

Gevrey spaces related to Lie algebras of operators

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GEVREY SPACES
RELATED TO
LIE ALGEBRAS OF OPERATORS

A.F.M. TER ELST

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prof. dr. ir. J. de Graaf

en

prof. dr. A.C.M. van Rooij

Copromotor:

dr. ir. S.J.L. van Eijndhoven

Aan mijn ouders

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Introduction

In the last century the concepts of real analytic function and of infinitely differentiable function have been introduced. It is well-known that each real analytic function is infinitely differentiable, but the converse is clearly not true. Indeed, an infinitely differentiable function f on \mathbb{R} is real analytic if and only if for all compact subsets K of \mathbb{R} there exist constants $C, t > 0$ such that

$$\forall_{n \in \mathbb{N}_0} \forall_{x \in K} [|f^{(n)}(x)| \leq C t^n n!]. \quad (I.1)$$

So the space of all real analytic functions on \mathbb{R} is much smaller than the space of all infinitely differentiable functions on \mathbb{R} . In 1918, Gevrey introduced a scale of spaces of infinitely differentiable functions, which are not necessarily real analytic, now known as Gevrey spaces. Roughly speaking, Gevrey replaced the factor $n!$ in (I.1) by $n!^\lambda$, where λ is a fixed number greater than 1.

Here we present the definition of the Gevrey space $\mathcal{E}_\lambda(\Omega)$, where Ω is an open subset of \mathbb{R}^d and $\lambda \geq 1$. For any compact subset K of Ω , the space $D_\lambda(K)$ consists of restrictions to K of infinitely differentiable functions φ on Ω for which there exist constants $C, t > 0$ such that

$$\forall_{i \in \mathbb{N}_0^d} \forall_{x \in K} [|(D_1^{i_1} \dots D_d^{i_d} \varphi)(x)| \leq C t^{|i|} |i|!^\lambda]. \quad (I.2)$$

Here D_k denotes the partial derivative with respect to the k -th coordinate and $|i| = i_1 + \dots + i_d$. Now $\mathcal{E}_\lambda(\Omega)$ is the space of all infinitely differentiable functions φ on Ω such that for every compact subset K of Ω , the restriction of φ to K is an element of $D_\lambda(K)$. For $\lambda > 1$, the space $\mathcal{E}_\lambda(\Omega)$ has been introduced by Gevrey, [Gev]. The space $\mathcal{E}_1(\Omega)$ is just the space of all real analytic functions on Ω .

Since $|i|!^\lambda$ on the right hand side of (I.2) can be replaced by $i_1!^\lambda \cdot \dots \cdot i_d!^\lambda$ without changing the definition, we see that Gevrey treats the coordinates x_1, \dots, x_d symmetrically. Roumieu, [Rou], introduced Gevrey type spaces in which the coordinates are not treated symmetrically. In fact, Roumieu considered spaces in which the factor $|i|!^\lambda$ in (I.2) is replaced by $i_1!^{\lambda_1} \cdot \dots \cdot i_d!^{\lambda_d}$, where $\lambda_1, \dots, \lambda_d$ are fixed numbers, which are not necessarily the same.

Another generalization of the spaces $\mathcal{E}_\lambda(\Omega)$ can be obtained by replacing Ω by a real analytic manifold M . Moreover, instead of complex valued functions, one may consider functions which take their values in a Hilbert space H . An interesting situation occurs if

M is taken to be a Lie group G . Let π be a continuous (unitary) representation of G in H . A vector $u \in H$ is called infinitely differentiable (resp. analytic) for π if and only if the function $x \mapsto \pi_x u$ from G into H is infinitely differentiable (resp. analytic). The space of all infinitely differentiable vectors for π is dense in H . (See Gårding, [Går1].) The space of analytic vectors for π is also dense in H . This result has been proved by Nelson, [Nel] on the basis of the following concept of analytic vector relative to a finite set of operators in a Hilbert space. Let A_1, \dots, A_d be (possibly unbounded) operators in a Hilbert space H . A vector $u \in H$ is an analytic vector relative to $\{A_1, \dots, A_d\}$ if there exist $C, t > 0$ such that for all $n \in \mathbb{N}$ and for all $i_1, \dots, i_n \in \{1, \dots, d\}$:

$$u \in D(A_{i_1} \circ \dots \circ A_{i_n})$$

and

$$\|A_{i_1} \circ \dots \circ A_{i_n} u\| \leq C t^n n!. \quad (I.3)$$

Now given an arbitrary basis $\{X_1, \dots, X_d\}$ in the Lie algebra \mathfrak{g} of G , Nelson has proved that the space of analytic vectors for the representation π is equal to the space of analytic vectors relative to the set of operators $\{d\pi(X_1), \dots, d\pi(X_d)\}$, where for each $X \in \mathfrak{g}$ the operator $d\pi(X)$ is the infinitesimal generator of the one parameter group $t \mapsto \pi_{\exp(tX)}$. Moreover, the latter space is dense in H .

Likewise, Gevrey vectors for a representation π of G and Gevrey vectors relative to a finite set $\{A_1, \dots, A_d\}$ of operators can be introduced (cf. Goodman and Wallach, [GW]), by replacing the factor $n!$ by $n!^\lambda$ both in the definition of analytic vectors for π and in Nelson's definition (I.3).

As a special case we mention the Heisenberg group. Let Q be the operator of multiplication by the function $x \mapsto x$ in $L^2(\mathbb{R})$ and let D be the (skew-adjoint) differentiation operator in $L^2(\mathbb{R})$. There exist a representation π of the Heisenberg group G in $L^2(\mathbb{R})$ and a basis X, Y, Z in the Lie algebra of G such that $d\pi(X) = iQ$, $d\pi(Y) = D$ and $d\pi(Z) = iI$. It turns out that the space of all infinitely differentiable vectors for π is equal to Schwartz' space $S(\mathbb{R})$ which consists of all infinitely differentiable functions φ on \mathbb{R} which, together with their derivatives, vanish faster than any polynomial at infinity, i.e.

$$\forall k, l \in \mathbb{N}_0 \left[\sup\{|x^k \varphi^{(l)}(x)| : x \in \mathbb{R}\} < \infty \right].$$

Moreover, for all $\lambda \geq 1$, one can see that the space of Gevrey vectors of order λ relative to $\{d\pi(X), d\pi(Y), d\pi(Z)\}$ is equal to the Gelfand-Shilov space S_λ^λ , which consists of all infinitely differentiable functions φ on \mathbb{R} for which there exist constants $C, t > 0$ such that

$$\forall k, l \in \mathbb{N}_0 \forall x \in \mathbb{R} \left[|x^k \varphi^{(l)}(x)| \leq C t^{k+l} k!^\lambda l!^\lambda \right].$$

Gelfand-Shilov spaces are defined more generally for all $\alpha \geq 0$ and $\beta \geq 0$, so also for $0 \leq \beta < 1$ and $\alpha \neq \beta$. Indeed, for all $\alpha, \beta \geq 0$ the Gelfand-Shilov space S_α^β is defined as the space of all infinitely differentiable functions φ on \mathbb{R} for which there exist constants $C, t > 0$ such that

$$\forall_{k,l \in \mathbb{N}_0} \forall_{x \in \mathbb{R}} \left[|x^k \varphi^{(l)}(x)| \leq C t^{k+l} k!^\alpha l!^\beta \right].$$

In the definition of Gevrey vectors of order λ relative to $\{A_1, \dots, A_d\}$, the operators A_1, \dots, A_d are treated in a symmetric way. Following Roumieu, in this thesis we introduce the space $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ where A_1, \dots, A_d are operators in a Hilbert space H and $\lambda_1, \dots, \lambda_d \geq 0$. A vector $u \in H$ belongs to $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ and is called a Gevrey vector of order $(\lambda_1, \dots, \lambda_d)$ relative to (A_1, \dots, A_d) if and only if there exist constants $C, t > 0$ such that for all $n \in \mathbb{N}_0$ and $i_1, \dots, i_n \in \{1, \dots, d\}$:

$$u \in D(A_{i_1} \circ \dots \circ A_{i_n})$$

and

$$\|A_{i_1} \circ \dots \circ A_{i_n} u\| \leq C t^n n_1!^{\lambda_1} \dots n_d!^{\lambda_d},$$

where $n_k := \text{card}\{l \in \{1, \dots, n\} : i_l = k\}$, with $k \in \{1, \dots, d\}$. In Section 1.1 we introduce a locally convex topology for the space $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$. In Section 1.2 we examine some topological properties: we characterize the bounded subsets, we present a condition which implies that this space is complete and we consider continuous linear maps between spaces of type $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$. In Section 1.3 we investigate the space of analytic vectors and Gevrey vectors for a unitary representation of a Lie group. We introduce the space of weak Gevrey vectors of order λ for π and we characterize its bounded subsets. In Section 1.4 we consider Gevrey vectors relative to operators A_1, \dots, A_d , which commute strongly, commute on a common domain or commute in another sense. In the particular case that the operators A_1, \dots, A_d are strongly commuting self-adjoint operators, it is proved that the space $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ is of type $S_{\pi, C}$, where $S_{\pi, C}$ denotes the space of smoothed vectors relative to a representation π of a locally compact Abelian group G and a subset C of $L^1(G)$. A summary of the important definitions and theorems concerning the space $S_{\pi, C}$ is presented in Appendix B.

Every Gelfand-Shilov space S_α^β is a Gevrey space of our type:

$$S_\alpha^\beta = S_{\alpha, \beta}(Q, D).$$

Clearly $S_{\alpha, \beta}(Q, D)$ is contained in $S_\alpha(Q)$ and also $S_{\alpha, \beta}(Q, D)$ is contained in $S_\beta(D)$. Thus $S_{\alpha, \beta}(Q, D) \subset S_\alpha(Q) \cap S_\beta(D)$. One might hope that the reverse inclusion is also valid. (Cf. Hartogs' theorem.) For all $\alpha, \beta > 0$ such that $\alpha + \beta \geq 1$, Van Eijndhoven, [vE] has proved that indeed

$$S_{\alpha, \beta}(Q, D) = S_\alpha(Q) \cap S_\beta(D)$$

as sets. Thus the following interesting problem comes up: Find conditions on $\lambda_1, \dots, \lambda_d$ and A_1, \dots, A_d such that

$$S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d) = \bigcap_{k=1}^d S_{\lambda_k}(A_k) \quad (\text{I.4})$$

or

$$S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d) = S_{\lambda_1, \dots, \lambda_n}(A_1, \dots, A_n) \cap S_{\lambda_{n+1}, \dots, \lambda_d}(A_{n+1}, \dots, A_d) \quad (\text{I.5})$$

for some $n \in \{2, \dots, d-1\}$. In Chapter 2 several mild conditions are presented such that the equality (I.4) or (I.5) hold. For a summary of these conditions we refer to the introduction of Chapter 2.

In Chapter 3 we present a detailed study of some examples. We consider Gevrey spaces relative to $(d\pi(X_1), \dots, d\pi(X_d))$, where π is a representation of a Lie group G and X_1, \dots, X_d is a basis in the Lie algebra of G . In Section 3.1 we consider a classical infinite dimensional representation of the Heisenberg group $A(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$. The corresponding Gevrey spaces are the Gelfand-Shilov spaces $S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n}$. For several combinations of $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n$ we prove that $S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n}$ is equal to the Gevrey space relative to a single operator. In Section 3.2 we consider an irreducible representation of the $ax + b$ group in $L^2(\mathbb{R})$. We determine all corresponding non-trivial Gevrey spaces and we show that almost each of these non-trivial spaces is equal to the Gevrey space relative to a particular self-adjoint operator. In the proofs, one of the intersection theorems of Chapter 2 plays an essential role. Finally, in Section 3.3 we consider a non-irreducible representation of the real unimodular group $SL(2, \mathbb{R})$ in $L^2(\mathbb{T})$. If A, X, Y is a suitable basis in $sl(2, \mathbb{R})$, we show that the Gevrey space of order λ relative to $\{d\pi(A), d\pi(X), d\pi(Y)\}$ is equal to the Gevrey space of order λ relative to $d\pi(A)$. This type of reduction can be proved for general semisimple Lie groups; they depend on the Casimir element in the complex universal enveloping algebra of the semisimple Lie algebra.

Some notations

Let A be a set and let V be a subset of A . By $A \setminus V$ we denote the *complement* of V in A , so $A \setminus V = \{a \in A : a \notin V\}$. By 1_V we denote the *characteristic function* of V , thus $1_V : A \rightarrow \mathbb{R}$

$$1_V(a) = \begin{cases} 1 & \text{if } a \in V \\ 0 & \text{if } a \notin V \end{cases} \quad (a \in A).$$

Let f be a complex valued bounded function on A . Then $\|f\|_\infty := \sup\{|f(a)| : a \in A\}$. For every set A we denote by I the *identity map* from A onto A . The *entier function* is the function $[\]$ from \mathbb{R} into \mathbb{Z} such that

$$[x] := \max\{k \in \mathbb{Z} : k \leq x\}$$

for all $x \in \mathbb{R}$. \mathbb{N} is the set of positive integers, $\mathbb{N} := \{1, 2, \dots\}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $n \in \mathbb{N}_0$ and $\lambda \in [0, \infty)$. In order to avoid clutter we write

$$n!^\lambda := (n!)^\lambda.$$

Let X be a topological space and let V be a subset of X . Then $\text{clo}V = \bar{V}$ denotes the *closure* of V . Let f be a complex valued function on X . The *support* of f is $\text{supp}f := \text{clo}\{x \in X : f(x) \neq 0\}$. By $C_c(X)$ we denote the set of complex valued continuous functions on X with compact support. Further, $C_0(X)$ denotes the Banach space of all complex valued continuous functions on X which vanish at infinity, i.e. for all $\varepsilon > 0$ there exists a compact subset K of X such that $|f(x)| < \varepsilon$ for all $x \in X \setminus K$. By $C(X)$ we denote the set of all complex valued continuous functions on X .

The abbreviation a.e. means *almost everywhere* with respect to some measure.

Let G be a locally compact topological group with a (left) Haar measure μ . For $x \in G$ and $f \in L^1(G)$ we define $L_x f \in L^1(G)$ by

$$(L_x f)(y) := f(x^{-1}y) \quad (\text{a.e. } y \in G).$$

Similarly we define $L_x f \in L^2(G)$ for all $x \in G$ and $f \in L^2(G)$. Let π be a (continuous unitary) representation of G in a Hilbert space H . For $f \in L^1(G)$ we denote by $\pi(f)$ the continuous operator on H such that

$$(\pi(f)u, v) = \int_G f(x)(\pi_x u, v) d\mu(x)$$

for all $u, v \in H$. Moreover, $\|f\|_1$ denotes the *norm* of f and for $g \in L^1(G)$ we denote by $f * g$ the *convolution* of f and g . Now suppose G is Abelian. The *dual group* of G is \hat{G} . By \hat{f} we denote the *Fourier transform* of f and by \mathbb{F} we denote the *Fourier transform* from $L^2(G)$ onto $L^2(\hat{G})$. We normalize the Haar measure λ on \mathbb{R} such that $\lambda([0, 1]) = \frac{1}{\sqrt{2\pi}}$. The Haar measure on \mathbb{T} is normalized to 1, the Haar measure on \mathbb{Z} is the counting measure. For $y \in \mathbb{R}$ define $\gamma_y : \mathbb{R} \rightarrow \mathbb{C}$ by

$$\gamma_y(x) := e^{ixy} \quad (x \in \mathbb{R}).$$

We identify $\widehat{\mathbb{R}}$ with \mathbb{R} via the map $y \mapsto \gamma_y$. For $k \in \mathbb{Z}$ define $\omega_k : \mathbb{T} \rightarrow \mathbb{C}$ by

$$\omega_k(z) := z^k \quad (z \in \mathbb{T}).$$

We identify $\widehat{\mathbb{T}}$ with \mathbb{Z} via the map $k \mapsto \omega_k$.

Let G be a Lie group. We define its *Lie algebra* \mathfrak{g} by the tangent space at the identity of G and the *commutator* $[X, Y]$ of two elements of \mathfrak{g} corresponds to the commutator of the two corresponding left invariant vector fields on G . We denote by \tilde{G} the *universal covering group* of a connected Lie group G . For the terminology of Lie groups we refer to Helgason, [Hel] and Varadarajan, [Var1].

Let H be a Hilbert space. The *inner product* in H is denoted by (\cdot, \cdot) and the *norm* by $\|\cdot\|$. Let T be a (not necessarily densely defined) operator from H into a Hilbert space. Then $D(T)$ denotes the *domain* of T and \bar{T} denotes the *closure* of T . If T is densely defined, we denote by T^* the *adjoint* of T . The operator T is called *Hermitian* if $(Tx, y) = (x, Ty)$ for all $x, y \in D(T)$. T is called *symmetric* if T is Hermitian and densely defined and T is called *self-adjoint* if T is densely defined and $T = T^*$. A symmetric operator T is *essentially self-adjoint* if \bar{T} is self-adjoint. The operator T is called *skew-Hermitian*, *skew-symmetric*, *essentially skew-adjoint* and *skew-adjoint* if iT is Hermitian, symmetric, essentially self-adjoint and self-adjoint respectively. Let T be an injective linear map from H into H . We complete the vector space $T(H)$ with an inner product such that the map T from H onto $T(H)$ is a unitary map. Thus $T(H)$ becomes a Hilbert space. Let (X, \mathcal{B}, m) be a *locally finite* measure space, i.e. for every $A \in \mathcal{B}$ with $m(A) > 0$ there exists $A_1 \in \mathcal{B}$ with $A_1 \subset A$ and $0 < m(A_1) < \infty$. Let h be a measurable function on X . The *multiplication operator* by the function h in the Hilbert space $L^2(m) := L^2(X, m)$ is the operator T in $L^2(m)$ with $D(T) := \{f \in L^2(m) : hf \in L^2(m)\}$ and $Tf := hf$ for all $f \in D(T)$. As usual we identify a function $f \in L^2(m)$ with its equivalence class in $L^2(m)$. Suppose X is also a topological space and \mathcal{B} is the Borel σ -algebra of X . Suppose every non-empty open subset of X has positive measure. Then every element of $L^2(m)$ has at most one continuous representative and we identify an element of $L^2(m)$ with its continuous representative if it has one. We denote by Q the multiplication operator by the identity function $x \mapsto x$ in $L^2(\mathbb{R})$. Let \mathbb{F} be the Fourier transform on $L^2(\mathbb{R})$. Define

$$P := \mathbb{F}Q\mathbb{F}^{-1}.$$

Let A be a totally ordered set and for each $\alpha \in A$ let $(E_\alpha, \|\cdot\|_{E_\alpha})$ and $(F_\alpha, \|\cdot\|_{F_\alpha})$ be normed spaces, which are, as vector spaces, subspaces of a fixed vector space. Suppose $E_\alpha \subset E_\beta$ and $F_\alpha \subset F_\beta$ for all $\alpha, \beta \in A$ with $\alpha \leq \beta$. Let $\alpha \in A$. Define the norm $\|\cdot\|_\alpha$ on $E_\alpha \cap F_\alpha$ by

$$\|x\|_\alpha := \|x\|_{E_\alpha} + \|x\|_{F_\alpha} \quad (x \in E_\alpha \cap F_\alpha).$$

Let $E := \bigcup_{\alpha \in A} E_\alpha$ and $F := \bigcup_{\alpha \in A} F_\alpha$. The (standard) topology for $E \cap F$ is the inductive limit topology generated by the normed spaces $E_\alpha \cap F_\alpha$ with $\alpha \in A$.

Let A, B be totally ordered sets. For all $\alpha \in A$ and $\beta \in B$ let X_α and Y_β be locally convex separated topological vector spaces. Suppose X_{α_1} is continuously embedded in X_{α_2} for all $\alpha_1, \alpha_2 \in A$ with $\alpha_1 \leq \alpha_2$ and, similarly, suppose Y_{β_1} is continuously embedded in Y_{β_2} for all $\beta_1, \beta_2 \in B$ with $\beta_1 \leq \beta_2$. Let $X := \bigcup_{\alpha \in A} X_\alpha$ and $Y := \bigcup_{\beta \in B} Y_\beta$. The topologies for X and Y are the (natural) inductive limit topologies. We call $X = Y$ as *locally convex spaces with equivalent spectra* if for all $\alpha \in A$ there exists $\beta \in B$ such that $X_\alpha \subset Y_\beta$ and the embedding map is continuous and, secondly, for all $\beta \in B$ there exists $\alpha \in A$ such that $Y_\beta \subset X_\alpha$ and the embedding map is continuous. If $X = Y$ as locally convex spaces with equivalent spectra, then $X = Y$ as locally convex topological vector spaces. Conversely, if X and Y are both regular inductive limits and $X = Y$ as locally convex topological vector spaces, then $X = Y$ as locally convex spaces with equivalent spectra. In case X and Y are regular inductive limits, we even use the terminology "as locally convex spaces with equivalent spectra".

We finish with some trivial definitions. Let $n, k \in \mathbb{Z}$. Then

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n, \\ 0 & \text{else.} \end{cases}$$

Let $a_1, a_2, \dots \in \mathbb{C}$. Then $\sum_{n=1}^0 a_n = 0$ and $\prod_{n=1}^0 a_n = 1$. Let A_1, A_2, \dots be operators in a Hilbert space. Then $A_k \circ \dots \circ A_1 = I$ if $k = 0$. In a vector space we define $\text{span}(\emptyset) := \{0\}$.

Chapter 1

Gevrey spaces

In this chapter we introduce the concept of Gevrey space and prove some topological properties of Gevrey spaces. Finally we consider Gevrey spaces corresponding to commuting symmetric operators.

1.1 Multi-indices and Gevrey spaces

Let V be a non-empty set. We define the set $M(V)$ by

$$M(V) := \bigcup_{n \in \mathbb{N}_0} V^n.$$

Here V^0 denotes the set with one element, called the *empty sequence*, which is denoted by $()$. The elements of $M(V)$ are called *multi-indices* (over V). For $v \in V$ define the v -length $\|\cdot\|_v : M(V) \rightarrow \mathbb{N}_0$ by

$$\|()\|_v := 0$$

$$\|(j_1, \dots, j_n)\|_v := \text{card} \{i \in \{1, \dots, n\} : j_i = v\} \quad (n \in \mathbb{N}, j_1, \dots, j_n \in V).$$

For $\alpha \in M(V)$ define the *length* $\|\alpha\|$ of α by $\|\alpha\| = n$, where $n \in \mathbb{N}_0$ is the unique number such that $\alpha \in V^n$. So

$$\|\alpha\| = \sum_{v \in V} \|\alpha\|_v$$

for all $\alpha \in M(V)$.

In a natural way we define an operation on $M(V)$: for $\alpha, \beta \in M(V)$ define the *concatenation* $\langle \alpha, \beta \rangle$ of α and β by

$$\langle (), () \rangle := ()$$

$$\langle (), (j_1, \dots, j_n) \rangle := (j_1, \dots, j_n)$$

$$\langle (j_1, \dots, j_n), () \rangle := (j_1, \dots, j_n)$$

$$\langle (j_1, \dots, j_n), (k_1, \dots, k_m) \rangle := (j_1, \dots, j_n, k_1, \dots, k_m)$$

for all $n, m \in \mathbb{N}$ and all $j_1, \dots, j_n, k_1, \dots, k_m \in V$. So $M(V)$ is a monoid with identity $(\)$. For $p \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_p \in M(V)$ define the *concatenation* $\langle \alpha_1, \dots, \alpha_p \rangle$ of $\alpha_1, \dots, \alpha_p$ by $\langle \alpha_1, \dots, \alpha_p \rangle := \alpha_1$ if $p = 1$ and inductively

$$\langle \alpha_1, \dots, \alpha_p \rangle := \langle \langle \alpha_1, \dots, \alpha_{p-1} \rangle, \alpha_p \rangle$$

if $p > 1$.

At the moment we finish our discussion on multi-indices with the reverse operation on $M(V)$. For $\alpha \in M(V)$ define the *reverse* α^r of α by

$$\begin{aligned} (\)^r &:= (\) \\ (j_1, \dots, j_n)^r &:= (j_n, \dots, j_1). \end{aligned}$$

Then $\alpha \rightarrow \alpha^r$ is a bijection from $M(V)$ onto $M(V)$.

In the next chapter we shall introduce some more operations on $M(V)$.

Let H be a Hilbert space and let A_1, \dots, A_d be d (unbounded) operators in H . Let $V := \{1, \dots, d\}$. For $\alpha \in M(V)$ define the operator A_α by

$$\begin{aligned} A_{(\)} &:= I \\ A_{(j_1, \dots, j_n)} &:= A_{j_1} \circ \dots \circ A_{j_n} \quad (n \in \mathbb{N}, j_1, \dots, j_n \in V). \end{aligned}$$

We define the *joint C^∞ -domain* $D^\infty(A_1, \dots, A_d)$ of the operators A_1, \dots, A_d by

$$D^\infty(A_1, \dots, A_d) := \bigcap_{\alpha \in M(V)} D(A_\alpha).$$

Here, $D(T)$ denotes the domain of the operator T . We emphasize that the space $D^\infty(A_1, \dots, A_d)$ may be trivial, i.e. $D^\infty(A_1, \dots, A_d) = \{0\}$. For all $\alpha \in M(V)$ define the seminorm $\| \cdot \|_{A_1, \dots, A_d; \alpha}$ for $D^\infty(A_1, \dots, A_d)$ by

$$\|u\|_{A_1, \dots, A_d; \alpha} := \|A_\alpha u\| \quad (u \in D^\infty(A_1, \dots, A_d)).$$

We drop the indices A_1, \dots, A_d in the seminorm $\| \cdot \|_{A_1, \dots, A_d; \alpha}$ if the meaning is clear from the context. So $\| \cdot \|_\alpha = \| \cdot \|_{A_1, \dots, A_d; \alpha}$. The topology for $D^\infty(A_1, \dots, A_d)$ is the locally convex topology generated by the seminorms $\| \cdot \|_\alpha$, with $\alpha \in M(V)$. Since the identity map from $D^\infty(A_1, \dots, A_d)$ into H is continuous, the topology for $D^\infty(A_1, \dots, A_d)$ is Hausdorff. Clearly $D^\infty(A_1, \dots, A_d)$ is metrizable.

Similarly to the space $D^\infty(A_1, \dots, A_d)$ we define the *joint ordered C^∞ -domain* of the operators A_1, \dots, A_d by

$$D_{\text{ord}}^\infty(A_1, \dots, A_d) := \bigcap_{n_1, \dots, n_d \in \mathbb{N}_0} D(A_1^{n_1} \circ \dots \circ A_d^{n_d}).$$

For all $n_1, \dots, n_d \in \mathbb{N}_0$ we define the seminorm $\| \cdot \|_{n_1, \dots, n_d}$ (or to be more precise, $\| \cdot \|_{A_1, \dots, A_d; n_1, \dots, n_d}$) by

$$\|u\|_{n_1, \dots, n_d} := \|A_1^{n_1} \circ \dots \circ A_d^{n_d} u\| \quad (u \in D_{\text{ord}}^\infty(A_1, \dots, A_d)).$$

Also the topology for $D_{\text{ord}}^\infty(A_1, \dots, A_d)$ is the locally convex topology for $D_{\text{ord}}^\infty(A_1, \dots, A_d)$ generated by the seminorms $\|\cdot\|_{n_1, \dots, n_d}$, with $n_1, \dots, n_d \in \mathbb{N}_0$.

We want to define subspaces of $D^\infty(A_1, \dots, A_d)$ and $D_{\text{ord}}^\infty(A_1, \dots, A_d)$ similar to the Gelfand–Shilov spaces S_α^β and $S_{\alpha, A}^{\beta, B}$. (See [GS], Section IV.3.3.) Let $\lambda_1, \dots, \lambda_d \geq 0$. For $t > 0$ define the *unordered* function

$$\begin{aligned} & \| \cdot \|_{A_1, \dots, A_d; \lambda_1, \dots, \lambda_d; t} : D^\infty(A_1, \dots, A_d) \rightarrow [0, \infty) \\ & \|u\|_{A_1, \dots, A_d; \lambda_1, \dots, \lambda_d; t} := \sup_{\alpha \in \mathcal{M}(V)} \frac{\|A_\alpha u\|}{t^{|\alpha|} \|\alpha\|_1^{\lambda_1} \dots \|\alpha\|_d^{\lambda_d}} \quad (u \in D^\infty(A_1, \dots, A_d)). \end{aligned}$$

and the *ordered* function

$$\begin{aligned} & \| \cdot \|_{A_1, \dots, A_d; \lambda_1, \dots, \lambda_d; t, \text{ord}} : D_{\text{ord}}^\infty(A_1, \dots, A_d) \rightarrow [0, \infty) \\ & \|u\|_{A_1, \dots, A_d; \lambda_1, \dots, \lambda_d; t, \text{ord}} := \sup_{n \in \mathbb{N}_0^d} \frac{\|A_1^{n_1} \circ \dots \circ A_d^{n_d} u\|}{t^{n_1 + \dots + n_d} n_1!^{\lambda_1} \dots n_d!^{\lambda_d}} \quad (u \in D_{\text{ord}}^\infty(A_1, \dots, A_d)). \end{aligned}$$

Let

$$S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d) := \{u \in D^\infty(A_1, \dots, A_d) : \|u\|_{A_1, \dots, A_d; \lambda_1, \dots, \lambda_d; t} < \infty\}.$$

$S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ is a normed space with norm $\|\cdot\|_{A_1, \dots, A_d; \lambda_1, \dots, \lambda_d; t}$. Similarly define the normed space $S_{\lambda_1, \dots, \lambda_d; t}^{\text{ord}}(A_1, \dots, A_d)$. Again we drop the indices A_1, \dots, A_d when no confusion can arise. So

$$\begin{aligned} & \| \cdot \|_{\lambda_1, \dots, \lambda_d; t} = \| \cdot \|_{A_1, \dots, A_d; \lambda_1, \dots, \lambda_d; t} \\ & \| \cdot \|_{\lambda_1, \dots, \lambda_d; t, \text{ord}} = \| \cdot \|_{A_1, \dots, A_d; \lambda_1, \dots, \lambda_d; t, \text{ord}}. \end{aligned}$$

Define the *unordered Gevrey space*

$$S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d) := \bigcup_{t > 0} S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$$

and the *ordered Gevrey space*

$$S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(A_1, \dots, A_d) := \bigcup_{t > 0} S_{\lambda_1, \dots, \lambda_d; t}^{\text{ord}}(A_1, \dots, A_d).$$

The topology σ_{ind} for $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ is the inductive limit topology generated by the normed spaces $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ with $t > 0$. Similarly the topology for

$S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(A_1, \dots, A_d)$ is the inductive limit topology generated by the normed spaces $S_{\lambda_1, \dots, \lambda_d; t}^{\text{ord}}(A_1, \dots, A_d)$ with $t > 0$ and it is denoted by σ_{ind} also. For the terminology of locally convex topological vector space theory we refer to Appendix A.

In the following lemma we summarize some elementary norm inequalities and corresponding inclusions. The proof is omitted.

Lemma 1.1 *Let $\lambda_1, \dots, \lambda_d \geq 0, \mu_1, \dots, \mu_d \geq 0$ and let $s, t > 0$. Suppose $\lambda_i \leq \mu_i$ for all $i \leq d$ and $s \leq t$. Then*

$$\|u\|_{\mu_1, \dots, \mu_d; t} \leq \|u\|_{\lambda_1, \dots, \lambda_d; s} \text{ for all } u \in D^\infty(A_1, \dots, A_d),$$

$$\|u\|_{\mu_1, \dots, \mu_d; t, \text{ord}} \leq \|u\|_{\lambda_1, \dots, \lambda_d; s, \text{ord}} \text{ for all } u \in D_{\text{ord}}^\infty(A_1, \dots, A_d),$$

$$\|u\|_{\mu_1, \dots, \mu_d; t, \text{ord}} \leq \|u\|_{\lambda_1, \dots, \lambda_d; s} \text{ for all } u \in D^\infty(A_1, \dots, A_d).$$

The embedding maps

$$S_{\lambda_1, \dots, \lambda_d; s}(A_1, \dots, A_d) \hookrightarrow S_{\mu_1, \dots, \mu_d; t}(A_1, \dots, A_d),$$

$$S_{\lambda_1, \dots, \lambda_d; s}^{\text{ord}}(A_1, \dots, A_d) \hookrightarrow S_{\mu_1, \dots, \mu_d; t}^{\text{ord}}(A_1, \dots, A_d),$$

$$S_{\lambda_1, \dots, \lambda_d; s}(A_1, \dots, A_d) \hookrightarrow S_{\mu_1, \dots, \mu_d; t}^{\text{ord}}(A_1, \dots, A_d),$$

$$S_{\lambda_1, \dots, \lambda_d; s}(A_1, \dots, A_d) \hookrightarrow D^\infty(A_1, \dots, A_d),$$

$$S_{\lambda_1, \dots, \lambda_d; s}^{\text{ord}}(A_1, \dots, A_d) \hookrightarrow D_{\text{ord}}^\infty(A_1, \dots, A_d),$$

$$S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d) \hookrightarrow D^\infty(A_1, \dots, A_d),$$

$$S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(A_1, \dots, A_d) \hookrightarrow D_{\text{ord}}^\infty(A_1, \dots, A_d),$$

$$D^\infty(A_1, \dots, A_d) \hookrightarrow D_{\text{ord}}^\infty(A_1, \dots, A_d),$$

$$D^\infty(A_1, \dots, A_d) \hookrightarrow H,$$

$$D_{\text{ord}}^\infty(A_1, \dots, A_d) \hookrightarrow H$$

are continuous. The topologies for $D^\infty(A_1, \dots, A_d), D_{\text{ord}}^\infty(A_1, \dots, A_d), S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ and $S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(A_1, \dots, A_d)$ are Hausdorff.

Example 1.2

Let A be a positive self-adjoint operator in a Hilbert space H . For all $t > 0$ the continuous operator e^{-tA} is injective, so there exists a unique norm $\|\cdot\|_t$ on the vector space $e^{-tA}(H)$ such that $e^{-tA}(H)$ becomes a Hilbert space and the map e^{-tA} from H into $e^{-tA}(H)$ is a unitary map. The *analyticity space* $S_{H,A}$ is defined by $S_{H,A} := \bigcup_{t>0} e^{-tA}(H)$. The topology σ for $S_{H,A}$ is the inductive limit topology generated by the Hilbert spaces $e^{-tA}(H)$. In the monograph [EG] the space $S_{H,A}$ is introduced for any separable Hilbert space H and any positive self-adjoint operator A in H . A lot of examples are included in Chapter II of the book [EG]. The assumed separability of the Hilbert space H is an superfluous restriction which we do not assume here.

Let H be a Hilbert space, A a self-adjoint operator in H and let $\lambda > 0$. We shall prove that $S_\lambda(A) = S_{H,|A|^{1/\lambda}}$, i.e.

$$\{u \in D^\infty(A) : \exists M > 0 \exists t > 0 \forall n \in \mathbb{N}_0 : \|A^n u\| \leq M t^n n!^\lambda\} = \bigcup_{t > 0} e^{-t|A|^{1/\lambda}}(H)$$

as locally convex spaces with equivalent spectra. In the proof we need some elementary inequalities which we present in the following lemma. This lemma will be used frequently in the remaining part of this thesis.

Lemma 1.3

- I. Let $k, m \in \mathbb{N}_0$. Then $m^k \leq k!e^m$.
- II. Let $n \in \mathbb{N}_0$. Then $e^{-n}n^n \leq n! \leq n^n \leq n!e^n$.
- III. Let $m, n \in \mathbb{N}_0$ and $\lambda \geq 0$. Then $(m+n)!^\lambda \leq 2^{\lambda(m+n)}m!^\lambda n!^\lambda$ and, if $n \leq m$, $(m-n)!^\lambda \leq m!^\lambda n!^{-\lambda}$.
- IV. Let $n \in \mathbb{N}_0$ and $\lambda > 0$. Then

$$\left(\left[\frac{n}{\lambda}\right] + 1\right)!^\lambda \leq \left\{\left(\left[\frac{n}{\lambda}\right] + 1\right)e\right\}^\lambda \left(\frac{e}{\lambda}\right)^n n!$$

(Here $[\]$ denotes the entier function.)

- V. Let $d \in \mathbb{N}$, $n_1, \dots, n_d \in 2\mathbb{N}_0$ and let $\lambda \geq 0$. Let $n := n_1 + \dots + n_d$. Then

$$n!^\lambda \leq (2^{\lambda d})^n \left[\left(\frac{n_1}{2}\right)!^\lambda \dots \left(\frac{n_d}{2}\right)!^\lambda\right]^2.$$

Proof. I. $e^m = \sum_{n=0}^{\infty} n!^{-1} m^n \geq k!^{-1} m^k$.

II. Take $k = m = n$ in I.

III. $(m+n)!^\lambda = \binom{m+n}{n}^\lambda m!^\lambda n!^\lambda \leq (2^{m+n})^\lambda m!^\lambda n!^\lambda$. If $n \leq m$, then $(m-n)!^\lambda =$

$$\binom{m}{m-n}^{-\lambda} m!^\lambda n!^{-\lambda} \leq m!^\lambda n!^{-\lambda}.$$

IV. $\left(\left[\frac{n}{\lambda}\right] + 1\right)!^\lambda \leq \left\{\left(\left[\frac{n}{\lambda}\right] + 1\right)\left(\left[\frac{n}{\lambda}\right] + 1\right)^{[n/\lambda]}\right\}^\lambda \leq \left\{\left(\left[\frac{n}{\lambda}\right] + 1\right)\left(\frac{n}{\lambda} + 1\right)^{n/\lambda}\right\}^\lambda \leq$

$$\left\{\left(\left[\frac{n}{\lambda}\right] + 1\right)e\left(\frac{n}{\lambda}\right)^{n/\lambda}\right\}^\lambda = \left\{\left(\left[\frac{n}{\lambda}\right] + 1\right)e\right\}^\lambda \left(\frac{1}{\lambda}\right)^n n^n \leq \left\{\left(\left[\frac{n}{\lambda}\right] + 1\right)e\right\}^\lambda \left(\frac{e}{\lambda}\right)^n n!.$$

V. This inequality follows from III. □

We continue with Example 1.2.

Theorem 1.4 Let A be a self-adjoint operator in a Hilbert space H and let $\lambda > 0$. Then $S_\lambda(A) = S_{H, |A|^{1/\lambda}}$ as locally convex spaces with equivalent spectra.

Proof. Let $t > 0$. Let $s := \frac{1}{2}\lambda e^{-1}t^{-1/\lambda}$. We prove that $S_{\lambda, t}(A)$ is continuously embedded in $e^{-s|A|^{1/\lambda}}(H)$. Let

$$C := e^s + \sum_{n=0}^{\infty} \left\{\left(\left[\frac{n}{\lambda}\right] + 1\right)e\right\}^\lambda t \left(\frac{1}{2}\right)^n < \infty.$$

Let $u \in S_{\lambda;t}(A)$. Then for all $n \in \mathbb{N}_0$ we obtain by Lemma 1.3.IV :

$$\begin{aligned}
 \| |A|^{n/\lambda} u \| &\leq \| |A|^{n/\lambda} 1_{[1,\infty)}(|A|)u \| + \| |A|^{n/\lambda} 1_{[0,1)}(|A|)u \| \\
 &\leq \| |A|^{[n/\lambda]+1} 1_{[1,\infty)}(|A|)u \| + \| u \| \\
 &\leq \| |A|^{[n/\lambda]+1} u \| + \| u \| \\
 &\leq t^{[n/\lambda]+1} \left(\left[\frac{n}{\lambda} \right] + 1 \right)!^\lambda \| u \|_{\lambda;t} + \| u \|_{\lambda;t} \\
 &\leq t^{n/\lambda+1} \left\{ \left(\left[\frac{n}{\lambda} \right] + 1 \right) e \right\}^\lambda \left(\frac{e}{\lambda} \right)^n n! \| u \|_{\lambda;t} + \| u \|_{\lambda;t} \\
 &\leq \left\{ \left(\left[\frac{n}{\lambda} \right] + 1 \right) e \right\}^\lambda t \left(\frac{1}{2} \right)^n s^{-n} n! \| u \|_{\lambda;t} + \| u \|_{\lambda;t}.
 \end{aligned}$$

So

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} \| |A|^{n/\lambda} u \| \leq C \| u \|_{\lambda;t} < \infty.$$

Let $v := \sum_{n=0}^{\infty} \frac{s^n}{n!} |A|^{n/\lambda} u \in H$. Then $u = e^{-s|A|^{1/\lambda}} v \in e^{-s|A|^{1/\lambda}}(H)$ and $\| u \|_s = \| v \| \leq C \| u \|_{\lambda;t}$.

It remains to prove that for all $t > 0$ there exists $s > 0$ such that $e^{-t|A|^{1/\lambda}}(H)$ is continuously embedded in $S_{\lambda;s}(A)$. So, let $t > 0$. Let $s := \left(\frac{\lambda}{t} \right)^\lambda$. Let $n \in \mathbb{N}$. Since the function $x \mapsto x^n e^{-tx^{1/\lambda}}$, defined on $[0, \infty)$, has the maximum value $\left(\frac{n\lambda}{t} \right)^{n\lambda} e^{-n\lambda}$, for all $u \in H$ we obtain:

$$\| A^n e^{-t|A|^{1/\lambda}} u \| \leq \left(\frac{n\lambda}{t} \right)^{n\lambda} e^{-n\lambda} \| u \| \leq s^n n!^\lambda \| u \|.$$

So $e^{-t|A|^{1/\lambda}} u \in S_{\lambda;s}(A)$ and $\| e^{-t|A|^{1/\lambda}} u \|_{\lambda;s} \leq \| u \| = \| e^{-t|A|^{1/\lambda}} u \|_t$. □

Corollary 1.5 *Let A be an operator in a Hilbert space which has a self-adjoint extension. Let $d \in \mathbb{N}$ and let $\lambda > 0$. Then*

$$S_\lambda(A) = S_{\lambda d}(A^d)$$

as locally convex spaces with equivalent spectra.

Proof. Clearly for all $t > 0$ the space $S_{\lambda;t}(A)$ is continuously embedded in $S_{\lambda d;s}(A^d)$, where $s := 2^{(d-1)d\lambda} t^d$.

Let B be a self-adjoint extension of A . Let $t > 0$. By Theorem 1.4 there exist $t_1, t_2 > 0$ such that the following inclusions are continuous:

$$S_{\lambda d;t}(A^d) \hookrightarrow S_{\lambda d;t}(B^d) \hookrightarrow e^{-t_1|B^d|^{1/(\lambda d)}}(H) = e^{-t_1|B|^{1/\lambda}}(H) \hookrightarrow S_{\lambda;t_2}(B).$$

Since $S_{\lambda d;t}(A^d) \subset D^\infty(A)$ as sets, it follows that in fact $S_{\lambda d;t}(A^d)$ is continuously embedded in $S_{\lambda;t_2}(A)$. □

Example 1.6

Let A be an operator in a Hilbert space H . Then $S_0(A)$ is the set of all *bounded vectors* for A . (See Faris, [Far], §16.)

Example 1.7

Let A_1, \dots, A_d be d operators in a Hilbert space H and let $\lambda \geq 0$. Then $S_{\lambda, \dots, \lambda}(A_1, \dots, A_d)$ is equal to the set of *Gevrey vectors of order λ relative to $\{A_1, \dots, A_d\}$* . (See Goodman-Wallach, [GW], Section 1.) In particular, for $\lambda = 1$, the set $S_{1, \dots, 1}(A_1, \dots, A_d)$ is equal to the set of *analytic vectors* for $\{A_1, \dots, A_d\}$. (See Goodman, [Goo1], Section 2 or Nelson, [Nel], Section 2.)

Example 1.8

Let Q be the multiplication operator by the function $x \mapsto x$ in $L^2(\mathbb{R})$. Let \mathbb{F} be the Fourier transform on $L^2(\mathbb{R})$ such that

$$(\mathbb{F}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y)e^{-ixy} dy \quad a.e. x \in \mathbb{R}$$

for all $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Let $P := \mathbb{F}Q\mathbb{F}^{-1}$. Then P and Q are self-adjoint operators in $L^2(\mathbb{R})$. The dense joint C^∞ -domain $D^\infty(P, Q)$ is equal to $D_{\text{ord}}^\infty(Q, P)$ and the latter space is equal to *Schwartz' space* $S(\mathbb{R})$, i.e. the space of all infinitely differentiable functions φ defined on \mathbb{R} such that

$$\sup\{|x^k \varphi^{(l)}(x)| : x \in \mathbb{R}\} < \infty$$

for all $k, l \in \mathbb{N}_0$. (See [Goo2], page 65.)

Let $\alpha, \beta \geq 0$. Gelfand and Shilov define the space S_α^β by

$$S_\alpha^\beta := \{\varphi \in S(\mathbb{R}) : \exists_{n \in \mathbb{N}} \sup_{\substack{k, l \in \mathbb{N}_0 \\ x \in \mathbb{R}}} |x^k \varphi^{(l)}(x)| n^{-k-l} k^{-k\alpha} l^{-l\beta} < \infty\}.$$

In order to introduce topologies for the space S_α^β , for $n \in \mathbb{N}$ we define the normed space

$$\tilde{S}_\infty(n; \alpha, \beta) := \{\varphi \in S(\mathbb{R}) : \sup_{\substack{k, l \in \mathbb{N}_0 \\ x \in \mathbb{R}}} |x^k \varphi^{(l)}(x)| n^{-k-l} k^{-k\alpha} l^{-l\beta} < \infty\}$$

with norm

$$\|\varphi\|_{\tilde{S}_\infty(n; \alpha, \beta)} := \sup\{|x^k \varphi^{(l)}(x)| n^{-k-l} k^{-k\alpha} l^{-l\beta} : k, l \in \mathbb{N}_0, x \in \mathbb{R}\}.$$

The *Gelfand-Shilov space* S_α^β equals

$$S_\alpha^\beta := \bigcup_{n=1}^{\infty} \tilde{S}_\infty(n; \alpha, \beta)$$

and we provide the space S_α^β with the inductive limit topology generated by the normed spaces $\tilde{S}_\infty(n; \alpha, \beta)$, with $n \in \mathbb{N}$. (See [GS], Section IV.3.3 and [Wlo], §29.5. Wloka uses the notation $\tilde{S}_\infty(m^n)$ instead of $\tilde{S}_\infty(n)$.)

Remark. In [GS] no topology is defined for the vector space S_α^β , but only the concept of converging sequence occurs. Using Theorem 1.9 and Corollary 1.13.I infra, it is not too hard to show that a sequence in S_α^β converges to 0 in the sense of Gelfand-Shilov if and only if it converges to 0 with respect to the inductive limit topology.

In the following theorem we prove that the Gelfand-Shilov space S_α^β can be described as an ordered Gevrey space.

Theorem 1.9 *Let $\alpha, \beta \geq 0$. Then $S_\alpha^\beta = S_{\alpha, \beta}^{\text{ord}}(Q, P)$ as locally convex spaces with equivalent spectra.*

Proof. For $n \in \mathbb{N}$ let

$$S_2(n; \alpha, \beta) := \{ \varphi \in S(\mathbb{R}) : \sum_{k, l \in \mathbb{N}_0} \int_{\mathbb{R}} |(1+x^2)^k \varphi^{(l)}(x) n^{-2k-l} (2k)^{-2k\alpha} l^{-l\beta}|^2 dx < \infty \}$$

with norm

$$\| \varphi \|_{S_2(n; \alpha, \beta)} := \left(\sum_{k, l \in \mathbb{N}_0} \int_{\mathbb{R}} |(1+x^2)^k \varphi^{(l)}(x) n^{-2k-l} (2k)^{-2k\alpha} l^{-l\beta}|^2 dx \right)^{\frac{1}{2}}.$$

Then $S_2(n; \alpha, \beta)$ is a Hilbert space. (See [Wlo], page 4.) Let

$$E := \bigcup_{n=1}^{\infty} S_2(n; \alpha, \beta).$$

The topology for E is the inductive limit topology. By [Wlo], §29.5 and §29.3 the spaces E and S_α^β are equal as locally convex spaces with equivalent spectra.

It remains to prove that E and $S_{\alpha, \beta}^{\text{ord}}(Q, P)$ are equal as locally convex spaces with equivalent spectra. Let $n \in \mathbb{N}$. Let $t := 2^\alpha e^\alpha e^{\beta n} > 0$. Let $\varphi \in S_2(n; \alpha, \beta)$. Then $\varphi \in S(\mathbb{R}) = D_{\text{ord}}^\infty(Q, P)$. Let $k, l \in \mathbb{N}_0$. Let $k_1 \in \mathbb{N}_0$ be such that $2k_1 - 1 \leq k \leq 2k_1$. Then

$$\begin{aligned} \| Q^k P^l \varphi \| &\leq \| (1+Q^2)^{k_1} P^l \varphi \| \\ &\leq \| \varphi \|_{S_2(n; \alpha, \beta)} n^{2k_1+l} (2k_1)^{2k_1\alpha} l^{l\beta} \\ &\leq \| \varphi \|_{S_2(n; \alpha, \beta)} n^{k+1+l} (k+1)^{(k+1)\alpha} l^{l\beta} \\ &\leq \| \varphi \|_{S_2(n; \alpha, \beta)} n^{k+1+l} (k+1)!^\alpha l!^\beta e^{(k+1)\alpha} e^{l\beta} \\ &\leq n e^\alpha \| \varphi \|_{S_2(n; \alpha, \beta)} t^{k+l} k!^\alpha l!^\beta. \end{aligned}$$

So $\varphi \in S_{\alpha, \beta; t}^{\text{ord}}(Q, P)$ and $\| \varphi \|_{\alpha, \beta; t, \text{ord}} \leq n e^\alpha \| \varphi \|_{S_2(n; \alpha, \beta)}$.

Now let $t > 0$. Let $n \in \mathbb{N}$ be such that $n \geq 2(1+t)$. Let $\varphi \in S_{\alpha,\beta;t}^{\text{ord}}(Q, P)$ and let $c := \|\varphi\|_{\alpha,\beta;t,\text{ord}}$. Then $\varphi \in D_{\text{ord}}^{\infty}(Q, P) = S(\mathbb{R})$. Moreover, for all $k, l \in \mathbb{N}_0$ we have

$$\begin{aligned} \|(1+Q^2)^k P^l \varphi\| &\leq \sum_{m=0}^k \binom{k}{m} \|Q^{2m} P^l \varphi\| \\ &\leq \sum_{m=0}^k \binom{k}{m} c t^{2m+l} (2m)!^{\alpha} l^{\beta} \\ &\leq c(1+t^2)^k t^l (2k)!^{\alpha} l^{\beta} \\ &\leq c(1+t)^{2k+l} (2k)^{2k\alpha} l^{\beta} \\ &\leq c\left(\frac{1}{2}\right)^{2k+l} n^{2k+l} (2k)^{2k\alpha} l^{\beta}. \end{aligned}$$

So

$$\sum_{k,l \in \mathbb{N}_0} \left(\|(1+Q^2)^k P^l \varphi\| n^{-(2k+l)} (2k)^{-2k\alpha} l^{-l\beta} \right)^2 \leq \frac{64}{45} c^2.$$

Hence $\varphi \in S_2(n; \alpha, \beta)$ and $\|\varphi\|_{S_2(n; \alpha, \beta)} \leq \sqrt{\frac{64}{45}} \|\varphi\|_{\alpha,\beta;t,\text{ord}}$. This proves the theorem. \square

1.2 Topological properties of Gevrey spaces

The ordered and unordered Gevrey spaces are introduced as inductive limits of normed spaces. In general, an inductive limit of normed spaces is not regular, not complete and so on. In this section we prove that an (un)ordered Gevrey space is regular and, with some additional assumptions, that the unordered Gevrey space is complete. We give a rich class of examples which satisfy these assumptions.

Let A_1, \dots, A_d be d fixed operators in a Hilbert space H . Let $V := \{1, \dots, d\}$.

Lemma 1.10 *Let $\lambda_1, \dots, \lambda_d \geq 0$. Then $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ and $S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(A_1, \dots, A_d)$ are bornological.*

Proof. Any separated inductive limit of metrizable locally convex topological vector spaces is bornological. (See [Wil], Example 4-4-7 and Theorem 13-1-13.) \square

In the next theorem we prove that the inductive limit spaces $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ and $S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(A_1, \dots, A_d)$ are both regular.

Theorem 1.11 *Let $\lambda_1, \dots, \lambda_d \geq 0$ and let B be a subset of $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$. Then B is bounded in $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ if and only if there exists $t > 0$ such that B is a bounded subset of $S_{\lambda_1, \dots, \lambda_d;t}(A_1, \dots, A_d)$.*

Similar results hold for $S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(A_1, \dots, A_d)$.

Proof. For completeness we include the proof of this theorem, although it is completely similar to the proof of [Goo3], Lemma 1.2.

Suppose B is bounded in $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$. We have to prove that there exists $t > 0$ such that $\sup_{u \in B} \|u\|_{\lambda_1, \dots, \lambda_d;t} < \infty$. Suppose this were not the case. Then for all $t > 0$ we would have

$$\sup_{u \in B} \sup_{\alpha \in M(V)} \frac{\|A_\alpha u\|}{t^{|\alpha|} M_\alpha} = \infty,$$

where $M_\alpha := \|\alpha\|_1^{\lambda_1} \cdots \|\alpha\|_d^{\lambda_d}$. Since the identity map from $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ into H is continuous, the set B is bounded in H and $\sup_{u \in B} \frac{\|A_\alpha u\|}{t^{|\alpha|} M_\alpha} < \infty$. Hence for all $k \in \mathbb{N}$ there exist $u_k \in B$ and $\alpha_k \in M(V)$ such that

$$\|A_{\alpha_k} u_k\| \geq k^{|\alpha_k|} M_{\alpha_k} \quad (1.1)$$

and $\|\alpha_k\| \geq 1$. Define $\eta : (0, \infty) \rightarrow (0, \infty)$ by

$$\eta(t) := \inf\{(k^{1/2} t^{-1})^{|\alpha_k|} : k \in \mathbb{N}\} \quad (t > 0)$$

and let U be the set of all $v \in S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ of the form $v = \sum_{t>0} c_t v_t$, where $\sum_{t>0} |c_t| \leq 1$ (finite sum), $v_t \in S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ and $\|v_t\|_{\lambda_1, \dots, \lambda_d; t} \leq \eta(t)$ for all $t > 0$. Then U is absolutely convex and $U \cap S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ is a neighborhood of 0 in $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ for all $t > 0$. So U is a neighborhood of 0 in $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ by [Wil], Theorem 13-1-11. Since B is bounded, there exists $\delta > 0$ such that $B \subset \delta U$. But for all $k \in \mathbb{N}$ we obtain that

$$\|A_{\alpha_k} u_k\| \leq \delta \sup_{t>0} t^{|\alpha_k|} \eta(t) M_{\alpha_k} \leq \delta \sup_{t>0} t^{|\alpha_k|} (k^{1/2} t^{-1})^{|\alpha_k|} M_{\alpha_k} = \delta k^{|\alpha_k|/2} M_{\alpha_k}.$$

This contradicts (1.1) for k sufficiently large, since $\|\alpha_k\| \geq 1$. \square

Only for the unordered Gevrey space we can prove that bounded subsets behave well. In the proof of the theorem we feel inspired by Liu, [Liu], Theorem I.2.5 2).

Theorem 1.12 *Suppose the operators A_1, \dots, A_d are Hermitian or skew-Hermitian. Let $\lambda_1, \dots, \lambda_d \geq 0$ and let $t > 0$. Then there exists $s \geq t$ such that for all bounded sets $B \subset S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ we have $(B, \sigma_{\text{ind}}) = (B, \|\cdot\|_{\lambda_1, \dots, \lambda_d; s})$ as topological spaces.*

Proof. Let $s := 2^{\lambda_1 + \dots + \lambda_d} t$. First we prove the interpolation inequality

$$\|u\|_{\lambda_1, \dots, \lambda_d; s} \leq \|u\|^{1/2} \|u\|_{\lambda_1, \dots, \lambda_d; t}^{1/2} \quad (1.2)$$

for all $u \in S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$. Let $u \in S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ and let $\alpha \in M(V)$. Then by Lemma 1.3.III we obtain that

$$\begin{aligned} \left(\frac{\|A_\alpha u\|}{s^{|\alpha|} \|\alpha\|_1^{\lambda_1} \cdots \|\alpha\|_d^{\lambda_d}} \right)^2 &= \frac{|(u, A_{(\alpha^r, \alpha)} u)|}{s^{|\langle \alpha^r, \alpha \rangle|} \|\alpha\|_1^{2\lambda_1} \cdots \|\alpha\|_d^{2\lambda_d}} \\ &\leq \frac{\|u\| \|A_{(\alpha^r, \alpha)} u\| 2^{2\lambda_1} \|\alpha\|_1 \cdots 2^{2\lambda_d} \|\alpha\|_d}{s^{|\langle \alpha^r, \alpha \rangle|} \|\alpha\|_1^{\lambda_1} \cdots \|\alpha\|_d^{\lambda_d}} \\ &\leq \frac{\|u\| \|A_{(\alpha^r, \alpha)} u\| (2^{\lambda_1 + \dots + \lambda_d})^{|\langle \alpha^r, \alpha \rangle|}}{s^{|\langle \alpha^r, \alpha \rangle|} \|\alpha\|_1^{\lambda_1} \cdots \|\alpha\|_d^{\lambda_d}} \\ &= \|u\| \cdot \frac{\|A_{(\alpha^r, \alpha)} u\|}{t^{|\langle \alpha^r, \alpha \rangle|} \|\alpha\|_1^{\lambda_1} \cdots \|\alpha\|_d^{\lambda_d}} \\ &\leq \|u\| \|u\|_{\lambda_1, \dots, \lambda_d; t}. \end{aligned}$$

This proves inequality (1.2).

Let B be a bounded subset of $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$. Clearly the identity map from $(B, \|\cdot\|_{\lambda_1, \dots, \lambda_d; s})$ onto (B, σ_{ind}) is continuous. Let $M > 0$ be such that $\|u\|_{\lambda_1, \dots, \lambda_d; t} \leq M$ for all $u \in B$. Let $(u_i)_{i \in I}$ be a σ_{ind} -convergent net in B with limit $u \in B$. Then $\lim_i u_i = u$ in H , so $\lim_i \|u_i - u\| = 0$. Hence

$$\begin{aligned} \limsup_i \|u_i - u\|_{\lambda_1, \dots, \lambda_d; s} &\leq \limsup_i \|u_i - u\|^{\frac{1}{2}} \|u_i - u\|_{\lambda_1, \dots, \lambda_d; t}^{\frac{1}{2}} \\ &\leq \sqrt{2M} \limsup_i \|u_i - u\|^{\frac{1}{2}} = 0. \end{aligned}$$

So $\lim_i u_i = u$ in $S_{\lambda_1, \dots, \lambda_d; s}(A_1, \dots, A_d)$. Therefore the identity map from (B, σ_{ind}) onto $(B, \|\cdot\|_{\lambda_1, \dots, \lambda_d; s})$ is continuous. \square

Corollary 1.13 *Suppose the operators A_1, \dots, A_d are Hermitian or skew-Hermitian. Let $\lambda_1, \dots, \lambda_d \geq 0$. Let B be a subset of $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$. Then:*

- I. B is bounded in $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ if and only if there exists $t > 0$ such that B is a bounded subset of $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ and $(B, \sigma_{\text{ind}}) = (B, \|\cdot\|_{\lambda_1, \dots, \lambda_d; t})$.
- II. B is compact in $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ if and only if B is a compact subset of $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ for some $t > 0$.
- III. B is sequentially compact in $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ if and only if B is a sequentially compact subset of $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ for some $t > 0$.
- IV. B is sequentially compact in $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ if and only if B is compact in $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$.

The joint C^∞ -domain $D^\infty(A_1, \dots, A_d)$ and the joint ordered C^∞ -domain $D_{\text{ord}}^\infty(A_1, \dots, A_d)$ are metrizable locally convex spaces. They may fail to be Fréchet spaces. Also, not every Gevrey space $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ is complete. We give an example.

Example 1.14 *Let $\mu \geq 0$. Then there exists a symmetric operator A in a Hilbert space H such that $S_0(A)$ is dense in H , the space $S_\lambda(A)$ is complete for all $\lambda \in [0, \mu]$ and for all $\lambda > \mu$ the space $S_\lambda(A)$ is not sequentially complete. In particular, $D^\infty(A)$ is not complete.*

Proof. Let A be the restriction of the operator Q to the space $S_\mu(Q)$. Then A is symmetric. For all $\lambda \in [0, \mu]$ we have $S_\lambda(Q) \subset S_\mu(Q)$, so $S_\lambda(A) = S_\lambda(Q)$. Hence $S_0(A)$ is dense in H . Moreover, by Theorem 1.4 and [EG], Theorem I.1.13(i) it follows that $S_\lambda(Q)$ is complete for all $\lambda > 0$, so $S_\lambda(A)$ is complete for all $\lambda \in (0, \mu]$. An elementary proof shows that the space $S_0(A) = S_0(Q)$ is complete. (Cf. infra, Corollary 1.18.)

It remains to prove that for all $\lambda > \mu$ the spaces $S_\lambda(A)$ and $D^\infty(A)$ are not sequentially complete. So let $\lambda > \mu$. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) := \exp(-\frac{1}{2}|x|^{1/\lambda}) \quad (x \in \mathbb{R}).$$

For $N \in \mathbb{N}_0$ let

$$f_N := 1_{(-N,N)} \cdot f$$

We prove that the sequence f_1, f_2, \dots is a Cauchy sequence in $S_\lambda(A)$. Let $g := \sqrt{f}$ and $g_N := \sqrt{f_N}$ for all $N \in \mathbb{N}_0$. Let $N, M \in \mathbb{N}_0$. Then for all $n \in \mathbb{N}_0$:

$$\begin{aligned} \|A^n(f_N - f_M)\| &= \|Q^n g(g_N - g_M)\| \leq \sup_{x \in \mathbb{R}} |x^n g(x)| \|g_N - g_M\| \\ &= (4n\lambda)^{n\lambda} e^{-\lambda n} \|g_N - g_M\| \leq [(4e\lambda)^\lambda e^{-\lambda}]^n n!^\lambda \|g_N - g_M\|. \end{aligned}$$

So $f_N = f_N - f_0 \in S_\lambda(A)$ for all $N \in \mathbb{N}$. Moreover, since $\lim_{N, M \rightarrow \infty} \|g_N - g_M\| = 0$, we obtain that f_1, f_2, \dots is a Cauchy sequence in $S_\lambda(A)$ and also in $D^\infty(A)$.

Suppose the Cauchy sequence f_1, f_2, \dots converges to some element h in $S_\lambda(A)$. Then by Lemma 1.1 we would have in $L^2(\mathbb{R})$: $f = \lim f_N = h \in S_\lambda(A) \subset D^\infty(A) = S_\mu(Q)$. However, for all $n \in \mathbb{N}$ with $2n\lambda \geq 2$ we obtain with $k := [2n\lambda] + 1$:

$$\begin{aligned} \sqrt{2\pi} \|Q^n f\|^2 &= \int_{\mathbb{R}} x^{2n} \exp(-|x|^{1/\lambda}) dx = 2\lambda \int_0^\infty y^{(2n+1)\lambda-1} e^{-y} dy = \\ &2\lambda \Gamma((2n+1)\lambda) \geq 2\lambda \Gamma(k-1) = 2\lambda(k-2)! \geq 2\lambda k^{-2} k^k e^{-k} \geq \\ &2\lambda 4^{-k} e^{-k} (2n\lambda)^{2n\lambda} \geq 2\lambda (4e)^{-(2n\lambda+1)} (2\lambda)^{2n\lambda} n^{2n\lambda} \geq \lambda (2e)^{-1} [(\lambda(2e)^{-1})^\lambda]^{2n} n!^{2\lambda}. \end{aligned}$$

So

$$\|Q^n f\| \geq C t^n n!^\lambda$$

for all $n \in \mathbb{N}$, where $t = (\frac{\lambda}{2e})^\lambda$ and $C > 0$ is some constant. Hence $f \notin S_\mu(Q)$ since $\mu < \lambda$.

By the same argument we can show that the Cauchy sequence f_1, f_2, \dots is not convergent in $D^\infty(A)$. \square

Theorem 1.15 *Suppose the operators A_1, \dots, A_d are closed. Then $D^\infty(A_1, \dots, A_d)$ and $D_{\text{ord}}^\infty(A_1, \dots, A_d)$ are Fréchet spaces.*

Proof. Let $(u_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $D^\infty(A_1, \dots, A_d)$. Then for all $\alpha \in M(V)$ also $(A_\alpha u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in H , so $u_\alpha := \lim_{n \rightarrow \infty} A_\alpha u_n$ exists in H . Let $u := u_{(\cdot)}$. For $N \in \mathbb{N}_0$ hypothesis $P(N)$ states:

$$\text{For all } \alpha \in V^N \text{ we have } u \in D(A_\alpha) \text{ and } u_\alpha = A_\alpha u.$$

Obviously hypothesis $P(0)$ is valid. Let $N \in \mathbb{N}_0$ and suppose hypothesis $P(N)$ is valid. Let $\alpha \in V^N$ and let $v \in V$. Then we obtain in the Hilbert space $H \times H$:

$$\lim_{n \rightarrow \infty} (A_\alpha u_n, A_v A_\alpha u_n) = (u_\alpha, u_{(v,\alpha)}) = (A_\alpha u, u_{(v,\alpha)}).$$

Since A_v is a closed operator, it follows that $A_\alpha u \in D(A_v)$ and $A_v A_\alpha u = u_{(v,\alpha)}$. So $u \in D(A_{(v,\alpha)})$ and $A_{(v,\alpha)} u = u_{(v,\alpha)}$. This proves hypothesis $P(N+1)$.

By induction, $u \in \bigcap_{\alpha \in M(V)} D(A_\alpha) = D^\infty(A_1, \dots, A_d)$ and $\lim_{n \rightarrow \infty} A_\alpha u_n = A_\alpha u$ for all $\alpha \in M(V)$. So $D^\infty(A_1, \dots, A_d)$ is complete.

Similarly, $D_{\text{ord}}^\infty(A_1, \dots, A_d)$ is complete.

The completeness of the spaces $D^\infty(A_1, \dots, A_d)$ and $D_{\text{ord}}^\infty(A_1, \dots, A_d)$ has the following consequences for the Gevrey spaces.

Theorem 1.16 *Let $\lambda_1, \dots, \lambda_d \geq 0$ and let $t > 0$. If $D^\infty(A_1, \dots, A_d)$ is a Fréchet space then $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ is a Banach space. If $D_{\text{ord}}^\infty(A_1, \dots, A_d)$ is a Fréchet space then $S_{\lambda_1, \dots, \lambda_d; t}^{\text{ord}}(A_1, \dots, A_d)$ is a Banach space.*

Proof. Suppose $D^\infty(A_1, \dots, A_d)$ is a Fréchet space. Let $(u_n)_{n \in \mathbb{N}}$ be a Cauchy sequence in $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$. Since the identity map from $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ into $D^\infty(A_1, \dots, A_d)$ is continuous, also $(u_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $D^\infty(A_1, \dots, A_d)$. Hence $u := \lim_{n \rightarrow \infty} u_n$ exists in $D^\infty(A_1, \dots, A_d)$.

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $\|u_k - u_l\|_{\lambda_1, \dots, \lambda_d; t} \leq \varepsilon$ for all $k, l \geq N$. Hence for all $k, l \geq N$ and all $\alpha \in M(V)$ we have

$$\|A_\alpha(u_k - u_l)\| \leq \varepsilon t^{|\alpha|} \|\alpha\|_1^{\lambda_1} \dots \|\alpha\|_d^{\lambda_d}.$$

Let $l \geq N$. Since $\lim_{n \rightarrow \infty} u_n = u$ in $D^\infty(A_1, \dots, A_d)$ we obtain for all $\alpha \in M(V)$:

$$\begin{aligned} \|A_\alpha(u - u_l)\| &= \lim_{k \rightarrow \infty} \|A_\alpha(u_k - u_l)\| = \limsup_{k \rightarrow \infty} \|A_\alpha(u_k - u_l)\| \\ &\leq \varepsilon t^{|\alpha|} \|\alpha\|_1^{\lambda_1} \dots \|\alpha\|_d^{\lambda_d}. \end{aligned}$$

So in particular, $u - u_N \in S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ and

$$u = u_N + (u - u_N) \in S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d).$$

Moreover, $\|u - u_l\|_{\lambda_1, \dots, \lambda_d; t} \leq \varepsilon$ for all $l \geq N$ and $\lim_{l \rightarrow \infty} u_l = u$ in $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$.

Similarly, $S_{\lambda_1, \dots, \lambda_d; t}^{\text{ord}}(A_1, \dots, A_d)$ is complete if $D_{\text{ord}}^\infty(A_1, \dots, A_d)$ is a Fréchet space. \square

Corollary 1.17 *Suppose $D^\infty(A_1, \dots, A_d)$ is a Fréchet space and each of the operators A_1, \dots, A_d is Hermitian or skew-Hermitian. Let $\lambda_1, \dots, \lambda_d \geq 0$. Then $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ is complete.*

Proof. Let B be a bounded closed subset of $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$. By Corollary 1.13.I, there exists $t > 0$ such that $B \subset S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ and $(B, \sigma_{\text{ind}}) = (B, \|\cdot\|_{\lambda_1, \dots, \lambda_d; t})$. Let $(u_i)_{i \in I}$ be a Cauchy net in (B, σ_{ind}) . Then $(u_i)_{i \in I}$ is a Cauchy net in $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$, whence $u := \lim_i u_i$ exists in $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$. Then also $u = \lim_i u_i$ in $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$. Since B is closed, $u \in B$ and $u = \lim_i u_i$ in (B, σ_{ind}) . So $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ is boundedly complete.

By Lemma 1.1, $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ is the inductive limit of the spaces $S_{\lambda_1, \dots, \lambda_d; n}(A_1, \dots, A_d)$ with $n \in \mathbb{N}$. By Theorem 1.16 each $S_{\lambda_1, \dots, \lambda_d; n}(A_1, \dots, A_d)$ is a Banach space. So $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ is a boundedly complete LB space. By Floret, [Flo], Satz 4.3, $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ is complete. \square

Corollary 1.18 *Suppose the operators A_1, \dots, A_d are closed and suppose each of the operators A_1, \dots, A_d is Hermitian or skew-Hermitian. Let $\lambda_1, \dots, \lambda_d \geq 0$. Then $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ is complete.*

Proof. Theorem 1.15 and Corollary 1.17. □

Corollary 1.19 *Let $\lambda_1, \dots, \lambda_d \geq 0$. Suppose $D^\infty(A_1, \dots, A_d)$ is a Fréchet space. Then $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ is barrelled.*

Similarly, $S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(A_1, \dots, A_d)$ is barrelled if $D_{\text{ord}}^\infty(A_1, \dots, A_d)$ is a Fréchet space.

Proof. Suppose $D^\infty(A_1, \dots, A_d)$ is complete. By Theorem 1.16 the normed space $S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d)$ is complete for all $t > 0$, hence barrelled. Now the corollary follows by [Wil], Theorem 13-1-13. □

We finish this section with an examination of continuous linear maps between spaces of type $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ and $S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(A_1, \dots, A_d)$. By the bornologicality of the spaces $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ and $S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(A_1, \dots, A_d)$, a linear map between spaces of type $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ and $S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(A_1, \dots, A_d)$ is continuous if and only if it maps bounded subsets into bounded subsets. Here we present a useful condition which implies that a linear map is continuous.

Lemma 1.20 *Let E be a Fréchet space and suppose the operators A_1, \dots, A_d are closable. Let $\lambda_1, \dots, \lambda_d \geq 0$ and let T be a linear map from E into $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$. Let i denote the identity map from $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ into H . Suppose the map $i \circ T$ is continuous from E into H . Then T is continuous from E into $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$.*

Proof. We first prove that the map $A_\alpha \circ T$ is continuous from E into H for all $\alpha \in M(V)$. The proof is by induction to $\|\alpha\|$. If $\alpha = ()$, then by assumption the map $A_\alpha \circ T = i \circ T$ is continuous from E into H . Now let $\alpha \in M(V)$ and $v \in V$ and suppose the map $A_\alpha \circ T$ is continuous from E into H . Let B be the closure of the operator A_v . Then the map $B \circ A_\alpha \circ T$ from E into H has a closed graph, hence it is continuous by the closed graph theorem. Then in particular the map $A_{(v, \alpha)} \circ T = A_v A_\alpha T$ is continuous from E into H .

For $N \in \mathbb{N}$ let

$$Z_N := \{x \in E : \forall \alpha \in M(V) \left[\|A_\alpha T x\| \leq N^{|\alpha|+1} \|\alpha\|_1^{\lambda_1} \dots \|\alpha\|_d^{\lambda_d} \right]\}.$$

Since all the maps $A_\alpha \circ T$ are continuous from E into H , the set Z_N is closed in E . Clearly $E = \bigcup_{N \in \mathbb{N}} Z_N$. By the Baire category theorem there exists $N \in \mathbb{N}$ such that Z_N has a non-empty interior. There exist $x_0 \in Z_N$ and a neighborhood U of 0 in E such that $x_0 + U \subset Z_N$. Then $\|Tx\|_{\lambda_1, \dots, \lambda_d; N} \leq 2N$ for all $x \in U$. It follows that the map T is bounded from E into $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$. Since the space E is bornological, the map T is continuous from E into $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$. □

Theorem 1.21 *Let H_1 and H_2 be Hilbert spaces. Let $d_1, d_2 \in \mathbb{N}$. Let A_1, \dots, A_{d_1} be closed operators in H_1 and let B_1, \dots, B_{d_2} be closable operators in H_2 . Let $\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2} \geq 0$. Let T be a linear map from $S_{\lambda_1, \dots, \lambda_{d_1}}(A_1, \dots, A_{d_1})$ into $S_{\mu_1, \dots, \mu_{d_2}}(B_1, \dots, B_{d_2})$. Suppose the operator T is closable as an operator from H_1 into H_2 . Then T is continuous. Similar results are valid if $S_{\lambda_1, \dots, \lambda_{d_1}}(A_1, \dots, A_{d_1})$ or $S_{\mu_1, \dots, \mu_{d_2}}(B_1, \dots, B_{d_2})$ is replaced by $S_{\lambda_1, \dots, \lambda_{d_1}}^{\text{ord}}(A_1, \dots, A_{d_1})$ resp. $S_{\mu_1, \dots, \mu_{d_2}}^{\text{ord}}(B_1, \dots, B_{d_2})$.*

Proof. For $t > 0$ let i_t be the identity map from $S_{\lambda_1, \dots, \lambda_{d_1}; t}(A_1, \dots, A_{d_1})$ into $S_{\lambda_1, \dots, \lambda_{d_1}}(A_1, \dots, A_{d_1})$ and let i and j be the identity maps from $S_{\lambda_1, \dots, \lambda_{d_1}}(A_1, \dots, A_{d_1})$ and $S_{\mu_1, \dots, \mu_{d_2}}(B_1, \dots, B_{d_2})$ into H_1 resp. H_2 . Let $t > 0$. Then the map $i \circ i_t$ is continuous from $S_{\lambda_1, \dots, \lambda_{d_1}; t}(A_1, \dots, A_{d_1})$ into H_1 and the map $j \circ T$ from H_1 into H_2 is closable. So the map $j \circ T \circ i_t = (j \circ T) \circ (i \circ i_t)$ is closable from $S_{\lambda_1, \dots, \lambda_{d_1}; t}(A_1, \dots, A_{d_1})$ into H_2 . By Theorems 1.15 and 1.16, the space $S_{\lambda_1, \dots, \lambda_{d_1}; t}(A_1, \dots, A_{d_1})$ is a Banach space. Therefore we obtain by the closed graph theorem that the map $j \circ T \circ i_t$ from $S_{\lambda_1, \dots, \lambda_{d_1}; t}(A_1, \dots, A_{d_1})$ into H_2 has a continuous extension. Hence the map $j \circ T \circ i_t$ is continuous from $S_{\lambda_1, \dots, \lambda_{d_1}; t}(A_1, \dots, A_{d_1})$ into H_2 .

From Lemma 1.20 it follows that the map $T \circ i_t$ is continuous from $S_{\lambda_1, \dots, \lambda_{d_1}; t}(A_1, \dots, A_{d_1})$ into $S_{\mu_1, \dots, \mu_{d_2}}(B_1, \dots, B_{d_2})$. Now the theorem follows by [Wil], Theorem 13-1-8. \square

Corollary 1.22 *Let $d_1, d_2 \in \mathbb{N}$. Let $A_1, \dots, A_{d_1}, B_1, \dots, B_{d_2}$ be closed operators in a Hilbert space H . Let $\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2} \geq 0$. Suppose $S_{\lambda_1, \dots, \lambda_{d_1}}(A_1, \dots, A_{d_1}) = S_{\mu_1, \dots, \mu_{d_2}}(B_1, \dots, B_{d_2})$ as sets. Then $S_{\lambda_1, \dots, \lambda_{d_1}}(A_1, \dots, A_{d_1}) = S_{\mu_1, \dots, \mu_{d_2}}(B_1, \dots, B_{d_2})$ as locally convex spaces with equivalent spectra.*

Proof. Let i be the identity map from $S_{\lambda_1, \dots, \lambda_{d_1}}(A_1, \dots, A_{d_1})$ into $S_{\mu_1, \dots, \mu_{d_2}}(B_1, \dots, B_{d_2})$. Then i is closable as an operator from H into H . Hence i is continuous by Theorem 1.21. Now let $t > 0$. Let

$$B := \{u \in S_{\lambda_1, \dots, \lambda_{d_1}}(A_1, \dots, A_{d_1}) : \|u\|_{A_1, \dots, A_{d_1}; \lambda_1, \dots, \lambda_{d_1}; t} \leq 1\}.$$

Then B is bounded in $S_{\lambda_1, \dots, \lambda_{d_1}}(A_1, \dots, A_{d_1})$, hence B is bounded in $S_{\mu_1, \dots, \mu_{d_2}}(B_1, \dots, B_{d_2})$, since i is continuous. By Theorem 1.11 there exist $s > 0$ and $M > 0$ such that

$$\|u\|_{B_1, \dots, B_{d_2}; \mu_1, \dots, \mu_{d_2}; s} \leq M$$

for all $u \in B$. So the space $S_{\lambda_1, \dots, \lambda_{d_1}; t}(A_1, \dots, A_{d_1})$ is continuously embedded in $S_{\mu_1, \dots, \mu_{d_2}; s}(B_1, \dots, B_{d_2})$.

Similarly, for all $t > 0$ there exists $s > 0$ such that $S_{\mu_1, \dots, \mu_{d_2}; t}(B_1, \dots, B_{d_2})$ is continuously embedded in $S_{\lambda_1, \dots, \lambda_{d_1}; s}(A_1, \dots, A_{d_1})$. The corollary follows. \square

1.3 Gevrey spaces relative to infinitesimal generators

In this section we introduce a rich class of examples of Gevrey spaces $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ relative to operators A_1, \dots, A_d such that $D^\infty(A_1, \dots, A_d)$ is a Fréchet space, all operators

A_1, \dots, A_d are skew-adjoint and $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ is dense in H for all $\lambda_1, \dots, \lambda_d \geq 1$. Further we present another characterization of these spaces which explains the terminology *Gevrey space*.

Let G be a real Lie group and let π be a (continuous unitary) representation of G in a Hilbert space H . (We only consider continuous unitary representations.) For the terminology of Lie group theory we refer to Helgason, [Hel] and Varadarajan, [Var1]. For every $u \in H$ define $\tilde{u} : G \rightarrow H$ by

$$\tilde{u}(g) := \pi_g u \quad (g \in G).$$

A vector $u \in H$ is called *infinitely differentiable* respectively *analytic* for π if and only if the function \tilde{u} is infinitely differentiable respectively (real) analytic from G into H . Let $H^\infty(\pi)$ resp. $H^\omega(\pi)$ denote the set of all infinitely differentiable resp. analytic vectors for π . It has been proved by Gårding, [Går1] that $H^\infty(\pi)$ is dense in H and by Cartier and Dixmier, [CD], Nelson, [Nel] and again Gårding, [Går2] that $H^\omega(\pi)$ is dense in H .

Let \mathfrak{g} be the Lie algebra of G . For $X \in \mathfrak{g}$ denote by $d\pi(X)$ the *infinitesimal generator* of the one-parameter unitary group $t \mapsto \pi_{\exp(tX)}$. So $d\pi(X)$ is skew-adjoint. Goodman has proved the following infinitesimal characterization of $H^\infty(\pi)$:

Theorem 1.23 *Let X_1, \dots, X_d be any basis in the Lie algebra \mathfrak{g} . Then*

$$H^\infty(\pi) = D^\infty(d\pi(X_1), \dots, d\pi(X_d)) = \bigcap_{k=1}^d D^\infty(d\pi(X_k))$$

as sets.

Proof. By [Goo4], Proposition 1.1, $H^\infty(\pi) = D_{\text{ord}}^\infty(d\pi(Y_1), \dots, d\pi(Y_d))$ for any basis Y_1, \dots, Y_d in \mathfrak{g} . So $H^\infty(\pi) = D^\infty(d\pi(X_1), \dots, d\pi(X_d))$. The remaining part of the theorem is proved in [Goo2], Theorem 1.1. \square

By Theorem 1.23 the space $H^\infty(\pi)$ is invariant under $d\pi(X)$ for all $X \in \mathfrak{g}$. Let $\partial\pi(X)$ denote the *restriction* of the operator $d\pi(X)$ to $H^\infty(\pi)$. Then the map $\partial\pi : X \mapsto \partial\pi(X)$ is a Lie algebra homomorphism from \mathfrak{g} into the set of all skew-symmetric operators defined on $H^\infty(\pi)$. (This can be proved similarly to a proof on page 209 of Harish-Chandra, [HC].) Hence $\partial\pi$ extends uniquely to an associative algebra homomorphism, denoted by $\partial\pi$ also, from the *complex universal enveloping algebra* $U(\mathfrak{g})$ of \mathfrak{g} into the set of all linear operators from $H^\infty(\pi)$ into $H^\infty(\pi)$. The following theorem is of special interest.

Theorem 1.24 *Let X_1, \dots, X_d be any basis in \mathfrak{g} . Let $\Delta := \sum_{k=1}^d X_k^2 \in U(\mathfrak{g})$. Then the operator $\partial\pi(I - \Delta)$ is essentially self-adjoint. The spaces $H^\infty(\pi)$ and $D^\infty(\overline{\partial\pi(I - \Delta)})$ are equal as sets. Moreover, the spaces $D^\infty(\overline{\partial\pi(I - \Delta)})$ and $D^\infty(d\pi(X_1), \dots, d\pi(X_d))$ are equal as locally convex spaces.*

For all $\lambda \geq 1$ we have

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = S_{2\lambda}(\overline{\partial\pi(I - \Delta)}) = S_{2\lambda}(\partial\pi(I - \Delta))$$

as locally convex spaces with equivalent spectra.

Let Y_1, \dots, Y_d be a second basis in \mathfrak{g} . Let $\lambda \geq 0$. Then

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = S_{\lambda, \dots, \lambda}(d\pi(Y_1), \dots, d\pi(Y_d))$$

as locally convex spaces with equivalent spectra.

Proof. The essential self-adjointness of the operator $\partial\pi(I - \Delta)$ has been proved by Nelson in Theorem 3 of [Nel]. From the proof of the same theorem it follows that $H^\infty(\pi) = D^\infty(\overline{\partial\pi(I - \Delta)})$ as sets. By [Goo4], Proposition 1.3 the spaces $D^\infty(\overline{\partial\pi(I - \Delta)})$ and $D^\infty(d\pi(X_1), \dots, d\pi(X_d))$ are equal as locally convex spaces. The spaces $S_{\lambda, \dots, \lambda}(\partial\pi(X_1), \dots, \partial\pi(X_d))$ and $S_{2\lambda}(\partial\pi(I - \Delta))$ are equal as locally convex spaces with equivalent spectra. (See Goodman-Wallach, [GW], Example following Theorem 1.7.) Since

$$D^\infty(d\pi(X_1), \dots, d\pi(X_d)) = D^\infty(\overline{\partial\pi(I - \Delta)}) = D^\infty(\partial\pi(I - \Delta)),$$

the spaces $S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d))$, $S_{2\lambda}(\overline{\partial\pi(I - \Delta)})$ and $S_{2\lambda}(\partial\pi(I - \Delta))$ are equal as locally convex spaces with equivalent spectra.

Let Y_1, \dots, Y_d be a second basis in \mathfrak{g} . For all $k \in \{1, \dots, d\}$ there exists $c_{k1}, \dots, c_{kd} \in \mathbb{R}$ such that $Y_k = \sum_{m=1}^d c_{km} X_m$. Let $M := \max(\{|c_{km}| : k, l \in \{1, \dots, d\}\})$. Let $n \in \mathbb{N}$ and let $j_1, \dots, j_n \in \{1, \dots, d\}$. Then $\partial\pi(Y_{j_1}) \circ \dots \circ \partial\pi(Y_{j_n})$ is a sum of d^n terms of the form $\gamma \partial\pi(X_{k_1}) \circ \dots \circ \partial\pi(X_{k_n})$, where $\gamma \in \mathbb{R}$, $|\gamma| \leq M^n$ and $k_1, \dots, k_n \in \{1, \dots, d\}$. So

$$\|u\|_{\partial\pi(Y_1), \dots, \partial\pi(Y_d); \lambda, \dots, \lambda; dMt2^{\lambda d}} \leq \|u\|_{\partial\pi(X_1), \dots, \partial\pi(X_d); \lambda, \dots, \lambda; t}$$

for all $u \in H^\infty(\pi)$ and $t > 0$. (See Lemma 1.3.III.) Therefore the embedding

$$S_{\lambda, \dots, \lambda; t}(\partial\pi(X_1), \dots, \partial\pi(X_d)) \hookrightarrow S_{\lambda, \dots, \lambda; dMt2^{\lambda d}}(\partial\pi(Y_1), \dots, \partial\pi(Y_d))$$

is continuous for all $t > 0$. Interchanging the roles of X_k and Y_m , it follows that

$$S_{\lambda, \dots, \lambda}(\partial\pi(Y_1), \dots, \partial\pi(Y_d)) = S_{\lambda, \dots, \lambda}(\partial\pi(X_1), \dots, \partial\pi(X_d))$$

as locally convex spaces with equivalent spectra.

Since $H^\infty(\pi) = D^\infty(d\pi(X_1), \dots, d\pi(X_d)) = D^\infty(d\pi(Y_1), \dots, d\pi(Y_d))$ as sets (Theorem 1.23), it follows that $S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = S_{\lambda, \dots, \lambda}(d\pi(Y_1), \dots, d\pi(Y_d))$ as locally convex spaces with equivalent spectra. \square

Also $H^\omega(\pi)$ admits an infinitesimal characterization.

Theorem 1.25 *Let X_1, \dots, X_d be any basis in \mathfrak{g} . Then*

$$H^\omega(\pi) = S_{1, \dots, 1}(d\pi(X_1), \dots, d\pi(X_d)) = S_{1, \dots, 1}^{\text{ord}}(d\pi(X_1), \dots, d\pi(X_d))$$

as sets. Moreover,

$$S_{1, \dots, 1}(d\pi(X_1), \dots, d\pi(X_d)) = S_{1, \dots, 1}^{\text{ord}}(d\pi(X_1), \dots, d\pi(X_d))$$

as locally convex spaces with equivalent spectra.

Proof. The first equality has been proved by Nelson, [Nel], Lemma 7.1. The other equalities can be proved with similar arguments as in the proof of [Nel], Lemma 7.1. \square

Corollary 1.26 *Let $Y_1, \dots, Y_m \in \mathfrak{g}$ and suppose $\mathfrak{g} = \text{span}(\{Y_1, \dots, Y_m\})$. Then*

$$H^\infty(\pi) = D^\infty(d\pi(Y_1), \dots, d\pi(Y_m)) = \bigcap_{k=1}^m D^\infty(d\pi(Y_k))$$

as sets and

$$D^\infty(d\pi(Y_1), \dots, d\pi(Y_m)) = D_{\text{ord}}^\infty(d\pi(Y_1), \dots, d\pi(Y_m))$$

as locally convex spaces and both spaces are Fréchet spaces. Let $\lambda_1, \dots, \lambda_m \geq 0$ and let $t > 0$. Then

$$S_{\lambda_1, \dots, \lambda_m; t}(d\pi(Y_1), \dots, d\pi(Y_m)),$$

$$S_{\lambda_1, \dots, \lambda_m; t}^{\text{ord}}(d\pi(Y_1), \dots, d\pi(Y_m))$$

and

$$S_{\lambda_1, \dots, \lambda_m}(d\pi(Y_1), \dots, d\pi(Y_m))$$

are complete. Moreover, if in addition $\lambda_1, \dots, \lambda_m \geq 1$, then

$$S_{\lambda_1, \dots, \lambda_m}(d\pi(Y_1), \dots, d\pi(Y_m))$$

is dense in H .

Proof. There exist $1 \leq i_1 < i_2 < \dots < i_d \leq m$ such that Y_{i_1}, \dots, Y_{i_d} is a basis in \mathfrak{g} . Let $X_k := Y_{i_k}$ for all $k \in \{1, \dots, d\}$. By Theorem 1.23, $H^\infty(\pi)$ is invariant under $d\pi(Y_j)$ for all $j \in \{1, \dots, m\}$. So

$$\begin{aligned} H^\infty(\pi) &\subset D^\infty(d\pi(Y_1), \dots, d\pi(Y_m)) \subset D_{\text{ord}}^\infty(d\pi(Y_1), \dots, d\pi(Y_m)) \subset \\ &\subset \bigcap_{k=1}^m D^\infty(d\pi(Y_k)) \subset \bigcap_{k=1}^d D^\infty(d\pi(X_k)) = H^\infty(\pi). \end{aligned}$$

Therefore $D^\infty(d\pi(Y_1), \dots, d\pi(Y_m)) = D_{\text{ord}}^\infty(d\pi(Y_1), \dots, d\pi(Y_m)) = \bigcap_{k=1}^m D^\infty(d\pi(Y_k))$ as sets. Since the operators $d\pi(Y_1), \dots, d\pi(Y_m)$ are skew-adjoint, it follows from Theorem 1.15 that the spaces $D^\infty(d\pi(Y_1), \dots, d\pi(Y_m))$ and $D_{\text{ord}}^\infty(d\pi(Y_1), \dots, d\pi(Y_m))$ are Fréchet spaces. By the closed graph theorem, $D^\infty(d\pi(Y_1), \dots, d\pi(Y_m))$ is equal to $D_{\text{ord}}^\infty(d\pi(Y_1), \dots, d\pi(Y_m))$ as locally convex spaces.

The completeness of the three spaces follows from Theorem 1.16 and Corollary 1.17.

With elementary counting arguments it can be shown that

$$S_{1, \dots, 1}(d\pi(Y_1), \dots, d\pi(Y_m)) = S_{1, \dots, 1}(d\pi(X_1), \dots, d\pi(X_d))$$

as sets. (Actually equality as topological vector spaces holds. See Corollary 1.22.) So

$$H^\omega(\pi) \subset S_{\lambda_1, \dots, \lambda_m}(d\pi(Y_1), \dots, d\pi(Y_m))$$

when $\lambda_1, \dots, \lambda_m \geq 1$. Since $H^\omega(\pi)$ is dense in H , the corollary follows. \square

Nelson has proved a characterization for the analyticity space $S_{1, \dots, 1}(d\pi(X_1), \dots, d\pi(X_d))$, namely:

$$S_{1, \dots, 1}(d\pi(X_1), \dots, d\pi(X_d)) = H^\omega(\pi) := \{u \in H : \tilde{u} \text{ is an analytic map}\}.$$

(See Theorem 1.25.) A characterization of the same type exists for the Gevrey space $S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d))$ in case $\lambda > 1$.

Let Ω be an open subset of \mathbb{R}^d , let $\lambda \geq 1$ and let K be a compact subset of Ω . Let the space $D_\lambda(K)$ consists of restrictions to K of all infinitely differentiable functions φ on Ω for which there exist constants $C, t > 0$ such that

$$\forall_{n \in \mathbb{N}_0} \forall_{i_1, \dots, i_n \in \{1, \dots, d\}} \forall_{x \in K} \left[|(D_{i_1} \circ \dots \circ D_{i_n} \varphi)(x)| \leq C t^n n!^\lambda \right].$$

Here D_i denotes the partial differentiation with respect to the i -th coordinate. In 1918, Gevrey introduced the space $\mathcal{E}_\lambda(\Omega)$ of all infinitely differentiable functions φ on Ω such that for every compact subset K of Ω , the restriction of φ to K is an element of $D_\lambda(K)$. (See [Gev].)

This idea of Gevrey leads to the following definitions. Let G be a Lie group and let H be a Hilbert space. (To start with, we only need that G is a real analytic manifold.) Let $\lambda \geq 1$. Let (U, x) be a chart on G , let K be a non-empty compact subset of U and let $t > 0$. The normed space $G_{\lambda, t}(H, K, U, x)$ denotes the space of restrictions to K of all infinitely differentiable functions φ from G into H such that

$$\|\varphi|_K\|_{\lambda, t, U, x} := \sup_{n \in \mathbb{N}_0} \sup_{i_1, \dots, i_n \in \{1, \dots, d\}} \sup_{p \in K} \frac{\| [D_{i_1} \circ \dots \circ D_{i_n} (\varphi \circ x^{-1})](x(p)) \|}{t^n n!^\lambda} < \infty.$$

Here D_i denotes the partial differentiation with respect to the i -th coordinate. Let

$$G_\lambda(H, K, U, x) := \bigcup_{t > 0} G_{\lambda, t}(H, K, U, x).$$

The topology for $G_\lambda(H, K, U, x)$ is the inductive limit topology generated by the normed spaces $G_{\lambda, t}(H, K, U, x)$ with $t > 0$. Let (V, y) be another chart on G such that $K \subset V$. Then we can deduce from [Nel], Theorem 2 and [GW], Theorem 1.1, that $G_\lambda(H, K, U, x) = G_\lambda(H, K, V, y)$ as locally convex spaces with equivalent spectra. Let $G_\lambda(H)$ be the space of all infinitely differentiable functions φ from G into H such that for every chart (U, x) on G and every non-empty compact subset K of U , the restriction of φ to K is an element of $G_\lambda(H, K, U, x)$. The topology for $G_\lambda(H)$ is the projective limit topology generated by the spaces $G_\lambda(H, K, U, x)$.

Now let Y_1, \dots, Y_d be analytic vector fields on G which are linearly independent at each point of G . Let K be a non-empty compact subset of G . Let $t > 0$. The normed space

$G_{\lambda,t}(H, K, Y_1, \dots, Y_d)$ denotes the space of restrictions to K of all infinitely differentiable functions φ from G into H such that

$$\|\varphi|_K\|_{\lambda,t,Y_1,\dots,Y_d} := \sup_{n \in \mathbb{N}_0} \sup_{i_1, \dots, i_n \in \{1, \dots, d\}} \sup_{p \in K} \frac{\|(Y_{i_1} \circ \dots \circ Y_{i_n} \varphi)(p)\|}{t^n n!^\lambda} < \infty.$$

Let

$$G_\lambda(H, K, Y_1, \dots, Y_d) := \bigcup_{t>0} G_{\lambda,t}(H, K, Y_1, \dots, Y_d).$$

We endow $G_\lambda(H, K, Y_1, \dots, Y_d)$ with the inductive limit topology. In case (U, x) is a chart on G such that K is a subset of U , it follows again from [Nel], Theorem 2 and [GW], Theorem 1.1 that $G_\lambda(H, K, Y_1, \dots, Y_d) = G_\lambda(H, K, U, x)$ as locally convex spaces with equivalent spectra. Let $G_\lambda(H, Y_1, \dots, Y_d)$ be the space of all infinitely differentiable functions φ from G into H such that for every non-empty compact subset K of G , the restriction of φ to K is an element of $G_\lambda(H, K, Y_1, \dots, Y_d)$. The topology for $G_\lambda(H, Y_1, \dots, Y_d)$ is the projective limit topology generated by the spaces $G_\lambda(H, K, Y_1, \dots, Y_d)$ with K a non-empty compact subset of G . Since for all compact subsets K of G there exist finitely many charts (U_i, x_i) on G and compact subsets $K_i \subset U_i$, $i \in \{1, \dots, n\}$, such that $K = \bigcup_{i=1}^n K_i$, it follows easily that the spaces $G_\lambda(H)$ and $G_\lambda(H, Y_1, \dots, Y_d)$ are equal as locally convex spaces. Let also Z_1, \dots, Z_d be analytic vector fields on G which are linearly independent at each point of G . Let K be a non-empty compact subset of G . Then it follows similarly that $G_\lambda(H, K, Y_1, \dots, Y_d) = G_\lambda(H, K, Z_1, \dots, Z_d)$ as locally convex spaces with equivalent spectra and that $G_\lambda(H, Y_1, \dots, Y_d) = G_\lambda(H, Z_1, \dots, Z_d)$ as locally convex topological vector spaces.

Let π be a representation of G in a Hilbert space H and let $\lambda \geq 1$. Let (U, x) be a chart on G , let K be a non-empty compact subset of U and let $t > 0$. Define the normed space

$$H_{\lambda,t}(\pi, K, U, x) := \{u \in H^\infty(\pi) : \tilde{u} \in G_{\lambda,t}(H, K, U, x)\}$$

with norm

$$\|u\|_{\lambda,t,\pi,K,U,x} := \|\tilde{u}|_K\|_{\lambda,t,U,x} \quad (u \in H_{\lambda,t}(\pi, K, U, x)).$$

Define

$$H_\lambda(\pi, K, U, x) := \bigcup_{t>0} H_{\lambda,t}(\pi, K, U, x),$$

$$H_\lambda(\pi) := \bigcap_{K,(U,x)} H_\lambda(\pi, K, U, x),$$

$H_{\lambda,t}(\pi, K, Y_1, \dots, Y_d)$, $H_\lambda(\pi, K, Y_1, \dots, Y_d)$ and $H_\lambda(\pi, Y_1, \dots, Y_d)$, with their natural topologies. The elements of $H_\lambda(\pi)$ are called *Gevrey vectors of order λ for π* . Observe that

$$H_1(\pi) = H^\omega(\pi)$$

as sets.

Let \mathfrak{g} be the Lie algebra of G . Let $X \in \mathfrak{g}$. Let \widetilde{X} be the corresponding left invariant real analytic vector field on G . So

$$(\widetilde{X}f)(g) = \left. \frac{d}{dt} \right|_0 f(g \exp tX)$$

for every infinitely differentiable function f on G and all $g \in G$. Now we present a global description of Gevrey vectors.

Theorem 1.27 *Let π be a representation of a Lie group G in a Hilbert space H . Let X_1, \dots, X_d be a basis in the Lie algebra \mathfrak{g} of G . Let $\lambda \geq 1$ and let K be a non-empty compact subset of G . Then*

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = H_\lambda(\pi, K, \widetilde{X}_1, \dots, \widetilde{X}_d)$$

as locally convex spaces with equivalent spectra. Moreover,

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = H_\lambda(\pi)$$

as locally convex topological vector spaces.

Proof. Let $u \in H^\infty(\pi) = D^\infty(d\pi(X_1), \dots, d\pi(X_d))$. According to the proof of Proposition 1.1 in [Goo4], $(\widetilde{X}\tilde{u})(g) = (d\pi(X)u)^\sim(g)$ for all $X \in \mathfrak{g}$ and $g \in G$. Let $n \in \mathbb{N}_0$ and let $i_1, \dots, i_n \in \{1, \dots, d\}$. Then for all $p \in K$:

$$\begin{aligned} \|(\widetilde{X}_{i_1} \circ \dots \circ \widetilde{X}_{i_n} \tilde{u})(p)\| &= \|(d\pi(X_{i_1}) \circ \dots \circ d\pi(X_{i_n})u)^\sim(p)\| \\ &= \|\pi_p d\pi(X_{i_1}) \circ \dots \circ d\pi(X_{i_n})u\| \\ &= \|d\pi(X_{i_1}) \circ \dots \circ d\pi(X_{i_n})u\|. \end{aligned}$$

Hence the equality of the spaces $S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d))$ and $H_\lambda(\pi, K, \widetilde{X}_1, \dots, \widetilde{X}_d)$ as locally convex spaces with equivalent spectra follows easily.

The remaining part of the theorem is trivial now. \square

Let $u, v \in H$. Define the function $(\tilde{u}, v) : G \rightarrow \mathbb{C}$ by

$$[(\tilde{u}, v)](g) := (\tilde{u}(g), v) \quad (g \in G).$$

Poulsen ([Pou]) presented the following weak description of infinitely differentiable vectors for the representation π :

Theorem 1.28 *Let π be a representation of a Lie group G in a Hilbert space H . Let $u \in H$. Then u is an infinitely differentiable vector for π if and only if for all $v \in H$ the function (\tilde{u}, v) is infinitely differentiable from G into \mathbb{C} .*

Proof. Cf. [Pou], Lemma 1.2 and Proposition 1.1. \square

Also there exists a weak description of analytic vectors for π .

Theorem 1.29 *Let π be a representation of a Lie group G in a Hilbert space H . Let $u \in H$. Then u is an analytic vector for π if and only if for all $v \in H$ the function (\tilde{u}, v) is analytic from G into \mathbb{C} .*

Proof. See [Var2], page 303. □

In the same way as in [Var2], page 303 a weak characterization of Gevrey vectors for π can be given: let $\lambda \geq 1$ and let $u \in H$. Then $u \in H_\lambda(\pi)$ if and only if for all $v \in H$ the function (\tilde{u}, v) is an element of $G_\lambda(\mathbb{C})$.

In the remaining part of this section we introduce the locally convex *topological* vector space of weak Gevrey vectors of order $\lambda \geq 1$ for the representation π . We present a characterization of its bounded subsets.

Let π be a representation of a Lie group G in a Hilbert space H and let $\lambda \geq 1$. Let (U, x) be a chart on G , let K be a non-empty compact subset of U , let $v \in H$ and let $t > 0$. Define the normed space

$$H_{\lambda,t}(\pi, K, U, x; v) := \{u \in H : (\tilde{u}, v)|_K \in G_{\lambda,t}(\mathbb{C}, K, U, x)\}$$

with norm

$$\|u\|_{\lambda,t,\pi,K,U,x;v} := \|(\tilde{u}, v)|_K\|_{\lambda,t,U,x} \quad (u \in H_{\lambda,t}(\pi, K, U, x; v)).$$

Let

$$H_\lambda(\pi, K, U, x; v) := \bigcup_{t>0} H_{\lambda,t}(\pi, K, U, x; v).$$

The topology for $H_\lambda(\pi, K, U, x; v)$ is the inductive limit topology generated by the normed spaces $H_{\lambda,t}(\pi, K, U, x; v)$ with $t > 0$. Define the space of *weak Gevrey vectors of order λ for π* by

$$H_\lambda^\pi(\pi) := \bigcap H_\lambda(\pi, K, U, x; v)$$

where the intersection is over all charts (U, x) on G , all non-empty compact subsets K of U and all $v \in H$. The topology for $H_\lambda^\pi(\pi)$ is the corresponding projective limit topology. For analytic vector fields Y_1, \dots, Y_d on G define similarly the spaces $H_{\lambda,t}(\pi, K, Y_1, \dots, Y_d; v)$, $H_\lambda(\pi, K, Y_1, \dots, Y_d; v)$ and $H_\lambda^\pi(\pi, Y_1, \dots, Y_d)$, with their natural topologies. Obviously the spaces $H_\lambda(\pi, K, U, x; v)$ and $H_\lambda(\pi, K, Y_1, \dots, Y_d; v)$ are equal as locally convex spaces with equivalent spectra if both spaces are properly defined.

Theorem 1.30 *Let π be a representation of a Lie group G in a Hilbert space H . Let $\lambda \geq 1$. Then, as sets, the space of Gevrey vectors of order λ for π is equal to the space of weak Gevrey vectors of order λ for π . Moreover, let B be a subset of $H_\lambda(\pi)$. Then B is bounded in $H_\lambda(\pi)$ if and only if B is bounded in $H_\lambda^\pi(\pi)$.*

Proof. Let (U, x) be a chart on G , let K be a compact subset of U , let $v \in H$ and let $t > 0$. Since every infinitely differentiable function from G into H is weakly differentiable, and since

$$\|u\|_{\lambda;t,\pi,K,U,x;v} \leq \|u\|_{\lambda;t,\pi,K,U,x} \|v\| < \infty$$

for all $u \in H_{\lambda;t}(\pi, K, U, x)$, we obtain that $H_{\lambda;t}(\pi, K, U, x) \subset H_{\lambda;t}(\pi, K, U, x; v)$ where the embedding is continuous. Thus $H_\lambda(\pi)$ is continuously embedded in $H_\lambda^\#(\pi)$. In particular, every bounded subset in $H_\lambda(\pi)$ is a bounded subset in $H_\lambda^\#(\pi)$.

Now let B be a bounded subset of $H_\lambda^\#(\pi)$. For all $u \in B$ and $v \in H$ the function (\tilde{u}, v) is infinitely differentiable, so $u \in H^\infty(\pi)$ by Theorem 1.28. Let X_1, \dots, X_d be a basis in the Lie algebra \mathfrak{g} of G . Then $B \subset D^\infty(d\pi(X_1), \dots, d\pi(X_d))$. (See Theorem 1.23.) So $\tilde{X}(\tilde{u}, v) = ([d\pi(X)u]^\sim, v)$ for all $u \in B$ and $v \in H$. Let e be the identity in G and let $K := \{e\}$. Since $H_\lambda^\#(\pi) = H_\lambda^\#(\pi, \tilde{X}_1, \dots, \tilde{X}_d)$ as locally convex spaces, the set B is bounded in $H_\lambda(\pi, K, \tilde{X}_1, \dots, \tilde{X}_d; v)$ for all $v \in H$. It follows similarly to the proof of Theorem 1.11 that there exists $t > 0$ (depending a priori on v) such that $B \subset H_{\lambda;t}(\pi, K, \tilde{X}_1, \dots, \tilde{X}_d; v)$ and B is bounded therein. So there exists $M > 0$ (depending on v) such that for all $n \in \mathbb{N}_0$, $i_1, \dots, i_n \in \{1, \dots, d\}$ and $u \in B$ we have

$$|(d\pi(X_{i_1}) \circ \dots \circ d\pi(X_{i_n})u, v)| = |[\tilde{X}_{i_1} \circ \dots \circ \tilde{X}_{i_n}(\tilde{u}, v)](e)| \leq Mt^n n!^\lambda.$$

For $N \in \mathbb{N}$ let

$$E_N := \left\{ v \in H : \forall_{n \in \mathbb{N}_0} \forall_{i_1, \dots, i_n \in \{1, \dots, d\}} \forall_{u \in B} \left[|(d\pi(X_{i_1}) \circ \dots \circ d\pi(X_{i_n})u, v)| \leq N^{n+1} n!^\lambda \right] \right\}.$$

Then E_N is closed in H and $H = \bigcup_{N=1}^\infty E_N$. By the Baire category theorem there exist $N \in \mathbb{N}$, $v_0 \in H$ and $\varepsilon > 0$ such that $\{v \in H : \|v - v_0\| \leq \varepsilon\} \subset E_N$. Then for all $n \in \mathbb{N}_0$, $i_1, \dots, i_n \in \{1, \dots, d\}$ and $u \in B$ we obtain that

$$\|d\pi(X_{i_1}) \circ \dots \circ d\pi(X_{i_n})u\| \leq \frac{2}{\varepsilon} NN^n n!^\lambda.$$

By Lemma 1.3.III, B is a bounded subset of $S_{\lambda, \dots, \lambda, 2^\lambda d N}(d\pi(X_1), \dots, d\pi(X_d))$. So B is a bounded subset of $H_\lambda(\pi)$ by Theorem 1.27. \square

Let X_1, \dots, X_d be a basis in \mathfrak{g} and let $\psi : \mathbb{R}^d \rightarrow G$ be defined by

$$\psi(t_1, \dots, t_d) := \exp(t_1 X_1 + \dots + t_d X_d) \quad (t_1, \dots, t_d \in \mathbb{R}).$$

Let $u \in H$. By definition, u is infinitely differentiable for π , i.e. $u \in H^\infty(\pi)$, if and only if the map \tilde{u} is infinitely differentiable from G into H . Since the left translations on G are analytic maps, we obtain that u is infinitely differentiable if and only if $\tilde{u} \circ \psi$ is infinitely differentiable in a neighborhood of $0 \in \mathbb{R}^d$. By the identity

$$H^\infty(\pi) = \bigcap_{k=1}^d D^\infty(d\pi(X_k)),$$

$u \in H^\infty(\pi)$ if and only if the map $t \mapsto \tilde{u} \circ \psi(te_k)$ is infinitely differentiable from \mathbb{R} into H for all $k \in \{1, \dots, d\}$, where e_k is the k -th standard basis vector in \mathbb{R}^d . So $\tilde{u} \circ \psi$ is infinitely

differentiable in a neighborhood of $0 \in \mathbb{R}^d$ if and only if $\tilde{u} \circ \psi$ is infinitely differentiable in each coordinate.

Hartogs' theorem states that every complex valued function f defined on an open subset U of \mathbb{C}^d is analytic if and only if it is separately analytic, i.e. the function $t \mapsto f(z + te_k)$ is analytic in a neighborhood of 0 for each $z \in U$ and each $k \in \{1, \dots, d\}$. (See Hörmander, [Hör], Theorem 2.2.8.) However, a similar theorem for real analytic functions instead of analytic functions is not valid. Since $u \in S_{1, \dots, 1}(d\pi(X_1), \dots, d\pi(X_d))$ iff $u \in H^\omega(\pi)$ iff the map $\tilde{u} \circ \psi$ is real analytic at 0, and since $\bigcap_{k=1}^d S_1(d\pi(X_k))$ is the space of all $u \in H$ such that $\tilde{u} \circ \psi$ is separately real analytic at 0, it is not clear whether

$$S_{1, \dots, 1}(d\pi(X_1), \dots, d\pi(X_d)) = \bigcap_{k=1}^d S_1(d\pi(X_k)). \quad (1.3)$$

The description of the elements of the set on the right hand side is in general much easier than the description of the elements of the set on the left hand side. It is not even clear whether there exists $n \in \{1, \dots, d-1\}$ such that

$$S_{1, \dots, 1}(d\pi(X_1), \dots, d\pi(X_d)) = S_{1, \dots, 1}(d\pi(X_1), \dots, d\pi(X_n)) \cap S_{1, \dots, 1}(d\pi(X_{n+1}), \dots, d\pi(X_d)). \quad (1.4)$$

General operators and parameters make matters worse: there arises the following question.

Let A_1, \dots, A_d be operators in a Hilbert space H and let $\lambda_1, \dots, \lambda_d \geq 0$. Under which conditions can we prove that:

$$S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d) = \bigcap_{k=1}^d S_{\lambda_k}(A_k) \quad (1.5)$$

or

$$S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d) = S_{\lambda_1, \dots, \lambda_n}(A_1, \dots, A_n) \cap S_{\lambda_{n+1}, \dots, \lambda_d}(A_{n+1}, \dots, A_d) \quad (1.6)$$

for some $n \in \{1, \dots, d-1\}$?

These types of questions have been posed before, only for spaces of analytic vectors. The following results have been derived:

- If $\text{span}(\{X_1, \dots, X_n\})$ is a Lie subalgebra of \mathfrak{g} and $\text{span}(\{X_{n+1}, \dots, X_d\})$ is a Lie ideal in \mathfrak{g} , then Goodman has proved equality (1.4). (See [Goo2], Theorem 3.1.)
- If $\mathcal{L}_i := \text{span}(\{X_1, \dots, X_i\})$ is a subalgebra of \mathfrak{g} and \mathcal{L}_i is an ideal in \mathcal{L}_{i+1} for all $i \in \{1, \dots, d-1\}$, then (1.3) holds. (See [Goo2], Corollary 3.1.)
- If $\text{span}(\{X_1, \dots, X_n\})$ and $\text{span}(\{X_{n+1}, \dots, X_d\})$ are subalgebras of \mathfrak{g} , then (1.4) holds. (See Flato and Simon, [FS], Theorem 2.)

- There exists a basis X_1, \dots, X_d in \mathfrak{g} such that (1.3) holds. (See [FS], Theorem 3.)
- If G is reductive, then (1.3) holds for every basis X_1, \dots, X_d for \mathfrak{g} . (See Simon, [Sim], Proposition 2.)

In this connection, we also mention the following result, valid for the operators P and Q .

- If $\alpha, \beta > 0$ and $\alpha + \beta \geq 1$, then

$$S_{\alpha, \beta}(Q, P) = S_{\alpha}(Q) \cap S_{\beta}(P)$$

as sets. (See Van Eijndhoven, [vE], Theorems 4.5(iii) and 4.3(iii).)

In Chapter 2 of this thesis we present several generalizations of the above results. A simple situation occurs if all operators X_1, \dots, X_d are commuting. We consider this case in the following section.

1.4 Gevrey spaces relative to commuting operators

In this section we consider Gevrey spaces $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ for which some or all operators commute. In the interesting case that all operators A_1, \dots, A_d strongly commute, we give a characterization of $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ in terms of "smoothed" vectors corresponding to a representation of a locally compact Abelian topological group. (See Appendix B.) We start with an intersection result for a collection of operators which "commute in two parts."

Theorem 1.31 *Let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ be operators in a Hilbert space defined on a common invariant domain. Suppose all operators $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ are Hermitian or skew-Hermitian and suppose*

$$X_i Y_j = Y_j X_i$$

for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. Let $\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2} \geq 0$. Then

$$S_{\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}}(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}) = S_{\lambda_1, \dots, \lambda_{d_1}}(X_1, \dots, X_{d_1}) \cap S_{\mu_1, \dots, \mu_{d_2}}(Y_1, \dots, Y_{d_2})$$

as locally convex spaces with equivalent spectra.

Proof. Let $Z_1 := X_1, \dots, Z_{d_1} := X_{d_1}, Z_{d_1+1} := Y_1, \dots, Z_{d_1+d_2} := Y_{d_2}$, let $d := d_1 + d_2, V_1 := \{1, \dots, d_1\}, V_2 := \{1, \dots, d_2\}$ and let $V := \{1, \dots, d\}$. Let

$$M := \max(\{\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}\}).$$

Let $t > 1$. Let $u \in S_{\lambda_1, \dots, \lambda_{d_1}, t}(X_1, \dots, X_{d_1}) \cap S_{\mu_1, \dots, \mu_{d_2}, t}(Y_1, \dots, Y_{d_2})$. Let

$$c_1 := \|u\|_{X_1, \dots, X_{d_1}; \lambda_1, \dots, \lambda_{d_1}; t} \text{ and } c_2 := \|u\|_{Y_1, \dots, Y_{d_2}; \mu_1, \dots, \mu_{d_2}; t}.$$

We shall prove that $u \in S_{\lambda_1, \dots, \mu_2; 2M_t}(Z_1, \dots, Z_d)$ and that u has norm less than or equal to $c_1 + c_2$.

Let $\gamma \in M(V)$. There exist $p \in \mathbb{N}_0, \alpha_1, \dots, \alpha_p \in M(V_1)$ and $\beta_1, \dots, \beta_p \in M(V_2)$ such that $Z_\gamma = X_{\alpha_1} Y_{\beta_1} \circ \dots \circ X_{\alpha_p} Y_{\beta_p}$. Let

$$\alpha := \langle \alpha_p^t, \dots, \alpha_1^t, \alpha_1, \dots, \alpha_p \rangle,$$

$$\beta := \langle \beta_p^t, \dots, \beta_1^t, \beta_1, \dots, \beta_p \rangle.$$

Then we obtain by Schwarz' inequality:

$$\begin{aligned} \|Z_\gamma u\|^2 &= |(Y_{\beta_p} X_{\alpha_p} \circ \dots \circ Y_{\beta_1} X_{\alpha_1} X_{\alpha_1} Y_{\beta_1} \circ \dots \circ X_{\alpha_p} Y_{\beta_p} u, u)| \\ &= |(X_\alpha Y_\beta u, u)| = |(Y_\beta u, X_\alpha u)| \leq \|Y_\beta u\| \|X_\alpha u\| \\ &\leq c_1 c_2 t^{|\alpha| + |\beta|} \|\alpha\|_1^{\lambda_1} \dots \|\alpha\|_{d_1}^{\lambda_{d_1}} \|\beta\|_1^{\mu_1} \dots \|\beta\|_{d_2}^{\mu_{d_2}} \\ &\leq [(c_1 + c_2)(2^{M_t})^{|\gamma|} \|\gamma\|_1^{\lambda_1} \dots \|\gamma\|_{d_1}^{\lambda_{d_1}} \|\gamma\|_{d_1+1}^{\mu_1} \dots \|\gamma\|_d^{\mu_{d_2}}]^2. \end{aligned}$$

This proves the theorem. \square

Corollary 1.32 Let X_1, \dots, X_d be commuting operators in a Hilbert space defined on a common invariant domain. Suppose all operators X_1, \dots, X_d are Hermitian or skew-Hermitian. Let $\lambda_1, \dots, \lambda_d \geq 0$. Then

$$S_{\lambda_1, \dots, \lambda_d}(X_1, \dots, X_d) = S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(X_1, \dots, X_d) = \bigcap_{k=1}^d S_{\lambda_k}(X_k)$$

as locally convex spaces with equivalent spectra.

Proof. By Theorem 1.31 we obtain for all $m \in \{1, \dots, d-1\}$ that

$$S_{\lambda_1, \dots, \lambda_m}(X_1, \dots, X_m) \cap S_{\lambda_{m+1}}(X_{m+1}) = S_{\lambda_1, \dots, \lambda_{m+1}}(X_1, \dots, X_{m+1})$$

as locally convex spaces with equivalent spectra. So

$$S_{\lambda_1, \dots, \lambda_m}(X_1, \dots, X_m) \cap \bigcap_{k=m+1}^d S_{\lambda_k}(X_k) = S_{\lambda_1, \dots, \lambda_{m+1}}(X_1, \dots, X_{m+1}) \cap \bigcap_{k=m+2}^d S_{\lambda_k}(X_k)$$

as locally convex spaces with equivalent spectra for all $m \in \{1, \dots, d-1\}$. By induction on m it follows that

$$S_{\lambda_1, \dots, \lambda_d}(X_1, \dots, X_d) = \bigcap_{k=1}^d S_{\lambda_k}(X_k)$$

as locally convex spaces with equivalent spectra.

Obviously $S_{\lambda_1, \dots, \lambda_d}(X_1, \dots, X_d) = S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(X_1, \dots, X_d)$ as locally convex spaces with equivalent spectra. \square

In the remaining part of this section we deal with *strongly commuting* self-adjoint operators, i.e. self-adjoint operators whose spectral projections commute, or equivalently,

self-adjoint operators whose Cayley transforms commute. In Appendix B the smoothed space $S_{\pi,C}$ is introduced for every locally compact Abelian group G , for every representation π in a Hilbert space H and for every subset C of $L^1(G)$ such that the pair (C, π) satisfies certain conditions (i.e. at least $P1$ and $P2$.) We shall prove that for all strongly commuting self-adjoint operators A_1, \dots, A_d in a Hilbert space H and for all $\lambda_1, \dots, \lambda_d > 0$ there exists a unitary representation π of some Abelian locally compact topological group G in the Hilbert space H and a set $C \subset L^1(G)$ such that the pair (C, π) satisfies conditions $P1'$, $P2$, $P3$ and $P4$ (see Appendix B) and

$$S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d) = S_{\pi, C}$$

as locally convex spaces with equivalent spectra. We start with some lemmas.

Lemma 1.33 *Let $\nu > 0$. Define for all $t > 0$ the function $h_t : \mathbb{R} \rightarrow \mathbb{R}$ by*

$$h_t(x) := e^{-t|x|^\nu} \quad (x \in \mathbb{R}).$$

Then for all $t > 0$ there exists $f_t \in L^1(\mathbb{R})$ such that $\hat{f}_t = h_t$.

Proof. Since $h_t \in L^1(\mathbb{R})$ and h_t is continuous, by the Fourier inversion theorem it is sufficient to show that $\hat{h}_t \in L^1(\mathbb{R})$. We distinguish two cases.

Case I. Suppose $\nu > \frac{1}{2}$.

Then for all $t > 0$ we have for all $x \in \mathbb{R} \setminus \{0\}$ that $h'_t(x) = -t\nu|x|^{\nu-1}h(x)\text{sgn}x$. So $h_t \in D(P)$ and $\mathbb{F}h_t \in D(Q)$. Then $(1 + |Q|)\hat{h}_t \in L^2(\mathbb{R})$. Since

$$\hat{h}_t(x) = \frac{1}{1 + |x|} \cdot [(1 + |x|)\hat{h}_t(x)]$$

for all $x \in \mathbb{R}$ we obtain by Hölder's inequality that $\hat{h}_t \in L^1(\mathbb{R})$.

Case II. Suppose $\nu \leq 1$.

Let $t > 0$. For all $x > 0$ we have $h'_t(x) = -t\nu e^{-x^\nu} x^{\nu-1}$, so h'_t is an increasing function on $(0, \infty)$. Let $k \in \mathbb{N}_0$. Define $\psi_k : [0, \frac{\pi}{2}] \rightarrow \mathbb{R}$ by

$$\psi_k(x) := h_t(2k\pi + x) - h_t((2k+1)\pi - x) - h_t((2k+1)\pi + x) + h_t((2k+2)\pi - x)$$

Then ψ_k is continuous, $\psi_k(\frac{\pi}{2}) = 0$ and for all $x \in (0, \frac{\pi}{2})$ we have $\psi'_k(x) = h'_t(2k\pi + x) + h'_t((2k+1)\pi - x) - h'_t((2k+1)\pi + x) - h'_t((2k+2)\pi - x) \leq 0$. So $\psi_k(x) \geq 0$ for all $x \in [0, \frac{\pi}{2}]$.

From this it follows that

$$\begin{aligned} \sqrt{2\pi}\hat{h}_t(1) &= \int_{-\infty}^{\infty} h_t(x)e^{-ix} dx = 2 \int_0^{\infty} h_t(x) \cos x dx = 2 \sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+2)\pi} h_t(x) \cos x dx \\ &= 2 \sum_{k=0}^{\infty} \int_0^{\frac{\pi}{2}} [h_t(2k\pi + x) - h_t((2k+1)\pi - x) - h_t((2k+1)\pi + x) + \\ &\quad + h_t((2k+2)\pi - x)] \cos x dx \\ &= 2 \sum_{k=0}^{\infty} \int_0^{\frac{\pi}{2}} \psi_k(x) \cos x dx \geq 0. \end{aligned}$$

So $\hat{h}_t(1) \geq 0$ for all $t > 0$. Since $\hat{h}_t(y) = |y|^{-1} \hat{h}_s(1)$, where $s = t|y|^{-\nu}$, for all $y \in \mathbb{R} \setminus \{0\}$ and $\hat{h}_t(0) \geq 0$, we obtain that $\hat{h}_t \geq 0$ for all $t > 0$. Because \hat{h}_t is bounded, we can use Theorem 31.42 of Hewitt and Ross ([HR2]) and conclude that $\hat{h}_t \in L^1(\mathbb{R})$. \square

Lemma 1.34 *Let A be a self-adjoint operator in a Hilbert space H . For $x \in \mathbb{R}$ define the unitary operator π_x on H by $\pi_x := e^{-ixA}$. Let $f \in L^1(\mathbb{R})$. Then $\pi(f) = \hat{f}(A)$.*

Proof. By the spectral theorem ([MP], Theorem A7, page 497) there exist a measure space (Y, \mathcal{B}, m) , a real valued measurable function h on Y and a unitary map W from H onto $L^2(m)$ such that $A = W^{-1}M_hW$ with M_h the multiplication operator by h in $L^2(m)$. Without loss of generality we may assume that $H = L^2(m)$ and that W is the identity map. Let $\xi \in L^2(m)$. For all $n \in \mathbb{N}$ let $Y_n := \{y \in Y : |\xi(y)| \geq \frac{1}{n}\}$. Then $m(Y_n) < \infty$ for all $n \in \mathbb{N}$. By Lebesgue's dominated convergence theorem and Fubini's theorem, for all $\eta \in L^2(m)$ we have :

$$\begin{aligned} (\pi(f)\xi, \eta) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) (\pi_x \xi, \eta) dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_Y \lim_{n \rightarrow \infty} 1_{Y_n}(y) f(x) e^{-ixh(y)} \xi(y) \overline{\eta(y)} dm(y) dx \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_Y f(x) e^{-ixh(y)} \xi(y) \overline{\eta(y)} d(1_{Y_n} m)(y) dx \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2\pi}} \int_Y \int_{\mathbb{R}} f(x) e^{-ixh(y)} \xi(y) \overline{\eta(y)} dx d(1_{Y_n} m)(y) \\ &= \lim_{n \rightarrow \infty} \int_Y \hat{f}(h(y)) \xi(y) \overline{\eta(y)} dx d(1_{Y_n} m)(y) \\ &= (\hat{f}(A)\xi, \eta). \end{aligned}$$

This proves the lemma. \square

Theorem 1.35 *Let A_1, \dots, A_d be d strongly commuting self-adjoint operators in a Hilbert space H and let $\lambda_1, \dots, \lambda_d > 0$. Then there exists an Abelian Lie group G , a representation π of G in H , and a set $C \subset L^1(G)$ such that the pair (C, π) has Properties P1', P2, P3 and P4 of Appendix B and*

$$S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d) = S_{\pi, C}$$

as locally convex spaces with equivalent spectra. Moreover, there exists a basis X_1, \dots, X_d in the Lie algebra \mathfrak{g} of G such that $d\pi(X_k) = -iA_k$ for all $k \in \{1, \dots, d\}$.

Proof. Let $G := \mathbb{R}^d$ and let y be the canonical chart on G , i.e. take Cartesian coordinates. For $(x_1, \dots, x_d) \in \mathbb{R}^d$ define $\pi_{(x_1, \dots, x_d)} : H \rightarrow H$ by

$$\pi_{(x_1, \dots, x_d)} := e^{-ix_1 A_1} \circ \dots \circ e^{-ix_d A_d}.$$

Then π is a representation of G in H . For $k \in \{1, \dots, d\}$ let $X_k := \frac{\partial}{\partial y_k} |_{(0, \dots, 0)}$. Then $d\pi(X_k) = -iA_k$. For all $t \in \mathbb{R}, t > 0$ and all $k \in \{1, \dots, d\}$ define $h_{k,t} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$h_{k,t}(x) := e^{-t|x|^{1/\lambda_k}} \quad (x \in \mathbb{R}).$$

By Lemma 1.33 there exists $f_{k,t} \in L^1(\mathbb{R})$ such that $\hat{f}_{k,t} = h_{k,t}$. For $t > 0$ define $f_t \in L^1(\mathbb{R}^d)$ by $f_t(x_1, \dots, x_d) = f_{1,t}(x_1) \cdots f_{d,t}(x_d)$, a.e. $(x_1, \dots, x_d) \in G$. Then by Lemma 1.34 we obtain for all $t > 0$:

$$\pi(f_t) = \hat{f}_{1,t}(A_1) \circ \dots \circ \hat{f}_{d,t}(A_d) = h_{1,t}(A_1) \circ \dots \circ h_{d,t}(A_d).$$

Let $C := \{f_t : t > 0\}$. Because $f_t * f_s = f_{s+t}$ for all $s, t > 0$, the pair (C, π) has Property $P1'$. Obviously the pair (C, π) has Properties $P2$ and $P3$.

Let $t > 0$. Similarly to the second part of the proof of Theorem 1.4 it follows that there exists $s > 0$ such that $\pi(f_t)(H)$ is continuously embedded in $S_{\lambda_1, \dots, \lambda_d; s}(A_1, \dots, A_d)$.

It remains to prove that for all $t > 0$ there exists $s > 0$ such that $S_{\lambda_1, \dots, \lambda_d; s}(A_1, \dots, A_d)$ is continuously embedded in $\pi(f_s)(H)$. So let $t > 0$. Let

$$s := \frac{1}{2e} \min(\{\lambda_k t^{-1/\lambda_k} : k \in \{1, \dots, d\}\})$$

and for $k \in \{1, \dots, d\}$ let

$$C_k := e^s + \sum_{n=0}^{\infty} \left\{ \left(\left[\frac{n}{\lambda_k} \right] + 1 \right) e \right\}^{\lambda_k} t \left(\frac{1}{2} \right)^n < \infty.$$

We shall prove that for all $u \in S_{\lambda_1, \dots, \lambda_d; s}(A_1, \dots, A_d)$ we have $u \in \pi(f_s)(H)$ and $\|u\|_f \leq C_1 \cdots C_d \|u\|_{\lambda_1, \dots, \lambda_d; s}$. So let $u \in S_{\lambda_1, \dots, \lambda_d; s}(A_1, \dots, A_d)$. For typographical convenience we write $c = \|u\|_{\lambda_1, \dots, \lambda_d; s}$. For $k \in \{1, \dots, d+1\}$ the hypothesis $P(k)$ states:

For all $n_1, \dots, n_d \in \mathbb{N}_0$ we have

$$A_k^{n_k} \circ \dots \circ A_d^{n_d} u \in D \left(e^{s|A_{k-1}|^{1/\lambda_{k-1}}} \circ \dots \circ e^{s|A_1|^{1/\lambda_1}} \right)$$

and

$$\begin{aligned} \|e^{s|A_{k-1}|^{1/\lambda_{k-1}}} \circ \dots \circ e^{s|A_1|^{1/\lambda_1}} A_k^{n_k} \circ \dots \circ A_d^{n_d} u\| &\leq \\ &\leq c C_1 \cdots C_{k-1} t^{n_k + \dots + n_d} n_k!^{\lambda_k} \cdots n_d!^{\lambda_d}. \end{aligned}$$

Obviously hypothesis $P(1)$ is true, this is just by definition of $\|u\|_{\lambda_1, \dots, \lambda_d; s}$. Let $k \in \{1, \dots, d\}$ and suppose $P(k)$ is true. Because the operators A_1, \dots, A_d strongly commute, we obtain for all $n_1, \dots, n_d \in \mathbb{N}_0$ that

$$\begin{aligned} A_k^{n_k} e^{s|A_{k-1}|^{1/\lambda_{k-1}}} \circ \dots \circ e^{s|A_1|^{1/\lambda_1}} A_{k+1}^{n_{k+1}} \circ \dots \circ A_d^{n_d} u &= \\ = e^{s|A_{k-1}|^{1/\lambda_{k-1}}} \circ \dots \circ e^{s|A_1|^{1/\lambda_1}} A_k^{n_k} \circ \dots \circ A_d^{n_d} u. \end{aligned}$$

Let $n_1, \dots, n_{k-1}, n_{k+1}, \dots, n_d \in \mathbb{N}_0$ be fixed. Then for all $n_k \in \mathbb{N}_0$ we obtain by Lemma 1.3.IV:

$$\begin{aligned} & \| |A_k|^{n_k/\lambda_k} e^{s|A_{k-1}|^{1/\lambda_{k-1}}} \circ \dots \circ e^{s|A_1|^{1/\lambda_1}} A_{k+1}^{n_{k+1}} \circ \dots \circ A_d^{n_d} u \| \leq \\ & \leq \| |A_k|^{[n_k/\lambda_k]+1} e^{s|A_{k-1}|^{1/\lambda_{k-1}}} \circ \dots \circ e^{s|A_1|^{1/\lambda_1}} A_{k+1}^{n_{k+1}} \circ \dots \circ A_d^{n_d} u \| + \\ & \quad + \| e^{s|A_{k-1}|^{1/\lambda_{k-1}}} \circ \dots \circ e^{s|A_1|^{1/\lambda_1}} A_{k+1}^{n_{k+1}} \circ \dots \circ A_d^{n_d} u \| \\ & \leq cC_1 \dots C_{k-1} t^{n_{k+1}+\dots+n_d} n_{k+1}!^{\lambda_{k+1}} \dots n_d!^{\lambda_d} \left(t^{[n_k/\lambda_k]+1} \left(\left[\frac{n_k}{\lambda_k} \right] + 1 \right)!^{\lambda_k + 1} \right) \\ & \leq cC_1 \dots C_{k-1} t^{n_{k+1}+\dots+n_d} n_{k+1}!^{\lambda_{k+1}} \dots n_d!^{\lambda_d} \cdot \\ & \quad \cdot \left(\left\{ \left(\left[\frac{n_k}{\lambda_k} \right] + 1 \right) e \right\}^{\lambda_k} \cdot t \left(\frac{1}{2} \right)^{n_k} s^{-n_k} n_k! + 1 \right). \end{aligned}$$

So

$$\begin{aligned} & \sum_{n_k=0}^{\infty} \frac{s^{n_k}}{n_k!} \| |A_k|^{n_k/\lambda_k} e^{s|A_{k-1}|^{1/\lambda_{k-1}}} \circ \dots \circ e^{s|A_1|^{1/\lambda_1}} A_{k+1}^{n_{k+1}} \circ \dots \circ A_d^{n_d} u \| \leq \\ & \leq cC_1 \dots C_{k-1} t^{n_{k+1}+\dots+n_d} n_{k+1}!^{\lambda_{k+1}} \dots n_d!^{\lambda_d}. \end{aligned}$$

Now the validity of hypothesis $P(k+1)$ follows.

So, by induction, hypothesis $P(d+1)$ turns out to be true. Hence

$$u \in D \left(e^{s|A_d|^{1/\lambda_d}} \circ \dots \circ e^{s|A_1|^{1/\lambda_1}} \right) = \pi(f_s)(H)$$

and $\|u\|_{f_s} = \|e^{s|A_d|^{1/\lambda_d}} \circ \dots \circ e^{s|A_1|^{1/\lambda_1}} u\| \leq C_1 \dots C_d \|u\|_{\lambda_1, \dots, \lambda_d; t}$.

Thus $S_{\pi, C}$ and $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ are equal as locally convex spaces with equivalent spectra. By Corollary 1.18 the space $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ and hence $S_{\pi, C}$ is complete. Then by [tE], Theorem 3.12, the pair (C, π) has Property P4. Thus the theorem is proved. \square

Corollary 1.36 *Let G be an Abelian Lie group with Lie algebra \mathfrak{g} . Let $X_1, \dots, X_d \in \mathfrak{g}$. Let π be a representation of G in a Hilbert space H . Let $\lambda_1, \dots, \lambda_d > 0$. Then there exist an Abelian Lie group K , a representation σ of K in H and a set $C \subset L^1(K)$ such that the pair (C, σ) has Properties P1', P2, P3 and P4 of Appendix B and*

$$S_{\lambda_1, \dots, \lambda_d}(d\pi(X_1), \dots, d\pi(X_d)) = S_{\sigma, C}$$

as locally convex spaces with equivalent spectra.

Proof. It is well known that the operators $id\pi(X_1), \dots, id\pi(X_d)$ are strongly commuting self-adjoint operators in H . (A proof follows from [HR2], Theorem 33.9 and the fact that the dual group of \mathbb{R}^d is isomorphic with \mathbb{R}^d as topological group.) Now the corollary follows from Theorem 1.35. \square

Note that we use another group (K) and another representation (σ), to describe $S_{\lambda_1, \dots, \lambda_d}(d\pi(X_1), \dots, d\pi(X_d))$ in Corollary 1.36. It would be interesting if the same group and representation can be used. That this can be done will be proved in case all λ_k , $k \in \{1, \dots, d\}$ are equal.

Theorem 1.37 *Let G be an Abelian Lie group with Lie algebra \mathfrak{g} . Let X_1, \dots, X_d be a basis in \mathfrak{g} . Let π be a representation of G in a Hilbert space and let $\lambda > 0$. Then there exist a set $C \subset L^1(G)$ such that the pair (C, π) has Properties P1', P2, P3 and P4 of Appendix B and*

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = S_{\pi, C}$$

as locally convex spaces with equivalent spectra.

Proof. There exist $d_1, d_2 \in \{0, \dots, d\}$ and a discrete Lie group K such that $d_1 + d_2 = d$ and G is isomorphic with $\mathbb{T}^{d_1} \times \mathbb{R}^{d_2} \times K$. (See [SW], page 155.) So we may assume that $G = \mathbb{T}^{d_1} \times \mathbb{R}^{d_2} \times K$ as Lie groups. Let e_3 be the identity in K . Define $y : (\mathbb{T} \setminus \{-1\})^{d_1} \times \mathbb{R}^{d_2} \times \{e_3\} \rightarrow \mathbb{R}^d$ by

$$y(e^{ix_1}, \dots, e^{ix_{d_1}}, x_{d_1+1}, \dots, x_d, e_3) := (x_1, \dots, x_d)$$

$(x_1, \dots, x_{d_1} \in (-\pi, \pi), x_{d_1+1}, \dots, x_d \in \mathbb{R})$. Then $((\mathbb{T} \setminus \{-1\})^{d_1} \times \mathbb{R}^{d_2} \times \{e_3\}, y)$ is a chart on G . Let e be the identity in G and for $n \in \{1, \dots, d\}$ let

$$Y_n := \left. \frac{\partial}{\partial y_n} \right|_e.$$

Then Y_1, \dots, Y_d is a basis in \mathfrak{g} . So $S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = S_{\lambda, \dots, \lambda}(d\pi(Y_1), \dots, d\pi(Y_d))$ as locally convex spaces with equivalent spectra. (This is the crucial point where we use that all lambdas are equal.) Let μ_1, μ_2, μ_3 be the Haar measures on \mathbb{T}, \mathbb{R} and K such that $\mu_1(\mathbb{T}) = 1$, $\mu_2([0, 1]) = \frac{1}{\sqrt{2\pi}}$ and $\mu_3(\{e_3\}) = 1$. Let μ be the product measure on $G = \mathbb{T}^{d_1} \times \mathbb{R}^{d_2} \times K$. We have introduced the identifications between the dual group $\widehat{\mathbb{R}}$ and \mathbb{R} , and between the dual group $\widehat{\mathbb{T}}$ and \mathbb{Z} . Using these identifications, we identify the dual group \widehat{G} of G with $\mathbb{Z}^{d_1} \times \mathbb{R}^{d_2} \times \widehat{K}$ in the natural way.

Let $(A, m, I, A_i, \tau_i, W)$ be a Stone-representative for π . (See Appendix B.) For $n \in \{1, \dots, d\}$ define $h_n : A \rightarrow \mathbb{R}$ by

$$h_n(\tau_i(k_1, \dots, k_{d_1}, x_1, \dots, x_{d_2}, \varphi)) := \begin{cases} k_n & \text{if } n \leq d_1 \\ x_{n-d_1} & \text{if } n > d_1 \end{cases}$$

$(i \in I, k_1, \dots, k_{d_1} \in \mathbb{Z}, x_1, \dots, x_{d_2} \in \mathbb{R}, \varphi \in \widehat{K})$. Define $p : A \rightarrow \widehat{K}$ by

$$p(\tau_i(k_1, \dots, k_{d_1}, x_1, \dots, x_{d_2}, \varphi)) := \varphi$$

$(i \in I, k_1, \dots, k_{d_1} \in \mathbb{Z}, x_1, \dots, x_{d_2} \in \mathbb{R}, \varphi \in \widehat{K})$. As in the proof of [HR2], Theorem 33.8, for all $\xi \in L^2(m)$, for all $x_1, \dots, x_d \in \mathbb{R}$ and for all $z \in K$ we obtain for a.e. $a \in A$:

$$\left[W \pi_{(e^{ix_1}, \dots, e^{ix_{d_1}}, x_{d_1+1}, \dots, x_d, z)} W^{-1} \xi \right] (a) = e^{-ix_1 h_1(a)} \dots e^{-ix_d h_d(a)} [p(a)](z) \xi(a).$$

So $d\pi(Y_n) = W^{-1} M_{-ih_n} W$ for all $n \in \{1, \dots, d\}$, where M_h denotes the multiplication operator by h in $L^2(m)$ for every complex valued function h on A .

Let $t > 0$. Let $n \in \{1, \dots, d_1\}$. Define $g_{n,t} : \mathbb{Z} \rightarrow \mathbb{R}$ by

$$g_{n,t}(k) := e^{-t|k|^{1/\lambda}} \quad (k \in \mathbb{Z}).$$

Then $g_{n,t} \in \ell^2(\mathbb{Z})$ and since \mathbb{T} is compact, there exists $f_{n,t} \in L^1(\mathbb{T})$ such that $\hat{f}_{n,t} = g_{n,t}$. Let $n \in \{d_1 + 1, \dots, d\}$. Define $g_{n,t} : \mathbb{R} \rightarrow \mathbb{R}$ by

$$g_{n,t}(x) := e^{-t|x|^{1/\lambda}} \quad (x \in \mathbb{R}).$$

By Lemma 1.33 there exists $f_{n,t} \in L^1(\mathbb{R})$ such that $\hat{f}_{n,t} = g_{n,t}$. Define $f_t \in L^1(G)$ by

$$f_t(x_1, \dots, x_d, z) := f_{1,t}(x_1) \cdots f_{d,t}(x_d) 1_{\{e_3\}}(z)$$

for a.e. $(x_1, \dots, x_d, z) \in G$. Let

$$C := \{f_t : t > 0\}.$$

Clearly the pair (C, π) has Properties $P1'$ and $P3$. By definition of the concept of Stone-representative we obtain that for all $t > 0$ and all $\xi \in L^2(m)$:

$$W\pi(f_t)W^{-1}\xi = e^{-t|h_1|^{1/\lambda} - \dots - t|h_d|^{1/\lambda}} \xi.$$

Hence the pair (C, π) has Property $P2$ by Lebesgue's theorem on dominated convergence. Similarly to the proof of Theorem 1.35 it follows that $S_{\pi,C} = S_{\lambda, \dots, \lambda}(d\pi(Y_1), \dots, d\pi(Y_d))$ as locally convex spaces with equivalent spectra. Moreover, by the same argument it follows that the pair (C, π) has Property $P4$. \square

Chapter 2

Intersection of Gevrey spaces

In this chapter we prove that some Gevrey spaces can be written as the intersection of Gevrey spaces relative to a reduced number of operators. We give a summary of results.

Let $d_1, d_2 \in \mathbb{N}$ and let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ be skew-Hermitian operators defined on a common invariant domain in a Hilbert space. Suppose that

$$[X_i, Y_j] \in \text{span}(\{X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}\})$$

for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. So $\text{span}(\{X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}\})$ need not be a Lie algebra. Let $\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2} \geq 0$. We prove that

$$\begin{aligned} S_{\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}}(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}) &= \\ &= S_{\lambda_1, \dots, \lambda_{d_1}}(X_1, \dots, X_{d_1}) \cap S_{\mu_1, \dots, \mu_{d_2}}(Y_1, \dots, Y_{d_2}) \end{aligned}$$

as locally convex spaces with equivalent spectra in the following cases:

- I. $[X_i, Y_j] = 0$ for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. (See Theorem 1.31.)
- II. $\lambda_1 = \dots = \lambda_{d_1} \geq 1$ and $\mu_1 = \dots = \mu_{d_2} \geq 1$. (See Theorem 2.2.)
- III. $\lambda_1, \dots, \lambda_{d_1} \geq 1$, $\mu_1 = \dots = \mu_{d_2} \geq 0$ and $[X_i, Y_j] \in \text{span}(\{Y_1, \dots, Y_{d_2}\})$ for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. (See Theorem 2.16.I.)
- IV. $\lambda_1, \dots, \lambda_{d_1} \geq 1$, $\mu_1 \geq \dots \geq \mu_{d_2} \geq 0$ and $[X_i, Y_j] \in \text{span}(\{Y_j, \dots, Y_{d_2}\})$ for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. (See Theorem 2.16.II.)

Moreover, let \mathfrak{g} be a real Lie algebra of skew Hermitian operators defined on a common invariant domain in a Hilbert space. Let $X_1, \dots, X_d \in \mathfrak{g}$ and suppose $\mathfrak{g} = \text{span}(\{X_1, \dots, X_d\})$. Let $\lambda_1, \dots, \lambda_d \geq 0$. We prove that

$$S_{\lambda_1, \dots, \lambda_d}(X_1, \dots, X_d) = \bigcap_{k=0}^d S_{\lambda_k}(X_k)$$

as locally convex spaces with equivalent spectra in the following cases:

- V. $[X_i, X_j] = 0$ for all $i, j \in \{1, \dots, d\}$. (See Corollary 1.32.)
- VI. $\lambda_1 \geq 1$, $\lambda_2 \geq \dots \geq \lambda_{d-1} \geq \max(1, \lambda_d) \geq \lambda_d \geq 0$ and

$$[X_i, X_j] \in \text{span}(\{X_{\max(i,j)}, \dots, X_d\})$$

for all $i, j \in \{1, \dots, d\}$. (See Corollary 2.18.)

We also consider non-invariant domains. Let G be a Lie group with Lie algebra \mathfrak{g} and let π be a representation of G in a Hilbert space. Let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2} \in \mathfrak{g}$ and suppose $\mathfrak{g} = \text{span}(\{X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}\})$. Let $\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2} \geq 0$. Then

$$\begin{aligned} S_{\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}}(d\pi(X_1), \dots, d\pi(X_{d_1}), d\pi(Y_1), \dots, d\pi(Y_{d_2})) = \\ = S_{\lambda_1, \dots, \lambda_{d_1}}(d\pi(X_1), \dots, d\pi(X_{d_1})) \cap S_{\mu_1, \dots, \mu_{d_2}}(d\pi(Y_1), \dots, d\pi(Y_{d_2})) \end{aligned}$$

in the following cases:

I'. $[X_i, Y_j] = 0$ for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. (See Theorem 1.31.)

II'. $\lambda_1 = \dots = \lambda_{d_1} \geq 1$ and $\mu_1 = \dots = \mu_{d_2} \geq 1$. (See Corollary 2.3.)

III'. $\lambda_1, \dots, \lambda_{d_1} \geq 1$, $\mu_1 = \dots = \mu_{d_2} \geq 0$ and $[X_i, Y_j] \in \text{span}(\{Y_1, \dots, Y_{d_2}\})$ for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. (See Remark following Corollary 2.18.)

IV'. $\lambda_1, \dots, \lambda_{d_1} \geq 1$, $\mu_1 \geq \dots \geq \mu_{d_2} \geq 0$ and $[X_i, Y_j] \in \text{span}(\{Y_j, \dots, Y_{d_2}\})$ for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. (See Remark following Corollary 2.18.)

Note that in general the domains of the operators $d\pi(X_1), \dots, d\pi(Y_{d_2})$ do not equal their (joint) C^∞ -domains. (Cf. Example 1.14!) Similar results hold in cases V and VI.

In the proof of some intersection theorems we need some more operations on multi-indices. Therefore we include a section about multi-indices.

Let G be a nilpotent Lie group with Lie algebra \mathfrak{g} and let X_1, \dots, X_d be a basis in \mathfrak{g} such that

$$[X_i, X_j] \in \text{span}(\{X_{\max(i,j)+1}, \dots, X_d\})$$

for all $i, j \in \{1, \dots, d\}$. Let π be a representation of G in a Hilbert space H . We prove intersection results for the Gevrey space

$$S_{\lambda_1, \dots, \lambda_d}(d\pi(X_1), \dots, d\pi(X_d))$$

in case $\lambda_1 \geq 1$, $\lambda_2 \geq \dots \geq \lambda_{d-1} \geq \max(1, \lambda_d) \geq \lambda_d \geq 0$. From general theory, (Corollary 1.26), we know that this Gevrey space is dense in H if in addition $\lambda_d \geq 1$. For the above mentioned case, in Section 2.4 we even prove that the Gevrey space is dense in H if in addition $\lambda_d > 0$.

Let π be a representation of a Lie group G in a Hilbert space and let $\lambda \geq 1$. Let \mathfrak{g} be the Lie algebra of G . The Gevrey space $S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d))$ is independent of the choice of the basis X_1, \dots, X_d in \mathfrak{g} . In Section 2.5 we prove that there exists a basis X_1, \dots, X_d in \mathfrak{g} such that

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = \bigcap_{k=0}^d S_\lambda(d\pi(X_k))$$

as locally convex spaces with equivalent spectra. Thus extending a result of Flato and Simon.

At the end of this chapter we present some topological remarks concerning an equality of the form

$$S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d) = \bigcap_{k=1}^d S_{\lambda_k}(A_k).$$

2.1 Gevrey spaces relative to coupled sets of skew-Hermitian operators

Let $d_1, d_2 \in \mathbb{N}$ and let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ be skew-Hermitian operators defined on a common invariant domain in a Hilbert space. Suppose

$$[X_i, Y_j] \in \text{span}(\{X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}\})$$

for all $i \in \{1, \dots, d_1\}$ and all $j \in \{1, \dots, d_2\}$. Let $\lambda, \mu \geq 1$. In this section we prove the intersection result

$$S_{\lambda, \dots, \lambda, \mu, \dots, \mu}(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}) = S_{\lambda, \dots, \lambda}(X_1, \dots, X_{d_1}) \cap S_{\mu, \dots, \mu}(Y_1, \dots, Y_{d_2})$$

as locally convex spaces with equivalent spectra. Thus we extend a result of Flato and Simon ([FS], Theorem 2) in three directions at once. They proved the above intersection result in the following very special case: $\lambda = \mu = 1$; $\mathfrak{g} := \text{span}(\{X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}\})$ is an integrable Lie algebra and both $\text{span}(\{X_1, \dots, X_{d_1}\})$ and $\text{span}(\{Y_1, \dots, Y_{d_2}\})$ are subalgebras of \mathfrak{g} . However, the theorem of Flato and Simon is also valid for representations in a Banach space. The essence of the proof in [FS] is that to each element of $S_{1, \dots, 1}(X_1, \dots, X_{d_1}) \cap S_{1, \dots, 1}(Y_1, \dots, Y_{d_2})$ a function is constructed, which is separately real analytic and because of a result of Browder ([Bro]) this function can be shown to be jointly real analytic. The proof of our more general intersection theorem is based on totally different techniques.

Let $V_1 := \{1, \dots, d_1\}$, $V_2 := \{1, \dots, d_2\}$ and $V := \{1, \dots, d\}$, where $d := d_1 + d_2$. Let $Z_1 := X_1, \dots, Z_{d_1} := X_{d_1}, Z_{d_1+1} := Y_1, \dots, Z_d := Y_{d_2}$. For all $k, m \in \mathbb{N}_0$ we define the subset $U_{k,m}$ of $M(V)$ by

$$U_{k,m} := \{\gamma \in M(V) : \sum_{i=1}^{d_1} \|\gamma\|_i = k \text{ and } \sum_{i=d_1+1}^d \|\gamma\|_i = m\}.$$

Lemma 2.1 *There exists a constant $M \geq 1$ such that for all $k, m \in \mathbb{N}$ and all $\gamma \in U_{k,m}$ there exist $x \in V_1$, $\delta \in U_{k-1,m}$, $c_1, \dots, c_{d_1 m} \in \mathbb{R}$, $\theta_1, \dots, \theta_{d_1 m} \in U_{k,m-1}$, $b_1, \dots, b_{d_2 m} \in \mathbb{R}$ and $\eta_1, \dots, \eta_{d_2 m} \in U_{k-1,m}$ such that*

$$Z_\gamma = Z_\delta X_x + \sum_{p=1}^{d_1 m} c_p Z_{\theta_p} + \sum_{q=1}^{d_2 m} b_q Z_{\eta_q},$$

$|c_p| \leq M$ for all $p \in \{1, \dots, d_1 m\}$ and $|b_q| \leq M$ for all $q \in \{1, \dots, d_2 m\}$.

Proof. By assumption, for all $i \in V_1$ and $j \in V_2$ there exist $c_{i,j}^1, \dots, c_{i,j}^{d_1}, b_{i,j}^1, \dots, b_{i,j}^{d_2} \in \mathbb{R}$ such that

$$[X_i, Y_j] = \sum_{l=1}^{d_1} c_{i,j}^l X_l + \sum_{l=1}^{d_2} b_{i,j}^l Y_l.$$

Let $M := 1 + \max\{|c_{i,j}^l| : i \in V_1, j \in V_2\} + \max\{|b_{i,j}^l| : i \in V_1, j \in V_2, l \in V_2\}$.

Let $k, m \in \mathbb{N}$ and let $\gamma \in U_{k,m}$. We may assume that the last index of the multi-index γ is not an element of V_1 , since otherwise we can write $Z_\gamma = Z_\delta X_x$ for some $\delta \in U_{k-1,m}$ and $x \in V_1$. There exist $\beta \in M(V)$, $x \in V_1$, $n \in \{1, \dots, m\}$ and $j_1, \dots, j_n \in V_2$ such that $\gamma = \langle \beta, x, d_1 + j_1, \dots, d_1 + j_n \rangle$. Then

$$\begin{aligned} Z_\gamma &= Z_\beta Y_{j_1} \circ \dots \circ Y_{j_n} X_x + Z_\beta \text{ad} X_x (Y_{j_1} \circ \dots \circ Y_{j_n}) \\ &= Z_\beta Y_{j_1} \circ \dots \circ Y_{j_n} X_x + Z_\beta \sum_{i=1}^n Y_{j_1} \circ \dots \circ Y_{j_{i-1}} \left(\sum_{l=1}^{d_1} c_{x,j_i}^l X_l \right) Y_{j_{i+1}} \circ \dots \circ Y_{j_n} + \\ &\quad + Z_\beta \sum_{i=1}^n Y_{j_1} \circ \dots \circ Y_{j_{i-1}} \left(\sum_{l=1}^{d_2} b_{x,j_i}^l Y_l \right) Y_{j_{i+1}} \circ \dots \circ Y_{j_n}. \end{aligned}$$

This proves the lemma. \square

Theorem 2.2 Let $d_1, d_2 \in \mathbb{N}$ and let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ be skew-Hermitian operators in a Hilbert space defined on a common invariant domain. Suppose

$$[X_i, Y_j] \in \text{span}(\{X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}\})$$

for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. Let $\lambda, \mu \geq 1$. Then

$$S_{\lambda, \dots, \lambda, \mu, \dots, \mu}(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}) = S_{\lambda, \dots, \lambda}(X_1, \dots, X_{d_1}) \cap S_{\mu, \dots, \mu}(Y_1, \dots, Y_{d_2})$$

as locally convex spaces with equivalent spectra.

Proof. Let $V_1, V_2, V, d, U_{k,m}, Z_i$ be as above and let $M \geq 1$ be as in Lemma 2.1. Let $b := 3 \cdot 2^{\lambda d_1 + \mu d_2}$. Let $t \geq Md$ and let $u \in S_{\lambda, \dots, \lambda; t}(X_1, \dots, X_{d_1}) \cap S_{\mu, \dots, \mu; t}(Y_1, \dots, Y_{d_2})$. We shall prove that $u \in S_{\lambda, \dots, \lambda, \mu, \dots, \mu; bt}(Z_1, \dots, Z_d)$ and that

$$\|u\|_{Z_1, \dots, Z_d; \lambda, \dots, \lambda, \mu, \dots, \mu; bt} \leq \|u\|_{X_1, \dots, X_{d_1}; \lambda, \dots, \lambda; t} + \|u\|_{Y_1, \dots, Y_{d_2}; \mu, \dots, \mu; t}.$$

Let $c_1 := \|u\|_{X_1, \dots, X_{d_1}; \lambda, \dots, \lambda; t}$ and $c_2 := \|u\|_{Y_1, \dots, Y_{d_2}; \mu, \dots, \mu; t}$. For $N \in \mathbb{N}_0$ hypothesis $P(N)$ states:

$$|(Y_\beta Z_\gamma X_\alpha u, u)| \leq c_1 c_2 3^{\|\gamma\| t} t^{\|\alpha\| + \|\beta\| + \|\gamma\|} (\|\alpha\| + k)^\lambda (\|\beta\| + m)^\mu \quad (2.1)$$

for all $k, m \in \mathbb{N}_0$, $\alpha \in M(V_1)$, $\beta \in M(V_2)$ and $\gamma \in U_{k,m}$ such that $k + m = N$.

If $k = m = 0$, $\alpha \in M(V_1)$, $\beta \in M(V_2)$ and $\gamma \in U_{k,m}$, then $\gamma = ()$. Therefore by Schwarz' inequality and the definitions of c_1 and c_2 we obtain that

$$|(Y_\beta Z_\gamma X_\alpha u, u)| = |(X_\alpha u, Y_\beta u)| \leq \|X_\alpha u\| \|Y_\beta u\| \leq c_1 t^{|\alpha|} \|\alpha\|!^\lambda c_2 t^{|\beta|} \|\beta\|!^\mu.$$

So hypothesis $P(0)$ holds.

Let $N \in \mathbb{N}$ and suppose hypothesis $P(N-1)$ holds. Let $k, m \in \mathbb{N}_0$, $\alpha \in M(V_1)$, $\beta \in M(V_2)$, $\gamma \in U_{k,m}$ and suppose $k+m = N$. If $k=0$ or $m=0$, inequality (2.1) follows by hypothesis $P(0)$. So we may assume that $k \neq 0$ and $m \neq 0$.

Suppose $k \geq m$. By Lemma 2.1 there exist $x \in V_1$, $\delta \in U_{k-1,m}$, $c_1, \dots, c_{d_1 m} \in \mathbb{R}$, $\theta_1, \dots, \theta_{d_1 m} \in U_{k,m-1}$, $b_1, \dots, b_{d_2 m} \in \mathbb{R}$ and $\eta_1, \dots, \eta_{d_2 m} \in U_{k-1,m}$ such that

$$Z_\gamma = Z_\delta X_x + \sum_{p=1}^{d_1 m} c_p Z_{\theta_p} + \sum_{q=1}^{d_2 m} b_q Z_{\eta_q}$$

and $|c_p| \leq M$ and $|b_q| \leq M$ for all p, q . Now we obtain by induction hypothesis $P(N-1)$ and the inequality $dM \leq t$:

$$\begin{aligned} |(Y_\beta Z_\gamma X_\alpha u, u)| &\leq \\ &\leq |(Y_\beta Z_\delta X_{(x,\alpha)} u, u)| + \sum_{p=1}^{d_1 m} |c_p| |(Y_\beta Z_{\theta_p} X_\alpha u, u)| + \sum_{q=1}^{d_2 m} |b_q| |(Y_\beta Z_{\eta_q} X_\alpha u, u)| \\ &\leq c_1 c_2 3^{|\gamma|-1} t^{|\alpha|+|\beta|+|\gamma|} (\|\alpha\|+k)!^\lambda (\|\beta\|+m)!^\mu + \\ &\quad + d_1 m M c_1 c_2 3^{|\gamma|-1} t^{|\alpha|+|\beta|+|\gamma|-1} (\|\alpha\|+k)!^\lambda (\|\beta\|+m-1)^\mu + \\ &\quad + d_2 m M c_1 c_2 3^{|\gamma|-1} t^{|\alpha|+|\beta|+|\gamma|-1} (\|\alpha\|+k-1)!^\lambda (\|\beta\|+m)!^\mu \\ &\leq c_1 c_2 3^{|\gamma|-1} t^{|\alpha|+|\beta|+|\gamma|} (\|\alpha\|+k)!^\lambda (\|\beta\|+m)!^\mu \cdot \\ &\quad \cdot \left(1 + \frac{m}{(\|\beta\|+m)^\mu} + \frac{m}{(\|\alpha\|+k)^\lambda} \frac{(\|\alpha\|+k)^\lambda}{(\|\alpha\|+k)^\lambda} \right) \\ &\leq c_1 c_2 3^{|\gamma|} t^{|\alpha|+|\beta|+|\gamma|} (\|\alpha\|+k)!^\lambda (\|\beta\|+m)!^\mu. \end{aligned}$$

In case $k \leq m$ a similar argument can be used by decomposing $Z_\gamma = Y_\gamma Z_\delta +$ "small terms". This proves hypothesis $P(N)$.

In particular, for all $k, m \in \mathbb{N}_0$ and $\gamma \in U_{k,m}$ we obtain that $|(Z_\gamma u, u)| \leq c_1 c_2 (3t)^{|\gamma|} k!^\lambda m!^\mu$. Now let $\gamma \in M(V)$. Let $k, m \in \mathbb{N}_0$ be such that $\gamma \in U_{k,m}$. Then $\langle \gamma^t, \gamma \rangle \in U_{2k, 2m}$ and we obtain by Lemma 1.3.V:

$$\begin{aligned} \|Z_\gamma u\|^2 &= |(Z_{\langle \gamma^t, \gamma \rangle} u, u)| \leq c_1 c_2 (3t)^{2|\gamma|} (2k)!^\lambda (2m)!^\mu \\ &\leq \left[(c_1 + c_2) (bt)^{|\gamma|} \|\gamma\|_1!^\lambda \cdot \dots \cdot \|\gamma\|_{d_1}!^\lambda \|\gamma\|_{d_1+1}!^\mu \cdot \dots \cdot \|\gamma\|_d!^\mu \right]^2. \end{aligned}$$

So $u \in S_{\lambda, \dots, \lambda, \mu, \dots, \mu; bt}(Z_1, \dots, Z_d)$ and $\|u\|_{Z_1, \dots, Z_d; \lambda, \dots, \lambda, \mu, \dots, \mu; bt} \leq c_1 + c_2$.

Since the identity map from $S_{\lambda, \dots, \lambda, \mu, \dots, \mu; t}(Z_1, \dots, Z_d)$ into $S_{\lambda, \dots, \lambda; t}(X_1, \dots, X_{d_1}) \cap S_{\mu, \dots, \mu; t}(Y_1, \dots, Y_{d_2})$ is continuous for all $t > 0$, the theorem follows. \square

Corollary 2.3 *Let G be a (real) Lie group with Lie algebra \mathfrak{g} . Let $d_1, d_2 \in \mathbb{N}$ and let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2} \in \mathfrak{g}$. Suppose*

$$\mathfrak{g} = \text{span}(\{X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}\}).$$

Let π be a representation of G in a Hilbert space H . Let $\lambda, \mu \geq 1$. Then

$$\begin{aligned} S_{\lambda, \dots, \lambda, \mu, \dots, \mu}(d\pi(X_1), \dots, d\pi(X_{d_1}), d\pi(Y_1), \dots, d\pi(Y_{d_2})) &= \\ &= S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_{d_1})) \cap S_{\mu, \dots, \mu}(d\pi(Y_1), \dots, d\pi(Y_{d_2})) \end{aligned}$$

as locally convex spaces with equivalent spectra.

Proof. The operators $\partial\pi(X_1), \dots, \partial\pi(X_{d_1}), \partial\pi(Y_1), \dots, \partial\pi(Y_{d_2})$ are all skew-symmetric and admit $H^\infty(\pi)$ as their invariant domain. By Corollary 1.26 we have

$$\begin{aligned} D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1})) \cap D^\infty(d\pi(Y_1), \dots, d\pi(Y_{d_2})) & \\ \subset \bigcap_{k=1}^{d_1} D^\infty(d\pi(X_k)) \cap \bigcap_{k=1}^{d_2} D^\infty(d\pi(Y_k)) = H^\infty(\pi) & \\ = D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1}), d\pi(Y_1), \dots, d\pi(Y_{d_2})) & \\ \subset D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1})) \cap D^\infty(d\pi(Y_1), \dots, d\pi(Y_{d_2})). & \end{aligned}$$

So

$$\begin{aligned} D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1}), d\pi(Y_1), \dots, d\pi(Y_{d_2})) &= \\ = H^\infty(\pi) = D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1})) \cap D^\infty(d\pi(Y_1), \dots, d\pi(Y_{d_2})) & \end{aligned}$$

as sets. With these equalities, Theorem 2.2 implies that the following spaces are equal as locally convex spaces with equivalent spectra:

$$\begin{aligned} S_{\lambda, \dots, \lambda, \mu, \dots, \mu}(d\pi(X_1), \dots, d\pi(X_{d_1}), d\pi(Y_1), \dots, d\pi(Y_{d_2})) & \\ S_{\lambda, \dots, \lambda, \mu, \dots, \mu}(\partial\pi(X_1), \dots, \partial\pi(X_{d_1}), \partial\pi(Y_1), \dots, \partial\pi(Y_{d_2})) & \\ S_{\lambda, \dots, \lambda}(\partial\pi(X_1), \dots, \partial\pi(X_{d_1})) \cap S_{\mu, \dots, \mu}(\partial\pi(Y_1), \dots, \partial\pi(Y_{d_2})) & \\ S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_{d_1})) \cap S_{\mu, \dots, \mu}(d\pi(Y_1), \dots, d\pi(Y_{d_2})). & \end{aligned}$$

□

Remark. The replacement of $\partial\pi(X_k)$ by $d\pi(X_k)$ as shown in this corollary can be carried through in all forthcoming results of this chapter. (Namely in 2.5, 2.6, 2.16, 2.17, 2.18, 2.32.) We will not give further explicit proofs in this matter.

The last equality in the following corollary has been firstly proved by Flato and Simon. (See [FS], Theorem 2.)

Corollary 2.4 *Let G be a real Lie group with Lie algebra \mathfrak{g} . Let \mathfrak{g}_1 and \mathfrak{g}_2 be subalgebras of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$. (Not necessarily a direct sum.) Let G_1 and G_2 be subgroups of G which have Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 respectively. Let π be a representation of G in a Hilbert space H , and let π_1 and π_2 be the restrictions of π to G_1 and G_2 respectively. Let Z_1, \dots, Z_d be a basis in \mathfrak{g} , let X_1, \dots, X_{d_1} be a basis in \mathfrak{g}_1 and let Y_1, \dots, Y_{d_2} be a basis in \mathfrak{g}_2 . Let $\lambda \geq 1$. Then*

$$\begin{aligned} S_{\lambda, \dots, \lambda}(d\pi(Z_1), \dots, d\pi(Z_d)) &= \\ &= S_{\lambda, \dots, \lambda}(d\pi_1(X_1), \dots, d\pi_1(X_{d_1})) \cap S_{\lambda, \dots, \lambda}(d\pi_2(Y_1), \dots, d\pi_2(Y_{d_2})) \end{aligned}$$

as locally convex spaces with equivalent spectra. In particular,

$$H_\lambda(\pi) = H_\lambda(\pi_1) \cap H_\lambda(\pi_2)$$

as sets and

$$H^\omega(\pi) = H^\omega(\pi_1) \cap H^\omega(\pi_2)$$

as sets.

Corollary 2.5 *Let \mathfrak{g} be a 2-dimensional real Lie algebra of skew-Hermitian operators in a Hilbert space defined on a common invariant domain. Let X, Y be any basis in \mathfrak{g} and let $\lambda, \mu \geq 1$. Then*

$$S_{\lambda, \mu}(X, Y) = S_\lambda(X) \cap S_\mu(Y) = S_{\lambda, \mu}^{\text{ord}}(X, Y)$$

as locally convex spaces with equivalent spectra.

Corollary 2.6 *Let \mathfrak{g} be a solvable real Lie algebra of skew-Hermitian operators defined on a common invariant domain. Let X_1, \dots, X_d be a basis in \mathfrak{g} such that $\mathcal{L}_i := \text{span}\{\{X_1, \dots, X_i\}\}$ is a subalgebra of \mathfrak{g} and \mathcal{L}_i is an ideal in \mathcal{L}_{i+1} for all $i \in \{1, \dots, d\}$. Let $\lambda \geq 1$. Then*

$$S_{\lambda, \dots, \lambda}(X_1, \dots, X_d) = \bigcap_{k=1}^d S_\lambda(X_k) = S_{\lambda, \dots, \lambda}^{\text{ord}}(X_1, \dots, X_d)$$

as locally convex spaces with equivalent spectra.

Proof. By induction to d it follows that $S_{\lambda, \dots, \lambda}(X_1, \dots, X_d) = \bigcap_{k=1}^d S_\lambda(X_k)$ as locally convex spaces with equivalent spectra. (Cf. the proof of Corollary 1.32.) Let $t > 0$. There exists $s > 0$ such that the identity map from $\bigcap_{k=1}^d S_{\lambda; t}(X_k)$ into $S_{\lambda, \dots, \lambda; s}(X_1, \dots, X_d)$ is continuous. Then the inclusions

$$S_{\lambda, \dots, \lambda; t}(X_1, \dots, X_d) \subset S_{\lambda, \dots, \lambda; t}^{\text{ord}}(X_1, \dots, X_d) \subset \bigcap_{k=1}^d S_{\lambda; t}(X_k) \subset S_{\lambda, \dots, \lambda; s}(X_1, \dots, X_d)$$

are continuous. So the spaces $S_{\lambda, \dots, \lambda}(X_1, \dots, X_d)$ and $S_{\lambda, \dots, \lambda}^{\text{ord}}(X_1, \dots, X_d)$ are equivalent as locally convex spaces with equivalent spectra. \square

Remark. A Lie group version for the last two corollaries can be formulated similarly to Corollary 2.3. (See the Remark following Corollary 2.3.) The Lie group version in case $\lambda = 1$ of Corollary 2.6 has been proved first by Goodman, [Goo2], Corollary 3.1.

2.2 Multi-indices (Part 2)

In the proofs of intersection theorems of the form

$$\begin{aligned} S_{\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}}(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}) &= \\ &= S_{\lambda_1, \dots, \lambda_{d_1}}(X_1, \dots, X_{d_1}) \cap S_{\mu_1, \dots, \mu_{d_2}}(Y_1, \dots, Y_{d_2}) \end{aligned}$$

where $[X_i, Y_j] \in \text{span}(\{Y_1, \dots, Y_{d_2}\})$, we want to write $X_\alpha Y_\beta$ as a sum of terms of the form $cY_\delta X_\gamma$. In this section we introduce some more operations on multi-indices in order to develop tools to calculate which γ and δ occur in this sum.

Let $n \in \mathbb{N}$ and let $k \in \mathbb{N}_0, k \leq n$. We define the subset P_k^n of the symmetric group S_n by

$$P_k^n := \{\sigma \in S_n : \sigma(n) < \sigma(n-1) < \dots < \sigma(k+1) \text{ and } \sigma(k) < \sigma(k-1) < \dots < \sigma(1)\}$$

if $k \notin \{0, \dots, n\}$ and we define

$$P_0^n := P_n^n := \left\{ \begin{pmatrix} 1 & 2 & \dots & n \\ n & n-1 & \dots & 1 \end{pmatrix} \right\}.$$

Note that in these definitions P_0^0 does not occur. Therefore, let σ_0 be any object which is not an element of $\bigcup_{n=1}^{\infty} \bigcup_{k=0}^n P_k^n$. We define

$$P_0^0 := \{\sigma_0\}.$$

Let V be a fixed non-empty set and as in Chapter 1, let $M(V)$ denote the set of all multi-indices over V . Let $n, k \in \mathbb{N}_0$ be such that $k \leq n$ and let $\alpha \in M(V)$ be such that $\|\alpha\| = n$. Let $\sigma \in P_k^n$. We define multi-indices $\tilde{\sigma}(\alpha)$ and $\hat{\sigma}(\alpha)$ over V by

$$\tilde{\sigma}(\alpha) := \hat{\sigma}(\alpha) := () \text{ if } n = 0,$$

and if $n \neq 0$ and $\alpha = (j_1, \dots, j_n)$:

$$\tilde{\sigma}(\alpha) := (j_{\sigma(k)}, \dots, j_{\sigma(1)}) \quad \text{if } k \neq 0,$$

$$\tilde{\sigma}(\alpha) := () \quad \text{if } k = 0,$$

$$\hat{\sigma}(\alpha) := (j_{\sigma(n)}, \dots, j_{\sigma(k+1)}) \quad \text{if } k \neq n,$$

$$\hat{\sigma}(\alpha) := () \quad \text{if } k = n.$$

(These definitions are inspired by the formulation of Lemma 2.11.)

We summarize some elementary facts.

Lemma 2.7 *Let V be a non-empty set, let $n, k \in \mathbb{N}_0, k \leq n$, let $\sigma \in P_k^n$ and let $\alpha \in M(V)$ with $\|\alpha\| = n$. Then*

$$\text{card } P_k^n = \binom{n}{k},$$

$$\|\tilde{\sigma}(\alpha)\| = k,$$

$$\|\hat{\sigma}(\alpha)\| = n - k.$$

Furthermore, there exists a function $l : V \rightarrow \mathbb{N}_0$ such that

$$\sum_{v \in V} l(v) = k$$

and

$$\|\hat{\sigma}(\alpha)\|_v = \|\alpha\|_v - l(v) \quad \text{for all } v \in V.$$

In the remaining part of this section let V be a non-empty totally ordered set with ordering \leq . A *positive mutation* (on V) is a function $\mu : V \times V \rightarrow \mathbb{N}_0$ such that

$$\mu(v, w) = 0 \quad \text{for all } v, w \in V \text{ with } v > w.$$

Let $\alpha, \beta \in M(V)$ and $k \in \mathbb{N}_0$. We say that α is *connected with β via a positive mutation of length k* if there exists a positive mutation μ on V such that

$$\left[\begin{array}{l} \|\alpha\|_v + \sum_{w \in V} [\mu(w, v) - \mu(v, w)] = \|\beta\|_v \quad \text{for all } v \in V, \\ \sum_{v, w \in V} \mu(v, w) = k. \end{array} \right.$$

(This definition is inspired by the formulation of Lemma 2.12.) Then

$$\|\alpha\|_v + \sum_{w < v} \mu(w, v) - \sum_{w > v} \mu(v, w) = \|\beta\|_v$$

for all $v \in V$.

Remark. If α is connected with β via a positive mutation of length k , then in general it is not true that β is connected with α via a positive mutation of length k . Also the positive mutation μ is not unique and it is well possible that α is connected with β via a positive mutation of length l with $l \in \mathbb{N}_0$, $l \neq k$.

Lemma 2.8

- I. Let $\alpha, \beta \in M(V)$ and let $k \in \mathbb{N}_0$. Suppose α is connected with β via a positive mutation of length k . Then $\|\alpha\| = \|\beta\|$.
- II. Let $n \in \mathbb{N}$, $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \in M(V)$ and let $k_1, \dots, k_n \in \mathbb{N}_0$. Suppose α_i is connected with β_i via a positive mutation of length k_i for all $i \in \{1, \dots, n\}$. Then $(\alpha_1, \dots, \alpha_n)$ is connected with $(\beta_1, \dots, \beta_n)$ via a positive mutation of length $\sum_{i=1}^n k_i$.
- III. Let $\alpha, \beta, \gamma \in M(V)$ and let $k, l \in \mathbb{N}_0$. Suppose α (resp. β) is connected with β (resp. γ) via a positive mutation of length k (resp. l). Then α is connected with γ via a positive mutation of length $k + l$. (Transitivity.)

Proof. I: trivial, II: induction, III: trivial. (Take $\mu = \mu_1 + \mu_2$.) □

2.3 Gevrey spaces, Lie algebras of operators and their ideals

Let $d_1, d_2 \in \mathbb{N}$ and let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ be skew-Hermitian operators in a Hilbert space defined on a common invariant domain. Suppose

$$[X_i, Y_j] \in \text{span}(\{X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}\})$$

for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. In Section 2.1 we proved that

$$S_{\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}}(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}) = S_{\lambda_1, \dots, \lambda_{d_1}}(X_1, \dots, X_{d_1}) \cap S_{\mu_1, \dots, \mu_{d_2}}(Y_1, \dots, Y_{d_2})$$

for all $\lambda, \mu \geq 1$. In this section we consider the stronger assumption that

$$[X_i, Y_j] \in \text{span}(\{Y_1, \dots, Y_{d_2}\})$$

for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. (For example: the operators $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ span a Lie algebra \mathfrak{g} and $\text{span}(\{Y_1, \dots, Y_{d_2}\})$ is an ideal in \mathfrak{g} .) Under this stronger condition we prove that

$$S_{\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}}(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}) = S_{\lambda_1, \dots, \lambda_{d_1}}(X_1, \dots, X_{d_1}) \cap S_{\mu_1, \dots, \mu_{d_2}}(Y_1, \dots, Y_{d_2})$$

for all $\lambda_1, \dots, \lambda_{d_1} \geq 1$ and all $\mu \geq 0$.

In the second part of this section we consider nilpotent Lie algebras and certain solvable Lie algebras. Namely, let \mathfrak{g} be a Lie algebra and let X_1, \dots, X_d be a basis in \mathfrak{g} . The basis X_1, \dots, X_d is called an *ordered basis* in \mathfrak{g} if

$$[X_i, X_j] \in \text{span}(\{X_{\max(i,j)}, \dots, X_d\})$$

for all $i, j \in \{1, \dots, d\}$. Not every solvable Lie algebra has an ordered basis; but every Lie algebra which has an ordered basis is solvable. Every nilpotent Lie algebra has an ordered basis. Let \mathfrak{g} be a real Lie algebra of skew-Hermitian operators in a Hilbert space defined on a common invariant domain and let X_1, \dots, X_d be an ordered basis in \mathfrak{g} . Let $\lambda_1 \geq 1$, $\lambda_2 \geq \dots \geq \lambda_{d-1} \geq \max(\lambda_d, 1) \geq \lambda_d \geq 0$. We shall prove that

$$S_{\lambda_1, \dots, \lambda_d}(X_1, \dots, X_d) = S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(X_1, \dots, X_d) = \bigcap_{k=1}^d S_{\lambda_k}(X_k)$$

as locally convex spaces with equivalent spectra.

Because the proof of the following special case is much shorter than the proof of the general case, we present the following theorem. For technical reasons, we interchange the role of the operators X_k and Y_l .

Theorem 2.9 *Let $d_1, d_2 \in \mathbb{N}$ and let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ be skew-Hermitian operators in a Hilbert space defined on a common invariant domain. Suppose*

$$[X_i, Y_j] \in \text{span}(\{X_1, \dots, X_{d_1}\})$$

for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. Let $\lambda \geq 0$ and $\mu \geq 1$. Then

$$S_{\lambda, \dots, \lambda, \mu, \dots, \mu}(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}) = S_{\lambda, \dots, \lambda}(X_1, \dots, X_{d_1}) \cap S_{\mu, \dots, \mu}(Y_1, \dots, Y_{d_2})$$

as locally convex spaces with equivalent spectra.

We emphasize that μ may be taken smaller than 1.

Proof. Because the set-up of this proof is the same as in Theorem 2.2, we only present a sketch. Let V_1, V_2, d, Z_i and $U_{k,m}$ be as in Section 2.1. Because the coefficients $b_{i,j}^l$ in the proof of Lemma 2.1 can be taken equal to zero, we obtain that there exists a constant $M \geq 1$ such that for all $k, m \in \mathbb{N}$ and all $\gamma \in U_{k,m}$ there exist $x \in V_1, \delta \in U_{k-1,m}, c_1, \dots, c_{d_1 m} \in \mathbb{R}$ and $\theta_1, \dots, \theta_{d_1 m} \in U_{k,m-1}$ such that

$$Z_\gamma = Z_\delta X_x + \sum_{p=1}^{d_1 m} c_p Z_{\theta_p} \tag{2.2}$$

and $|c_p| \leq M$ for all $p \in \{1, \dots, d_1 m\}$.

Let $b := 3 \cdot 2^{\lambda d_1 + \mu d_2}$. Let $t \geq Md$ and let $u \in S_{\lambda, \dots, \lambda; t}(X_1, \dots, X_{d_1}) \cap S_{\mu, \dots, \mu; t}(Y_1, \dots, Y_{d_2})$. Let c_1, c_2 and the hypotheses $P(N), N \in \mathbb{N}_0$ be as in the proof of Theorem 2.2. Again hypothesis $P(0)$ holds.

Let $N \in \mathbb{N}$ and suppose hypothesis $P(N-1)$ holds. Let $k, m \in \mathbb{N}_0, \alpha \in M(V), \beta \in M(V_2), \gamma \in U_{k,m}$ and suppose $k+m = N$. We may assume that $k \neq 0$ and $m \neq 0$. By decomposing Z_γ as in equality (2.2) we now obtain:

$$\begin{aligned} |(Y_\beta Z_\gamma X_\alpha u, u)| &\leq \\ &\leq |(Y_\beta Z_\delta X_{(x,\alpha)} u, u)| + \sum_{p=1}^{d_1 m} |c_p| |(Y_\beta Z_{\theta_p} X_\alpha u, u)| \\ &\leq c_1 c_2 3^{\|\gamma\| - 1} t^{\|\alpha\| + \|\beta\| + \|\gamma\|} (\|\alpha\| + k)!^\lambda (\|\beta\| + m)!^\mu + \\ &\quad + d_1 m M c_1 c_2 3^{\|\gamma\| - 1} t^{\|\alpha\| + \|\beta\| + \|\gamma\| - 1} (\|\alpha\| + k)!^\lambda (\|\beta\| + m - 1)!^\mu \\ &\leq c_1 c_2 3^{\|\gamma\| - 1} t^{\|\alpha\| + \|\beta\| + \|\gamma\|} (\|\alpha\| + k)!^\lambda (\|\beta\| + m)!^\mu \left(1 + \frac{m}{(\|\beta\| + m)^\mu} \right) \\ &\leq c_1 c_2 3^{\|\gamma\|} t^{\|\alpha\| + \|\beta\| + \|\gamma\|} (\|\alpha\| + k)!^\lambda (\|\beta\| + m)!^\mu. \end{aligned}$$

This proves hypothesis $P(N)$. The remaining part of the proof is the same as the corresponding remaining part of the proof of Theorem 2.2. \square

For the proof of the general case we have to make a lot of preparations. Let $d_1, d_2 \in \mathbb{N}$ and let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ be skew-Hermitian operators in a Hilbert space defined on a common invariant domain D . Suppose

$$[X_i, Y_j] \in \text{span}(\{Y_1, \dots, Y_{d_2}\})$$

for all $i \in V_1 := \{1, \dots, d_1\}$ and $j \in V_2 := \{1, \dots, d_2\}$.

Let $\text{Hom}(D)$ be the vector space of all linear maps from D into D . The following lemma is due to Nelson ([Nel], Lemma 2.1.)

Lemma 2.10 (Nelson.) *Let $n \in \mathbb{N}$ and let $Z_1, \dots, Z_n \in \text{Hom}(D)$. Let $W \in \text{Hom}(D)$. Then*

$$Z_1 \circ \dots \circ Z_n W = \sum_{k=0}^n \sum_{\sigma \in P_k^n} [\text{ad} Z_{\sigma(k)} \dots \text{ad} Z_{\sigma(1)}(W)] Z_{\sigma(n)} \circ \dots \circ Z_{\sigma(k+1)}.$$

This lemma leads to the following definition. Let $V_1 := \{1, \dots, d_1\}$, $V_2 := \{1, \dots, d_2\}$ and $\beta \in M(V_2)$. Define $D^\alpha(Y_\beta) \in \text{Hom}(D)$ by

$$D^\alpha(Y_\beta) := \begin{cases} \text{ad} X_{j_1} \dots \text{ad} X_{j_k}(Y_\beta) & \text{if } \|\alpha\| \neq 0 \text{ and } \alpha = (j_1, \dots, j_n), \\ Y_\beta & \text{if } \|\alpha\| = 0. \end{cases}$$

We rewrite Lemma 2.10 in our notations. The lemma is also true for $\alpha = ()$.

Lemma 2.11 *Let $\alpha \in M(V_1)$ and $\beta \in M(V_2)$. Then*

$$X_\alpha Y_\beta = \sum_{j=0}^{\|\alpha\|} \sum_{\sigma \in P_j^{\|\alpha\|}} D^{\tilde{\sigma}(\alpha)}(Y_\beta) X_{\delta(\alpha)}.$$

Lemma 2.12

I. *There exist constants $M \geq 1$ and $c_{\beta,\gamma}^\alpha \in \mathbb{R}$, where $\alpha \in M(V_1)$ and $\beta, \gamma \in M(V_2)$ such that*

$$D^\alpha(Y_\beta) = \sum_{\gamma \in M(V_2)} c_{\beta,\gamma}^\alpha Y_\gamma \quad \text{for all } \alpha \in M(V_1) \text{ and } \beta \in M(V_2), \quad (2.3)$$

$$c_{\beta,\gamma}^\alpha = 0 \quad \text{for all } \alpha \in M(V_1) \text{ and } \beta, \gamma \in M(V_2) \text{ with } \|\beta\| \neq \|\gamma\|, \quad (2.4)$$

$$|c_{\beta,\gamma}^\alpha| \leq (M\|\beta\|)^{\|\alpha\|} \quad \text{for all } \alpha \in M(V_1) \text{ and } \beta, \gamma \in M(V_2) \text{ with } \|\beta\| = \|\gamma\|. \quad (2.5)$$

II. *Suppose $[X_i, Y_j] \in \text{span}(\{Y_j, \dots, Y_{d_2}\})$ for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. Then the constants $c_{\beta,\gamma}^\alpha$ in I can be chosen such that in addition: for all $\alpha \in M(V_1)$ and $\beta, \gamma \in M(V_2)$ with $c_{\beta,\gamma}^\alpha \neq 0$ we have that β is connected with γ via a positive mutation of length $\|\alpha\|$.*

Here, the ordering for V_2 is the natural one.

Proof. There exist (possible non-unique) constants $c_{i,j,k} \in \mathbb{R}$, where $i \in V_1$ and $j, k \in V_2$ such that

$$[X_i, Y_j] = \sum_{k=1}^{d_2} c_{i,j,k} Y_k \quad (2.6)$$

for all $i \in V_1$ and $j \in V_2$. Let

$$M_0 := 1 + \max\{|c_{i,j,k}| : i \in V_1, j, k \in V_2\}.$$

For all $\beta, \gamma \in M(V_2)$ take $c_{\beta, \gamma}^{()} := 1$ if $\beta = \gamma$ and $c_{\beta, \gamma}^{()} := 0$ if $\beta \neq \gamma$. Take $c_{\beta, \gamma}^\alpha := 0$ for all $\alpha \in M(V_1) \setminus V_1^0$ and $\beta, \gamma \in M(V_2)$ with $\beta = ()$ or $\gamma = ()$. Then (2.3), (2.4) and (2.5) hold for all $\alpha \in M(V_1)$ and $\beta, \gamma \in M(V_2)$ such that $\alpha = ()$ or $\beta = ()$ or $\gamma = ()$. We define $c_{\beta, \gamma}^\alpha := 0$ for all $\alpha \in M(V_1)$ and $\beta, \gamma \in M(V_2)$ with $\|\beta\| \neq \|\gamma\|$.

Let $m \in \mathbb{N}$. Now we want to choose suitable constants $c_{\beta, \gamma}^\alpha$ such that (2.3) and (2.5) hold for all $\alpha \in M(V_1)$ with $\alpha \neq ()$ and $\beta, \gamma \in M(V_2)$ with $\|\beta\| = \|\gamma\| = m$. Let $i \in V_1$, $p \in \{1, \dots, m\}$ and let $k \in V_2$. For $\delta = (j_1, \dots, j_m) \in V_2^m \subset M(V_2)$ define

$$f_{p, k}(\delta) := (j_1, \dots, j_{p-1}, k, j_{p+1}, \dots, j_m)$$

and

$$g_{i, p, k}(\delta) := c_{i, j_p, k}.$$

Let $n \in \mathbb{N}$, $\alpha \in V_1^n$ and let $\beta \in V_2^m$. Let $\alpha = (i_1, \dots, i_n)$. It follows by induction to n that

$$\begin{aligned} D^\alpha(Y_\beta) &= \sum_{p_1=1}^m \sum_{k_1=1}^{d_2} \dots \sum_{p_n=1}^m \sum_{k_n=1}^{d_2} g_{i_n, p_n, k_n}(\beta) \cdot g_{i_{n-1}, p_{n-1}, k_{n-1}}(f_{p_n, k_n}(\beta)) \cdot \dots \\ &\quad \dots \cdot g_{i_1, p_1, k_1}(f_{p_2, k_2} \circ \dots \circ f_{p_n, k_n}(\beta)) Y_{f_{p_1, k_1} \circ \dots \circ f_{p_n, k_n}(\beta)}. \end{aligned}$$

Now the definitions of $c_{\beta, \gamma}^\alpha$, with $\gamma \in V_2^m$ speak for themselves. For $\gamma \in V_2^m$ we define

$$c_{\beta, \gamma}^\alpha := \sum g_{i_n, p_n, k_n}(\beta) \cdot g_{i_{n-1}, p_{n-1}, k_{n-1}}(f_{p_n, k_n}(\beta)) \cdot \dots \cdot g_{i_1, p_1, k_1}(f_{p_2, k_2} \circ \dots \circ f_{p_n, k_n}(\beta))$$

where the sum is over all $p_1, \dots, p_n \in \{1, \dots, m\}$ and $k_1, \dots, k_n \in V_2$ such that $\gamma = f_{p_1, k_1} \circ \dots \circ f_{p_n, k_n}(\beta)$. This proves (2.3), and clearly $\|\beta\| = \|\gamma\|$ and $|c_{\beta, \gamma}^\alpha| \leq m^n d_2^n M_0^n = (d_2 M_0 \|\beta\|)^{\|\alpha\|}$.

II. Now suppose that $[X_i, Y_j] \in (\{Y_j, \dots, Y_{d_2}\})$ for all $i \in V_1$ and $j \in V_2$. Then the constants in (2.6) can be chosen such that $c_{i, j, k} = 0$ for all $j, k \in V_2$ with $j > k$. We define $c_{\beta, \gamma}^\alpha$ as in I. Let $\alpha \in M(V_1)$, $\beta, \gamma \in M(V_2)$ and suppose $c_{\beta, \gamma}^\alpha \neq 0$. We may as well assume that $\alpha \neq ()$, $\beta \neq ()$ and $\gamma \neq ()$. Let $m := \|\beta\| = \|\gamma\|$ and let $i_1, \dots, i_n \in V_1$ be such that $\alpha = (i_1, \dots, i_n)$. Let $f_{p, k}$ and $g_{i, p, k}$ be as in I.

Let $\delta = (j_1, \dots, j_m) \in V_2^m$, let $i \in V_1$, $p \in \{1, \dots, m\}$, $k \in V_2$ and suppose $g_{i, p, k}(\delta) \neq 0$. Then $c_{i, j_p, k} \neq 0$. This implies that $j_p \leq k$ and so $\delta = (j_1, \dots, j_p, \dots, j_m)$ is connected with $f_{p, k}(\delta) = (j_1, \dots, j_{p-1}, k, j_{p+1}, \dots, j_m)$ via a positive mutation of length 1. (Take as mutation $\mu(v, w) := 1$ if $v = j_p$ and $w = k$ and $\mu(v, w) := 0$ else.)

Since $c_{\beta, \gamma}^\alpha \neq 0$ there exists $p_1, \dots, p_n \in \{1, \dots, m\}$ and $k_1, \dots, k_n \in V_2$ such that

$$g_{i_n, p_n, k_n}(\beta) \cdot g_{i_{n-1}, p_{n-1}, k_{n-1}}(f_{p_n, k_n}(\beta)) \cdot \dots \cdot g_{i_1, p_1, k_1}(f_{p_2, k_2} \circ \dots \circ f_{p_n, k_n}(\beta)) \neq 0$$

and

$$\gamma = f_{p_1, k_1} \circ \dots \circ f_{p_n, k_n}(\beta).$$

Then $g_{i_n, p_n, k_n}(\beta) \neq 0$ and ... and $g_{i_1, p_1, k_1}(f_{p_2, k_2} \circ \dots \circ f_{p_n, k_n}(\beta)) \neq 0$. Because $g_{i_n, p_n, k_n}(\beta) \neq 0$ we obtain that β is connected with $f_{p_n, k_n}(\beta)$ via a positive mutation of length 1. Because $g_{i_{n-1}, p_{n-1}, k_{n-1}}(f_{p_n, k_n}(\beta)) \neq 0$ we obtain that $f_{p_n, k_n}(\beta)$ is connected with $f_{p_{n-1}, k_{n-1}}(f_{p_n, k_n}(\beta))$ via a positive mutation of length 1. Then by Lemma 2.8.III, β is connected with $f_{p_{n-1}, k_{n-1}}(f_{p_n, k_n}(\beta))$ via a positive mutation mutation of length 2. By induction, β is connected with $f_{p_1, k_1} \circ \dots \circ f_{p_n, k_n}(\beta) = \gamma$ via a positive mutation of length $n = \|\alpha\|$. \square

Lemma 2.13 *Let $\beta, \gamma \in M(V_2)$, let $j \in \mathbb{N}_0$ and let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{d_2} \geq 0$. Suppose β is connected with γ via a positive mutation of length j . Then*

$$\|\gamma\|_1!^{\mu_1} \dots \|\gamma\|_{d_2}!^{\mu_{d_2}} \leq (2^{d_2 \mu_2})^{|\beta|+j} \|\beta\|_1!^{\mu_1} \dots \|\beta\|_{d_2}!^{\mu_{d_2}}.$$

Proof. In this lemma we write $d := d_2$. Let τ be a positive mutation on V_2 such that

$$\|\beta\|_v + \sum_{w \in V_2} [\tau(w, v) - \tau(v, w)] = \|\gamma\|_v \quad \text{for all } v \in V_2$$

and

$$\sum_{v, w \in V_2} \tau(v, w) = j.$$

We write $\tau_{v, w} := \tau(v, w)$. for all $v, w \in V_2$. So

$$\|\beta\|_v + \sum_{w < v} \tau_{w, v} - \sum_{w > v} \tau_{v, w} = \|\gamma\|_v$$

for all $v \in V_2$. Then by Lemma 1.3:

$$\begin{aligned} & \|\gamma\|_1!^{\mu_1} \dots \|\gamma\|_d!^{\mu_d} = \\ & = (\|\beta\|_1 - \tau_{1,2} - \dots - \tau_{1,d})!^{\mu_1} \cdot (\|\beta\|_2 + \tau_{1,2} - \tau_{2,3} - \dots - \tau_{2,d})!^{\mu_2} \dots \\ & \quad \dots \cdot (\|\beta\|_d + \tau_{1,d} + \dots + \tau_{d-1,d})!^{\mu_d} \\ & \leq \|\beta\|_1!^{\mu_1} \tau_{1,2}!^{-\mu_1} \dots \tau_{1,d}!^{-\mu_1} \cdot (2^{\mu_2})^{|\beta|+2+\tau_{1,2}} \|\beta\|_2!^{\mu_2} \tau_{1,2}!^{\mu_2} \tau_{2,3}!^{-\mu_2} \dots \tau_{2,d}^{-\mu_2} \cdot \\ & \quad \cdot (2^{\mu_3})^{2(|\beta|+3+\tau_{1,3}+\tau_{2,3})} \|\beta\|_3!^{\mu_3} \tau_{1,3}!^{\mu_3} \tau_{2,3}!^{\mu_3} \tau_{3,4}!^{-\mu_3} \dots \tau_{3,d}^{-\mu_3} \dots \\ & \quad \dots \cdot (2^{\mu_d})^{(d-1)(|\beta|+d+\tau_{1,d}+\dots+\tau_{d-1,d})} \|\beta\|_d!^{\mu_d} \tau_{1,d}!^{\mu_d} \dots \tau_{d-1,d}^{\mu_d} \\ & \leq (2^{\mu_2})^{d(|\beta|+2+\dots+|\beta|+d+\sum_{v,w} \tau_{v,w})} \|\beta\|_1!^{\mu_1} \dots \|\beta\|_d!^{\mu_d} \prod_{\substack{v, w \in V_2 \\ v < w}} \frac{\tau_{v,w}!^{\mu_w}}{\tau_{v,w}!^{\mu_v}} \\ & \leq (2^{d\mu_2})^{|\beta|+j} \|\beta\|_1!^{\mu_1} \dots \|\beta\|_d!^{\mu_d}. \end{aligned}$$

Note that the ordering of the μ 's is only used in the last inequality. \square

Lemma 2.14 *Let $p \in \mathbb{N}$ and let $m_1, \dots, m_p \in \mathbb{N}_0$. Let $j \in \mathbb{N}_0$. Then*

$$\sum_{\substack{j_1, \dots, j_p \in \mathbb{N}_0 \\ j_1 + \dots + j_p = j}} \frac{m_1^{j_1} \dots m_p^{j_p}}{j_1! \dots j_p!} = \frac{m^j}{j!}$$

where $m := m_1 + \dots + m_p$.

Proof. Let $z \in \mathbb{C}$. Then $\frac{m^{jk}}{j^k!}$ is the coefficient of z^{jk} in the power series of $e^{m_k z}$. Then

$$\sum_{\substack{j_1, \dots, j_p \in \mathbb{N}_0 \\ j_1 + \dots + j_p = j}} \frac{m_1^{j_1} \cdot \dots \cdot m_p^{j_p}}{j_1! \cdot \dots \cdot j_p!}$$

is the coefficient of z^j in $e^{m_1 z} \cdot \dots \cdot e^{m_p z} = e^{mz}$ and this coefficient is $\frac{m^j}{j!}$. □

Lemma 2.15 *Let $p \in \mathbb{N}$ and let $k_1, \dots, k_p, j_1, \dots, j_p \in \mathbb{N}_0$. Let $k := k_1 + \dots + k_p$ and $j := j_1 + \dots + j_p$. Suppose $j \leq k$. Then*

$$\binom{k_1}{j_1} \cdot \binom{k_1 + k_2 - j_1}{j_2} \cdot \dots \cdot \binom{k_1 + \dots + k_p - j_1 - \dots - j_{p-1}}{j_p} \leq \frac{k!}{(k-j)!} \cdot \frac{1}{j_1! \cdot \dots \cdot j_p!}$$

Proof. We may assume that $j_1 \leq k_1, \dots, j_p \leq k_1 + \dots + k_p - j_1 - \dots - j_{p-1}$. Then

$$\begin{aligned} & \binom{k_1}{j_1} \cdot \dots \cdot \binom{k_1 + \dots + k_p - j_1 - \dots - j_{p-1}}{j_p} = \\ &= \frac{k_1!}{j_1!(k_1 - j_1)!} \cdot \frac{(k_1 + k_2 - j_1)!}{j_2!(k_1 + k_2 - j_1 - j_2)!} \cdot \dots \cdot \frac{(k_1 + \dots + k_p - j_1 - \dots - j_{p-1})!}{j_p!(k - j)!} \\ &= \frac{k_1!}{j_1!} \binom{k_1 + k_2 - j_1}{k_2} \frac{k_2!}{j_2!} \binom{k_1 + k_2 + k_3 - j_1 - j_2}{k_3} \cdot \\ & \quad \dots \cdot \frac{k_{p-1}!}{j_{p-1}!} \binom{k - j_1 - \dots - j_{p-1}}{k_p} \frac{k_p!}{j_p!(k - j)!} \\ &\leq \frac{k_1!}{j_1!} \binom{k_1 + k_2}{k_2} \frac{k_2!}{j_2!} \binom{k_1 + k_2 + k_3}{k_3} \cdot \dots \cdot \frac{k_{p-1}!}{j_{p-1}!} \binom{k}{k_p} \frac{k_p!}{j_p!(k - j)!} \\ &= \frac{k!}{(k-j)!} \cdot \frac{1}{j_1! \cdot \dots \cdot j_p!} \end{aligned}$$

□

We now prove the main theorem of this section.

Theorem 2.16 *Let $d_1, d_2 \in \mathbb{N}$ and let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ be skew-Hermitian operators in a Hilbert space defined on a common invariant domain. Suppose*

$$[X_i, Y_j] \in \text{span}(\{Y_1, \dots, Y_{d_2}\})$$

for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$. Let $\lambda_1, \dots, \lambda_{d_1} \geq 1$ and let $\mu_1 \geq \dots \geq \mu_{d_2} \geq 0$. Then

$$\begin{aligned} S_{\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}}(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}) = \\ = S_{\lambda_1, \dots, \lambda_{d_1}}(X_1, \dots, X_{d_1}) \cap S_{\mu_1, \dots, \mu_{d_2}}(Y_1, \dots, Y_{d_2}) \end{aligned}$$

as locally convex spaces with equivalent spectra, in the following two cases:

- I. $\mu_1 = \dots = \mu_{d_2}$.
- II. $[X_i, Y_j] \in \text{span}(\{Y_j, \dots, Y_{d_2}\})$ for all $i \in \{1, \dots, d_1\}$ and $j \in \{1, \dots, d_2\}$.

Note that μ_j may be taken smaller than 1.

Proof. Let $V_1 := \{1, \dots, d_1\}$, $V_2 := \{1, \dots, d_2\}$ and let $M \geq 1$ and let $c_{\beta, \gamma}^\alpha$ be as in Lemma 2.12.

Let $t \geq 1$ and let $u \in S_{\lambda_1, \dots, \lambda_{d_1}; t}(X_1, \dots, X_{d_1}) \cap S_{\mu_1, \dots, \mu_{d_2}; t}(Y_1, \dots, Y_{d_2})$. Write $c_1 := \|u\|_{X_1, \dots, X_{d_1}; \lambda_1, \dots, \lambda_{d_1}; t}$ and $c_2 := \|u\|_{Y_1, \dots, Y_{d_2}; \mu_1, \dots, \mu_{d_2}; t}$. Then

$$\begin{aligned} \|X_\alpha u\| &\leq c_1 t^{|\alpha|} \|\alpha\|_1!^{\lambda_1} \dots \|\alpha\|_{d_1}!^{\lambda_{d_1}} \quad \text{for all } \alpha \in M(V_1), \\ \|Y_\beta u\| &\leq c_2 t^{|\beta|} \|\beta\|_1!^{\mu_1} \dots \|\beta\|_{d_2}!^{\mu_{d_2}} \quad \text{for all } \beta \in M(V_2). \end{aligned}$$

We shall prove that $\|u\|_{X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}; \lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}; t} \leq c_1 + c_2$ for some constant b independent of u (and in fact, also independent of t). Let $p' \in \mathbb{N}$, $\alpha'_1, \dots, \alpha'_{p'} \in M(V_1)$ and $\beta'_1, \dots, \beta'_{p'} \in M(V_2)$. Then

$$\begin{aligned} \|X_{\alpha'_1} Y_{\beta'_1} \circ \dots \circ X_{\alpha'_{p'}} Y_{\beta'_{p'}} u\|^2 &= \\ &= |(Y_{(\beta'_{p'})^r} X_{(\alpha'_{p'})^r} \circ \dots \circ Y_{(\beta'_1)^r} X_{(\alpha'_1)^r} X_{\alpha'_1} Y_{\beta'_1} \circ \dots \circ X_{\alpha'_{p'}} Y_{\beta'_{p'}} u, u)|. \end{aligned}$$

We introduce the following notation. Let $p := 2p' + 1$. Let

$$\begin{array}{ll} \alpha_1 := () & \alpha_{p'+1} := (\alpha'_1)^r \\ \beta_1 := (\beta'_{p'})^r & \beta_{p'+1} := () \\ \alpha_2 := (\alpha'_{p'})^r & \alpha_{p'+2} := \alpha'_1 \\ \beta_2 := (\beta'_{p'-1})^r & \beta_{p'+2} := \beta'_1 \\ \vdots & \vdots \\ \alpha_{p'} := (\alpha'_2)^r & \alpha_p := \alpha'_{p'} \\ \beta_{p'} := (\beta'_1)^r & \beta_p := \beta'_{p'}. \end{array}$$

By Lemmas 2.11 and 2.12 we obtain that

$$X_\alpha Y_\beta = \sum_{j=0}^{|\alpha|} \sum_{\sigma \in P_j^{|\alpha|}} \sum_{\substack{\gamma \in M(V_2) \\ \|\gamma\| = \|\beta\|}} c_{\beta, \gamma}^{\tilde{\sigma}(\alpha)} Y_\gamma X_{\tilde{\sigma}(\alpha)}$$

for all $\alpha \in M(V_1)$ and $\beta \in M(V_2)$. So $Z := Y_{(\beta'_{p'})^r} X_{(\alpha'_{p'})^r} \circ \dots \circ Y_{(\beta'_1)^r} X_{(\alpha'_1)^r} X_{\alpha'_1} Y_{\beta'_1} \circ \dots \circ X_{\alpha'_{p'}} Y_{\beta'_{p'}} = X_{\alpha_1} Y_{\beta_1} \circ \dots \circ X_{\alpha_p} Y_{\beta_p}$ is a linear combination of monomials $Y_\beta X_\alpha$. By induction to p it follows that Z is the sum of

$$\tilde{c}_{\beta_1, \gamma_1}^{\tilde{\sigma}_1(\delta_1)} \dots \tilde{c}_{\beta_p, \gamma_p}^{\tilde{\sigma}_p(\delta_p)} Y_{(\gamma_1, \dots, \gamma_p)} X_{\tilde{\sigma}_p(\delta_p)}$$

where the sum is over all $j_1 \in \mathbb{N}_0$, $j_1 \leq \|\delta_1\|$ with $\delta_1 := \alpha_1$, all $\sigma_1 \in P_{j_1}^{|\delta_1|}$, all $\gamma_1 \in M(V_2)$ with $\|\gamma_1\| = \|\beta_1\|$,

over all $j_2 \in \mathbb{N}_0$, $j_2 \leq \|\delta_2\|$ with $\delta_2 := (\tilde{\sigma}_1(\delta_1), \alpha_2)$, all $\sigma_2 \in P_{j_2}^{|\delta_2|}$, all $\gamma_2 \in M(V_2)$ with

$$\|\gamma_2\| = \|\beta_2\|, \dots$$

...

over all $j_p \in \mathbb{N}_0$, $j_p \leq \|\delta_p\|$ with $\delta_p := \langle \hat{\sigma}_{p-1}(\delta_{p-1}), \alpha_p \rangle$, all $\sigma_p \in P_{j_p}^{[\delta_p]}$, all $\gamma_p \in M(V_2)$ with $\|\gamma_p\| = \|\beta_p\|$. Consider one term of the sum

$$c_{\beta_1, \gamma_1}^{\tilde{\sigma}_1(\delta_1)} \cdots c_{\beta_p, \gamma_p}^{\tilde{\sigma}_p(\delta_p)} Y_{(\gamma_1, \dots, \gamma_p)} X_{\hat{\sigma}_p(\delta_p)}$$

which is *not zero* and which corresponds to the tuple $j_1, \delta_1, \sigma_1, \gamma_1, \dots, j_p, \delta_p, \sigma_p, \gamma_p$. Let $k_i := \|\alpha_i\|$ and $m_i := \|\beta_i\| = \|\gamma_i\|$ for all $i \in \{1, \dots, p\}$. Let $k := k_1 + \dots + k_p$, $m := m_1 + \dots + m_p$ and $j := j_1 + \dots + j_p$. Let $\alpha := \langle \alpha_1, \dots, \alpha_p \rangle$, $\beta := \langle \beta_1, \dots, \beta_p \rangle$ and $\gamma := \langle \gamma_1, \dots, \gamma_p \rangle$. Then $j_i \leq \|\delta_i\| = k_1 + \dots + k_i - j_1 - \dots - j_{i-1}$ for all $i \in \{1, \dots, p\}$. Moreover, $k = \|\alpha\|$ and $m = \|\beta\| = \|\gamma\|$. By Schwarz' inequality we obtain that

$$\begin{aligned} |(Y_{(\gamma_1, \dots, \gamma_p)} X_{\hat{\sigma}_p(\delta_p)} u, u)| &= |(X_{\hat{\sigma}_p(\delta_p)} u, Y_{\gamma^r} u)| \leq \|X_{\hat{\sigma}_p(\delta_p)} u\| \|Y_{\gamma^r} u\| \leq \\ &\leq c_1 c_2 t^{\|\hat{\sigma}_p(\delta_p)\| + m} \|\hat{\sigma}_p(\delta_p)\|_1^{\lambda_1} \cdots \|\hat{\sigma}_p(\delta_p)\|_{d_1}^{\lambda_{d_1}} \cdot \|\gamma\|_1^{\mu_1} \cdots \|\gamma\|_{d_2}^{\mu_{d_2}}. \end{aligned}$$

Let $l(v) := \|\alpha\|_v - \|\hat{\sigma}_p(\delta_p)\|_v$ for all $v \in V_1$. Since $\sigma_i \in P_{j_i}^{[\delta_i]}$ for all i , it follows by induction to p that $l(1), \dots, l(d_1) \in \mathbb{N}_0$ and that $\sum_{v \in V_1} l(v) = j$. (See Lemma 2.7.) Then by Lemma 1.3:

$$\begin{aligned} \|\hat{\sigma}_p(\delta_p)\|_1^{\lambda_1} \cdots \|\hat{\sigma}_p(\delta_p)\|_{d_1}^{\lambda_{d_1}} &\leq \|\alpha\|_1^{\lambda_1} \cdots \|\alpha\|_{d_1}^{\lambda_{d_1}} (l(1))^{-\lambda_1} \cdots (l(d_1))^{-\lambda_{d_1}} \\ &\leq \|\alpha\|_1^{\lambda_1} \cdots \|\alpha\|_{d_1}^{\lambda_{d_1}} (l(1))^{-1} \cdots (l(d_1))^{-1} \\ &\leq \|\alpha\|_1^{\lambda_1} \cdots \|\alpha\|_{d_1}^{\lambda_{d_1}} 2^{d_1 j} j^{-1}. \end{aligned}$$

This is the only place where we use that $\lambda_1, \dots, \lambda_{d_1} \geq 1$.

Next we estimate the factor $\|\gamma\|_1^{\mu_1} \cdots \|\gamma\|_{d_2}^{\mu_{d_2}}$ in the cases I and II.

Case I. Suppose $\mu_1 = \dots = \mu_{d_2}$.

Then $\|\gamma\|_1^{\mu_1} \cdots \|\gamma\|_{d_2}^{\mu_{d_2}} \leq \|\gamma\|^{\mu_1} = \|\beta\|^{\mu_1} \leq 2^{\mu_1 d_2 m} \|\beta\|_1^{\mu_1} \cdots \|\beta\|_{d_2}^{\mu_{d_2}}$ by Lemma 1.3.

Case II. Suppose $[X_i, Y_j] \in \text{span}(\{Y_j, \dots, Y_{d_2}\})$ for all $i \in V_1$ and $j \in V_2$.

This case needs more care. Now we can use Lemma 2.12.II and so we may assume that the constants $c_{\beta, \gamma}^\alpha$ are as in Lemma 2.12.II. Recall that we are considering a non-zero term of a large sum. So the coefficients $c_{\beta_i, \gamma_i}^{\tilde{\sigma}_i(\delta_i)}$ are not zero for all $i \in \{1, \dots, p\}$. Hence by Lemma 2.12.II we obtain that β_i is connected with γ_i via a positive mutation of length $\|\tilde{\sigma}_i(\delta_i)\| = j_i$. (See Lemma 2.7.) Therefore by Lemma 2.8.II, $\beta = \langle \beta_1, \dots, \beta_p \rangle$ is connected with $\gamma = \langle \gamma_1, \dots, \gamma_p \rangle$ via a positive mutation of length $j = j_1 + \dots + j_p$. But then by Lemma 2.13,

$$\begin{aligned} \|\gamma\|_1^{\mu_1} \cdots \|\gamma\|_{d_2}^{\mu_{d_2}} &\leq (2^{d_2 \mu_2})^{m+j} \|\beta\|_1^{\mu_1} \cdots \|\beta\|_{d_2}^{\mu_{d_2}} \\ &\leq (2^{d_2 \mu_2})^{m+k} \|\beta\|_1^{\mu_1} \cdots \|\beta\|_{d_2}^{\mu_{d_2}}. \end{aligned}$$

Having estimated $\|\gamma\|_1^{\mu_1} \cdots \|\gamma\|_{d_2}^{\mu_{d_2}}$ in the two cases we obtain that there exists $b_1 \geq 1$, independent of u (and t) such that

$$\|\gamma\|_1!^{\mu_1} \cdots \|\gamma\|_{d_2}!^{\mu_{d_2}} \leq b_1^{k+m} \|\beta\|_1!^{\mu_1} \cdots \|\beta\|_{d_2}!^{\mu_{d_2}}.$$

Then

$$\begin{aligned} & |(Y_{\langle \gamma_1, \dots, \gamma_p \rangle} X_{\partial_p(\delta_p)} u, u)| \leq \\ & \leq c_1 c_2 (b_2 t)^{k+m} j!^{-1} \|\alpha\|_1!^{\lambda_1} \cdots \|\alpha\|_{d_1}!^{\lambda_{d_1}} \cdot \|\beta\|_1!^{\mu_1} \cdots \|\beta\|_{d_2}!^{\mu_{d_2}}, \end{aligned}$$

where $b_2 := 2^{d_1} b_1$.

For brevity, we write $C := \|\alpha\|_1!^{\lambda_1} \cdots \|\alpha\|_{d_1}!^{\lambda_{d_1}} \cdot \|\beta\|_1!^{\mu_1} \cdots \|\beta\|_{d_2}!^{\mu_{d_2}}$. We count the number of terms in the sum. Since $\sigma_i \in P_{j_i}^{|\delta_i|}$ and $\|\delta_i\| = k_1 + \dots + k_i - j_1 - \dots - j_{i-1}$, there are $\binom{k_1 + \dots + k_i - j_1 - \dots - j_{i-1}}{j_i}$ permutations σ_i for all $i \in \{1, \dots, p\}$. There are $d_2^{m_i}$ multi-indices $\gamma_i \in M(V_2)$ with $\|\gamma_i\| = \|\beta_i\| = m_i$. Furthermore, $\left| c_{\beta_i, \gamma_i}^{\tilde{\sigma}_i(\delta_i)} \right| \leq (M m_i)^{\|\tilde{\sigma}_i(\delta_i)\|} = (M m_i)^{j_i}$ for all $i \in \{1, \dots, p\}$. Hence we obtain by the triangle inequality:

$$\begin{aligned} |(Zu, u)| & \leq \sum_{j_1=0}^{\infty} \cdots \sum_{j_p=0}^{\infty} \binom{k_1}{j_1} \cdots \binom{k_1 + \dots + k_p - j_1 - \dots - j_{p-1}}{j_p} d_2^{m_1 + \dots + m_p} \\ & \quad \cdot (M m_1)^{j_1} \cdots (M m_p)^{j_p} c_1 c_2 (b_2 t)^{k+m} (j_1 + \dots + j_p)!^{-1} C \\ & \leq c_1 c_2 (b_2 d_2 M t)^{k+m} C \sum_{j=0}^k \sum_{\substack{j_1, \dots, j_p \in \mathbb{N}_0 \\ j_1 + \dots + j_p = j}} \\ & \quad \binom{k_1}{j_1} \cdots \binom{k_1 + \dots + k_p - j_1 - \dots - j_{p-1}}{j_p} m_1^{j_1} \cdots m_p^{j_p} \cdot j!^{-1} \\ & \leq c_1 c_2 (b_2 d_2 M t)^{k+m} C \sum_{j=0}^k \sum_{\substack{j_1, \dots, j_p \in \mathbb{N}_0 \\ j_1 + \dots + j_p = j}} \frac{k!}{(k-j)! j!} \cdot \frac{m_1^{j_1} \cdots m_p^{j_p}}{j_1! \cdots j_p!}. \end{aligned}$$

The last inequality is due to Lemma 2.15. So by Lemma 2.14:

$$\begin{aligned} |(Zu, u)| & \leq \\ & \leq c_1 c_2 (b_2 d_2 M t)^{k+m} C \sum_{j=0}^k \binom{k}{j} \frac{m^j}{j!} \\ & \leq c_1 c_2 (2eb_2 d_2 M t)^{k+m} C \\ & \leq \left[(c_1 + c_2) (b_3 t)^{\frac{k+m}{2}} \left(\frac{\|\alpha\|_1}{2} \right)!^{\lambda_1} \cdots \left(\frac{\|\alpha\|_{d_1}}{2} \right)!^{\lambda_{d_1}} \left(\frac{\|\beta\|_1}{2} \right)!^{\mu_1} \cdots \left(\frac{\|\beta\|_{d_2}}{2} \right)!^{\mu_{d_2}} \right]^2 \end{aligned}$$

where $b_3 := 2eb_2 d_2 M 2^\tau$, with $\tau := \max\{\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}\}$. Note that

$\|\langle \alpha'_1, \dots, \alpha'_p \rangle\|_v = \frac{1}{2} \|\alpha\|_v$ for all $v \in V_1$, $\|\langle \beta'_1, \dots, \beta'_p \rangle\|_v = \frac{1}{2} \|\beta\|_v$ for all $v \in V_2$ and $\|\langle \alpha'_1, \dots, \alpha'_p \rangle\| + \|\langle \beta'_1, \dots, \beta'_p \rangle\| = \frac{1}{2}(k+m)$. So

$$u \in S_{\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}; b_3 t}(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2})$$

and $\|u\|_{\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}; b_3 t} \leq c_1 + c_2 = \|u\|_{\lambda_1, \dots, \lambda_{d_1}; t} + \|u\|_{\mu_1, \dots, \mu_{d_2}; t}$, where the constant b_3 is independent of u (and t). This finishes the proof of the theorem since the space

$S_{\lambda_1, \dots, \lambda_{d_1}, \mu_1, \dots, \mu_{d_2}; t}(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2})$ is always continuously embedded in

$S_{\lambda_1, \dots, \lambda_{d_1}; t}(X_1, \dots, X_{d_1}) \cap S_{\mu_1, \dots, \mu_{d_2}; t}(Y_1, \dots, Y_{d_2})$ for all $t > 0$. \square

Corollary 2.17 *Let \mathfrak{g} be a real Lie algebra of skew-Hermitian operators in a Hilbert space defined on a common invariant domain. Let $d_1, d_2 \in \mathbb{N}$ and let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2} \in \mathfrak{g}$. Suppose*

$$\mathfrak{g} = \text{span}(\{X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}\})$$

and suppose that $\text{span}(\{Y_1, \dots, Y_{d_2}\})$ is an ideal in \mathfrak{g} . Let $\lambda_1, \dots, \lambda_{d_1} \geq 1$ and let $\mu \geq 0$. Then

$$S_{\lambda_1, \dots, \lambda_{d_1}, \mu, \dots, \mu}(X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}) = S_{\lambda_1, \dots, \lambda_{d_1}}(X_1, \dots, X_{d_1}) \cap S_{\mu, \dots, \mu}(Y_1, \dots, Y_{d_2})$$

as locally convex spaces with equivalent spectra.

Corollary 2.18 *Let \mathfrak{g} be a real Lie algebra of skew-Hermitian operators in a Hilbert space defined on a common invariant domain. Let X_1, \dots, X_d be an ordered basis in \mathfrak{g} and let $\lambda_1 \geq 1$ and $\lambda_2 \geq \dots \geq \lambda_{d-1} \geq \max(\lambda_d, 1) \geq \lambda_d \geq 0$. Then*

$$S_{\lambda_1, \dots, \lambda_d}(X_1, \dots, X_d) = S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(X_1, \dots, X_d) = \bigcap_{k=1}^d S_{\lambda_k}(X_k)$$

as locally convex spaces with equivalent spectra.

N.B. Recall the Remark following Corollary 2.3.

2.4 Non-triviality of certain Gevrey spaces

Let G be a nilpotent Lie group with Lie algebra \mathfrak{g} . Let X_1, \dots, X_d be a basis in \mathfrak{g} . The basis X_1, \dots, X_d is called a *strictly ordered basis* in \mathfrak{g} if

$$[X_i, X_j] \in \text{span}(\{X_{\max(i,j)+1}, \dots, X_d\})$$

for all $i, j \leq d$. Then X_1, \dots, X_d is a strictly ordered basis in \mathfrak{g} if and only if X_d, \dots, X_1 is an ordered Jordan-Hölder basis in \mathfrak{g} . (See [Goo5].) Let π be a representation of G in a Hilbert space H . In the previous section we considered the Gevrey space

$$S_{\lambda_1, \dots, \lambda_d}(d\pi(X_1), \dots, d\pi(X_d))$$

where X_1, \dots, X_d is an ordered basis in \mathfrak{g} and $\lambda_1 \geq 1$ and $\lambda_2 \geq \dots \geq \lambda_{d-1} \geq \max(\lambda_d, 1) \geq \lambda_d \geq 0$. We know that this Gevrey space is dense in the Hilbert space H in case $\lambda_d \geq 1$. (See Corollary 1.26.) In this section we prove that this Gevrey space is dense in H if X_1, \dots, X_d is a strictly ordered basis in \mathfrak{g} , $\lambda_1, \dots, \lambda_{d-1} \geq 1$ and $\lambda_d > 0$.

In case the representation π is irreducible and X_1, \dots, X_d is a strictly ordered basis in \mathfrak{g} , it is easy to show that even $S_{1, \dots, 1, 0}(d\pi(X_1), \dots, d\pi(X_d))$ is dense in H . Indeed, because X_d belongs to the center of \mathfrak{g} and $\partial\pi(X_d)$ is closable, it follows by Taylor, [Tay], Chapter 0 Propositions 4.3 and 4.5 that there exists $\alpha \in \mathbb{C}$ such that $\partial\pi(X_d) = \alpha I$. Then

$S_{1,\dots,1,0}(d\pi(X_1), \dots, d\pi(X_d)) = S_{1,\dots,1,1}(d\pi(X_1), \dots, d\pi(X_d))$ is dense in H . Note that in general, for non-irreducible representations π , the operator $d\pi(X_d)$ is not bounded.

So let X_1, \dots, X_d be a fixed strictly ordered basis in \mathfrak{g} and let π be a (not necessarily irreducible) representation of G in a Hilbert space H . Let $\mathfrak{g}_\mathbb{C}$ be the complexification of \mathfrak{g} and let $G_\mathbb{C}$ be a connected simply connected complex Lie group with Lie algebra $\mathfrak{g}_\mathbb{C}$. (See Varadarajan, [Var1], Theorem 3.15.1.) Let \exp denote the exponential map from $\mathfrak{g}_\mathbb{C}$ onto $G_\mathbb{C}$. Without loss of generality we may assume that $G = \exp(\mathfrak{g})$. For all $k \in \{1, \dots, d\}$ define $g_k : \mathbb{C} \rightarrow G_\mathbb{C}$ by

$$g_k(z) := \exp(zX_k) \quad (z \in \mathbb{C}).$$

Define $g : \mathbb{C}^d \rightarrow G_\mathbb{C}$ by

$$g(z_1, \dots, z_d) := g_d(z_d) \cdots g_1(z_1) \quad (z_1, \dots, z_d \in \mathbb{C}).$$

By [Var1], Theorem 3.18.11, the map g is an analytic diffeomorphism from \mathbb{C}^d onto $G_\mathbb{C}$ and the map $g|_{\mathbb{R}^d}$ is an analytic diffeomorphism from \mathbb{R}^d into G . As usual, we start with some lemmas.

Lemma 2.19 *Let $k \in \{1, \dots, d\}$. Then there exists polynomials $P_{k,j} : \mathbb{C}^{j-k} \rightarrow \mathbb{C}$, where $j \in \{k+2, \dots, d\}$ such that for all $z \in \mathbb{C}$ and all $t_1, \dots, t_d \in \mathbb{C}$ we have*

$$g_k(z)g(t_d, \dots, t_1) = g(s_d, \dots, s_1)$$

with

$$\begin{aligned} s_j &= t_j + P_{k,j}(z, t_{k+1}, \dots, t_{j-1}) && \text{if } j > k+1, \\ s_{k+1} &= t_{k+1}, \\ s_k &= t_k + z, \\ s_j &= t_j && \text{if } j < k. \end{aligned}$$

Proof. See [Goo5], Lemma 5.1. □

Lemma 2.20 *Let $X \in \mathfrak{g}$, let $\lambda \in (0, 1)$ and let $u \in H$. Define $F : \mathbb{R} \rightarrow H$ by*

$$F(t) := \pi_{\exp(tX)} u \quad (t \in \mathbb{R}).$$

Suppose the function F extends holomorphically to an entire function from \mathbb{C} into H of exponential order $\leq (1-\lambda)^{-1}$. Then $u \in S_\lambda(d\pi(X))$.

Proof. We denote the extension of F also by F . By assumption, there exist $A, B > 0$ such that

$$\|F(z)\| \leq A \exp(B|z|^{(1-\lambda)^{-1}})$$

for all $z \in \mathbb{C}$. By [Goo2], Propositions 4.1 and 2.2 we obtain that $u \in D^\infty(d\pi(X))$ and

$$F(z) = \sum_{n=0}^{\infty} z^n n!^{-1} d\pi(X)^n u$$

for all $z \in \mathbb{C}$. So in particular, the series converges absolutely in H . Then for all $n \in \mathbb{N}_0$ and all $v \in H$ we obtain for all $R > 0$:

$$\begin{aligned} |(d\pi(X)^n u, v)| &= \left| \left(\frac{d}{dz} \right)^n (F(z), v) \Big|_{z=0} \right| = \left| \frac{n!}{2\pi i} \int_{\Gamma_R} \frac{(F(\xi), v)}{\xi^{n+1}} d\xi \right| \\ &\leq \frac{n!}{2\pi} \int_{\Gamma_R} \frac{\|F(\xi)\|}{|\xi|^{n+1}} d|\xi| \cdot \|v\|. \end{aligned}$$

So

$$\|d\pi(X)^n u\| \leq \frac{n!}{2\pi} \int_{\Gamma_R} \frac{\|F(\xi)\|}{|\xi|^{n+1}} d|\xi|$$

for all $n \in \mathbb{N}_0$. Choosing R suitably, the lemma follows similarly to the proof of [GS], Section IV.7.5 Theorem 3. \square

Lemma 2.21 *Let $X \in \mathfrak{g}$, $\lambda \in (0, 1)$, $u \in H$ and $f \in L^1(G)$. Suppose the map $t \mapsto L_{\exp(tX)} f$ from \mathbb{R} into $L^1(G)$ extends holomorphically to an entire function from \mathbb{C} into $L^1(G)$ of exponential order $\leq (1 - \lambda)^{-1}$. Then $\pi(f)u \in S_\lambda(d\pi(X))$.*

Proof. Define $F : \mathbb{R} \rightarrow H$ by

$$F(t) := \pi_{\exp(tX)} \pi(f)u \quad (t \in \mathbb{R}).$$

Then $F(t) = \pi(L_{\exp(tX)} f)u$ for all $t \in \mathbb{R}$. Clearly F extends to an entire function from \mathbb{C} into H of exponential order $\leq (1 - \lambda)^{-1}$. So by Lemma 2.20, $\pi(f)u \in S_\lambda(d\pi(X))$. \square

Theorem 2.22 *Let G be a nilpotent Lie group with Lie algebra \mathfrak{g} . Let π be a representation of G in a Hilbert space H and let X_1, \dots, X_d be a strictly ordered basis in \mathfrak{g} . Let $\lambda_d \in \mathfrak{g}$. Then $S_{1, \dots, 1, \lambda_d}(d\pi(X_1), \dots, d\pi(X_d))$ is a dense subspace of H .*

Proof. We may suppose that G is connected and simply connected. Let \mathfrak{g} , G_c , g_k and g be as in the beginning of this section. Let the polynomials $P_{k,j}$ be as in Lemma 2.19. Let m be the maximum of the degrees of these polynomials $P_{k,j}$. For $k \in \{1, \dots, d\}$ let

$$p_k := (1 + m)^{d-k} (1 - \lambda_d)^{-1}$$

and let $p_{d+1} := p_{d+2} := 0$. For $k \in \{1, \dots, d\}$ let

$$q_k := \max(p_k, mp_{k+2})$$

and let

$$\lambda_k := 1 - q_k^{-1}$$

for all $k \in \{1, \dots, d-1\}$. Then also $\lambda_d = 1 - q_d^{-1}$ and $q_k \geq p_k \geq (1 - \lambda_d)^{-1} > 1$ for all $k \in \{1, \dots, d-1\}$, hence $\lambda_k > 0$. We shall prove that $\bigcap_{k=1}^d S_{\lambda_k}(d\pi(X_k))$ is dense in H . Since all $\lambda_k < 1$, then also $\bigcap_{k=1}^{d-1} S_1(d\pi(X_k)) \cap S_{\lambda_d}(d\pi(X_d))$ is dense in H . Hence by Corollary 2.18 and the Remark following Corollary 2.3, $S_{1, \dots, 1, \lambda_d}(d\pi(X_1), \dots, d\pi(X_d)) = \bigcap_{k=1}^{d-1} S_1(d\pi(X_k)) \cap S_{\lambda_d}(d\pi(X_d))$ is dense in H .

We shall prove the following assertion.

There exists a dense set Z_2 of $L^1(G)$ such that $\pi(f)(H) \subset \bigcap_{k=1}^d S_{\lambda_k}(d\pi(X_k))$ for all $f \in Z_2$.

Because the representation π is continuous, $\bigcup_{f \in Z_2} \pi(f)(H)$ is dense in H .

Let Z be the set of all entire functions F on \mathbb{C}^d for which there exist constants $A, B, C > 0$ (depending on F) such that

$$|F(z_1, \dots, z_d)| \leq C \exp[-A \sum_{k=1}^d |\operatorname{Re} z_k|^{p_k} + B \sum_{k=1}^d |\operatorname{Im} z_k|^{p_k}]$$

for all $(z_1, \dots, z_d) \in \mathbb{C}^d$. Since $p_1, \dots, p_d > 1$, the set $\{F|_{\mathbb{R}^d} : F \in Z\}$ is dense in $L^1(\mathbb{R}^d)$ according to [GS], Section IV.9. Let

$$Z_1 := \{f \in \mathcal{O}^{G_c} : f \circ g \in Z\}.$$

Finally, let

$$Z_2 := \{f|_G : f \in Z_1\}.$$

By Pukanszky, [Puk], page 90, the map $f \mapsto \int_{\mathbb{R}^d} f(g(z)) dz$ is a Haar integral on G , hence the set Z_2 is dense in $L^1(G)$. Similarly, Z_1 is dense in $L^1(G_c)$ and in particular, a subset of $L^1(G_c)$.

Let $f \in Z_1$, let $F := f \circ g$ and let A, B, C be constants corresponding to F . Let $k \in \{1, \dots, d\}$. There exists $C_1 > 0$ such that for all $j \in \{1, \dots, d\}$ with $j > k + 1$, all $t_1, \dots, t_d \in \mathbb{R}$ and all $z \in \mathbb{C}$ we have

$$|\operatorname{Im} s_j| \leq C_1 \left(1 + \sum_{l=k+1}^{j-1} |t_l|^m + |z|^m \right)$$

and

$$|\operatorname{Re} s_j| \geq |t_j| - C_1 \left(1 + \sum_{l=k+1}^{j-1} |t_l|^m + |z|^m \right)$$

with $s_j := t_j + P_{k,j}(z, t_{k+1}, \dots, t_{j-1})$. Using the inequalities $|a + b|^p \leq 2^p(|a|^p + |b|^p)$ and $-|a - b|^p \leq -2^{-p}|a|^p + |b|^p$ for all $a, b \in \mathbb{R}$ and $p \geq 1$, and using the fact that $p_1 \geq \dots \geq p_d \geq 1$, we obtain by Lemma 2.19 that for all $t_1, \dots, t_d \in \mathbb{R}$ and all $z \in \mathbb{C}$:

$$\begin{aligned} & |f(g_k(z)g(t_1, \dots, t_d))| \leq \\ & \leq C \exp \left[-A \sum_{j=1}^{k-1} |t_j|^{p_j} - A|t_k + \operatorname{Re} z|^{p_k} - A|t_{k+1}|^{p_{k+1}} \right. \\ & \quad \left. - A \sum_{j=k+2}^d \left| |t_j| - C_1 \left(1 + \sum_{l=k+1}^{j-1} |t_l|^m + |z|^m \right) \right|^{p_j} \right. \\ & \quad \left. + B|\operatorname{Im} z|^{p_k} + B \sum_{j=k+2}^d \left| C_1 \left(1 + \sum_{l=k+1}^{j-1} |t_l|^m + |z|^m \right) \right|^{p_j} \right] \leq \\ & \leq C_2 \exp \left[-A_1 \sum_{j=1}^d |t_j|^{p_j} + B_1 \sum_{j=1}^d |t_j|^{m p_{j+1}} + C_3 |z|^{q_k} \right] \end{aligned}$$

where A_1, B_1, C_2, C_3 are positive constants which depend only on A, B, C, C_1 and p_1, \dots, p_d . For $z \in \mathbb{C}$ define $T_z f : G \rightarrow \mathbb{C}$ by

$$(T_z f)(x) := f(g_k(z)x) \quad (x \in G).$$

Since $mp_{j+1} < p_j$ for all $j \in \{1, \dots, d\}$, we have $T_z f \in L^1(G)$ for all $z \in \mathbb{C}$. Hence the map $t \mapsto T_t f$ from \mathbb{R} into $L^1(G)$ extends holomorphically to an entire function from \mathbb{C} into $L^1(G)$ of exponential order $\leq q_k$. So the map $t \mapsto L_{\exp(tX_k)}(f|_G) = T_{-t} f$ from \mathbb{R} into $L^1(G)$ extends holomorphically to an entire function from \mathbb{C} into $L^1(G)$ of exponential order $\leq q_k = (1 - \lambda_k)^{-1}$. Then by Lemma 2.21, $\pi(f|_G)u \in S_{\lambda_k}(d\pi(X_k))$ for all $u \in H$. This proves the assertion. \square

Corollary 2.23 *Let π be a representation of a nilpotent Lie group G in a Hilbert space H . Let X_1, \dots, X_d be a strictly ordered basis in the Lie algebra of G . Let $\lambda_1, \dots, \lambda_{d-1} \geq 1$ and let $\lambda_d > 0$. Then $S_{\lambda_1, \dots, \lambda_d}(d\pi(X_1), \dots, d\pi(X_d))$ is dense in H .*

Proof. Theorem 2.22 and Lemma 1.1. \square

There exists another class of Lie groups for which every Gevrey space of infinitesimal operators of a representation of such Lie group is dense in H , viz. the class of compact Lie groups.

Theorem 2.24 *Let π be a representation of a compact Lie group G in a Hilbert space H . Let X_1, \dots, X_d be any basis in the Lie algebra of G . Let $\lambda_1, \dots, \lambda_d \geq 0$. Then $S_{\lambda_1, \dots, \lambda_d}(d\pi(X_1), \dots, d\pi(X_d))$ is dense in H .*

Proof. By a well-known theorem ([HR2], Theorem 27.44) there exists an index set I and for all $i \in I$ there exists a finite dimensional π -invariant subspace H_i of H such that $H = \bigoplus_{i \in I} H_i$. Let $\pi_i := \pi|_{H_i}$. Then π_i is a finite dimensional representation of G in H_i . So each infinitesimal operator $d\pi_i(X_k)$ is densely defined, hence everywhere defined and therefore continuous, for all $i \in I$ and $k \in \{1, \dots, d\}$. Hence

$$\begin{aligned} H_i &= S_{0, \dots, 0}(d\pi_i(X_1), \dots, d\pi_i(X_d)) \\ &\subset S_{0, \dots, 0}(d\pi(X_1), \dots, d\pi(X_d)) \\ &\subset S_{\lambda_1, \dots, \lambda_d}(d\pi(X_1), \dots, d\pi(X_d)) \end{aligned}$$

for all $i \in I$. Since $\text{span}(\bigcup_{i \in I} H_i)$ is dense in H , the Gevrey space $S_{\lambda_1, \dots, \lambda_d}(d\pi(X_1), \dots, d\pi(X_d))$ is dense in H . \square

2.5 Separate and joint Gevrey vectors for a representation of a Lie group

Let G be a real Lie group with Lie algebra \mathfrak{g} . Let $\lambda \geq 1$ and let π be a representation of G in a Hilbert space. We know that the Gevrey space $S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d))$ does not depend on the choice of a basis X_1, \dots, X_d in \mathfrak{g} . Also we have

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) \subset \bigcap_{k=1}^d S_{\lambda}(d\pi(X_k))$$

for any basis X_1, \dots, X_d in \mathfrak{g} . In this section we shall prove that there exists a basis X_1, \dots, X_d in \mathfrak{g} such that

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = \bigcap_{k=1}^d S_{\lambda}(d\pi(X_k)).$$

In the special cases that $\lambda = 1$, the existence of such basis has been proved first by Flato and Simon. (See [FS], Theorem 3.)

Lemma 2.25 *Let A be a Hermitian or skew-Hermitian operator in a Hilbert space which has an invariant domain D . Let $\lambda \geq 0$. Let $t \geq 1$. Then there exists $s \geq t$ with the following property: for all $C > 0$ we have*

$$\{u \in D : \forall m \in \mathbb{N}_0 \left[\|A^{2^m} u\| \leq Ct^{2^m} (2^m)!^\lambda \right]\} \subset \{u \in D : \|u\|_{A; \lambda; s} \leq \max(C, \|u\|)\}.$$

Proof. The corresponding statement for $\lambda = 1$ can be found in [FS], Lemma 1, but there the proof is based on different arguments.

Let $s := 4^\lambda t$. Let $C > 0$, let $u \in D$ and suppose that $\|A^{2^m} u\| \leq Ct^{2^m} (2^m)!^\lambda$ for all $m \in \mathbb{N}_0$. For $n \in \mathbb{N}$ hypothesis $P(n)$ states

$$\|A^k u\| \leq Cs^k k!^\lambda \quad \text{for all } k \in \{1, \dots, n\}.$$

Clearly hypotheses $P(1)$ and $P(2)$ are valid. Let $n \in \mathbb{N}$, $n \geq 2$ and suppose $P(n-1)$ is valid. If ${}^2\log n \in \mathbb{N}$ then hypothesis $P(n)$ holds. Suppose ${}^2\log n \notin \mathbb{N}$. There exist unique $m, k \in \mathbb{N}_0$ such that $n = 2^m + k$ and $1 \leq k < 2^m$. Then $2k < 2^m + k = n$, hence $2k \leq n-1$. So by assumption and hypothesis $P(n-1)$ we obtain:

$$\begin{aligned} \|A^n u\|^2 &= |(A^{2^{m+1}} u, A^{2k} u)| \leq \|A^{2^{m+1}} u\| \|A^{2k} u\| \\ &\leq C^2 t^{2^{m+1}} s^{2k} (2^{m+1})!^\lambda (2k)!^\lambda \leq C^2 t^{2^{m+1}} s^{2k} 2^{\lambda 2^{m+1}} 2^{2\lambda k} (2^m)!^{2\lambda} k!^{2\lambda} \\ &\leq C^2 t^{2^{m+1}} s^{2k} (2^\lambda)^{2(2^m+k)} (2^m + k)!^{2\lambda} \\ &\leq [Cs^n n!^\lambda]^2. \end{aligned}$$

This proves the lemma. □

Theorem 2.26 *Let G be a compact Lie group with Lie algebra \mathfrak{g} . Let $\lambda \geq 1$ and let π be a representation of G in a Hilbert space H . Let X_1, \dots, X_d be any basis in \mathfrak{g} . Then*

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = \bigcap_{k=1}^d S_{\lambda}(d\pi(X_k))$$

as locally convex spaces with equivalent spectra.

Proof. Cf. the proof of Flato and Simon [FS], Theorem 1, for the case $\lambda = 1$.

The compactness of G insures that there exists a positive definite invariant real symmetric bilinear form β on $\mathfrak{g} \times \mathfrak{g}$. (See [Hoc], Theorem XIII.1.1.) Let Y_1, \dots, Y_d be a basis in \mathfrak{g} such that $\beta(Y_i, Y_j) = \delta_{i,j}$ for all $i, j \in \{1, \dots, d\}$. By [Nel], Lemma 6.1 there exists a constant $M_1 \geq 1$ such that for all $i, j \in \{1, \dots, d\}$ and all $u \in H^\infty(\pi)$ we have

$$\|\partial\pi(X_i X_j)u\| \leq M_1 \|\partial\pi(I - \Delta_X)u\|,$$

where $\Delta_X := \sum_{k=1}^d X_k^2 \in U(\mathfrak{g})$. So there exists a constant $M \geq 1$ such that for all $u \in H^\infty(\pi)$:

$$\|\partial\pi(I - \Delta_Y)u\| \leq \frac{M}{d+1} \|\partial\pi(I - \Delta_X)u\|,$$

where $\Delta_Y := \sum_{k=1}^d Y_k^2 \in U(\mathfrak{g})$.

Now let $t \geq 1$. Let $u \in \bigcap_{k=1}^d S_{\lambda, t}(d\pi(X_k))$. Then $u \in \bigcap_{k=1}^d D^\infty(d\pi(X_k)) = H^\infty(\pi) = D^\infty(d\pi(X_1), \dots, d\pi(X_d))$. (See Theorem 1.23.) Since G is compact, there exists an index set I and for all $\alpha \in I$ there exists a π -invariant subspace H_α of H such that $\pi_\alpha := \pi|_{H_\alpha}$ is irreducible and $H = \bigoplus_{\alpha \in I} H_\alpha$. Let $u_\alpha \in H_\alpha$ be the projection of u on H_α . Note that $H_\alpha \subset H^\infty(\pi)$.

Let $\alpha \in I$. By [Bou], Chapter I §3.7 Proposition 11, Δ_Y belongs to the center of $U(\mathfrak{g})$. Since π_α is irreducible, it follows by [Tay], Chapter 0 Propositions 4.3 and 4.5 that there exists $\delta_\alpha \in \mathbb{C}$ such that $\partial\pi_\alpha(\Delta_Y) = -\delta_\alpha I$. Because the operator $\partial(\Delta_Y)$ is negative, we obtain that $\delta_\alpha \geq 0$. Then $(1 + \delta_\alpha)\|u_\alpha\| = \|\partial\pi_\alpha(I - \Delta_Y)u_\alpha\| = \|\partial\pi(I - \Delta_Y)u_\alpha\| \leq \frac{M}{d+1} \|\partial\pi(I - \Delta_X)u_\alpha\| \leq \frac{M}{d+1} \sum_{k=1}^d \|\partial\pi(X_k)^2 u_\alpha\|$, where $X_0 := I \in \mathbb{C} \subset U(\mathfrak{g})$. So there exists $k_\alpha \in \{0, \dots, d\}$ such that $(1 + \delta_\alpha)\|u_\alpha\| \leq M \|\partial\pi(X_{k_\alpha})^2 u_\alpha\|$.

For all $m \in \mathbb{N}_0$ let hypothesis $P(m)$ state:

$$(1 + \delta_\alpha)^{2m} \|u_\alpha\| \leq M^{2m} \|\partial\pi(X_{k_\alpha})^{2m+1} u_\alpha\|.$$

We have already proved hypothesis $P(0)$. Let $m \in \mathbb{N}_0$ and suppose $P(m)$ holds. Then

$$\begin{aligned} (1 + \delta_\alpha)^{2m+1} \|u_\alpha\| \|u_\alpha\| &= \left[(1 + \delta_\alpha)^{2m} \|u_\alpha\| \right]^2 \leq M^{2m+1} |(\partial\pi(X_{k_\alpha})^{2m+2} u_\alpha, u_\alpha)| \\ &\leq M^{2m+1} \|\partial\pi(X_{k_\alpha})^{2m+2} u_\alpha\| \|u_\alpha\|. \end{aligned}$$

So $P(m)$ is valid for all $m \in \mathbb{N}_0$.

For $k \in \{1, \dots, d\}$ let $c_k := \|u\|_{d\pi(X_k); \lambda; t}$ and let $c_0 := \|u\|$. Let $m \in \mathbb{N}_0$. Then

$$\begin{aligned} \|\partial\pi(I - \Delta_Y)^{2m} u\|^2 &= \sum_{\alpha \in I} \|\partial\pi(I - \Delta_Y)^{2m} u_\alpha\|^2 = \sum_{\alpha \in I} \left[(1 + \delta_\alpha)^{2m} \|u_\alpha\| \right]^2 \\ &\leq \sum_{\alpha \in I} \left[M^{2m} \|\partial\pi(X_{k_\alpha})^{2m+1} u_\alpha\| \right]^2 \\ &\leq \sum_{\alpha \in I} \sum_{k=0}^d \left[M^{2m} \|\partial\pi(X_k)^{2m+1} u_\alpha\| \right]^2 \end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^d \left[M^{2m} \|\partial\pi(X_k)^{2^{m+1}} u\| \right]^2 \leq \sum_{k=0}^d \left[c_k (Mt^2)^{2^m} (2^{m+1})!^\lambda \right]^2 \\
&\leq \left[\sum_{k=0}^d c_k (2^{2\lambda} Mt^2)^{2^m} (2^m)!^{2\lambda} \right]^2.
\end{aligned}$$

So

$$\|\partial\pi(I - \Delta_Y)^{2^m} u\| \leq \left(\sum_{k=0}^d c_k \right) (2^{2\lambda} Mt^2)^{2^m} (2^m)!^{2\lambda}$$

for all $m \in \mathbb{N}_0$. Hence by Lemma 2.25 there exists $s \geq 2^{2\lambda} Mt^2$, independent of c_0, \dots, c_d such that $u \in S_{2\lambda, s}(\partial\pi(I - \Delta_Y))$ and $\|u\|_{\partial\pi(I - \Delta_Y); 2\lambda, s} \leq \max\left(\sum_{k=0}^d c_k, \|u\|\right) = \sum_{k=0}^d c_k \leq 2 \sum_{k=0}^d \|u\|_{d\pi(X_k); \lambda, t}$. Since

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = S_{\lambda, \dots, \lambda}(d\pi(Y_1), \dots, d\pi(Y_d))$$

as locally convex spaces with equivalent spectra, the theorem follows by Theorem 1.24. (Here we use that $\lambda \geq 1$.) \square

We arrive at the main theorem of this section.

Theorem 2.27 *Let G be a Lie group with Lie algebra \mathfrak{g} . Then there exists a basis X_1, \dots, X_d in \mathfrak{g} such that for all $\lambda \geq 1$ and all representations π of G we have*

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = \bigcap_{k=1}^d S_\lambda(d\pi(X_k))$$

as locally convex space with equivalent spectra.

Proof. We prove the theorem by induction to $\dim \mathfrak{g}$. If $\dim \mathfrak{g} = 1$ then nothing has to be proved. For $d \in \mathbb{N}$ let hypothesis $P(d)$ state:

For any Lie group G with Lie algebra \mathfrak{g} and $d_1 := \dim \mathfrak{g} \leq d$ there exists a basis X_1, \dots, X_{d_1} in \mathfrak{g} such that for all $\lambda \geq 1$ and all representations π of G we have

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_{d_1})) = \bigcap_{k=1}^{d_1} S_\lambda(d\pi(X_k))$$

as locally convex spaces with equivalent spectra.

Let $d \in \mathbb{N}$, $d \geq 2$ and suppose hypothesis $P(d-1)$ is valid. Let G be a Lie group with Lie algebra \mathfrak{g} . Suppose $\dim \mathfrak{g} = d$. We shall prove:

Assertion 1: There exists a basis X_1, \dots, X_d in \mathfrak{g} such that for all $\lambda \geq 1$ and all representations π of G we have

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) = \bigcap_{k=1}^d S_\lambda(d\pi(X_k))$$

as locally convex spaces with equivalent spectra.

First we prove the following assertion:

Assertion 2: Let $\mathfrak{g}_1, \mathfrak{g}_2$ be subalgebras of \mathfrak{g} such that \mathfrak{g} is the direct sum of \mathfrak{g}_1 and \mathfrak{g}_2 .

Suppose $\dim \mathfrak{g}_1 \geq 1$ and $\dim \mathfrak{g}_2 \geq 1$. Then Assertion 1 holds.

Proof of Assertion 2. Let G_1 and G_2 be subgroups of G which have Lie algebras \mathfrak{g}_1 and \mathfrak{g}_2 respectively. By induction hypothesis $P(d-1)$ there exist a basis X_1, \dots, X_{d_1} in \mathfrak{g}_1 and a basis Y_1, \dots, Y_{d_2} in \mathfrak{g}_2 such that for every representation π_1 of G_1 and for every representation π_2 of G_2 and all $\lambda \geq 1$ we have

$$S_{\lambda, \dots, \lambda}(d\pi_1(X_1), \dots, d\pi_1(X_{d_1})) = \bigcap_{k=1}^{d_1} S_\lambda(d\pi_1(X_k))$$

and

$$S_{\lambda, \dots, \lambda}(d\pi_2(Y_1), \dots, d\pi_2(Y_{d_2})) = \bigcap_{k=1}^{d_2} S_\lambda(d\pi_2(Y_k))$$

as locally convex spaces with equivalent spectra. Then $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ is a basis in \mathfrak{g} .

Now let π be a representation of G in a Hilbert space H and let $\lambda \geq 1$. Let π_1 and π_2 be the restrictions of π to G_1 and G_2 respectively. Then $d\pi(X_k) = d\pi_1(X_k)$ for all $k \in \{1, \dots, d_1\}$ and $d\pi(Y_k) = d\pi_2(Y_k)$ for all $k \in \{1, \dots, d_2\}$. So by Corollary 2.4 we obtain that

$$\begin{aligned} S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(Y_{d_2})) &= \\ &= S_{\lambda, \dots, \lambda}(d\pi_1(X_1), \dots, d\pi_1(X_{d_1})) \cap S_{\lambda, \dots, \lambda}(d\pi_2(Y_1), \dots, d\pi_2(Y_{d_2})) \\ &= \bigcap_{k=1}^{d_1} S_\lambda(d\pi_1(X_k)) \cap \bigcap_{k=1}^{d_2} S_\lambda(d\pi_2(Y_k)) \\ &= \bigcap_{k=1}^{d_1} S_\lambda(d\pi(X_k)) \cap \bigcap_{k=1}^{d_2} S_\lambda(d\pi(Y_k)) \end{aligned}$$

as locally convex spaces with equivalent spectra. This proves Assertion 2.

Now we prove Assertion 1. If \mathfrak{g} is solvable, then Assertion 1 follows by Corollary 2.6 and the Remark following Corollary 2.3. So we may assume that \mathfrak{g} is not solvable. Let \mathfrak{q} be the radical of \mathfrak{g} . By [Var1], Theorem 3.14.1 there exists a semisimple subalgebra \mathfrak{m} of \mathfrak{g} such that \mathfrak{g} is the direct sum of \mathfrak{q} and \mathfrak{m} . (This is a Levi decomposition of \mathfrak{g} .) If $\dim \mathfrak{q} \geq 1$, then Assertion 1 follows by Assertion 2.

So we may assume that $\dim \mathfrak{q} = 0$. Then $\mathfrak{g} = \mathfrak{m}$ is semisimple. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be an Iwasawa decomposition of \mathfrak{g} . (See Helgason, [Hel], Theorem VI.3.4.) Let $\mathfrak{s} := \mathfrak{a} + \mathfrak{n}$. Then \mathfrak{k} and \mathfrak{s} are subalgebras of \mathfrak{g} , \mathfrak{s} is solvable and \mathfrak{g} is the direct sum of \mathfrak{k} and \mathfrak{s} . Since \mathfrak{g} is semisimple, always $\dim \mathfrak{k} \geq 1$. In case $\dim \mathfrak{s} \geq 1$, Assertion 1 follows again by Assertion 2.

So we may assume that $\dim \mathfrak{s} = 0$. Then $\mathfrak{g} = \mathfrak{k}$. So the Lie algebra \mathfrak{g} is compact. But the Lie group G need not be compact and we cannot immediately apply Theorem 2.26. Corresponding to the Lie algebra \mathfrak{g} there exists a connected simply connected Lie group G_1 with Lie algebra \mathfrak{g} . (See [Var1], Theorem 3.15.1.) Then G_1 is compact by Wallach [Wal], Theorem 3.6.6. Let X_1, \dots, X_d be any basis in \mathfrak{g} . Let $\lambda \geq 1$ and let π be a representation of G in a Hilbert space H . Now $X \mapsto \partial\pi(X)$ is a representation of the Lie algebra \mathfrak{g} by skew-symmetric operators in H and the operator $\partial\pi(X_1)^2 + \dots + \partial\pi(X_d)^2$ is essentially self-adjoint. (See Theorem 1.24.) So by [Nel], Corollary 9.1 there exists a representation σ of G_1 such that $d\sigma(X) = d\pi(X)$ for all $X \in \mathfrak{g}$. Therefore we obtain by Theorem 2.26 that

$$\begin{aligned} S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_d)) &= S_{\lambda, \dots, \lambda}(d\sigma(X_1), \dots, d\sigma(X_d)) \\ &= \bigcap_{k=1}^d S_{\lambda}(d\sigma(X_k)) = \bigcap_{k=1}^d S_{\lambda}(d\pi(X_k)) \end{aligned}$$

as locally convex spaces with equivalent spectra. This proves the theorem. \square

Corollary 2.28 *Let π be a representation of a Lie group G . Let \mathfrak{g} be the Lie algebra of G and let $\lambda \geq 1$. Then*

$$H_{\lambda}(\pi) = \bigcap_{X \in \mathfrak{g}} S_{\lambda}(d\pi(X))$$

as sets.

Let π be a representation of a Lie group G in a Hilbert space H . Now we present another description for the infinitely differentiable vectors for π and the Gevrey vectors of order λ for π in terms of the positive definite function (\tilde{u}, u) corresponding to a vector $u \in H$. More precisely, we shall prove that

$$H^{\infty}(\pi) = \{u \in H : \text{the function } (\tilde{u}, u) \text{ is infinitely differentiable on } G\}$$

and

$$H_{\lambda}(\pi) = \{u \in H : (\tilde{u}, u) \in G_{\lambda}(\mathbb{C})\}$$

for all $\lambda \geq 1$. We need a well-known lemma.

Lemma 2.29 *Let A be a self-adjoint operator in a Hilbert space H . Let $u \in H$. Define $F : \mathbb{R} \rightarrow \mathbb{C}$ by*

$$F(t) := (e^{itA}u, u) \quad (t \in \mathbb{R}).$$

Let V be an open neighborhood of 0. Suppose the restriction $F|_V$ is infinitely differentiable. Then $u \in D^{\infty}(A)$. Moreover, for all $n \in \mathbb{N}_0$ we have $F^{(2n)}(0) = (-1)^n \|A^n u\|^2$.

Proof. We may assume that A is the multiplication operator by the function h in the Hilbert space $H = L^2(Y, m)$ for some measure space (Y, \mathcal{B}, m) . Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(t) := \operatorname{Re} F(t) = \int \cos(th) |u|^2 dm, t \in \mathbb{R}$. Since f is an even function and infinitely differentiable on V , we obtain that $f'(0) = 0$. Then by Fatou's lemma:

$$\begin{aligned} \int h^2 |u|^2 dm &= 2 \int \liminf_{n \rightarrow \infty} n^2 \left(1 - \cos\left(\frac{1}{n}h\right)\right) |u|^2 dm \\ &\leq 2 \liminf_{n \rightarrow \infty} \int n^2 \left(1 - \cos\left(\frac{1}{n}h\right)\right) |u|^2 dm \\ &= -2 \liminf_{n \rightarrow \infty} n^2 \left(f\left(\frac{1}{n}\right) - f(0) - \frac{1}{n}f'(0)\right) \\ &= -f''(0) < \infty. \end{aligned}$$

So $u \in D(A)$ and $f''(0) = -\|Au\|^2$. Hence by Lebesgue's theorem on dominated convergence we obtain that F is twice differentiable on V and $F''(t) = -(e^{itA} Au, Au)$ for all $t \in V$. By induction, the lemma follows. \square

Theorem 2.30 *Let π be a representation of a Lie group G in a Hilbert space H . Let $\lambda \geq 1$. Then*

$$H^\infty(\pi) = \{u \in H : (\tilde{u}, u) \in C^\infty(G)\}$$

and

$$H_\lambda(\pi) = \{u \in H : (\tilde{u}, u) \in G_\lambda(\mathbb{C})\}.$$

In particular,

$$H^\omega(\pi) = \{u \in H : \text{the function } x \mapsto (\pi_x u, u) \text{ from } G \text{ into } \mathbb{C} \text{ is real analytic}\}.$$

Proof. Let $u \in H$. Suppose $(\tilde{u}, u) \in C^\infty(G)$. Let X_1, \dots, X_d be a basis in the Lie algebra \mathfrak{g} of G . Let $k \in \{1, \dots, d\}$. Then the function $t \mapsto (e^{td\pi(X_k)} u, u) = [(\tilde{u}, u)](\exp tX_k)$ from \mathbb{R} into \mathbb{C} is infinitely differentiable, so by Lemma 2.29 we obtain that $u \in D^\infty(d\pi(X_k))$. Therefore, $u \in \bigcap_{k=1}^d D^\infty(d\pi(X_k)) = H^\infty(\pi)$ by Theorem 1.23.

Now let $\lambda \geq 1$, let $u \in H$ and suppose $(\tilde{u}, u) \in G_\lambda(\mathbb{C})$. Since $G_\lambda(\mathbb{C}) \subset C^\infty(G)$, by the previous part we obtain that $u \in H^\infty(\pi)$. Let K be a compact neighborhood of the identity e in G . Let $X \in \mathfrak{g}, X \neq 0$. There exists a basis X_1, \dots, X_d in \mathfrak{g} such that $X_1 = X$. Then $(\tilde{u}, u) \in G_\lambda(\mathbb{C}) \subset G_\lambda(\mathbb{C}, K, \tilde{X}_1, \dots, \tilde{X}_d)$. So there exist $C, t > 0$ such that for all $n \in \mathbb{N}_0$, all $i_1, \dots, i_n \in \{1, \dots, d\}$ and all $p \in K$ we have $|\tilde{X}_{i_1} \circ \dots \circ \tilde{X}_{i_n}(\tilde{u}, u)(p)| \leq Ct^n n!^\lambda$. Define $F : \mathbb{R} \rightarrow \mathbb{C}$ by $F(t) := (e^{td\pi(X)} u, u), t \in \mathbb{R}$. Then by Lemma 2.29 we obtain for all $n \in \mathbb{N}_0$:

$$\begin{aligned} \| [d\pi(X)]^n u \|^2 &= |F^{(2n)}(0)| = \left| \left(\frac{d}{dt} \right)^{2n} (\tilde{u}, u)(\exp tX) \Big|_{t=0} \right| \\ &= |[\tilde{X}^{2n}(\tilde{u}, u)](e)| \leq Ct^{2n} (2n)!^\lambda \leq C(2t)^{2n} n!^{2\lambda}. \end{aligned}$$

Hence $u \in S_\lambda(d\pi(X))$. Therefore $u \in \bigcap_{X \in \mathfrak{g}} S_\lambda(d\pi(X)) = H_\lambda(\pi)$ by Corollary 2.28. \square

Remark. Note that the previous theorem gives a new proof for Theorems 1.28 and 1.29.

Let π be a representation of a Lie group G and let \mathfrak{g} be the Lie algebra of G . By Theorem 1.25 we know that

$$S_{1,\dots,1}(d\pi(X_1), \dots, d\pi(X_d)) = S_{1,\dots,1}^{\text{ord}}(d\pi(X_1), \dots, d\pi(X_d))$$

for any basis X_1, \dots, X_d in \mathfrak{g} . Now we can partly extend this equality for $\lambda \geq 1$.

Corollary 2.31 *Let G be a Lie group with Lie algebra \mathfrak{g} . Then there exists a basis X_1, \dots, X_d in \mathfrak{g} such that for all $\lambda \geq 1$ and all representations π of G we have:*

$$S_{\lambda,\dots,\lambda}(d\pi(X_1), \dots, d\pi(X_d)) = S_{\lambda,\dots,\lambda}^{\text{ord}}(d\pi(X_1), \dots, d\pi(X_d))$$

as locally convex spaces with equivalent spectra.

However, for all $\lambda \geq 2$ we can prove that also

$$S_{\lambda,\dots,\lambda}(d\pi(X_1), \dots, d\pi(X_d)) = S_{\lambda,\dots,\lambda}^{\text{ord}}(d\pi(X_1), \dots, d\pi(X_d))$$

as locally convex spaces with equivalent spectra for any basis X_1, \dots, X_d in \mathfrak{g} . This follows from the following more general theorem and the Remark following Corollary 2.3.

Theorem 2.32 *Let \mathfrak{g} be a real Lie algebra of (not necessarily skew-Hermitian) operators defined on a common invariant domain in a Hilbert space. Let $X_1, \dots, X_d \in \mathfrak{g}$ and suppose*

$$\mathfrak{g} = \text{span}(\{X_1, \dots, X_d\}).$$

Then for all $\lambda \geq 2$ we have:

$$S_{\lambda,\dots,\lambda}(X_1, \dots, X_d) = S_{\lambda,\dots,\lambda}^{\text{ord}}(X_1, \dots, X_d)$$

as locally convex spaces with equivalent spectra.

Proof. Let $V := \{1, \dots, d\}$. By assumption, for all $i, j \in \{1, \dots, d\}$ there exist $c_{i,j}^1, \dots, c_{i,j}^d \in \mathbb{R}$ such that

$$[X_i, X_j] = \sum_{k=1}^d c_{i,j}^k X_k.$$

Let $M := 1 + d \max\{|c_{i,j}^k| : i, j, k \in \{1, \dots, d\}\}$, let $b_0 := 1 + Md$ and $b := b_0 2^{\lambda d}$.

Let $t \geq 1$. Let $u \in S_{\lambda,\dots,\lambda;t}^{\text{ord}}(X_1, \dots, X_d)$. We shall prove that $u \in S_{\lambda,\dots,\lambda;t}$ and that

$$\|u\|_{X_1,\dots,X_d;\lambda,\dots,\lambda;t} \leq \|u\|_{X_1,\dots,X_d;\lambda,\dots,\lambda;t,\text{ord}}.$$

In order to avoid clutter we write $c := \|u\|_{X_1,\dots,X_d;\lambda,\dots,\lambda;t,\text{ord}}$. Then

$$\|X_1^{n_1} \circ \dots \circ X_d^{n_d} u\| \leq c t^{n_1 + \dots + n_d} (n_1 + \dots + n_d)!^{\lambda}$$

for all $n \in \mathbb{N}_0^d$. The proof is by induction. For $N \in \mathbb{N}$ hypothesis $P(N)$ states:

$$|(X_\alpha u, u)| \leq c^2 (b_0 t)^{\|\alpha\|} \|\alpha\|!^\lambda \quad \text{for all } \alpha \in M(V) \text{ with } \|\alpha\| \leq N.$$

By Schwarz' inequality, hypothesis $P(1)$ holds. Let $N \in \mathbb{N}$ and suppose $P(N)$ holds. For all $j_1, \dots, j_{N+1} \in V$ and all $k \in \{1, \dots, N\}$ we have

$$\begin{aligned} X_{j_1} \circ \dots \circ X_{j_k} X_{j_{k+1}} \circ \dots \circ X_{j_{N+1}} &= \\ &= X_{j_1} \circ \dots \circ X_{j_{k-1}} X_{j_{k+1}} X_{j_k} X_{j_{k+2}} \circ \dots \circ X_{j_{N+1}} + \\ &+ \sum_{l=1}^d c_{j_k, j_{k+1}}^l X_{j_1} \circ \dots \circ X_{j_{k-1}} X_l X_{j_{k+2}} \circ \dots \circ X_{j_{N+1}}. \end{aligned}$$

Let $\alpha \in M(V)$, $\|\alpha\| = N + 1$. In order to get the $N + 1$ indices of α in a prescribed order, at most $\binom{N+1}{2}$ commutations are needed. So there exist $n_1, \dots, n_d \in \mathbb{N}_0$ and, further, for all $i \in \{1, \dots, \binom{N+1}{2} d\}$ there exist $c_i \in \mathbb{R}$ and $\alpha_i \in M(V)$ such that $n_1 + \dots + n_d = N + 1$, $\|\alpha_i\| = N$ and $|c_i| \leq M$ for all i and

$$X_\alpha = X_1^{n_1} \circ \dots \circ X_d^{n_d} + \sum_{i=1}^{\binom{N+1}{2} d} c_i X_{\alpha_i}.$$

Thus we arrive at the estimations

$$\begin{aligned} |(X_\alpha u, u)| &\leq |(X_1^{n_1} \circ \dots \circ X_d^{n_d} u, u)| + \sum_{i=1}^{\binom{N+1}{2} d} |c_i (X_{\alpha_i} u, u)| \\ &\leq c^2 t^{N+1} (N+1)!^\lambda + \binom{N+1}{2} d M c^2 (b_0 t)^N N!^\lambda \\ &\leq c^2 t^{N+1} (N+1)!^\lambda + c^2 d M b_0^N t^N (N+1)^2 N!^\lambda \\ &\leq c^2 t^{N+1} (N+1)!^\lambda + c^2 d M b_0^N t^N (N+1)!^\lambda \\ &\leq c^2 (b_0 t)^{N+1} (N+1)!^\lambda. \end{aligned}$$

This proves hypothesis $P(N + 1)$.

Now let $\alpha \in M(V)$. Then by Lemma 1.3.V:

$$\begin{aligned} \|Z_\alpha u\|^2 &= |(X_{\langle \alpha^r, \alpha \rangle} u, u)| \leq c^2 (b_0 t)^{\|\langle \alpha^r, \alpha \rangle\|} \|\langle \alpha^r, \alpha \rangle\|!^\lambda \\ &\leq [c (b t)^{\|\alpha\|} \|\alpha\|_1!^\lambda \cdot \dots \cdot \|\alpha\|_d!^\lambda]^2. \end{aligned}$$

So $u \in S_{\lambda, \dots, \lambda; b t}(X_1, \dots, X_d)$ and $\|u\|_{X_1, \dots, X_d; \lambda, \dots, \lambda; b t} \leq c = \|u\|_{X_1, \dots, X_d; \lambda, \dots, \lambda; t, \text{ord}}$.

This proves the theorem, since the identity map from $S_{\lambda, \dots, \lambda; t}(X_1, \dots, X_d)$ into $S_{\lambda, \dots, \lambda; t, \text{ord}}(X_1, \dots, X_d)$ is continuous for all $t > 0$. □

2.6 An application of intersection results: topological properties

Let A_1, \dots, A_d be Hermitian or skew-Hermitian operators in a Hilbert space H . Let $\lambda_1, \dots, \lambda_d \geq 0$. Suppose

$$S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d) = \bigcap_{k=1}^d S_{\lambda_k}(A_k) \quad (2.7)$$

as locally convex spaces with equivalent spectra. So for all $t > 0$ there exists $s \geq t$ such that the embeddings

$$S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d) \hookrightarrow \bigcap_{k=1}^d S_{\lambda_k; t}(A_k) \hookrightarrow S_{\lambda_1, \dots, \lambda_d; s}(A_1, \dots, A_d)$$

are continuous. Here $\bigcap_{k=1}^d S_{\lambda_k; t}(A_k)$ is a normed space with norm

$$\| \cdot \|_{A_1; \lambda_1; t} + \dots + \| \cdot \|_{A_d; \lambda_d; t}.$$

The topology σ_{ind} for the space $\bigcap_{k=1}^d S_{\lambda_k}(A_k)$ at the right hand side of (2.7) is the inductive limit topology, generated by the normed spaces $\bigcap_{k=1}^d S_{\lambda_k; t}(A_k)$, with $t > 0$.

However, we can define a second topology for $\bigcap_{k=1}^d S_{\lambda_k}(A_k)$, namely the *projective limit topology* τ_{proj} generated by the locally convex spaces $S_{\lambda_k}(A_k)$, with $k \in \{1, \dots, d\}$. We prove that σ_{ind} and τ_{proj} determine the same bounded sets.

Lemma 2.33 *Let B be a subset of $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$. Then B is σ_{ind} -bounded if and only if B is τ_{proj} -bounded. Moreover, if B is bounded, with relative topology,*

$$(B, \sigma_{\text{ind}}) = (B, \tau_{\text{proj}})$$

as topological spaces.

Proof. By Theorem 1.11 the assertion

$$B \text{ is } \sigma_{\text{ind}}\text{-bounded}$$

is equivalent to

$$\text{There exists } t > 0 \text{ such that } B \text{ is boundedly contained in } S_{\lambda_1, \dots, \lambda_d; t}(A_1, \dots, A_d).$$

By assumption (2.7), this is equivalent to

$$\text{There exists } s > 0 \text{ such that } B \text{ is a bounded subset of } \bigcap_{k=1}^d S_{\lambda_k; s}(A_k).$$

By definition, this is equivalent to

$$\text{There exists } s > 0 \text{ such that for all } k \in \{1, \dots, d\} \text{ the set } B \text{ is a bounded subset of } S_{\lambda_k; s}(A_k).$$

Again by Theorem 1.11, this is equivalent to

For all $k \in \{1, \dots, d\}$ the set B is bounded in $S_{\lambda_k}(A_k)$.

By [Wil], Theorem 4-4-5, this is equivalent to

B is τ_{proj} -bounded.

Now suppose B is bounded. Let $(u_i)_{i \in I}$ be a τ_{proj} -convergent net in B with limit u . Then for all $k \in \{1, \dots, d\}$, B is bounded in $S_{\lambda_k}(A_k)$ and $\lim_i u_i = u$ in $S_{\lambda_k}(A_k)$. By Corollary 1.13.I there exists $t > 0$ such that B is a bounded subset in $S_{\lambda_k;t}(A_k)$ and $\lim_i u_i = u$ in $S_{\lambda_k;t}(A_k)$ for all $k \in \{1, \dots, d\}$. By assumption (2.7) there exists $s \geq t$ such that $\lim_i u_i = u$ in $S_{\lambda_1, \dots, \lambda_d; s}(A_1, \dots, A_d)$. Then $\lim_i u_i = u$ in (B, σ_{ind}) . So the identity map from (B, τ_{proj}) onto (B, σ_{ind}) is continuous.

For all $k \in \{1, \dots, d\}$ the identity map from $(S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d), \sigma_{\text{ind}})$ into $S_{\lambda_k}(A_k)$ is continuous, so the identity map from $(S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d), \sigma_{\text{ind}})$ into $(\bigcap_{k=1}^d S_{\lambda_k}(A_k), \tau_{\text{proj}})$ is continuous. In particular the identity map from (B, σ_{ind}) into (B, τ_{proj}) is continuous. \square

Let τ_{proj}^b be the locally convex vector topology for $\bigcap_{k=1}^d S_{\lambda_k}(A_k)$ such that $(\bigcap_{k=1}^d S_{\lambda_k}(A_k), \tau_{\text{proj}}^b)$ is the bornological space associated with the locally convex space $(\bigcap_{k=1}^d S_{\lambda_k}(A_k), \tau_{\text{proj}})$. (See [Sch], Chapter II §8.) We have the following relations between the topologies σ_{ind} , τ_{proj} and τ_{proj}^b for $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$.

Theorem 2.34

- I. $\sigma_{\text{ind}} = \tau_{\text{proj}}^b$.
- II. $\tau_{\text{proj}} \subset \sigma_{\text{ind}}$.
- III. The identity map from $(\bigcap_{k=1}^d S_{\lambda_k}(A_k), \tau_{\text{proj}})$ into $(\bigcap_{k=1}^d S_{\lambda_k}(A_k), \sigma_{\text{ind}})$ is sequentially continuous.

Proof. I. By Lemma 2.33, σ_{ind} and τ_{proj} determine the same bounded subsets of $\bigcap_{k=1}^d S_{\lambda_k}(A_k)$. Since τ_{proj}^b is the finest locally convex topology τ' for $\bigcap_{k=1}^d S_{\lambda_k}(A_k)$ such that $(\bigcap_{k=1}^d S_{\lambda_k}(A_k), \tau')$ has the same bounded subsets as $(\bigcap_{k=1}^d S_{\lambda_k}(A_k), \tau_{\text{proj}})$, we obtain that $\sigma_{\text{ind}} \subset \tau_{\text{proj}}^b$.

In fact, $(\bigcap_{k=1}^d S_{\lambda_k}(A_k), \tau_{\text{proj}}^b)$ has the same bounded subsets as $(\bigcap_{k=1}^d S_{\lambda_k}(A_k), \tau_{\text{proj}})$ has, which is the same set as for $(\bigcap_{k=1}^d S_{\lambda_k}(A_k), \sigma_{\text{ind}})$. (See Lemma 2.33.) So the identity map from $(\bigcap_{k=1}^d S_{\lambda_k}(A_k), \sigma_{\text{ind}})$ into $(\bigcap_{k=1}^d S_{\lambda_k}(A_k), \tau_{\text{proj}}^b)$ is bounded. Since σ_{ind} is bornological, this map is continuous. Therefore $\tau_{\text{proj}}^b \subset \sigma_{\text{ind}}$ and we obtain that $\tau_{\text{proj}}^b = \sigma_{\text{ind}}$.

II. By I: $\tau_{\text{proj}} \subset \tau_{\text{proj}}^b = \sigma_{\text{ind}}$. (For a direct proof, see the proof of Lemma 2.33.)

III. This follows from the second part of Lemma 2.33 since every convergent sequence is bounded. \square

Remark. It is an open problem whether $\tau_{\text{proj}} = \sigma_{\text{ind}}$.

Chapter 3

Examples of Gevrey spaces

In this chapter we study Gevrey spaces relative to infinitesimal operators corresponding to representations of the following Lie groups: the Heisenberg group, the $ax + b$ group and $SL(2, \mathbb{R})$.

3.1 The Heisenberg group

Let $n \in \mathbb{N}$. The *Heisenberg group* $A(\mathbb{R}^n)$ is a Lie group which, as a manifold, is equal to $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{T}$ and in which the group operation is given by

$$(a_1, b_1, z_1) \circ (a_2, b_2, z_2) := (a_1 + a_2, b_1 + b_2, z_1 z_2 e^{ia_1 \cdot b_2})$$

for all $a_1, a_2, b_1, b_2 \in \mathbb{R}^n$ and $z_1, z_2 \in \mathbb{T}$. Here $a_1 \cdot b_2$ denotes the inner product of a_1 and b_2 in \mathbb{R}^n . Define

$$y : \mathbb{R}^n \times \mathbb{R}^n \times (\mathbb{T} \setminus \{-1\}) \rightarrow \mathbb{R}^n \times \mathbb{R}^n \times (-\pi, \pi)$$

by

$$y(a, b, e^{i\varphi}) := (a, b, \varphi)$$

for all $a, b \in \mathbb{R}^n$ and $\varphi \in (-\pi, \pi)$. Then $(\mathbb{R}^n \times \mathbb{R}^n \times (\mathbb{T} \setminus \{-1\}), y)$ is a chart on $A(\mathbb{R}^n)$. Let $e := (0, 0, 1)$ denote the identity in $A(\mathbb{R}^n)$. For $k \in \{1, \dots, n\}$ let

$$X_k := \left. \frac{\partial}{\partial y_k} \right|_e, \quad Y_k := \left. \frac{\partial}{\partial y_{n+k}} \right|_e, \quad Z := \left. \frac{\partial}{\partial y_{2n+1}} \right|_e.$$

Let \mathfrak{g} be the Lie algebra of $A(\mathbb{R}^n)$. Then $X_1, \dots, X_n, Y_1, \dots, Y_n, Z$ is a basis in \mathfrak{g} and $[X_k, Y_k] = -[Y_k, X_k] = Z$ for all $k \in \{1, \dots, n\}$, all other commutators are 0.

Let R denote the right regular representation of \mathbb{R}^n on $L^2(\mathbb{R}^n)$. So

$$(R_a f)(x) = f(x + a) \quad \text{a.e. } x \in \mathbb{R}^n$$

for all $a \in \mathbb{R}^n$ and $f \in L^2(\mathbb{R}^n)$. For $b \in \mathbb{R}^n$ define the multiplication operator M_b on $L^2(\mathbb{R}^n)$ by

$$(M_b f)(x) := e^{ib \cdot x} f(x) \quad \text{a.e. } x \in \mathbb{R}^n$$

for all $f \in L^2(\mathbb{R}^n)$. For $(a, b, z) \in A(\mathbb{R}^n)$ define the operator $U_{(a,b,z)}$ on the Hilbert space $H := L^2(\mathbb{R}^n)$ by

$$U_{(a,b,z)} := z M_b R_a.$$

then $(a, b, z) \mapsto U_{(a,b,z)}$ is an irreducible unitary representation of $A(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$. (See for example [HR2], (33.31).) For $k \in \{1, \dots, n\}$ let Q_k be the multiplication operator by the function $x \mapsto x_k$ on $L^2(\mathbb{R}^n)$. Here x_k is the k -th coordinate of x . For $k \in \{1, \dots, n\}$ let \mathbb{F}_k be the unique unitary operator from $L^2(\mathbb{R}^n)$ onto $L^2(\mathbb{R}^n)$ such that

$$(\mathbb{F}_k f)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x_1, \dots, x_{k-1}, y, x_{k+1}, \dots, x_n) e^{-ix_k y} dy \quad \text{a.e. } x \in \mathbb{R}^n$$

for all $f \in C_c(\mathbb{R}^n)$. Let

$$P_k = \mathbb{F}_k Q_k \mathbb{F}_k^{-1}.$$

Then $dU(X_k) = -iP_k$, $dU(Y_k) = iQ_k$ and $dU(Z) = iI$ for all $k \in \{1, \dots, n\}$.

Mainly on the basis of a Sobolev inequality it can be proved that the space $H^\infty(U)$ of all infinitely differentiable vectors for U is precisely the Schwartz space $S(\mathbb{R}^n)$ which consists of all infinitely differentiable functions φ on \mathbb{R}^n such that

$$\sup\{|x^k D_{i_1} \dots D_{i_m} \varphi(x)| : x \in \mathbb{R}^n\} < \infty$$

for all $k, m \in \mathbb{N}_0$ and $i_1, \dots, i_m \in \{1, \dots, n\}$. Moreover, by Theorem 1.23:

$$S(\mathbb{R}^n) = \bigcap_{k=1}^n D^\infty(Q_k) \cap \bigcap_{k=1}^n D^\infty(P_k).$$

Let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \geq 0$. We want to define the Gelfand-Shilov space $S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n}$ together with its topology. Let $N \in \mathbb{N}$. Define the normed space

$$\begin{aligned} \tilde{S}_\infty(N; \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n) := \{ \varphi \in S(\mathbb{R}^n) : \exists_{c \geq 0} \forall_{k, m \in \mathbb{N}_0^n} \forall_{x \in \mathbb{R}^n} \\ [|x_1^{k_1} \dots x_n^{k_n} D_1^{m_1} \dots D_n^{m_n} \varphi(x) | \leq \\ c N^{k_1 + \dots + k_n + m_1 + \dots + m_n} k_1^{\alpha_1} \dots k_n^{\alpha_n} m_1^{\beta_1} \dots m_n^{\beta_n}] \} \end{aligned}$$

with natural norm $\| \cdot \|_{\tilde{S}_\infty(N; \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n)}$. The Gelfand-Shilov space $S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n}$ is defined to be

$$S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n} := \bigcup_{N=1}^\infty \tilde{S}_\infty(N; \alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n).$$

The topology for $S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n}$ is the natural inductive limit topology. It follows similarly to Theorem 1.9 that ,

$$S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n} = S_{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n}^{\text{ord}}(Q_1, \dots, Q_n, P_1, \dots, P_n)$$

as locally convex spaces with equivalent spectra. We observe that the space $S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n}$ is trivial in case $\alpha_k + \beta_k < 1$ or $(\alpha_k, \beta_k) = (1, 0)$ or $(\alpha_k, \beta_k) = (0, 1)$ for some $k \in \{1, \dots, n\}$. In all other cases $S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n}$ is dense in $L^2(\mathbb{R}^n)$. (See [GS], Section IV.9.) We shall prove the equality

$$S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n} = S_{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n}(Q_1, \dots, Q_n, P_1, \dots, P_n) = \bigcap_{k=1}^n S_{\alpha_k}(Q_k) \cap \bigcap_{k=1}^n S_{\beta_k}(P_k)$$

as locally convex spaces with equivalent spectra, for all $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \geq 0$ such that $\alpha_k + \beta_k \geq 1$ for all $k \in \{1, \dots, n\}$.

Lemma 3.1 *Let X, Y be skew-Hermitian operators in a Hilbert space defined on a common invariant domain. Suppose*

$$[X, Y] = iI.$$

Let $\lambda, \mu \geq 0$ and suppose $\lambda + \mu \geq 1$. Then

$$S_{\lambda, \mu}(X, Y) = S_{\lambda}(X) \cap S_{\mu}(Y)$$

as locally convex spaces with equivalent spectra.

Proof. (Cf. the proof of Theorem 2.2.) Let $Z_1 := X, Z_2 := Y, d := 2$ and $V := \{1, 2\}$. Using the equality $XY = YX + iI$, we obtain that for all $\gamma \in M(V)$ with $\|\gamma\|_1 \neq 0$ and $\|\gamma\|_2 \neq 0$ there exist $\delta \in M(V), l \in \{0, \dots, \|\gamma\|_2\}$ and $\theta_1, \dots, \theta_l \in M(V)$ such that

$$Z_{\gamma} = Z_{\delta}X + \sum_{p=1}^l iZ_{\theta_p}, \tag{3.1}$$

$\|\delta\|_1 = \|\gamma\|_1 - 1, \|\delta\|_2 = \|\gamma\|_2, \|\theta_p\|_1 = \|\gamma\|_1 - 1$ and $\|\theta_p\|_2 = \|\gamma\|_2 - 1$ for all $p \in \{1, \dots, l\}$.

Let $b := 2^{\lambda+\mu+1}$. Let $t \geq 1$ and let $u \in S_{\lambda, t}(X) \cap S_{\mu, t}(Y)$. Let $c_1 := \|u\|_{X, \lambda, t}$ and $c_2 := \|u\|_{Y, \mu, t}$. We shall prove that $u \in S_{\lambda, \mu, bt}(X, Y)$ and that $\|u\|_{X, Y, \lambda, \mu, bt} \leq c_1 + c_2$. For $N \in \mathbb{N}_0$ hypothesis $P(N)$ states:

$$|(Y^m Z_{\gamma} X^k u, u)| \leq c_1 c_2 2^{\|\gamma\|_1 k + m + \|\gamma\|_2} (\|\gamma\|_1 + k)!^{\lambda} (\|\gamma\|_2 + m)!^{\mu}$$

for all $k, m \in \mathbb{N}_0, \gamma \in M(V)$ such that $\|\gamma\| = N$.

Clearly hypothesis $P(0)$ holds. Let $N \in \mathbb{N}$ and suppose hypothesis $P(N - 1)$ holds. Let $k, m \in \mathbb{N}_0, \gamma \in M(V)$ and suppose $\|\gamma\| = N$. We may assume that $\|\gamma\|_1 \neq 0$ and $\|\gamma\|_2 \neq 0$. Suppose $\|\gamma\|_1 \geq \|\gamma\|_2$. (The case $\|\gamma\|_1 \leq \|\gamma\|_2$ runs similarly.) By decomposing Z_{γ} as in equality (3.1) we obtain that

$$\begin{aligned} |(Y^m Z_{\gamma} X^k u, u)| &\leq \\ &\leq |(Y^m Z_{\delta} X^{k+1} u, u)| + \sum_{p=1}^l |(Y^m Z_{\theta_p} X^k u, u)| \\ &\leq c_1 c_2 2^{\|\gamma\|_1^{-1} k + m + \|\gamma\|_2} (\|\gamma\|_1 + k)!^{\lambda} (\|\gamma\|_2 + m)!^{\mu} + \end{aligned}$$

$$\begin{aligned}
& + lc_1c_22^{\|\gamma\|-2}t^{k+m+\|\gamma\|-2}(\|\gamma\|_1+k-1)!^\lambda(\|\gamma\|_2+m-1)^\mu \\
\leq & c_1c_22^{\|\gamma\|-1}t^{k+m+\|\gamma\|}(\|\gamma\|_1+k)!^\lambda(\|\gamma\|_2+m)^\mu \cdot \\
& \cdot \left[1 + \frac{l}{\|\gamma\|_2^{\lambda+\mu}} \left(\frac{\|\gamma\|_2}{\|\gamma\|_1} \right)^\lambda \left(\frac{\|\gamma\|_1}{\|\gamma\|_1+k} \right)^\lambda \left(\frac{\|\gamma\|_2}{\|\gamma\|_2+m} \right)^\mu \right] \\
\leq & c_1c_22^{\|\gamma\|}t^{k+m+\|\gamma\|}(\|\gamma\|_1+k)!^\lambda(\|\gamma\|_2+m)^\mu.
\end{aligned}$$

This proves hypothesis $P(N)$.

Now let $\gamma \in M(V)$. Then

$$\begin{aligned}
\|Z_\gamma u\|^2 & = |(Z_{(\gamma,\gamma)}u, u)| \leq c_1c_22^{2\|\gamma\|}t^{2\|\gamma\|}(2\|\gamma\|_1)^\lambda(2\|\gamma\|_2)^\mu \\
& \leq [(c_1+c_2)(bt)^{\|\gamma\|}\|\gamma\|_1!^\lambda\|\gamma\|_2!^\mu]^2.
\end{aligned}$$

Therefore $u \in S_{\lambda,\mu;bt}(X, Y)$ and $\|u\|_{X,Y;\lambda,\mu;bt} \leq c_1+c_2$. This proves the lemma. \square

Van Eijndhoven, [vE] has proved the following equalities for the Gelfand-Shilov space S_α^β in case $\alpha > 0$, $\beta > 0$ and $\alpha + \beta \geq 1$:

$$S_\alpha^\beta = S_{\alpha,\beta}(Q, P) = S_{\alpha,\beta}^{\text{ord}}(Q, P) = S_\alpha(Q) \cap S_\beta(P)$$

as sets. Now we extend these results in three directions, namely we include the case that $\alpha = 0$ or $\beta = 0$, we consider higher dimensions and we prove an equality as topological spaces.

Theorem 3.2 *Let $n \in \mathbb{N}$, let $\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n \geq 0$ and suppose that $\alpha_k + \beta_k \geq 1$ for all $k \in \{1, \dots, n\}$. Then*

$$S_{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n}^{\beta_1, \dots, \beta_n} = S_{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n}(Q_1, \dots, Q_n, P_1, \dots, P_n) = \bigcap_{k=1}^n S_{\alpha_k}(Q_k) \cap \bigcap_{k=1}^n S_{\beta_k}(P_k)$$

as locally convex spaces with equivalent spectra.

Proof. Let $G := A(\mathbb{R}^n)$, let U be the representation of G in the Hilbert space $L^2(\mathbb{R}^n)$ and let the basis $X_1, \dots, X_n, Y_1, \dots, Y_n, Z$ in the Lie algebra \mathfrak{g} of G be as at the beginning of this section. For all $k \in \{1, \dots, n\}$ we have $[\partial U(X_k), \partial U(Y_k)] = \partial U(Z) = iI$, so by Lemma 3.1 the spaces $S_{\alpha_k, \beta_k}(\partial U(X_k), \partial U(Y_k))$ and $S_{\alpha_k}(\partial U(X_k)) \cap S_{\beta_k}(\partial U(Y_k))$ are equal as locally convex spaces with equivalent spectra. Moreover,

$$H^\infty(U) = D^\infty(Q_1, \dots, Q_n, P_1, \dots, P_n) = \bigcap_{k=1}^n D^\infty(Q_k) \cap D^\infty(P_k)$$

as sets. Finally, $[\partial U(X_k), \partial U(X_m)] = [\partial U(Y_k), \partial U(Y_m)] = [\partial U(X_k), \partial U(Y_m)] = 0$ for all $k, m \in \{1, \dots, n\}$ with $k \neq m$. Then by Theorem 1.31 and induction we obtain that the following spaces are equal as locally convex spaces with equivalent spectra:

$$S_{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n}(Q_1, \dots, Q_n, P_1, \dots, P_n),$$

$$S_{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n}(\partial U(X_1), \dots, \partial U(X_n), \partial U(Y_1), \dots, \partial U(Y_n)),$$

$$\begin{aligned}
 & S_{\alpha_1, \beta_1}(\partial U(X_1), \partial U(Y_1)) \cap S_{\alpha_2, \dots, \alpha_n, \beta_2, \dots, \beta_n}(\partial U(X_2), \dots, \partial U(X_n), \partial U(Y_2), \dots, \partial U(Y_n)), \\
 & \dots, \\
 & \bigcap_{k=1}^n S_{\alpha_k, \beta_k}(\partial U(X_k), \partial U(Y_k)), \\
 & \bigcap_{k=1}^n [S_{\alpha_k}(\partial U(X_k)) \cap S_{\beta_k}(\partial U(Y_k))], \\
 & \bigcap_{k=1}^n S_{\alpha_k}(Q_k) \cap \bigcap_{k=1}^n S_{\beta_k}(P_k).
 \end{aligned}$$

Let $t > 0$. Since the following embeddings are continuous for some $s \geq t$

$$\begin{aligned}
 & S_{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n; t}(Q_1, \dots, Q_n, P_1, \dots, P_n) \hookrightarrow S_{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n; t}^{\text{ord}}(Q_1, \dots, Q_n, P_1, \dots, P_n) \hookrightarrow \\
 & \hookrightarrow \bigcap_{k=1}^n S_{\alpha_k; t}(Q_k) \cap \bigcap_{k=1}^n S_{\beta_k; t}(P_k) \hookrightarrow S_{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n; s}(Q_1, \dots, Q_n, P_1, \dots, P_n)
 \end{aligned}$$

it follows that the spaces $S_{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n}(Q_1, \dots, Q_n, P_1, \dots, P_n)$ and $S_{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n}^{\text{ord}}(Q_1, \dots, Q_n, P_1, \dots, P_n)$ are equal as locally convex spaces with equivalent spectra. Because we know already that the spaces $S_{\alpha_1, \dots, \alpha_n, \beta_1, \dots, \beta_n}^{\text{ord}}(Q_1, \dots, Q_n, P_1, \dots, P_n)$ and $S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n}$ are equal as locally convex spaces with equivalent spectra, the theorem follows. \square

Let $\Delta := \sum_{k=1}^n X_k^2 + \sum_{k=1}^n Y_k^2 \in U(\mathfrak{g})$. By Theorem 1.24, the operator $\partial U(I - \Delta)$ is essentially self-adjoint and

$$S(\mathbb{R}^n) = H^\infty(U) = D^\infty(\overline{\partial U(I - \Delta)})$$

as sets. So $S(\mathbb{R}^n)$ is equal to the (joint) C^∞ -domain of one single operator. Similarly, for all $\alpha \geq 1$ we obtain that

$$S_{\alpha, \dots, \alpha}^{\alpha, \dots, \alpha} = S_{2\alpha}(\overline{\partial U(I - \Delta)})$$

as locally convex spaces with equivalent spectra. (See Theorems 1.24 and 3.2.) So $S_{\alpha, \dots, \alpha}^{\alpha, \dots, \alpha}$ equals the Gevrey space relative to a single operator. The question arises whether a Gelfand-Shilov space $S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n}$ can be written as the Gevrey space relative to one single operator. Earlier results in this direction are available only in case $n = 1$. Let P_0 and Q_0 denote the restrictions of P and Q to Schwartz' space $S(\mathbb{R})$, respectively. Then the following results are known.

$$\begin{aligned}
 S_\alpha^\alpha &= S_{2\alpha}(P_0^2 + Q_0^2), \quad \alpha \geq \frac{1}{2}, \\
 S_{\frac{k}{k+1}}^{\frac{k}{k+1}} &= S_{\frac{2k}{k+1}}(P_0^2 + Q_0^{2k}), \quad k \in \mathbb{N}, \\
 S_{\frac{k}{k+1}}^{\frac{k}{k+1}} &= S_{\frac{2k}{k+1}}(P_0^{2k} + Q_0^2), \quad k \in \mathbb{N}, \\
 S_{\frac{1}{k}}^1 &= S_{2m}(P_0^{2m} + Q_0^{2km}), \quad k, m \in \mathbb{N}, \\
 S_1^{\frac{1}{k}} &= S_{2m}(P_0^{2km} + Q_0^{2m}), \quad k, m \in \mathbb{N}.
 \end{aligned}$$

as sets. Moreover, the operators $P_0^2 + Q_0^2, P_0^2 + Q_0^{2k}, P_0^{2k} + Q_0^2, P_0^{2m} + Q_0^{2km}$ and $P_0^{2km} + Q_0^{2m}$ are essentially self-adjoint and may be replaced by their self-adjoint closures. Proofs can be found in Zhang, [Zha], Van Eijndhoven-De Graaf-Pathak, [EGP] and Goodman, [Goo6], Theorem 6.1.

We extend these results in the following cases. For $\alpha, \beta > 0$ with

$$\begin{aligned} \alpha &\geq 1, \beta \geq 1 \text{ and } \frac{\alpha}{\beta} \in \mathbb{Q}, \\ \alpha &\geq 1 \text{ and } \frac{\alpha}{\beta} \in \mathbb{N}, \\ \beta &\geq 1 \text{ and } \frac{\beta}{\alpha} \in \mathbb{N}, \\ \alpha &= \beta \geq \frac{1}{2} \end{aligned}$$

we show that there exist $\lambda \geq 1$ (depending on α and β) and a symmetric differential operator A in $L^2(\mathbb{R})$ such that $S_\alpha^\beta = S_\lambda(A)$ as locally convex spaces with equivalent spectra. In particular, for all $p, q \in \mathbb{N}$ define $\rho_{p,q}$ by $\rho_{p,q} := \max(\frac{1}{p}, \frac{1}{q}), \rho_{p,1} := \frac{1}{p}$ and $\rho_{1,q} := \frac{1}{q}$ if $p, q \geq 2$ and $\rho_{1,1} := \frac{1}{2}$. Then for all $\rho \geq \rho_{p,q}$ we prove that

$$S_{q\rho}^{p\rho} = S_{2pq\rho}(P_0^{2q} + Q_0^{2p} + I) \tag{3.2}$$

as locally convex spaces with equivalent spectra. Equality (3.2) as sets has already been proved in [tEvE]. From this equality it follows by an elementary counting argument that

$$S_{q\rho}^{p\rho} = S_{2pq\rho}(P_0^{2q} + Q_0^{2p})$$

as locally convex spaces with equivalent spectra. If, in addition, it *could* be proved that the operator $P_0^{2q} + Q_0^{2p}$ is essentially self-adjoint and strictly positive, it follows by [Nel], Lemma 5.2 and inequalities (3.6) and (3.7), *infra*, that

$$S(\mathbb{R}) = D^\infty(\overline{P_0^{2q} + Q_0^{2p}})$$

as sets. Moreover, under these two additional assumptions, it follows by Theorem 1.4 that for all $\rho \geq \rho_{p,q}$

$$S_{q\rho}^{p\rho} = S_{L^2(\mathbb{R}), B^{\frac{1}{2pq\rho}}} \tag{3.3}$$

as locally convex spaces with equivalent spectra. Here $B := \overline{P_0^{2q} + Q_0^{2p}}$. Equality (3.3) is in the same spirit as Conjecture II.2.7 in [EG], which states that for all $\rho \in (0, 1]$:

$$S_{\left(\frac{1}{(p+q)\rho}\right)}^{\left(\frac{p}{(p+q)\rho}\right)} = S_{L^2(\mathbb{R}), B_{p,q}^{\varepsilon_{p,q}(\rho)}} \tag{3.4}$$

as sets, where

$$B_{p,q} := \left(-\frac{d^2}{dx^2}\right)^p + (x^2)^q$$

and $\varepsilon_{p,q}(\rho) = \frac{(p+q)\rho}{2pq}$. However, in [EG] the authors allege that the operator $B_{p,q}$ is self-adjoint by referring to the monograph of Müller-Pfeiffer, [MüPf]. But Müller-Pfeiffer only

proves that all self-adjoint extensions of a suitable restriction of the operator $B_{p,q}$ have a discrete spectrum. So it is not clear whether the right hand side of (3.4) makes sense.

Equality (3.2) can be extended to higher dimensions. (See infra, Theorem 3.13.) An essential tool in the proof is the following theorem of Goodman and Wallach, [GW], Theorem 1.3.

Theorem 3.3 *Let $d \in \mathbb{N}$. Let A, X_1, \dots, X_d be operators in a Hilbert space defined on a common invariant domain D . Suppose there exist $\delta \in \mathbb{N}$, $\lambda \geq 1$ and $M > 0$ such that*

$$\|X_{i_1} \circ \dots \circ X_{i_k} u\| \leq M \|Au\|,$$

$$\|\text{ad}X_{j_1} \dots \text{ad}X_{j_n}(A)u\| \leq M^n n!^\lambda \|Au\|$$

for all $k \in \{0, \dots, \delta\}$, all $i_1, \dots, i_k \in \{1, \dots, d\}$, all $n \in \mathbb{N}_0$, all $j_1, \dots, j_n \in \{1, \dots, d\}$ and all $u \in D$. Then for all $t > 0$ there exists $s > 0$ such that

$$S_{\lambda\delta;t}(A) \subset S_{\lambda, \dots, \lambda; s}(X_1, \dots, X_d),$$

where the inclusion is continuous.

Since the higher dimensional case is to be treated further on, we now take a more abstract point of view. Note that $[-iP_0, iQ_0] = iI$.

Theorem 3.4 *Let X, Y be skew-Hermitian operators in a Hilbert space H , defined on a common invariant domain D . Suppose*

$$[X, Y] = iI.$$

Let $p, q \in \mathbb{N}$. Let

$$V := \{X^k : k \in \{0, \dots, p\}\} \cup \{Y^k : k \in \{0, \dots, q\}\}$$

and

$$L := \text{span}(\{W_1 W_2 : W_1, W_2 \in V\}).$$

Let $A \in L$ be a Hermitian operator which satisfies the following condition:

$$\forall W \in V \forall u \in D \left[\|Wu\|^2 \leq (Au, u) \right]. \tag{3.5}$$

Then

$$S_{q\rho, p\rho}(X, Y) = S_{2pq\rho}(A)$$

as locally convex spaces with equivalent spectra in the following cases:

- $\rho \geq \max(\frac{1}{p}, \frac{1}{q})$
- $\rho \geq \frac{1}{p}$ and $q = 1$ and X has a skew-adjoint extension.

- $\rho \geq \frac{1}{2}$ and $p = q = 1$ and both X and Y have skew-adjoint extensions.

An example of an operator A which satisfies Condition (3.5) is

$$A = I - \sum_{k=1}^p X^{2k} - \sum_{k=1}^q Y^{2k}$$

and, if the operators X and Y have skew-adjoint extensions, an example is:

$$A = I - X^{2p} - Y^{2q}.$$

The proof of Theorem 3.4 is a compilation of a number of auxiliary results. In fact, Theorem 3.4 follows from Theorems 3.10 and 3.11.

Let X, Y, H, D, p, q, V, L and A be as in Theorem 3.4, and suppose Condition (3.5) is satisfied.

Lemma 3.5 *Let $W \in L$. Then there exists $c > 0$ such that for all $u \in D$ the inequality $|(Wu, u)| \leq c(Au, u)$ holds.*

Proof. It suffices to show that for all $W_1, W_2 \in V$ and all $u \in D$: $|(W_1W_2u, u)| \leq (Au, u)$. So let $W_1, W_2 \in V$. Then for all $u \in D$ we obtain:

$$|(W_1W_2u, u)| = |(W_2u, W_1u)| \leq \|W_2u\| \|W_1u\| \leq \left[\sqrt{(Au, u)} \right]^2 = (Au, u).$$

□

Lemma 3.6 *Let W be any Hermitian or skew-Hermitian operator in H with domain D and suppose there exists $c > 0$ such that $|(Wu, u)| \leq c(Au, u)$ for all $u \in D$. Then for all $u, v \in D$:*

$$|(Wu, v)| \leq 3c\sqrt{(Au, u)}\sqrt{(Av, v)}.$$

Proof. We may as well assume that W is Hermitian. So $((cA - W)u, u) \geq 0$ for all $u \in D$. Then by Schwarz' inequality ([Wei], Theorem 1.4) we obtain for all $u, v \in D$:

$$\begin{aligned} |(Wu, v)| &\leq |((cA - W)u, v)| + c|(Au, v)| \\ &\leq \sqrt{((cA - W)u, u)}\sqrt{((cA - W)v, v)} + c\sqrt{(Au, u)}\sqrt{(Av, v)} \\ &\leq 3c\sqrt{(Au, u)}\sqrt{(Av, v)}. \end{aligned}$$

□

Corollary 3.7 *Let $n \in \mathbb{N}$. Then there exists $c > 0$ such that for all $u, v \in D$:*

$$|((\text{ad}X)^n(A)u, v)| \leq c\sqrt{(Au, u)}\sqrt{(Av, v)}$$

and

$$|((\text{ad}Y)^n(A)u, v)| \leq c\sqrt{(Au, u)}\sqrt{(Av, v)}.$$

Proof. It can be readily checked that $(\text{ad}X)^n(V) \subset \text{span}(V)$. Therefore $\text{ad}X(L) \subset L$ and in particular $(\text{ad}X)^n(A) \in L$. Now $(\text{ad}X)^n(A)$ is an Hermitian operator. So the first inequality follows from Lemmas 3.5 and 3.6. The proof of the second inequality runs similarly. \square

Lemma 3.8

I. There exists a constant $M > 0$ such that for all $u \in D$

$$\|X^k u\| \leq M \|Au\|, \quad k = 0, \dots, 2p$$

and

$$\sqrt{(AX^k u, X^k u)} \leq M \|Au\|, \quad k = 0, \dots, p.$$

II. There exists a constant $N > 0$ such that for all $u \in D$

$$\|Y^k u\| \leq N \|Au\|, \quad k = 0, \dots, 2q$$

and

$$\sqrt{(AY^k u, Y^k u)} \leq N \|Au\|, \quad k = 0, \dots, q.$$

Proof. We proof I, the proof of II runs similarly. The proof is by induction. For $r \in \{0, \dots, p\}$ assertion $P(r)$ states:

There exists $M_r > 0$ such that for all $u \in D$ and $k \in \{0, \dots, p+r\}$:

$$\|X^k u\| \leq M_r \|Au\|$$

and for all $u \in D$ and $k \in \{0, \dots, r\}$:

$$\sqrt{(AX^k u, X^k u)} \leq M_r \|Au\|.$$

Let $r = 0$ and let $k \in \{0, \dots, p\}$. Then $X^k \in V$. So by Condition (3.5), $\|X^k u\|^2 \leq (Au, u)$ for all $u \in D$. Since $(Au, u) \leq \|Au\| \|u\| \leq \|Au\|^2$, we get

$$\|X^k u\| \leq \|Au\|$$

and

$$\sqrt{(Au, u)} \leq \|Au\|$$

for all $u \in D$. This proves assertion $P(0)$.

Assume assertion $P(r-1)$ holds for some constant $M_{r-1} > 0$ and $r \in \{1, \dots, p\}$. Under this assumption we shall prove $P(r)$. By Corollary 3.7 there exists a constant $N > 0$ such that for all $u, v \in D$ and all $n \in \{1, \dots, r\}$:

$$|((\text{ad}X)^n(A)u, v)| \leq N \sqrt{(Au, u)} \sqrt{(Av, v)}.$$

Using the standard commutation formula

$$X^r A - A X^r = \sum_{n=1}^r \binom{r}{n} (\text{ad} X)^n(A) X^{r-n}$$

(see Lemma 2.10) we obtain for all $u \in D$:

$$\begin{aligned} (A X^r u, X^r u) &= (X^r A u, X^r u) - \sum_{n=1}^r \binom{r}{n} ((\text{ad} X)^n(A) X^{r-n} u, X^r u) \\ &\leq |(A u, X^{2r} u)| + N \sum_{n=1}^r \binom{r}{n} \sqrt{(A X^{r-n} u, X^{r-n} u)} \sqrt{(A X^r u, X^r u)}. \end{aligned}$$

By assumption $P(r-1)$ we get

$$(A X^r u, X^r u) \leq \|A u\| \|X^{2r} u\| + 2^r M_{r-1} N \|A u\| \sqrt{(A X^r u, X^r u)}.$$

This is a quadratic inequality of the form $x^2 \leq ax + b$ with $x = \sqrt{(A X^r u, X^r u)}$ and $a, b \geq 0$. It follows that $x \leq \frac{1}{2}(a + \sqrt{a^2 + 4b}) \leq a + \sqrt{b}$ and hence that $x^2 \leq 2a^2 + 2b$. Thus we obtain

$$(A X^r u, X^r u) \leq 2^{2r+1} M_{r-1}^2 N^2 \|A u\|^2 + 2 \|A u\| \|X^{2r} u\|.$$

Let $t \in \{1, \dots, p\}$. Since the operator A satisfies Condition (3.5) we have

$$\|X^{t+r} u\|^2 \leq (A X^r u, X^r u) \leq 2^{2r+1} M_{r-1}^2 N^2 \|A u\|^2 + 2 \|A u\| \|X^{2r} u\|.$$

Again for $t = r$ we get a quadratic inequality, so

$$\|X^{2r} u\| \leq (2^{r+1/2} M_{r-1} N + 2) \|A u\|$$

and

$$(A X^r u, X^r u) \leq (2^{r+1/2} M_{r-1} N + 2)^2 \|A u\|^2.$$

Finally, for $t = p$ we derive

$$\|X^{p+r} u\|^2 \leq (2^{r+1/2} M_{r-1} N + 2)^2 \|A u\|^2.$$

This proves assertion $P(r)$ with $M_r := 2^{r+1/2} M_{r-1} N + 2$. □

Lemma 3.9 *Let $W \in L$. Then there exists $d > 0$ such that for all $u \in D$:*

$$\|W u\| \leq d \|A u\|.$$

Proof. The operators

$$X^k, Y^l, \quad k \in \{0, \dots, 2p\}, l \in \{0, \dots, 2q\}$$

and

$$X^k Y^l, Y^l X^k, \quad k \in \{0, \dots, p\}, l \in \{0, \dots, q\}$$

span the space L . So we only need to prove the lemma for these operators. If $W = X^k$ or $W = Y^l$ for some $k \in \{0, \dots, 2p\}$ or $l \in \{0, \dots, 2q\}$, the result follows from Lemma 3.8.

Let $k \in \{0, \dots, p\}$ and $l \in \{0, \dots, q\}$. By Lemmas 3.5 and 3.8 there exists $M > 0$ such that for all $u \in D$:

$$\begin{aligned} |(X^{2k}u, u)| &\leq M(Au, u) & |(Y^{2l}u, u)| &\leq M(Au, u) \\ (AX^k u, X^k u) &\leq M\|Au\|^2 & (AY^l u, Y^l u) &\leq M\|Au\|^2. \end{aligned}$$

Then for all $u \in D$:

$$\|X^k Y^l u\|^2 = |(X^{2k} Y^l u, Y^l u)| \leq M(A Y^l u, Y^l u) \leq M^2 \|Au\|^2$$

and, similarly,

$$\|Y^l X^k u\|^2 \leq M^2 \|Au\|^2.$$

□

Because of the previous lemma it makes sense to define the following norm on L :

$$\| \|W\| \| := \inf \{ M > 0 : \forall u \in D [\|Wu\| \leq M \|Au\|] \}.$$

We arrive at the following theorem.

Theorem 3.10 *Let X, Y, H, D, p, q, V, L and A be as in Theorem 3.4 and suppose Condition (3.5) is satisfied. Then in the following cases*

- I. $\rho \geq \max(\frac{1}{p}, \frac{1}{q})$,
- II. $\rho \geq \frac{1}{p}$ and $q = 1$ and X has a skew-adjoint extension,
- III. $\rho \geq \frac{1}{2}$ and $p = q = 1$ and both X and Y have skew-adjoint extensions,

we obtain that for all $t > 0$ there exists $s > 0$ such that the inclusion

$$S_{2pq\rho; t}(A) \subset S_{q\rho, p\rho; s}(X, Y)$$

is continuous.

Proof. (Cf. the proof of Lemma 6.2 in [Nel].)

I. Since $\text{ad}X$ is a linear operator from the finite dimensional vector space L into L , the operator $\text{ad}X$ is continuous and $\|\text{ad}X\| < \infty$. So for all $n \in \mathbb{N}$:

$$\| \|(\text{ad}X)^n(A)\| \| \leq \|\text{ad}X\|^n \| \|A\| \| = \|\text{ad}X\|^n.$$

It follows from the definition of $\| \| \|$ that for all $u \in D$ and $n \in \mathbb{N}$:

$$\| \|(\text{ad}X)^n(A)u\| \| \leq \|\text{ad}X\|^n \| \|Au\|. \tag{3.6}$$

Also, by Lemma 3.8 there exists $M > 0$ such that for all $u \in D$ and $k \in \{0, \dots, 2p\}$:

$$\|X^k u\| \leq M \|Au\|. \quad (3.7)$$

According to Theorem 3.3 with $\lambda = \rho q \geq 1$ and $\delta = 2p$ we obtain that there exists $s_1 > 0$ such that the inclusion

$$S_{2pq\rho;t}(A) \subset S_{q\rho;s_1}(X)$$

is continuous.

Since also $\text{ad}Y$ is a linear operator from L into L we obtain similarly that there exists $s_2 > 0$ such that the inclusion

$$S_{2pq\rho;t}(A) \subset S_{p\rho;s_2}(Y)$$

is continuous. Let $s_0 := \max(s_1, s_2)$. We may assume that $s_1 = s_2 = s_0$. By Lemma 3.1 there exists $s > 0$ such that the inclusion

$$S_{q\rho;s_0}(X) \cap S_{p\rho;s_0}(Y) \subset S_{q\rho,p\rho;s}(X, Y)$$

is continuous. Then $S_{2pq\rho;t}(A)$ is continuously embedded in $S_{q\rho,p\rho;s}(X, Y)$.

II. As in I it follows that there exists $s_2 > 0$ such that the inclusion

$$S_{2pq\rho;t}(A) \subset S_{p\rho;s_2}(Y)$$

is continuous. Also, it can be readily checked that $\text{ad}(X^p)(V) \subset \text{span}(V)$. So $\text{ad}(X^p)$ is a linear map from L into L . As explained in I it follows that for all $u \in D$ and $n \in \mathbb{N}$:

$$\|(\text{ad}(X^p))^n(A)u\| \leq \|\text{ad}(X^p)\|^n \|Au\|.$$

By Lemma 3.8 there exists $M > 0$ such that for all $u \in D$:

$$\|X^p u\| \leq M \|Au\| \text{ and } \|X^{2p} u\| \leq \|Au\|.$$

So by Theorem 3.3 with $\lambda = p\rho$ and $\delta = 2$ we obtain that there exists $s_1 > 0$ such that the inclusion

$$S_{2p\rho;t}(A) \subset S_{p\rho;s_1}(X^p)$$

is continuous.

Since the operator X has a skew-adjoint extension, it follows from Corollary 1.5 that the Gevrey spaces $S_{p\rho}(X^p)$ and $S_\rho(X)$ are equal as locally convex spaces with equivalent spectra. So there exists $s_3 > 0$ such that the inclusion

$$S_{p\rho;s_1}(X^p) \subset S_{\rho;s_3}(X)$$

is continuous. Then by Lemma 3.1 there exists $s > 0$ such that $S_{2p\rho;t}(A)$ is continuously embedded in $S_{\rho,p\rho;s}(X, Y)$.

III. Both $\text{ad}(X^2)$ and $\text{ad}(Y^2)$ are linear operators from L into L . So with $\delta = 1$ and $\lambda = 2\rho \geq 1$ we obtain that there exists $s_1 > 0$ such that the inclusion

$$S_{2\rho;t}(A) \subset S_{2\rho;s_1}(X^2) \cap S_{2\rho;s_1}(Y^2)$$

is continuous. As in II it now follows that the inclusion

$$S_{2\rho;t}(A) \subset S_{\rho,\rho;s}(X, Y)$$

is continuous for some $s > 0$. □

In order to complete the proof of Theorem 3.4 we present the following general result.

Theorem 3.11 *Let X, Y, H, D, p, q, V and L be as in Theorem 3.4 and let $W \in L$. Let $\rho \geq 0$. Then for all $t \geq 1$ there exists $s > 0$ such that the inclusion*

$$S_{q\rho,p\rho;t}(X, Y) \subset S_{2pq\rho;s}(W)$$

is continuous.

Proof. There exist constants a_k, b_l, c_{kl} and d_{kl} such that

$$W = \sum_{k=0}^{2p} a_k X^k + \sum_{l=0}^{2q} b_l Y^l + \sum_{k=1}^p \sum_{l=1}^q [c_{kl} X^k Y^l + d_{kl} Y^l X^k].$$

Let $M := 1 + \max\{|a_k|, |b_l|, |c_{kl}|, |d_{kl}|\}$. Let

$$s := 2(p+1)(q+1)Mt^{2(p+q)}(2pqe)^{2pq\rho}.$$

Let $u \in S_{q\rho,p\rho;t}(X, Y)$. Let $n \in \mathbb{N}_0$. Then W^n is a sum of $[2(p+1)(q+1)]^n$ terms of the form $\gamma W_1 \circ \dots \circ W_{2n}$ with $\gamma \in \mathbb{C}$, $|\gamma| \leq M^n$ and $W_i \in V$ for all $i \in \{1, \dots, 2n\}$. Consider one such term $\gamma W_1 \circ \dots \circ W_{2n}$. Let

$$n_p := \text{card}\{i : W_i \in \{I, X, \dots, X^p\}\},$$

$$n_q := \text{card}\{i : W_i \in \{Y, \dots, Y^q\}\}.$$

Then $n_p + n_q = 2n$. So with $c := \|u\|_{X,Y;q\rho,p\rho;t}$ we obtain:

$$\begin{aligned} \|\gamma W_1 \circ \dots \circ W_{2n} u\| &\leq M^n c t^{2n(p+q)} (pn_p)!^{q\rho} (qn_q)!^{p\rho} \\ &\leq M^n c t^{2n(p+q)} (2pn)^{pq\rho n_p} (2qn)^{pq\rho n_q} \\ &\leq c [Mt^{2(p+q)}(2pqe)^{2pq\rho}]^n n!^{2pq\rho}. \end{aligned}$$

Now the theorem follows, because

$$\|W^n u\| \leq c s^n n!^{2pq\rho}.$$

□

With this, Theorem 3.4 is proved completely. □

We extend Theorem 3.4 to higher dimensions. We need a lemma. The conditions of the following lemma can be weakened, but these conditions are sufficient for our purpose.

Lemma 3.12 Let $n \in \mathbb{N}$. Let $A_1, \dots, A_n, B_1, \dots, B_n$ be strictly positive operators in a Hilbert space H , defined on a common invariant domain D . Suppose there exist positive self-adjoint extensions $A'_1, \dots, A'_n, B'_1, \dots, B'_n$ of the operators $A_1, \dots, A_n, B_1, \dots, B_n$ such that for all $k, l \in \{1, \dots, n\}$ with $k \neq l$ the operators A'_k and A'_l commute strongly, the operators A'_k and B'_l commute strongly and the operators B'_k and B'_l commute strongly. Let $\lambda \geq 0$. Then

$$S_\lambda(A_1 + \dots + A_n + B_1 + \dots + B_n) = \bigcap_{k=1}^n S_\lambda(A_k + B_k)$$

as locally convex spaces with equivalent spectra.

Proof. Let $k, l \in \{1, \dots, n\}$, $k \neq l$ and let $u \in D$. Then $A'_k A'_l$ is a positive operator (Spectral Theorem), so

$$(A_k u, A_l u) = (A_k u, A'_l u) = (A'_l A_k u, u) = (A'_l A'_k u, u) \geq 0.$$

Similarly $(A_k u, B_l u) \geq 0$ and $(B_k u, B_l u) \geq 0$. Hence for all $u \in D$:

$$\begin{aligned} \|(A_1 + \dots + A_n + B_1 + \dots + B_n)u\|^2 &= \\ &= \sum_{k=1}^n \sum_{l=1}^n ((A_k + B_k)u, (A_l + B_l)u) \\ &= \sum_{k=1}^n \|(A_k + B_k)u\|^2 + \sum_{\substack{k, l \in \{1, \dots, n\} \\ k \neq l}} (A_k u, A_l u) + 2(A_k u, B_l u) + (B_k u, B_l u) \\ &\geq \sum_{k=1}^n \|(A_k + B_k)u\|^2. \end{aligned}$$

So

$$\|(A_k + B_k)u\| \leq \|(A_1 + \dots + A_n + B_1 + \dots + B_n)u\|$$

for all $k \in \{1, \dots, n\}$ and $u \in D$. Let $k \in \{1, \dots, n\}$. Then for all $u \in D$ and $m \in \mathbb{N}$ it follows by induction that

$$\|(A_k + B_k)^m u\| \leq \|(A_1 + \dots + A_n + B_1 + \dots + B_n)^m u\|,$$

since $[A_k + B_k, A_1 + \dots + A_n + B_1 + \dots + B_n] = 0$. So for all $t > 0$ we obtain that the embedding

$$S_{\lambda, t}(A_1 + \dots + A_n + B_1 + \dots + B_n) \hookrightarrow S_{\lambda, t}(A_k + B_k)$$

is continuous and hence the embedding

$$S_{\lambda, t}(A_1 + \dots + A_n + B_1 + \dots + B_n) \hookrightarrow \bigcap_{k=1}^n S_{\lambda, t}(A_k + B_k)$$

is continuous.

Since for all $t > 0$ there exists $s > 0$ such that the embedding

$$\bigcap_{k=1}^n S_{\lambda,t}(A_k + B_k) \hookrightarrow S_{\lambda,\dots,\lambda;s}(A_1 + B_1, \dots, A_n + B_n)$$

is continuous (see Corollary 1.32), it follows by an elementary counting argument that for all $t > 0$ there exists $s > 0$ such that the embedding

$$\bigcap_{k=1}^n S_{\lambda,t}(A_k + B_k) \hookrightarrow S_{\lambda;s}(A_1 + \dots + A_n + B_1 + \dots + B_n)$$

is continuous. □

Let $n \in \mathbb{N}$. For $k \in \{1, \dots, n\}$ let \tilde{Q}_k and \tilde{P}_k denote the restrictions of Q_k and P_k to Schwartz' space $S(\mathbb{R}^n)$, respectively.

Theorem 3.13 *Let $n \in \mathbb{N}$, let $p_1, \dots, p_n, q_1, \dots, q_n \in \mathbb{N}$ and let $\lambda \geq 1$. Suppose for all $k \in \{1, \dots, n\}$ at least one of the following four conditions is satisfied:*

- $\lambda \geq \max(2p_k, 2q_k)$,
- $\lambda \geq 2$ and $p_k = 1$,
- $\lambda \geq 2$ and $q_k = 1$,
- $p_k = q_k = 1$.

Then we have the following characterization for a Gelfand-Shilov space:

$$S_{\frac{\lambda}{2p_1}, \dots, \frac{\lambda}{2q_n}} = S_{\lambda}(\tilde{Q}_1^{2p_1} + \dots + \tilde{Q}_n^{2p_n} + \tilde{P}_1^{2q_1} + \dots + \tilde{P}_n^{2q_n} + I)$$

as locally convex spaces with equivalent spectra.

Proof. By Theorem 3.4 we obtain that

$$S_{\frac{\lambda}{2p_k}, \frac{\lambda}{2q_k}}(\tilde{Q}_k, \tilde{P}_k) = S_{\lambda}(\tilde{Q}_k^{2p_k} + \tilde{P}_k^{2q_k} + \frac{1}{n}I)$$

as locally convex spaces with equivalent spectra for all $k \in \{1, \dots, n\}$. So by Theorem 3.2 and Lemma 3.12 we obtain that the following spaces are equal as locally convex spaces with equivalent spectra:

$$S_{\frac{\lambda}{2q_1}, \dots, \frac{\lambda}{2q_n}, \frac{\lambda}{2p_1}, \dots, \frac{\lambda}{2p_n}}$$

$$\bigcap_{k=1}^n S_{\frac{\lambda}{2p_k}, \frac{\lambda}{2q_k}}(Q_k, P_k),$$

$$\bigcap_{k=1}^n S_{\frac{\lambda}{2p_k}, \frac{\lambda}{2q_k}}(\tilde{Q}_k, \tilde{P}_k),$$

$$\bigcap_{k=1}^n S_{\lambda}(\tilde{Q}_k^{2p_k} + \tilde{P}_k^{2q_k} + \frac{1}{n}I)$$

and

$$S_\lambda(\tilde{Q}_1^{2p_1} + \dots + \tilde{Q}_n^{2p_n} + \tilde{P}_1^{2q_1} + \dots + \tilde{P}_n^{2q_n} + I).$$

This proves the theorem. \square

Remark. The previous theorem can also be proved by copying the proof of Theorem 3.4 with

$$V := \bigcup_{k=1}^n \{I, \tilde{Q}_k, \dots, \tilde{Q}_k^{p_k}, \tilde{P}_k, \dots, \tilde{P}_k^{q_k},\}$$

and

$$L := \text{span}(\{W_1 W_2 : W_1, W_2 \in V\})$$

followed by obvious modifications.

We finish this section with a Gevrey space relative to two unbounded closed linear operators A, B ; the operator A is not essentially self-adjoint; $S_{\lambda, \mu}(A, B) = S_\lambda(A) \cap S_\mu(B)$ for all $\lambda, \mu \geq 0$ and the space $S_{\lambda, \mu}(A, B)$ is dense in H for all $\lambda > 1$ and $\mu \geq 0$.

Let $H := L^2([0, \infty))$. For every function f on $[0, \infty)$ define $\tilde{f} : \mathbb{R} \rightarrow \mathbb{C}$ by

$$\tilde{f}(x) := \begin{cases} f(x) & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

The map $f \mapsto \tilde{f}$ induces a map from $L^2([0, \infty))$ into $L^2(\mathbb{R})$, also denoted by $f \mapsto \tilde{f}$. Define the operator A in $L^2([0, \infty))$ by

$$\begin{aligned} D(A) &:= \{f \in L^2([0, \infty)) : f \text{ is continuous, } f(0) = 0 \text{ and } \tilde{f} \in D(P)\} \\ Af &:= Pf \quad (f \in D(A)), \end{aligned}$$

and let B be the multiplication operator by the function $x \mapsto x$ in $L^2([0, \infty))$. By [Wei], Section 8.2 Example 1, the operator A is closed, symmetric and not essentially self-adjoint. Note that

$$D^\infty(A) = \{f \in L^2([0, \infty)) : \tilde{f} \in D^\infty(P)\}.$$

Lemma 3.14 For the joint C^∞ -domain $D^\infty(A, B)$ we have

$$D^\infty(A, B) = D^\infty(A) \cap D^\infty(B) = \{f \in L^2([0, \infty)) : \tilde{f} \in S(\mathbb{R})\}.$$

Proof. Let $H_0 := \{f \in H : \tilde{f} \in S(\mathbb{R})\}$. Then

$$\begin{aligned} D^\infty(A, B) &\subset D^\infty(A) \cap D^\infty(B) = \{f \in H : \tilde{f} \in D^\infty(P)\} \cap \{f \in H : \tilde{f} \in D^\infty(Q)\} \\ &= \{f \in H : \tilde{f} \in D^\infty(P) \cap D^\infty(Q)\} = H_0. \end{aligned}$$

Clearly $H_0 \subset D(A)$, $A(H_0) \subset H_0$, $H_0 \subset D(B)$ and $B(H_0) \subset H_0$. So $H_0 \subset D^\infty(A, B)$. \square

Lemma 3.15 Let $\lambda, \mu \geq 0$. Then

$$S_{\lambda,\mu}(A, B) = \{f \in H : \tilde{f} \in S_\mu^\lambda\}.$$

Proof. Trivial. □

Theorem 3.16 *Let $\lambda, \mu \geq 0$. If $\lambda \leq 1$ then $S_{\lambda,\mu}(A, B) = \{0\}$. If $\lambda > 1$ then $S_{\lambda,\mu}(A, B)$ is dense in $L^2([0, \infty))$.*

Proof. Suppose $\lambda \leq 1$. Let $f \in S_{\lambda,\mu}(A, B)$. Then $\tilde{f} \in S_\mu^\lambda$. By [GS], Section IV.2.3, the function \tilde{f} can be extended to an analytic function. But $\tilde{f}(x) = 0$ for all $x < 0$, so $\tilde{f} = 0$ and $f = 0$.

Suppose $\lambda > 1$. Let $f \in L^2([0, \infty))$ and suppose $(f, g) = 0$ for all $g \in S_{\lambda,\mu}(A, B)$. Let $x \in (0, \infty)$. By [GS], Section IV.8.1, there exists $h \in S_0^\lambda$ such that $\text{supp } h \subset (0, \infty)$ and $h(x) \neq 0$. Let $y \in \mathbb{R}$. Then the function $z \mapsto h(z)e^{izy}$, $z \in \mathbb{R}$ is an element of S_0^λ . Let g be the restriction of this function to $[0, \infty)$. Then $g \in S_{\lambda,0}(A, B) \subset S_{\lambda,\mu}(A, B)$. Hence

$$0 = (f, g) = \int_{-\infty}^{\infty} e^{izy} \tilde{f}(z) h(z) dz.$$

Then $(\tilde{f} \cdot h)^\wedge = 0$, and $\tilde{f} \cdot h = 0$ a.e. Since $h(x) \neq 0$ and h is continuous, it follows that $\tilde{f} = 0$ a.e. on a neighborhood of x . So $\tilde{f} = 0$ a.e. on $(0, \infty)$. Then $f = 0$ in $L^2([0, \infty))$. □

Theorem 3.17 *Let $\lambda, \mu \geq 0$. Then $S_{\lambda,\mu}(A, B) = S_\lambda(A) \cap S_\mu(B)$ as locally convex spaces with equivalent spectra.*

Proof. If $\lambda \leq 1$ then $S_\lambda(A) = \{0\}$ since every $\tilde{f} \in S_1(P)$ can be extended to an analytic function. (See Paley-Wiener, [PW], Theorem I.) So $S_{\lambda,\mu}(A, B) = \{0\} = S_\lambda(A) \cap S_\mu(B)$ if $\lambda \leq 1$.

Now suppose $\lambda \geq 1$. Let A_0 and B_0 be the restrictions of A and B to $D^\infty(A, B)$ respectively. By Lemma 3.1 we obtain that $S_{\lambda,\mu}(A_0, B_0) = S_\lambda(A_0) \cap S_\mu(B_0)$ as locally convex spaces with equivalent spectra. Since $D^\infty(A, B) = D^\infty(A) \cap D^\infty(B)$, also $S_{\lambda,\mu}(A, B) = S_\lambda(A) \cap S_\mu(B)$ as locally convex spaces with equivalent spectra. □

3.2 The $ax + b$ group

Let $G := (0, \infty) \times \mathbb{R}$ with the induced topology of \mathbb{R}^2 . Define a multiplication on G by

$$(a, b) \cdot (c, d) := (ac, ad + b) \quad ((a, b), (c, d) \in G).$$

So G is a Lie group which is isomorphic with the group of matrices of the form $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ with $a > 0$ and $b \in \mathbb{R}$. Then G is a solvable Lie group. Let y denote the embedding from G into \mathbb{R}^2 . Then (G, y) is a chart on G . By $e := (1, 0)$ we denote the identity in G . Define

$$X := \left. \frac{\partial}{\partial y_1} \right|_e, \quad Y := \left. \frac{\partial}{\partial y_2} \right|_e.$$

Then X, Y is a basis in the Lie algebra \mathfrak{g} of G and $[X, Y] = Y$. So by Theorem 2.16.II and the Remark following Corollary 2.3 we obtain for every representation π in a Hilbert space, all $\lambda \geq 1$ and all $\mu \geq 0$ that

$$S_{\lambda, \mu}(d\pi(X), d\pi(Y)) = S_\lambda(d\pi(X)) \cap S_\mu(d\pi(Y))$$

as locally convex spaces with equivalent spectra.

Let \exp denote the exponential map from \mathfrak{g} into G . Then

$$\exp(\alpha X + \beta Y) = \begin{cases} (e^\alpha, \beta \frac{e^\alpha - 1}{\alpha}) & \text{if } \alpha \neq 0, \\ (1, \beta) & \text{if } \alpha = 0. \end{cases}$$

The $ax + b$ group has up to unitary equivalence two irreducible infinite dimensional representations. For $(a, b) \in G$ there exists a unique unitary map $U_{(a,b)}^\pm$ from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ such that

$$[U_{(a,b)}^\pm f](x) = e^{\pm i b e^x} f(x + \log a) \quad \text{a.e. } x \in \mathbb{R}$$

for all $f \in C_c(\mathbb{R})$. Then $(a, b) \mapsto U_{(a,b)}^\pm$ is a (continuous unitary) irreducible representation of G in $L^2(\mathbb{R})$. Let E be the multiplication operator in $L^2(\mathbb{R})$ of multiplication by the function $x \mapsto e^x, x \in \mathbb{R}$. Then

$$dU^\pm(X) = -iP,$$

$$dU^\pm(Y) = \pm iE.$$

For $\lambda, \mu \geq 0$ we shall consider the Gevrey space $S_{\lambda, \mu}(dU^\pm(X), dU^\pm(Y)) = S_{\lambda, \mu}(P, E)$. Since $S_{\lambda, \mu}(P, E) \subset D^\infty(P)$, every element of $S_{\lambda, \mu}(P, E)$ is infinitely differentiable.

Lemma 3.18 *Let $\lambda, \mu \geq 0$. Let $f \in S_{\lambda, \mu}(P, E)$. Then there exist $C, t > 0$ such that for all $k, l \in \mathbb{N}_0$:*

$$\|E^k P^l f\|_\infty \leq C t^{k+l} k!^\mu l!^\lambda.$$

Proof. There exist $C, t > 0$ such that $\|f\|_{P, E; \lambda, \mu; t} \leq C$. Then by a classical Sobolev inequality we obtain for all $k, l \in \mathbb{N}_0$:

$$\begin{aligned} \|E^k P^l f\|_\infty &\leq \sqrt[2]{\frac{\pi}{2}} \|E^k P^l f\|_2 + \sqrt[2]{\frac{\pi}{2}} \|P E^k P^l f\|_2 \\ &\leq \sqrt[2]{\frac{\pi}{2}} C t^{k+l} k!^\mu l!^\lambda + \sqrt[2]{\frac{\pi}{2}} C t^{k+l+1} k!^\mu (l+1)!^\lambda \\ &\leq \sqrt[2]{\frac{\pi}{2}} C (1+t) (2^\lambda t)^{k+l} k!^\mu l!^\lambda. \end{aligned}$$

□

Paley and Wiener ([PW], Theorems I and IV) have given a characterization of the space $S_1(P)$ in terms of analytic functions. Because of our intersection results, this is useful in the characterization of the spaces $S_{1, \mu}(P, E)$ where $\mu > 0$.

Theorem 3.19 Let $f \in L^2(\mathbb{R})$. Then $f \in S_1(P)$ if and only if there exist $t > 0$ such that f can be extended to an analytic function F on the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < 2t\}$ and

$$\sup_{y \in (-t, t)} \int_{-\infty}^{\infty} |F(x + iy)|^2 dx < \infty.$$

Theorem 3.20 Let $\mu > 0$ and $f \in L^2(\mathbb{R})$. The following conditions are equivalent:

- I. $f \in S_{1, \mu}(P, E)$,
- II. $f \in S_1(P)$ and there exist $C, \sigma, t > 0$ such that f can be extended to an analytic function F on the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < 2t\}$ and

$$|F(x + iy)| \leq C e^{-\sigma e^{\frac{x}{\mu}}}$$

for all $x \in \mathbb{R}$ and all $y \in (-t, t)$.

Proof. I \Rightarrow II. By Lemma 3.18 there exist $C, s > 0$ such that $\|E^k P^l f\|_{\infty} \leq C s^{k+l} k!^{\mu} l!$ for all $k, l \in \mathbb{N}_0$. Then we have for each $N \in \mathbb{N}$ and all $x, h \in \mathbb{R}$ the following estimate for the remainder term of the Taylor formula:

$$\left| f(x+h) - f(x) - hf'(x) - \dots - \frac{h^{N-1} f^{(N-1)}(x)}{(N-1)!} \right| = \left| \frac{h^N}{N!} f^{(N)}(x + \theta h) \right| \leq C |h|^N s^N.$$

So $f(x+h) = \sum_{l=0}^{\infty} h^l l!^{-1} f^{(l)}(x)$ for all $x \in \mathbb{R}$ and all $h \in \mathbb{R}$ with $|h| < 2t$, where $t := (2s)^{-1}$. Hence f can be extended to an analytic function F on the strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < 2t\}$.

For the remaining part of the proof I \Rightarrow II we proceed as in [GS], Section IV.2. Let $x \in \mathbb{R}$ and $y \in (-t, t)$. Then for all $k \in \mathbb{N}_0$:

$$\begin{aligned} |e^{kx} F(x + iy)| &\leq \left| \sum_{l=0}^{\infty} e^{kx} \frac{(iy)^l}{l!} f^{(l)}(x) \right| \\ &\leq \sum_{l=0}^{\infty} C |y|^l s^{k+l} k!^{\mu} \\ &\leq 2C s^k k^{k\mu}. \end{aligned}$$

So $|F(x + iy)| \leq 2C \inf\{\frac{k^{\mu}}{(s^{-1}e^x)^k} : k \in \mathbb{N}_0\}$. Then by [GS], inequality IV.2.1.3 we obtain that

$$|F(x + iy)| \leq 2C e^{\frac{\mu x}{2}} e^{-\frac{\mu}{2}(s^{-1}e^x)^{\frac{1}{2}}}.$$

II \Rightarrow I. Let $k \in \mathbb{N}$. Then

$$\begin{aligned} \int_{\mathbb{R}} |(e^{\frac{x}{\mu}})^k f(x)|^2 dx &\leq \sqrt{2\pi} \|f\|_2^2 + C^2 \int_0^{\infty} e^{2\frac{x}{\mu}k} e^{-2\sigma e^{\frac{x}{\mu}}} dx \\ &= \sqrt{2\pi} \|f\|_2^2 + \mu C^2 \int_0^{\infty} \left(\frac{y}{2\sigma}\right)^{2k} e^{-y} y^{-1} dy \\ &= \sqrt{2\pi} \|f\|_2^2 + \mu C^2 (2\sigma)^{-2k} (2k-1)! \\ &\leq \sqrt{2\pi} \|f\|_2^2 + \mu C^2 \sigma^{-2k} k!^2 < \infty. \end{aligned}$$

So $f \in D(|E|^{k/\mu})$ and $\|(|E|^{1/\mu})^k f\| \leq \|f\|^2 + \frac{\sqrt{\mu}C}{\sqrt{2\pi}} \sigma^{-k} k!$ for all $k \in \mathbb{N}$. Hence $f \in S_1(|E|^{1/\mu}) = S_\mu(E)$ by Theorem 1.4. So, by assumption, $f \in S_1(P) \cap S_\mu(E)$. Now we use the intersection Theorem 2.16.II and the Remark following Corollary 2.3, in order to conclude that $f \in S_{1,\mu}(P, E)$. \square

Corollary 3.21 *Let $\mu > 0$. Define W from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ by*

$$[Wf](x) := \frac{1}{\sqrt{\mu}} f\left(\frac{x}{\mu}\right) \quad (\text{a.e. } x \in \mathbb{R}).$$

for all $f \in L^2(\mathbb{R})$. Then W is a unitary map in $L^2(\mathbb{R})$ which maps $S_{1,1}(P, E)$ onto $S_{1,\mu}(P, E)$.

Corollary 3.22 *Let $\mu > 0$. Then the Gevrey space $S_{1,\mu}(P, E)$ is dense in $L^2(\mathbb{R})$.*

Proof. By Corollary 1.26 the space $S_{1,1}(P, E) = S_{1,1}(dU^\pm(X), dU^\pm(Y)) = H^\omega(U^\pm)$ is dense in $L^2(\mathbb{R})$, so the corollary follows from Corollary 3.21. \square

Lemma 3.23 *The space $S_{1,0}(P, E)$ is trivial.*

Proof. Let $f \in S_{1,0}(P, E)$. By Lemma 3.18 there exist $C, t > 0$ such that $\|E^k P^l f\|_\infty \leq C t^{k+l} l!$ for all $k, l \in \mathbb{N}_0$. So f can be extended to an analytic function F on the strip $\{z \in \mathbb{C} : |\text{Im } z| < t^{-1}\}$.

Let $x \in \mathbb{R}$ be such that $e^x > t$. Then for all $k \in \mathbb{N}_0$: $e^{kx} |f(x)| \leq C t^k$, so $|F(x)| = |f(x)| \leq \inf\{C(t e^{-x})^k : k \in \mathbb{N}_0\} = 0$. Hence $F = 0$ and $f = 0$. \square

Lemma 3.24 *Let $\lambda \in [0, 1)$ and $\mu > 0$. Then the space $S_{\lambda,\mu}(P, E)$ is trivial.*

Proof. Let $f \in S_{\lambda,\mu}(P, E)$. By Lemma 3.18 there exist $C, t > 0$ such that $\|E^k P^l f\|_\infty \leq C t^{k+l} k!^\mu l!^\lambda$ for all $k, l \in \mathbb{N}_0$. Similarly to [GS], Section IV.2.2, it follows that f can be extended to an entire function F for which there exist $\tau, C_1 > 0$ such that

$$|e^{kx} F(x + iy)| \leq C_1 t^k k!^\mu \exp(\tau |y|^{1-\lambda})$$

for all $x, y \in \mathbb{R}$ and $k \in \mathbb{N}_0$. Similarly to the proof of Theorem 3.20 there exist $\sigma, C_2 > 0$ such that

$$|F(x + iy)| \leq C_2 e^{-\sigma e^{x/\mu} + \tau |y|^{1-\lambda}}$$

for all $x, y \in \mathbb{R}$.

Let $z = x + iy \in \mathbb{C}$. Then

$$|F(z) \cdot F(iz) \cdot F(-z) \cdot F(-iz)| \leq C_2^4 e^{-\sigma e^{x/\mu} - \sigma e^{-x/\mu} + 2\tau |x|^{1-\lambda}} e^{-\sigma e^{y/\mu} - \sigma e^{-y/\mu} + 2\tau |y|^{1-\lambda}}.$$

Hence $\lim_{|z| \rightarrow \infty} F(z) \cdot F(iz) \cdot F(-z) \cdot F(-iz) = 0$. By the Liouville theorem, $F(z) \cdot F(iz) \cdot F(-z) \cdot F(-iz) = 0$ for all $z \in \mathbb{C}$. Then also $F = 0$ and $f = 0$. \square

At this point it is not known whether the spaces $S_{\lambda,0}(P, E)$ are trivial if $\lambda > 1$. This problem will be solved in the following lemma.

Lemma 3.25 *Let $\lambda > 1$. Then the space $S_{\lambda,0}(P, E)$ is dense in $L^2(\mathbb{R})$. In particular, the Gelfand-Shilov space S_0^λ is a subspace of $S_{\lambda,0}(P, E)$.*

Proof. Let $f \in S_0^\lambda$. Then there exist $C, t > 0$ such that $|x|^k |f^{(l)}(x)| \leq Ct^{k+l} l!^\lambda$ for all $x \in \mathbb{R}$ and $k, l \in \mathbb{N}_0$. Then for all $x \in \mathbb{R} \setminus \{0\}$: $|f(x)| \leq \inf\{Ct^k |x|^{-k} : k \in \mathbb{N}_0\}$, so $f(x) = 0$ for all $x \in \mathbb{R}$, $|x| > t$. Therefore, for all $k \in \mathbb{N}_0$:

$$\int_{\mathbb{R}} |(e^x)^k f(x)|^2 dx \leq (e^t)^{2k} \sqrt{2\pi} \|f\|_2^2,$$

hence $f \in S_0(E)$.

Obviously, $f \in S_0^\lambda \subset S_\lambda(P)$. Therefore $f \in S_\lambda(P) \cap S_0(E)$. Using again the intersection result Theorem 2.16.II, we obtain that $f \in S_{\lambda,0}(P, E)$. \square

We summarize the results of the previous lemmas and corollary.

Theorem 3.26 *Let $\lambda, \mu \geq 0$. Then the space $S_{\lambda,\mu}(P, E)$ is dense in $L^2(\mathbb{R})$ if and only if*

$$\lambda > 1 \text{ and } \mu \geq 0$$

or

$$\lambda = 1 \text{ and } \mu > 0.$$

If π is a representation of an arbitrary Lie group K in a Hilbert space H and Y_1, \dots, Y_d is a basis in the Lie algebra of K , then Bruhat, [Bru], has proved that the space $H^\infty(\pi) = D^\infty(d\pi(Y_1), \dots, d\pi(Y_d))$ is K -invariant, i.e. $\pi_g(H^\infty(\pi)) \subset H^\infty(\pi)$ for all $g \in K$ and the restriction representation of π to $H^\infty(\pi)$ is a continuous representation of K in the locally convex topological vector space $H^\infty(\pi)$. Later, in [Goo3], Theorem 2.1, Goodman has proved similar facts in case the space $H^\infty(\pi)$ is replaced by the smaller space $S_{\lambda, \dots, \lambda}(d\pi(Y_1), \dots, d\pi(Y_d))$, where any $\lambda \geq 0$ may be taken. We shall prove that for all $\mu > 1$ similar statements do not hold for the space $S_{1,\mu}(dU^\pm(X), dU^\pm(Y))$. In fact, these spaces are not even G -invariant.

Theorem 3.27 *Let $\mu > 1$. Then the space $S_{1,\mu}(dU^\pm(X), dU^\pm(Y))$ is not G -invariant.*

Proof. We have to show that there exist $f \in S_{1,\mu}(P, E)$ and $g \in G$ such that $U_g^\pm f \notin S_{1,\mu}(P, E)$. Define $f : \mathbb{R} \rightarrow \mathbb{R}$ by

$$f(x) := e^{-e^{\frac{1}{h}\sqrt{1+x^2}}} \quad (x \in \mathbb{R}).$$

By Theorem 3.20 and a theorem of Paley-Wiener, Theorem 3.19, we obtain that $f \in S_{1,\mu}(P, E)$.

Let $t > 0$ and let $g := \exp(tY)$. We shall prove that $U_g^\pm f \notin S_1(P)$ and therefore $U_g^\pm f \notin S_{1,\mu}(P, E)$.

Suppose $U_g^\pm f \in S_1(P)$. By the same theorem of Paley-Wiener, there exist $s > 0$ and an extension h of $U_g^\pm f$ defined on the strip $\{z \in \mathbb{C} : |\text{Im } z| < 2s\}$ such that

$$\sup_{\nu \in (-s, s)} \int_{-\infty}^{\infty} |h(x + iy)|^2 dx < \infty.$$

For all $x \in \mathbb{R}$ we have

$$h(x) = (U_{\exp(iy)}^{\pm} f)(x) = (U_{(1,t)}^{\pm} f)(x) = e^{\pm ite^x} f(x) = e^{\pm ite^x - e^{\frac{1}{\mu}\sqrt{1+x^2}}}.$$

So $h(z) = e^{\pm ite^z - e^{\frac{1}{\mu}\sqrt{1+z^2}}}$ for all $z \in \mathbb{C}$ with $|\operatorname{Im} z| < \min(2s, 1)$. Let $y = \mp \frac{1}{2} \min(s, 1)$. (The sign corresponds to U^{\pm} .) Let $\tau > 1$ be such that $\tau < \mu$. Let $M > 0$ be so large that $1 + |x + iy|^2 \leq \tau^2 x^2$ and $e^{(\tau/\mu-1)x} \leq \frac{1}{2}t|\sin y|$ for all $x \geq M$. Then for all $x \geq M$:

$$e^{-x} \operatorname{Re} e^{\frac{1}{\mu}\sqrt{1+(x+iy)^2}} \leq e^{\frac{1}{\mu}\sqrt{1+(x+iy)^2} - x} \leq e^{\frac{1}{\mu}\sqrt{1+|x+iy|^2} - x} \leq e^{\frac{1}{\mu}\tau x - x} \leq \frac{1}{2}t|\sin y|.$$

So

$$\operatorname{Re} \left[\pm ite^{x+iy} - e^{\frac{1}{\mu}\sqrt{1+(x+iy)^2}} \right] \geq t|\sin y|e^x - \frac{1}{2}t|\sin y|e^x = \frac{1}{2}t|\sin y|e^x.$$

Hence

$$|h(x + iy)| = e^{\operatorname{Re} \left[\pm ite^{x+iy} - e^{\frac{1}{\mu}\sqrt{1+(x+iy)^2}} \right]} \geq e^{\frac{1}{2}t|\sin y|e^x}$$

for all $x \geq M$ and therefore the function $x \mapsto h(x + iy)$ is not an element of $L^2(\mathbb{R})$. Contradiction. \square

Also in this section we consider the problem whether a Gevrey space $S_{\lambda, \mu}(P, E)$ is equal to the Gevrey space relative to one single operator. Let $\lambda \geq 1$ and $\mu > 0$. We shall prove that there exist $\tau > 0$ and a positive self-adjoint operator A in $L^2(\mathbb{R})$ such that $S_{\lambda, \mu}(P, E) = S_{\tau}(A)$ as locally convex spaces with equivalent spectra. More precisely, for $\nu > 0$ let E_0^{ν} be the restriction of the operator E^{ν} to the space $D^{\infty}(P, E) = H^{\infty}(U^{\pm})$. Let P_0 be the restriction of P to $D^{\infty}(P, E)$. Let $q \in \mathbb{N}$. Let $A_0 := P_0^{2q} + E_0^{2q\frac{\lambda}{\mu}} + I$. Then A_0 is essentially self-adjoint and $S_{\lambda, \mu}(P, E) = S_{2q\lambda}(\overline{A_0})$ as locally convex spaces with equivalent spectra.

Let $\nu > 0$. Define the unitary operator W_{ν} from $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ by

$$[W_{\nu} f](x) := \frac{1}{\sqrt{\nu}} f\left(\frac{x}{\nu}\right) \quad (\text{a.e. } x \in \mathbb{R})$$

for all $f \in L^2(\mathbb{R})$. (Cf. Corollary 3.21.) For all $f \in H^{\infty}(U^{\pm})$ we have $W_{\nu} f \in D^{\infty}(P) = D^{\infty}(dU^{\pm}(X))$ and $W_{\nu} f \in D^{\infty}(E) = D^{\infty}(dU^{\pm}(Y))$, so $W_{\nu} f \in H^{\infty}(U^{\pm})$ by Theorem 1.23. Let P_0 and E_0 be the restrictions of P and E to $H^{\infty}(U^{\pm})$ respectively. Let

$$E_0^{\nu} := W_{\nu}^{-1} E_0 W_{\nu}.$$

Note that $(E_0^{\nu} f)(x) = e^{\nu x} f(x)$ for all $x \in \mathbb{R}$ and $f \in H^{\infty}(U^{\pm})$. Moreover

$$\nu^{-1} P_0 = W_{\nu}^{-1} P_0 W_{\nu}.$$

Theorem 3.28 *Let $\lambda \geq 1$, $\mu > 0$ and $q \in \mathbb{N}$. Let*

$$A_0 := P_0^{2q} + E_0^{2q\frac{\lambda}{\mu}} + I.$$

Then the operator A_0 is essentially self-adjoint and $S_{\lambda,\mu}(P, E) = S_{2q\lambda}(\overline{A_0})$ as locally convex spaces with equivalent spectra.

Proof. Let $\nu := \frac{\lambda}{\mu}$. Then $A_0 = W_\nu^{-1} \left((\nu P_0)^{2q} + E_0^{2q} + I \right) W_\nu$, so A_0 is unitarily equivalent with the operator

$$B_0 := (\nu P_0)^{2q} + E_0^{2q} + I.$$

By [Goo6], Corollary 4.1, the operator B_0 is essentially self-adjoint and $H^\infty(U^\pm) = D^\infty(\overline{B_0})$. So A_0 is essentially self-adjoint and $D^\infty(\overline{A_0}) = D^\infty(W_\nu^{-1}\overline{B_0}W_\nu) = W_\nu^{-1}D^\infty(\overline{B_0}) = W_\nu^{-1}H^\infty(U^\pm) = H^\infty(U^\pm)$. Moreover, by the same Corollary it follows that there exists $M > 0$ such that for all $k \in \{0, \dots, 2q\}$, for all $X_1, \dots, X_k \in \{P_0, E_0\}$ and all $u \in H^\infty(U^\pm)$ we have

$$\|X_1 \circ \dots \circ X_k u\| \leq M \|B_0 u\|.$$

Since $\text{span}(\{iP_0, iE_0\})$ is a Lie algebra, now it follows that there exists $N > 0$ such that for all $n \in \mathbb{N}$, all $X_1, \dots, X_n \in \{P_0, E_0\}$ and all $u \in H^\infty(U^\pm)$ we have

$$\|\text{ad}X_1 \dots \text{ad}X_n(B_0)u\| \leq N^n \|B_0 u\|.$$

Hence by Theorem 3.3 we obtain that

$$S_{2q\lambda}(B_0) \subset S_{\lambda,\lambda}(P_0, E_0)$$

as sets. By an elementary counting argument we obtain also that

$$S_{\lambda,\lambda}(P_0, E_0) \subset S_{2q\lambda}(B_0).$$

Therefore

$$S_{2q\lambda}(B_0) = S_{\lambda,\lambda}(P_0, E_0) = S_\lambda(P_0) \cap S_\lambda(E_0)$$

as sets, by Theorem 2.16.II.

As sets, we obtain by Theorem 1.4: $S_\lambda(E_0^\nu) = S_\lambda(E^\nu) \cap H^\infty(U^\pm) = S_\mu(E) \cap H^\infty(U^\pm) = S_\mu(E_0)$. So

$$W_\nu S_{2q\lambda}(A_0) = S_{2q\lambda}(W_\nu A_0 W_\nu^{-1}) = S_{2q\lambda}(B_0) = S_\lambda(P_0) \cap S_\lambda(E_0)$$

and

$$\begin{aligned} S_{2q\lambda}(A_0) &= W_\nu^{-1} S_\lambda(P_0) \cap W_\nu^{-1} S_\lambda(E_0) \\ &= S_\lambda(W_\nu^{-1} P_0 W_\nu) \cap S_\lambda(W_\nu^{-1} E_0 W_\nu) \\ &= S_\lambda(\nu^{-1} P_0) \cap S_\lambda(E_0^\nu) \\ &= S_\lambda(P_0) \cap S_\mu(E_0) \\ &= S_{\lambda,\mu}(P_0, E_0). \end{aligned}$$

The last equality is again by Theorem 2.16.II. Since $D^\infty(\overline{A_0}) = H^\infty(U^\pm) = D^\infty(P, E)$, we obtain that $S_{2q\lambda}(\overline{A_0}) = S_{\lambda,\mu}(P, E)$ as sets. Because the operators P, E and $\overline{A_0}$ are closed, then also $S_{2q\lambda}(\overline{A_0}) = S_{\lambda,\mu}(P, E)$ as locally convex spaces with equivalent spectra. (See Corollary 1.22.) \square

3.3 The real unimodular group $SL(2, \mathbb{R})$

The group $SL(2, \mathbb{R})$ consists of all 2×2 real matrices with determinant 1. It is common practice to identify the Lie algebra of $SL(2, \mathbb{R})$ with the Lie algebra $sl(2, \mathbb{R})$ of all 2×2 real matrices with trace 0. Let

$$A := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X := \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y := \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then A, X, Y is a basis in $sl(2, \mathbb{R})$ and

$$[A, X] = 2Y, \quad [A, Y] = -2X, \quad [X, Y] = -\frac{1}{2}A.$$

Define in the complex universal enveloping algebra of $sl(2, \mathbb{R})$ the *Casimir element*

$$C := \frac{1}{8}(4X^2 + 4Y^2 - A^2).$$

Then C commutes with A, X and Y .

For $k \in \mathbb{Z}$ define $\gamma_k : \mathbb{T} \rightarrow \mathbb{C}$ by

$$\gamma_k(z) := z^k \quad (z \in \mathbb{T}).$$

Let $H_0 := \text{span}(\{\gamma_k : k \in \mathbb{Z}\})$. Then H_0 is a dense subspace of $L^2(\mathbb{T})$. Let $s \in \mathbb{R}$ be fixed. Define the operators A_0, X_0 and Y_0 on H_0 by

$$\begin{aligned} A_0 \gamma_k &:= -ik \gamma_k, \\ X_0 \gamma_k &:= -\frac{1}{4}(k+1+is)\gamma_{k+2} + \frac{1}{4}(k-1-is)\gamma_{k-2}, \\ Y_0 \gamma_k &:= \frac{i}{4}(k+1+is)\gamma_{k+2} + \frac{i}{4}(k-1-is)\gamma_{k-2}, \end{aligned}$$

where $k \in \mathbb{Z}$. Then

$$\begin{aligned} (A_0 f)(e^{i\theta}) &= -\frac{d}{d\theta} f(e^{i\theta}), \\ (X_0 f)(e^{i\theta}) &= -\frac{1}{2} \sin 2\theta \frac{d}{d\theta} f(e^{i\theta}) - \frac{1+is}{2} \cos 2\theta f(e^{i\theta}), \\ (Y_0 f)(e^{i\theta}) &= \frac{1}{2} \cos 2\theta \frac{d}{d\theta} f(e^{i\theta}) - \frac{1+is}{2} \sin 2\theta f(e^{i\theta}), \end{aligned}$$

for all $f \in H_0$ and $\theta \in \mathbb{R}$. Moreover, A_0, X_0 and Y_0 are skew-symmetric operators which satisfy the same commutation relations as A, X and Y , namely $[A_0, X_0] = 2Y_0$, $[A_0, Y_0] = -2X_0$ and $[X_0, Y_0] = -\frac{1}{2}A_0$. So the map

$$aA + xX + yY \mapsto aA_0 + xX_0 + yY_0 \quad (a, x, y \in \mathbb{R})$$

from $sl(2, \mathbb{R})$ into the real vector space of skew-symmetric operators in $L^2(\mathbb{T})$ with domain H_0 , is a representation of $sl(2, \mathbb{R})$ by skew-symmetric operators on H_0 . Let $\Delta_0 := A_0^2 + X_0^2 + Y_0^2$. Then $\Delta_0 \gamma_k = -\frac{1}{4}(1 + s^2 + 5k^2)\gamma_k$ for all $k \in \mathbb{Z}$. So

$$H_0 = S_0(\Delta_0) \subset S_1(\Delta_0).$$

Since H_0 is dense in $L^2(\mathbb{T})$, it follows from [Nel], Lemma 5.1 that the operator Δ_0 is essentially self-adjoint. Hence by [Nel], Theorem 5 there exists a unique unitary representation U of the universal covering group $[SL(2, \mathbb{R})]^\sim$ of $SL(2, \mathbb{R})$, such that

$$dU(A) = \overline{A_0}, \quad dU(X) = \overline{X_0}, \quad dU(Y) = \overline{Y_0}.$$

(Actually, U can be restricted to a unitary representation of $SL(2, \mathbb{R})$, see Van Dijk, [vD], §2.) Since $\Delta_0 \gamma_k = -\frac{1}{4}(1 + s^2 + 5k^2)\gamma_k$ and $A_0 \gamma_k = -ik\gamma_k$ for all $k \in \mathbb{Z}$, it follows that for all $\lambda \geq 0$

$$S_{2\lambda}(\overline{\Delta_0}) = S_\lambda(\overline{A_0})$$

as locally convex spaces with equivalent spectra. So by Theorem 1.24 we obtain for all $\lambda \geq 1$:

$$S_{\lambda, \lambda, \lambda}(dU(A), dU(X), dU(Y)) = S_\lambda(\overline{A_0}) = S_\lambda(dU(A))$$

as locally convex spaces with equivalent spectra. In particular, $H^\omega(U) = S_1(\overline{A_0})$ is equal to the set of all real analytic functions on \mathbb{T} . We shall prove that also for all $\lambda \geq 1$

$$H_\lambda(U) = S_{\lambda, \lambda, \lambda}(dU(A), dU(X), dU(Y)) = S_{\lambda, \lambda}(dU(X), dU(Y))$$

as locally convex topological vector spaces and the last two spaces are equal as locally convex spaces with equivalent spectra. In the proof the Casimir element plays an essential role. Because $\frac{1}{8}(4X_0^2 + 4Y_0^2 - A_0^2) = -\frac{1}{8}(1 + s^2)I$, we have $\partial U(C) = -\frac{1}{8}(1 + s^2)I$. We can put this in a more general setting. We need a theorem on "analytic" dominance.

Theorem 3.29 *Let $d, d_1 \in \mathbb{N}$. Let Z_1, \dots, Z_d and X_1, \dots, X_{d_1} be operators in a Hilbert space which are defined on a common invariant domain D . Set*

$$\| |u| \| := \max(\|u\|, \|X_1 u\|, \dots, \|X_{d_1} u\|) \quad (u \in D).$$

Suppose there exist $\lambda \geq 1$ and $M > 0$ such that for all $u \in D$, all $n \in \mathbb{N}$, all $j_1, \dots, j_n \in \{1, \dots, d\}$, all $j \in \{1, \dots, d\}$ and $k \in \{1, \dots, d_1\}$:

$$\|Z_j u\| \leq M \| |u| \|,$$

$$\| \text{ad} Z_{j_1} \dots \text{ad} Z_{j_n} (X_k) u \| \leq M^n n!^\lambda \| |u| \|.$$

Then for all $t > 0$ there exists $s > 0$ such that

$$S_{\lambda, \dots, \lambda; t}(X_1, \dots, X_{d_1}) \subset S_{\lambda, \dots, \lambda; s}(Z_1, \dots, Z_d)$$

and the canonical inclusion is continuous.

Proof. See [GW], Theorem 1.1. □

Theorem 3.30 *Let G be a Lie group with Lie algebra \mathfrak{g} . Let π be a representation of G in a Hilbert space H . Let $d_1, d_2 \in \mathbb{N}$ and let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ be a basis in \mathfrak{g} . Let*

$$C := X_1^2 + \dots + X_{d_1}^2 - Y_1^2 - \dots - Y_{d_2}^2 \in U(\mathfrak{g}).$$

Suppose C belongs to the center of $U(\mathfrak{g})$ and suppose there exists $\tau \in \mathbb{R}$ such that

$$\partial\pi(C) = \tau I.$$

Let $\lambda \geq 1$. Then

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_{d_1}), d\pi(Y_1), \dots, d\pi(Y_{d_2})) = S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_{d_1}))$$

as locally convex spaces with equivalent spectra.

Proof. First we prove that $S_{\lambda, \dots, \lambda}(\partial\pi(X_1), \dots, \partial\pi(X_{d_1}), \partial\pi(Y_1), \dots, \partial\pi(Y_{d_2})) = S_{\lambda, \dots, \lambda}(\partial\pi(X_1), \dots, \partial\pi(X_{d_1}))$ as locally convex spaces with equivalent spectra.

Let $u \in H^\infty(\pi)$. Then

$$\begin{aligned} \tau \|u\|^2 &= (\tau u, u) = (\partial\pi(C)u, u) = \\ &= (\partial\pi(X_1)^2 u, u) + \dots + (\partial\pi(X_{d_1})^2 u, u) - (\partial\pi(Y_1)^2 u, u) - \dots - (\partial\pi(Y_{d_2})^2 u, u) \\ &= -\|\partial\pi(X_1)u\|^2 - \dots - \|\partial\pi(X_{d_1})u\|^2 + \|\partial\pi(Y_1)u\|^2 + \dots + \|\partial\pi(Y_{d_2})u\|^2. \end{aligned}$$

So for all $j \in \{1, \dots, d_2\}$ we obtain

$$\begin{aligned} \|\partial\pi(Y_j)u\| &\leq \left[\tau \|u\|^2 + \|\partial\pi(X_1)u\|^2 + \dots + \|\partial\pi(X_{d_1})u\|^2 \right]^{\frac{1}{2}} \\ &\leq (\sqrt{|\tau|} + \sqrt{d_1}) \max(\|u\|, \|\partial\pi(X_1)u\|, \dots, \|\partial\pi(X_{d_1})u\|) \\ &=: (\sqrt{|\tau|} + \sqrt{d_1}) \|u\|. \end{aligned}$$

and clearly for all $k \in \{1, \dots, d_1\}$

$$\|\partial\pi(X_k)u\| \leq (\sqrt{|\tau|} + \sqrt{d_1}) \|u\|.$$

Let $Z_1 := X_1, \dots, Z_{d_1} := X_{d_1}, Z_{d_1+1} := Y_1, \dots, Z_d := Y_{d_2}$, where $d := d_1 + d_2$. Since \mathfrak{g} is a Lie algebra, we obtain for all $n \in \mathbb{N}$, all $j_1, \dots, j_n \in \{1, \dots, d\}$, all $k \in \{1, \dots, d_1\}$ and all $u \in H^\infty(\pi)$ that $\text{ad}Z_{j_1} \dots \text{ad}Z_{j_n}(X_k)$ is a linear combination of Z_1, \dots, Z_d and that

$$\begin{aligned} \|\text{ad}\partial\pi(Z_{j_1}) \dots \text{ad}\partial\pi(Z_{j_n})(\partial\pi(X_k))u\| &\leq M^n d^{n-1} (\|\partial\pi(Z_1)u\| + \dots + \|\partial\pi(Z_d)u\|) \\ &\leq M^n d^n (\sqrt{|\tau|} + \sqrt{d_1}) \|u\|, \end{aligned}$$

where M is the maximum of the absolute values of the structure constants of \mathfrak{g} with respect to the basis Z_1, \dots, Z_d .

So by Theorem 3.29 for all $t > 0$ there exists $s > 0$ such that the embedding

$$S_{\lambda, \dots, \lambda; s}(\partial\pi(X_1), \dots, \partial\pi(X_{d_1})) \hookrightarrow S_{\lambda, \dots, \lambda; s}(\partial\pi(X_1), \dots, \partial\pi(X_{d_1}), \partial\pi(Y_1), \dots, \partial\pi(Y_{d_2}))$$

is continuous. Hence

$$S_{\lambda, \dots, \lambda}(\partial\pi(X_1), \dots, \partial\pi(X_{d_1})) = S_{\lambda, \dots, \lambda}(\partial\pi(X_1), \dots, \partial\pi(X_{d_1}), \partial\pi(Y_1), \dots, \partial\pi(Y_{d_2})) \quad (3.8)$$

as locally convex spaces with equivalent spectra.

We shall prove that

$$\begin{aligned} H^\infty(\pi) &= D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1})) \\ &= D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1}), d\pi(Y_1), \dots, d\pi(Y_{d_2})). \end{aligned}$$

Then we can replace $\partial\pi(X_k)$ by $d\pi(X_k)$ and $\partial\pi(Y_j)$ by $d\pi(Y_j)$ in (3.8) and we have proved the theorem. By the first part of Theorem 1.23 we already know that $H^\infty(\pi) = D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1}), d\pi(Y_1), \dots, d\pi(Y_{d_2}))$. So it remains to prove that

$$H^\infty(\pi) = D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1}))$$

as sets. This is stated in the following theorem.

Theorem 3.31 *Let G be a Lie group with Lie algebra \mathfrak{g} . Let π be a representation of G in a Hilbert space H . Let $d_1, d_2 \in \mathbb{N}$ and let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ be a basis in \mathfrak{g} . Let*

$$C := X_1^2 + \dots + X_{d_1}^2 - Y_1^2 - \dots - Y_{d_2}^2 \in U(\mathfrak{g}).$$

Suppose C belongs to the center of $U(\mathfrak{g})$ and suppose there exists $\tau \in \mathbb{R}$ such that

$$\partial\pi(C) = \tau I.$$

Then

$$H^\infty(\pi) = D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1}))$$

as sets.

Proof. We may assume that G is connected. Let $Z_1 := X_1, \dots, Z_{d_1} := X_{d_1}, Z_{d_1+1} := Y_1, \dots, Z_d := Y_{d_2}$, where $d := d_1 + d_2$. Let

$$\begin{aligned} \Delta &:= Z_1^2 + \dots + Z_d^2 \in U(\mathfrak{g}), \\ \Delta_1 &:= X_1^2 + \dots + X_{d_1}^2 \in U(\mathfrak{g}), \\ \tilde{\Delta} &:= \tilde{Z}_1^2 + \dots + \tilde{Z}_d^2, \\ \tilde{\Delta}_1 &:= \tilde{X}_1^2 + \dots + \tilde{X}_{d_1}^2. \end{aligned}$$

Here \tilde{Z} denotes the left invariant vector field on G which corresponds to Z . Let $u \in D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1}))$ be fixed. Let $m \in \mathbb{N}$. Let $v \in H^\infty(\pi)$. Then $(\tilde{u}, v)(g) =$

$(\pi_g u, v) = (u, \pi_{g^{-1}} v)$ for all $g \in G$, so (\tilde{u}, v) is an infinitely differentiable function from G into \mathbb{C} . Let $n \in \mathbb{N}$ and let $j_1, \dots, j_n \in \{1, \dots, d\}$. Then for all $g \in G$:

$$\begin{aligned}
 [\tilde{Z}_{j_1} \circ \dots \circ \tilde{Z}_{j_n}(\tilde{u}, v)](g) &= \\
 &= \frac{\partial}{\partial t_1} \Big|_0 \dots \frac{\partial}{\partial t_n} \Big|_0 (\pi_g \pi_{\exp(t_1 Z_{j_1})} \dots \pi_{\exp(t_n Z_{j_n})} u, v) \\
 &= \frac{\partial}{\partial t_1} \Big|_0 \dots \frac{\partial}{\partial t_n} \Big|_0 (\pi_g u, \pi_{\exp(-t_n \text{Ad}(g) Z_{j_n})} \dots \pi_{\exp(-t_1 \text{Ad}(g) Z_{j_1})} v) \\
 &= (-1)^n (\pi_g u, \partial \pi(\text{Ad}(g) Z_{j_n}) \dots \partial \pi(\text{Ad}(g) Z_{j_1}) v) \\
 &= (-1)^n (\pi_g u, \partial \pi(\text{Ad}(g)(Z_{j_n} \dots Z_{j_1})) v).
 \end{aligned}$$

Let $W \in \mathfrak{g}$. Then $\text{Ad}(\exp W)(C) = e^{\text{ad}W}(C) = C$, because C belongs to the center of $U(\mathfrak{g})$. Since G is connected, $\text{Ad}(g)(C) = C$ for all $g \in G$. Note that $\Delta = 2\Delta_1 - C$. So we obtain for all $g \in G$:

$$\begin{aligned}
 [\tilde{\Delta}^m(\tilde{u}, v)](g) &= (\pi_g u, \partial \pi(\text{Ad}(g)(\Delta^m)) v) \\
 &= \sum_{k=0}^m (-1)^k \binom{m}{k} (\pi_g u, [\partial \pi(\text{Ad}(g)(C))]^k \partial \pi(\text{Ad}(g)(2\Delta_1)^{m-k}) v) \\
 &= \sum_{k=0}^m (-1)^k \binom{m}{k} (\pi_g u, [\partial \pi(C)]^k \partial \pi(\text{Ad}(g)(2\Delta_1)^{m-k}) v) \\
 &= \sum_{k=0}^m (-1)^k \binom{m}{k} (\pi_g u, \tau^k \partial \pi(\text{Ad}(g)(2\Delta_1)^{m-k}) v) \\
 &= (\pi_g u, \partial \pi(\text{Ad}(g)((2\Delta_1 - \tau)^m)) v) \\
 &= [(2\tilde{\Delta}_1 - \tau)^m(\tilde{u}, v)](g) \\
 &= (\tilde{w}, v)(g),
 \end{aligned}$$

where

$$w := \left(2 \sum_{k=1}^{d_1} [d\pi(X_k)]^2 - \tau I \right)^m u.$$

(Recall that $u \in D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1}))$.)

Now let $\varphi \in C_c^\infty(G)$. Let λ be a right Haar measure on G . Then

$$\begin{aligned}
 \int_G \tilde{\Delta}^m \varphi(g)(\tilde{u}, v)(g) d\lambda(g) &= \int_G \varphi(g) [\tilde{\Delta}^m(\tilde{u}, v)](g) d\lambda(g) \\
 &= \int_G \varphi(g)(\tilde{w}, v)(g) d\lambda(g).
 \end{aligned}$$

Since $H^\infty(\pi)$ is dense in H , it follows from Lebesgue's theorem on dominated convergence that

$$\int_G \tilde{\Delta}^m \varphi(g)(\tilde{u}, v)(g) d\lambda(g) = \int_G \varphi(g)(\tilde{w}, v)(g) d\lambda(g)$$

for all $v \in H$.

Let $v \in H$. Then the function (\tilde{u}, v) is a weak solution of the equation $\tilde{\Delta}^m f = (\tilde{w}, v)$. Since (\tilde{w}, v) is a continuous function and $\tilde{\Delta}^m$ is an elliptic operator of order $2m$, it follows from the local regularity theorem for elliptic operators that (\tilde{u}, v) has locally L^2 derivatives of order $\leq 2m$. (See Folland, [Fol], Theorem 6.30.) Hence by [Fol], Lemma 6.9 (the Sobolev lemma), the function (\tilde{u}, v) is $2m - d$ times continuously differentiable. Therefore (\tilde{u}, v) is infinitely differentiable for all $v \in H$. By Theorem 1.28 (Poulsen), it follows that $u \in H^\infty(\pi)$. This proves the theorems. \square

Corollary 3.32 *Let G be a Lie group with Lie algebra \mathfrak{g} . Let π be an irreducible representation of G in a Hilbert space. Let $d_1, d_2 \in \mathbb{N}$ and let $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}$ be a basis in \mathfrak{g} . Let*

$$C := X_1^2 + \dots + X_{d_1}^2 - Y_1^2 - \dots - Y_{d_2}^2 \in U(\mathfrak{g}).$$

Suppose C belongs to the center of $U(\mathfrak{g})$. Let $\lambda \geq 1$. Then

$$S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_{d_1}), d\pi(Y_1), \dots, d\pi(Y_{d_2})) = S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_{d_1}))$$

as locally convex spaces with equivalent spectra.

Proof. Since π is irreducible, by [Tay], Chapter 0 Propositions 4.3 and 4.5 there exists $\tau \in \mathbb{C}$ such that $\partial\pi(C) = \tau I$. \square

Corollary 3.33 *Let G be a semisimple Lie group with Lie algebra \mathfrak{g} . Let π be a representation of G in a Hilbert space H . Let $C \in U(\mathfrak{g})$ be the Casimir element. Suppose there exists $\tau \in \mathbb{C}$ such that $\partial\pi(C) = \tau I$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition of \mathfrak{g} and let K be a subgroup of G with Lie algebra \mathfrak{k} . Then*

$$H^\infty(\pi) = H^\infty(\pi|_K)$$

as sets. Moreover, for all $\lambda \geq 1$ we have

$$H_\lambda(\pi) = H_\lambda(\pi|_K)$$

as locally convex spaces. In particular, $H^\omega(\pi) = H^\omega(\pi|_K)$.

Proof. Let B denote the Killing form of \mathfrak{g} . Let X_1, \dots, X_{d_1} be a basis in \mathfrak{k} and Y_1, \dots, Y_{d_2} be a basis in \mathfrak{p} such that $B(X_i, X_j) = -\delta_{i,j}$ and $B(Y_i, Y_j) = \delta_{i,j}$ for all i, j . Then $C = \sum_{k=1}^{d_1} Y_k^2 - \sum_{k=1}^{d_2} X_k^2$. So by Theorems 3.31 and 1.23 we obtain that

$$H^\infty(\pi) = D^\infty(d\pi(X_1), \dots, d\pi(X_{d_1})) = H^\infty(\pi|_K)$$

as sets. Moreover, by Theorems 3.30 and 1.27 we obtain that

$$\begin{aligned} H_\lambda(\pi) &= S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(Y_{d_2})) \\ &= S_{\lambda, \dots, \lambda}(d\pi(X_1), \dots, d\pi(X_{d_1})) \\ &= H_\lambda(\pi|_K) \end{aligned}$$

as locally convex spaces. \square

Example 3.34

Let $n \in \mathbb{N}$, $n \geq 2$. Let π be a representation of $SL(2, \mathbb{R})$ in a Hilbert space H . Let $K := SO(n, \mathbb{R}) \subset SL(n, \mathbb{R})$. Let $C \in U(\mathfrak{sl}(n, \mathbb{R}))$ be the Casimir element. Suppose there exists $\tau \in \mathbb{C}$ such that $\partial\pi(C) = \tau I$. (For example, suppose π is irreducible.) Then $H^\infty(\pi) = H^\infty(\pi|_K)$, $H^\omega(\pi) = H^\omega(\pi|_K)$ and $H_\lambda(\pi) = H_\lambda(\pi|_K)$ for all $\lambda \geq 1$.

Let A, X, Y be the basis in $\mathfrak{sl}(2, \mathbb{R})$ as in the beginning of this section. Let $C := \frac{1}{8}(4X^2 + 4Y^2 - A^2)$.

Corollary 3.35 *Let π be a representation of $[SL(2, \mathbb{R})]^\sim$ in a Hilbert space and suppose there exists $\tau \in \mathbb{R}$ such that $\partial\pi(C) = \tau I$. Let $\lambda_1, \lambda_2, \lambda_3 \geq 1$. Then*

- I. $S_{\lambda_1, \lambda_2, \lambda_3}(d\pi(A), d\pi(X), d\pi(Y)) = S_{\lambda_1}(d\pi(A)) \cap S_{\lambda_2}(d\pi(X)) \cap S_{\lambda_3}(d\pi(Y)) = S_{\lambda_1}(d\pi(A))$ if $\lambda_1 \leq \lambda_2$ and $\lambda_1 \leq \lambda_3$,
- II. $S_{\lambda_1, \lambda_2, \lambda_3}(d\pi(A), d\pi(X), d\pi(Y)) = S_{\lambda_1}(d\pi(A)) \cap S_{\lambda_2}(d\pi(X)) \cap S_{\lambda_3}(d\pi(Y)) = S_{\lambda_1}(d\pi(A)) \cap S_{\lambda_2}(d\pi(X))$ if $\lambda_1 \leq \lambda_3$,

as locally convex spaces with equivalent spectra.

Proof. I. This follows immediately from Theorem 3.30 and Lemma 1.1.

II. By I and Corollary 2.3 we obtain that for all $t > 0$ there exists $t_2 \geq t_1 \geq t$ such that the following inclusions are continuous:

$$\begin{aligned}
 S_{\lambda_1, \lambda_2, \lambda_3; t}(d\pi(A), d\pi(X), d\pi(Y)) &\subset S_{\lambda_1; t}(d\pi(A)) \cap S_{\lambda_2; t}(d\pi(X)) \\
 &\subset S_{\lambda_1, \lambda_1, \lambda_1; t_1}(d\pi(A), d\pi(X), d\pi(Y)) \cap S_{\lambda_2; t}(d\pi(X)) \\
 &\subset S_{\lambda_1, \lambda_1; t_1}(d\pi(A), d\pi(Y)) \cap S_{\lambda_2; t_1}(d\pi(X)) \\
 &\subset S_{\lambda_1, \lambda_2, \lambda_1; t_2}(d\pi(A), d\pi(X), d\pi(Y)) \\
 &\subset S_{\lambda_1, \lambda_2, \lambda_3; t_2}(d\pi(A), d\pi(X), d\pi(Y)).
 \end{aligned}$$

Now the corollary follows. □

Appendix A

Topological vector spaces

Most of the following definitions are taken from the monograph of Wilansky, [Wil]. The scalar field is \mathbb{C} .

Let X be a vector space and let A be a subset of X . Then

$$\text{span}A := \begin{cases} \{\sum_{n=1}^N \lambda_n a_n : N \in \mathbb{N}, a_1, \dots, a_N \in A, \lambda_1, \dots, \lambda_N \in \mathbb{C}\} & \text{if } A \neq \emptyset, \\ \{0\} & \text{if } A = \emptyset, \end{cases}$$

denotes the *span* of A . A is *convex* if $\lambda A + (1 - \lambda)A \subset A$ for all $\lambda \in [0, 1]$ and A is *balanced* if $\lambda A \subset A$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. A is called *absolutely convex* if $A \neq \emptyset$, A is convex and balanced. A is called *absorbing* if for every $x \in X$ there exists $\varepsilon > 0$ such that $\lambda x \in A$ for all $\lambda \in \mathbb{C}$ with $|\lambda| < \varepsilon$. Let B be a subset of X . Then A *absorbs* B if there exists $M > 0$ such that $B \subset \lambda A$ for all $\lambda \in \mathbb{C}$ with $|\lambda| > M$.

Let X be a vector space. A map p from X into \mathbb{R} is a *seminorm* if for all $x, y \in X$ and $\lambda \in \mathbb{C}$:

$$\begin{cases} p(x) \geq 0, \\ p(x + y) \leq p(x) + p(y), \\ p(\lambda x) = |\lambda|p(x). \end{cases}$$

A *topological vector space* (TVS) is a vector space with a topology such that vector addition and scalar multiplication are continuous. Two topological vector spaces X, Y are called *isomorphic as topological vector spaces* if there exists a bijection from X onto Y which is linear and a topological homeomorphism. Let P be a set of seminorms on a vector space X . The *P -topology* for X is the smallest topology τ for X such that (X, τ) is a TVS and such that all elements of P are continuous seminorms. P *separates* the points of X if for every $x \in X$, $x \neq 0$ there exists $p \in P$ such that $p(x) \neq 0$. A LCSTVS is a *locally convex separated topological vector space*. Let X be a LCSTVS. A *local base of neighborhoods of 0* in X is a set B of neighborhoods of 0 such that for each neighborhood U of 0 there exists $V \in B$ with $V \subset U$.

Let (X, τ) be a LCSTVS and let A be subset of X . A is *bounded* if every neighborhood of 0 absorbs A . A is a *bornivore* if A absorbs every bounded subset of A . A is a *barrel* if A is an absolutely convex absorbing closed set. X is called *bornological* resp. *barrelled* if every absolutely convex bornivore resp. every barrel is a neighborhood of 0. Let τ^b be the

finest locally convex topology for X such that (X, τ^b) has the same bounded subsets as (X, τ) . We call (X, τ^b) the *bornological space associated with* (X, τ) .

Let X be a TVS. A *Cauchy net* in X is a net $(x_\alpha)_{\alpha \in I}$ such that for each neighborhood U of 0 there exists $\alpha_0 \in I$ such that $\alpha \geq \alpha_0$ and $\beta \geq \alpha_0$ implies $x_\alpha - x_\beta \in U$. A sequence $(x_k)_{k \in \mathbb{N}}$ is a *Cauchy sequence* if $(x_k)_{k \in \mathbb{N}}$ is a Cauchy net in X . X is called *complete*, *boundedly complete* and *sequentially complete* if every Cauchy net, every bounded Cauchy net respectively every Cauchy sequence in X is convergent. A *Fréchet space* is a metrizable complete LCSTVS.

Let $(X_\alpha)_{\alpha \in A}$ be a family of LCSTVS's, let X be a vector space and for each $\alpha \in A$ let $u_\alpha : X_\alpha \rightarrow X$ be a linear map. Suppose $X = \text{span}(\{u_\alpha(x) : \alpha \in A, x \in X_\alpha\})$. The *inductive limit topology* for X is the finest topology τ for X such that (X, τ) is a locally convex TVS and each u_α , $\alpha \in A$ is continuous. An inductive limit $X = \bigcup_{\alpha \in A} X_\alpha$ is called *regular* if A is a directed set, X_α is a LCSTVS which is a vector subspace of X for all $\alpha \in A$, for all $\alpha, \beta \in A$ with $\alpha \leq \beta$ the space X_α is continuously embedded in X_β , the topology for X is the inductive limit topology generated by the spaces X_α , $\alpha \in A$ and, moreover, for every bounded subset B of X there exists $\alpha \in A$ such that B is a bounded subset of X_α . X is called an *LB-space* if $A = \mathbb{N}$, for all $k \in \mathbb{N}$, X_k is a Banach space and $X_k \subset X$ as a vector space, X_k is continuously embedded in X_l if $k \leq l$ and the topology for X is the inductive limit topology.

Let A, B be totally ordered sets. For all $\alpha \in A$ and $\beta \in B$ let X_α and Y_β be LCSTVS's. Suppose $X_{\alpha_1} \subset X_{\alpha_2}$ and the embedding map is continuous for all $\alpha_1, \alpha_2 \in A$ with $\alpha_1 \leq \alpha_2$ and, similarly, suppose $Y_{\beta_1} \subset Y_{\beta_2}$ and the embedding map is continuous for all $\beta_1, \beta_2 \in B$ with $\beta_1 \leq \beta_2$. Let $X := \bigcup_{\alpha \in A} X_\alpha$ and $Y := \bigcup_{\beta \in B} Y_\beta$. The topologies for X and Y are the (natural) inductive limit topologies. We call $X = Y$ as *locally convex spaces with equivalent spectra* if for all $\alpha \in A$ there exists $\beta \in B$ such that $X_\alpha \subset Y_\beta$ and the embedding map is continuous, and secondly, for all $\beta \in B$ there exists $\alpha \in A$ such that $Y_\beta \subset X_\alpha$ and the embedding map is continuous. It follows then that $X = Y$ as locally convex topological vector spaces. (See [Wil], Theorem 13-1-8.)

Let $(X_\alpha)_{\alpha \in A}$ be a family of LCSTVS's, let X be a vector space and for each $\alpha \in A$ let $u_\alpha : X_\alpha \rightarrow X$ be a linear map. The *projective limit topology* for X is the weakest topology τ for X such that each u_α , $\alpha \in A$ is continuous. Then (X, τ) is a locally convex TVS.

Appendix B

Spaces of type $S_{\pi, C}$

In this appendix we give a summary of the report [tE] where the smoothed spaces $S_{\pi, C}$ are introduced. These spaces establish a generalization of the analyticity spaces $S_{H, A}$ of De Graaf, [Gra]. (See Example 1.2.) Let G be a locally compact Abelian topological group with a Haar measure μ and let π be a (continuous unitary) representation of G in a Hilbert space H . For all $f \in L^1(G)$ define the continuous operator $\pi(f)$ on H by

$$(\pi(f)u, v) := \int_G f(x)(\pi_x u, v) d\mu(x) \quad (u, v \in H).$$

Then $\pi(f * g) = \pi(f)\pi(g)$ for all $f, g \in L^1(G)$. Let C be a fixed subset of $L^1(G)$. Suppose the pair (C, π) possesses the following properties:

P1. For all $f, g \in C$ there exists $h \in C$ such that 1 and 2 hold:

- 1) $f = h$ or there exists $f_1 \in L^1(G)$ such that $f = h * f_1$,
- 2) $g = h$ or there exists $g_1 \in L^1(G)$ such that $g = h * g_1$.

P2. There exists a net $(f_\lambda)_{\lambda \in J}$ in C such that for all $u \in H$ we have $\lim_\lambda \pi(f_\lambda)u = u$.

Throughout this appendix we suppose that the pair (C, π) has Properties *P1* and *P2*.

Remark. These conditions are weak, but sufficiently strong to enable us to construct the space $S_{\pi, C}$. In practice one meets sets C which have much more stronger properties such as:

P1'. For all $f, g \in C$ there exist $h \in C$ and $f_1, g_1 \in L^1(G)$ such that $f = h * f_1$ and $g = h * g_1$.

P2'. There exists an $L^1(G)$ -bounded net $(f_\lambda)_{\lambda \in J}$ in C such that for all $u \in H$ we have $\lim_\lambda \pi(f_\lambda)u = u$.

P2''. There exists an $L^1(G)$ -bounded net $(f_\lambda)_{\lambda \in J}$ in C such that for all $g \in L^1(G)$ we have $\lim_\lambda f_\lambda * g = g$ in $L^1(G)$.

Of course: *P1'* implies *P1*, *P2''* implies *P2'* and *P2'* implies *P2*. For $f \in L^1(G)$ define

$$\begin{aligned} N_f &:= \{u \in H : \pi(f)u = 0\}, && \text{the kernel of } \pi(f), \\ R_f &:= \pi(f)(H), && \text{the range of } \pi(f), \\ \Omega_f &:= \pi(f)|_{N_f^\perp} : N_f^\perp \rightarrow R_f. \end{aligned}$$

Then Ω_f is a bijection. Since N_f^\perp is a Hilbert subspace of H , there exists a unique norm $\| \cdot \|_f$ on R_f such that R_f becomes a Hilbert space and Ω_f is a unitary map. So for all $f, g \in L^1(G)$ with $f = g * h$ we obtain that R_f is continuously embedded in R_g . Define

$$S_{\pi,C} := \bigcup_{f \in C} R_f.$$

By Property P1, $S_{\pi,C}$ is a linear vector space. The topology σ_{ind} for $S_{\pi,C}$ is the inductive limit topology generated by the Hilbert spaces R_f , with $f \in C$.

By Property P2 it follows that $S_{\pi,C}$ is dense in H . Moreover, $S_{\pi,C}$ is continuously embedded in H and the topology σ_{ind} is Hausdorff.

In order to describe the topology σ_{ind} for $S_{\pi,C}$ we need some structure theory for unitary representations of locally compact Abelian groups. For this we introduce a so called Stone-representative $(A, m, I, A_i, \tau_i, W)$ for π . To this end, let A be a locally compact Hausdorff space, m a measure on A , defined on the Borel σ -algebra of A , let I be an index set, for all $i \in I$ let A_i be an open subset of A with induced topology and let $\tau_i : \hat{G} \rightarrow A_i$ be a topological homeomorphism. Let W be a unitary operator from H onto $L^2(m)$. The tuple $(A, m, I, A_i, \tau_i, W)$ is called a *Stone-representative* for π if and only if:

$$\left[\begin{array}{l} A_i \cap A_j = \emptyset \text{ if } i \neq j \quad (i, j \in I), \\ A = \bigcup_{i \in I} A_i, \\ \text{The map } Y \mapsto m(\tau_i(Y)), \text{ } Y \text{ a Borel measurable subset of } \hat{G}, \text{ is a finite} \\ \text{regular measure on } \hat{G} \quad (i \in I), \\ 0 < m(A_i) < \infty \quad (i \in I), \\ m(Z) = \sum_{i \in I} m(Z \cap A_i) \quad (Z \subset A \text{ Borel measurable}), \\ \text{For all } f \in L^1(G) \text{ let } \hat{f} \text{ be the continuous function on } A \text{ such that } \hat{f}(\tau_i(\gamma)) = \hat{f}(\gamma) \\ \text{for all } i \in I \text{ and } \gamma \in \hat{G}. \text{ Then } W\pi(f)W^{-1}\xi = \hat{f} \cdot \xi \text{ for all } f \in L^1(G) \\ \text{and } \xi \in L^2(m). \end{array} \right.$$

By [HR2], Remark 33.6 and [HR1], Theorem C.37, there exists a Stone-representative for π .

Let $Bor(\hat{G}, \mathbb{C})$ be the set of all complex valued Borel measurable functions on \hat{G} and let $Bor_b(\hat{G}, \mathbb{C})$ be the subset of all bounded elements of $Bor(\hat{G}, \mathbb{C})$. With the aid of a Stone-representative for π we can extend the set of operators $\pi(f)$ with $f \in L^1(G)$ to a set of operators $\pi[F]$ with $F \in Bor(\hat{G}, \mathbb{C})$. Let $(A, m, I, A_i, \tau_i, W)$ be a Stone-representative for π and let F be a Borel measurable function on \hat{G} . Define the Borel measurable function \mathbf{F} on A by

$$\mathbf{F}(\tau_i(\gamma)) := F(\gamma) \quad i \in I, \gamma \in \hat{G}.$$

For every $F \in \text{Bor}(\widehat{G}, \mathbb{C})$ define the normal operator $\pi[F, A]$ on H by $\pi[F, A] := W^{-1}M_F W$, with M_F the multiplication operator by F on $L^2(m)$. We prove that the operator $\pi[F, A]$ does not depend on A . For a detailed proof we refer to the report [tE], Lemma 2.5, Theorem 2.6 and Lemma 2.9, here we present a sketch of the proof. Let $(A, m, I, A_i, \tau_i, W)$ be a Stone-representative for π . For every uniformly bounded sequence $(F_n)_{n \in \mathbb{N}}$ in $\text{Bor}_b(\widehat{G}, \mathbb{C})$ such that $F(\gamma) := \lim_{n \rightarrow \infty} F_n(\gamma)$ exists for all $\gamma \in \widehat{G}$ we obtain by Lebesgue's theorem on dominated convergence that $\pi[F, A] = s - \lim_{n \rightarrow \infty} \pi[F_n, A]$. Since $\{\hat{f} : f \in L^1(G)\}$ is dense in $C_0(\widehat{G})$ and $\pi[\hat{f}, A] = \pi(f)$ does not depend on the Stone-representative for all $f \in L^1(G)$, it follows that $\pi[F, A]$ does not depend on the Stone-representative for all $F \in C_0(\widehat{G})$. Now let V be an open subset of \widehat{G} and let

$$X := \text{clo}\{\pi[F, A](H) : F \in C_0(\widehat{G}), 0 \leq F \leq 1_V\}.$$

Mainly by the regularity and finiteness of the measure $Y \mapsto m(\tau_i(Y))$ on \widehat{G} , it follows that $\pi[1_V, A]$ is the projection of H onto X . So $\pi[1_V, A]$ does not depend on the Stone-representative. Now it is easy to show that the operator $\pi[F, A]$ does not depend on the Stone-representative for all $F \in \text{Bor}_b(\widehat{G}, \mathbb{C})$ and even for all $F \in \text{Bor}(\widehat{G}, \mathbb{C})$.

Let C be the fixed subset of $L^1(G)$ which has Properties P1 and P2. Corresponding to the set C we define the two subsets of $\text{Bor}(\widehat{G}, \mathbb{C})$ and seminorms on $S_{\pi, C}$:

$$\begin{aligned} C^\# &:= \{F \in \text{Bor}(\widehat{G}, \mathbb{C}) : \text{for all } f \in C \text{ the function } \hat{f} \cdot F \text{ is bounded}\}, \\ C_p^\# &:= \{F \in C^\# : \exists \varepsilon > 0 \forall \gamma \in \widehat{G} [|F(\gamma)| > \varepsilon]\}. \end{aligned}$$

For all $F \in C^\#$ define $s_F : S_{\pi, C} \rightarrow \mathbb{R}$

$$s_F(\varphi) := \|\pi[F]\varphi\| \quad (\varphi \in S_{\pi, C}).$$

Then s_F is a σ_{ind} -continuous seminorm on $S_{\pi, C}$. Because $1_{\widehat{G}} \in C_p^\#$, the set of seminorms $\{s_F : F \in C_p^\#\}$ separates the points of $S_{\pi, C}$. So $\{s_F : F \in C_p^\#\}$ defines a Hausdorff topology for $S_{\pi, C}$, which is denoted by σ_{proj} . Then $\sigma_{\text{proj}} \subset \sigma_{\text{ind}}$, but not necessarily $\sigma_{\text{ind}} = \sigma_{\text{proj}}$. (For a counter example, see [tE], Corollary 4.7.) In [tE] a necessary and sufficient condition has been presented for the equality $\sigma_{\text{ind}} = \sigma_{\text{proj}}$. Also a sufficient condition is presented:

P3. *There exist a sequence of Borel measurable disjoint sets Q_1, Q_2, \dots in \widehat{G} and a sequence of positive real numbers b_1, b_2, \dots such that $\widehat{G} = \bigcup_{n=1}^\infty Q_n$ and $\sum_{n=1}^\infty b_n^{-1} < \infty$ and for all $f \in C$ there exists $g \in C$ and $\delta > 0$ such that for all $n \in \mathbb{N}$:*

$$b_n \sup\{|\hat{f}(\gamma)| : \gamma \in Q_n\} \leq \delta \inf\{|\hat{g}(\gamma)| : \gamma \in Q_n\}.$$

If the pair (C, π) has Property P3, then $\sigma_{\text{ind}} = \sigma_{\text{proj}}$ as locally convex spaces. (See [tE], Corollary 2.31.)

Finally we introduce two more properties.

P4. $\forall_{F \in \text{Bor}(\widehat{G}, \mathbb{C})} [(\forall_{K \in C^\#} [F \cdot K \text{ is bounded}]) \Rightarrow \exists_{f \in C} \exists_{\varepsilon > 0} [\pi[1_{\{\gamma \in \widehat{G} : |F(\gamma)| > \varepsilon\}} \hat{f}(\gamma)]] = 0]]$.

$$P4'. \quad \forall_{F \in \text{Bor}(\widehat{G}, \mathfrak{C})} [(\forall_{K \in C^*} [F \cdot K \text{ is bounded}]) \Rightarrow \exists_{f \in C} \exists_{c > 0} [|F| \leq c|\hat{f}|]].$$

Clearly Property $P4'$ implies Property $P4$.

Now suppose the pair (C, π) has Property $P3$. Then the following conditions are equivalent.

- I. The pair (C, π) has Property $P4$.
- II. $S_{\pi, C} = \bigcap_{F \in C^*} D(\pi[F])$.
- III. $S_{\pi, C}$ is complete.
- IV. $S_{\pi, C}$ is sequentially complete.
- V. Every bounded sequence in $S_{\pi, C}$ has a weakly convergent subsequence.
- VI. For every bounded subset B of $S_{\pi, C}$ there exist $f \in C$ and a bounded subset B_0 of H such that $B = \pi(f)(B_0)$.
- VII. $S_{\pi, C}$ is reflexive.

For a proof, see [tE], Theorem 3.12.

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$(A, m, I, A_i, \tau_i, W)$	110	Q	8
A_α	12	Q_k	78
$c_{\beta, \gamma}^\alpha$	54	\tilde{Q}_k	91
$D(T)$	8	Q_0	81
$D^\alpha(Y_\beta)$	54	$S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$	13
$d\pi(X)$	26	$S_{\lambda_1, \dots, \lambda_d}^{\text{ord}}(A_1, \dots, A_d)$	13
$D^\infty(A_1, \dots, A_d)$	12	$S_{\lambda_1, \dots, \lambda_d; \text{dit}}(A_1, \dots, A_d)$	13
$D_{\text{ord}}^\infty(A_1, \dots, A_d)$	12	$S_{\lambda_1, \dots, \lambda_d; \text{dit}}^{\text{ord}}(A_1, \dots, A_d)$	13
E	94	$S_{H, A}$	14
\hat{f}	7	$S(\mathbb{R})$	17
\tilde{f}	92	$S(\mathbb{R}^n)$	78
\mathbb{F}	7	S_α^β	17
\mathbb{F}_k	78	$S_{\alpha_1, \dots, \alpha_n}^{\beta_1, \dots, \beta_n}$	78
$f * g$	7	$S_{\pi, C}$	110
g	62	T^*	8
g_k	62	\tilde{u}	26
G	26	(\tilde{u}, v)	31
G_c	62	U^\pm	94
$G_\lambda(H)$	29	$U(\mathfrak{g})$	26
$G_\lambda(H, K, Y_1, \dots, Y_d)$	30	V	83
H	12	$[X, Y]$	8
$H_\lambda(\pi)$	30	\bar{X}	31
$H_\lambda^\omega(\pi)$	32	α^r	12
$H^\omega(\pi)$	26	$\langle \alpha_1, \dots, \alpha_p \rangle$	12
$H^\infty(\pi)$	26	$\langle \alpha, \beta \rangle$	11
I	7	$\ \alpha\ $	11
L	83	$\ \alpha\ _v$	11
$L_x f$	7	$\partial\pi$	26
$M(V)$	11	$\partial\pi(X)$	26
P	8	Δ	26
P_k	78	π	26
\tilde{P}_k	91		

$\pi(f)$	7
σ_{ind}	13
$\hat{\sigma}(\alpha)$	50
$\tilde{\sigma}(\alpha)$	50
σ_0	50
τ_{proj}	74
\mathfrak{g}	26
\mathfrak{g}_c	62
1_V	7
$()$	11
$[]$	7
$\ \ A_1, \dots, A_d; \alpha$	12
$\ \ A_1, \dots, A_d; n_1, \dots, n_d$	12
$\ \ A_1, \dots, A_d; \lambda_1, \dots, \lambda_d; t$	13
$\ \ A_1, \dots, A_d; \lambda_1, \dots, \lambda_d; t, \text{ord}$	13
$\ \ f$	110
$\ \ n_1, \dots, n_d$	12
$\ \ t$	14
$\ \ \alpha$	12
$\ \ \lambda_1, \dots, \lambda_d; t$	13
$\ \ \lambda_1, \dots, \lambda_d; t, \text{ord}$	13
$\ \ \ $	87
$\ \ \infty$	7

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Samenvatting

In dit proefschrift wordt een klasse van lokaal konvexe topologische vektorruimten ingevoerd en bestudeerd, namelijk de klasse der zogenaamde Gevreyruimten. Elke Gevreyruimte wordt vastgelegd door een eindig aantal operatoren A_1, \dots, A_d en niet-negatieve getallen $\lambda_1, \dots, \lambda_d$ en wordt aangegeven door $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$. Voorbeelden van Gevreyruimten zijn de klassieke Gelfand-Shilov ruimten S_α^β , de ruimten van analytische vectoren $D^\omega(A_1, \dots, A_d)$ van Nelson en de ruimte van symmetrische Gevrey vektoren $G_\lambda(A_1, \dots, A_d)$ van Goodman-Wallach. Er geldt namelijk dat $S_\alpha^\beta = S_{\alpha, \beta}(Q, P)$, $D^\omega(A_1, \dots, A_d) = S_{1, \dots, 1}(A_1, \dots, A_d)$ en $G_\lambda(A_1, \dots, A_d) = S_{\lambda, \dots, \lambda}(A_1, \dots, A_d)$. De definitie van $G_\lambda(A_1, \dots, A_d)$ is geïnspireerd op de klassieke definitie van Gevreyfunctie door Gevrey, terwijl de definitie van $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ is geïnspireerd op Roumieu's definitie van (niet-symmetrische) Gevreyfunctie. Een grote klasse van niet triviale voorbeelden ontstaat door voor de operatoren A_1, \dots, A_d infinitesimaal generatoren van unitaire representaties van Lie groepen te nemen. De Gevreyruimten worden voorzien van een natuurlijke inductieve limiet topologie. Voor deze topologie wordt een gedetailleerde karakterisatie van eigenschappen als volledigheid, kompaktheid, begrensdsheid, enz. afgeleid.

Een belangrijk gedeelte van dit proefschrift is gewijd aan voorwaarden op de operatoren A_1, \dots, A_d en de getallen $\lambda_1, \dots, \lambda_d$ waaronder $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ gelijk is aan de doorsnede van de Gevreyruimten $S_{\mu_1, \dots, \mu_{d_1}}(B_1, \dots, B_{d_1})$ en $S_{\nu_1, \dots, \nu_{d_2}}(C_1, \dots, C_{d_2})$ met $d_1 + d_2 \leq d$? In het bijzonder worden er voorwaarden gegeven waaronder $S_{\lambda_1, \dots, \lambda_d}(A_1, \dots, A_d)$ gelijk aan de doorsnede van Gevreyruimten gerelateerd aan één operator? Dit is bijvoorbeeld het geval als de operatoren A_1, \dots, A_d een Lie algebra opspannen.

Concrete uitwerkingen worden gegeven voor representaties van de Lie algebras van de Heisenberggroep, de $ax + b$ groep en de unimodulaire groep $SL(2, \mathbb{R})$.

Curriculum Vitae

De schrijver van dit proefschrift is geboren op 4 mei 1962 te Utrecht. Op 29 mei 1980 behaalde hij het eindexamen Atheneum B aan het Dominicus College te Nijmegen. Daarna studeerde hij wis- en natuurkunde aan de Katholieke Universiteit te Nijmegen. Op 30 januari 1986 deed hij doktoraal examen in wiskunde en op 13 februari 1986 in natuurkunde. (Beide met lof.) Vanaf 1 maart 1986 is hij als assistent onderzoeker in dienst van de Technische Universiteit Eindhoven.

STELLINGEN
 behorende bij het proefschrift
GEVREY SPACES RELATED TO LIE ALGEBRAS OF OPERATORS
 door A.F.M. ter Elst

1.

Zij $\mathcal{L}(H)$ de verzameling der continue operatoren in een (niet noodzakelijk separabele) Hilbertruimte H . Voor een niet lege deelverzameling \mathcal{M} van $\mathcal{L}(H)$ en $T \in \mathcal{L}(H)$ definieer de afstand $d(T, \mathcal{M}) := \inf\{\|T - M\| : M \in \mathcal{M}\}$. Zij \mathcal{N} , $\mathcal{U}n$, \mathcal{C} resp. $\mathcal{I}nv$ de deelverzameling van $\mathcal{L}(H)$ bestaande uit de normale, unitaire, kompakte en inverteerbare operatoren. Een operator $T \in \mathcal{L}(H)$ heet *antinormaal* als $d(T, \mathcal{N}) = \|T\|$. Voor elk oneindig kardinaalgetal ω zij

$$\mathcal{C}_\omega := \text{afsl}\{S \in \mathcal{L}(H) : \dim \overline{S(H)} < \omega\}.$$

Zij $T \in \mathcal{L}(H)$ en stel $\dim \text{Ker } T < \dim \text{Ker } T^*$. Zij $\omega := \max(\aleph_0, \dim \text{Ker } T^*)$ en zij

$$n(T) := \inf\{x > 0 : \dim 1_{[0,x]}(|T|) \geq \max(\aleph_0, \dim \text{Ker } T^*)\}.$$

Dan zijn de volgende uitspraken equivalent:

- I. T is antinormaal.
- II. T is essentieel antinormaal, d.w.z. $d(T, \mathcal{N} + \mathcal{C}) = \|T\|$.
- III. $d(T, \mathcal{N} + \mathcal{C}_\omega) = \|T\|$.
- IV. $\|T\| = n(T)$.
- V. Er zijn $\alpha > 0$, een niet surjektieve isometrie W en een positieve Hermitische kontraktie $K \in \mathcal{C}_{\omega'}$, waarbij $\omega' = \max(\aleph_0, \dim \text{Ker } W^*)$, zo dat $T = \alpha W(I - K)$.
- VI. Voor alle $U \in \mathcal{U}n$ geldt $\sigma(UT) = \{z \in \mathbb{C} : |z| \leq \|T\|\}$.
- VII. $d(T, \mathcal{U}n) = 1 + \|T\|$.
- VIII. $d(T, \mathcal{U}n + \mathcal{C}) = 1 + \|T\|$.
- IX. $d(T, \mathcal{U}n + \mathcal{C}_\omega) = 1 + \|T\|$.
- X. $d(T, \mathcal{I}nv) = \|T\|$.
- XI. $d(T, \mathcal{I}nv + \mathcal{C}) = \|T\|$.
- XII. $d(T, \mathcal{I}nv + \mathcal{C}_\omega) = \|T\|$.

Literatuur: [E1], [I].

2.

Zij H een (niet noodzakelijk separabele) Hilbertruimte en zij $T \in \mathcal{L}(H)$. Dan geldt voor de afstand $d(T, \mathcal{U}n)$ van T tot de verzameling $\mathcal{U}n$ der unitaire operatoren in H :

$$d(T, \mathcal{U}_n) = \begin{cases} \max(1 - m(T), \|T\| - 1) & \text{als } \dim \text{Ker } T = \dim \text{Ker } T^*, \\ \max(1 + n(T), \|T\| - 1) & \text{als } \dim \text{Ker } T < \dim \text{Ker } T^*. \end{cases}$$

Hierbij is

$$m(T) := \inf \sigma(|T|)$$

en $n(T)$ als in Stelling 1.

Literatuur: [E2], [R].

3.

Zij H een (niet noodzakelijk separabele) Hilbertruimte en zij $\mathcal{F} \subset \mathcal{L}(H)$ een (niet noodzakelijk aftelbare) kollektie Hermitische operatoren met $AB = BA$ voor alle $A, B \in \mathcal{F}$. Dan bestaan er een lokaal eindige maatruimte (X, \mathcal{B}, m) en een unitaire afbeelding U van H op $L^2(m)$ zo dat voor alle $A \in \mathcal{F}$ de operator UAU^{-1} een vermenigvuldigungsoperator op $L^2(m)$ is.

4.

In [EGK] is het tweede deel van voorwaarde A.I overbodig. De symmetrie konditie A.IV kan vervangen worden door de zwakkere konditie A.IV'

$$\forall \zeta \in \Phi \exists \varphi \in \Phi \exists c > 0 \left[1_{\{\lambda \in \mathbb{R}^n : \zeta(\lambda) > c\varphi(\lambda)\}}(A) = 0 \right].$$

Onder aanname van voorwaarden A.I, A.II en A.III zijn de volgende beweringen equivalent:

- I. Het paar (Φ, A) heeft eigenschap A.IV'.
- II. $(T_{\Phi(A)}, \tau_{\text{proj}}) = (S_{\Phi(A)}, \sigma_{\text{ind}})$ als topologische vektorruimten.
- III. $(T_{\Phi(A)}, \tau_{\text{proj}})$ is bornologisch.
- IV. $(T_{\Phi(A)}, \tau_{\text{proj}})$ is getond.
- V. $(T_{\Phi(A)}, \tau_{\text{proj}})$ is reflexief.
- VI. $(S_{\Phi(A)}, \sigma_{\text{ind}})$ is volledig.
- VII. $(S_{\Phi(A)}, \sigma_{\text{ind}})$ is rijvolledig.
- VIII. $(S_{\Phi(A)}, \sigma_{\text{ind}})$ is zwak begrensd volledig.
- IX. Elke begrensde rij in $(S_{\Phi(A)}, \sigma_{\text{ind}})$ heeft een zwak konvergente deelrij.
- X. Voor elke begrensde verzameling B in $S_{\Phi(A)}$ zijn er $\varphi \in \Phi$ en een begrensde verzameling B_0 in X zo dat $B = \varphi(A) \bullet B_0$.
- XI. $(S_{\Phi(A)}, \sigma_{\text{ind}})$ is reflexief.

Literatuur: [E3].

5.

Zij π een standaard irreducibele representatie van de Heisenberg groep $A(\mathbb{R})$ in de Hilbertruimte $H := L^2(\mathbb{R})$. Zij $\alpha, \beta > 0$ en stel $\alpha + \beta \geq 1$. Dan bestaat er een deelverzameling $C_{\alpha, \beta}$ van $L^1(A(\mathbb{R}))$ zo dat

$$S_{\alpha}^{\beta} = \bigcup_{f \in C_{\alpha, \beta}} \pi(f)(H)$$

als verzameling in de volgende gevallen:

- I. $\beta \geq 1$,
- II. $\beta^{-1} \in 2\mathbb{N}$,
- III. $\alpha = \beta$,
- IV. $\alpha + \beta = 1$ en $\frac{\alpha}{\beta} \in \mathbb{N}$.

en in de overeenkomstige gevallen met α en β verwisseld.

Ook bestaan er deelverzamelingen C_1 en C_2 van $L^1(A(\mathbb{R}))$ zo dat

$$S(\mathbb{R}) = \bigcup_{f \in C_1} \pi(f)(H)$$

en

$$D(\mathbb{R}) = \bigcup_{f \in C_2} \pi(f)(H).$$

6.

Zij D_0 de differentiatieoperator en Q_0 de operator van vermenigvuldiging met de functie $x \mapsto x$ op de ruimte $S(\mathbb{R})$ van Schwartz. Zij $A_0 := i(I + 2Q_0D_0)$. Dan is de operator A_0 essentieel zelfgeadjungeerd in $L^2(\mathbb{R})$. De analyticieteitsruimte van de afsluiting $\overline{A_0}$ bestaat uit alle functies f op \mathbb{R} waarvoor een $\varphi_0 \in (0, \frac{\pi}{2})$ bestaat zo dat f uit te breiden is tot een analytische functie F op $\{re^{i\varphi} : r \in \mathbb{R} \setminus \{0\}, \varphi \in (-\varphi_0, \varphi_0)\}$ en

$$\sup_{\varphi \in (-\varphi_0, \varphi_0)} \int_{\mathbb{R}} |F(re^{i\varphi})|^2 dr < \infty.$$

7.

Zij E, F twee lokaal convexe Hausdorffruimten waarbij de topologie op E en F wordt voortgebracht door verzamelingen halfnormen \mathcal{P} en \mathcal{Q} resp. Veronderstel bovendien dat de halfnormen in \mathcal{P} en \mathcal{Q} afkomstig zijn van halfinproducten. Dan is op de tensorproductruimte $E \otimes F$ op een natuurlijke wijze een lokaal convexe Hausdorfftopologie τ te definiëren door middel van halfnormen die afkomstig zijn van halfinproducten. De topologie hangt niet af van de keuze van \mathcal{P} en \mathcal{Q} . Als E en F Hilbertruimten zijn, dan is de completering van $(E \otimes F, \tau)$ gelijk aan het Hilbertruimte tensorproduct van E en F . Als A en B positieve zelfgeadjungeerde operatoren zijn in Hilbertruimten X en Y resp., dan is de completering van $(T_{X,A} \otimes T_{Y,B}, \tau)$ gelijk aan $TT_{X \otimes Y, A \otimes I, I \otimes B}$.

Literatuur: [E4], Appendix C en Chapter 8 en [EG], Theorem III.6.5.

8.

Zij \mathfrak{g} een Lie algebra met universele overdekkings algebra $U(\mathfrak{g})$. Zij $d_1, d_2 \in \mathbb{N}$ en zij $X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2} \in \mathfrak{g}$. Stel $\mathfrak{g} = \text{span}(\{X_1, \dots, X_{d_1}, Y_1, \dots, Y_{d_2}\})$. Zij $V_1 := \{1, \dots, d_1\}$ en $V_2 := \{1, \dots, d_2\}$. We gebruiken dezelfde notaties als in dit proefschrift. Voor alle $\alpha \in M(V_1)$ definieer $X_\alpha \in U(\mathfrak{g})$ op de voor de hand liggende wijze analoog aan blz. 12 van dit proefschrift. Definieer net zo $Y_\beta \in U(\mathfrak{g})$ voor alle $\beta \in M(V_2)$. Dan zijn er konstanten $M, s \geq 0$ en voor alle $\eta, \gamma \in M(V_1)$ en $\beta, \delta \in M(V_2)$ met $\|\gamma\| + \|\delta\| \leq \|\beta\|$ is er $c_{\eta, \delta}^{\gamma, \delta} \in \mathbb{C}$ zo dat voor alle $\alpha \in M(V_1)$ en $\beta \in M(V_2)$ geldt

$$X_\alpha Y_\beta = \sum_{j=0}^{\|\alpha\|} \sum_{\sigma \in P_j^{\|\alpha\|}} \sum_{\substack{\gamma \in M(V_1) \\ \delta \in M(V_2) \\ \|\gamma\| + \|\delta\| \leq \|\beta\|}} c_{\sigma(\alpha), \beta}^{\gamma, \delta} Y_\delta X_{(\gamma, \delta(\alpha))}$$

en bovendien geldt

$$|c_{\eta, \beta}^{\gamma, \delta}| \leq s^{\|\beta\|} (M \|\beta\|)^{\|\eta\| + \|\beta\| - \|\gamma\| - \|\delta\|}$$

voor alle voorkomende $\eta, \beta, \gamma, \delta$.

9.

De schriftelijke examens op zaterdag zullen snel worden afgeschaft indien ook hoogleraren als surveillant worden ingezet.

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