The shape of a rotating fluid drop

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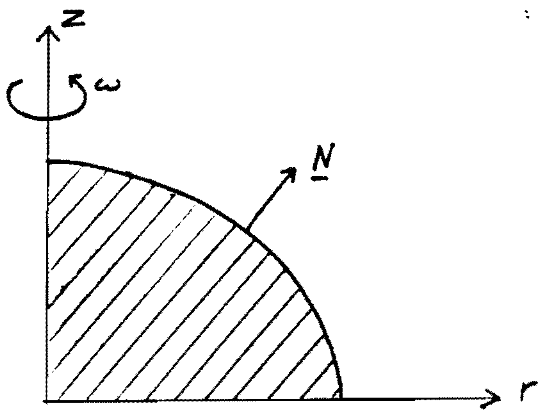
1. Summary

In this article the problem of a rotating fluid drop, held together by surface-tension, will be studied. A differential equation for the shape of the fluid surface is derived and the solution of this differential equation, dependent on characteristic parameters is calculated numerically.

2. Formulation of the problem

A fluid drop, held together by surface-tension rotates around a fixed axis.
The following figure sketches the situation: Known parameters:



ω angular velocity
γ surface-tension
ρ density of the fluid
V volume

Such problems are conveniently described in cylindrical coordinates, to utilize the rotational symmetry in the (x,y)-plane. The surface of the drop will be described by a curve in the (r,z)-plane rotating around the z-axis. Thus the curve in the first quadrant of the (r,z)-plane can be written in the form

$$z = F(r) \text{ with } r = \sqrt{x^2+y^2}.$$

In order to apply the boundary-condition

(2.1) $$p = \gamma \cdot \operatorname{div} \underline{N}$$

where

p denotes the excess pressure at the surface and

$\underline{N}$ the outward normal field of the surface,

we want to find equations for p and div $\underline{N}$ in terms of $F(r)$.
The following expression for $\underline{N}$ is given in reference [1]

(2.2) $$\underline{N}(x,y,z) = \frac{1}{\sqrt{1+(F')^2}} \left[\frac{-x}{\sqrt{x^2+y^2}} F', \frac{-y}{\sqrt{x^2+y^2}} F', +1 \right].$$

It should be mentioned, that this is only one of the many possible extensions of the normal field on the surface S given by $z = F(r)$. If we calculate the divergence of $\underline{N}$ and restrict it to S, the result is not dependent on the choice of the extension of $\underline{N}$. For the divergence we get:

(2.3) $$\operatorname{div} \underline{N} = -\frac{1}{\sqrt{1+(F')^2}} \left[\frac{1}{\sqrt{x^2+y^2}} F' - \frac{(F')^2 F''}{1+(F')^2} \right].$$

Restricting this to S we get:

(2.4) $$\operatorname{div} \underline{N} = -\frac{F'}{r\sqrt{1+(F')^2}} - \frac{F''}{\sqrt{1+(F')^2}^{\,3}}.$$

Next we want to obtain an expression for the pressure p. Consider in the (x,y)-plane

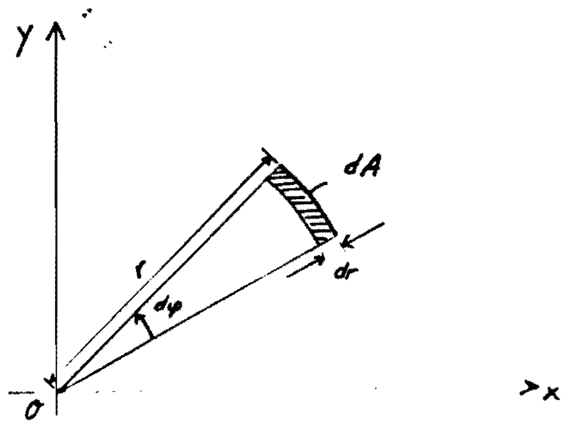


Fig. 2.2.

The differential force (caused by the differential mass-element dm filling the volume $dV = dr \cdot dA$) is given by

(2.5) $$dF = a \cdot dm$$

where a is the centrifugal accelleration due to the rotation. It is given by

(2.6) $$a(r\,;\omega) = \omega^2 \cdot r.$$

Using $dm = \rho\, dV = \rho\, r\, d\phi\, dz\, dr$ we get for dF

$$dF = \omega^2 r \cdot \rho\, r\, d\phi\, dz\, dr$$

and finally

(2.7) $$dp = \frac{dF}{dA} = \rho\, \omega^2 r\, dr\,,$$

which means, after integration

$$p(r\,;\omega) = \int_0^r dp = \int_0^r \rho\, \omega^2 r\, dr$$

(2.8) $$= p_0(\omega) + ½\,\omega^2 \rho\, r^2.$$

The integration 'constant' p_0 will be calculated later. Note that we expect a 'smooth' transition of $p(r;\omega)$ for $\omega \to 0$ to the case $\bar{p}(r)$ for $\omega = 0$.
Then the fluid forms a ball. The divergence of the normal field of a sphere with radius R equals

(2.9) $$\operatorname{div} \underline{N} = \frac{2}{R}.$$

This means, that $\bar{p} = \gamma \cdot \frac{2}{R}$ on the surface of the ball. In this case R is given by $R = 3/4\pi \cdot V^{1/3}$.
Gathering all details, we arrive at a differential equation for $F = F(r)$

(2.10) $$p_0 + ½\,\omega^2 \rho\, r^2 = -\gamma \left[\frac{F'}{r(1+(F')^2)^{½}} + \frac{F''}{(1+(F')^2)^{3/2}} \right]$$

$$r \in [0, r_1]$$

with boundary values

(2.11a) $F'(0) = 0$

(2.11b) $F(r_1) = 0$

(2.11c) $F'(r_1) = \lim_{r \uparrow r_1} F'(r) = -\infty$

(2.11d) $V = 4\pi \int_0^{r_1} r \cdot F(r)\, dr.$

The constants ω, ρ, V are given and r_1 and p_0 are upto now unknown. This is consistent with the second order differential equation and the four boundary conditions.

3. Derivation of the final equation

Now we define $g := F'$ and notice, that

$$\frac{g'}{\sqrt{1+g^2}^{\,3}} + \frac{g}{\sqrt{1+g^2}} = \frac{1}{r}\,\frac{d}{dr}\left[\frac{rg}{\sqrt{1+g^2}}\right].$$

The differential equation (2.10) is thus transformed into

(3.1) $$-\left[\frac{p_0}{\gamma}r + \frac{\omega^2 \rho}{2\gamma} r^3\right] = \frac{d}{dr}\left[\frac{rg}{\sqrt{1+g^2}}\right].$$

This can be easily integrated

(3.2) $$\frac{rg}{\sqrt{1+g^2}} = -\left[\frac{p_0}{2\gamma}r^2 + \frac{\omega^2 \rho}{8\gamma}r^4\right] + K_0.$$

The integration constant K_0 can be calculated by applying boundary condition (2.11a)

$$g(0) = 0 \;\Rightarrow\; 0 = K_0.$$

It means that g is given by:

(3.3) $$g = \frac{-\left[\frac{p_0}{2\gamma}r + \frac{\omega^2 \rho}{8\gamma}r^3\right]}{\left\{1 - \left[\frac{p_0}{2\gamma}r + \frac{\omega^2 \rho}{8\gamma}r^3\right]^2\right\}^{½}}.$$

If we introduce the constants

(3.4) $$K_1 = \frac{p_0}{2\gamma} \; ; \; K_2 = \frac{\omega^2 \rho}{8\gamma},$$

then g can be written in the following form:

(3.5) $$g = \frac{-(K_1 r + K_2 r_3)}{\{(1-(K_1 r + K_2 r^3))\cdot(1+(K_1 r + K_2 r^3))\}^{½}}.$$

Boundary condition (2.11c) says, that for a certain r_1 the relation

$$g(r_1) = \lim_{r \uparrow r_1} g(r) = -\infty \;\text{ holds.}$$

If we restrict ourselves to $K_1 > -3/2(2K_2)^{1/3}$, then $1+(K_1 r + K_2 r^3)$ has no positive zeros. This means, that $p_0 > -3\cdot\left[\frac{\rho\omega^2\gamma^2}{4}\right]^{1/3}$. Hence r_1 is the lowest positive zero of $1-(K_1 r + K_2 r^3)$, i.e. $1 = K_1 r_1 + K_2 r_1^3$. Now p_0 can be calculated;

(3.6) $$p_0(\omega) = \frac{2\gamma}{r_1} - \frac{\omega^2 \rho r_1^2}{4}.$$

This, indeed, gives the desired result for $\omega = 0$ (compare (2.9)):

(3.7) $$p_0(0) = \frac{2\gamma}{r_1} = \bar{p} \;\text{ (the pressure of the drop at rest).}$$

We now define

(3.8) $$C := K_2 r_1^3 = \frac{\omega^2 \rho}{8\gamma}\cdot r_1^3$$

and so

(3.9) $$1 - C = K_1 r_1 = \frac{p_0 r_1}{2\gamma}.$$

The restriction on K_1 means, that

$$K_1 r_1 > -3/2\, (2K_2 r_1^3)^{1/3}$$

$$\Rightarrow 1 - C > -3/2\, (2C)^{1/3}$$

and so $C < 4$.

The constant C has also a physical interpretation.

(3.10) $$C = \frac{\frac{1}{4}\omega^2 \rho r_1^2}{2\gamma / r_1} = \text{ratio} \frac{\text{additional pressure due to rotation}}{\text{pressure of a drop of the same diameters at rest}}$$

$$\text{since} : p(r_1) = \frac{2\gamma}{r_1} + \frac{1}{4}\omega^2 \rho r_1^2.$$

Our first goal was to say something explicitly about the *shape* of the drop. We could therefore as well scale the whole problem with the radius r_1 (which is still unknown):

$$r = r_1 r^* \quad F = r_1 F^*.$$

We get

(3.11) $$g^* = F^{*\prime} = \frac{-(K_1 r_1 r^* + K_2 r_1^3 r^{*3})}{\{1 - (K_1 r_1 r^* + K_2 r_1^3 r^{*3})^2\}^{1/2}}.$$

If we now drop the $*$ and use the definition of C we get

(3.12) $$F' = \frac{-((1-C) r + C r^3)}{\sqrt{1 - ((1-C) r + C r^3)^2}}.$$

This is exactly the same equation, that Chandresekhar gave in his appendix to [1].

We now integrate (3.12) from v to 1 and obtain:

$$F(1) - F(r) = \int_r^1 \frac{-((1-C)x + C x^3)}{\sqrt{1 - ((1-C)x + C x^3)^2}}.$$

Application of boundary condition (2.11b) yields $F(r_1) = F^*(1) = 0$, which results in the following equation for F, with the only parameter C

(3.13) $$F(r) = \int_r^1 \frac{(1-C)x + C x^3}{\sqrt{1 - ((1-C)x + C x^3)^2}}\, dx$$

C is implicitly given by

(3.14)
$$\tilde{V} = \int_0^1 r\, F(r;C)\, dr$$
$$\tilde{V} = V \cdot C \cdot \frac{8\gamma}{\omega^2 \rho}$$

This equation is also given in Chandresekhar's paper [1].

Analysis of the equation (3.13)

In [1] the integral (3.13) is transformed by a substitution and rather complicated definitions for the boundaries of the integral. In the end Chandraseklar gets a sum of elliptic integrals of first and second kind for F.
We did not want to do this. However we want to say something about the integral by more elementary considerations.

There are four different regions for C

a) $C = 0$

b) $0 < C < 1$

c) $C = 1$

d) $1 < C$

The case $C = 0$ which corresponds to $\omega = 0$ gives us another possibility to check the equation:

(4.1)
$$F_0(r) = \int_r^1 \frac{x}{\sqrt{1-x^2}}\, dx = [-\sqrt{1-x^2}\,]_r^1$$
$$= \sqrt{1-r^2}.$$

This means that for $C = 0 (\omega = 0)$ we get back the ball.

Now we look at the other three cases.
The equations are:

(4.2)
$$F(r) = \int_r^1 \frac{(1-C)x + C x^3}{\sqrt{1-((1-C)x + C x^3)^2}}\, dx$$

(4.3)
$$F'(r) = -\frac{(1-C)r + C r^3}{\sqrt{1-((1-C)r + C r^3)^2}}\, dx$$

(4.4)
$$F''(r) = -\frac{(1-C) + 3C r^2}{\sqrt{1-((1-C)r + (r^3)^2}} - \frac{\{(1-C)r + C r^3\} \cdot \{(1-C) + 3C r^2\}}{\sqrt{1-((1-C)r + C r^3)^2}}.$$

We look for extremal points of $F(r)$ for

$$r \in [0;1] : F'(r) = 0 = (1-C)\,r + C\,r^3$$

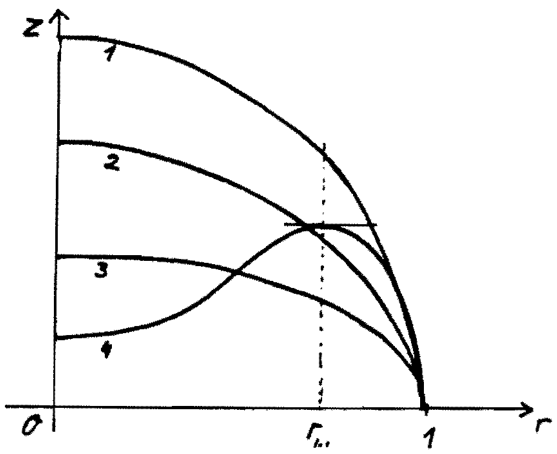
$$\Rightarrow\; r = 0 \text{ of } r = \pm\sqrt{1 - 1/C}\,.$$

- In the case $0 < C < 1$ the last two solutions are no real numbers and $F'(r) < 0 \;\; \forall r \in (0,1)$.
- For $C = 1$ all zeros coincide. I.e. $F'(r) < 0$, $r \in (0,1)$. If we evaluate $F''(0)$ in this case, then we get $F''(0) = -(1-C) = 0$, meaning that for $C = 1$ the drop is flat at the top.
- If $C > 1$ we get two zeros of F' that are relevant, viz. $r = 0$ and $r_\infty = \sqrt{1 - \frac{1}{C}}$ with $r_\infty \in (0,1)$.
  Evaluating $F''(r_\infty)$ we get

  $$F''(r_\infty) = -4(C-1) < 0\,,$$

  which means that r_∞ is a maximum.

The graph of $F(r)$ is qualitatively plotted in the following diagram:



1 : $C = 0$
2 : $0 < C < 1$
3 : $C = 1$
4 : $C > 1$

Fig. 4.1

5. Numerical results

From Fig. 4.1 the question arises:
For which value of C is $F(0) = 0$?
In order to get this, we tried to integrate equation (4.2) numerically, and it turned out, that a modified Simpson-method worked quite well. So we could test for which C it happens that $F(0) = 0$.

As a first approximation we got $C_{max} = 2{,}32$. This is also the number given in [1] for the maximum C, if the drop is to enclose the origin.

Apparently, for $C > C_{max}$ the drop surface becomes 'inverted' and therefore unphysical. Maybe, beyond $C = 4$ other shapes become possible again. This is still an interesting subject of study.

Plots of some typical drop shapes can be seen on the following figures. The corresponding numerical results are listed in Tab. 5.1.

NUMERIC RESULTS

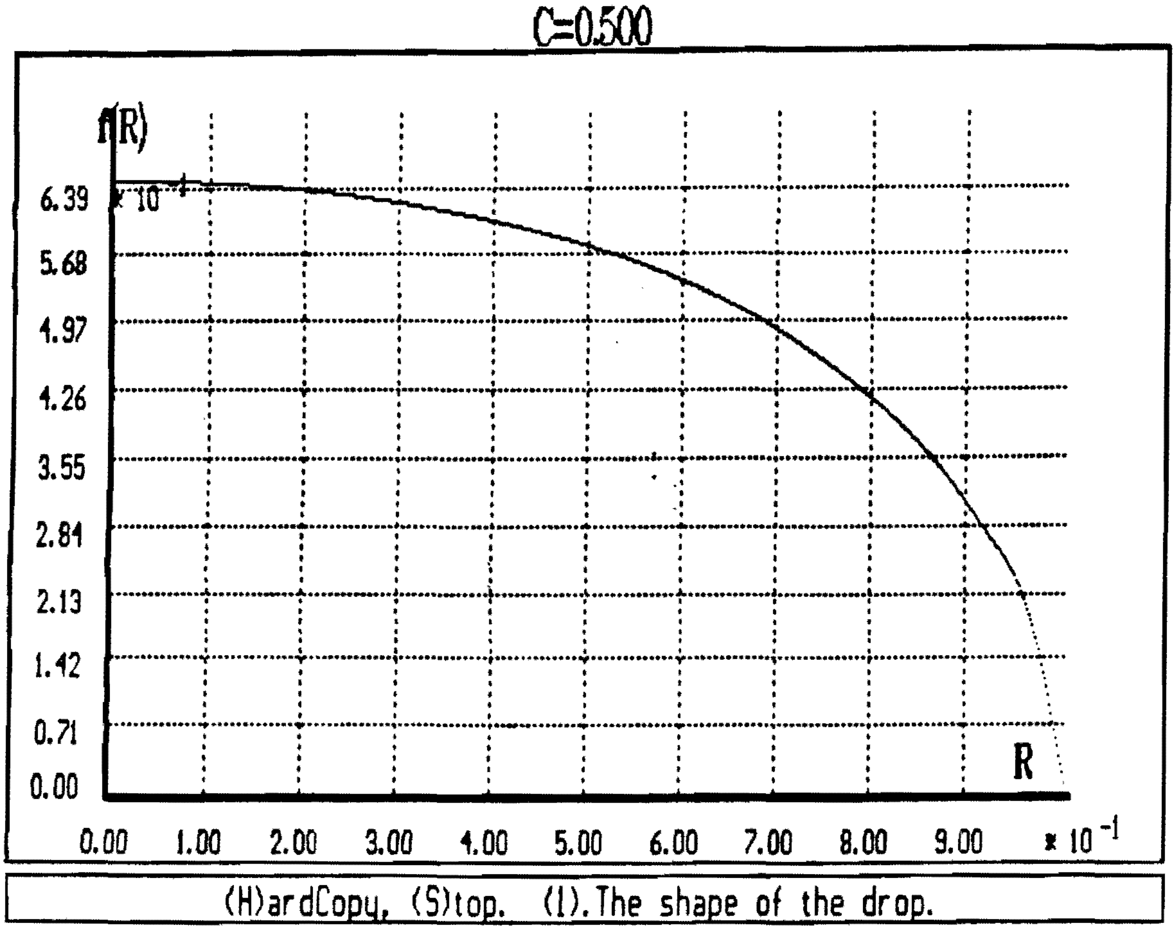
R	$f(R)$ $C = 0.500$	$C = 1.00$	$C = 1.700$	$C = 2.320$
0.0000	0.6450	0.4333	0.1969	0.0044
0.0800	0.6435	0.4328	0.1986	0.0083
0.1600	0.6387	0.4321	0.2047,	0.0204
0.2400	0.6305	0.4309	0.2144	0.0402
0.3200	0.6184	0.4286	0.2268	0.0667
0.4000	0.6018	0.4244	0.2404	0.0977
0.4800	0.5801	0.4171	0.2532	0.1304
0.5600	0.5522	0.4052	0.2628	0.1609
0.6100	0.5165	0.3870	0.2666	0.1847
0.7200	0.4706	0.3600	0.2614	0.1979
0.8000	0.4105	0.3204	0.2434	0.1963
0.8800	0.3276	0.2606	0.2056	0.1735
0.9600	0.1939	0.1569	0.1278	0.1114
1.0000	0.0000	0.0000	0.0000	0.0000

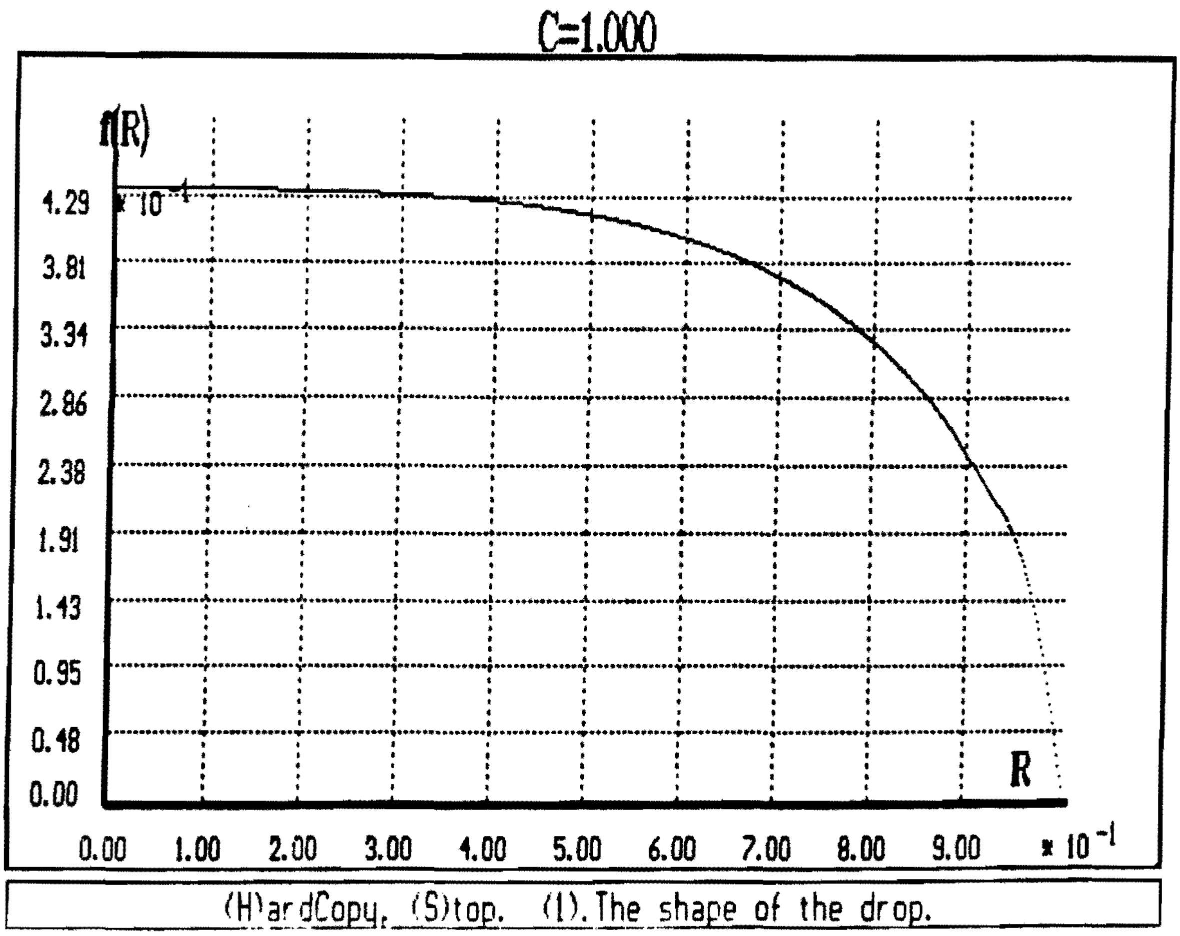
Tab. 5.1.

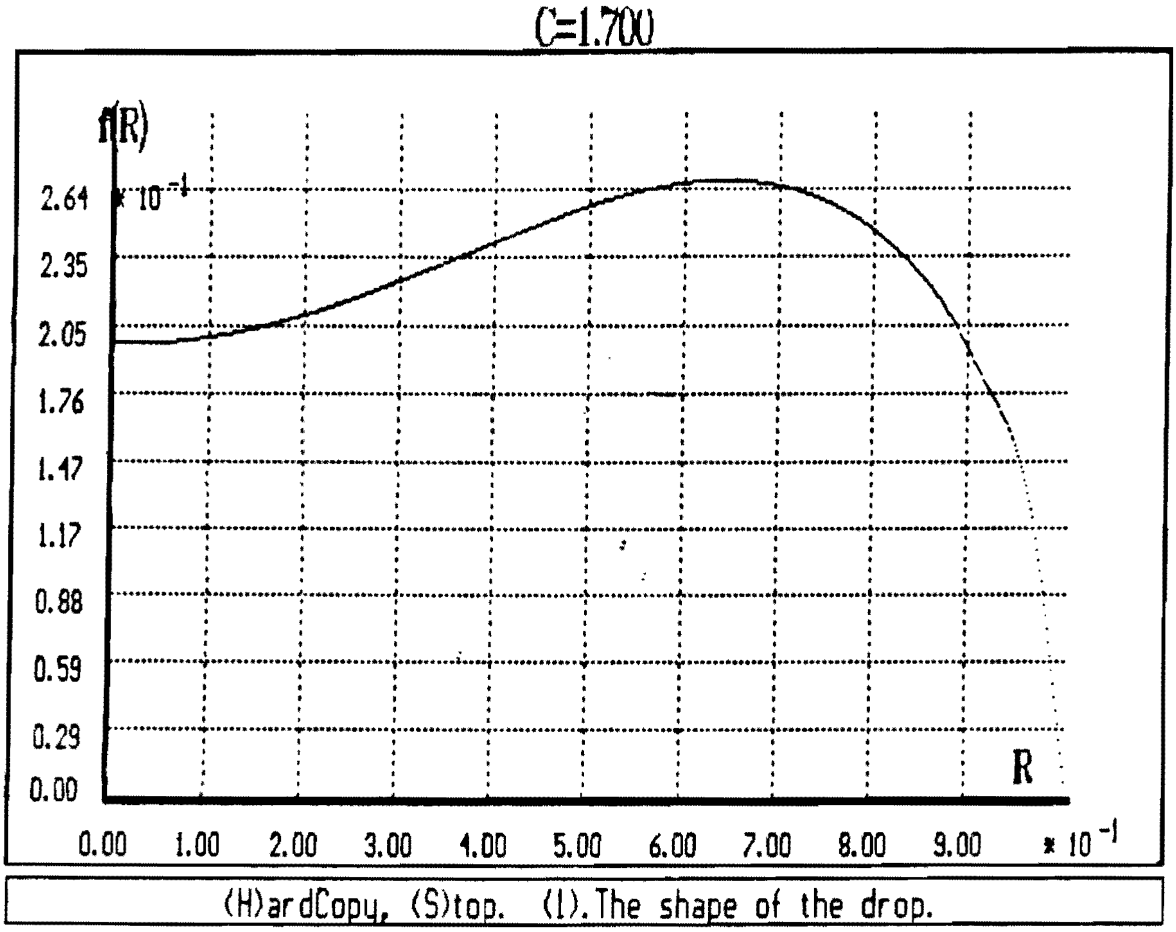
References

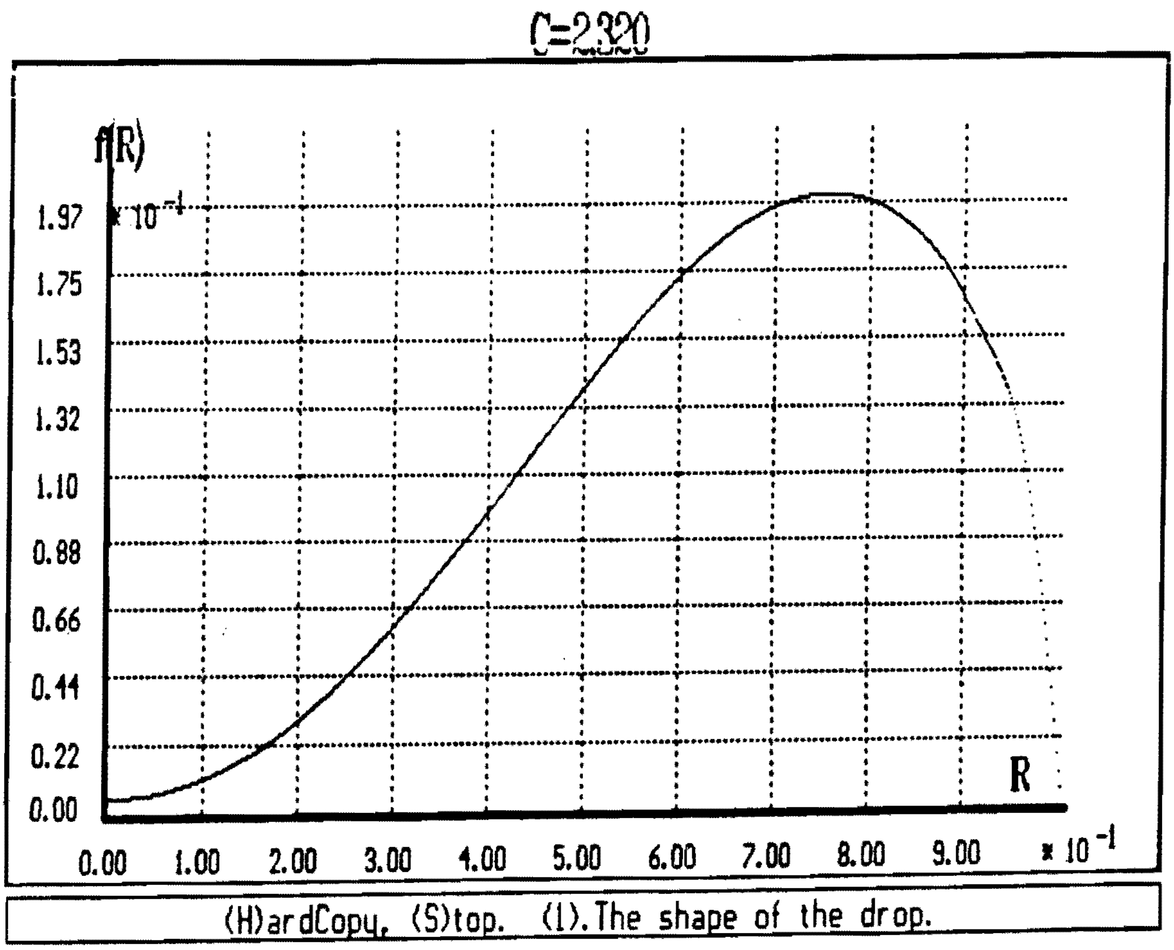
[1] Chandrasekhar, S. (1965).
The stability of a rotating liquid drop, Proc. Roy. Soc. (London) A286, 1-26.

[2] Lord Rayleigh (1914).
The equilibrium of revolving liquid, Phil. Mag. 28, p. 161-170.


C=0.500
f(R)
x 10^-1
6.39
5.68
4.97
4.26
3.55
2.84
2.13
1.42
0.71
0.00
R
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
x 10^-1
(H)ardCopy, (S)top. (1).The shape of the drop.


C=1.000
f(R)
x 10^-1
4.29
3.81
3.34
2.86
2.38
1.91
1.43
0.95
0.48
0.00
R
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
x 10^-1
(H)ardCopy, (S)top. (1).The shape of the drop.


C=1.700
f(R)
$\times 10^{-1}$
2.64
2.35
2.05
1.76
1.47
1.17
0.88
0.59
0.29
0.00
R
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
$\times 10^{-1}$
(H)ardCopy, (S)top. (1).The shape of the drop.


C=2.320
f(R)
$\times 10^{-1}$
1.97
1.75
1.53
1.32
1.10
0.88
0.66
0.44
0.22
0.00
R
0.00
1.00
2.00
3.00
4.00
5.00
6.00
7.00
8.00
9.00
$\times 10^{-1}$
(H)ardCopy, (S)top. (1).The shape of the drop.