hat{L}$ in such a group is plausible (but not proven) if a question mark appears in the first column of the corresponding line).

TABLE 2

Finite quasisimple (but not simple) groups $\hat{L}$ and their embeddability in $G = E_8(\mathbf{C})$

status	$\hat{L}$	type of $C_G(Z(\hat{L}))$	an intermediate Lie subgroup
+	$2 \cdot \mathrm{Alt}_n$ $(n \leq 17)$	D_8	$\mathrm{HSpin}(16, \mathbf{C})$
+	$3 \cdot \mathrm{Alt}_6$	A_2E_6	$SL(3, \mathbf{C})$
+	$3 \cdot \mathrm{Alt}_7$	A_2E_6	$SL(6, \mathbf{C})$
+	$6 \cdot \mathrm{Alt}_6$	$A_5A_2A_1$	$SL(6, \mathbf{C})$
+	$6 \cdot \mathrm{Alt}_7$	$A_5A_2A_1$	$SL(6, \mathbf{C})$
+	$SL(2, 7)$	D_8	$SL(4, \mathbf{C})$
+	$SL(2, 11)$	D_8	$SL(6, \mathbf{C})$
+	$SL(2, 13)$	D_8	$SL(6, \mathbf{C})$
+	$SL(2, 17)$	D_8	$SL(8, \mathbf{C})$
?	$SL(2, 19)$	A_1E_7	$2 \cdot E_7(\mathbf{C})$
+	$SL(2, 25)$	D_8	$\mathrm{Spin}(13, \mathbf{C})$
?	$SL(2, 29)$	A_1E_7	$2 \cdot E_7(\mathbf{C})$
?	$SL(2, 37)$	A_1E_7	$2 \cdot E_7(\mathbf{C})$
+	$2 \cdot L(3, 4)$	A_1E_7	$4 \cdot PSL(8, \mathbf{C})$
+	$4 \cdot L(3, 4)$	A_7A_1	$SL(8, \mathbf{C})$
+	$6 \cdot L(3, 4)$	$A_5A_2A_1$	$SL(6, \mathbf{C})$
+	$2 \cdot PSU(4, 2)$	D_8	$SL(4, \mathbf{C})$
+	$6 \cdot PSU(4, 3)$	$A_5A_2A_1$	$SL(6, \mathbf{C})$
+	$2 \cdot PS\Omega(7, 2)$	D_8	$SL(8, \mathbf{C})$
+	$2 \cdot PS\Omega^+(8, 2)$	D_8	$SL(8, \mathbf{C})$
+	$2^2 \cdot PS\Omega^+(8, 2)$	D_4D_4	$\mathrm{Spin}(8, \mathbf{C})$
+	$\mathrm{Sp}(4, 5)$	D_8	$\mathrm{Spin}(13, \mathbf{C})$
+	$2 \cdot M_{12}$	D_8	$\mathrm{Spin}(12, \mathbf{C})$
+	$2 \cdot J_2$	D_8	$SL(6, \mathbf{C})$

We deal with central extensions of finite simple groups because suitable classifications of finite subgroups of proper closed Lie subgroups of $E_8(\mathbf{C})$ of strictly positive dimension do not exist for all cases. Notably, the adjoint complex Lie group of type E_7 does not (regularly) embed in G while its simply connected cover does, so a finite simple group L will have an embedding in the adjoint Lie group of type E_7 if and only if it has a central extension L_1 with center $Z(L_1)$ of order at most 2 which can be embedded in a closed Lie subgroup of G of type E_7. Thus, our main result contributes to filling the gaps in those classifications (cf. Corollary 1.2).

Another reason for encompassing central covers in our study is that various finite simple groups contain central covers of a smaller simple group whose nonembeddability in $E_8(\mathbf{C})$ is readily established. By this criterium, several finite simple groups are eliminated as candidates for an embedding in G.

1.2. COROLLARY. *If L is a nonabelian finite subgroup of $E_7(\mathbf{C})$ then L is isomorphic to one of* Alt_n $(n \leq 9)$, $L(2, q)$ $(q = 7, 8, 11, 13, 17, 19, 25, 27, 29)$, $L(3, 3)$, $L(3, 4)$, $PSU(3, 3)$, $PSU(3, 8)$, $PSU(4, 2)$, $PS\Omega(7, 2)$, $PS\Omega^+(8, 2)$, ${}^3D_4(2)$, ${}^2F_4(2)'$, M_{11}, M_{12}, J_2.

Except for $L(2, 19)$, $L(2, 29)$, $PSU(3, 8)$, and J_2, we know that all isomorphism types occur. $2 \cdot M_{12}$ embeds via $\mathrm{Spin}(12, \mathbf{C})$ in the simply connected Lie subgroup K of G of type E_7 in such a way that $Z(K)$ coincides with the center of this cover; however, $2 \cdot \mathrm{Alt}_{13}$ embeds in K via $\mathrm{Spin}(12, \mathbf{C})$ in such a way that the centers of these groups do not coincide.

We would like to see the question marks in Tables 1 and 2 settled. In order to finish the full problem of determining isomorphism classes of finite subgroups, knowledge about normalizers and centralizers of nonabelian simple subgroups L of G is required. In this paper, we do not tend to this problem at all. We know of very few criteria to show nonmaximality in case $\chi|_L$ does not involve the trivial character. As a consequence, the following list of maximal finite simple subgroups of G may well be longer than the actual one.

1.3. COROLLARY. *If L is a finite simple subgroup of G which is not contained in a positive-dimensional closed Lie subgroup of G, then L is isomorphic to one of the following groups:* Alt_5 [*with* $\chi|_L$ *as in Table* 1], Alt_6, Alt_7, Alt_{10}, $L(2,q)$ $(q \in \{7,8,11,13,16,17,19,25,29,31,32,61\})$, $L(3,5)$, $PSU(3,3)$, $PSU(4,2)$, $PS\Omega(7,2)$, M_{11}.

1.4. REMARK. Let H be a group of exceptional Lie type with $Z(H) = 1$ and let h be its Coxeter number. Kostant observed that $2h+1$ is a prime power and that H is likely to contain $L \cong PSL(2, 2h+1)$. More generally, if d is a regular number (cf. Springer [**25**]) such that $2d+1$ is a prime power, it seems interesting to ask whether H contains a subgroup isomorphic to $PSL(2, 2d+1)$. According to our main theorem, the answer is negative for $H = E_8(\mathbf{C})$ and $d = 20$ or 24. Nevertheless, the special choice $d = h/2$ leads to an interesting list, including $L(2,19)$ and $L(2,31)$ for $E_7(\mathbf{C})$ and $E_8(\mathbf{C})$, respectively.

1.5. NOTATION. Our notation is reasonably standard. For general group theory and finite simple groups, see Huppert [**19**], Gorenstein [**16**]. For a subset S of a group, $S^\#$ denotes $S - \{1\}$. We write $\mathrm{Frob}_{a \cdot b}$ for a Frobenius group with kernel of order a and complement of order b, and Alt_n, Sym_n for the alternating and symmetric group on n letters, respectively. Furthermore, Dih_n stands for the dihedral group of order n. Let p be a prime. We sometimes use p^n to denote the elementary abelian p-group $\mathbf{Z}_p^n$ and p^{1+2n} for an extraspecial p-group; when $p = 2$, we may write 2_ε^{1+2n} where $\varepsilon = +$ if p^{n+1} embeds in the extraspecial group, and $\varepsilon = -$ if not. For groups A and B, $A.B$ denotes an extension with normal subgroup A and quotient B. The action of $A.B$ on A is meant to be understood from the context. Also $A : B$ and $A \cdot B$ denote split and nonsplit extensions, respectively (again, there is ambiguity in general, but it should be resolved in context).

2. Generalities on finite subgroups of $E_8(\mathbf{C})$. The following division of cases shows how to relate the above result to the classification of all finite subgroups.

2.1. PROPOSITION. *Let G be a semisimple Lie group with trivial center. For a finite group L embedded in G, one of the following holds:*

(i) *L is contained in a proper Lie subgroup of G of positive dimension;*

(ii) $L \leq N_G(E)$ *where* E *is a nontrivial elementary abelian subgroup of* G *such that* $C_G(A)$ *is finite for every nontrivial* $N_G(E)$*-invariant subgroup* A *of* E;

(iii) L *has a nonabelian normal subgroup* K *which is a direct product of isomorphic finite simple groups such that* $C_G(K)$ *is finite.*

PROOF. Suppose (i) does not hold. If K is a nontrivial normal subgroup of L, then $C_G(K)$ is normalized by L, whence finite. If moreover, K is a minimal normal subgroup of L, then, by standard theory (cf. Gorenstein [**16**]), K is a direct product of isomorphic simple groups. If K is abelian, then (ii) readily follows by taking $E = K$ and reduction to $L = N_G(E)$; otherwise the simple factors of K are nonabelian and (iii) follows. □

2.2. REMARKS. If $G = E_8(\mathbf{C})$ and $L \leq G$ is as in (ii) but not as in (i) of the proposition, the subgroup E is known to be conjugate to one of two particular elementary abelian subgroups of G of order 2^5 and 5^3, respectively, and $N_G(E)$ is isomorphic to a group of shape

$$2^{5+10}.SL(5,2) \quad \text{or} \quad 5^3.SL(3,5),$$

respectively (cf. Alekseevskii [**2**]). It may be worth making a few remarks about these groups. The subgroup L of $G = E_8(\mathbf{C})$ of shape $2^{5+10}.SL(5,2)$ is nonsplit. (This follows from the main theorem.) The elementary abelian normal subgroup $E = [O_2(L), O_2(L)]$ of order 2^5 has 31 involutions from a single conjugacy class in G. Its centralizer $C_G(E) = C_L(E)$ is special of order 2^{5+10}, and its normalizer L contains a subgroup of shape $E \cdot SL(5,2)$, the famous nonsplit Dempwolff extension (cf. Thompson [**27**] and Griess [**17**]).

The subgroup L of G of shape $5^3.SL(3,5)$ is split. Unpublished work of McLaughlin shows that $H^2(SL(3,5), \mathbf{F}_5^3) \cong \mathbf{F}_5$ and that in a nonsplit extension

$$1 \to E \to L \to SL(3,5) \to 1$$

with $E \cong \mathbf{F}_5^3$ the group $X = C/D$, where D is a hyperplane of E and $C = C_L(E/D)$ satisfies $O_5(X) \cong 5^{1+2}$ (extraspecial). It follows (cf. Curtis and Reiner [**13**, Corollary (50.7)]) that any faithful character of L has degree at least $5 \cdot 124 = 620$, whence the group $L \cong E.SL(3,5)$ in G is a split extension. A Sylow 5-group of L has exponent 5.

The following result is essentially due to Jacobson (cf. Chevalley [**7**]).

2.3. PROPOSITION. *Let* L *be a group operating as a group of automorphisms on the finite dimensional Lie algebra* $\mathbf{g}$ *and acting completely reducibly (e.g.,* L *finite). If* $\mathbf{g}$ *is reductive, then so is* $C_{\mathbf{g}}(L)$.

PROOF. Write $\mathbf{a} = C_{\mathbf{g}}(L)$ and let F be the sum of all nontrivial L-submodules of $\mathbf{g}$. Then $\mathbf{a}$ is a subalgebra of $\mathbf{g}$ such that $\mathbf{g}$ is the direct sum of $\mathbf{a}$ and F, and $[\mathbf{a}, F] \subset F$. Therefore, by [**7**, Chapter V, §2, no. 7], $\mathbf{a}$ is reductive. □

The induction hypothesis with respect to closed Lie subgroups will primarily be used through the following

2.4. COROLLARY. *If $\hat{L}$ is a finite quasisimple subgroup of the complex Lie group G of type E_8, then L fixes no nonzero vector of the Lie algebra* $\mathbf{g}$ *of G or $L = \hat{L}/Z(\hat{L})$ can be embedded in a Lie group whose type is one of A_i $(1 \le i \le 7)$, D_j $(5 \le j \le 7)$, E_k $(6 \le k \le 7)$.*

PROOF. Suppose $\hat{L}$ fixes a vector in $\mathbf{g}$. In view of Proposition 2.3 there is a nonzero toral element fixed by $\hat{L}$. Therefore $\hat{L}$ centralizes a torus T in G. Since $C_G(T)$ is reductive and connected, has rank at most $8 - \dim T$ and is generated by root groups from a system for a maximal torus containing T, we are done. □

2.5. LEMMA. *If L is a finite subgroup of G, then $\chi|_L$, where χ is the character of G on the adjoint module* $\mathbf{g}$, *is a sum of (not necessarily irreducible) real characters, and*

$$\sum_{g\in L}(\chi(g)^3 - 3\chi(g^2)\chi(g) + 2\chi(g^3)) > 0.$$

PROOF. As L is compact, it can be embedded in a maximal compact subgroup of G; therefore, L acts as a group of automorphisms on a real compact form $\mathbf{g}_{\mathbf{R}}$ of $\mathbf{g}$ and preserves the nondegenerate real anisotropic symmetric bilinear Killing form $\kappa_{\mathbf{R}}$ on $\mathbf{g}_{\mathbf{R}}$. The inequality can be written as $(\bigwedge^3 \chi|_L, 1) > 0$, and as such expresses the fact that L preserves an alternating trilinear form on $\mathbf{g}$, viz., $(x,y,z) \mapsto \kappa([x,y],z)$ $(x,y,z \in \mathbf{g})$. □

2.6. PROPOSITION. *Let G be a simple adjoint Lie group of type X, say, where X is one of A_i $(1 \le i \le 8)$, D_i $(5 \le i \le 8)$, or E_6. If L is a nonabelian finite simple subgroup of G, then L is one of the groups in Table 3. (Each L is given only once, namely on the first line containing the type X of a group in which it occurs.)*

TABLE 3

All nonabelian finite simple groups L embeddable in a closed Lie subgroup of type X

X	L
A_1	Alt_5
A_2	$\mathrm{Alt}_6, L(2,7)$
A_3	$\mathrm{Alt}_7, PSU(4,2)$
A_4	$L(2,11)$
A_5	$\mathrm{Alt}_5, L(2,13), L(3,4), PSU(3,3), PSU(4,3), J_2$
A_6	$\mathrm{Alt}_8, L(2,8), PS\Omega(7,2)$
A_7	$\mathrm{Alt}_9, L(2,17), L(3,4), PS\Omega^+(8,2)$
A_8	$\mathrm{Alt}_{10}, L(2,19)$
D_5	$\mathrm{Alt}_{11}, M_{11}$
D_6	$\mathrm{Alt}_{12}, \mathrm{Alt}_{13}, L(3,3), M_{12}$
D_7	$\mathrm{Alt}_{14}, \mathrm{Alt}_{15}, L(2,25), P\,\mathrm{Sp}(4,5), G_2(3)$
D_8	$\mathrm{Alt}_{16}, \mathrm{Alt}_{17}, L(2,16), L(2,29)$
E_6	${}^3D_4(2), {}^2F_4(2)', L(2,27)$

PROOF. As for all but the last type X, this is straightforward from standard representation theory and the classification of finite simple groups (cf. Landazuri and Seitz [**21**]). As for the case $X = E_6$, see Cohen and Wales [**11**]. □

Note that the pair L, X does not always correspond to an embedding of the simple group L in $G = E_8(\mathbf{C})$; for instance $L(3,4)$ has a central extension by a group of order 4 which embeds in $SL(8, \mathbf{C})$ and hence in G, but, according to the Theorem 1.1, there is no subgroup in G isomorphic to $L(3,4)$.

3. Small subgroups of $E_8(\mathbf{C})$. Throughout the rest of the paper, G shall denote $E_8(\mathbf{C})$, $\mathbf{g}$ the Lie algebra of G (of dimension 248), and χ the character of G on $\mathbf{g}$. According to Moody and Patera [**24**], the number of conjugacy classes in G of elements of order a divisor of i is the coefficient of t^i in the power series expansion of

$$\frac{1}{(1-t)(1-t^2)^2(1-t^3)^2(1-t^4)^2(1-t^5)(1-t^6)}.$$

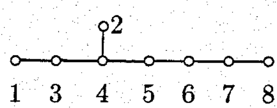
This series begins as follows:

$$1 + t + 3t^2 + 5t^3 + 10t^4 + 15t^5 + 27t^6 + 30t^7 + 63t^8 + 90t^9 + 135t^{10} + 187t^{11} + 270t^{12} + 364t^{13} + \cdots.$$

Thus, the number of conjugacy classes of elements of order i $(1 \le i \le 13)$ is the coefficient of t^i in

$$t + 2t^2 + 4t^3 + 7t^4 + 14t^5 + 20t^6 + 38t^7 + 53t^8 + 85t^9 + 118t^{10} + 186t^{11} + 236t^{12} + 363t^{13} + \cdots.$$

Fix a Cartan subspace $\mathbf{h}$ of $\mathbf{g}$ and a root system $\Phi \subset \mathbf{h}^*$ of $\mathbf{h}$. We shall index the roots of a fundamental set in Φ (with respect to some ordering coming from a Borel) as in Bourbaki [**6**]:



Each semisimple automorphism of $\mathbf{g}$ is conjugate in $G = \operatorname{aut} \mathbf{g}$ to an element $h_\alpha(\varsigma) \in G$ acting trivially on $\mathbf{h}$ and by the scalar multiplication with $\varsigma^{\alpha\cdot\beta}$ on the root space associated with $\beta \in \Phi$. (Here, $\cdot$ denotes the usual dot product with respect to the fundamental basis of roots in Φ.)

3.1. PROPOSITION (*Feasibility of characters on* $\mathbf{g}$). *The elements of order* $k = 2, 3, 4, 5, 6,$ *and* 7 *in* $G = E_8(\mathbf{C})$ *are given explicitly (up to a conjugacy) in Table* 4. *The values of* χ *on these elements are as listed in the last column of this table.*

Each line in Table 4 corresponds to the element $h_\alpha(e^{2\pi i m_j/k})$ of G (for $m_1 = 1$), and nonconjugate powers (for $m_2, \ldots > 1$). The first column contains a label the first digit of which indicates the order k of this element; between square brackets are the exponents $m_2, \ldots$ of nontrivial powers representing distinct conjugacy classes of elements of the same order and letters indicating the labels of conjugacy classes of powers by prime divisors of k (in decreasing order of the power).

TABLE 4

Representatives $h_\alpha(e^{2\pi m_j i/k})$ of conjugacy classes in $E_8(\mathbf{C})$ of elements of order $k = 2, 3, 4, 5, 6,$ and 7

label [exps, powers]	α	centralizer	multiplicity of $e^{2\pi ij/k}$ for $j = 0, \ldots, [(k+1)/2]$	trace on $\mathfrak{g}$
2A	$(0,0,0,0,0,0,0,1)$	A_1E_7	136 112	24
2B	$(1,0,0,0,0,0,0,0)$	D_8	120 128	-8
3A	$(0,1,0,0,0,0,0,0)$	A_8	80 84	-4
3B	$(0,0,0,0,0,0,1,0)$	A_2E_6	86 81	5
3C	$(1,0,0,0,0,0,0,0)$	D_7T_1	92 78	14
3D	$(1,0,2,0,1,2,0,1)$	E_7T_1	134 57	77
4A[A]	$(0,0,0,1,0,0,0,0)$	A_7A_1	66 56 70	-4
4B[B]	$(1,0,1,0,0,0,0,0)$	A_7T_1	64 64 56	8
4C[B]	$(0,0,0,0,0,1,0,0)$	A_3D_5	60 64 60	0
4D[A]	$(2,0,,0,0,0,0,0,1)$	$A_1D_6T_1$	70 56 66	4
4E[A]	$(0,0,0,0,0,0,1,0)$	$A_1E_6T_1$	82 56 54	28
4F[B]	$(2,1,0,0,0,0,0,0)$	D_7T_1	92 64 28	64
4G[A]	$(0,0,0,0,0,0,0,1)$	E_7T_1	134 56 2	132
5A[2]	$(2,2,0,0,0,0,0,0)$	A_7T_1	64 56 36	$8+20\tau$
5B[2]	$(0,1,0,0,0,0,0,1)$	$A_6A_1T_1$	52 49 49	3
5C	$(1,2,1,4,0,1,2,3)$	A_4A_4	48 50 50	-2
5D[2]	$(0,0,0,0,0,1,0,0$	$A_2D_5T_1$	54 51 46	$3+5\tau$
5E[2]	$(0,0,0,0,0,0,1,0)$	$A_1E_6T_1$	82 54 29	$28+25\tau$
5F[2]	$(3,1,0,0,0,0,0,0)$	D_7T_1	92 64 14	$28+50\tau$
5G	$(1,0,0,0,0,0,0,1)$	D_6T_2	68 45 45	23
5H[2]	$(0,0,0,0,0,0,0,1)$	E_7T_1	134 56 1	$78+55\tau$
6A[B, A]	$(0,1,0,0,0,0,0,0)$	A_7T_1	64 56 28 16	76
6B[B, D]	$(1,2,0,0,0,0,0,0)$	A_7T_1	64 29 28 70	-5
6C[A, A]	$(0,0,1,0,0,0,0,0)$	$A_6A_1T_1$	54 42 42 28	24
6D[B, C]	$(1,0,3,0,0,0,0,0)$	A_6T_2	50 43 35 42	16
6E[A, C]	$(1,1,1,0,0,0,0,0)$	A_6T_2	50 35 43 42	0
6F[A, B]	$(0,0,0,1,0,0,0,0)$	$A_5A_2A_1$	46 36 45 40	-3
6G[A, A]	$(0,1,0,0,0,0,1,0)$	$A_5A_2T_1$	44 38 46 36	0
6H[B, B]	$(1,2,3,5,4,3,2,1)$	$A_5A_1A_1T_1$	42 42 39 44	1
6I[B, A]	$(0,0,0,1,0,0,0,1)$	$A_4A_3T_1$	40 44 40 40	4
6J[A, C]	$(1,2,3,1,2,3,1,2)$	$A_3D_4T_1$	44 40 38 48	-2
6K[B, B]	$(0,0,0,0,0,1,0,0)$	$A_2D_5T_1$	54 48 33 32	37
6L[B, C]	$(1,2,3,4,5,0,1,2)$	$A_1A_1D_5T_1$	52 36 42 40	6
6M[B, D]	$(1,0,0,0,0,0,0,2)$	$A_1D_6T_1$	70 32 25 64	13
6N[A, D]	$(2,0,0,0,0,0,0,1)$	$A_1D_6T_1$	70 24 33 64	-3
6O[A, B]	$(1,0,0,0,0,0,1,0)$	$A_1D_5T_2$	50 38 43 36	9
6P[A, B]	$(0,0,0,0,0,0,1,0)$	$A_1E_6T_1$	82 54 27 4	105
6Q[B, C]	$(4,1,0,0,0,0,0,0)$	D_7T_1	92 64 14 0	142
6R[A, C]	$(4,0,0,0,0,0,0,1)$	D_6T_2	68 44 34 24	54
6S[A, D]	$(0,0,0,0,0,0,0,1)$	E_7T_1	134 56 1 0	189
6T[A, D]	$(0,0,0,0,0,0,1,1)$	E_6T_2	80 29 28 54	27
7A[2, 4]	$(0,1,0,0,0,0,0,0)$	A_7T_1	64 56 28 8	$16+48\sigma+20\sigma^2$
7B[2, 4]	$(2,4,0,0,0,0,0,0)$	$A_6A_1T_1$	52 35 21 42	$52-7\sigma-21\sigma^2$
7C[2, 4]	$(4,4,0,0,0,0,0,0)$	A_6T_2	50 42 29 28	$20+14\sigma+\sigma^2$
7D[2, 4]	$(4,1,5,2,6,0,0,0)$	$A_5A_1T_2$	40 34 37 33	$-1+\sigma+4\sigma^2$
7E[2, 4]	$(2,0,0,2,0,0,0,0)$	$A_4A_3T_1$	40 40 34 30	$2+10\sigma+4\sigma^2$

label [exps, powers]	α	centralizer	multiplicity of $e^{2\pi ij/k}$ for $j=0,\ldots,[(k+1)/2]$				trace on $\mathbf{g}$
7F[2, 4]	$(0,0,0,2,0,0,0,0)$	$A_4A_2A_1T_1$	36	35	35	36	$2-\sigma-\sigma^2$
7G[2, 4]	$(1,2,0,0,0,0,1,0)$	$A_2D_5T_1$	54	48	30	19	$13+29\sigma+11\sigma^2$
7H	$(1,1,0,0,1,1,0,0)$	$A_2D_4T_2$	38	35	35	35	3
7I[2, 4]	$(0,0,0,0,0,1,0,1)$	$A_1D_5T_2$	50	35	28	36	$30-\sigma-8\sigma^2$
7J[2, 4]	$(0,0,0,0,0,0,2,4)$	$A_1E_6T_1$	82	54	27	2	$30+52\sigma+25\sigma^2$
7K[2, 4]	$(1,0,0,0,0,0,0,0)$	D_7T_1	92	64	14	0	$64+64\sigma+14\sigma^2$
7L[2, 4]	$(1,0,0,0,0,0,0,1)$	D_6T_2	68	44	33	13	$15+31\sigma+20\sigma^2$
7M[2, 4]	$(1,6,0,0,0,0,0,0)$	E_7T_1	134	56	1	0	$132+56\sigma+\sigma^2$
7N	$(1,2,3,4,0,0,0,0)$	E_6T_2	80	28	28	28	52

TABLE 4 (con't)

Here, $\tau=(1+\sqrt{5})/2$ and $\sigma=2\cos(2\pi/7)$. [So $\tau^2=\tau+1; \sigma^3=-\sigma^2+2\sigma+1$.]

PROOF. Straightforward computation for each α gives the indicated centralizer type and eigenvalue multiplicities of the elements $h_\alpha(e^{2\pi i/k})$ listed. In general these invariants distinguish as many conjugacy classes as there should be in view of the above formula, and representatives for all classes have been found. The only difficulty occurs for the elements of order 5 with trace 3. It can be argued as follows that such an element, x say, cannot be conjugate to its square. First of all, note that, since $-\mathrm{id}\in W(E_8)$, each semisimple element of G is conjugate to its inverse. Since the centralizer $C_G(x)$ of x is a connected closed Lie subgroup of type $A_6A_1T_1$, there is a unique 1-dimensional semisimple subalgebra t_1 centralized by $C_G(x)$. The normalizer of $\langle x\rangle$ acts rationally on this torus so the action group is $\mathbf{Z}_2$ at most (and, since $-1\in W(E_8)$, actually coincides with $\mathbf{Z}_2$). Now, x is the exponential of some $u\in t_1$, so it is conjugate to its inverse, but not to any other power. □

3.2. REMARK. Let ψ be a character of L of degree 248. A necessary condition for L to be embeddable in G with $\chi|_L=\psi$ is that, for each $g\in L$ of order $k\leq 7$, there is an element $x\in G$ of the same order with $\chi(x)=\psi(g)$. We shall say that ψ is *feasible on* $\mathbf{g}$ if it satisfies this condition. In the sequel, many groups L will be eliminated as possible subgroups of G by this feasibility condition. The following character of $Sz(8)$, however, is feasible $14_a+14_b+64_a+65_*+91_a$ (its character values are: $\chi(2A)=-8, \chi(4A)=0, \chi(4B)=0, \chi(5A)=-2, \chi(7A)=\sigma+1$, where the notation for elements of $Sz(8)$ follows the Atlas [**12**]), while $Sz(8)$ does not appear in the main theorem. Thus, sometimes, a more detailed analysis is required.

It is of interest to know the precise structure of the centralizers of these elements of small order. It is well known that they are connected (cf. Springer and Steinberg [**26**] or Chevalley [**7**]). See also Borel and Tits [**5**] for further information concerning the centers of these centralizers.

3.3. LEMMA. *Let* $y\in G$.

(i) *If* y *is an involution with trace* -8 *on* $\mathbf{g}$, *then its centralizer is isomorphic to the half-spin group* HSpin$(16,\mathbf{C})$, *i.e., the quotient of the spin group* Spin$(16,\mathbf{C})$

by a central involution outside the kernel of the natural map $\mathrm{Spin}(16, \mathbf{C}) \to SO(16, \mathbf{C})$.

(ii) *If y is an involution with trace 24 on $\mathbf{g}$, then $\langle y \rangle$ is the full center of $C = C_G(y)$ of type A_1E_7, and occurs in the commutator subgroup $[C_1, C_1]$ of either central product factor C_1 of C.*

(iii) *If y has order 3 and trace -4 on $\mathbf{g}$, then the centralizer of y has a center of order 3; it is the quotient $3 \cdot PSL(9, \mathbf{C})$ of $SL(9, \mathbf{C})$ by its central subgroup of order 3.*

(iv) *Suppose that y is an element of order 4 such that $C_G(y^2)$ has type D_8 and $C_G(y)$ has a component of type A_7. Then $C_G(y) \cong GL(8, \mathbf{C})$. Also, $N_G(\langle y \rangle) = C_G(y)\langle \gamma \rangle$, where γ has order 2 and acts on $C_G(y) \cong GL(8, \mathbf{C})$ by the inverse transpose.*

PROOF. (i) Since the group contains a central involution and a full maximal torus, it must be a nonsplit central extension of $PSO(16, \mathbf{C})$; by identification of its representation on the -1 eigenspace of the central involution as a half-spin representation and on the fixed point space as the adjoint representation (again, by use of the root system), it must be the half-spin group.

(ii), (iii). Similar arguments work.

(iv) Let $\pi\colon C_G(y^2) \to PSO(16, \mathbf{C})$ be an epimorphism with kernel $\langle y^2 \rangle$, and let $\rho\colon SO(16, \mathbf{C}) \to PSO(16, \mathbf{C})$ be the natural quotient map. We may assume $\pi(y) = \rho(u)$, where $u \in SO(16, \mathbf{C})$ is a diagonal element with eigenvalues i and $-i$, both having multiplicities 8. Therefore the connected group $C_G(y)$ has type A_7T_1 and we must specify the center of the A_7 factor and how the two facters intersect. We may assume $C_G(y^2)$ corresponds to the subset $\{2, 3, \ldots, 8, 0\}$ of the extended Dynkin diagram. $C_G(y^2)$ corresponds to the stabilizer of a direct sum decomposition of $\mathbf{C}^{16}$ into a pair of maximal isotropic subspaces, i.e., the i and $-i$ eigenspaces of $\rho(u)$. All such stabilizers are conjugate in $O(16, \mathbf{C})$ but fall into two orbits under $SO(16, \mathbf{C})$. Representatives of these two classes of $GL(8, \mathbf{C})$ subgroups of $SO(16, \mathbf{C})$ correspond to subsets $\{3, \ldots, 8, 0\}$ and $\{2, 3, 4, \ldots, 8, 0\}$ of the set of nodes $\{2, 3, \ldots, 8, 0\}$ corresponding to our D_8. Now, we may assume the semisimple factor S of $C_G(y)$ corresponds to the subset $\{3, \ldots, 8, 0\}$ of the extended Dynkin diagram. (It seems a priori possible that S would correspond to $\{2, 4, \ldots, 8, 0\}$. However, S would then commute with $P \cong SL(2, \mathbf{C})$ corresponding to node 1; then SP has a root lattice of type A_7A_1 of determinant 16 and with cokernel $\mathbf{Z}_4$. Since this lattice is embedded in the E_8 lattice, $Z(SP) \cong \mathbf{Z}_4$, and y^2 is the involution of $Z(S) \cong \mathbf{Z}_4$; therefore, $\pi(S) \cong SL(8, \mathbf{C})/Z_0$, where $Z_0 \cong \mathbf{Z}_4$, which is certainly false as $\pi(S)$ should be $SL(8, \mathbf{C})$.) We observe that the group H corresponding to $\{1, 3, \ldots, 8, 0\}$ is isomorphic to $SL(9, \mathbf{C})/Z$, where $Z \cong \mathbf{Z}_3$ is a group of scalar matrices and H contains a subgroup $R \cong GL(8, \mathbf{C})$ (via the standard embedding of $GL(8, \mathbf{C})$ in $SL(9, \mathbf{C})$ putting the 8×8-matrix in the upper left corner and its inverse determinant as the $(9, 9)$-entry). We may take $S \leq R$. Certainly, $C_G(S)$ is reductive of Lie rank 1, so without loss of generality, we may take $y \in Z(R)$, a

1-dimensional torus in $C_G(S)$. Since $\dim R = \dim C_G(y) = 64$, and $R \leq C_G(y)$, $C_G(y)/R$ is finite. Connectedness of $C_G(y)$ forces $R = C_G(y)$, and the first statement is proved. As for the last statement, notice that $N_{H_1} \cong GL(8, \mathbf{C})\langle\gamma\rangle$, where $H_1 = N_G(\langle v\rangle)$, $\langle v\rangle = Z(H) \cong \mathbf{Z}_3$. □

3.4. COROLLARY. *Suppose $\hat{L}$ is a quasisimple finite group with simple quotient L, and z is an element of order a in $Z(\hat{L})$. If $\hat{L} \leq G$, then $\hat{L}$ is embeddable in a connected closed Lie subgroup of G of type X, where X is one of D_8, E_7 if $a = 2$; A_8, E_6 if $a = 3$; A_7, D_5 if $a = 4$; A_4 if $a = 5$; A_5 if $a = 6$.*

PROOF. The relevant Lie subgroup is a factor of the central product decomposition of $C_G(z)$. The proof comes down to verifying (by means of Table 4) that the centralizer C of z in G with $z \in [C, C]$ has central product factors which are embeddable in a group of type X. □

By Borel and Serre [**3**], every supersolvable subgroup normalizes a maximal torus. In particular, finite p-subgroups of G belong to $N_G(T)$ for some maximal torus T of G. Since the structure of $N_G(T)$, of shape $T \cdot W(E_8)$ where $W(E_8)$ denotes the Weyl group of type E_8, is well known, this provides strong information on the structure of such a p-subgroup:

3.5. LEMMA. *If L is a finite p-subgroup of G, then either $p > 7$ and L is abelian of rank at most 8, or there is a power q of p such that L can be embedded in a group of shape*

(i) $\mathbf{Z}_q^8.2^8.2^3\,\mathrm{Dih}_8$ *for* $p = 2$;
(ii) $\mathbf{Z}_q^8.((\mathbf{Z}_3 \operatorname{wr} \mathbf{Z}_3) \times \mathbf{Z}_3)$ *for* $p = 3$;
(iii) $\mathbf{Z}_q^8.5^2$ *for* $p = 5$;
(iv) $\mathbf{Z}_q^8.7$ *for* $p = 7$.

For a proof of this and the following result, the reader is referred to Cohen and Seitz [**9**].

3.6. PROPOSITION (COHEN AND SEITZ [**9**]). *If p is a prime, the p-rank of G (i.e., the maximal number a for which an elementary abelian p-subgroup of G of order p^a exists) is equal to 8 unless $p = 2$, in which case it is 9. Moreover, there is a unique conjugacy class of elementary abelian p-subgroups in G of maximal order.*

Adams [**1**] has classified all elementary abelian 2-groups in G which are maximal with respect to containment. (Up to conjugacy, there are precisely two, one, the above, has rank 9, the other has rank 8.)

3.7 LEMMA. *If E is a Klein four-group embedded in G, then up to conjugacy, E is one of the following four groups:*

TABLE 5

Klein four-subgroups of G

$E^{\#}$	centralizer	automizer	multiplicities
BBB	$D_4D_4.\mathbf{Z}_2$	Sym_3	56 64 64 64
BBA	$A_7T_1.\mathbf{Z}_2$	$\mathbf{Z}_2$	64 56 56 72
BAA	$A_1A_1D_6$	$\mathbf{Z}_2$	72 56 64 64
AAA	$E_6T_2.\mathbf{Z}_2$	Sym_3	80 56 56 56

In the first case (type BBB*), the centralizer C of E contains an involution interchanging the two central product factors isomorphic to* $\mathrm{Spin}(8, \mathbf{C})$*, and $N_G(E)$ splits over C° (i.e., $N_G(E)$ is a semidirect product of the normal subgroup C° and a subgroup isomorphic to* $\mathrm{Sym}_3 \times \mathbf{Z}_2$*).*

PROOF. By Springer and Steinberg [**26**] all elementary abelian groups of order 4 are embeddable in a maximal torus T of G. The action of $W(E_8) \cong N_G(T)/T$ on the 2-torsion V of T is that of $PSO^+(8,2) \cong W(E_8)/Z(W(E_8))$ on its standard module. Involutions in T of type 2A (cf. Table 4) in G correspond to nonisotropic points of this module. Applying Witt's Theorem to the orthogonal space underlying V, we find four classes of Klein four-subgroups in T, corresponding to totally isotropic, hyperbolic, degenerate, and elliptic lines, respectively, with $E^{\#}$ containing precisely $3, 2, 1, 0$ involutions of type 2B (cf. Table 4) in the respective cases.

In order to determine the structure of $N = N_G(E)$, let $x, y \in V$ generate E. The centralizer $J = C_G(x)$ is a quotient of its universal cover $\hat{J}$ by an involution. Since the centralizer of a semisimple element in the simply connected group $\hat{J}$ is connected, the index of the connected group C° in $C = C_J(y)$ is 2 or 1 according to whether there does or does not exist an element in $\hat{J}$ interchanging the two inverse images of y.

Let x be of type 2A and write $J = SK$, where $S \cong SL(2, \mathbf{C})$ and $K \cong 2 \cdot E_7(\mathbf{C})$. If $E^{\#}$ has type BAA, then we can take $y \in K$ of type B, and so, as K is simply connected, $C_J(y) = C_K(y)S$ is connected. If $E^{\#}$ has type BBA or AAA, we can choose $y_1 \in K$ and $y_2 \in S$ of order 4 such that y_1y_2 and $y_1^2 = y_2^2 = x$. Elements of order 4 in K and S are rational, so there are $g_1 \in K$ and $g_2 \in S$ such that $y_i^{g_i} = y_i^{-1} = xy_i$. In particular, g_1g_2 commutes with y, and, as its inverse image in the simply connected cover of SK does not centralize the inverse image of y, it must lie in $C - C^\circ$.

Now suppose that x, y, and xy are of type 2B. Then $J \cong \mathrm{HSpin}(16, \mathbf{C})$, and y can be chosen as the element whose inverse image in $\hat{J}$ maps to an element $\bar{y}$ with eigenvalue distribution $-1^8, 1^8$ in $SO(16, \mathbf{C})$. Take v to be an element of J whose inverse image in $\hat{J}$ maps to an involution $\bar{v}$ in $SO(16, \mathbf{C})$ interchanging the two eigenspaces of $\bar{y}$. Then v is an involution in J interchanging the two central product factors of C of type D_4. Therefore, $N = N_G(E)$ contains a subgroup N_0 of index 2 stabilizing the two D_4's; N_0 is complemented by $\langle v \rangle$. Additionally, from the Weyl group structure (using that E is a totally isotropic

2-space in V) we find that N induces the full linear group $GL(2,2) \cong \mathrm{Sym}_3$ on E. Finally $C^\circ = N_0 \cap C \cong \mathrm{Spin}(8,\mathbf{C}) \circ \mathrm{Spin}(8,\mathbf{C})$ (with center E), and $N = N_0 J$, so $N/C^\circ \cong N_0/C^\circ \times C/C^\circ \cong N/C \times N/N_0 \cong \mathrm{Sym}_3 \times \mathbf{Z}_2$.

Let H be the normalizer in N of $\langle v, E\rangle$, let K be the stabilizer in H of each of the two central product factors J_1, J_2 of J, and let S be the subgroup of K consisting of all $xvxv^{-1}$ for $x \in J_1$. Then projection onto J_1/E shows that SE/E is isomorphic to $PSO(8,\mathbf{C})$; also, $S \cap E = 1$ and $[v, K] \le E$, so that SE is the direct product of S and E and is contained in K. By a Frattini argument applied to the involution vE/E of the normal subgroup J/E of N/E (observe that J/E has a unique class of involutions outside J_1J_2/E), we must have $N = JH$ and hence $N = JK$. Thus K induces Sym_3 on E and so, by knowledge of $\mathrm{Aut}\, D_4(\mathbf{C})$, there is a subgroup L of K containing E such that K/E is the semidirect product of $SE/E \cong PSO(8,\mathbf{C})$ and $L/E \cong \mathrm{Sym}_3$. Now L must be isomorphic to Sym_4 (it permutes the 4 elements of Sym_4 faithfully) so contains a subgroup M, say, isomorphic to Sym_3 and complementing E. We conclude that N is the semidirect product of C and M, and hence also the semidirect product of C° and $\langle v\rangle \times M$. □

In the next two propositions we continue with the investigation of elementary abelian 2-groups all of whose involutions are of the same class.

3.8. PROPOSITION. *Let T be a maximal torus of G and denote by N its normalizer in G. The group N/T is isomorphic to $W(E_8)$ and acts on the 2-torsion V of T as on the natural module for $\Omega^+(8,2) \cong W(E_8)/Z(W(E_8))$. The singular and nonsingular elements of V are involutions in G of trace -8 and 24, respectively. Fix a maximal singular subspace Z_4 of V and denote by Z_5 the subgroup of N generated by Z_4 and a fixed element $z \in N$ mapping to $-\mathrm{id} \in W(E_8)$ under the natural morphism $N \to N/T$. Then Z_5 is as described in Remark 2.2; thus, $Z_5^\#$ consists of 31 involutions of trace -8 on $\mathfrak{g}$, and any two subgroups of Z_5 of the same order are conjugate in G. For $m = 1,2,3$ denote by Z_m a fixed subgroup of Z_4 of order 2^m. Then, if E is an elementary abelian subgroup of G of order 2^k, for some $k \ge 2$, all of whose involutions have type 2B (cf. Table 4), then $k \le 5$ and E is conjugate to Z_k. If N and C denote the normalizer and centralizer of E, respectively, then C/C° has order $2^{2^{k-1}-1}$ and $N/C \cong L(k,2)$.*

(i) *If $k=3$, then $J = C^\circ$ is the central product of eight fundamental $SL(2,\mathbf{C})$'s, say J_i $(1 \le i \le 8)$, with center $Z = Z(J) \cong 2^4$ such that E is the unique hyperplane in Z disjoint from each $Z(J_i)^\#$. Furthermore, $C/J \cong 2^3$ acts regularly on $Z - E$, and the actions of N on $Z - E$ and on $\{J_i \mid 1 \le i \le 8\}$ are the same.*

(ii) *If $k = 4$, then C° is a maximal torus of G and the group C/C° has shape 2_+^{1+6}.*

(iii) *If $k=5$, then $C^\circ = 1$ and C has shape $2^5.2^{10}$ (nilpotency class 2).*

PROOF. In view of the preceding lemma we may restrict to $E \ge Z_2$ of rank at least 3. First, suppose E has rank 3. Computing inner products of characters, we get $\dim C_{\mathfrak{g}}(E) = \frac{1}{8}(248 + 7(-8)) = 24$. Choose $z \in E - Z_2$. If z

were to interchange the two central product factors of $C_G(Z_2)^\circ$, we would have $\dim C_{\mathfrak{g}}(E) = 28$, a contradiction. Therefore, $z \in C_G(Z_2)^\circ$, and so E is contained in a maximal torus of G. From the previous lemma it is clear that E is (conjugate to) a singular subspace of V of dimension 3, and from the Weyl group action we obtain that N/C is the full linear group $GL(3,2)$ on E. Having established uniqueness up to conjugacy, we may construct a convenient example of E (cf. [**9**]) in order to further analyze its normalizer. To this end, fix a maximal torus T and take the following set of eight pairwise orthogonal roots, written in terms of coefficients with respect to the fundamental roots as indexed in §3: 22343210, 01122210, 00000010, 01121000, 00001000, 01000000, 00100000, and 23465432. Let J_i $(1 \le i \le 8)$ be the fundamental $SL(2,\mathbf{C})$ with respect to the ith root from this set of 8, set $J = J_1 \cdots J_8$, and write e_i for the generator of $Z(J_i)$. Then $Z = \langle e_i | 1 \le i \le 8 \rangle$, the center of J, is an elementary abelian group of order 16. More precisely, the e_i satisfy the relations $\prod_{i \in I} e_i = 1$, where I is one of the tuples

$$\begin{array}{lccccccc} & 4567, & 2367, & 1256, & 1247 & 2345, & 1357, & 1346, \\ (1) & 1238, & 1458, & 3478, & 3568, & 1678, & 2468, & 2578, \\ & 12345678. & & & & & & \end{array}$$

(These words constitute the so-called extended Hamming code $[8,4,4]$, cf. Sloane and MacWilliams [**22**, p. 27].) Thus, $Z_3 = \{e_ie_j | 1 \le i < j \le 8\}$, and $Z - Z_3 = \{e_i | 1 \le i \le 8\}$. Now the eight elements $(e_i)_{1 \le i \le 8}$ are all of type 2A, and the seven elements $(e_ie_j)_{1 \le i < j \le 8}$ are all of type 2B; the subgroup of all 'even elements' (of type 2B or 1) is a conjugate of E. Taking E to be equal to this group, we find that $J = C^\circ$ (recall that the dimension of $C_{\mathfrak{g}}(E)$ is 24). From connectedness of J, it follows that Z can be embedded in a maximal torus as well. The Weyl group induces a group of shape $2^3.GL(3,2)$ on Z, and so N/J is isomorphic to this group, and C is the inverse image in N of $O_2(N/J)$ under projection mod J. Also, from the Weyl group action on Z, it is clear that $C/J \cong O_2(N/J) \cong 2^3$ acts regularly on $Z - E$. Finally, since $J_i = C_G(C_G(e_i)^\circ)^\circ$, the actions of N/J on the e_i and on the J_i are the same.

Next assume E has rank 4 and contains Z_3. Then $\dim C_{\mathfrak{g}}(E) = 8$, so element $z \in E - Z_2$ normalizes each J_i of $C_G(Z_3)$ with a 1-dimensional fixed point subgroup. In particular, $z \in J$ can be embedded in a maximal torus T of G contained in J, so that E is conjugate to Z_4 indeed. Also, C° coincides with T. From the Weyl group action we obtain a group of shape $2_+^{1+6}.GL(4,2)$ normalizing E, and so $C \cong T.2_+^{1+6}$, $N/C \cong GL(4.2) \cong \mathrm{Alt}_8$.

Suppose E has rank 5 and contains Z_4. Then $\dim C_{\mathfrak{g}}(E) = 0$, so N is a finite group and $z \in E - Z_4$ inverts T. Consequently, zT corresponds to a generator of $Z(W(E_8))$, and we may take $z = z_1 \cdots z_8$ with $z_i \in J_i$ of order 4 inverting the 1-dimensional torus $T \cap J_i$ of J_i. The coset zT for such a z is readily seen to be a single $N_G(T)$-class of involutions (for instance, use the structure of the individual $J_i \cong SL(2,\mathbf{C})$ to show that any two $z \in E - Z_4$ must be conjugate). Since z sends each root space of T to its opposite and acts on the Lie algebra of T by scalar multiplication with -1, it must have trace -8 and be of type 2B, indeed. (See Remark 2.2 for other comments on existence.) The remaining statements for $k=5$ follow from Griess [**17**].

Finally, the existence of E in G with rank 6 would imply $(1, \chi|_L) = \frac{1}{64}(248 + 63(-8)) < 0$, which is absurd. □

3.9. PROPOSITION. *Let T, N, and V be as above. Take E_2 to be a 2-dimensional anisotropic subspace of V. Then there is a 2-dimensional torus S containing E_2 and contained in a subgroup $R \cong G_2(\mathbf{C})$ of a fundamental subgroup of type E_6 (fixing pointwise an 8-dimensional subspace of a 27-dimensional irreducible for $3 \cdot E^6(\mathbf{C})$). Take involution $e \in N_R(S)$ mapping to the nontrivial central element of $N_R(S)/S \cong W(G_2)$. Then $E_3 = \langle E_2, e\rangle$ is an elementary abelian subgroup of G of order 8 all of whose involutions have type 2A; its centralizer C in G is isomorphic to $F_4(\mathbf{C}) \times E$ and $N_G(E) \cong 2^3 \cdot L(3,2) \times F_4(\mathbf{C})$, while $C_G(C_G(E)^\circ) \cong G_2(\mathbf{C})$. Furthermore, if E is an elementary abelian subgroup of G of order 2^k for some $k \geq 3$, all of whose involutions have trace 24, then E is conjugate to E_3.*

PROOF. As for existence of E_3, observe that there is a unique class of involutions and a unique class of elementary abelian subgroups of order 8 in R. Now the trace of an involution in R on a 27-dimensional module for $3 \cdot E_6(\mathbf{C})$ is 3. From the embedding of $3 \cdot E_6(\mathbf{C})$ in G it is then clear that the involutions in this group have trace 24. (The trace of such an involution is 3 on the 27-dimensional module and -2 on the adjoint for E_6, whence $8 \cdot 1 + (-2) + 6 \cdot 3 = 24$ on $\mathbf{g}$.) Also, from the embedding of E in $G_2(\mathbf{C})$ we see that N induces $GL(3,2)$ on E (cf. Cohen and Wales [**10**]).

Let $E \geq E_2$ be of rank 3, and take $z \in E - E_2$. We have $\dim C_{\mathbf{g}}(E) = 52$. If z were contained in $C_G(E)^\circ$, then E would be embeddable in a maximal torus of G, which is impossible as V does not contain totally anisotropic spaces of dimension 3. Therefore, z lies outside $C_G(E)^\circ$ and induces an outer automorphism on Y, the central product factor of type E_6 and inverts the toral factor S. The only 52-dimensional subalgebra arising as the fixed points of an involutory outer automorphism of Y is well known to be of type F_4, and all such automorphisms of this kind are conjugate (cf. Helgason [**18**, Chapter X]). This establishes that E is unique up to conjugacy and that $C \cong F_4(\mathbf{C}) \times E$ (recall that z inverts S). Thus $C^\circ \cong F_4(\mathbf{C})$ is entirely contained in Y, so $C_G(C^\circ)$ contains $C_G(Y)$, a fundamental $SL(2, \mathbf{C})$ containing S, the existence of E in $C_G(C^\circ)$, and the fact that $\dim C_{\mathbf{g}}(C^\circ) = 14$ (the representation of $F_4(\mathbf{C})$ on each of the six 27-dimensional irreducible subrepresentations of $\mathbf{g}$ for Y decomposes into $1 + 26$, and the adjoint for Y into $52+26$, so that $\dim C_{\mathbf{g}}(C^\circ) = \dim C_{\mathbf{g}}(Y) + 6 \cdot 1 = 14$), imply that $C_G(C^\circ) \cong G_2(\mathbf{C})$.

Finally, suppose that $E \geq E_3$ has rank 3. Then there must be an element $y \in C_G(E_3)^\circ \cong F_4(\mathbf{C})$ such that yE_3 consists of involutions of type 2A only, and $\dim C_{\mathbf{g}}(E) = \frac{1}{16}(248 + 15 \times 24) = 38$. But there are only two classes of involutions in $F_4(\mathbf{C})$ with centralizer types A_1C_3 and B_4, respectively, leading to $\dim C_{\mathbf{g}}(E) = \dim C_{\mathbf{f}}(y) = 17, 28$, in the respective cases (where $\mathbf{f}$ denotes the Lie algebra of $C_G(E_3)$), a contradiction. □

The maximal elementary abelian subgroup of order 2^8 found by Adams [**1**] is conjugate to $E \times F$, where $E = E_3$ is as above and F is the elementary abelian subgroup of $C_G(E)^\circ \cong F_4(\mathbf{C})$ generated by the 2-torsion of a maximal torus of $F_4(\mathbf{C})$ and an involution in the normalizer of that torus mapping to the central involution of the Weyl group of type F_4 under the projection modulo the torus.

4. The module M for $2 \cdot E_7(\mathbf{C})$. It can be seen from the relevant involution in G that $K \cong 2 \cdot E_7(\mathbf{C})$ occurs as a closed connected Lie subgroup of G centralizing a fundamental subgroup $C \cong SL(2, \mathbf{C})$. The restriction to K of the G-representation on $\mathbf{g}$ decomposes into a 3-dimensional fixed point space on which C acts adjointly, an adjoint module of dimension 133, and two irreducibles of dimension 56. We shall now be concerned with such an irreducible K-constituent M of dimension 56.

By the same procedure as for E_8, we can determine representatives of conjugacy classes of elements of small order in $2 \cdot E_7(\mathbf{C})$ (cf. Cohen and Wales [**11**] for similar results concerning $3 \cdot E_6(\mathbf{C})$). Thus from Moody and Patera [**24**], we obtain that the number of conjugacy classes in G of elements of order a divisor of i is the coefficient of t^i in the power series expansion of

$$\frac{1}{(1-t)(1-t^2)^2(1-t^3)(1-t^4)(1-yt)(1-yt^2)(1-yt^3)},$$

with respect to the base $(y^j t^i)_{0 \le j \le 1, 0 \le i}$ where $y^2 = 1$. Consequently, the number of conjugacy classes of elements of order i $(1 \le i \le 13)$ is the coefficient of t^i in

$$t + 3t^2 + 5t^3 + 11t^4 + 21t^5 + 35t^6 + 63t^7 \\ + 97t^8 + 153t^9 + 229t^{10} + 351t^{11} + 474t^{12} + 714t^{13} + \cdots,$$

and so (with the conventions of the previous proposition) we obtain

4.1. PROPOSITION (*Feasibility of characters on M*). *Let K be the pointwise stabilizer in G of the root space corresponding to the longest root* 23465432 *of* Φ. *Then $K \le C_G(e)$ and $K \cong \langle e \rangle \cdot E_7(\mathbf{C})$, where e is the involution $h_\beta(-1)$ of G with $\beta = (0,0,0,0,0,0,0,1)$. The elements of order $k = 2, 3, 4, 5, 6,$ and 7 in K are given explicitly (up to conjugacy) in Table* 6, *where $\alpha = (\alpha_i) \in \mathbf{Z}^8$ satisfies* $2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8 \equiv 0 \pmod{k}$.

TABLE 6

Representatives $h_\alpha(e^{2\pi m_j i/k})$ of conjugacy classes of elements in $2 \cdot E_7(\mathbf{C})$ of order $k = 2, 3, 4, 5, 6$, and 7

label [exps, powers]	α	centralizer in K	multiplicity of $e^{2\pi ij/k}$ on M for $j = 0, \ldots, k-1$	class in G
2A	$(0,0,0,0,0,0,0,1)$	E_7	0 56	2A
2B	$(1,0,0,0,0,0,0,1)$	A_1D_6	32 24	2A
2C	$(1,0,0,0,0,0,0,0)$	A_1D_6	24 32	2B
3A	$(0,1,0,0,0,0,0,0)$	A_6T_2	14 21 21	3A
3B	$(0,0,0,0,0,0,1,0)$	E_6T_1	2 27 27	3B
3C	$(0,0,0,1,0,0,0,0)$	A_5A_2	20 18 18	3B
3D	$(1,1,1,0,0,0,0,0)$	$A_1D_5T_1$	20 18 18	3C
3E	$(1,0,1,0,0,0,0,0)$	D_6T_2	32 12 12	3D
4A[A]	$(0,2,0,0,0,0,0,1)$	A_7	0 28 0 28	4A
4B[B]	$(0,0,1,0,0,0,0,0)$	$A_5A_1T_1$	12 12 20 12	4A
4C[C]	$(1,1,0,0,0,0,1,0)$	A_5T_2	12 16 12 16	4B
4D[C]	$(0,1,0,0,1,0,0,0)$	$A_3A_3A_1$	12 16 12 16	4C
4E[C]	$(0,0,0,0,0,1,0,0)$	$A_1D_5T_1$	4 16 20 16	4C
4F[B]	$(0,0,1,0,0,1,0,0)$	$A_1A_1D_4T_1$	16 12 16 12	4D
4G[B]	$(1,0,0,0,0,0,0,1)$	D_6T_1	0 12 32 12	4D
4H[A]	$(2,2,0,0,0,0,0,1)$	E_6T_1	0 28 0 28	4E
4I[B]	$(1,0,0,1,0,0,0,0)$	$A_5A_1T_1$	20 12 12 12	4E
4J[C]	$(0,0,0,0,0,1,0,2)$	$A_1D_5T_1$	20 16 4 16	4F
4K[B]	$(2,0,1,0,0,0,0,0)$	D_6T_1	32 12 0 12	4G
5A[2]	$(1,1,0,0,0,0,0,0)$	A_5T_2	12 7 15 15 7	5A
5B[2]	$(0,0,0,1,1,1,0,0)$	$A_4A_1T_2$	10 12 11 11 12	5B
5C[2]	$(2,0,0,0,2,1,0,1)$	A_6T_1	0 21 7 7 21	5B
5D[2]	$(0,0,0,0,1,0,0,0)$	$A_4A_2T_1$	6 10 15 15 10	5C
5E[2]	$(0,0,0,1,0,1,0,0)$	$A_3A_2A_1$	12 10 12 12 10	5D
5F[2]	$(1,0,0,0,0,0,1,0)$	D_5T_2	2 10 17 17 10	5D
5G[2]	$(0,0,1,1,0,0,0,0)$	$A_5A_1T_1$	20 6 12 12 6	5E
5H[2]	$(0,0,0,0,0,0,1,1)$	E_6T_1	0 1 27 27 1	5E
5I[2]	$(0,0,0,0,0,1,0,3)$	$A_1D_5T_1$	20 16 2 2 16	5F
5J	$(1,1,1,1,0,0,0,0)$	$A_1D_4T_2$	16 10 10 10 10	5G
5K[2]	$(1,0,2,0,0,0,0,0)$	D_6T_1	32 0 12 12 0	5H
6A[C, A]	$(2,0,0,0,1,3,1,0)$	A_5T_2	12 15 6 2 6 15	6A
6B[C, E]	$(1,0,1,0,0,0,0,0)$	A_5T_2	12 6 6 20 6 6	6B
6C[A, A]	$(2,0,0,0,2,0,0,5)$	A_6T_1	0 21 0 14 0 21	6C
6D[B, A]	$(2,0,0,0,1,0,1,0)$	$A_4A_1T_2$	10 10 11 4 11 10	6C
6E[C, D]	$(2,0,0,0,1,1,1,4)$	A_4T_3	10 11 7 10 7 11	6D
6F[B, D]	$(2,0,0,0,1,1,1,1)$	A_4T_3	10 7 11 10 11 7	6E
6G[A, C]	$(2,0,0,0,2,0,2,5)$	A_5A_2	0 18 0 20 0 18	6F
6H[B, B]	$(1,0,0,0,2,1,0,1)$	$A_5A_1T_1$	2 12 15 0 15 12	6F
6I[B, C]	$(0,0,0,1,0,0,0,0)$	$A_3A_2A_1T_1$	8 6 12 12 12 6	6F
6J[B, A]	$(2,0,0,0,1,4,5,4)$	A_5T_2	2 6 15 12 15 6	6G
6K[B, A]	$(1,0,0,0,2,0,0,3)$	$A_3A_2T_2$	8 9 12 6 12 9	6G
6L[C, C]	$(1,0,0,0,1,3,1,4)$	$A_5A_1T_1$	0 6 12 20 12 6	6H
6M[C, B]	$(1,0,0,0,2,1,0,4)$	$A_5A_1T_1$	0 15 12 2 12 15	6H
6N[C, C]	$(2,0,0,0,1,1,3,4)$	$A_3A_1A_1T_2$	8 10 8 12 8 10	6H
6O[C, A]	$(2,0,0,0,1,0,1,3)$	$A_4A_1T_2$	4 11 10 10 10 11	6I

label [exps, powers]	α	centralizer in K	multiplicity of $e^{2\pi ij/k}$ on M for $j = 0, \ldots, k-1$	class in G
6P[C, A]	(1,0,0,0,2,0,0,0)	$A_3A_2T_2$	6 12 9 8 9 12	6I
6Q[C, D]	(2,0,0,0,2,1,0,0)	$A_3A_1A_1A_1T_2$	8 10 8 12 8 10	6J
6R[C, D]	(2,0,0,0,0,1,4,2)	$A_1D_4T_2$	4 8 10 16 10 8	6J
6S[C, C]	(1,0,0,0,2,2,0,2)	$A_3A_2A_1T_1$	12 12 6 8 6 12	6K
6T[C, B]	(2,0,0,0,0,1,2,2)	D_5T_2	2 16 11 0 11 16	6K
6U[B, D]	(1,0,0,0,3,0,1,2)	$A_3A_1A_1A_1T_1$	12 8 10 8 10 8	6L
6V[B, D]	(2,0,0,0,0,0,2,1)	$A_1D_5T_1$	0 18 0 20 0 18	6L
6W[B, D]	(0,0,0,0,0,1,0,1)	$A_1D_5T_1$	0 2 16 20 16 2	6L
6X[C, E]	(2,0,0,0,0,1,0,2)	$A_1A_1D_4T_1$	16 8 4 16 4 8	6M
6Y[C, E]	(1,0,0,0,0,0,0,2)	D_6T_1	0 0 12 32 12 0	6M
6Z[B, E]	(2,0,0,0,0,1,0,5)	$A_1A_1D_4T_1$	16 4 8 16 8 4	6N
6AA[A, E]	(2,0,0,0,0,0,0,1)	D_6T_1	0 12 0 32 0 12	6N
6BB[B, C]	(2,0,0,0,1,1,3,1)	$A_3A_1A_1T_2$	12 8 10 8 10 8	6O
6CC[B, B]	(2,0,0,0,0,1,2,5)	D_5T_2	0 11 16 2 16 11	6O
6DD[B, C]	(1,0,0,0,1,3,1,1)	$A_5A_1T_1$	20 12 6 0 6 12	6P
6EE[A, B]	(2,0,0,0,0,0,4,1)	E_6T_1	0 27 0 2 0 27	6P
6FF[C, D]	(2,0,0,0,0,5,0,0)	$A_1D_5T_1$	20 16 2 0 2 16	6Q
6GG[B, D]	(2,0,0,0,0,1,4,5)	$A_1D_4T_2$	16 10 8 4 8 10	6R
6HH[B, E]	(1,0,0,0,0,0,0,5)	D_6T_1	32 12 0 0 0 12	6S
6II[B, E]	(2,0,0,0,1,2,1,2)	A_5T_2	20 6 6 12 6 6	6T
7A[2, 4]	(1,0,0,0,3,1,2,4)	A_5T_2	12 6 1 15 15 1 6	7A
7B[2, 4]	(1,0,0,0,3,0,0,2)	A_6T_1	0 7 21 0 0 21 7	7B
7C[2, 4]	(1,0,0,0,1,1,4,6)	$A_4A_1T_2$	10 10 10 3 3 10 10	7B
7D[2, 4]	(1,0,0,0,1,1,1,0)	A_4T_3	10 6 6 11 11 6 6	7C
7E[2, 4]	(1,0,0,0,2,2,0,4)	A_5T_2	0 15 6 7 7 6 15	7D
7F[2, 4]	(1,0,0,0,2,0,1,3)	$A_3A_1T_3$	8 8 9 7 7 9 8	7D
7G[2, 4]	(1,0,0,0,1,1,3,4)	$A_4A_1T_2$	4 5 10 11 11 10 5	7E
7H[2, 4]	(1,0,0,0,2,0,3,0)	$A_3A_2T_2$	6 12 6 7 7 6 12	7E
7I[2, 4]	(1,0,0,0,3,0,1,4)	$A_4A_2T_1$	0 10 15 3 3 15 10	7F
7J[2, 4]	(1,0,0,0,1,0,3,6)	$A_4A_1T_2$	2 5 10 12 12 10 5	7F
7K[2, 4]	(1,0,0,1,1,0,0,4)	$A_2A_2A_1T_2$	6 9 7 9 9 7 9	7F
7L[2, 4]	(1,0,0,0,0,2,2,6)	D_5T_2	2 10 1 16 16 1 10	7G
7M[2, 4]	(1,0,0,1,2,0,3,4)	$A_2A_3A_1T_1$	12 6 4 12 12 4 6	7G
7N	(1,0,0,0,0,1,1,6)	D_4T_3	2 9 9 9 9 9 9	7H
7O	(1,0,0,1,0,0,1,5)	$A_2A_1A_1A_1T_2$	8 8 8 8 8 8 8	7H
7P[2, 4]	(1,0,0,0,0,0,1,1)	D_5T_2	0 1 10 17 17 10 1	7I
7Q[2, 4]	(1,0,0,0,1,1,0,5)	$A_3A_1A_1T_2$	12 6 8 8 8 8 6	7I
7R[2, 4]	(1,0,0,2,0,0,0,0)	$A_5A_1T_1$	20 0 12 6 6 12 0	7J
7S[2, 4]	(1,0,0,0,0,0,6,4)	E_6T_1	0 0 1 27 27 1 0	7J
7T[2, 4]	(1,0,0,0,0,0,5,2)	$A_1D_5T_1$	20 16 2 0 0 2 16	7K
7U[2, 4]	(1,0,0,0,0,1,0,4)	$A_1D_4T_2$	16 10 8 2 2 8 10	7L
7V[2, 4]	(1,0,0,0,0,0,0,6)	D_6T_1	32 12 0 0 0 0 12	7M
7W	(1,0,0,0,2,2,2,1)	A_5T_2	20 6 6 6 6 6 6	7N

TABLE 6 (con't)

The proof is as in §3.1. In order to compute the multiplicities on M, we have taken for M the module spanned by the root spaces of $\mathfrak{g}$ with respect to $\gamma \in \Phi$ such that γ has inner product 1 with the longest root. Let ϕ denote the character of K on M. A character ψ of a group L will be called *feasible on M* if, for each element $g \in L$ of order ≤ 7, there is an element $x \in K$ of the same order such that $\phi(x) = \psi(g)$.

As a corollary of the above proposition we obtain

4.2. COROLLARY. *Let T be a maximal torus in $K \cong 2 \cdot E_7(\mathbf{C})$.*

(i) *The element* $-\mathrm{id} \in W(E_7) = N_K(T)/T$ *lifts to elements of order* 4 *in* $N_K(T) \leq K$. *All such elements of order* 4 *are conjugate by an element of* T.

(ii) *The perfect central extension* $4 \cdot PSL(8, \mathbf{C})$ *occurs as a subgroup of* K, *whence* $P\Omega^+(8,2) \leq 4 \cdot PSL(8, \mathbf{C}) \leq K \leq G$. □

We shall now consider M. The next result is a slightly adapted version of Kantor and Skopets [20].

4.3. LEMMA. *On M there exist a K-invariant nondegenerate alternating bilinear form $(\cdot,\cdot)$ and a K-invariant symmetric quadrilinear form $(\cdot,\cdot,\cdot,\cdot)$ such that, for $x, y, z \in M$, writing xyz for the unique element of M satisfying*

$$(u, x, y, z) = (u, xyz) \qquad (u \in M),$$

we have the identity

$$(xxx)xy = (y, x, x, x)x - (x, y)xxx.$$

PROOF. Let $S \leq G$ be the fundamental $SL(2, \mathbf{C})$ centralizing K. Take a torus T in S and an element $\tau \in S$ of order 4 inverting T. The Lie algebra $\mathbf{g}$ has a grading by T-eigenspaces: $\mathbf{g} = \sum_{-2 \leq i \leq 2} \mathbf{g}_i$, where, choosing a surjective morphism $\lambda \mapsto t_\lambda$ from $\mathbf{C}^\#$ to T, we have $t\lambda(x) = \lambda^i x$ whenever $x \in \mathbf{g}_i$. Now $\dim \mathbf{g}_{\pm 2} = 1$; in fact, for $i = \pm 2$, there are $e_i \in \mathbf{g}_i$ with $h = [e_2, e_{-2}]$ such that $\langle e_i \rangle = \mathbf{g}_i$ and $\langle e_{\pm 2}, h \rangle$ is the adjoint for S. The adjoint $\mathbf{k}$ for K complements $\langle h \rangle$ in $\mathbf{g}_0$. We identify M with $\mathbf{g}_{-1}$ and write $e = e_{-2}$. The element τ interchanges $\mathbf{g}_i$ and $\mathbf{g}_{-i}$, and, as it is contained in $S \leq C_G(K)$, induces the identity on $\mathbf{k}$. By inspection of the adjoint for S, we can take τ and $e_{\pm 2}$ so that $\tau(e_{-2}) = -e_2$, and $\tau(e_2) = e_{-2}$. Thus, $\tau h = -h$. Observe that $[x, h] = x$ after a suitable sign change in the $e_{\pm 2}$'s. Define the bilinear map $x, y \mapsto (x, y)$ on M to $\mathbf{C}$ and the trilinear map $x, y, z \mapsto xyz$ on M into M by

$$[x, y] = 2(x, y)e_{-2}$$

and

$$xyz = [[x, [e_{+2}, y]], z] + (z, y)x + (z, x)y + (x, y)z \qquad (x, y, z \in M).$$

Then $(\cdot,\cdot)$ is obviously bilinear and alternating and readily seen to be nondegenerate. The trilinear map can be shown to be symmetric by intensive use of the Jacobi identity, and similarly for the symmetry of the quadrilinear map $u, x, y, z \mapsto (u, x, y, z)$, $u, x, y, z \in M$. Finally, the identity also follows from such computations. □

The above identity shows that M has the structure of a so-called *Freudenthal triple system*. It can be shown that the triple system on M determines $\mathbf{g}$ up to isomorphism (cf. [20]). The following result, due to Mars [23], describes the orbit structure of K on M. Here, $R_u(X)$, for X a Lie group, stands for the unipotent radical of X.

4.4. PROPOSITION. *There are four K-orbits on the collection of* 1*-spaces* $\langle x\rangle$ *in* M:

(a) $(x,x,x,x)\neq 0$, *with pointwise stabilizer* K_x *isomorphic to* $3\cdot E_6(\mathbf{C})$;
(b) $xxx\neq 0$, $(x,x,x,x)=0$, *with* $K_x\cong F_4(\mathbf{C})$;
(c) $xxx=0$, $xxM\neq\langle x\rangle$, *with* $K_x/R_u(K_x)\cong \mathrm{Spin}(11,\mathbf{C})$;
(d) $xxx=0$, $xxM=\langle x\rangle$, *with* $K_x/R_u(K_x)\cong E_6(\mathbf{C})$.

5. Finite simple subgroups of G. Recall that $G=E_8(\mathbf{C})$ and χ is the character afforded by the adjoint representation of G, i.e., the natural representation on its Lie algebra $\mathbf{g}$. In the remainder of this paper, K is a closed Lie subgroup of G isomorphic to $2\cdot E_7(\mathbf{C})$, M is a K-module as in §4. Furthermore, $\hat{L}$ is a finite quasisimple group with nonabelian simple factor group L.

5.1. *Alternating subgroups of G.* The characters of Alt_{10} of degree at most 248 are given in the Atlas [**12**]. From this and Proposition 3.1 it is straightforward to derive that $L\leq G$, $L\cong \mathrm{Alt}_{10}$ implies

$$\chi|_L = 9_a+35_a+36_a+2\times 84_a \tag{1}$$

(with character values: $\chi(2A)=24, \chi(2B)=-8, \chi(3A)=77, \chi(3B)=14, \chi(3C)=5, \chi(4A)=4, \chi(4B)=4, \chi(4C)=0, \chi(5A)=23, \chi(5B)=-2, \chi(6A)=-3, \chi(6B)=6, \chi(6C)=-2, \chi(7A)=3$). This character is realized by the embedding $\mathrm{Alt}_{10}\leq 3\cdot PSL(9,\mathbf{C})\leq E_8(\mathbf{C})$. The restriction of χ to $3\cdot PSL(9,\mathbf{C})$ decomposes into the adjoint representation (of degree 80) of $PSL(9,\mathbf{C})$, the third exterior power of the natural representation and the contragredient of the latter (both of degree 84) (see Dynkin [**15**, Table 25, p. 205]). Since no 248-dimensional character of Alt_{11} has a restriction to a subgroup Alt_{10} equal to the character in (1), there is no subgroup in G isomorphic to Alt_{11}. As $\mathrm{Alt}_m\leq \mathrm{Alt}_n$ for $m\leq n$, we have found

5.1.1. Alt_n *embeds into G if and only if $n\leq 10$.*

5.1.2. *If $L\cong \mathrm{Alt}_n$ and $Z(\hat{L})\neq 1$, then $\hat{L}$ embeds into G if and only if $n\leq 17$.*

PROOF. For $n\leq 17$, observe that $2\cdot\mathrm{Alt}_n\leq 2\cdot \mathrm{Alt}_{17}\leq \mathrm{Spin}(16,\mathbf{C})$ and that $2\cdot\mathrm{Alt}_{17}$ maps to Alt_{17} under the natural map $\mathrm{Spin}(16,\mathbf{C})\to SO(16,\mathbf{C})$, so the center does not vanish under the natural map from $\mathrm{Spin}(16,\mathbf{C})$ to the half-spin group of type D_8. If $n\geq 18$ and $\hat{L}$ is a subgroup of G isomorphic to $2\cdot\mathrm{Alt}_n$, we have $\hat{L}\leq C_G(Z(\hat{L}))$, so that $\hat{L}$ is contained in the centralizer of an involution, say e. By the feasibility condition (Proposition 2.7), this involution has centralizer type A_1E_7, so that we must have $\mathrm{Alt}_n\leq E_7(\mathbf{C})$. But this is absurd in view of traces of involutions (cf. Table 6).

Finally, the statements concerning 3- and 6-fold covers of Alt_6 and Alt_7 are trivial consequences of known representations of these groups. □

5.1.3. *(fpf) If $L\leq G, L\cong \mathrm{Alt}_n$ and $\chi|_L$ has no nonzero fixed vectors in $\mathbf{g}$, then $n\in\{5,6,7,10\}$. If $n=5$, or 10, the character is as given in Table* 1. *Also,* *for $n = 5$, this character is realized via an embedding in the unique $SL(4, \mathbf{C})$ acting without fixed points (see Dynkin* [**15**, *p.* 202]).

PROOF. By the above, we may restrict attention to $5 \leq n \leq 9$.
Suppose $n = 5$. Then

$$\begin{aligned}(1, \chi|_L) &= \frac{248}{60} + \frac{1}{4}\chi(2A) + \frac{1}{3}\chi(3A) + \frac{1}{5}\chi(5A) + \frac{1}{5}\chi(5B) \\ &\geq 4 + \frac{2}{15} + \frac{1}{4}(-8) + \frac{1}{3}(-4) + \frac{1}{5}(-4) \geq 0,\end{aligned}$$

and equality occurs only if $\chi(2A) = -8$, $\chi(3A) = -4$, and $\chi(5A) = \chi(5B) = -4$. This determines $\chi|_L$ as claimed [the restriction of χ to $SL(4, \mathbf{C})$ decomposes into a sum of two distinct irreducibles of degree 45 (with highest weights 210 and 012 in Dynkin's notation), two of degree 15 (weight 101), and two of degree 64 (weight 111)].

Next, let $n = 8$, and suppose L has no fixed points in $\mathbf{g}$. Observe that every character has degree 0 mod 7, except for the pair of complex conjugate ones of degree 45, the one of degree 64, and the one of degree 20. Our elements of order 7 are rational. Let a be the number of times $45_a + 45_b$ occurs, let b be the number of times 64 occurs, and let c be the number of times 20_a occurs. We have $90a + 64b + 20c \leq 248$. Thus, $a \leq 2, b \leq 3$, and $c \leq 12$. Also, since $248 \equiv 3 \pmod 7$, $6a+b+6c \equiv 90a+64b+20c \equiv 3 \pmod 7$. The traces on the adjoint module for rational elements of order 7 are 3 and 52. Only 3 is possible here. Therefore, $b \geq 3$, and so $b = 3$. We conclude that $a + c \equiv 0 \pmod 7$. Since only $248 - 192 = 56$ dimensions remain, we conclude that $a + c = 0$. Thus, these 56 dimensions must be made up with characters of degrees 7, 14, 21, 28, 35, and 56. There are several ways to make the 2-central involution $2A$ have trace -8 (obviously, trace 24 is impossible here). A glance at the table shows that we may not tolerate an irreducible ψ with $\psi(2A) \neq -\psi(1)/7$, so we are left with degrees 7, 21, and 28. Then the non-2-central involution has positive trace unequal to 24, unless the degree 7 character occurs with multiplicity 8, in which case the trace on an element of type $3A$ is $32 + 3 \times 4 = 44$, a contradiction with Proposition 3.1. This proves that every subgroup isomorphic to Alt_8 in G has a nonzero fixed vector in $\mathbf{g}$.

Finally, the proof for Alt_9 is similar. Since a straightforward but tedious check of the character table of Alt_9 against Table 5 will also give the result, we shall dispense with details. □

5.1.4. (E_7) Alt_n *embeds in* $E_7(\mathbf{C})$ *if and only if* $n \leq 9$.

PROOF. Since, by §4.2, $\mathrm{Alt}_9 \leq 4 \cdot PSL(8, \mathbf{C}) \leq K$ and, by the above, $\mathrm{Alt}_{10} \leq E_7(\mathbf{C})$ would imply an embedding of the nonsplit $2 \cdot \mathrm{Alt}_{10}$ in K, which is absurd as the only faithful irreducible character degrees ≤ 56 of $2 \cdot \mathrm{Alt}_{10}$ are 32 and 16. □

5.2. *The two-dimensional linear groups.* Throughout this section, $L = L(2, q)$ and $S = SL(2, q)$. We shall employ the character table of S (p odd) as given in, e.g., Dornhoff [**14**].

5.2.1. *If* $n = 2$, *then*
(i) $L \leq G$ *implies* $q \in \{2, 3, 4, 5, 7, 8, 9, 11, 13, 16, 17, 19, 25, 27, 29, 31, 32, 61\}$;
(ii) $S \leq G$ *and* q *odd imply* $q \in \{3, 5, 7, 9, 11, 13, 17, 19, 25, 27, 29, 31, 37\}$.

PROOF. First, suppose q even. Both S and L have an elementary abelian subgroup U, say, of order $q = 2^m$ for some m, all of whose nontrivial elements belong to a single conjugacy class. Suppose that S or L is a subgroup of G and consider the restriction of χ to U. If an involution $u \in U$ has trace $\chi(u) = -8$, then $(\chi|_U, 1) \geq 0$ gives $0 \leq 248 + (q-1)(-8)$, so that $q \leq 32$. On the other hand, if an involution in U has trace 24, then integrality of $(1, \chi|_U)$ yields the divisibility condition $248 + (q-1)24 = 0 \pmod{q}$, whence, again, $q \leq 32$. So, in view of Table 4, $q \leq 32$.

Next, suppose q is an odd prime. Then the Borel subgroup of L is supersolvable, so by Borel and Serre [3], we have that $(q-1)/2$ is the order of an element in $W(E_8)$. It follows that q is one of the following primes: 3,5,7,11,13,17,19,29,31, 37, 41, 61. (We use that W has no element of order 21 to eliminate $q = 43$.)

We next show that $q = 41$ does not occur. The degrees of the irreducibles of $S = SL(2, 41)$ are $\{1, 21, 40, 41, 42\}$. If S or L is embedded in a proper Lie subgroup of G of positive dimension, then, by Proposition 2.6 we may take this group to be $2 \cdot E_7(\mathbf{C})$. On the 56-dimensional module M for $2 \cdot E_7(\mathbf{C})$, $\dim C_M(S) \geq 14$, whence $C_{\mathfrak{g}}(S)$ has dimension at least $3 + 14 + 14 = 31$ and so contains a toral subalgebra of rank at least 4. It follows that S is embedded in a fundamental Lie subgroup of rank at most 4, but a look at the degrees of the irreducibles for S rules this out.

Now, if $q = p^a$ $(a > 1)$ is an odd prime power, then we must have $\frac{1}{2}(q-1) \leq 248$, whence $q \leq 497$, and it remains to eliminate $q = p^a \in \{3^4, 3^5, 5^3, 7^2, 7^3\}$.

Suppose $S \leq G$ for $q \in \{3^4, 3^5, 5^3, 7^2, 7^3\}$. By Proposition 3.1 we must have $S \leq 2 \cdot E_7(\mathbf{C})$, and S has a nontrivial representation of degree ≤ 56. Therefore $q = 3^4$ or 7^2, and S fixes (pointwise) a subspace of M, a 56-dimensional space for $2 \cdot E_7(\mathbf{C})$ of dimension at least 6, whence $\dim C_{\mathfrak{g}}(S) \geq 2 \cdot 6 + 3 = 15$. By Proposition 2.3, $C_{\mathfrak{g}}(S)$ contains a toral subalgebra $\mathfrak{t}$ of rank at least 3; but then the Lie group of $\mathfrak{t}$ contains an elementary abelian subgroup of order 2^3 in its torus, hence an involution with centralizer type D_8, so that $S/Z(S)$ embeds into $PSO(16, \mathbf{C})$, a contradiction.

Finally, assume $L \leq G$ for $q \in \{3^4, 3^5, 5^3, 7^2, 7^3\}$. By Proposition 3.1 and the above paragraph, we see that L fixes no nonzero vector in $\mathfrak{g}$. The degrees ≤ 248 of nontrivial irreducible representations of L are 41, 80, 81, 82, if $q = 3^4$, 121, 242, 243, 244 if $q = 3^5$, 63, 124, 125, 126 if $q = 5^3$, 25, 48, 49, 50 if $q = 7^2$, and 171 if $q = 7^3$. Since no linear combination with nonnegative integral coefficients of these degrees can be formed yielding 248 if $q = 3^4, 3^5$, or 7^3, we are left with $q \in \{5^3, 7^2\}$. If $q = 5^3$, then $\chi|_L$ is the sum of two characters of degree 124. Since the trace of an involution on such a character is 0, the value of χ on an involution of L is 0, contradicting Proposition 2.6.

Assume $q = 7^2$. Let $b \in L$ be an element of order 25. A look at the character table of L and Proposition 3.1 readily yields that $\chi|_L$ is the sum of four irreducible characters of degree 50 and one irreducible character, ψ say, of degree 48, with trace -8 on an involution and $\chi(b) = \psi(b) = -2\cos(2\pi j/25)$ for some j with $j \not\equiv 0 \pmod{25}$. We derive $\chi(b^5) = \psi(b^5) = -2\cos(2\pi j/5)$, and (using Proposition 3.1) $\chi(b^5) = -2$ and $j \equiv 0 \pmod 5$. Consequently,

$$\dim C_{\mathbf{g}}(b) = \frac{1}{25}\left(248 + 4(-2) + \sum_{l \not\equiv 0 \pmod 5} \chi(b^l)\right)$$

$$= 9 + \frac{1}{25}\left(15 - \sum_{l \not\equiv 0 \pmod 5} 2\cos\left(\frac{2\pi l j}{25}\right)\right) = 9 + 1 = 10,$$

where $l \in \{0, \ldots, 24\}$. As $C_G(b^5)$ has type A_4A_4 by Proposition 3.1, the following lemma establishes that $\langle b \rangle$, and therefore L, cannot be embedded in G.

5.2.2. *G contains no element b of order 25 with $\chi(b) \in \mathbf{Q}(2\cos(2\pi/5))$, $C_{\mathbf{g}}(b^5) \cong \mathbf{sl}_5 \oplus \mathbf{sl}_5$ and $\dim C_{\mathbf{g}}(b) \leq 10$.*

PROOF. Since b acts on a complement of $C_{\mathbf{g}}(b^5)$ with eigenvalues $\exp(2\pi i l/25)$ ($l \not\equiv 0 \pmod 5$) occurring in ten algebraically conjugate 20-tuples, the trace of b on $C_{\mathbf{g}}(b^5)$ must be a sum of $48 = \dim C_{\mathbf{g}}(b^5)$ elements of the form $\exp(2\pi i l/5)$ ($0 \leq l \leq 5$), the nontrivial ones occurring in complex conjugate pairs and the trivial ones being precisely $d := \dim C_{\mathbf{g}}(b)$ in number. It follows that, on $C_{\mathbf{g}}(b^5)$, the element b has spectrum

$$d \times 1,\ c \times \exp(2\pi i/5),\ a \times \exp(4\pi i/5),\ a \times \exp(\pi i/5),\ c \times \exp(3\pi i/5),$$

for certain $a, c \in \mathbf{Z}_{\geq 0}$ with $d + 2a + 2c = 48$. On the other hand, we know that the centralizer in G of b^5 has type A_4A_4, whence, by a simple check is isomorphic to $SL(5, \mathbf{C}) \circ SL(5, \mathbf{C})$, a central product with center $\langle b^5 \rangle$, represented by the diagonal $\exp(2\pi i/5)$ in either factor. Moreover, the action of this group on $C_{\mathbf{g}}(b^5)$ is the adjoint action on $\mathbf{sl}_5(\mathbf{C}) \oplus \mathbf{sl}_5(\mathbf{C})$. In particular, b can be written in the form $b_1 b_2$ where each b_j is a diagonal element of $SL(5, \mathbf{C})$ with nonzero entries $\exp(2\pi i(3 + 5m_{jr})/25)$ ($1 \leq r \leq 5$), where $0 \leq m_{jr} \leq 4$. In its action on the adjoint b has eigenvalues 1 with multiplicity 8 (from the main diagonals) while the remaining eigenvalues are of the form

$$\exp(2\pi i(m_{jk} - m_{jl})/25)$$

for $k \neq l$ ($1 \leq k, l \leq 5$) and $j = 1, 2$. As $\dim C_{\mathbf{a}}(b) \leq 10$, there are at most $10 - 8 = 2$ eigenvalues equal to 1 among them. But such an eigenvalue will only occur for one value of j, say $j = 1$, because such occurrences come in pairs (namely k, l pairs with l, k). Therefore, the numbers $(m_{2l})_{1 \leq l \leq 5}$ are all distinct. But then $\{m_{2l} | 1 \leq l \leq 5\} = \{0, 1, 2, 3, 4\}$ and b_2 has determinant $\prod_l \exp(2\pi i(3 + 5m_{2l})/25) \neq 1$, contradicting that b_2 belongs to $SL(5, \mathbf{C})$. □

5.2.3. *Let $q = 16$. Then L does not fix a nontrivial subspace of* $\mathbf{g}$*, and* $\chi|_L$ *is as indicated in Table* 1 *(with* $\chi(2A) = -8$*,* $\chi(3A) = -4$*, and* $\chi(5A) = -2$*). Moreover,* $L(2,16)$ *is not embeddable in the simple group* $E_7(\mathbf{C})$*.*

PROOF. Let B be a Borel subgroup of L. Since all elements of $O_2(B)^\#$ are from the same conjugacy class, $O_2(B)$ is conjugate to the group Z_4 of Proposition 3.8. Thus, involutions of L have trace -8 on $\mathbf{g}$ and an element $h \in B$ of order 15 is contained in the normalizer of the maximal torus $T = C_G(O_2(B))^\circ$. According to Springer [**25**], h permutes the root spaces of T on $\mathbf{g}$ in orbits of length 15 and the eigenvalues of its action on the Lie algebra of T are the 8 primitive 15th roots of unity. Therefore h, h^3, h^5 have trace $1, -4, -2$ on $\mathbf{g}$. In particular $(\chi|_B, 1) = 0$, so L has no nonzero fixed vectors in $\mathbf{g}$. A look at the character table shows that $\chi|_L$ has shape $12 \times 15_* + 17_d + 17_e + 17_f + 17_{\mathbf{g}}$, and the last statement readily follows. □

5.2.4. *Let $q = 25$. Then G contains both S and L.*

PROOF. L is contained in $E_6(\mathbf{C})$ (cf. Cohen and Wales [**11**]) and S is contained in $\mathrm{Spin}(13, \mathbf{C})$. □

5.2.5. *Let $q = 27$. Then G does not contain $SL(2,27)$ and if $L \leq G$, it lies in a natural F_4 (centralizing a subgroup E_3 as in Proposition* 3.9*).*

PROOF. Let $\hat{L} \leq G, \hat{L} \cong S$ or L. Straightforward use of Table 4 and the character table of $\hat{L}$ shows that $\chi|_{\hat{L}}$ involves the trivial character. Therefore (cf. Corollary 2.4 and Proposition 2.6), $\hat{L} \leq K$ (up to conjugacy). Thus, setting $\mathbf{b} = C_{\mathbf{g}}(\hat{L})$, we have $\dim \mathbf{b} \geq 1$. We prove that $\mathbf{b}$ has Lie rank at least two. Suppose not. Then, as $\mathbf{sl}_2 \cong C_{\mathbf{g}}(K) \subset \mathbf{b}$, we must have $\mathbf{b} \cong \mathbf{sl}_2$, and the character ϕ of $\hat{L}$ on the Lie subalgebra $\mathbf{k}$ of K has no trivial constituents. Also, since $SL(2,27)$ is not contained in $\mathrm{Spin}(16, \mathbf{C})$, the center of $\hat{L}$ must be contained in the center of K, so $Z(\hat{L})$ coincides with the kernel of ϕ. Therefore, ϕ has constituents of degrees 13, 26, 27, and 28 only. Now $u \in U^\#$ has trace 5 on $\mathbf{g}$, and so, by Table 6, trace -25 or 2 on M. Consequently, its trace $\phi(u)$ on $\mathbf{k}$ equals $5 - 3 - 2 \times (-25 \text{ or } 2) = 52$ or -2. By inspection of the character table of $L(2,27)$, we obtain $\phi(u) = m_{28} - m$, where m_i denotes the number of irreducibles of degree i, and $m = m_{26} + \frac{1}{2}m_{13}$ (as the irreducibles of degree 13 are nonreal, they occur in pairs). Thus, $\phi(u) = -2 = m_{28} - m$. Together with the degree equation $133 = 28m_{28} + 27m_{27} + 26m$ this yields $7 = 3m + m_{27}$. In particular, $m \leq 2$. Let a be the sum of half the number of characters of degree 13 and the number of characters having value 2 at an involution $h \in \hat{L} - Z(\hat{L})$. Then, according to the character table of L, $\phi(h) = 2a + (m - a)(-2) - m_{27}$, but according to Table 6, this value is equal to 5, so $a \geq 3$. But then $m \geq 3$, a contradiction. Thus, $\mathbf{b}$ has Lie rank at least 2, indeed.

Now let $\mathbf{t}$ be a Cartan subalgebra of $\mathbf{b}$. Then $G_1 = C_G(\mathbf{t})$ is semisimple with factors of type A_i $(1 \leq i \leq 6)$, D_i $(1 \leq i \leq 6)$, E_6, and contains $\hat{L}$. The character table of $\hat{L}$ implies that E_6 is at hand. By considering an irreducible 27-dimensional submodule of $\mathbf{g}$ for $G_1 \cong 3 \cdot E_6(\mathbf{C})$, we get $\hat{L} \cong L(2,27)$ and, as the trace of an involution of G_1 on the 27-dimensional module is -5 or 3, that $\hat{L} = L$ fixes a nonzero vector; since the minimal degree of a faithful representation exceeds 10, knowledge of vector stabilizers in such a 27-dimensional module implies that L embeds in a natural closed Lie subgroup of G_1 isomorphic to $F_4(\mathbf{C})$ (cf. Cohen and Wales [**11**]). □

5.2.6. *Let $q = 29$. There is a subgroup in* Spin(15, C) *isomorphic to $L(2, 29)$, whence also in G. Furthermore, if $S \leq G$, then S lies in a conjugate of K. Finally, if $L \leq E_7(\mathbf{C})$, then L lifts to $SL(2, 29)$ in $2 \cdot E_7(\mathbf{C})$.*

PROOF. As involutions of L have eigenvalue pattern $1^7, -1^8$ in the orthogonal 15-dimensional representation, $L \leq \mathrm{HSpin}(15, \mathbf{C})$, and S cannot be embedded in HSpin(16, C). Therefore, the first and the second statements follow (use §3.3 and §3.4).

Suppose now that $L \leq K \cong 2 \cdot E_7(\mathbf{C})$. There can be no fixed points in M in view of Proposition 4.4. The nontrivial irreducible degrees for L are 15, 28, 29, 30. Letting m_i be the number of irreducibles of degree i, we obtain from the degree equation on M that $m_{15} + m_{29} + 2m_{30} \equiv 0 \pmod 7$. But $m_{15} \leq 3, m_{29} \leq 1$, and $m_{30} \leq 1$. Therefore $m_{15} = m_{29} = m_{30} = 0$, whence $m_{28} = 2$. This forces the trace of an involution in L on M to be 0, which contradicts Proposition 4.1. □

5.2.7. *Let $q = 31$. Then S is not contained in G. If L is in G, $\chi|_L$ as indicated in Table 1 (in particular, $\chi|_L$ is rational valued) and the subgroups of L listed in Table 7 have the indicated centralizers in G.*

TABLE 7

Subgroups of $L \cong L(2, 31)$ with centralizers in G for $L \leq G$

$\mathbf{Z}_2$	HSpin(16, C)	Dih_8	A_1^4
$\mathbf{Z}_4$	$GL(8, \mathbf{C})$	Dih_{16}	T_4
$\mathbf{Z}_8$	$A_3^2 T_2$	Dih_{32}	T_4
$\mathbf{Z}_{16}$	$A_1^4 T_2$	Dih_6	B_4
$\mathbf{Z}_3$	$3 \cdot PSL(9, \mathbf{C})$	Dih_{10}	$B_2 B_2$
$\mathbf{Z}_5$	$A_4 A_4$	Dih_{30}	T_4
$\mathbf{Z}_{15}$	$A_1^4 T_4$	Alt_4	$A_2 A_2$
$\mathbf{Z}_{31}$	T_8	Sym_4	A_2
$\mathbf{Z}_2^2$	$D_4 D_4 . \mathbf{Z}_2$	Alt_5	1
$\mathrm{Frob}_{31 \cdot 15}$	1		

PROOF. The degrees of irreducibles are 1, 15, 16, 30, 31, 32; the characters of degrees 15 and 16 are nonreal, so must occur in pairs. If S is embedded in G, it must lie in a subgroup K of the form $2 \cdot E_7(\mathbf{C})$ with $Z(K) = Z(S)$. On M, S has fixed point subspace of dimension at least 8, a contradiction with $Z(K) = Z(S)$. Consider the possibility that $L \leq G$. If L lies in a proper Lie subgroup of strictly positive dimension, reasoning as above shows that the latter must be conjugate to K and $\mathbf{a} = C_{\mathbf{g}}(L)$ has dimension at least $3 + 8 + 8 = 19$. Consequently, $\mathbf{a}$ contains a total subalgebra of dimension at least 3, and L is embedded in a Lie group of rank at most 5, which is impossible. In particular, the trivial character is no constituent of $\chi|_L$.

Take h and u to be elements of order 15 and 31 in L such that $B = \langle h\rangle\langle u\rangle = N_L(\langle u\rangle)$ is a Borel subgroup of L, and set $H = \langle h\rangle$, $U = \langle u\rangle$. Observe that u is rational in G as every semisimple element in G is real and h normalizes U. Since the trivial constituent does not occur, $\chi(u) = 0$, whence $\mathbf{a} = C_{\mathbf{g}}(U)$ is a Cartan subalgebra of $\mathbf{g}$. Since H acts on $\mathbf{a}$ as a subgroup of order 15 in the Weyl group, h must be a regular element (cf. Springer [**25**]). Therefore, it permutes the 240 roots in cycles of length 15 and acts on the Cartan subalgebra $\mathbf{a}$ with eigenvalues the 8 primitive 15th roots of unity. In particular, $\chi(h) = 1$, $\chi(h^5) = -4$, and $\chi(h^3) = -2$. Let m_i be the number of irreducibles of degree i occurring in $\chi|_L$. Then, as we have seen above, m_{15} is even; set $m = (m_{15}/2) + m_{30}$. Now $-m + m_{32} \equiv 248 \equiv 0 \pmod{31}$, while $m \le 8$ and $m_{32} \le 7$. Therefore, $m = m_{32}$. Thus, $62m + 31m_{31} = 248$, whence

$$(2) \qquad m_{31} + 2m = 8, \quad m_{31} \le 8, \quad \text{and} \quad m \le 4.$$

Suppose a is the number of irreducible constituents of $\chi|_L$ of degree 32 with value -1 at h^5. Then there are $m - a$ irreducibles of degree 32 with value 2 at h^5, and, using (2), we see $-4 = 4\chi(h^5) = m_{31} - a + 2(m - a) = 8 - 3a$, whence $a = 4, m = 4$, and $m_{31} = 0$. In particular, $\chi|_L$ has shape

$$(4 - m_{30})(15_a + 15_b) + m_{30} \times 30_* + 4 \times 32_{**}.$$

Now the restriction $\chi(h) = 0$ forces the degree 32 constituents to be $32_d, 32_e, 32_f$, and 32_g. It remains to determine the trace on $\mathbf{g}$ of elements of order a power of two. Let f_4 be an element of order 16 in L; denote by f_i $(1 \le i \le 3)$ its 2^{4-i}th power, and by t an involution of L inverting f_4. Thus f_1 is an involution; inspection of the character table of L shows that $\chi(f_1) = -8$. Consequently (cf. Lemma 3.7), $\langle f_1, t\rangle$ is a Klein four-group with centralizer, D say, of type D_4D_4. Now, f_2 normalizes $\langle f_1, t\rangle$ but does not centralize it.

Suppose f_2 stabilizes the two D_4's. Then $\bar{f}_2$, a preimage in $SO(16, \mathbf{C})$ of the image of $f_2 \in C_G(f_1) \cong \mathrm{HSpin}(16, \mathbf{C})$ in $PSO(16, \mathbf{C})$, stabilizes the two 8-dimensional eigenspaces of an involution $\bar{t} \in SO(16, \mathbf{C})$ corresponding to $t \in C_G(f_1)$, so $\bar{f}_2$ and $\bar{t}$ commute. If $\bar{f}_2$ has order 4, it squares to the center of $SO(16, \mathbf{C})$, and must have eigenvalue pattern $i^8, -i^8$, so that $\det \bar{f}_2 = 1$. On the other hand, f_2 induces a nontrivial action on $Z(D)$, whence $\det \bar{f}_2 = -1$, a contradiction. Thus $\bar{f}_2$ is an involution. Let m_i for $i = 0, 1$ be the multiplicity of the $\bar{f}_2$-eigenvalue -1 on the $\bar{t}$-eigenspace with respect to $(-1)^i$. Then the multiplicities of the eigenvalues -1 of $\bar{f}_2$ and $\bar{f}_2\bar{t}$ are $m_0 + m_1$ and $m_0 + 8 - m_1$, respectively. Since f_2 and f_2t have orders 4 and 2, respectively, we must have $m_0 + m_1 \equiv 2 \pmod 4$, and $m_0 - m_1 \equiv 0 \pmod 4$. Thus, $m_0 \equiv m_1 \equiv 1$ or 3 (mod 4). Multiplying by a suitable central element of $SO(16, \mathbf{C})$, we may take $m_0 = 1$ or 3. Thus (m_0, m_1) is one of $(1, 1), (1, 5), (3, 3), (3, 7)$, and the type of $C_{\mathbf{g}}(\langle f_2, t\rangle)$ is readily seen to be $B_3B_3T_2$, $A_1B_2B_3T_1$, $A_1A_1B_2B_2$, $A_1B_3B_2T_1$ of dimension 44, 36, 26, 35, in the respective cases. On the other hand, since f_2

squares to an involution in G with a centralizer of type D_8, it must have trace $\chi(f_2) \in \{8, 0, 64\}$ (cf. Table 4), and so the dimension of the centralizer in $\mathbf{g}$ of $P = \langle f_2, t \rangle$ is

$$(\chi|_P, 1) = \frac{1}{8}(248 + 5(-8) + 2\chi(f_2)) \in \{26, 28, 42\}.$$

Clearly, only dimension 26 fits; so $\chi(f_2) = 0$ (and $C_{\mathbf{g}}(P)$ has type $A_1A_1B_2B_2$). As 31_a does not occur in $\chi|_L$ and degree 32 irreducibles vanish at nonidentity 2-elements, $\chi(f_2) = 0$ implies that $\chi(y) = 0$ for all $y \in L$ of order 2^k, $k \geq 2$. Thus,

$$\dim C_{\mathbf{g}}(\langle f_4, t \rangle) = \frac{1}{32}(248 + 17(-8)) \notin \mathbf{Z},$$

again a contradiction.

Consequently, f_2 interchanges the two central product factors of D, whence $\dim C_{\mathbf{g}}(\langle f_2, t \rangle) = 28$. On the other hand, f_2 is rational, so the same number can be computed as $(\chi|_{<f_2>}, 1) = \frac{1}{8}(248 + 5(-8) + 2\chi(f_2))$. Consequently, $\chi(f_2) = 8$, and, by §3.1 and §3.3, f_2 has centralizer $C \cong GL(8, \mathbf{C})$ in G. Let b be the number of irreducibles of degree 30 with value 2 at f_2. Then, evaluating the decomposition of $\chi|_L$ at f_2 yields $(4 - m_{30})(-2) + (-2(m_{30} - b) + 2b) = 8$, whence $b = 4$, $m_{30} = 4$, and $\chi|_L$ has shape $c \times 30_b + (4-c) \times 30_c + 32_d + 32_e + 32_f + 32_g$ for some integer c. It follows that $\chi(f_3) = 0$, and $C_{\mathbf{g}}(f_3) = \frac{1}{8}(248 + (-8) + 2 \cdot 8 + 4 \cdot 0) = 32$. Therefore, as an element in $C \cong GL(8, \mathbf{C})$, the eigenvalue distribution of f_3 must be $\{\varepsilon, -\varepsilon\}^4$, where ε is a prime 8th root of unity (observe that its centralizer must have type $A_3A_3T_2$, whence be isomorphic to $GL(4, \mathbf{C}) \times GL(4, \mathbf{C})$). By inspection of the decomposition of $\chi|_L$ established so far, we obtain that $\sum_{\gcd(i,16)=1} \chi(f_4^i) = 0$. Consequently, $\dim_{\mathbf{g}}(f_4) = \frac{1}{16}(248 - 8 + 2 \cdot 8) = 16$. It follows that the centralizer of f_4 in C is of type $A_1A_1A_1A_1$, with eigenvalue distribution $\{\varsigma^1, \varsigma^5, \varsigma^9, \varsigma^{13}\}^2$, where ς is a square root of ε. In particular, f_4 is rational in $N_G(C) = N_G(f_2)$ (cf. Lemma 3.3). Finally,

$$0 = \sum_{\gcd(i,16)=1} \chi(f_4^i) = 8 \times \chi(f_4),$$

so $\chi(f_4) = 0$ and, evaluating $\chi(f_4)$ on the character decomposition, we find $0 = \chi(f_4) = (4 - 2c)\sqrt{2}$, whence $c = 2$, and the character is shown to be as in Table 1. □

5.2.8. REMARK. Let $\bigwedge^2 \chi$ be the character of G on $\bigwedge^2 \mathbf{g}$. In $\bigwedge^2 \chi|_L$, 30_b occurs with multiplicity 72 and 32_d with multiplicity 65. Therefore, the character of G on the fundamental module $\ker\{\bigwedge^2 \mathbf{g} \to \mathbf{g}\}$ of dimension $\binom{248}{2} - 248 = 30380$ restricts to a character of L. (Of course, if it did not, we would have eliminated the possibility of an embedding $L \leq G$.)

5.2.9. *If $q = 32$, then $L \leq G$ implies that $\chi|_L$ is a sum of 8 degree 31 irreducible characters. If a and b denote the number of constituents ϕ which satisfy $\phi(x) = -2$ and 1, respectively, for $x \in L$ of order 3, we have $(a, b) = (4, 4)$ or $(1, 7)$. In particular, $\chi|_L$ is not rational.*

PROOF. Take $L \leq G$. If $L \leq K$, the trace on M of an involution in L must be 8 or -8; impossible. Consequently (cf. Corollary 2.4), L has no nonzero fixed points in $\mathbf{g}$. The character table implies that $\chi|_L$ is the sum of 8 irreducibles of degree 31. Here $\chi(x) = \{-4, 5\}$, so (a, b) must be one of $(4,4)$ or (1.7). □

5.2.10. *Suppose $q = 37$. Then L does not embed in G and if $S \leq G$, it lies in a conjugate of $K \cong 2 \cdot E_7(\mathbf{C})$.*

PROOF. Suppose $L \leq G$. We argue as follows that L has nonzero fixed points in $\mathbf{g}$. Assume not. Let a, b, c, d, respectively, be the sum of the multiplicities of the characters of degrees $19, 36, 37, 38$. Then $248 = 19a + 36b + 37c + 38d$, whence $a \leq 13, b \leq 6, c \leq 6, d \leq 6$. Using the above degree equation we get $1 \equiv -2b - c \pmod{19}$, $a + c \equiv 0 \pmod 2$, and $19a - b + d \equiv 26 \pmod{37}$. The last congruence implies $a \geq 1$ in view of the bounds on a, b, and d. But then $c \geq 1$ is odd, whence $c \in \{1, 3, 5\}$, and so $b \equiv (-1 - c)/2 \equiv -1, -2$, or $-3 \pmod{19}$, a contradiction with $b \leq 6$. Therefore, the trivial character is involved in $\chi|_L$, and so, by Corollary 2.4 we may assume $L \leq K$, where $K \cong 2 \cdot E_7(\mathbf{C})$.

We next eliminate the possibility of L in K. Consider a Frobenius subgroup $F \cong \mathrm{Frob}_{37 \cdot 18}$ of L. As F is supersolvable, there is a maximal torus T of K with $F \leq N_K(T)$. Then $F \cap T$ is of order 37 and $F/(F \cap T)$ of order 18. The involution of the latter group commutes with a cyclic group of order 9 in $W(E_7)$, hence is $-\mathrm{id}$ in the usual reflection representation of $W(E_7)$. By Corollary 4.2 the T-coset of this element in $N_K(T)$ consists of elements of order 4 only. On the other hand, the structure of F implies the existence of an element of order 2 in the coset, a contradiction.

Finally, if $S \leq G$, then obviously $Z(S)^{\#}$ cannot have centralizer type D_8, so $S \leq C_G(Z(S)) \cong SL(2, \mathbf{C}) \circ K$, whence $S \leq K$ (up to conjugacy). □

5.2.11. *If $q = 61$, then S cannot be embedded in G and $L \leq G$ implies that $\chi|_L = 62_a + 62_b + 62_c + 62_d$ is a sum of four algebraically conjugate distinct irreducible characters.*

The proof is a straightforward feasibility test.

5.2.12. *If $L \cong L(2,7)$ acts without fixed points as a group of automorphism on $\mathbf{g}$, then $\chi|_L$ is as indicated in Table 1.*

PROOF. By the same argument as for Alt_5, §5.1.3, we obtain $\chi(2A) = -8, \chi(3A) = -4, \chi(4A) = 0$, and $\chi(7A) = 3$. □

5.2.13. (max) *If $q = 31, 32, 61$, then L is not contained in a proper closed Lie subgroup of G of positive dimension.*

PROOF. Suppose H is a closed Lie subgroup of G containing L. Since L is simple, we can (and shall) take H to be quasisimple. Suppose $q = 31$ or 61. Embedding a Borel subgroup B of L in the normalizer of a maximal torus T of H, we find that $O_p(B)$ is a subgroup of T and that the quotient $B/O_p(B)$ can be embedded in $N_H(T)/T$. Since the Weyl group of H contains an element of order 15, and H has rank at most 8, the type of H must be E_8. Next, let $q = 32$. The Lie algebra $\mathbf{h}$ of H must be of rank at most 8 and have dimension a multiple of 31 strictly smaller than 248; but according to the classification of simple Lie algebras this is impossible. □

5.2.14. (E_7) *If L can be embedded in $E_7(\mathbf{C})$, then* $q \in \{4, 5, 7, 8, 9, 11, 13, 17, 19, 25, 27, 29, 37\}$.

PROOF. Direct from §§5.2.1, 5.2.3 for $q = 16$ and §5.2.13 for $q = 31, 32, 61$. □

5.3. *The linear groups $L(n, q)$ with $n \geq 3$.* In this section, $\hat{L}$ is a (possibly trivial) central extension of $L = L(n, q)$ with $n \geq 3$.

5.3.1. *Supose $\hat{L}$ can be embedded in G.*

(i) *If $\hat{L} = L$ is simple, then* $(n, q) = \{(3, 2), (3, 3), (3, 5), (4, 2)\}$.

(ii) *If $\hat{L}$ is quasisimple but not simple, then it is one of* $2 \cdot L(3, 2) \cong SL(2, 7)$, $2 \cdot L(3, 4), 4 \cdot L(3, 4)$ (*the extension with a faithful* 8*-dimensional representation*), $6 \cdot L(3, 4), 2 \cdot L(4, 2) \cong 2 \cdot \mathrm{Alt}_8$.

PROOF. As its p-Sylow is nonabelian, we must have $p \in \{2, 3, 5, 7\}$. Also, by Lemma 3.5, $n \leq 3$ if $p \geq 5$, $n \leq 4$ if $p = 3$, and $n \leq 6$ if $p = 2$. Furthermore, $SL(2, q) \leq \hat{L}$, and so from the previous section we obtain $q \in \{2, 3, 4, 5, 7, 8, 9, 16, 25, 27\}$. By Landazuri and Seitz [**21**], $L(n, q)$ has degrees $\geq q^{n-1} - 1$ if $q \geq 5$. Thus $q \leq \sqrt{248 + 1}$, and $q \in \{2, 3, 4, 5, 7, 8, 9\}$ remain. Suppose $q = 8, 9$. The degrees ≤ 248 of nontrivial representations of $SL(3, q)$ are $q^2 + q, q^2 + q + 1$, and so, by Proposition 2.6, an embedding of $\hat{L}$ into G leads to an embedding in $2 \cdot E_7(\mathbf{C})$, which in turn is contradicted by the fact that the degrees are ≥ 56. Thus $q \leq 7$.

Let us first consider $n = 3$. For $SL(3, 7)$, the nontrivial characters of degrees ≤ 248 have degrees 56, 57, 152, so again, $L \cong L(3, 7), SL(3, 7)$ would imply $\hat{L} \leq K \cong 2 \cdot E_7(\mathbf{C})$ but then the existence of a nontrivial representation on M implies that $\hat{L} = L$ is simple and the restriction to L of the character of K on M must be irreducible of degree 56. Hence an involution z in L has trace 8 on M, and so $\chi(z) = -8$ (cf. Tables 4 and 6). But there is no character of $L(3, 7)$ of degree 248 having value -8 on z. We conclude $q \leq 5$.

Since $L(3, 2) \cong L(2, 7)$, we are left with $L(3, 4)$. The character table yields that $L(3, 4)$ has no fixed point free character which is feasible on $\mathbf{g}$, whence by Proposition 3.1 an embedding in G leads to an embedding in $E_7(\mathbf{C})$. A glance at Table 6 shows that the trace of an element of order 5 on M must be 6; but this is incompatible with the characters of $L(3, 4)$.

The covers of $L(3, 4)$ are more intricate. The Schur multiplier of $L(3, 4)$ is isomorphic to $\mathbf{Z}_4 \times \mathbf{Z}_4 \times \mathbf{Z}_3$. Perfect groups $2 \cdot L(3, 4), 6 \cdot L(3, 4)$ and $4 \cdot L(3, 4)$ embed in G via subgroups of G isomorphic to $4 \cdot PSL(8, \mathbf{C})$, $SL(6, \mathbf{C})$, $SL(8, \mathbf{C})$, respectively. A nonsplit central extension $\hat{L}$ whose center has order a multiple of 3 distinct from 6 cannot be embedded in G since otherwise, by Corollary 3.4, $\hat{L}$ would be embeddable in $3 \cdot E_6(\mathbf{C})$ or $SL(9, \mathbf{C})$, a contradiction.

Now let $n = 4$. If $q = 3$, then a direct check of the character table against Table 5 shows that $\hat{L}$ cannot act fixed point freely on $\mathbf{g}$. Thus, by Proposition 2.6, we must have (up to isomorphism) $L \le K/Z(K)$, but then the trace on M of an involution in L cannot be ± 8. Hence $L(4,3)$ and $SL(4,3)$ cannot be embedded in G, and $p = 2$.

If $\hat{L} = L \cong L(4,4)$, then $SL(3,4) \le L$ leads to a contradiction. Therefore, $q = 2$.

Next consider $n = 5$. Then, by the above, $q = 2$. The character table forces each involution to have trace 24 on $\mathbf{g}$. There is an elementary abelian 2-group A in $\hat{L}$ of order 2^6. Then

$$(\chi|_A, 1) = \frac{1}{64}(248 + 63 \times 24) \notin \mathbf{Z},$$

a contradiction. Consequently, no quasisimple group with simple factor isomorphic to $L(5,2)$ can be embedded in G. Finally, this implies the same result for $L(n,q)$ with $n \ge 6$. □

We record the result of a straightforward feasibility check:

5.3.2. *If $SL(3,5) \cong L(3,5) \cong L \le G$, then $\chi|_L$ is as described in Table* 1. □

By Alekseevskii [**2**] this embedding is realized.

5.3.3. (*fpf*) *Suppose $L \le G$ acts on* $\mathbf{g}$ *without fixed points. Then $(n,q) = (3,2)$ or $(3,5)$ and $\chi|_L$ is as in Table* 1, *or $(n,q) = (3,3)$.*

5.3.4. (E_7). *Suppose $L \le E_7(\mathbf{C})$. Then $L \cong L(3,2), L(3,3), L(4,2)$, or $L(3,4)$, and each of these groups occurs.*

PROOF. Straightforward; observe that $L(3,4)$ lifts to $2{\cdot}L(3,4)$ in $4{\cdot}PSL(3,4)$, while the latter Lie group embeds in K. □

5.4. *The unitary groups.* Let $\hat{L}$ be a quasisimple group with simple factor $L = PSU(n,q)$, where $q = p^a$ is a power of the prime number p and $n \ge 3$. Suppose $\hat{L}$ is a subgroup of G. Since, by Lemma 3.5, Sylow r-subgroups of L must be abelian for $r \ge 11$, we have $p \le 7$.

Until further notice, assume $n = 3$. Then $\hat{L}$ is a quotient of $SU(3,q)$. We list some useful properties of L, or rather its Borel subgroup.

5.4.1. LEMMA. *Let B be a Borel subgroup of $\hat{L}$. Set $U = O_p(B)$ and let $H \cong \mathbf{Z}_m$ complement U in B; then $m = (q^2-1)/\gcd(3,q+1)$ and $|U| = q^3$. Also, $|C_H(Z(U))| = m_0 := (q+1)/\gcd(3,q+1)$ and the subgroup of order $q-1$ in H operates regularly on $Z(U)^{\#}$; thus, elements of $Z(U)$ are rational. If $x \in U - Z(U)$, $|C_U(x)| = q^2$ and $[U,x] = Z(U)$. Suppose ϕ is an irreducible character of B which is nontrivial on $Z(U)$. Then $\phi(1) = q(q-1)$ and ϕ is induced from a character ρ of degree q of UH_0, where $H_0 \le H, |H_0| = m_0$, and $\ker\rho$ is a subgroup of order p^{a-1} of $Z(U)$. So, if $x \in Z(U)$ and ψ is any character of B, then there is an integer $k \ge 0$ such that $\psi(x) = \psi(1) - kq(q-1) - kq = \psi(1) - kq^2$. In particular, $\psi(x) \equiv \psi(1) \pmod{p^{a+1}}$. Moreover, if $q = 2$, then*

$\Omega_1(U) = Z(U) \cong \mathbf{F}_q$, every $x \in U - Z(U)$ has order 4, and $C_G(x) \cong \mathbf{Z}_4^a$. On the other hand, if q is odd, then U has exponent p. □

5.4.2. *Let B be a Borel subgroup of $U(3,4)$. Then B is not embeddable in G.*

PROOF. Take U, B, H as above, and write $Z = Z(U)$. Then $Z \leq [C, C]$, where $C = C_G(Z)$, and $Z^\#$ is contained in a single conjugacy class of G. By Lemma 3.7 this yields that the involutions in Z have trace -8 on $\mathbf{g}$, and that C is a central product of two groups $C_i \cong \mathrm{Spin}(8, \mathbf{C})$ $(i = 1, 2)$ with center Z. Let π_i be the natural map $C \to C_i|Z \cong PS\Omega(8, \mathbf{C})$, and denote by U_i the inverse image of $\pi_i(U)$ in C_i. Since U is nonabelian and $U \leq U_1 U_2$, at least one U_i is, say for $i = 1$. Since $B \leq N_G(Z)$ and B' has odd index in B, it normalizes both U_i; since U_1/Z is a nontrivial B-module and a quotient of the irreducible $\mathbf{F}_2 B$-module U/Z, we must have $U_1/Z \cong U/Z \cong 2^4$. Also, U_1 is nonabelian, so $U_1 - Z$ must contain an element of order 4, and by the action of B, it follows that all elements in $U_1 - Z$ have order 4, and that $Z = [U_1, U_1]$.

Let ρ: $C_1 \to SO(8, \mathbf{C})$ be the natural map. Then $\mathbf{Z}_2 \cong \ker \rho \leq Z$, and so $\rho(W)$ is an extraspecial group 2_-^{1+4} (the type is minus as $\rho(U_1)$ has an automorphism of order 5 coming from B). Let $x \in U_1 - Z$ with $x^2 \in \ker \rho$. Then, by representation theory of 2_-^{1+4}, the element $\rho(x)$ is an involution with eigenvalue pattern $-1^4, 1^4$ and hence lifts to involutions in C_1, a contradiction with $|x| = 4$.

5.4.3. *Let $n = 3$. If $\hat{L}$ is a subgroup of G, then $q \in \{3, 8\}$ and $Z(\hat{L}) = 1$. Moreover, if $L = PSU(3,8)$ embeds in G, it embeds in K and $\chi|_L$ is as given in Table 1.*

PROOF. Let x be an element of order p in a Sylow p-center of $\hat{L}$. It is rational and its trace on $\mathbf{g}$ is of the form $\chi(x) = -248 - mq^2$ for some integer m. Now, §5.4.1 implies that $(q-1)q \leq 248$, whence $q \leq 16$. As before, let B be a Borel of $\hat{L}$, and set $U = O_p(B)$.

Suppose $p = 7$. Then $q = 7$ and $\hat{L} \cong L$. If the trivial character appears in $\chi|_L$, Corollary 2.4 and Proposition 2.6 force $L \leq K$ (up to conjugacy). But the degrees ≤ 248 of irreducible characters are 1, 42, and 43, so the adjoint for K picks up fixed points as well, and $L \leq E_6(\mathbf{C})$, which is absurd in view of character degrees. So we may assume $(\chi|_L, 1) = 0$. In view of Table 4, the permissible values of $\chi(x)$ are 3 and 52. But the character table reveals only negative values on nontrivial characters of degree at most 248 at x. Therefore $p \leq 5$.

Suppose $p = 5$. Then $q = 5$. If $\hat{L} \cong SU(3,5)$, the existence of a central element of order 3 in $[\hat{L}, \hat{L}]$ implies that $\hat{L}$ is embedded in a Lie subgroup of E_6 or A_8. The former is ruled out by Cohen and Wales **[11]** and the latter is absurd as $SU(3,5)$ has no 9-dimensional nontrivial character. So we may assume $L \cong PSU(3,5)$ is in G. An element of order 7 in L is conjugate to its square, hence is rational in G and has trace either 3 or 52. From the character table of L it follows that such a character value cannot be achieved with a character of L of degree 248 unless it involves the trivial character. Thus, L has nonzero fixed vectors. In view of Proposition 2.6, this implies that L may be embedded in K But a look at Table 6 shows that L does not possess a feasible character on M. This contradiction shows that $PSU(3,5)$ is not contained in G. We conclude that $p = 2$ or 3.

Suppose $p = 3$. Then $q = 3$ or 9. If $q = 9$, the character table quickly forces the trivial character to appear, with multiplicity $c \geq 248 - 3 \times 73 = 29$. This implies that an element of order 3 has trace at least 29, hence 77; but this is impossible to achieve. So $q = 3$. [There are embeddings of $U(3,3)$ in G via, for instance, $A_6(\mathbf{C})$ or $B_3(\mathbf{C})$, and we do not try to classify them at this time.]

Finally, let $p = 2$. Since $U(3,2)$ is solvable, we are left with $q = 4, 8, 16$. Now $q = 4$ is eliminated in §5.4.2 and it follows from $PSU(3,4) \leq PSU(3,16)$ that q cannot be 16. We are left with $q = 8$. In this case, the character table easily proves that $SU(3,8)$ does not embed in G and that if $PSU(3,8)$ does, it embeds in K and has character decomposition $3 \times 1_a + 133_* + 2 \times 56_a$. □

REMARK. There are several embeddings of the Borel of $PSU(3,8)$ in G (in fact, in $N_G(Z_3)$ of §3.8).

5.4.4. *Let* $n \geq 4$. *If* $\hat{L} \leq G$, *then* $\hat{L} \cong PSU(4,2)$, $2 \cdot PSU(4,2)$, *or* $6 \cdot PSU(4,3)$.

PROOF. First consider $n = 4$. Since $SU(3,q)$ is contained in $SU(4,q)$, we have $q \in \{2,3,8\}$. As $q = 8$ leads to a contradiction with Landazuri and Seitz [**21**] and for $q = 2$ both L and its central cover $2 \cdot L$ are embeddable (via $SL(6,\mathbf{C})$ and $SL(4,\mathbf{C})$, respectively), we may assume $q = 3$.

It is readily checked that $L \cong PSU(4,3)$ has no feasible character on $\mathbf{g}$ without fixed vectors. So, by Proposition 2.5, L must be embeddable in K. But it is readily seen that L does not afford a feasible character on M.

Next, we consider the covers $\hat{L} \leq G$ of $L \cong PSU(4,3)$. The Schur multiplier of $PSU(4,3)$ is $\mathbf{Z}_{12} \times \mathbf{Z}_3$. Put $Z = Z(\hat{L})$ and assume $Z > 1$. If $|Z|$ is divisible by 3, then $\hat{L}$ lies in a Lie subgroup of type A_8 or E_6 (cf. Corollary 3.4). If $\hat{L} \leq 3 \cdot PSL(9,\mathbf{C})$, the character of $\hat{L}$ on the natural module for $SL(9,\mathbf{C})$ is forced to break up as $3 \times 1_a + 6_*$ giving $|Z| = 6$. In $3 \cdot E_6(\mathbf{C})$ there are two classes of involutions with centralizers of type A_5A_1 and D_5T_1, respectively. If $\hat{L} \leq 3 \cdot E_6(\mathbf{C})$, the character table for $\hat{L}$ quickly shows that $|Z| = 6$ and that $\hat{L}$ embeds in the central product factor of the involution centralizer of type A_5A_1 isomorphic to $SL(6,\mathbf{C})$. Assume, therefore, that $|Z|$ divides 4. Then $\hat{L}$ lies in K since it has no nontrivial irreducible representation of degree ≤ 16; moreover $|Z| = 2$, since the character table of $4 \cdot L$ does not allow for a faithful feasible character on M. Furthermore, the trace of an involution of $\hat{L}$ on M forces the restriction to be a 56-dimensional irreducible, yielding trace 1 on an element of order 5, in conflict with Table 6. We conclude that Z must have order 6.

Finally, consider $n \geq 5$. By the embedding of a nontrivial quotient of $SU(4,q)$ in $\hat{L}$, we must have $q = 2$. The character table of $\hat{L} = L$ then forces involutions to have trace 24. Since L contains an elementary abelian 2-group of order 2^4, this conflicts with Proposition 3.9. The conclusion is that, for $n = 5$, and hence for $n \geq 5$, the group $\hat{L}$ does not embed in G. □

5.5. *The orthogonal and symplectic groups.*

5.5.1. *Let $L \cong P\Omega^-(2n, q)$, $n \geq 4$. Then $\hat{L}$ cannot be embedded in G.*

PROOF. By Landazuri and Seitz [**21**], all projective nontrivial representations of $PS\Omega^-(2n, q)$, $n \geq 4$, have degree

$$\geq (q^{n-1} + 1)(q^{n-2} - 1) > q^{2n-3} + q^{n-1} \geq q^5 + q^3,$$

so the existence of a central extension in G implies $q^5 + q^3 \leq 248$, i.e., $q = 2$. From the character table in the Atlas [**12**] it is readily seen that no central extension of $PS\Omega^-(8, 2)$ and therefore neither of $PS\Omega^-(2n, 2)$, $n \geq 4$, has a feasible character on **g**. □

5.5.2. *Let $L \cong PS\Omega(2n + 1, q)$, $n \geq 2$. If $\hat{L} \leq G$, then $\hat{L} \cong PS\Omega(7, 2)$, $2 \cdot PS\Omega(7, 2)$, or $2 \cdot PS\Omega(5, 5) \cong \mathrm{Sp}(4, 5)$.*

PROOF. $\hat{L}$ contains a central extension of $PS\Omega^-(2n, q)$, so by the previous paragraph $n \leq 3$.

Suppose first $n = 3$. Now $L \cong PS\Omega^+(7, q)$ contains a central extension of $PS\Omega^-(6, q) \cong PSU(4, q)$, so by §5.4.4, $q = 2$ or 3. If $q = 3$, then $|Z(\hat{L})|$ must divide 3, for otherwise there is no faithful character of $\hat{L}$ of degree at most 248. Also, if the center has order 3, the only nontrivial character degree ≤ 248 is 27, again leading to a contradiction with Proposition 2.6. Thus, $\hat{L} = L$, and $2 \cdot PSU(4, 3) \leq L$, which contradicts §5.4.4. The conclusion is that $q = 2$. Both $PS\Omega(7, 2)$ and its universal cover $2 \cdot PS\Omega(7, 2)$ embed in G, via $SL(7, \mathbf{C})$ and $SL(8, \mathbf{C})$, respectively.

Next, let $n = 2$. Then $PS\Omega(5, q) \cong P\mathrm{Sp}(4, q)$ contains an extension of $L(2, q^2)$, so by §5.2, $q \in \{2, 3, 4, 5\}$. Since $PS\Omega(5, 2) \cong \mathrm{Alt}_6$ and $PS\Omega(5, 3) \cong PSU(4, 2)$ have been dealt with, we may assume $q \in \{4, 5\}$.

If $\hat{L} \cong L \cong PS\Omega(5, 4)$, there is an elementary abelian subgroup of order 2^4 all of whose involutions are from a single conjugacy class (2C in the Atlas); the trace of such an involution on **g** in a feasible character must be 24 by the character table; on the other hand, Proposition 3.9 shows that there are no such elementary abelian subgroups of order 16. Thus, $q = 5$.

Let $L \cong PS\Omega(5, 5)$. Then a direct check shows that L has no feasible character on **g**, whence L cannot be embedded in G (cf. Proposition 3.1). Thus $\hat{L} \leq G$ implies that $|Z(\hat{L})| = 2$. [The group $2 \cdot PS\Omega(5, 5)$ embeds in G via $\mathrm{Spin}(13, \mathbf{C})$.] □

5.5.3. *Suppose $L \cong PS\Omega^+(2n, q)$, where $n \geq 4$. Then $\hat{L} \leq G$ implies $L \cong PS\Omega^+(8, 2)$, and all possibilities for $\hat{L}$ occur. Moreover, if $\hat{L} = L$, then $\chi|_L$ is as given in Table 1.*

PROOF. $\hat{L}$ contains a central extension of $L(n, q)$, so by §5.3.1, $(n, q) = (4, 2)$. Now $2^2 \cdot L$ embeds in each central product factor of $D_4(\mathbf{C}) \circ D_4(\mathbf{C})$ isomorphic to $\mathrm{Spin}(8, \mathbf{C})$, $2 \cdot L$ embeds in $SL(8, \mathbf{C})$, and L embeds in $4 \cdot SL(8, \mathbf{C})$. The final statement follows from the feasibility conditions. □

In view of the preceding, we need, for the symplectic case, only deal with q odd and $n \geq 3$.

5.5.4. *Suppose $L \cong P\mathrm{Sp}(2n, q)$ with q odd and $n \geq 3$. Then $\hat{L}$ cannot be embedded in G.*

PROOF. Again, from $L(2, q^n) \leq P\mathrm{Sp}(2n, q)$ and §4.2, we obtain that an embedding in G may only occur if $(n, q) = (3, 3)$. So take $L \cong P\mathrm{Sp}(6, 3)$. Then $\hat{L}$ has a subgroup isomorphic to $N : \mathrm{Sp}(4, 3)$, where N is an extraspecial group of shape 3^{1+4} and of exponent 3. Let z generate the center of N. By Corollary 3.4, and the fact that N does not have an embedding in a Lie group of type A_8, the centralizer in G of z must be a Lie group of type E_6. Thus $\chi(z) = 5$. Furthermore, if $y \in N - \langle z \rangle$, then

$$(\chi|_N, 1) = \frac{1}{243}(248 + 2 \times (5) + 240\chi(y)) \in \mathbf{Z},$$

so $\chi(y) = 5$. It follows that $C_G(y)$ is of type E_6 as well. Then, on a 27-dimensional module for the central product factor E of type E_6 in $C_G(z)$, the element y must have trace 0 (cf. Cohen and Wales [**11**]). Thus, if ψ denotes the character of E on this module,

$$(\psi|_N, 1) = \frac{1}{243}(248 - 27 + 240 \times 0) \notin \mathbf{Z},$$

a contradiction. □

5.6. *The remaining groups of Lie type.*

5.6.1. *Let L be a finite simple group of exceptional Lie type and untwisted (but not $G_2(2)' \cong PSU(3,3)$). If $\hat{L} \leq G$, then $\hat{L} \cong G_2(3)$. The latter group occurs in G via* $\mathrm{Spin}(14, \mathbf{C})$, *but not in K. Also, every embedding of $G_2(3)$ factors through a Lie subgroup isomorphic to* $\mathrm{Spin}(14, \mathbf{C})$, *and the character of $G_2(3)$ on $\mathbf{g}$ is as given in Table* 1.

PROOF. By Landazuri and Seitz [**21**] all nontrivial projective representations of $E_6(q), E_7(q)$, and $E_8(q)$ are of degree $\geq q^9(q^2 - 1) \geq 2^9 > 248 = \dim \mathbf{g}$.

$F_4(2)$ has only one degree ≤ 248, namely 52. As $248 \equiv 44 \pmod{52}$ an embedding of this group in G would lead to an embedding in a centralizer of a 3-dimensional torus, a contradiction with Proposition 2.6. If $q > 2$, then, by Landazuri and Seitz [**21**] the minimal degree of a projective representation of $F_4(q)$ is at least $q^4(q^6 - 1)$ if q is odd, and $\geq q^7(q^3 - 1)(q - 1)/2$ if q is even, in particular exceeding 248. Therefore neither $F_4(q)$ nor a central extension of it can be embedded in G.

Finally, suppose $L \cong G_2(q)$ $(q > 2)$ or a central extension is embedded in G. Since L contains subgroups isomorphic to $SL(3, q)$ and to $SU(3, q)$, §§5.3 and 5.4 yield that $q \in \{2, 3\}$. Clearly, $G_2(3) \leq \mathrm{Spin}(14, \mathbf{C}) \leq G$, and, if $\hat{L} \cong 3 \cdot G_2(3)$, we must have $L \leq E_6(\mathbf{C})$, a contradiction with Proposition 2.6. The only real feasible characters of $L \cong G_2(3)$ are $\psi_1 = 1_a + 2 \times 78_a + 91_a$ and

$\psi_2 = 1_a + 2 \times 14_a + 64_a + 64_b + 91_a$. Since there is a nonzero fixed vector in $\mathbf{g}$, Corollary 2.4 and Proposition 2.6, and nonfeasibility on M yield that L is contained in a Lie subgroup D of type D_7. But D centralizes a torus, hence an involution, and so is contained in D_8 with representation degrees 120 (adjoint) and 128 (half-spin). As D has an adjoint of dimension 91, there must be a D-submodule of $\mathbf{g}$ of dimension $120 - 91 = 29$. If $\chi|_L = \psi_1$, this is impossible, whence $\chi|_L = \psi_2$. Finally, since L does not leave invariant a 56-dimensional subspace of $\mathbf{g}$, it cannot be contained in K. □

We now proceed to the twisted case.

5.6.2. *Let $L \cong Sz(8) = {}^2B_2(8)$. The Borel subgroup of $\hat{L}$ does not embed in G, whence neither does $\hat{L}$.*

Let B be a Borel subgroup of L. The structure of B is $2^{3+3} : 7$, and $O_2(B)$ has order 64. Let $A = \langle a \rangle$ be a subgroup of B of order 7, and set $Z_0 = \Omega_1(O_2(B)) \cong 2^3$. Then $B - Z_0$ consists of elements of order 4 only. As $\mathbf{F}_2A$-modules, Z_0 and $O_2(B)/Z_0$ are dual (since Z_0 must be a quotient of, hence isomorphic to, $\bigwedge^2(O_2(B)/Z_0)$. Suppose now that B is a subgroup of G. From Propositions 3.8 and 3.9 it is clear that $Z_0^\#$ consists of 7 involutions with trace -8 on $\mathbf{g}$, and so, by Proposition 3.8, $C = C_G(Z_0)$ has a normal subgroup $J = J_1 \cdots J_8$, a central product of 8 copies J_i of $SL(2, \mathbf{C})$, such that $C/J \cong 2^3$ and $Z := Z(J) \cong 2^4$. Furthermore $N_G(Z_0) = N_G(Z) \cong C.SL(3,2) \cong J.2^3.SL(3,2)$, and $B \leq N_G(Z)$. We claim that $J \cap O_2(B) = O_2(B)$. For, otherwise, $N_G(Z) \cap O_2(B) - J \neq \emptyset$, whence $O_2(B)J/J \cong 2^3$, an irreducible $\mathbf{F}_2A$-module of dimension 3. But then $O_2(B)J/J \cong C/J$, which is an irreducible 3-dimensional $\mathbf{F}_2A$-module isomorphic to Z_0 (this can be seen by consideration of the nonsingular pairing $C/J \times \langle e_8 \rangle / Z_0 \to Z_0$ given by commutation), contradicting that $O_2(B)/Z_0$ is an $\mathbf{F}_2A$-module isomorphic to the dual of Z_0.

Thus, we must have $O_2(B) \leq J$. We shall show that this also leads to a contradiction. A permutes the seven elements in $Z - Z_0^\#$ cyclically, whence the seven centers $Z(J_i)$, and accordingly the seven groups J_i, so that A is not contained in J and $A \cap J = 1$. Now, without loss of generality, A can be taken so as to permute the J_i according to the permutation $(1,2,4,3,6,7,5)$ on the indices. Thus as an $\mathbf{F}_2A$-module, Z_0 is irreducible. (The element a acts with minimal polynomial $\lambda^3+\lambda+1$.) As $O_2(B) \leq J$, we have without loss of generality $y = y_1 \cdots y_8 \in J \cap O_2(B) - Z$ with $y_i \in J_i$. Now $\langle y^A \rangle Z/Z \cong 2^3$ must be the dual of Z_0 as an A-module. We can embed each y_i in a torus of J_i. Let $x_i = y_{i-1}^a$. Consider the module $D = \langle x_i, y_i \rangle_{1 \leq i \leq 8} Z/Z \cong 2^{16}$ for A. Let F be the subgroup of D consisting of all elements in a product of J_i for i ranging over the words in (1) of §3.8. Then F is an $\mathbf{F}_2A$-submodule of D of dimension $2 \times 4 = 8$. The quotient D/F is isomorphic to the sum of two copies of Z as an $\mathbf{F}_2A$-module, so $O_2(B)$ must be contained in F. But each element in $F - Z$ consists of involutions in view of (1) of §3.8, whereas it should contain elements of order 4 in order for F to contain $O_2(B)$. This contradiction shows that $O_2(B)$ does not embed in J

with the proper A-action; the conclusion is that B is not embeddable in $N_G(J)$, and hence not in G.

Now, choose a Borel subgroup $\hat{B}$ of $\hat{L}$ such that $\hat{B}$ is an extension of the Borel group B of L. Assume $\hat{B} \leq G$ and let $Y = Z(\hat{L})$. If Y contains an involution of trace -8, we embed a central extension of B into $SO(16, \mathbf{C})$, which is incompatible with the character table. So every involution in Y has trace 24. If $|Y| = 4$, then, by Lemma 3.7, $\hat{B} \leq E_6(\mathbf{C})$, which is absurd. So $|Y| = 2$. Then the involution in $Y^\#$ must have centralizer of type E_7. But from Table 6 we see that an involution on the 56-space has trace $8, -8$, or 56, whereas faithful representations of $\hat{B}$ have trace 0 or -16 on noncentral involutions, a final contradiction. □

5.6.3. *Suppose L is a finite simple group of twisted Chevalley type ($\neq {}^2D_n, \neq {}^2G_2(3)' = L(2,8), \neq Sz(2)'$). If $\hat{L} \leq G$, then $\hat{L} = {}^3D_4(2)$ or ${}^2F_4(2)'$. Both groups occur in G. Any embedding of the latter two groups factors through $E_6(\mathbf{C})$, and their characters on $\mathbf{g}$ are as given in Table* 1.

PROOF. Let $L = {}^3D_4(q)$. As $\hat{L}$ contains a central cover of $G_2(q)$, we conclude from the above that $q = 2$ or 3. If $q = 3$, let P be a parabolic of shape $3^{1+8}SL(2,27)\mathbf{Z}_2$. Representation theory of the extraspecial group of shape 3^{1+8} and Corollary 3.4 show that we must have $3^{1+8}SL(2,27) \cong P' \leq ES$, a central product of a group isomorphic to $3 \cdot E_6(\mathbf{C})$ and a group isomorphic to $SL(3, \mathbf{C})$; but then the projection onto the factor $S/Z(S)$ must be trivial, whence $P' \leq E$. However P' does not have a faithful character of degree 27. Hence $q = 2$. (See Cohen [**8**] for an embedding of Aut ${}^3D_4(2)$ in $F_4(\mathbf{C})$.) Beside the character in Table 1, there is one other feasible character of ${}^3D_4(2)$ on $\mathbf{g}$, viz. $52_a + 196_a$. Suppose $\chi|_L$ is the latter, and let U be an L-invariant subspace of $\mathbf{g}$ of dimension 52. In view of the decomposition $\bigwedge^2 52_a + 1274_a$, and $52_a \otimes 196_a = 637_a + 2184_a + 2457_a + 2457_b + 2457_c$, the space U must be an ideal of $\mathbf{g}$, a contradiction with the simplicity of $\mathbf{g}$. Therefore the character of ${}^3D_4(2)$ on $\mathbf{g}$ is as indicated in Table 1.

Let $q = 2^{2m+1}$, where $m \geq 0$ and let $\hat{L}$ be a (possibly trivial) central extension of the derived group of ${}^2F_4(q)$. Since, by Landazuri and Seitz [**21**], L has nontrivial projective representations of degree $\geq q^4(q-1)\sqrt{q/2}$ only, the latter number cannot exceed 248, whence $m = 0$ and $q = 2$. If $L \cong {}^2F_4(2)'$, the nontrivial character degrees ≤ 248 are 26, 27, 78. All of the corresponding characters have positive rational traces on elements of order 5. The feasible characters on $\mathbf{g}$ are $\psi_1 = 8 \times 1_a + 6 \times 27_* + 78_a$ and $\psi_2 = 14 \times 1_a + 6 \times 26_* + 78_a$. If $L \leq E_6(\mathbf{C})$, then, by [**11**], the character of L on the Lie algebra and on a 27-dimensional module for $E_6(\mathbf{C})$ must both be irreducible, and so $\chi|_L = \psi_1$. If L is not contained in $E_6(\mathbf{C})$, then, by Corollary 2.4, we must have $L \leq K$. But then, by the character table, L has fixed points on M, which is absurd because of Proposition 4.4. This proves the statements concerning the Tits group ${}^2F_4(2)'$.

Let $q = 2^{2m+1}$, where $m \geq 0$ and let L be a (possibly trivial) central extension of $Sz(q)$ which is embedded in G. By [**21**], L has nontrivial projective representations of degree $\geq (q-1)\sqrt{q/2}$ only, so $q \leq 32$. Then $q \in \{8, 32\}$. In the last case $\hat{L} \cong L \cong Sz(32)$, and $\chi|_L$ must be the sum of the two irreducibles of degree 124. The dimension of the centralizer in $\mathbf{g}$ of an element of order 25 is $\frac{1}{25}(248+4\times(-2)+20\times(-2)) = 8$, so is a Cartan subalgebra. On the other hand its fifth power has centralizer $\mathbf{sl}_5 \oplus \mathbf{sl}_5$ of dimension 48, and we have a contradiction with 5.2.2. [Alternately, one can reason that the only feasible character decomposition for $\chi|_L$ is $124_a + 124_b$ and calculate $(\bigwedge^3 \chi|_L, 1) = 0$ to finish by Lemma 2.5.] Thus, $q = 8$, which has been eliminated in 5.6.2.

Let $q = 3^{2m+1}$, where $m > 0$ and let $\hat{L}$ be a subgroup of G isomorphic to (a possibly trivial) central extension of ${}^2G_2(q)$. By [**21**], L has nontrivial projective representations of degree $\geq q(q-1)$ only, so $q < 27$, whence $m = 0$, a contradiction. Finally, ${}^2E_6(q)$ can be ruled out similarly. This ends the proof of the twisted case. □

5.7. *The sporadic groups. Let L be a finite sporadic simple group. If* $\hat{L} \leq G$, *then* $\hat{L}$ *is one of* $M_{11}, 2\cdot M_{12}, 2\cdot J_2$.

PROOF. Neither M_{22} nor any of its nonsplit central covers has a feasible character on $\mathbf{g}$. So, the only Mathieu groups $\hat{L}$ that could be contained in G or have a central extension in G are M_{11} and M_{12}. Indeed, $M_{11} \leq 2\cdot M_{12} \leq \mathrm{Spin}(11,\mathbf{C}) \leq G$.

We next prove that M_{12} cannot be embedded in G. Assume $L \leq G, L \cong M_{12}$. In view of Proposition 3.1, there are only two feasible character shapes for $\chi|_L$ (viz. $1_a + 2\times 11_* + 16_* + 2\times 55_* + 99_a$ and $2\times 11_* + 16_* + 45_a + 3\times 55_*$). In both events, there is a single irreducible constituent of degree 16. But the character of degree 16 is not real, so cannot occur in $\chi|_L$ with multiplicity one (cf. Lemma 2.5).

The remaining sporadic groups with a nontrivial projective character of degree at most 248 are $J_1, J_2, J_3, HS, McL, Suz, He, Ru, Fi_{22}$, and F_3. We observe that $2\cdot J_2$ is embedded in $SL(6,\mathbf{C})$. By feasibility of characters on $\mathbf{g}$, using the tables in the Atlas [**12**], all other possibilities are easily ruled out, except for (possibly) HS, He, and F_3. But M_{22} is a subgroup of HS, and $P\mathrm{Sp}(4,4)$ is a subgroup of He, so only F_3 remains to be considered. Now F_3 has trivial Schur multiplier. S. Norton has pointed out that the character in question does not admit an alternating trilinear form. Another way of seeing that F_3 is not a subgroup of G is to consider its subgroup isomorphic to $3\cdot G_2(3)$ centralizing an element in $e \in F_3$ of order 3. Since, in an embedding of F_3 in G, the element e must have trace 5, a look at Table 4 shows that containment of F_3 in G would imply containment of $G_2(3)$ in $C_G(e)/\langle e\rangle \cong E_6(\mathbf{C})$. But then $G_2(3)$ would be a subgroup of $E_6(\mathbf{C})$ which is not true according to Cohen and Wales [**11**]. □

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