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Abstract

In the on-line travelling salesman problem the points to be visited are not known in advance but given while the salesman is en route. This problem has been posed and studied by Ausiello, Feuerstein, Leonardi, Stougie, and Talamo (Algoritmica 29, 2001, 560-581). With respect to the competitive ratio of deterministic algorithms for this problem one case remained open, which concerned the problem on the real line. Ausiello et al. show a lower bound on the competitive ratio for any deterministic algorithm of $(9+\sqrt{17})/8 \approx 1.64$, whereas the best algorithm so far is 7/4-competitive. We close this gap by designing an algorithm that has competitive ratio $(9+\sqrt{17})/8$.

An alternative on-line model proposed by Blom, Krumke, de Paepe, and Stougie (INFORMS Journal on Computing 13, 2001, 138-148) introduces a notion of fairness. Also under this model the real line appeared to be the harder case to tackle. We derive a lower bound of $(5+\sqrt{57})/8 \approx 1.57$ on the competitive ratio of any deterministic algorithm for this version of the problem. In order not to leave any gap in the research on deterministic algorithms for the on-line travelling salesman problem, we complement this lower bound with an algorithm that has a matching competitive ratio.

1 Introduction

The travelling salesman problem is one of the most extensively studied problems in combinatorial optimization. Given a set of points in some metric space a shortest tour is to be found visiting all the points and returning to the departure point. A comprehensive survey of the numerous facets of this problem is found in [4] and [6]. The problem is NP-hard [5] in general metric spaces. It is easy on a tree, following any depth first search, and trivial to solve if the metric space is the real line: going first to the leftmost extreme, then to the rightmost extreme, and finally back to the origin.

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This triviality disappears if we consider the situation of the salesman wherein he does not have all information in advance. If the points to be visited are not known in advance but revealed while the salesman has started his tour already, the problem becomes the *on-line travelling salesman problem* (OLTSP). To be precise, we define the OLTSP as the problem of a server travelling in some metric space with a special point selected as the origin. The server is in the origin at time 0. The server can travel at maximum unit speed. Over time requests for visits (points in the metric space) are presented and the server has to make a tour to visit these points. While the server is on his tour, new requests may or may not arrive. Thus, at any time the server knows only the points requested in the past and does not know any future request, not even if there will be any future requests.

This problem has been posed and studied by Ausiello et al. [1], who called the problem in which the server is to return to the origin after having visited all requested points the *Homing*-OLTSP, as opposed to the *Nomadic*-OLTSP, in which the endpoint of his tour is free. In this paper we complete the map of research on deterministic algorithms for the Homing-OLTSP, which we will briefly call the OLTSP. Notice that the off-line version of this problem is actually a travelling salesman problem with individual release times of the points, i.e., specific moments in time at or after which the salesman must visit the points. This problem is of course also NP-hard for general metric spaces, but its complexity is unknown for trees (see [7]). It remains easy for the real line, but much less trivial than the TSP without release times [9].

In general, for on-line optimization problems there are no algorithms that attain the optimal off-line solution on every input sequence, irrespective of the computation time that is allowed. *Competitive analysis* is the most widely accepted way of measuring the performance of on-line algorithms and has been proposed for the first time in [8] (for a survey see [3]). The worst-case ratio between the objective value of an on-line algorithm and the objective value of an optimal off-line algorithm over all sequences of requests is called the *competitive ratio* of the on-line algorithm. Competitive analysis very often employs the notion of an adversary in a two-person game setting (see e.g. [3]). In this setting the adversary provides the sequence of requests and, knowing this sequence in advance, is able to serve the sequence in a tour that is optimal off-line.

For the OLTSP on general metric spaces a lower bound of 2 on the competitive ratio of any deterministic on-line algorithm is matched by a 2-competitive deterministic on-line algorithm in [1]. In the same paper the authors prove a lower bound on the competitive ratio of $(9+\sqrt{17})/8 \approx 1.64$ in case the metric space is the real-line. They also present a 7/4-competitive algorithm, which in terms of [2] is a so-called *zealous algorithm*; i.e., an algorithm that keeps the server active as long as there are unserved requests. In [2] it is shown that zealous algorithms cannot have competitive ratios lower than 7/4.

The question remained open if the lower bound of $(9+\sqrt{17})/8$ was too low or if there exists an on-line non-zealous algorithm with competitive ratio better than 7/4. In this paper we answer this question in favor of the latter possibility, by providing a *best possible* algorithm with competitive ratio $(9+\sqrt{17})/8$. The algorithm, presented in Section 2, is based on a minute study of the lower bound in [1] for deciding when and how long to wait. In Section 3 we prove $(9+\sqrt{17})/8$-competitiveness.

In [2] a notion of fairness is introduced in competitive analysis of on-line routing problems: at any point in time the off-line optimal tour, taken by the before mentioned adversary, up to that time is not allowed to move outside the convex hull of the points released so far. The authors give a lower bound on the competitive ratio of any deterministic algorithm for the case the metric space is a half-line, together with an algorithm with matching competitive ratio. This result shows that fair adversaries are weaker and therefore competitiveness under fairness can be better than if fairness is not imposed. They also showed that the introduction of fairness into the model does not have any effect on the competitiveness for the problem on general metric spaces. Also here the real line as a metric space was left as a blank spot on the map.

We colour this spot completely. In Section 4 we derive a lower bound of $(5+\sqrt{57})/8 \approx 1.57$ on the competitive ratio of any deterministic algorithm for the OLTSP under the fairness restriction on the real line. We then design an algorithm for this problem with a matching competitive ratio.

The results in this paper complete the picture of research on the competitiveness of deterministic algorithms for the on-line travelling salesman problem with imposed finish in the departure point. Research on randomized algorithms for this problem has hardly been pursued.

2 The algorithm

We present a best possible algorithm for the OLTSP on the real line with a competitive ratio of $(9+\sqrt{17})/8$. The algorithm is called WD (for Waiting Deliberately). WD is described completely by its behaviour at the moment a new request is given. The behaviour is determined only by the two unserved extreme requests, one on the positive halfline (*the rightmost extreme*) and one on the negative halfline (*the leftmost extreme*). All other unserved requests will be served while completing the tour and are therefore ignored. If a new request does not define a new extreme it is accordingly also ignored. We take the point 0 as the origin. If a new extreme is on the same side as the WD-server but closer to 0, then this new extreme will be served while completing the tour and is ignored as well. From now on we use the term extreme shortly for a leftmost or rightmost extreme request that is unserved and not ignored. Notice that any request can become extreme only at the moment it is presented.

First we introduce some notation. At any time t,

$$\begin{aligned} p_t &= \text{the position of the WD server,} \\ x_t &= \text{the leftmost extreme, having abcissa } -x_t, \\ y_t &= \text{the rightmost extreme, having abcissa } y_t, \\ X_t &= \text{the leftmost request ever presented until time t,} \\ Y_t &= \text{the rightmost request ever presented until time t.} \end{aligned}$$

Since we use the Euclidean metric on the real line, $d(v, 0) = |v|$ for any point v. We also define

$$\begin{aligned} r_v &= \text{the last time request } v \text{ is given,} \\ \hat{v} &= \max\{d(v,0), r_v\}, \\ \rho &= (9 + \sqrt{17})/8. \end{aligned}$$

If at time t there is no leftmost extreme (on the negative halfline), we set $x_t = \hat{x}_t = 0$, and similarly we set $y_t = \hat{y}_t = 0$ if there is no rightmost extreme (on the positive halfline). We denote the completion time of WD by Z^{WD} and that of the optimal solution by Z^*.

For notational convenience we will use x_t here, not only for the distance $d(-x_t, 0)$, but also to indicate the request, which actually is at point $-x_t$.

Before giving the precise description of WD, we explain the underlying ideas. Suppose that at time t, when a new request arrives, the position of the WD-server is to the left of the origin, i.e., $p_t \leq 0$ (the case in which $p_t \geq 0$ is symmetrical). WD has to decide which extreme to serve first. Clearly, serving x_t first gives the shortest possible tour at time t. Suppose for the time being that WD decides to serve x_t first. Let t_0 be the moment WD returns in the origin after having served x_t. There is a risk that at time t_0 a new leftmost extreme request arrives. First, this makes serving x_t before t_0 useless, and, second, WD may be too far away from the new request at t_0. If in the optimal off-line solution x_t is served before y_t, then serving x_t first should be a safe option.

Suppose that in the optimal off-line solution y_t is served before x_t. In this case $Z^* \geq \hat{y}_t + y_t + 2x_t$. We distinguish two situations. In the first one $t_0 \leq \rho(\hat{y}_t + y_t + 2x_t) - 2x_t - 2y_t$. Here t_0 is so low that serving x_t first should be a safe option, even if a new leftmost extreme would be presented at t_0.

The second situation occurs if $t_0 > \rho(\hat{y}_t + y_t + 2x_t) - 2x_t - 2y_t$. At t_0 a leftmost extreme at point $-t_0 + \hat{y}_t + y_t$ is given, which the off-line server may reach at t_0 after having served y_t, making $Z^* = t_0 + t_0 - \hat{y}_t - y_t = 2t_0 - \hat{y}_t - y_t$. WD still has to serve both extremes at time t_0, whence $Z^{WD} = t_0 + 2(t_0 - \hat{y}_t - y_t) + 2y_t = 3t_0 - 2\hat{y}_t$. Therefore, for WD to be ρ-competitive,

$$\frac{3t_0 - 2\hat{y}_t}{2t_0 - \hat{y}_t - y} \leq \rho \Leftrightarrow t_0 \geq \frac{\rho y_t - (2-\rho)\hat{y}_t}{2\rho - 3}. \quad (1)$$

This inequality shows the necessity to wait in some cases.

Now suppose WD waits and returns to the origin at $t_0 = \frac{\rho y_t - (2-\rho)\hat{y}_t}{2\rho-3}$. If no more requests are given, $Z^{WD} = \frac{\rho y_t-(2-\rho)\hat{y}_t}{2\rho-3} + 2y_t$. Since $Z^* \geq \hat{y}_t + y_t + 2x_t$, WD is ρ-competitive if

$$\begin{aligned} \frac{\rho y_t - (2-\rho)\hat{y}_t}{2\rho - 3} + 2y_t \leq \rho(\hat{y}_t + y_t + 2x_t) \ \Leftrightarrow \\ (8\rho - 2\rho^2 - 6)y_t \leq (2\rho^2 - 4\rho + 2)\hat{y}_t + (4\rho^2 - 6\rho)x_t \ \Leftrightarrow \\ (6\rho - 2\rho^2 - 4)y_t \leq (2\rho^2 - 3\rho)x_t. \end{aligned} \quad (2)$$

Since, by the choice of ρ, $6\rho - 2\rho^2 - 4 = 2\rho^2 - 3\rho$, inequality (2) holds if $x_t \geq y_t$. However, (2) does not hold if $x_t < y_t$. We notice that inequality (1) is based on the situation in which a new extreme would be presented at t_0. Inequality (2) is based on the situation in which at time t the last request was presented. Therefore, (2) must be satisfied if WD starts the shortest possible tour at time t first visiting x_t.

Basically, WD tries to satisfy both (1) and (2). Therefore, in view of (2), WD tries to follow the tour that visits the greater extreme first, starting in the origin at a moment such that it remains ρ-competitive and (1) and (2) are satisfied.

Inequality (1) shows that t_0 and therefore the specific moment to leave the origin depends on y_t and $\hat{y}_t$ only. However, to make the analysis of WD easier, we choose the specific moment to leave the origin to depend on x_t, $\hat{x}_t$, y_t and $\hat{y}_t$.

We come to the point now to be more precise about WD. We define

$$\begin{aligned} L_t^- &= \rho\hat{x}_t + (\rho - 2)x_t + (2\rho - 2)y_t, \\ L_t^+ &= \rho\hat{y}_t + (\rho - 2)y_t + (2\rho - 2)x_t. \end{aligned}$$

We notice that

$$x_t \geq y_t \Rightarrow \min\{L_t^-, L_t^+\} + 2x_t \geq (4\rho - 2)y_t = \frac{(2\rho-2)y_t}{2\rho - 3} \geq \frac{\rho y_t - (2-\rho)\hat{y}_t}{2\rho - 3}. \quad (3)$$

Thus, inequality (1) is satisfied if WD first serves x_t on a tour that leaves the origin not before time $\min\{L_t^-, L_t^+\}$. (The case $y_t \geq x_t$ is symmetrical.)

We distinguish two basic cases that may occur at time t: $L_t^- \leq L_t^+$ and $L_t^+ \leq L_t^-$ (breaking ties arbitrarily). Each basic case has seven different subcases making a total of fourteen cases. Given a basic case, the seven subcases form an ordered list. WD acts according to the first case in the list that fits its situation. We give the description of WD by listing the cases and the corresponding actions in Figure 1.

The tour that leaves the origin at time $\min\{L_t^-, L_t^+\}$ in the direction of the greater extreme, serves the extreme requests uninterruptedly at maximum speed, and returns to the origin is called the *preferred tour.* The situation in which WD can recover the preferred tour corresponds to cases **I1, I5, II1**, and **II5**.

In cases in which a preferred tour cannot be recovered WD will start an *enforced tour* starting at t in p_t, visiting the extremes uninterruptedly at maximum speed, and returning to the origin. If at time t WD is on the same side as the greater extreme, then WD starts an enforced tour first serving this greater extreme. This tour is the shortest possible tour and therefore inequality (2) should be satisfied. Inequality (1) is satisfied because WD cannot recover the preferred tour. This situation corresponds to cases **I2, I7, II2**, and **II7**.

If at time t WD is on the same side as the smaller extreme, then WD starts an enforced tour first serving this smaller extreme if certain requirements are met. This situation corresponds to cases **I3, I6, II3**, and **II6**. If these requirements are not met, then WD will cross the origin to serve the greater extreme first. This situation corresponds to cases **I4**, and **II4**.

3 WD is best possible

We state two preliminary lemmas.

Lemma 3.1 *If $L_t^- \leq L_t^+$, $y_t > 0$, and x_t is released at t then case* **I1** *or* **I5** *occurs. If $L_t^+ \leq L_t^-$, $|x_t| > 0$, and y_t is released at t then case* **II1** *or* **II5** *occurs.*

PROOF. We give the proof of the first statement only (the proof of the second statement is symmetric). If $x_t \geq y_t$, then $t + d(p_t, x_t) \leq \rho\hat{x}_t + (\rho - 2)x_t + (2\rho - 2)y_t + x_t$, since $d(p_t, x_t) \leq x_t + y_t$ and $t \leq \hat{x}_t$.

If $x_t < y_t$, then $t + d(p_t, y_t) \leq \rho\hat{x}_t + (\rho - 2)x_t + (2\rho - 2)y_t + y_t$, since $d(p_t, y_t) \leq x_t + y_t \leq 2y_t$ and using again $t \leq \hat{x}_t$. □

Lemma 3.2 *If $L_t^- \leq L_t^+$, then $Z^* \geq \hat{x}_t + x_t + 2y_t$. If $L_t^+ \leq L_t^-$, then $Z^* \geq \hat{y}_t + y_t + 2x_t$.*

PROOF. The lemma follows directly from the fact that a request can be served neither before its release time nor before its distance to the origin, together with the definitions of L_t^- and L_t^+. □

Theorem 3.1 *WD is ρ-competitive, with $\rho = (9 + \sqrt{17})/8$.*

PROOF. We prove the theorem by showing that, if WD is ρ-competitive before a new request is given at time t (which is true for $t = 0$), then WD is ρ-competitive after this new request. This is trivially true if the new request is an ignored request. Thus, we only have to be concerned if the new request is either a leftmost or rightmost unserved extreme. Without loss of generality we assume that the new request at time t is rigthmost extreme y_t. Trivial lower bounds on the optimal solution value are then $Z^* \geq t + y_t$ and $Z^* \geq 2X_t + 2Y_t$.

Clearly, WD is ρ-competitive if it can recover a preferred tour at time t (cases **I1, I5, II1** or **II5**). We disregard this situation from now on. If $x_t = 0$, then $Z^{WD} = t + d(p_t, y_t) + y_t \leq 3/2Z^*$, since $Z^* \geq t + y_t$ and $Z^*/2 \geq X_t + Y_t \geq$

Figure 1: WD

Case I $L_t^- \leq L_t^+$

I1 $x_t \geq y_t$ and $t + d(p_t, x_t) \leq L_t^- + x_t$. Go in the direction of the origin (or wait in the origin) until being on the preferred tour. At that moment start to follow the preferred tour first serving x_t.

I2 $x_t \geq y_t$ and $p_t \leq 0$. Follow the enforced tour first serving x_t.

I3 $x_t \geq y_t$, $t + 2y_t - p_t \geq (4\rho - 2)x_t$ and $p_t > 0$. Follow the enforced tour first serving y_t.

I4 $x_t \geq y_t$ and $p_t > 0$. Follow the enforced tour first serving x_t.

I5 $y_t > x_t$ and $t + d(p_t, y_t) \leq L_t^- + y_t$. Go in the direction of the origin (or wait in the origin) until being on the preferred tour. At that moment start to follow the preferred tour first serving y_t.

I6 $y_t > x_t$, and $p_t < 0$. Follow the enforced tour first serving x_t.

I7 $y_t > x_t$, and $p_t \geq 0$. Follow the enforced tour first serving y_t.

Case II $L_t^+ \leq L_t^-$ is symmetrical.

II1 $y_t \geq x_t$ and $t + d(p_t, y_t) \leq L_t^+ + y_t$. Go in the direction of the origin (or wait in the origin) until being on the preferred tour. At that moment start to follow the preferred tour first serving y_t.

II2 $y_t \geq x_t$ and $p_t \geq 0$. Follow the enforced tour first serving y_t.

II3 $y_t \geq x_t$, $t + 2x_t - |p_t| \geq (4\rho - 2)y_t$ and $p_t < 0$. Follow the enforced tour first serving x_t.

II4 $y_t \geq x_t$ and $p_t < 0$. Follow the enforced tour first serving y_t.

II5 $x_t > y_t$ and $t + d(p_t, x_t) \leq L_t^+ + x_t$. Go in the direction of the origin (or wait in the origin) until being on the preferred tour. At that moment start to follow the preferred tour first serving x_t.

II6 $x_t > y_t$, and $p_t > 0$. Follow the enforced tour first serving y_t.

II7 $x_t > y_t$, and $p_t \leq 0$. Follow the enforced tour first serving x_t.

$d(p_t, y_t)$. If $|x_t| > 0$, Case **II** at t is dismissed through Lemma 3.1. For the remaining cases, all having $L_t^- \leq L_t^+$ (Case I), we have to take the behaviour of the WD-server before t into account. The following claims for the time-interval $[r_{x_t}, t]$ are proved in appendix A.

Claim 3.1 *If there is a $t' \in (r_{x_t}, t]$, at which WD serves a rightmost extreme, then WD is ρ-competitive.*

Claim 3.2 *If there is a $t' \in [r_{x_t}, t]$, at which WD can recover the preferred tour, then WD is ρ-competitive.*

From now on we denote r_{x_t} by τ. Since x_t is still the leftmost unserved request at time t, no new leftmost request is given during the time interval $[\tau, t]$, $x_\tau = x_t$ and $p_\tau > -x_t$. At τ there may be a rightmost unserved extreme y_τ. We may assume that none of the premises of Lemma 3.1 and Claims 3.1 and 3.2 occurs during $[\tau, t]$, since this would make WD directly ρ-competitive. In particular, case **II** occurs at time τ, case **II** does not occur during $(\tau, t]$, cases **I1** and **I5** do not occur during $[\tau, t]$, and WD starts an enforced tour at time τ. We distinguish four main situations.

- *WD starts an enforced tour in the direction of x_t, not turning around before reaching x_t.*

  Thus, $Z^{WD} \leq \tau + d(p_\tau, x_t) + x_t + 2y_t$ and, using Lemma 3.2,

  $$\frac{Z^{WD}}{Z^*} \leq \frac{\tau + x_t + 2y_t}{Z^*} + \frac{d(p_\tau, x_t)}{Z^*} \leq 1 + \frac{X_t + Y_t}{Z^*} \leq \frac{3}{2}.$$

- *WD starts an enforced tour in the direction of x_t, turning around before reaching x_t.*

  To make WD turning around a new rightmost request y_1 must be given at some time $t' \in [\tau, t]$. WD starts to follow an enforced tour at time t' in the direction of y_1. Therefore, case **I3** or **I7** occurs at time t'. In both cases $p_{t'} \geq 0$ by definition.

  The first possibility is that the WD-server does not turn around before he reaches a rightmost extreme. Since we excluded that WD reaches a rightmost extreme before time t, y_t must be given before this rightmost extreme is reached. We note that $p_\tau > p_{t'} \geq 0$, so $Z^{WD} \leq \tau + 2y_t + 2x_t + p_\tau \leq 3/2 Z^*$ because $\tau + x_t + 2y_t \leq Z^*$ and $x_t + p_\tau \leq X_t + Y_t \leq Z^*/2$.

  The second possibility is that WD does turn around before reaching a rightmost extreme caused by the release of a new rightmost request y_2 at some time $t'' \in [t', t]$ at which WD sets out on an enforced tour in the direction of x_t. This excludes immediately cases **I3** and **I7** at t'', whereas $p_{t''} > p_{t'} \geq 0$ excludes cases **I2** and **I6**. If at time t' the situation was **I7**, then $y_2 > y_1 > x_t$ excludes case **I4** at t''. If at time t' the situation was **I3**, then $t'' + 2y_2 - p_{t''} > t' + 2y_1 - p_{t'}$ excludes case **I4** at t''. Thus, this possibility is excluded since we already assumed that cases **I1**,**I5** and **II** do not occur at presenting a new rightmost request in the interval $[\tau, t]$.

- *WD starts an enforced tour in the direction of y_τ, not turning around before reaching a rightmost extreme.*
  Since we excluded the premise of Claim 3.1, y_t is given before WD served a rightmost extreme and y_t must be this rightmost extreme. In all cases, since WD remains on enforced tours, $Z^{WD} \le \tau + 2y_t + |p_\tau| + 2x_t \le Z^* + x_t + |p_\tau|$, applying Lemma 3.2. If $|p_\tau| \le y_t$ then $x_t + |p_\tau| \le \frac{1}{2}Z^*$ and hence $Z^{WD} \le \frac{3}{2}Z^*$. This is directly true if $p_\tau \ge 0$. If $p_\tau < 0$, the only possible case at time τ in which WD starts an enforced tour in the direction of y_τ is **II4**, which by definition has $y_\tau \ge x_t \ge |p_\tau|$.

- *WD starts an enforced tour in the direction of y_τ, turning around before reaching a rightmost extreme.*
  Since we excluded the premise of Claim 3.1, a new rightmost extreme y_1 must be given at some time $t' \in [\tau, t]$ before WD reaches y_τ. If $y_1 > x_t$, then we have

  $$\begin{aligned} L^-_{t'} + y_1 &\ge \hat{x}_t + (2\rho - 3)(x_t + y_1) + 2y_1 \\ &> \tau + (2\rho - 3)(x_t + y_1) + x_t + y_1 \\ &> \tau + d(p_\tau, y_1) = t' + d(p_{t'}, y_1). \end{aligned}$$

  Thus, WD can recover the preferred tour and, using Claim 3.2, is ρ-competitive.

  If $y_1 \le x_t$, then $y_\tau < y_1 \le x_t$. This implies that case **II6** is the only possible case at τ, so $p_\tau > 0$ by definition. At t' WD can turn around and reach the origin before time $\tau + 2y_\tau - p_\tau$. At time τ case **II5** did not occur , so by definition $\tau + p_\tau > (2\rho - 2)(y_\tau + x_t)$. Using $4\rho^2 - 5\rho - 2 \ge 0$, we have

  $$\begin{aligned} L^-_{t'} &= \hat{x}_t + (\rho - 1)\hat{x}_t + (\rho - 2)x_t + (2\rho - 2)y_1 \\ &> \hat{x}_t + (\rho - 1)[(2\rho - 2)(y_\tau + x_t) - p_\tau] + (\rho - 2)x_t + (2\rho - 2)y_\tau \\ &= \hat{x}_t + 2y_\tau - (\rho - 1)p_\tau + (2\rho^2 - 2\rho - 2)y_\tau + (2\rho^2 - 3\rho)x_t \\ &> \hat{x}_t + 2y_\tau - (\rho - 1)p_\tau + (4\rho^2 - 5\rho - 2)y_\tau > \tau + 2y_\tau - p_\tau. \end{aligned}$$

  Thus, WD can recover the preferred tour and, using Claim 3.2, is ρ-competitive.

□

4 OLTSP on the line against a fair adversary

We start this section by presenting a lower bound on the competitive ratio of any deterministic algorithm for OLTSP on the real line against a fair adversary. On the real line an adversary is fair if he does not move outside the interval between the leftmost and rightmost request presented in the past.

Theorem 4.1 *Any σ-competitive algorithm for OLTSP on the real line against a fair adversary has $\sigma \geq (5 + \sqrt{57})/8$.*

PROOF. Consider any on-line server OL who is σ-competitive. The adversary starts the sequence with two requests at time 0, one at -1 and one at $+1$. At time 2 OL cannot have served both requests. Without loss of generality, we assume that at time 2 the position of OL is in the origin or on the negative halfline. At time 2 a request at point -1 is presented. Let t_0 denote the time at which OL returns to the origin after having served either the requests in -1 or the request in $+1$. The optimal off-line completion time $Z^* = 4$. Therefore $t_0 \leq 4\sigma - 2$. We distinguish two cases.

- If OL serves the requests at point -1 first, he cannot be back in the origin before time 3, implying $3 \leq t_0 \leq 4\sigma - 2$. At t_0 a request at point -1 is presented. OL cannot finish before time $t_0 + 4$, whereas $Z^* = t_0 + 1$. Therefore, $\sigma \geq (t_0 + 4)/(t_0 + 1)$.
- If OL serves the request at point $+1$ first, he cannot be back in the origin before time 4, implying $4 \leq t_0 \leq 4\sigma - 2$. At t_0 a request at point $+1$ is presented. OL cannot finish before time $t_0 + 4$, whereas $Z^* = t_0 + 1$. Also in this case, $\sigma \geq (t_0 + 4)/(t_0 + 1)$.

The ratio $(t_0 + 4)/(t_0 + 1)$ is monotonically decreasing in t_0, for $t_0 > 0$. Thus, $\sigma \geq (4\sigma + 2)/(4\sigma - 1)$, implying $\sigma \geq (5 + \sqrt{57})/8$.

□

We call the best possible algorithm for the OLTSP against a fair adversary WF (for Waiting under Fairness). WF is the same as WD in Section 2 only replacing ρ by $\sigma = (5 + \sqrt{57})/8$.

The proof of σ-competitiveness of WF is exactly the same as the proof of ρ-competitiveness of WD except for the first part of the proof of Claim 3.1. In this part of the proof we have to use the fact that the adversary does not move outside the interval between the leftmost and rightmost request presented in the past.We prove this part of Claim 3.1 for WF in appendix B.

5 Conclusions

The only open questions about deterministic algorithms for the on-line travelling salesman problem with return to the origin concerned the real line as a metric space [1], [2]. We answered those questions by designing a best possible algorithm that matches the lower bound in [1], and deriving a lower bound for the case the adversary has an imposed fairness restriction, together with designing an algorithm with matching competitive ratio. Improvements on competitive performance can now only be hoped for through randomized algorithms. Apart from some trivial lower bounds in [1], this is a virgin research topic.

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Appendix A

In this appendix we shall prove Claims 3.1 and 3.2. We need a part of Claim 3.1 to prove Claim 3.2 and we need Claim 3.2 to prove the remaining part of Claim 3.1.

Claim 3.1a: Suppose there is a $t' \in (r_{x_t}, t]$, at which WD serves a rightmost extreme and that the last case that occurred before time t' was not case **II6**. Then WD is ρ-competitive.

PROOF. We focus on the earliest possible moment WD can return to the origin after having served a rightmost extreme at time t', which we denote by $t_0^{y'}$. We abuse notation and denote the rightmost extreme served at t' by $y_{t'}$.

The last case that occurred before t' must have been case **I3, I5, I7, II1, II2, II4** or **II6**, since WD moved to the right. The definition of case **I3** implies that $t_0^{y'} \geq (4\rho-2)x_t$. We have excluded occurrence of case **II6**. In all remaining cases $y_{t'} \geq x_t$, implying (see (3), page 5), $\min\{L_{t'}^-, L_{t'}^+\} + 2y_{t'} \geq (4\rho-2)x_t$, and therefore, as in case **I3**, $t_0^{y'} \geq (4\rho-2)x_t$. A trivial upper bound on $t_0^{y'}$ is $t+|p_t|$. Thus, we always have

$$(4\rho-2)x_t \leq t_0^{y'} \leq t+|p_t|. \tag{4}$$

Now we consider the situation at time t. Case **I4** at t is excluded since $t+2y_t-|p_t| > t+|p_t| \geq t_0^{y'} \geq (4\rho-2)x_t$, implying that case **I3** would have occurred. Thus, WD does not cross the origin after time t before having served any of x_t or y_t. Therefore $Z^{WD} = t+2x_t+2y_t-|p_t|$ (unless the preferred tour can be recovered, in which case WD is ρ-competitive). Since $t+y_t \leq Z^*$, it suffices to prove that $2x_t+y_t-|p_t| \leq (\rho-1)Z^*$.

Suppose first that $y_t \leq \frac{(4-2\rho)x_t}{(2\rho-3)} + |p_t|$. Using $(4\rho-2) = (2\rho-2)/(2\rho-3)$ and (4), we have

$$\begin{aligned}
2x_t+y_t-|p_t| &\leq (2-\rho)(\frac{(4-2\rho)}{(2\rho-3)}x_t + |p_t| - y_t) + 2x_t + y_t - |p_t| \\
&= \frac{(2-\rho)(4-2\rho)}{(2\rho-3)}x_t + (2-\rho)|p_t| - (2-\rho)y_t + 2x_t + y_t - |p_t| \\
&= (\frac{(2\rho^2-4\rho+2)}{(2\rho-3)} - 2)x_t + (2-\rho)|p_t| + (\rho-2)y_t + 2x_t + y_t - |p_t| \\
&= (\rho-1)(\frac{(2\rho-2)}{(2\rho-3)}x_t - |p_t| + y_t) \\
&\leq (\rho-1)(t+y_t) \\
&\leq (\rho-1)Z^*.
\end{aligned}$$

Now suppose $y_t > \frac{(4-2\rho)x_t}{(2\rho-3)} + |p_t|$. Then

$$\begin{aligned}
2x_t+y_t-|p_t| &= (4-2\rho)x_t - (2\rho-3)y_t - |p_t| + (\rho-1)(2x_t+2y_t) \\
&< (4-2\rho)x_t - (4-2\rho)x_t - (2\rho-3)|p_t| - |p_t| + (\rho-1)(2x_t+2y_t) \\
&\leq (\rho-1)(2x_t+2y_t) \\
&\leq (\rho-1)Z^*.
\end{aligned}$$

□

Claim 3.2: If there is a $t' \in [r_{x_t}, t]$, at which WD can recover the preferred tour, then WD is ρ-competitive.

PROOF. We denote the *last* moment WD can recover the preferred tour by t'. If there is a rightmost extreme at t' we denote it by y_0. Obviously, WD is ρ-competitive if no new rightmost extremes are given after t', or if WD served a rightmost extreme before t. This is true by Claim 3.1a, since case **II6** occurs only at time τ (Lemma 3.1).

Thus, suppose at $t'' > t'$ a new rightmost extreme y_1 is given before WD reaches an extreme, which causes the WD-server to follow an enforced tour.

This excludes case **I1**, **I5** and **II** by Lemma 3.1. Clearly, $y_1 > y_0$.

Notice that $L^-_{t''} = L^-_{t'} + (2\rho - 2)(y_1 - y_0)$. Thus, if $L^+_{t'} \leq L^-_{t'}$, then $L^-_{t''} \geq (2\rho - 2)(y_1 - y_0) + L^+_{t'}$.

If at t' the WD-server would take action to serve y_0 first, then $t' + d(p_{t'}, y_0) \leq \min\{L^-_{t'}, L^+_{t'}\} + y_0$ and $y_0 \geq x_t$ by definition. Clearly, $y_1 > y_0 \geq x_t$ and $t'' + d(p_{t''}, y_1) = t' + d(p_{t'}, y_0) + (y_1 - y_0) < \min\{L^-_{t'}, L^+_{t'}\} + y_0 + (2\rho - 2)(y_1 - y_0) < L^-_{t''} + y_1$. Thus, WD can recover the preferred tour, which contradicts the assumption that t' is the last time before t at which a preferred tour can be recovered.

If at t' the WD-server would take action to serve x_t first, then $x_t \geq y_0$ by definition. If $y_1 \leq x_t$ the WD-server can recover the preferred tour because $t'' + d(p_{t''}, x_t) = t' + d(p_{t'}, x_t) \leq \min\{L^-_{t'}, L^+_{t'}\} + x_t < L^-_{t''} + x_t$. Again a contradiction.

If $y_1 > x_t$ we have to look at $p_{t''}$. In case $p_{t''} \geq 0$, then $t'' + p_{t''} = t' + p_{t'} \leq \min\{L^-_{t'}, L^+_{t'}\} < L^-_{t''}$. WD can recover the preferred tour, so a contradiction again.

If $p_{t''} < 0$ case **I7** is excluded by definition, whereas $y_1 > x_t$ excludes cases **I2**, **I3** and **I4**. Therefore, at t'' the only possible case is case **I6**. In this case WD starts an enforced tour first serving x_t and does not turn around unless the preferred tour can be recovered, which is excluded. At t' the WD-server was on the preferred tour or recovering the preferred tour first serving x_t, therefore $Z^{WD} \leq \min\{L^-_{t'}, L^+_{t'}\} + 2x_t + 2y_t$. Clearly WD is ρ-competitive. □

Claim 3.1b: If there is a $t' \in (r_{x_t}, t]$, at which WD serves a rightmost extreme while the last case was **II6**, then WD is ρ-competitive.

PROOF. Case **II6** can only occur at time τ. We therefore may exclude the premises of Lemma 3.1 and Claim 3.1a or 3.2 in the time interval $[\tau, t]$, since this would make WD directly ρ-competitive. We will argue that then $Z^{WD} \leq \tau + 2y_\tau + 2x_t + 2y_t - p_\tau$. This is immediately clear if between τ and t no new requests are given or only requests that make WD starting an enforced tour first serving x_t.

Suppose WD starts following an enforced tour in the direction of a new rightmost request y_1 at some time $t^1 \in [t', t]$, implying that case **I3** or **I7** must occur at t^1. In both cases $p_{t^1} \geq 0$. We assume that t^1 is the first time after t' at which WD goes in the direction of a rightmost extreme, so WD cannot have been to the left of the origin between t' and t^1.

The first possibility is that the WD-server does not turn around before reaching a rightmost extreme. Since we excluded this to occur before time t, y_t must be given before this rightmost extreme is reached. Therefore $Z^{WD} \leq \tau + 2y_\tau + 2x_t + 2y_t - p_\tau$.

The other possibility is that WD turns around before reaching a rightmost extreme caused by the release of a new rightmost request y_2 at some time $t^2 \in [t^1, t]$. This excludes immediately cases **I3** and **I7** at t^2, while $p_{t^2} > p_{t^1} \geq 0$ excludes cases **I2** and **I6**, leaving **I4** as the only possible one at t^2 (we have already excluded all other cases from the beginning). If at time t' the situation was **I7**, then $y_2 > y_1 > x_t$ excludes case **I4** at t_2. If at time t' the situation was **I3**, then $t_2 + 2y_2 - p_{t^2} > t' + 2y_1 - p_{t'}$ excludes case **I4** at t^2. Thus, this possibility is excluded.

We still have to prove that $Z^{WD} \leq \tau + 2y_\tau + 2x_t + 2y_t - p_\tau \leq \rho Z^*$. Using $0 \leq 4\rho^2 - 5\rho - 2$ and $y_\tau < x_t$, we derive the crucial inequality:

$$\begin{aligned} 2y_\tau + x_t - p_\tau &\leq 2y_\tau + x_t - p_\tau + (2-\rho)p_\tau + (4\rho^2 - 5\rho - 2)y_\tau \\ &< (2\rho^2 - 3\rho + 1)x_t + (2\rho^2 - 2\rho)y_\tau + (1-\rho)p_\tau \\ &= (\rho - 1)((2\rho - 2)(x_t + y_\tau) - p_\tau + x_t + 2y_\tau) \\ &< (\rho - 1)(\hat{x}_t + x_t + 2y_\tau). \end{aligned} \tag{5}$$

Suppose first that $y_\tau \leq y_t$. Applying this bound in (5) and using the fact that $\tau + x_t + 2y_t \leq Z^*$ yields $Z^{WD} \leq \rho Z^*$.

Now suppose that $y_t < y_\tau$. If in the optimal solution y_τ is served after x_t, then $Z^* \geq \hat{x}_t + x_t + 2y_\tau$. We notice that (5) also holds if y_τ is substituted by y_t, i.e., $2y_t + x_t - p_\tau \leq (\rho-1)(\hat{x}_t + x_t + 2y_\tau)$. Therefore, $2y_t + x_t - p_\tau \leq (\rho - 1)Z^*$. These observations together yield $Z^{WD} \leq \rho Z^*$.

If $y_t < y_\tau$ and in the optimal solution y_τ is served before x_t then $Z^* \geq t + y_t + 2x_t$, in case y_t is also served before x_t, or, if not,

$$Z^* \geq \hat{y}_\tau + y_\tau + 2x_t + 2y_t. \tag{6}$$

In the former case $Z^{WD} \leq \frac{3}{2}Z^*$, following easily from the observation that $\tau + y_\tau - p_\tau \leq t$. In the latter case we have to take the behavior of WD before τ into account, in particular on the time interval $[r_{y_\tau}, \tau]$. We denote r_{y_τ} by t^3 and the leftmost extreme at time t^3 by x_3. We note that in $(t^3, \tau]$ only new leftmost extremes can be given.

If during $[t^3, \tau]$ WD never moves to the left, then at t^3 WD either starts moving to the right until y_τ is reached or WD waits some time in the origin to recover the preferred tour. Therefore, $Z^{WD} \leq t^3 + d(p_{t^3}, y_\tau) + y_\tau + 2x_t + 2y_t = t^3 + y_\tau + 2x_t + 2y_t + d(p_{t^3}, y_\tau) \leq \frac{3}{2}Z^*$, since $t^3 + y_\tau + 2x_t + 2y_t \leq Z^*$ and $d(p_{t^3}, y_\tau) \leq X_t + Y_t \leq Z^*/2$, or $Z^{WD} \leq \min\{L^-_\tau, L^+_\tau\} + 2y_\tau + 2x_t + 2y_t \leq \rho Z^*$.

If WD does move to the left during $[t^3, \tau]$ we define time $t^l \in [t^3, \tau]$ as the last moment before time τ at which this happens.

First suppose that during the interval $[t^l, \tau]$ WD can recover the preferred tour. Let $t^p \in [t^l, \tau]$ be the last moment at which this is the case. If at t^p WD follows (or recovers) the preferred tour first serving y_τ, then he is still doing so until τ and $Z^{WD} \leq \min\{L^-_\tau, L^+_\tau\} + 2y_\tau + 2x_t + 2y_t \leq \rho Z^*$. If at t^p WD recovers the preferred tour first serving x_{t^p}, then $p_{t^p} \leq 0$ and $x_{t^p} \geq y_\tau$. This excludes $p_\tau > 0$ at time τ, conflicting the premises of the claim we are proving.

Thus, from now on we assume that during $[t^l, \tau]$ the preferred tour cannot be recovered. We consider first the case that $p_{t^l} \geq 0$. Since at t^l WD followed an enforced tour first serving the leftmost extreme, the last case that occurred before t^l must have been case **I4**. In case **I4** the premise is that $L^- \leq L^+$ and therefore Lemma 3.1 implies that the last request before t^l must have been y_τ. Therefore, $Z^{WD} \leq t^3 + p_{t^3} + 2y_\tau + 2x_t + 2y_t \leq 3/2\ Z^*$, since $p_{t^3} < y_\tau < x_t$.

Now consider the case that $p_{t^l} < 0$. We distinguish three situations.

- *WD serves x_{t^l} before time τ, while the last case was not case* **I6**.
  The symmetry of WD allows to use the same analysis used for Claim 3.1a to prove ρ-competitiveness, by substituting y_τ for x_t and x_t for y_t.

- *WD serves x_{t^l} before time τ or WD turns around before reaching x_{t^l} , while case* **I6** *is the last case before t^l.*
  Using the same arguments as before Lemma 3.1 implies that the last request must have been y_τ. By definition of case **I6**, $p_{t^3} < 0$, and hence $Z^{WD} \leq t^3 + 2x_{t^l} + 2y_\tau + 2x_t + 2y_t - |p_{t^3}|$. Since, $Z^* \geq \hat{y}_\tau + y_\tau + 2x_t + 2y_t$, we are left to prove that $2x_{t^l} + y_\tau - |p_{t^3}| \leq (\rho-1)Z^*$. At t^3 case **I5** did not occur, so by definition $t^3 + |p_{t^3}| > (2\rho - 2)(x_{t^l} + y_\tau)$. Using $x_t > y_\tau > x_{t^l}$ and $0 \leq 4\rho^2 - 5\rho - 2$, we have

$$\begin{aligned} 2x_{t^l} + y_\tau - |p_{t^3}| &\leq 2x_{t^l} + y_\tau - (\rho-1)|p_{t^3}| + (4\rho^2 - 5\rho - 2)x_{t^l} \\ &< (2\rho^2 - 4\rho + 2)(x_{t^l} + y_\tau) + (\rho - 1)(y_\tau + 2x_t - |p_{t^3}|) \\ &= (\rho - 1)((2\rho - 2)(x_{t^l} + y_\tau) - |p_{t^3}| + y_\tau + 2x_t) \\ &< (\rho - 1)(t^3 + y_\tau + 2x_t + 2y_t) \leq (\rho - 1)Z^*. \end{aligned}$$

- *WD turns around before reaching a leftmost extreme.*
  At some time $t^4 \in (t^l, \tau]$ a new leftmost request x_4 must be given such that WD starts an enforced tour in the direction of y_τ. This excludes cases **I**, **II1**, **II3**, **II5**, and **II7**. Case **II2** and **II6** at t^4 are excluded, since $p_{t^4} < 0$. If at t^4 case **II4** occurs, then by definition $y_\tau \geq x_4 > x_{t^l}$. This immediately excludes cases **I1**, **I2**, **I3**, **I4**, **II5**, **II6**, and **II7** as the last case before t^4. At t^l WD is going in the direction of the leftmost extreme. This excludes case **I5** **I7**, **II1**, **II2**, and **II4** as the last case before t^4. We already proved ρ-competitiveness if the last case before t^4 is case **I6**. If the last case before t^4 is case **II3**, then at t^4 case **II3** occurs instead of case **II4** since $t^4 + 2x_4 - |p_{t^4}| > t^l + 2x_{t^l} - |p_{t^l}|$.

□

Appendix B

In this appendix we shall prove the first part of Claim 3.1 for WF. We use this proof in the same way we use Claim 3.1a for WD.

Claim 3.1a: Suppose there is a $t' \in (r_{x_t}, t]$, at which WF serves a rightmost extreme and that the last case that occurred before time t' was not case **II6**. Then WF is σ-competitive.

PROOF. We denote the time before t at which WF serves a rightmost extreme by t'. We abuse notation and denote this rightmost extreme by $y_{t'}$. We denote $t_0^{y'}$ as the earliest possible moment WF can return to the origin after having served $y_{t'}$.

If at t' WF serves a rightmost extreme the last case that occurred before t' must have been case **I3, I5, I7, II1, II2, II4** or **II6**. If case **I3** is the last case before t', then by definition

$$t_0^{y'} \geq (4\sigma - 2)x_t. \tag{7}$$

We have excluded occurrence of case **II6**. Therefore, if case **I3** is not the last case before t', then $y_{t'} \geq x_t$ and

$$t_0^{y'} \geq \min\{L_{t'}^-, L_{t'}^+\} + 2y_{t'}. \tag{8}$$

We note that, if $y_{t'} \geq x_t$, then $\min\{L_{t'}^-, L_{t'}^+\} + 2y_{t'} \geq (2\sigma - 2)(x_t + y_{t'}) + 2y_{t'} \geq (4\sigma - 2)x_t$. Thus, we always have $t_0^{y'} \geq (4\sigma - 2)x_t$. This excludes occurrence of case **I4** after t'.

We now assume that t' is the *last* time before t at which WF serves a rightmost extreme while the last case that occurred before time t' was not case **II6**.

If WF can recover the preferred tour during the time interval $[t', t]$, then WF is σ-competitive. This follows from the proof of Claim 3.2, since in $[t', t]$ no extreme is served. We therefore assume WF cannot recover the preferred tour after t' for the remainder of the proof.

We denote the first request after t' by y^n. If at r_{y^n} WF starts an enforced tour first serving x_t, then case **I2** or case **I6** must occur. In both cases WF does not turn around before t. If at r_{y^n} WF starts an enforced tour first serving y^n, then case **I3** or case **I7** must occur. Also in these cases WF does not turn around before t (cf. proof of Claim 3.2). Thus, we have

$$Z^{WF} = r_{y^n} + 2x_t + 2y_t - |p_{t_{y^n}}| = t + 2x_t + 2y_t - |p_t|. \tag{9}$$

Since $t_0^{y'} - |p_t| + y_t \leq t + y_t \leq Z^*$, it suffices to prove that $2x_t + y_t \leq (\sigma - 1)(t_0^{y'} + y_t)$, or equivalently, $t_0^{y'} \geq \frac{2x_t + (2-\sigma)y_t}{(\sigma - 1)}$.

If $y_t \leq x_t$ we use (7) or (8) to obtain

$$t_0^{y'} \geq (4\sigma - 2)x_t \geq \frac{2x_t + (2-\sigma)y_t}{(\sigma - 1)}.$$

If $y_t > x_t$ and $y_{t'} \geq y_t$, then case **I3** is excluded as the last case before t', since $y_{t'} > x_t$. We use (8) to obtain

$$t_0^{y'} \geq (2\sigma - 2)(x_t + y_{t'}) + 2y_{t'} \geq \frac{2x_t + (2-\sigma)y_t}{(\sigma - 1)}.$$

If $y_t > x_t$ and $y_{t'} < y_t$, then we have to use the fact that the adversary is fair. We denote $t_0^{Y'}$ as the earliest possible moment WF can return to the origin after having served $Y_{t'}$. We distinguish three situations.

- $y_{t'} = Y_{t'}$.
  At r_{y^n} the position of the optimal server cannot be to the right of $Y_{t'}$. Therefore $Z^* \geq r_{y^n} + 2y_t - y_{t'}$ and inserting that into (9) shows that it suffices to prove that $2x_t + y_{t'} - |p_{t_{y^n}}| \leq (\sigma - 1)Z^*$ or, using $t_0^{y'} - |p_{t_{y^n}}| + y_t \leq t + y_t \leq Z^*$, to prove that $t_0^{y'} \geq \frac{2x_t + (2-\sigma)y_{t'}}{(\sigma-1)}$.
  If case **I3** is the last case before t', then $y_{t'} \leq x_t$. We use (7) to obtain

$$t_0^{y'} \geq (4\sigma - 2)x_t \geq \frac{2x_t + (2-\sigma)y_{t'}}{(\sigma - 1)}.$$

  If case **I3** is not the last case before t', then $y_{t'} \geq x_t$ and, using (8), we obtain

$$t_0^{y'} \geq (2\sigma - 2)(x_t + y_{t'}) + 2y_{t'} \geq \frac{2x_t + (2-\sigma)y_{t'}}{(\sigma - 1)}.$$

- $y_{t'} < Y_{t'}$ and $y_t \leq Y_{t'}$.
  We denote the time at which WF serves $Y_{t'}$ for the last time by t''. If $t'' > \tau$ then the last case before t'' cannot be case **I3**, since this would require $y_t > x_t \geq Y_{t'}$. We excluded case **II6** as the last case before serving a rightmost extreme after τ, so $Y_{t'} \geq x_t$ in all other cases. We use (8) to obtain

$$t_0^{y'} > t_0^{Y'} \geq (2\sigma - 2)(x_t + Y_{t'}) + 2Y_{t'} \geq \frac{2x_t + (2-\sigma)y_t}{(\sigma - 1)}.$$

  If $t'' \leq \tau$, then we focus on the last case before t''. If this is case **II6**, then $x_{t''} > Y_{t'}$ by definition. Since $x_{t''} > Y_{t'} \geq y_t > x_t$, WF must have served $x_{t''}$ before τ, therefore $\tau \geq 2Y_{t'} + x_{t''}$. We have

$$\begin{aligned} 2x_t + y_t &< (\sigma - 1)(4x_t + 2y_t) < (\sigma - 1)(2Y_{t'} + x_{t''} + x_t + 2y_t) \\ &\leq (\sigma - 1)(\tau + x_t + 2y_t) \leq (\sigma - 1)Z^*. \end{aligned}$$

If the last case before t'' is not case **II6**, then $t_0^{Y'} \geq 2\sigma Y_{t'}$ by definition. This can easily be verified by checking all cases. Since x_t and $y_{t'}$ are given after t'' we have

$$\tau \geq 2\sigma Y_{t'} - p_\tau, \tag{10}$$

and

$$r_{y_{t'}} \geq 2\sigma Y_{t'} - p_\tau. \tag{11}$$

We focus on the last case before t'. If case **I3** is not the last case before t', then $y_{t'} \geq x_t$. Using (8), and (10) or (11) we obtain

$$t_0^{y'} \geq (2\sigma - 2)(2\sigma - 1)Y_{t'} + 2\sigma y_{t'} \geq \frac{2x_t + (2-\sigma)y_t}{(\sigma - 1)}.$$

If case **I3** is the last case before t', then we look at the time interval $[\tau, t']$. Suppose WF can recover the preferred tour during the interval $[\tau, t']$. Let $t^p \in [\tau, t']$ be the last time at which this is the case. In the proof of Claim 3.2 we have seen that, if WF can recover the preferred tour first serving x_t, then WF does not turn around unless a preferred tour can be recovered. If WF can recover the preferred tour first serving y_p, then $t_0^{y'} > t_0^{y^p} \geq \min\{L^-_{t^p}, L^+_{t^p}\} + 2y_{t^p}$ and $y_p \geq x_t$. Since y_p is given after t'' we have $t_{y_p} \geq (2\sigma - 1)Y_{t'}$. Using the same analysis as for the situation that case **I3** is not the last case before t' we can show σ-competitiveness for WF. We therefore assume there is no $t^p \in [\tau, t']$.

Thus, suppose first that at τ WF starts an enforced tour first serving x_t. Clearly, WF must turn around before reaching x_t, so a new rightmost extreme y^q at some time $t_q \in [\tau, t']$ must be given such that WF starts an enforced tour first serving y^q, implying case **I3** or **I7** occurs at t_q. If at t_q case **I3** occurs, then p_{t_q} and $p_\tau > 0$. Case **I1** did not occur and $Y_{t'} \geq p_\tau$, so $\tau + Y_{t'} \geq \tau + p_\tau = t_q + p_{t_q} > \sigma\hat{x}_t + (\sigma-2)x_t + (2\sigma-2)y^q$ by definition. If we combine this with (10), we obtain

$$y^q < \frac{(3\sigma - 2\sigma^2)Y_{t'} + (2-\sigma)x_t}{2\sigma - 2}. \tag{12}$$

By definition of case **I3** $\tau + Y_{t'} + 2y^q > t_q + 2y^q - p_{t_q} \geq (4\sigma - 2)x_t$ and using (12) we have

$$\tau \;>\; (4\sigma - 2)x_t - Y_{t'} - 2\frac{(3\sigma - 2\sigma^2)Y_{t'} + (2-\sigma)x_t}{2\sigma - 2}. \tag{13}$$

Suppose first $Y_{t'} \leq 2x_t$. We use $4\sigma^2 - 5\sigma - 2 = 0$ and (13) to obtain

$$\begin{aligned}
2x_t + y_t &= 2x_t + y_t + (4\sigma^2 - 5\sigma - 2)Y_{t'} \\
&\leq 2x_t + y_t + (4\sigma^2 - 2\sigma - 6)x_t + (2\sigma^2 - 4\sigma + 1)Y_{t'} \\
&< (\sigma - 1)[(4\sigma - 2)x_t - Y_{t'} - 2[\frac{(3\sigma - 2\sigma^2)Y_{t'} + (2-\sigma)x_t}{2\sigma - 2}] + x_t + 2y_t] \\
&\leq (\sigma - 1)(\tau + x_t + 2y_t) \leq (\sigma - 1)Z^*.
\end{aligned}$$

If $Y_{t'} > 2x_t$, we use (10) to obtain

$$\begin{aligned}2x_t + y_t &< (4\sigma^2 - 5\sigma + 1)x_t + (\sigma - 1)2y_t \\ &= (\sigma - 1)((2\sigma - 1)2x_t + x_t + 2y_t) \\ &< (\sigma - 1)((2\sigma - 1)Y_{t'} + x_t + 2y_t) \\ &\leq (\sigma - 1)(\tau + x_t + 2y_t) \\ &\leq (\sigma - 1)Z^*.\end{aligned}$$

If at t_q case **I7** occurs, then $y^q > x_t$ and WF cannot recover the preferred tour. Therefore, using (10), we have

$$\begin{aligned}t_0^{y'} &> L_{t^q}^- + 2\max\{y_{t'}, y^q\} > (2\sigma - 2)(2\sigma - 1)Y_{t'} + 2\max\{y_{t'}, y^q\} \\ &> \frac{2x_t + (2-\sigma)y_t}{(\sigma - 1)}. \qquad (14)\end{aligned}$$

If at τ WF starts an enforced tour first serving y_τ, then, by Lemma 3.1, case **II** occurs. If $y_\tau \geq x_t$, then, similar to (14),

$$t_0^{y'} \geq L_\tau^+ + 2\max\{y_{t'}, y_\tau\} \geq \frac{2x_t + (2-\sigma)y_t}{(\sigma - 1)}. \qquad (15)$$

If $y_\tau < x_t$, then $\tau + p_\tau > \sigma\hat{y}_\tau + (\sigma - 2)y_\tau + (2\sigma - 2)x_t$ by definition of case **II**, which we assumed not to occur. We use (11) to obtain

$$\begin{aligned}\tau &> (2\sigma^2 - \sigma)Y_{t'} + (\sigma - 2)y_\tau + (2\sigma - 2)x_t - p_\tau \\ &> (2\sigma^2 - 3)Y_{t'} + (2\sigma - 2)x_t. \qquad (16)\end{aligned}$$

Using (16), $y_t \leq Y_{t'}$, and $2\sigma^3 - 4\sigma - 1 > 0$ we obtain

$$\begin{aligned}2x_t + y_t &< 2x_t + y_t + (2\sigma^3 - 4\sigma - 1)x_t \\ &< 2x_t + y_t + (2\sigma^3 - 2\sigma^2 - \sigma)y_t + (2\sigma^2 - 3\sigma - 1)x_t \\ &\leq (2\sigma^3 - 2\sigma^2 - 3\sigma + 3)Y_{t'} + (2\sigma^2 - 3\sigma + 1)x_t + (2\sigma - 2)y_t \\ &= (\sigma - 1)[(2\sigma^2 - 3)Y_{t'} + (2\sigma - 2)x_t + x_t + 2y_t] \\ &\leq (\sigma - 1)(\tau + x_t + 2y_t) \leq (\sigma - 1)Z^*. \qquad (17)\end{aligned}$$

- $y_{t'} < Y_{t'}$ and $y_t > Y_{t'}$.
  At t_{y^n} the position of the optimal server cannot be to the right of $Y_{t'}$. Therefore $Z^* \geq t_{y^n} + 2y_t - Y_{t'}$ and it suffices to prove that $2x_t + Y_{t'} - |p_{t_{y^n}}| \leq (\sigma - 1)Z^*$ or, using $t_0^{y'} - |p_{t_{y^n}}| + y_t \leq t + y_t \leq Z^*$, to prove that $t_0^{y'} \geq \frac{2x_t + (2-\sigma)Y_{t'}}{(\sigma - 1)}$.

We use the same analysis as for the previous situation ($y_{t'} < Y_{t'}$ and $y_t \leq Y_{t'}$). The only difference is that we now have to prove that $2x_t + Y_{t'} \leq (\sigma - 1)Z^*$ or $t_0^{y'} \geq \frac{2x_t+(2-\sigma)Y_{t'}}{(\sigma-1)}$, instead of having to prove that $2x_t + y_t \leq (\sigma - 1)Z^*$ or $t_0^{y'} \geq \frac{2x_t+(2-\sigma)y_t}{(\sigma-1)}$. This can be done by substituting $Y_{t'}$ for y_t in the two equations mentioned above, in the proof of the previous situation. In the proof of the previous situation we excluded some cases, using $y_t \leq Y_{t'}$. These cases we treat separately.

If $t'' > \tau$, and the last case before t'' is case **I3**, then $x_t \geq Y_{t'}$. We use (7) to obtain

$$t_0^{y'} > t_0^{Y'} \geq (4\sigma - 2)x_t \geq \frac{2x_t + (2-\sigma)Y_{t'}}{(\sigma - 1)}.$$

If $t'' \leq \tau$ and the last case before t'' is case **II6**, we excluded the possibility that $x_t \geq x_{t''}$. If $x_t \geq x_{t''}$ and case **I3** is the last case before t', then we use (7) to obtain

$$t_0^{y'} \geq (4\sigma - 2)x_t \geq \frac{2x_t + (2-\sigma)Y_{t'}}{(\sigma - 1)}.$$

If case **I3** is not the last case before t', then $y_{t'} \geq x_t$, which contradicts $x_t \geq x_{t''} > Y_{t'}$.

In inequality (17) we also used $y_t \leq Y_{t'}$. Using (16) and $2\sigma^3 - 4\sigma - 1 > 0$ we obtain

$$\begin{aligned}
2x_t + Y_{t'} &< 2x_t + Y_{t'} + (2\sigma^3 - 4\sigma - 1)\min\{x_t, Y_{t'}\} \\
&< (2\sigma^3 - 2\sigma^2 - 3\sigma + 3)Y_{t'} + (2\sigma^2 - 3\sigma + 1)x_t + (2\sigma - 2)y_t \\
&= (\sigma - 1)[(2\sigma^2 - 3)Y_{t'} + (2\sigma - 2)x_t + x_t + 2y_t] \\
&\leq (\sigma - 1)(\tau + x_t + 2y_t) \leq (\sigma - 1)Z^*.
\end{aligned}$$

□