EINDHOVEN UNIVERSITY OF TECHNOLOGY

Solution to Problem 61-10 : The expected value of a product

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Problem 61–10, The Expected Value of a Product, by L. A. SHEPP (Bell Telephone Laboratories).

Let E_n be the expected value of the product $x_1x_2x_3 \cdots x_n$, where x_1 is chosen at random (with a uniform distribution) in $(0, 1)$ and x_k is chosen at random (with a uniform distribution) in $(x_{k-1}, 1)$, $k = 2, 3, \cdots, n$. Show that

$$\lim_{n\to\infty} E_n = \frac{1}{e}.$$

Solution by J. VAN YZEREN (Technische Hogeschool Eindhoven, Netherlands).

The random variates $x_1, x_2, x_3, \cdots$ can be produced by

$$x_i = 1 - u_1u_2u_3 \cdots u_i$$

where the u are independent choices from a uniform distribution.

Consider the formal product

$$\prod_{i=1}^{\infty} (1 + tu_1u_2u_3 \cdots u_i) \tag{1}$$

The coefficient of t^n is the sum of all products $u_1^{\,n}u_2^{\,n_2}u_3^{\,n_3} \cdots$ with $n \geqq n_2 \geqq n_3 \geqq \cdots$. Evidently $E(u_1^{\,n}u_2^{\,n_2}u_3^{\,n_3} \cdots) = \dfrac{1}{n+1}\dfrac{1}{n_2+1}\dfrac{1}{n_3+1} \cdots$. The sum of all these expected values is

$$\frac{1/(n+1)}{1 - 1/(n+1)} \frac{1/n}{1 - 1/n} \cdots \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{1}{n!}.$$

Hence,

$$E\left\{\prod_{i=1}^{m} (1 + tu_1u_2u_3 \cdots u_i)\right\}$$

converges for $m \to \infty$ to e^t, for any value of t! Putting $t = -1$ gives the required result.

Solution by J. H. VAN LINT (Technological University, Eindhoven, Netherlands).

Let $E_n(x)$ be the expected value of the product $x_1x_2 \cdots x_n$, where x_1 is chosen at random (with a uniform distribution) in $(x, 1)$ and x_k is chosen at random (with a uniform distribution) in $(x_{k-1}, 1)$, $k = 2, 3, \cdots, n$.

The function $E_n(x)$ can be expressed as a multiple integral:

$$\begin{aligned} E_n(x) = \frac{1}{1-x}\int_x^1 \frac{x_1}{1-x_1}\,dx_1 \\ \int_{x_2}^1 \frac{x_2}{1-x_2}\,dx_2 \cdots \int_{x_{n-2}}^1 \frac{x_{n-1}}{1-x_{n-1}}\,dx_{n-1} \int_{x_{n-1}}^1 x_n\,dx_n. \end{aligned} \tag{1}$$

We define $E_0(x) = 1$, $E_1(x) = \dfrac{1+x}{2}$. We then have for $n = 1, 2, \cdots$ the differential equation

$$\{(1-x)E_n(x)\}' = -xE_{n-1}(x). \tag{2}$$

We now prove the inequality

$$e^{x-1} \leqq E_n(x) \leqq e^{x-1} + \frac{1-x}{2^n}. \tag{3}$$

For $n = 0$, the inequality is true. If the inequality is true for some value of n we find an inequality for E_{n+1} by applying (2). We find,

$$e^{x-1} \leqq E_{n+1}(x) \leqq e^{x-1} + \frac{1}{2^n}\left\{\frac{1}{6} + \frac{1}{6}x - \frac{1}{3}x^2\right\} \leqq \frac{1}{2^{n+1}}(1-x) + e^{x-1}.$$

By induction (3) is true for all n. A consequence of (3) is

$$\underline{\lim_{n\to\infty} E_n(x) = e^{x-1}} \tag{4}$$

(If we take $x = 0$, we find the theorem that was to be proved).

Also solved by L. L. Campbell (Assumption University of Windsor, Ontario, Canada), W. D. Fryer (Cornell Aeronautical Laboratory), J. S. Hicks and R. F. Wheeling (Socony Mobil Oil Company), R. Hines (ARCON Corporation), D. Rothman (Electronic Specialty Company), A. Van Gelder (Grumman Aircraft Engineering Corporation), and the proposer.

Problem 61–12, On a Least Square Approximation, by D. J. Newman (Yeshiva University).

$F(x)$ is given in the interval [0, 1] such that $|F^{(n)}(x)| < M$. $P(x)$ is the $(n-1)$st degree polynomial passing through the n points $(a_r, F(a_r))$, $(r = 1, 2, \cdots, n)$. If $\lambda(a_1, a_2, \cdots, a_n)$ is defined by the inequality

$$|F(x) - P(x)| \leqq \lambda(a_1, a_2, \cdots, a_n)M,$$

show that over all selection of the a_r's

$$\min \lambda(a_1, a_2, \cdots, a_n) = \frac{2}{4^n n!}.$$

The solutions by W. Fraser (University of Toronto) and Pierre Robert (Universite de Montreal) were virtually identical and is given below.

It is a well known result that the error term in the approximation of $F(x)$ by $P(x)$ is

$$(x-a_1)(x-a_2)\cdots(x-a_n)\frac{F^{(n)}(\xi)}{n!}$$

for $0 \leqq \xi \leqq 1$, if $0 \leqq x \leqq 1$ and $0 \leqq a_i \leqq 1$ for all i. (See Hildebrand: Introduction to Numerical Analysis, pp. 60–63.) Hence

$$|F(x) - P(x)| \leqq \frac{M}{n!}|(x-a_1)(x-a_2)\cdots(x-a_n)|.$$