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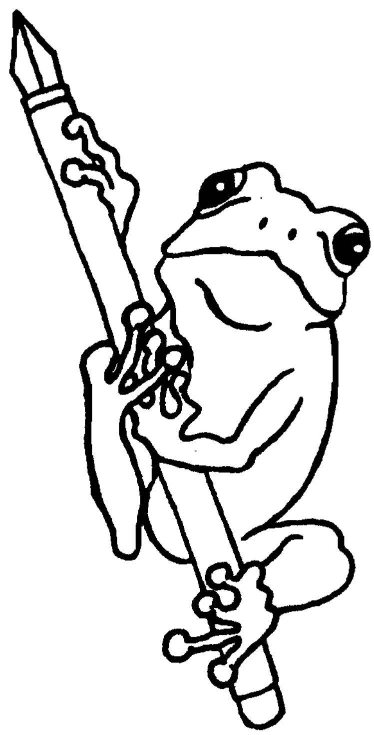

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Abstract

Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and $a_1, \ldots, a_{d'}$ an algebraic basis of $\mathfrak{g}$. Further let A_i denote the generators of left translations, acting on the L_p-spaces $L_p(G\,; dg)$ formed with left Haar measure dg, in the directions a_i. We consider second-order operators

$$H = -\sum_{i,j=1}^{d'} A_i\, c_{ij}\, A_j + \sum_{i=1}^{d'} (c_i\, A_i + A_i\, c_i') + c_0\, I$$

corresponding to a quadratic form with complex coefficients c_{ij}, c_i, c_i', $c_0 \in L_\infty$. The principal coefficients c_{ij} are assumed to be Hölder continuous and the matrix $C = (c_{ij})$ is assumed to satisfy the subellipticity condition

$$\Re C = 2^{-1}\Big(C + C^*\Big) \geq \mu I > 0$$

uniformly over G.

We discuss the hierarchy relating smoothness properties of the coefficients of H with smoothness of the kernel. Moreover, we establish Gaussian type bounds for the kernel and its derivatives.

Similar theorems are proved for operators

$$H' = -\sum_{i,j=1}^{d'} c_{ij}\, A_i\, A_j + \sum_{i=1}^{d'} c_i\, A_i + c_0\, I$$

in nondivergence form for which the principal coefficients are at least once differentiable.

1 Introduction

Subelliptic operators on a Lie group G generate semigroups whose action is determined by an integral kernel. The smoothness properties of the kernel as a function over $G \times G$ are related to the smoothness of the coefficients of the operator as functions over G. For example, in [ElR5] we proved that for subelliptic operators in divergence form with uniformly continuous coefficients the semigroup kernel is Hölder continuous of any order $\nu \in \langle 0, 1 \rangle$ jointly in each variable, i.e., there is an improvement of almost one degree of differentiability. Moreover, we established that the kernel and its Hölder derivatives satisfy Gaussian bounds with respect to the subelliptic geometry. Our aim is to establish that these results are a general phenomenon, the kernel is almost one degree more differentiable than the coefficients, and the kernel and its derivatives satisfy Gaussian bounds. In addition, if the operator is in non-divergence form the improvement in smoothness is almost two degrees, and the Gaussian bounds are still valid. In a recent paper [ElR6] we established these properties for strongly elliptic operators and then, by scaling, obtained some partial results for a particular class of subelliptic operators on a stratified Lie group. But the general subelliptic case, and in particular the results for differentiable coefficients, requires a quite different treatment and more sophisticated arguments.

The analysis of [ElR6] was based on De Giorgi estimates for solutions of the local elliptic equations associated with the strongly elliptic operators. Our current starting point is an idea of Xu and Zuily [XuZ], a particular transformation of vector fields, which allows us to establish the appropriate subelliptic estimates in a sufficiently small neighbourhood of the identity for operators with Hölder continuous coefficients. Then translation invariance gives De Giorgi estimates on the whole group and these can be turned into uniform Lipschitz bounds on the derivatives by the arguments of [ElR6]. Next, if the coefficients are at least once differentiable then the increased regularity of the kernel follows from a repeated use of local and global properties for operators with constant coefficients. In Xu [Xu] similar local estimates were deduced but in the present situation the proofs are simplified by use of global results. In the strongly elliptic situation one could then exploit Davies' exponential perturbation method to establish the Gaussian bounds but this is not possible in the subelliptic case if more than one derivative is involved. Therefore we introduce a different iterative technique. We use a Taylor expansion to interpolate between Gaussian bounds on the kernel and uniform bounds on its derivatives, and Hölder derivatives, to deduce Gaussian bounds on the intermediate derivatives. To be more precise we need to introduce some notation. In general we adopt the notation of [Rob] and [ElR5].

First we consider second-order operators in divergence form,

$$H = - \sum_{i,j=1}^{d'} A_i \, c_{ij} \, A_j + \sum_{i=1}^{d'} (c_i \, A_i + A_i \, c'_i) + c_0 \, I \quad , \tag{1}$$

with **complex** coefficients c_{ij}, c_i, c'_i, $c_0 \in L_\infty$. The A_i denote the generators, $A_i = dL(a_i)$, of left translations L on the L_p-spaces in the directions a_i of the Lie algebra $\mathfrak{g}$ of G where $a_1, \ldots, a_{d'}$ is an algebraic basis of $\mathfrak{g}$. Subellipticity corresponds to the assumption that the real part of the matrix $C = (c_{ij})$ of principal coefficients is strictly positive-definite, i.e.,

$$\Re C = 2^{-1}\big(C + C^*\big) \geq \mu I > 0 \quad , \tag{2}$$

in the sense of $d' \times d'$-matrices, uniformly over G. (If $a_1, \ldots, a_{d'}$ is a vector space basis of $\mathfrak{g}$ then subellipticity corresponds to strong ellipticity.) The least upper bound, μ_C, of the lower bound μ in (2) is called the ellipticity constant and we set $\|C\|_\infty = \sup_{g \in G} \|C(g)\|$ with $\|C(g)\|$ the l_2-norm of the matrix $C(g) = (c_{ij}(g))$.

The operator H, formally given by (1), is precisely defined as the sectorial operator associated with the form

$$\varphi \mapsto h(\varphi) = \sum_{i,j=1}^{d'} (A_i\varphi, c_{ij}A_j\varphi) + \sum_{i=1}^{d'} \Big((\overline{c_i}\varphi, A_i\varphi) - (A_i\varphi, c'_i\varphi) \Big) + (\varphi, c_0\varphi)$$

on $L_2 = L_2(G\,; dg)$, where dg denotes left invariant Haar measure, with domain $D(h) = L'_{2;1} = \bigcap_{i=1}^{d'} D(A_i)$. The form h is closed, H is maximal accretive and it generates a strongly continuous, holomorphic, semigroup S on L_2 (see, for example, [Kat], Chapter VI). If the principal coefficients c_{ij} are right uniformly continuous then S extends to a holomorphic semigroup on all the spaces $L_p(G\,; dg)$, $p \in [1, \infty]$. The extension is strongly continuous if $p \in [1, \infty\rangle$ and weakly* continuous if $p = \infty$ (see [ElR5], or [Aus] for the case $G = \mathbf{R}^d$). Moreover, H and S act on the spaces $L_{\hat{p}} = L_p(G\,; d\hat{g})$ formed with respect to right Haar measure $d\hat{g}$. The key to these interpolatory properties is the existence of a semigroup kernel $(g, h) \mapsto K_t(g\,; h)$ satisfying Gaussian bounds with respect to the appropriate subelliptic parameters.

Let $d'(\,\cdot\,;\,\cdot\,)$ be the **right** invariant distance canonically associated with the algebraic basis $a_1, \ldots, a_{d'}$. This distance has several equivalent characterizations but in particular it is given by

$$d'(g\,; h) = \sup\{\, |\psi(g) - \psi(h)| \,:\, \psi \in C_c^\infty(G)\,,\, \sum_{i=1}^{d'} |(A_i\psi)|^2 \leq 1 \,\} \tag{3}$$

where the ψ are real-valued ([Rob], Lemma IV.2.3, or [ElR4], Lemma 4.2). Next introduce the subelliptic modulus $g \mapsto |g|' = d'(g\,; e)$, where e denotes the identity of G, and let D' denote the local dimension, i.e., the integer for which the left Haar measure $|B'(g\,; r)|$ of the ball $B'(g\,; r) = \{\, h \in G \,:\, d'(g\,; h) < r \,\}$ satisfies bounds $c^{-1}r^{D'} \leq |B'(e\,; r)| \leq c\,r^{D'}$ for some $c > 0$ and all small r.

The principal result on the semigroup kernel K derived in [ElR5] can be summarized as follows.

Theorem 1.1 *Let H be a subelliptic operator in divergence form* (1) *and S the corresponding semigroup. Suppose the principal coefficients (c_{ij}) of H are right uniformly continuous and $c_i, c'_i, c_0 \in L_\infty(G)$. Then the action of S is given by an integral kernel K which satisfies bounds*

$$|K_t(g\,; h)| \leq a\,t^{-D'/2} e^{\omega t} e^{-b(|gh^{-1}|')^2 t^{-1}}$$

for some $a, b > 0$ and $\omega \geq 0$ uniformly for $g, h \in G$ and $t > 0$. The kernel is Hölder continuous and for all $\nu \in \langle 0, 1\rangle$, $\kappa > 0$ and $\tau \in \langle 0, 1\rangle$ there exist $a, b > 0$ and $\omega \geq 0$ such that

$$|K_t(k^{-1}g\,; l^{-1}h) - K_t(g\,; h)| \leq a\,t^{-D'/2} e^{\omega t} \left(\frac{|k|' + |l|'}{t^{1/2} + |gh^{-1}|'} \right)^\nu e^{-b(|gh^{-1}|')^2 t^{-1}}$$

for all $g, h, k, l \in G$ and $t > 0$ with $|k|' + |l|' \leq \kappa\, t^{1/2} + \tau\, |gh^{-1}|'$.

In [ElR5] a similar result was stated with $\tau = 2^{-1}$ but this particular choice was not essential.

This theorem provides the starting point of our discussion of more detailed smoothness properties. To formulate our results we need a multi-index notation suited to the definition of products. If $n \in \mathbf{N}_0$ we set

$$J_n(d') = \bigoplus_{k=0}^{n} \{1, \ldots, d'\}^k \quad \text{and} \quad J(d') = \bigcup_{n=0}^{\infty} J_n(d') \quad .$$

Then we define $A^\alpha = A_{i_1} \ldots A_{i_n}$ for $\alpha = (i_1, \ldots, i_n)$ and set $n = |\alpha|$. We also use the convention $A^\alpha = I$ if $|\alpha| = 0$. Further we set $L'_{p;n} = \bigcap_{\alpha \in J_n(d')} D(A^\alpha)$ in L_p with norm $\|\varphi\|'_{p;n} = \max_{\alpha \in J_n(d')} \|A^\alpha\|_p$ and seminorm $N'_{p;n}(\varphi) = \max_{|\alpha|=n} \|A^\alpha \varphi\|_p$. For the $L_{\hat{p}}$-spaces we use the notation $L'_{\hat{p};n}$, etc..

Next for $\nu \in \langle 0, 1 \rangle$ define the subelliptic Hölder space $C^{\nu\prime}(G)$ as the continuous, but not necessarily bounded, functions over G for which

$$|||\varphi|||_{C^{\nu\prime}} = \sup_{0 < |g|' \leq 1} (|g|')^{-\nu} \|(I - L(g))\varphi\|_\infty$$

is finite. Analogously, if $n \in \mathbf{N}_0$ and $\nu \in \langle 0, 1 \rangle$ introduce the space of bounded functions

$$C^{n+\nu\prime}(G) = \{\varphi \in L'_{\infty;n} : A^\alpha \varphi \in C^{\nu\prime}(G) \text{ for all } \alpha \in J_n(d')\} \quad ,$$

with norm

$$\|\varphi\|_{C^{n+\nu\prime}} = \max\Big(\|\varphi\|'_{\infty;n}, \max_{\alpha \in J_n(d')} |||A^\alpha \varphi|||_{C^{\nu\prime}}\Big) \quad .$$

Note that $L'_{\infty;n+1}(G) \subset C^{n+\nu\prime}(G)$ for all $n \in \mathbf{N}_0$ and $\nu \in \langle 0, 1 \rangle$.

The differentiability properties of the kernel, $(g, h) \mapsto K_t(g\,; h)$, involve derivatives with respect to both variables. Left derivatives with respect to the first variable will be denoted by $A_i K_t$ and left derivatives with respect to the second by $B_i K_t$. Multiple derivatives $A^\alpha K_t$, $B^\beta K_t$ etc. are expressed with the aid of multi-indices.

Our first result improves Theorem 1.1 whenever the coefficients are Hölder continuous. It is a direct extension of Theorem 1.1 in [ElR6] from strongly elliptic operators to subelliptic operators and makes precise the statement that the kernel is almost one degree more differentiable than the coefficients.

Theorem 1.2 *Let H be a subelliptic second-order operator in divergence form* (1) *and let $0 < \gamma < \nu < 1$. If $c_{ij}, c_i, c'_i \in C^{\nu\prime}(G)$ and $c_0 \in L_\infty$ then K_t is once left differentiable in the first variable and the derivatives are once left differentiable in the second. Moreover, for all $\alpha, \beta \in J_1(d')$, $\kappa > 0$ and $\tau \in \langle 0, 1 \rangle$ there exist $a, b > 0$ and $\omega \geq 0$ such that*

$$|(A^\alpha B^\beta K_t)(g\,; h)| \leq a\, t^{-(D'+|\alpha|+|\beta|)/2} e^{\omega t} e^{-b(|gh^{-1}|')^2 t^{-1}}$$

for $g, h \in G$ and $t > 0$ and

$$|(A^\alpha B^\beta K_t)(k^{-1}g\,; l^{-1}h) - (A^\alpha B^\beta K_t)(g\,; h)|$$
$$\leq a\, t^{-(D'+|\alpha|+|\beta|)/2} e^{\omega t} \Big(\frac{|k|' + |l|'}{t^{1/2} + |gh^{-1}|'}\Big)^\gamma e^{-b(|gh^{-1}|')^2 t^{-1}}$$

for all $g, h, k, l \in G$ and $t > 0$ with $|k|' + |l|' \leq \kappa\, t^{1/2} + \tau\, |gh^{-1}|'$.

Note that in the strongly elliptic case analyzed in [ElR6] the theorem is also valid with $\gamma = \nu$.

Our second result concerns operators with differentiable coefficients. But if the c_{ij} and c'_i are once differentiable then the operator can be rewritten in non-divergence form

$$H = -\sum_{i,j=1}^{d'} c_{ij}\, A_i A_j + \sum_{i=1}^{d'} c_i\, A_i + c_0\, I \tag{4}$$

and conversely an operator of the form (4) with the c_{ij} differentiable can be written in the divergence form (1) with the $c'_i = 0$. Therefore the result on kernel smoothness can be formulated for either form of operator. In fact the strongest statement is for non-divergence form operators for which there is an improvement of almost two degrees of differentiability in the first variable.

Theorem 1.3 *Let H be a second-order subelliptic operator in non-divergence form (4) with $c_{ij}, c_i, c_0 \in C^{n+\nu}{}'(G)$ for some $n \in \mathbf{N}$ and $0 < \nu < 1$. Then the semigroup kernel K_t is $(n+2)$-times differentiable in the first variable and the derivatives with respect to the first variable are n-times differentiable with respect to the second. Moreover, for all $\alpha \in J_{n+2}(d')$, $\beta \in J_n(d')$, $\gamma, \gamma^* \in \langle 0,1\rangle$, $\kappa > 0$ and $\tau \in \langle 0,1\rangle$ such that $|\alpha| + \gamma < n + 2 + \nu$ and $|\beta| + \gamma^* < n + \nu$ there exist $a, b > 0$ and $\omega \geq 0$ such that*

$$|(A^\alpha B^\beta K_t)(g\,;h)| \leq a\, t^{-(D'+|\alpha|+|\beta|)/2} e^{\omega t} e^{-b(|gh^{-1}|')^2 t^{-1}}$$

and

$$\begin{aligned}&|(A^\alpha B^\beta K_t)(k^{-1}g\,;l^{-1}h) - (A^\alpha B^\beta K_t)(g\,;h)| \\ &\leq a\, t^{-(D'+|\alpha|+|\beta|)/2} e^{\omega t} \left(\left(\frac{|k|'}{t^{1/2} + |gh^{-1}|'} \right)^{\gamma} + \left(\frac{|l|'}{t^{1/2} + |gh^{-1}|'} \right)^{\gamma^*} \right) e^{-b|gh^{-1}|^2 t^{-1}}\end{aligned}$$

uniformly for all $t > 0$, $g, h \in G$ and $k, l \in G$ such that $|k|' + |l|' \leq \kappa\, t^{1/2} + \tau |gh^{-1}|'$.

Similar statements are valid for $n = 1$, $\nu = 0$ and $c_{ij}, c_i, c_0 \in L'_{\infty;1}$.

The theorem extends a result, Theorem 1.5.III of [ElR6], previously obtained for strongly elliptic operators. As a corollary it follows that if H is in either divergence or non-divergence form and $c_{ij}, c_i, c_0 \in C_{b;\infty}(G)$ then the kernel is infinitely often differentiable and one has Gaussian estimates for all its derivatives.

2 Operators with $C^{\nu}{}'$-coefficients

In this section we consider operators in divergence form (1) with $c_{ij}, c'_i \in C^{\nu}{}'$, for some $\nu \in \langle 0,1\rangle$, and $c_i, c_0 \in L_\infty$. Our aim is to establish bounds on the semigroup S as an operator from L_2 into $C^{1+\nu}{}'$. In Section 4 we then use these bounds to derive Gaussian bounds on the semigroup kernel.

In [ElR6] similar bounds were established for strongly elliptic operators or for certain subelliptic operators on stratified groups. In the latter case we exploited the existence of special 'linear' functions on G, i.e., for each choice of $\xi_1, \ldots, \xi_{d'} \in \mathbf{C}$ there exists a $\tau \colon G \to \mathbf{C}$

such that $A_i\tau = \xi_i$ for all $i \in \{1,\ldots,d'\}$. But such functions do not exist, even locally, on a general group and for a general algebraic basis. Therefore quite different arguments are necessary. In a similar context Xu and Zuily [XuZ] circumvented this problem with an appropriately chosen transformation of vector fields. Set $B'(r) = B'(e\,;r)$ for all $r > 0$.

Lemma 2.1 *There exist $R_1 > 0$, real C^∞-vector fields $Y_1,\ldots,Y_{d'}$ on $B'(2R_1)$ and a C^∞-function $T\colon B'(2R_1) \to GL(d',\mathbf{R})$ such that*

I. $A_i|_g = \sum_{j=1}^{d'}[T(g)]_{ij}\,Y_j|_g$ *for all $i \in \{1,\ldots,d'\}$ and $g \in B'(R_1)$,*

II. $T(e) = I$, *and,*

III. *for all $\xi_1,\ldots,\xi_{d'} \in \mathbf{C}$ there exists a smooth function $\tau\colon B'(R_1) \to \mathbf{C}$ such that $(Y_i\tau)(g) = \xi_i$ uniformly for all $i \in \{1,\ldots,d'\}$ and $g \in B'(R_1)$.*

Proof This follows from [XuZ], Lemma 2.2 and Corollary 2.3. □

Throughout the following R_1, $Y_1,\ldots,Y_{d'}$ and T will be chosen as in Lemma 2.1. Our immediate aim is to estimate in various ways the effect of passing from the A_i, viewed locally as vector fields, to the Y_i. For this we need the following spaces. Let

$$H'_{2;1}(B'(g\,;r)) = \{\varphi \in L_2(B'(g\,;r)) : A_i\varphi \in L_2(B'(g\,;r)) \text{ for all } i \in \{1,\ldots,d'\}\} \quad ,$$

for each $g \in G$ and $r > 0$, where $A_i\varphi$ denotes the distributional derivative in $\mathcal{D}'(B'(g\,;r))$, with the norm

$$\varphi \mapsto \Big(\int_{B'(g;r)} dg\,|\varphi(g)|^2\Big)^{1/2} + \Big(\sum_{i=1}^{d'}\int_{B'(g;r)} dg\,|(A_i\varphi)(g)|^2\Big)^{1/2} \quad .$$

Note that for $r \leq R_1$ and $g = e$ one also has

$$H'_{2;1}(B'(r)) = \{\varphi \in L_2(B'(r)) : Y_i\varphi \in L_2(B'(r)) \text{ for all } i \in \{1,\ldots,d'\}\} \quad ,$$

where $Y_i\varphi$ denotes the distributional derivative in $\mathcal{D}'(B'(r))$, since

$$Y_i|_g = \sum_{j=1}^{d'}[(T(g))^{-1}]_{ij}\,A_j|_g$$

for all $i \in \{1,\ldots,d'\}$. We denote by $\overset{\circ}{H}{}'_{2;1}(B'(r))$ the closure of $C_c^\infty(B'(r))$ in $H'_{2;1}(B'(r))$.

Lemma 2.2

I. *There exists an $M_1 > 0$ such that*

$$M_1^{-1}\sum_{k=1}^{d'}\int_{B'(R)}|A_k\varphi|^2 \leq \sum_{k=1}^{d'}\int_{B'(R)}|Y_k\varphi|^2 \leq M_1\sum_{k=1}^{d'}\int_{B'(R)}|A_k\varphi|^2$$

uniformly for all $R \in \langle 0, R_1]$ and $\varphi \in H'_{2;1}(B'(R))$.

II. *There exists an $M_2 > 0$ such that*

$$\sum_{k=1}^{d'}\int_{B'(R)}|(Y_k - A_k)\varphi|^2 \leq M_2\,R^2\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\varphi|^2$$

uniformly for all $R \in \langle 0, R_1]$ and $\varphi \in H'_{2;1}(B'(R))$.

III. *There exists an $M_3 > 0$ such that*

$$|(Y_i\psi, Y_j\varphi) - (A_i\psi, A_j\varphi)| \le M_3\, R\Big(\sum_{k=1}^{d'} \int_{B'(R)} |Y_k\varphi|^2\Big)^{1/2} \Big(\sum_{k=1}^{d'} \int_{B'(R)} |Y_k\psi|^2\Big)^{1/2}$$

uniformly for all $R \in \langle 0, R_1]$, all $\varphi, \psi \in H'_{2;1}(B'(R))$ and $i, j \in \{1, \ldots, d'\}$.

IV. *There exists an $M_4 > 0$ and $R_2 \in \langle 0, R_1]$ such that*

$$\int_{B'(R)} |(Y_i + Y_i^*)\varphi|^2 \le M_4\, R^2 \sum_{k=1}^{d'} \int_{B'(R)} |Y_k\varphi|^2$$

uniformly for all $R \in \langle 0, R_2]$, $\varphi \in \overset{\circ}{H}{}'_{2;1}(B'(R))$ and $i \in \{1, \ldots, d'\}$.

Proof Statement I follows from Lemma 2.1.I and the smoothness of the function $g \mapsto (T(g))^{-1}$ on $B'(2R_1)$.

Secondly,

$$\sum_{k=1}^{d'} \int_{B'(R)} |(Y_k - A_k)\varphi|^2 \le d' \sum_{j,k=1}^{d'} \int_{B'(R)} dg\, |(\delta_{jk} - [T(g)]_{kj})(Y_j\varphi)(g)|^2$$

$$\le M_2\, R^2 \sum_{k=1}^{d'} \int_{B'(R)} |Y_k\varphi|^2$$

where we have used

$$|\delta_{jk} - [T(g)]_{kj}| \le M\, |g| \le M\, |g|'$$

for some $M > 0$, uniformly for all $g \in B'(R_1)$, which in turn follows because $T(e) = I$ and T is a C^∞-function on $B'(2R_1)$.

Statement III is an easy consequence of the inequality

$$|(Y_i\psi, Y_j\varphi) - (A_i\psi, A_j\varphi)| \le |(Y_i\psi, ((Y_j - A_j)\varphi)| + |((Y_i - A_i)\psi, A_j\varphi)| \quad ,$$

the Cauchy inequality and the previous two statements.

Finally, $Y_i + Y_i^*$ is a multiplication operator with a C^∞-function. Therefore

$$\int_{B'(R)} |(Y_i + Y_i^*)\varphi|^2 \le M \int_{B'(R)} |\varphi|^2$$

for a suitable $M > 0$ and Statement IV follows from the Dirichlet-type Poincaré inequality, [ElR5] Proposition 2.2. □

We next use the vector fields Y_i to prove Campanato-type De Giorgi inequalities for the A_i. First we examine the situation for operators with constant coefficients expressed in terms of the Y_i. If $g \in G$, $r > 0$ and $\varphi \in L_1(B'(g\,;r))$ let $\langle\varphi\rangle_{g,r}$ denote the mean value of φ over $B'(g\,;r)$ and for $g = e$ set $\langle\varphi\rangle_r = \langle\varphi\rangle_{e,r}$.

Lemma 2.3 *For all $M, \mu > 0$ there exist $c_{DG}, c'_{DG} > 0$ such that for all matrices of constant coefficients $C = (c_{ij})$ with $\mu_C \ge \mu$ and $\|C\|_\infty \le M$, all $R \in \langle 0, R_1]$ and for all $\varphi \in H'_{2;1}(B'(R))$ satisfying*

$$-\sum_{i,j=1}^{d'} c_{ij}\, Y_i\, Y_j\, \varphi = 0$$

weakly on $B'(R)$ one has

$$\sum_{k=1}^{d'} \int_{B'(r)} |Y_k\varphi - \langle Y_k\varphi\rangle_r|^2 \le c_{DG}(r/R)^{D'+2} \sum_{k=1}^{d'} \int_{B'(R)} |Y_k\varphi - \langle Y_k\varphi\rangle_R|^2$$
$$+ c'_{DG} R^2 \sum_{k=1}^{d'} \int_{B'(R)} |Y_k\varphi - \langle Y_k\varphi\rangle_R|^2$$

uniformly for all $r \in \langle 0, R]$.

Proof By Lemma 2.1.III there exist C^∞-functions $\chi_1, \ldots, \chi_{d'}: B'(R_1) \to \mathbf{C}$ such that $Y_i\chi_j = \delta_{ij}$. Let $R_0 \in \langle 0, 1]$ and $\sigma \in \langle 0, 1\rangle$ and let η_R denote the cut-off functions of [ElR5], Lemma 2.3. Fix $n > 1 + D'/2$. Let C be a constant matrix with $\mu_C \ge \mu$ and $\|C\|_\infty \le M$, then with $R \in \langle 0, R_0 \wedge R_1]$ let $\varphi \in H'_{2;1}(B'(2R))$ satisfy $-\sum_{i,j=1}^{d'} c_{ij}\, Y_i\, Y_j\, \varphi = 0$ weakly on $B'(R)$. Finally, let $r \in \langle 0, \sigma^2 R]$.

Set $b_i = \langle Y_i\varphi\rangle_R$ for all $i \in \{1, \ldots, d'\}$, $\psi = \varphi - \sum_{i=1}^{d'} b_i\, \chi_i$ and $\tau = \psi - \langle\psi\rangle_R$. Then $\psi, \tau \in H'_{2;1}(B'(R))$, $Y_i\tau = Y_i\varphi - \langle Y_i\varphi\rangle_R$, $\langle Y_i\tau\rangle_R = 0$ for all i and $-\sum_{i,j=1}^{d'} Y_i\, Y_j\, \tau = 0$ weakly on $B'(R)$. Next let $\zeta \in H'_{2;1}(B'(R))$ be such that

$$-\sum_{i,j=1}^{d'} c_{ij}\, A_i\, A_j\, \zeta = 0 \text{ weakly in } B'(R) \quad , \quad \chi = \tau - \zeta \in \mathring{H}'_{2;1}(B'(R)) \quad .$$

Then

$$\sum_{k=1}^{d'} \int_{B'(\sigma^{-1}r)} |Y_k\varphi - \langle Y_k\varphi\rangle_{\sigma^{-1}r}|^2 \le \sum_{k=1}^{d'} \int_{B'(\sigma^{-1}r)} |Y_k\tau - \langle A_k\zeta\rangle_{\sigma^{-1}r}|^2$$
$$\le 3 \sum_{k=1}^{d'} \int_{B'(\sigma^{-1}r)} |A_k\zeta - \langle A_k\zeta\rangle_{\sigma^{-1}r}|^2$$
$$+ 3 \sum_{k=1}^{d'} \int_{B'(\sigma^{-1}r)} |(Y_k - A_k)\tau|^2 + 3 \sum_{k=1}^{d'} \int_{B'(\sigma^{-1}r)} |A_k\chi|^2 \quad .$$

We estimate the three terms separately.

First

$$\sum_{k=1}^{d'} \int_{B'(\sigma^{-1}r)} |A_k\zeta - \langle A_k\zeta\rangle_{\sigma^{-1}r}|^2$$
$$\le |B'(\sigma^{-1}r)|^{-1} \sum_{k=1}^{d'} \int_{B'(\sigma^{-1}r)} dg \int_{B'(\sigma^{-1}r)} dh\, |(\eta_R A_k\zeta)(g) - (\eta_R A_k\zeta)(h)|^2$$
$$\le c\, r^{D'+2} \sum_{i,k=1}^{d'} \|A_i\eta_R A_k\zeta\|_\infty^2 \quad .$$

But the Sobolev embedding theorem gives

$$\|A_i\eta_R A_k\zeta\|_\infty = \|A_i\eta_R A_k(\zeta - \langle\zeta\rangle_R)\|_\infty$$
$$\le \varepsilon^{n-D'/2-1} N'_{2;n}(\eta_R A_k(\zeta - \langle\zeta\rangle_R)) + c\,\varepsilon^{-D'/2-1} \|\eta_R A_k(\zeta - \langle\zeta\rangle_R)\|_2 \quad ,$$

uniformly for all $\varepsilon \in \langle 0,1]$, for some $c > 0$, depending only on G, n and the algebraic basis. Setting $\varepsilon = R$ gives

$$\|A_i \eta_R A_k \zeta\|_\infty \leq R^{n-D'/2-1} N'_{2;n+1}(\eta_R(\zeta - \langle\zeta\rangle_R)) + R^{n-D'/2-1} N'_{2;n}((A_k\eta_R)(\zeta - \langle\zeta\rangle_R))$$
$$+ c\, R^{-D'/2-1} \|\eta_R A_k(\zeta - \langle\zeta\rangle_R)\|_2$$
$$\leq c_{\mu,M,n} R^{-D'/2-1} \Big(\sum_{j=1}^{d'} \int_{B'(R)} |A_j(\zeta - \langle\zeta\rangle_R)|^2 \Big)^{1/2} \quad ,$$

by Lemma 3.1.II of [ElR5], for some $c_{\mu,M,n} > 0$, depending only on μ, M and n, and for $R \in \langle 0, R_{\mu,M,n}]$, where $R_{\mu,M,n} > 0$ is a constant which depends only on μ, M and n. Therefore

$$\sum_{k=1}^{d'} \int_{B'(\sigma^{-1}r)} |A_k\zeta - \langle A_k\zeta\rangle_{\sigma^{-1}r}|^2 \leq c\,(r/R)^{D'+2} \sum_{k=1}^{d'} \int_{B'(R)} |A_k\zeta|^2$$
$$\leq 3c\,(r/R)^{D'+2} \sum_{k=1}^{d'} \int_{B'(R)} |Y_k\tau|^2$$
$$+ 3c \sum_{k=1}^{d'} \int_{B'(R)} |(Y_k - A_k)\tau|^2 + 3c \sum_{k=1}^{d'} \int_{B'(R)} |A_k\chi|^2$$
$$= 3c\,(r/R)^{D'+2} \sum_{k=1}^{d'} \int_{B'(R)} |Y_k\varphi - \langle Y_k\varphi\rangle_R|^2$$
$$+ 3c \sum_{k=1}^{d'} \int_{B'(R)} |(Y_k - A_k)\tau|^2 + 3c \sum_{k=1}^{d'} \int_{B'(R)} |A_k\chi|^2 \quad .$$

Secondly,

$$\sum_{k=1}^{d'} \int_{B'(R)} |(Y_k - A_k)\tau|^2 \leq M_2\, R^2 \sum_{k=1}^{d'} \int_{B'(R)} |Y_k\tau|^2 = M_2\, R^2 \sum_{k=1}^{d'} \int_{B'(R)} |Y_k\varphi - \langle Y_k\varphi\rangle_R|^2 \quad ,$$

by Lemma 2.2.II.

Thirdly,

$$\mu_C \sum_{k=1}^{d'} \int_{B'(R)} |A_k\chi|^2 \leq \mathrm{Re}(\chi, -\sum c_{ij}\, A_i\, A_j\, \chi) = \sum c_{ij}\ \mathrm{Re}(A_i\chi, A_j\,\tau)$$
$$\leq \sum c_{ij}\ \mathrm{Re}(Y_i\chi, Y_j\,\tau)$$
$$+ \|C\|_\infty M_3\, R\Big(\sum_{k=1}^{d'} \int_{B'(R)} |Y_k\chi|^2 \Big)^{1/2} \Big(\sum_{k=1}^{d'} \int_{B'(R)} |Y_k\tau|^2 \Big)^{1/2} \quad ,$$

where we have used Lemma 2.2.III. Next, if R is small enough,

$$\sum c_{ij}\ \mathrm{Re}(Y_i\chi, Y_j\,\tau) = -\sum c_{ij}\ \mathrm{Re}(Y_i^*\chi, Y_j\,\tau) + \sum c_{ij}\ \mathrm{Re}((Y_i + Y_i^*)\chi, Y_j\,\tau)$$
$$= \sum c_{ij}\ \mathrm{Re}((Y_i + Y_i^*)\chi, Y_j\,\tau)$$
$$\leq \|C\|_\infty M_1^{1/2}\, M_4^{1/2}\, R\Big(\sum_{k=1}^{d'} \int_{B'(R)} |A_k\chi|^2 \Big)^{1/2} \Big(\sum_{k=1}^{d'} \int_{B'(R)} |Y_k\tau|^2 \Big)^{1/2} \quad ,$$

since $-\sum c_{ij}\, Y_i\, Y_j\, \tau = 0$ weakly on $B'(R)$, and we have used Lemma 2.2.IV.

Combination of these estimates then gives

$$\sum_{k=1}^{d'} \int_{B'(R)} |A_k \chi|^2 \leq c\, R^2 \sum_{k=1}^{d'} \int_{B'(R)} |Y_k\varphi - \langle Y_k\varphi\rangle_R|^2$$

if $R \leq R_2$ where $R_2 > 0$ depends only on the vector fields and $c > 0$ also depends on μ and M.

Combination of all previous estimates then establishes the bounds

$$\sum_{k=1}^{d'} \int_{B'(\sigma^{-1}r)} |Y_k\varphi - \langle Y_k\varphi\rangle_{\sigma^{-1}r}|^2 \leq c_{DG}(r/R)^{D'+2} \sum_{k=1}^{d'} \int_{B'(R)} |Y_k\varphi - \langle Y_k\varphi\rangle_R|^2$$

$$+ c'_{DG} R^2 \sum_{k=1}^{d'} \int_{B'(R)} |Y_k\varphi - \langle Y_k\varphi\rangle_R|^2$$

uniformly for all $0 < R \leq R_0 \wedge R_1 \wedge R_2$, all $0 < r \leq \sigma^2 R$ and for all $\varphi \in H'_{2;1}(B'(2R))$ satisfying the equation $-\sum c_{ij}\, Y_i\, Y_j\, \varphi = 0$ weakly on $B'(R)$. From this the lemma follows as in the proof of Proposition 3.4 in [ElR5]. □

We are now in a position to derive Campanato-type De Giorgi estimates for subelliptic operators with $C^{\nu\prime}$-coefficients in divergence form. It is convenient to express these estimates uniformly for a large class of operators.

Let $\mathcal{E}^{\rm div}(\nu, \mu, M)$ be the set of all pure second-order subelliptic divergence form operators

$$H = -\sum_{i,j=1}^{d'} A_i\, c_{ij}\, A_j$$

with $c_{ij} \in C^{\nu\prime}$, where $\mu, M > 0$, $\nu \in \langle 0,1\rangle$ and one has $\mu_C \geq \mu$, $\|C\|_\infty \leq M$ and $|||c_{ij}|||_{C^{\nu\prime}} \leq M$ for all $i,j \in \{1,\ldots,d'\}$.

Proposition 2.4 *For all $M, \mu > 0$ and $\nu \in \langle 0,1\rangle$ there exist $c_{DG}, c'_{DG} > 0$ such that for all $H \in \mathcal{E}^{\rm div}(\nu,\mu,M)$, all $R \in \langle 0,1]$ and all $\varphi \in H'_{2;1}(B'(R))$ satisfying $H\varphi = 0$ weakly on $B'(R)$ one has*

$$\sum_{k=1}^{d'} \int_{B'(r)} |A_k\varphi - \langle A_k\varphi\rangle_r|^2 \leq c_{DG}(r/R)^{D'+2} \sum_{k=1}^{d'} \int_{B'(R)} |A_k\varphi - \langle A_k\varphi\rangle_R|^2$$

$$+ c'_{DG} R^{2\nu} \sum_{k=1}^{d'} \int_{B'(R)} |A_k\varphi|^2$$

uniformly for all $r \in \langle 0, R]$.

Proof Let c_{DG} and c'_{DG} be as in Lemma 2.3 and $M_1, \ldots, M_4$ and R_2 as in Lemma 2.2. Let $H \in \mathcal{E}^{\rm div}(\nu,\mu,M)$, $R \in \langle 0, R_1]$ and suppose $\varphi \in H'_{2;1}(B'(R))$ satisfies $H\varphi = 0$ weakly on $B'(R)$. Let $r \in \langle 0, R]$. Then

$$\sum_{k=1}^{d'} \int_{B'(r)} |A_k\varphi - \langle A_k\varphi\rangle_r|^2 \leq \sum_{k=1}^{d'} \int_{B'(r)} |A_k\varphi - \langle Y_k\varphi\rangle_r|^2$$

$$\leq 2\sum_{k=1}^{d'} \int_{B'(r)} |Y_k\varphi - \langle Y_k\varphi\rangle_r|^2 + 2\sum_{k=1}^{d'} \int_{B'(r)} |(A_k - Y_k)\varphi|^2$$

$$\leq 2\sum_{k=1}^{d'}\int_{B'(r)}|Y_k\varphi-\langle Y_k\varphi\rangle_r|^2+2M_1\,M_2\,R^2\sum_{k=1}^{d'}\int_{B'(r)}|A_k\varphi|^2 \quad .$$

Next, since

$$-\sum_{i,j=1}^{d'}c_{ij}(e)\,Y_i\,Y_j=\sum_{i,j=1}^{d'}c_{ij}(e)\,Y_i^*\,Y_j-\sum_{i,j=1}^{d'}c_{ij}(e)\,(Y_i+Y_i^*)\,Y_j$$

and $Y_i+Y_i^*$ is a multiplication operator with a C^∞-function it follows that there exists an $\eta\in H'_{2;1}(B'(R))$ such that

$$-\sum_{i,j=1}^{d'}c_{ij}(e)\,Y_i\,Y_j\,\eta=0\text{ weakly in }B'(R)\quad,\quad\chi=\varphi-\eta\in\mathring{H}'_{2;1}(B'(R))\quad .$$

Then arguing as in the proof of Proposition 2.6 in [ElR6] one deduces that

$$\sum_{k=1}^{d'}\int_{B'(r)}|Y_k\varphi-\langle Y_k\varphi\rangle_r|^2\leq 4c_{DG}(r/R)^{D'+2}\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\varphi-\langle Y_k\varphi\rangle_R|^2$$

$$+\,4c'_{DG}R^2\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\varphi|^2+(2+4c_{DG}+4c'_{DG})\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\chi|^2$$

$$\leq 8c_{DG}(r/R)^{D'+2}\sum_{k=1}^{d'}\int_{B'(R)}|A_k\varphi-\langle A_k\varphi\rangle_R|^2$$

$$+\,(8c_{DG}M_2+4c'_{DG})M_1\,R^2\sum_{k=1}^{d'}\int_{B'(R)}|A_k\varphi|^2$$

$$+\,(2+4c_{DG}+4c'_{DG})\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\chi|^2 \quad .$$

But

$$\mu\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\chi|^2\leq\operatorname{Re}\sum_{i,j=1}^{d'}c_{ij}(e)\,(Y_i\chi,Y_j\chi)$$

$$=-\operatorname{Re}\sum_{i,j=1}^{d'}c_{ij}(e)\,(Y_i^*\chi,Y_j\chi)+\operatorname{Re}\sum_{i,j=1}^{d'}c_{ij}(e)\,((Y_i+Y_i^*)\chi,Y_j\chi)$$

$$\leq-\operatorname{Re}\sum_{i,j=1}^{d'}c_{ij}(e)\,(Y_i^*\chi,Y_j\varphi)$$

$$+\,\|C\|_\infty M_4^{1/2}R\Big(\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\chi|^2\Big)^{1/2}\Big(\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\chi|^2\Big)^{1/2}$$

if $R\leq R_2$. So if, in addition, $R\leq 2^{-1}\|C\|_\infty^{-1}M_4^{-1/2}\mu$ then

$$\mu\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\chi|^2\leq-2\operatorname{Re}\sum_{i,j=1}^{d'}c_{ij}(e)\,(Y_i^*\chi,Y_j\varphi)$$

$$=2\operatorname{Re}\sum_{i,j=1}^{d'}c_{ij}(e)\,(Y_i\chi,Y_j\varphi)-2\operatorname{Re}\sum_{i,j=1}^{d'}c_{ij}(e)\,((Y_i+Y_i^*)\chi,Y_j\varphi) \quad .$$

But similarly one has

$$|((Y_i + Y_i^*)\chi, Y_j\varphi)| \le M_1^{1/2}\, M_4^{1/2}\, R\Big(\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\chi|^2\Big)^{1/2}\Big(\sum_{k=1}^{d'}\int_{B'(R)}|A_k\varphi|^2\Big)^{1/2}$$

and

$$|(Y_i\chi, Y_j\varphi) - (A_i\chi, A_j\varphi)| \le M_3\, R\Big(\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\chi|^2\Big)^{1/2}\Big(\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\varphi|^2\Big)^{1/2}$$

if $R \le R_2$. Therefore

$$\begin{aligned}\mathrm{Re}\sum_{i,j=1}^{d'} c_{ij}(e)\,(Y_i\chi, Y_j\varphi) &\le \mathrm{Re}\sum_{i,j=1}^{d'} c_{ij}(e)\,(A_i\chi, A_j\varphi)\\ &\quad + \|C\|_\infty (d')^2\, M_3\, R\Big(\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\chi|^2\Big)^{1/2}\Big(\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\varphi|^2\Big)^{1/2}\end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}\sum_{i,j=1}^{d'} c_{ij}(e)\,(A_i\chi, A_j\varphi) &= \mathrm{Re}\sum_{i,j=1}^{d'} (A_i\chi, (c_{ij}(e) - c_{ij})A_j\varphi)\\ &\le d' \sup_{i,j} |||c_{ij}|||_{C^{\nu\prime}} R^\nu \Big(\sum_{k=1}^{d'}\int_{B'(R)}|A_k\chi|^2\Big)^{1/2}\Big(\sum_{k=1}^{d'}\int_{B'(R)}|A_k\varphi|^2\Big)^{1/2}\\ &\le d'\, M\, M_1^{1/2}\, R^\nu\Big(\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\chi|^2\Big)^{1/2}\Big(\sum_{k=1}^{d'}\int_{B'(R)}|A_k\varphi|^2\Big)^{1/2}\end{aligned}$$

where we used the assumption $H\varphi = 0$ weakly on $B'(R)$. Combining these estimates then gives

$$\mu\sum_{k=1}^{d'}\int_{B'(R)}|Y_k\chi|^2 \le c\,R^{2\nu}\sum_{k=1}^{d'}\int_{B'(R)}|A_k\varphi|^2$$

if $R \le R_3$, for some $R_3 \in \langle 0, R_1]$ and $c > 0$, depending on μ and M and the proposition follows. □

As a corollary we obtain Campanato-type De Giorgi inequalities for balls with an arbitrary centre.

Proposition 2.5 *For all $M, \mu > 0$ and $\nu \in \langle 0, 1\rangle$ there exist $c_{DG}, c'_{DG} > 0$ such that for all $H \in \mathcal{E}^{\mathrm{div}}(\nu, \mu, M)$, all $g \in G$ all $R \in \langle 0, 1]$ and all $\varphi \in H'_{2;1}(B'(g\,; R))$ satisfying $H\varphi = 0$ weakly on $B'(g\,; R)$ one has*

$$\begin{aligned}\sum_{k=1}^{d'}\int_{B'(g;r)}|A_k\varphi - \langle A_k\varphi\rangle_{g,r}|^2 &\le c_{DG}(r/R)^{D'+2}\sum_{k=1}^{d'}\int_{B'(g\,;R)}|A_k\varphi - \langle A_k\varphi\rangle_{g,R}|^2\\ &\quad + c'_{DG}R^{2\nu}\sum_{k=1}^{d'}\int_{B'(g\,;R)}|A_k\varphi|^2\end{aligned}$$

uniformly for all $r \in \langle 0, R]$.

Proof This follows by right invariance. For all operators $H = -\sum_{i,j=1}^{d'} A_i\, c_{ij}\, A_j \in \mathcal{E}^{\rm div}(\nu,\mu,M)$ and $g \in G$ one has

$$-\sum_{i,j=1}^{d'} A_i \Big(R(g)c_{ij}\Big) A_j \in \mathcal{E}^{\rm div}(\nu,\mu,M)$$

and the A_i commute with the $R(g)$. □

The arguments of [ElR6], Section 3, now allow us to derive crossnorm estimates on the semigroups generated by divergence form operators with Hölder coefficients. For $\nu_2 \in \langle 0,1\rangle$, $\nu_1, \nu_1', \nu_0 \in [0,1\rangle$, $\mu > 0$ and $M > 0$ let $\mathcal{E}^{\rm div}(\nu_2,\nu_1,\nu_1',\nu_0,\mu,M)$ denote the set of all second-order subelliptic operators of the form

$$H = -\sum_{i,j=1}^{d'} A_i\, c_{ij}\, A_j + \sum_{i=1}^{d'} (c_i\, A_i + A_i\, c_i') + c_0\, I$$

such that $\|C\|_\infty \leq M$, $\mu_C \geq \mu$, $c_{ij} \in \mathcal{C}^{\nu_2\,\prime}$, $|||c_{ij}|||_{\mathcal{C}^{\nu_2\,\prime}} \leq M$, $\|c_i\|_\infty \leq M$, and $|||c_i|||_{\mathcal{C}^{\nu_1\,\prime}} \leq M$ if $\nu_1 > 0$, with similar conditions on the c_i' and c_0.

Proposition 2.6 *Let $\nu \in \langle 0,1\rangle$. Then for all $M, \mu > 0$ there exist $a, \omega > 0$ such that $S_t C_c^\infty(G) \subset \mathcal{C}^{1+\nu\,\prime}(G)$ for all $t > 0$ and*

$$\|S_t\varphi\|_{\mathcal{C}^{1+\nu\,\prime}} \leq a\, t^{-D'/4} t^{-(1+\nu)/2} e^{\omega t} \|\varphi\|_{\hat{2}}$$

uniformly for all $H \in \mathcal{E}^{\rm div}(\nu,0,\nu,0,\mu,M)$, $\varphi \in C_c^\infty(G)$ and $t > 0$.

Proof This follows precisely as in [ElR6], Section 3, up to Proposition 3.3, but now it is easier, since there is no Davies' exponential perturbation involved. □

In Section 4 these bounds will be used to deduce kernel bounds for K_t.

3 Operators with $\mathcal{C}^{n+\nu\,\prime}$-coefficients

The aim of this section is to establish an analogue of Proposition 2.6 for operators with smoother coefficients and in particular for coefficients which are at least once differentiable. If the coefficients c_{ij} and c_i' of the operator (1) are once differentiable then it can be rewritten in non-divergence form

$$H = -\sum_{i,j=1}^{d} c_{ij}\, A_i A_j + \sum_{i=1}^{d} c_i\, A_i + c_0\, I$$

with redefined c_i and c_0. There are two good reasons for considering non-divergence form operators. First one has improved smoothness properties for the kernel of the semigroup generated by non-divergence form operators. Secondly, these operators allow one to exploit perturbation arguments (see [Xu]). The important ingredient is an optimal regularity result for operators with constant coefficients on Hölder or Lipschitz spaces.

If $n \in \mathbf{N}_0$ and $\nu \in \langle 0,1\rangle$ then

$$\mathcal{C}^{n+\nu\,\prime} = (L_\infty, L'_{\infty;n+1})_{\frac{n+\nu}{n+1},\infty;\mathrm{K}} = (L'_{\infty;n}, L'_{\infty;n+1})_{\nu,\infty;\mathrm{K}}$$

with equivalent norms, where, in general, $(\mathcal{X},\mathcal{Y})_{\gamma,p;K}$ denotes the real interpolation space in the sense of Peetre. These identities follow from [ElR2], Theorems 2.1 and 3.2. Since $|||\varphi|||_{C^{\nu\,\prime}} \leq d' \|\varphi\|'_{\infty;1}$, one deduces that $\|\varphi\|_{C^{\gamma\,\prime}} \leq d' \|\varphi\|_{C^{\tau\,\prime}}$ for all $\gamma,\tau \in \langle 0,\infty\rangle\backslash\mathbf{N}$ with $\gamma \leq \tau$. By Leibniz' rule and Lemma 2.1.I of [ElR6] the spaces $C^{n+\nu\,\prime}$ are algebras and

$$\|\varphi\,\psi\|_{C^{n+\nu\,\prime}} \leq b_{n+\nu} \sum_{k=0}^{n} \Big(\|\varphi\|'_{\infty;k}\|\psi\|_{C^{n-k+\nu\,\prime}} + \|\varphi\|_{C^{k+\nu\,\prime}}\|\psi\|'_{\infty;n-k}\Big) \tag{5}$$

for some $b_{n+\nu} > 0$, uniformly for all $\varphi,\psi \in C^{n+\nu\,\prime}$.

We now use the notation of [ElR5]. In particular, let $R_0 \in \langle 0,1]$, $\sigma \in \langle 0,1\rangle$ and let η_R denote the cut-off functions of Lemma 2.3 in [ElR5]. Then $\|\eta_R\|'_{\infty;n} \leq c_n R^{-n}$ for all $n \in \mathbf{N}$ and $R \in \langle 0,R_0]$ and, by interpolation, $\|\eta_R\|_{C^{\gamma\,\prime}} \leq c_\gamma R^{-\gamma}$ for all $\gamma \in \langle 0,\infty\rangle\backslash\mathbf{N}$ and $R \in \langle 0,R_0]$. Moreover, $\eta_R = 1$ on $B'(\sigma R)$ and $\operatorname{supp}\eta_R \subset B'(R)$.

Lemma 3.1 *For all $n \in \mathbf{N}$ there exists a $c > 0$ such that*

$$\|\eta_R\varphi\|'_{\infty;k} \leq c\,R^{-k+1}\|\varphi\|'_{\infty;n}$$

uniformly for all $R \in \langle 0,R_0]$, $k \in \{0,\ldots,n\}$ and $\varphi \in L'_{\infty;n}$ with $\varphi(e) = 0$.

Proof If $|\alpha| = k$ then $\|(A^\alpha\eta_R)\varphi\|_\infty \leq \|A^\alpha\eta_R\|_\infty \sup_{g\in B'(R)} |\varphi(g)-\varphi(e)| \leq c\,R^{-k}\|\varphi\|'_{\infty;1}R$. Therefore

$$\begin{aligned}\|A^\alpha(\eta_R\varphi)\|_\infty &\leq \|(A^\alpha\eta_R)\varphi\|_\infty + \sum_{l=0}^{k-1}\binom{k}{l}\|\eta_R\|'_{\infty;l}\|\varphi\|'_{\infty;k-l}\\ &\leq c\,R^{-k+1}\|\varphi\|'_{\infty;1} + \sum_{l=0}^{k-1}\binom{k}{l}c'\,R^{-l}\|\varphi\|'_{\infty;n}\end{aligned}$$

and this immediately yields the desired bounds. □

The optimal regularity property for operators with constant coefficients is as follows.

Theorem 3.2 *For all $n \in \mathbf{N}_0$, $\nu \in \langle 0,1\rangle$ and $M,\mu > 0$ there exist $\Lambda > 0$ and $c > 0$ such that for all $H = -\sum_{i,j=1}^{d'} c_{ij}A_iA_j$ with constant coefficients $C = (c_{ij})$ satisfying $\mu_C \geq \mu$ and $\|C\|_\infty \leq M$ one has*

$$C^{n+2+\nu\,\prime} = \{\varphi \in L'_{\infty;2} : H\varphi \in C^{n+\nu\,\prime}\}\quad,$$

$$\|\varphi\|_{C^{n+\nu\,\prime}} \leq c\,\lambda^{-1}\|(H+\lambda I)\varphi\|_{C^{n+\nu\,\prime}} \tag{6}$$

and

$$\|\varphi\|_{C^{n+2+\nu\,\prime}} \leq c\,\|(H+\lambda I)\varphi\|_{C^{n+\nu\,\prime}}$$

uniformly for all $\lambda \geq \Lambda$ and $\varphi \in C^{n+2+\nu\,\prime}$.

Proof The bounds (6) follow since H generates a continuous semigroup on $C^{n+\nu\,\prime}$ with a kernel satisfying Gaussian bounds, and the parameters in the Gaussian bounds depend only on M and μ. The optimal regularity of the operator H follows from [ElR2], Theorem 4.1, where it has been proved that there exists a $c > 0$, depending on H, such that

$$\|\varphi\|_{\mathcal{C}^{n+2+\nu\prime}} \leq c\,(\|H\varphi\|_{\mathcal{C}^{n+\nu\prime}} + \|\varphi\|_{\mathcal{C}^{n+\nu\prime}}) \tag{7}$$

uniformly for all $\varphi \in \mathcal{C}^{n+2+\nu\prime}$. (Although [ElR2] only deals with real symmetric coefficients, the results are also valid for complex subelliptic operators, see, for example, [ElR3] Theorem 3.1.) But the set of coefficients of all constant coefficients operators with $\mu_C \geq \mu$ and $\|C\|_\infty \leq M$ is a compact subset of $\mathbf{C}^{(d')^2}$. Moreover,

$$\mathcal{C}^{n+2+\nu\prime} = \{\varphi \in L'_{\infty;2} : A_i A_j \varphi \in \mathcal{C}^{n+\nu\prime} \text{ for all } i,j \in \{1,\ldots,d'\}\} \quad .$$

Therefore the constant c in (7) must be uniform on compact sets by an argument similar to that used in the proof of Theorem 3.7 in [ElR1].

Finally, for all $\lambda \geq \Lambda$ one has

$$\begin{aligned}
\|\varphi\|_{\mathcal{C}^{n+2+\nu\prime}} &\leq c\,(\|H\varphi\|_{\mathcal{C}^{n+\nu\prime}} + \|\varphi\|_{\mathcal{C}^{n+\nu\prime}}) \\
&\leq c\,\|(H+\lambda I)\varphi\|_{\mathcal{C}^{n+\nu\prime}} + (c+\lambda)\,\|\varphi\|_{\mathcal{C}^{n+\nu\prime}} \\
&\leq \Big(c + (c+\lambda)c\lambda^{-1}\Big)\|(H+\lambda I)\varphi\|_{\mathcal{C}^{n+\nu\prime}} \\
&\leq c\,(2 + c\Lambda^{-1})\|(H+\lambda I)\varphi\|_{\mathcal{C}^{n+\nu\prime}}
\end{aligned}$$

for all $\varphi \in \mathcal{C}^{n+2+\nu\prime}$. □

Now we turn to operators with variable coefficients. For $N_2 \in \mathbf{N}$, $N_1, N_0 \in \mathbf{N}_0$, $\nu_2, \nu_1, \nu_0 \in \langle 0,1\rangle$, $\mu > 0$ and $M > 0$ let

$$\mathcal{E}^{\rm nondiv}(N_2+\nu_2, N_1+\nu_1, N_0+\nu_0, \mu, M)$$

denote the set of all second-order subelliptic operators of the form (4) such that $c_{ij} \in \mathcal{C}^{N_2+\nu_2\prime}$, $\|c_{ij}\|_{\mathcal{C}^{N_2+\nu_2\prime}} \leq M$, $\|C\|_\infty \leq M$, the ellipticity constant $\mu_C \geq \mu$ and similar conditions on the c_i and c_0.

If $n \in \mathbf{N}$, $\nu \in \langle 0,1\rangle$ and $M, \mu > 0$ then for all $R \leq R_0 \wedge (2(d')^2M)^{-1}\mu$ we can associate with each $H \in \mathcal{E}^{\rm nondiv}(n+\nu, n+\nu, n+\nu, \mu, M)$ the divergence form operator

$$H_R^w = -\sum_{i,j=1}^{d'} A_i\Big(c_{ij}(e)+\eta_R(c_{ij}-c_{ij}(e))\Big)A_j + \sum_{i=1}^{d'} \eta_R\, c_i\, A_i + \sum_{j=1}^{d'}\Big(A_i(\eta_R(c_{ij}-c_{ij}(e)))\Big)A_j + \eta_R\, c_0\, I \quad .$$

Formally, $H_R^w = H_0 + \eta_R(H - H_0)$, where $H_0 = -\sum_{i,j=1}^{d'} c_{ij}(e)\, A_i\, A_j$ is the pure second-order operator with coefficients fixed at the identity. The condition on R implies that the ellipticity constant of the operator H_R^w is at least $2^{-1}\mu$. If c is as in Lemma 3.1 then $\|A_i(\eta_R(c_{ij} - c_{ij}(e)))\|_\infty \leq 2cM$. Hence there exists a $\lambda_0 > 0$ such that

$$\mathrm{Re}(\varphi, (H_R^w + \lambda I)\varphi) \geq 2^{-1}\mu \sum_{i=1}^{d'} \|A_i\varphi\|_2 \tag{8}$$

uniformly for all $R \in \langle 0, R_0 \wedge (2(d')^2M)^{-1}\mu]$, $\varphi \in D(H_R^w)$ and $\lambda \geq \lambda_0$.

Lemma 3.3 *For all $n \in \mathbf{N}_0$, $\nu \in \langle 0,1\rangle$ and $M, \mu > 0$ there exist $R \in \langle 0, R_0 \wedge (2(d')^2 M)^{-1}\mu]$ and $c > 0$ such that for any $H \in \mathcal{E}^{\rm nondiv}((n \vee 1) + \nu, (n \vee 1) + \nu, (n \vee 1) + \nu, \mu, M)$, $\varphi \in D(H_R^w) \cap \mathcal{C}^{n+\nu\,\prime}$ and $\psi \in L'_{2;1} \cap \mathcal{C}^{n+\nu\,\prime}$ with*

$$H_R^w \varphi = \psi \quad ,$$

weakly on $L'_{2;1}$, one has $\varphi \in \mathcal{C}^{n+2+\nu\,\prime}$ and

$$\|\varphi\|_{\mathcal{C}^{n+2+\nu\,\prime}} \le c\,(\|\psi\|_{\mathcal{C}^{n+\nu\,\prime}} + \|\varphi\|_{\mathcal{C}^{n+\nu\,\prime}}) \quad .$$

Proof It follows from (5) and Lemma 3.1 that

$$\begin{aligned}\|\eta_R(c_{ij} - c_{ij}(e))\, A_i\, A_j \tau\|_{\mathcal{C}^{n+\nu\,\prime}} &\le b_{n+\nu} \sum_{k=0}^{n} \|\eta_R(c_{ij} - c_{ij}(e))\|'_{\infty;k} \|\tau\|_{\mathcal{C}^{n+2-k+\nu\,\prime}} \\ &\qquad + \|\eta_R(c_{ij} - c_{ij}(e))\|_{\mathcal{C}^{k+\nu\,\prime}} \|\tau\|'_{\infty;n+2-k} \\ &\le b_{n+\nu}\, c\, \|c_{ij}\|_{\mathcal{C}^{n+\nu\,\prime}} \sum_{k=0}^{n} R^{-k+1} \|\tau\|_{\mathcal{C}^{n+2-k+\nu\,\prime}} + R^{-k-\nu} \|\tau\|'_{\infty;n+2-k}\end{aligned}$$

for all $\tau \in \mathcal{C}^{n+2+\nu\,\prime}$. Therefore

$$\begin{aligned}\|\eta_R(c_{ij} - c_{ij}(e))\, A_i\, A_j \tau\|_{\mathcal{C}^{n+\nu\,\prime}} &\le b_{n+\nu}\, c\, M \sum_{k=0}^{n} R^{-k+1} \Big(\varepsilon^k \|\tau\|_{\mathcal{C}^{n+2+\nu\,\prime}} + c'\, \varepsilon^{-(n+2-k+\nu)} \|\tau\|_\infty\Big) \\ &\qquad + R^{-k-\nu} \Big(\delta_k^{k+\nu} \|\tau\|_{\mathcal{C}^{n+2+\nu\,\prime}} + c'\, \delta_k^{-(n+2-k)} \|\tau\|_\infty\Big) \\ &\le c''\, (R^\nu \|\tau\|_{\mathcal{C}^{n+2+\nu\,\prime}} + R^{-2(n+2)} \|\tau\|_\infty)\end{aligned}$$

for some $c'' > 0$, uniformly for all $R \in \langle 0, R_0]$ and $\tau \in \mathcal{C}^{n+2+\nu\,\prime}$ by choosing $\delta_k = R^{-(k+2\nu)/(k+\nu)}$ and $\varepsilon = R$. Similarly

$$\|\eta_R\, c_i\, A_i \tau\|_{\mathcal{C}^{n+\nu\,\prime}} \le c''\, (R^\nu \|\tau\|_{\mathcal{C}^{n+2+\nu\,\prime}} + R^{-2(n+2)} \|\tau\|_\infty)$$

and

$$\|\eta_R\, c_0\, \tau\|_{\mathcal{C}^{n+\nu\,\prime}} \le c''\, (R^\nu \|\tau\|_{\mathcal{C}^{n+2+\nu\,\prime}} + R^{-2(n+2)} \|\tau\|_\infty) \quad ,$$

possibly by increasing c''. Combining these estimates it follows that there exists a $c' > 0$ such that

$$\|\eta_R(H - H_0)\tau\|_{\mathcal{C}^{n+\nu\,\prime}} \le c'\, (R^\nu \|\tau\|_{\mathcal{C}^{n+2+\nu\,\prime}} + R^{-2(n+2)} \|\tau\|_{\mathcal{C}^{n+\nu\,\prime}})$$

uniformly for all $R \in \langle 0, R_0]$ and $\tau \in \mathcal{C}^{n+2+\nu\,\prime}$.

Now let $c \ge 1$ and $\Lambda > 0$ be as in Theorem 3.2. For clarity, let $H_R^{\mathcal{C}} = H_0 + \eta_R(H - H_0)$ denote the operator with domain $\mathcal{C}^{n+2+\nu\,\prime}$ and codomain $\mathcal{C}^{n+\nu\,\prime}$. Then

$$\begin{aligned}\|\eta_R(H - H_0)\tau\|_{\mathcal{C}^{n+\nu\,\prime}} &\le c'\, (R^\nu \|\tau\|_{\mathcal{C}^{n+2+\nu\,\prime}} + R^{-2(n+2)} \|\tau\|_{\mathcal{C}^{n+\nu\,\prime}}) \\ &\le c\, c'\, (R^\nu + \lambda^{-1}\, R^{-2(n+2)}) \|(H_0 + \lambda I)\tau\|_{\mathcal{C}^{n+\nu\,\prime}}\end{aligned}$$

for all $\tau \in \mathcal{C}^{n+2+\nu\,\prime}$ and $R \in \langle 0, R_0]$. Hence if $R \le (4^{-1} c c')^{1/\nu}$ and $\lambda \ge 4cc' R^{-2(n+2)}$ then the operator $H_R^{\mathcal{C}} + \lambda I \colon \mathcal{C}^{n+2+\nu\,\prime} \to \mathcal{C}^{n+\nu\,\prime}$ is invertible and

$$\|\tau\|_{\mathcal{C}^{n+2+\nu\,\prime}} \le 2c\, \|(H_R^{\mathcal{C}} + \lambda I)\tau\|_{\mathcal{C}^{n+\nu\,\prime}}$$

for all $\tau \in C^{n+2+\nu\,\prime}$.

Now we complete the proof of the lemma. Increasing λ and decreasing R, if necessary, the operator $H_R^{(2)} + \lambda I$ where $H_R^{(2)}: L'_{2;2} \to L_2$ denotes $H_0 + \eta_R(H - H_0)$ is also invertible. Then the perturbation series for the inverses gives $(H_R^{(2)}+\lambda I)^{-1}(\lambda\varphi+\psi) = (H_R^{C}+\lambda I)^{-1}(\lambda\varphi+\psi)$ as functions on G and as $(H_R^{C} + \lambda I)^{-1}(\lambda\varphi + \psi) \in C^{n+2+\nu\,\prime}$ it remains to show that $(H_R^{(2)} + \lambda I)^{-1}(\lambda\varphi + \psi) = \varphi$. Since H_R^w is an extension of $H_R^{(2)}$ it suffices to show that $H_R^w + \lambda I$ is injective on $L'_{2;1}$ if λ is large. But this is obvious from the ellipticity estimate (8), if λ is large enough. □

The next lemma provides the induction step for the general theorem.

Lemma 3.4 *Let $n \in \mathbf{N}_0$, $\nu \in \langle 0,1\rangle$, $M,\mu > 0$ and $a,\omega > 0$. Suppose for all $H \in \mathcal{E}^{\rm nondiv}((n \vee 1) + \nu, (n \vee 1) + \nu, (n \vee 1) + \nu, \mu, M)$ one has $S_t C_c^\infty(G) \subset C^{n+1+\nu\,\prime}$ and*

$$\|S_t\varphi\|_{C^{n+1+\nu\,\prime}} \leq a\, t^{-D'/4} t^{-(n+1+\nu)/2} e^{\omega t} \|\varphi\|_{\hat{2}}$$

for all $t > 0$ and $\varphi \in C_c^\infty(G)$.

Then there exist $a',\omega' > 0$ such that for all $H \in \mathcal{E}^{\rm nondiv}((n \vee 1) + \nu, (n \vee 1) + \nu, (n \vee 1) + \nu, \mu, M)$ one has $S_t C_c^\infty(G) \subset C^{n+2+\nu\,\prime}$ and

$$\|S_t\varphi\|_{C^{n+2+\nu\,\prime}} \leq a'\, t^{-D'/4} t^{-(n+2+\nu)/2} e^{\omega' t} \|\varphi\|_{\hat{2}}$$

for all $t > 0$ and $\varphi \in C_c^\infty(G)$.

Proof By density and interpolation with the L_∞-bounds (32) of [ElR5] it follows that for all $\gamma \in \langle 0, n+1+\nu] \backslash \mathbf{N}$ there exist $a_\gamma > 0$ and $\omega_\gamma \geq 0$ such that

$$\|S_t\varphi\|_{C^{\gamma\,\prime}} \leq a_\gamma\, t^{-D'/4} t^{-\gamma/2} e^{\omega_\gamma t} \|\varphi\|_{\hat{2}}$$

for all $t > 0$ and $\varphi \in C_c^\infty(G)$.

Let $R \in \langle 0, R_0]$ and $c > 0$ be as in Lemma 3.3. Then

$$\begin{aligned} H_R^w \eta_{\sigma R} S_t\varphi &= H\eta_{\sigma R} S_t\varphi = \eta_{\sigma R} H S_t\varphi + [H, \eta_{\sigma R}] S_t\varphi \\ &= \eta_{\sigma R} S_{t/2} H S_{t/2}\varphi + \sum_{\substack{|\beta|+|\gamma|\leq 2 \\ |\gamma|\leq 1}} c_{\beta,\gamma}\, (A^\beta \eta_{\sigma R}) A^\gamma S_t\varphi \end{aligned} \tag{9}$$

weakly on $L'_{2;1}$, for some $c_{\alpha,\beta} \in C^{n+\nu\,\prime}$ with $\|c_{\alpha,\beta}\|_{C^{n+\nu\,\prime}} \leq (d'+1)M$. But the right hand side is an element of $C^{n+\nu\,\prime}$. Hence

$$\begin{aligned} \|\eta_{\sigma R} S_t\varphi\|_{C^{n+2+\nu\,\prime}} &\leq c\, \|\eta_{\sigma R} S_{t/2} H S_{t/2}\varphi\|_{C^{n+\nu\,\prime}} \\ &\quad + c \sum_{\substack{|\beta|+|\gamma|\leq 2 \\ |\gamma|\leq 1}} 2(n+1)\, b_{n+\nu} \|c_{\beta,\gamma}\, (A^\beta \eta_{\sigma R})\|_{C^{n+\nu\,\prime}} \|A^\gamma S_t\varphi\|_{C^{n+\nu\,\prime}} \\ &\quad + c\, \|\eta_{\sigma R} S_t\varphi\|_{C^{n+\nu\,\prime}} \\ &\leq c' a_{n+\nu} t^{-D'/4} t^{-(n+\nu)/2} e^{\omega_{n+\nu} t} \|H S_{t/2}\varphi\|_{\hat{2}} \\ &\quad + c' a_{n+1+\nu} t^{-D'/4} t^{-(n+1+\nu)/2} e^{\omega_{n+1+\nu} t} \|\varphi\|_{\hat{2}} \\ &\quad + c' a_{n+\nu} t^{-D'/4} t^{-(n+\nu)/2} e^{\omega_{n+\nu} t} \|\varphi\|_{\hat{2}} \\ &\leq a' t^{-D'/4} t^{-(n+2+\nu)/2} e^{\omega' t} \|\varphi\|_{\hat{2}} \end{aligned}$$

for suitable $c', a' > 0$ and $\omega' \geq 0$, where we used inequality (8) of [ElR5] in the last step.

By interpolation it follows that $\eta_{\sigma R} S_t \varphi \in L'_{\infty;n+2}$ and

$$\|\eta_{\sigma R} S_t \varphi\|'_{\infty;n+2} \leq a'' t^{-D'/4} t^{-(n+2)/2} e^{\omega'' t} \|\varphi\|_{\hat{2}}$$

for some $a'' > 0$ and $\omega'' \geq 0$. We conclude that there exist $a, \omega > 0$ such that if $H \in \mathcal{E}^{\rm nondiv}((n \vee 1) + \nu, (n \vee 1) + \nu, (n \vee 1) + \nu, \mu, M)$, $t > 0$ and $\varphi \in L_{\hat{2}}$ then $S_t \varphi$ is $(n+2)$-times differentiable on $B'(\sigma^2 R)$,

$$|(A^\alpha S_t \varphi)(e)| \leq a\, t^{-D'/4} t^{-(n+2)/2} e^{\omega t} \|\varphi\|_{\hat{2}}$$

and

$$\left|\Big((I - L(g))(A^\alpha S_t \varphi)\Big)(e)\right| \leq a\,(|g|')^\nu t^{-D'/4} t^{-(n+2+\nu)/2} e^{\omega t} \|\varphi\|_{\hat{2}}$$

uniformly for all $g \in B'(\sigma^2 R)$ and $\alpha \in J_{n+2}(d')$. Then by right translation of the coefficients of H one deduces that $S_t \varphi$ is $(n+2)$-times differentiable on G,

$$|(A^\alpha S_t \varphi)(h)| \leq a\, t^{-D'/4} t^{-(n+2)/2} e^{\omega t} \|\varphi\|_{\hat{2}}$$

and

$$\left|\Big((I - L(g))(A^\alpha S_t \varphi)\Big)(h)\right| \leq a\,(|g|')^\nu t^{-D'/4} t^{-(n+2+\nu)/2} e^{\omega t} \|\varphi\|_{\hat{2}}$$

uniformly for all $h \in G$, $g \in B'(\sigma^2 R)$, $\alpha \in J_{n+2}(d')$, $H \in \mathcal{E}^{\rm nondiv}((n \vee 1) + \nu, (n \vee 1) + \nu, (n \vee 1) + \nu, \mu, M)$, $t > 0$ and $\varphi \in L_{\hat{2}}$. Finally it is easy to extend the condition $g \in B'(\sigma^2 R)$ to $g \in B'(1)$. □

Proposition 3.5 *Let $n \in \mathbf{N}$ and $\nu \in \langle 0, 1\rangle$. Then for all $M, \mu > 0$ there exist $a, \omega > 0$ such that for all $H \in \mathcal{E}^{\rm nondiv}(n + \nu, n + \nu, n + \nu, \mu, M)$ one has $S_t C_c^\infty(G) \subset C^{n+2+\nu'}$ and*

$$\|S_t \varphi\|_{C^{n+2+\nu'}} \leq a\, t^{-D'/4} t^{-(n+2+\nu)/2} e^{\omega t} \|\varphi\|_{\hat{2}}$$

for all $t > 0$ and $\varphi \in C_c^\infty(G)$.

Proof As $\mathcal{E}^{\rm nondiv}(1 + \nu, 1 + \nu, 1 + \nu, \mu, M) \subset \mathcal{E}^{\rm div}(\nu, \nu, \nu, \nu, \mu, (d' + 1)M)$ it follows from Proposition 2.6 that $S_t C_c^\infty(G) \subset C^{1+\nu'}$ and

$$\|S_t \varphi\|_{C^{1+\nu'}} \leq a\, t^{-D'/4} t^{-(1+\nu)/2} e^{\omega t} \|\varphi\|_{\hat{2}}$$

for some $a > 0$ and $\omega \geq 0$, uniformly for all $H \in \mathcal{E}^{\rm nondiv}(1+\nu, 1+\nu, 1+\nu, \mu, M)$, $\varphi \in C_c^\infty(G)$ and $t > 0$. Then Lemma 3.4 applied with $n = 0$ gives $S_t C_c^\infty(G) \subset C^{2+\nu'}$ and

$$\|S_t \varphi\|_{C^{2+\nu'}} \leq a\, t^{-D'/4} t^{-(2+\nu)/2} e^{\omega t} \|\varphi\|_{\hat{2}}$$

for some $a > 0$ and $\omega \geq 0$, uniformly for all $H \in \mathcal{E}^{\rm nondiv}(1+\nu, 1+\nu, 1+\nu, \mu, M)$, $\varphi \in C_c^\infty(G)$ and $t > 0$. Now the proposition follows by induction from Lemma 3.4. □

4 Gaussian bounds

In this section we convert the Gaussian bounds on the kernel K and the crossnorm bounds on S from $L_{\hat{2}}$ into $C^{\gamma'}$, together with similar bounds on the adjoint semigroup S^*, into Gaussian bounds on the derivatives and Hölder derivatives of the kernel up to order γ in the first variable. The proofs of Theorems 1.2 and 1.3 then follow easily. Although the conclusions are similar to those of [ElR6] in the strongly elliptic case the proofs at this point are quite different. Bounds on semigroup kernels and their derivatives follow in the strongly elliptic case by Davies' exponential perturbation technique but this method is not applicable to the derivatives in the subelliptic situation. We begin with a brief discussion of the limitations of the perturbation method in this respect.

The technique of Davies consists of replacing S by the perturbed semigroup $S^\rho = U_\rho S U_\rho^{-1}$ where $\rho \mapsto U_\rho$ is a family of bounded multiplication operators $U_\rho\varphi = e^{-\rho\psi}\varphi$ with $\psi \in C_c^\infty(G)$. This transformation effectively replaces A_i by $A_i^\rho = A_i + \rho(A_i\psi)$. Hence if S is generated by the divergence form subelliptic operator H then S^ρ is generated by the divergence form operator H_ρ obtained from H by the replacement of A_i by A_i^ρ. Thus H_ρ has the same principal part as H but first-order terms linear in ρ and zero-order term quadratic in ρ. It is also of crucial importance for Davies' argument that the coefficients of H_ρ depend on ψ only through the derivatives $A_i\psi$. Then one obtains kernel bounds by noting that

$$e^{-\rho(\psi(g)-\psi(h))}|K_t(g\,;h)| \leq \|S_t^\rho\|_{\hat{1}\to\infty} \quad . \tag{10}$$

Bounds on the crossnorm of S^ρ are inferred from bounds on the unperturbed S by tracing the dependence of the latter on the coefficients of the generator H. The quadratic property of H_ρ as a function of ρ leads to bounds on the crossnorm of the form $a(t)\, e^{\omega(1+\rho^2)t}$ and these bounds are uniform for $\psi \in C_c^\infty(G)$ satisfying $\sum_{i=1}^{d'} |(A_i\psi)|^2 \leq 1$ because ψ only enters the estimates through its derivatives $A_i\psi$. Thus minimizing the bounds on K, resulting from (10), with respect to ρ gives

$$|K_t(g\,;h)| \leq a(t)\, e^{\omega t} e^{-|\psi(g)-\psi(h)|^2/(4\omega t)^{-1}}$$

uniformly for $\psi \in C_c^\infty(G)$ with $\sum_{i=1}^{d'} |(A_i\psi)|^2 \leq 1$. Then minimizing with respect to ψ gives the desired Gaussian bounds because the subelliptic distance is characterized by (3). In principal the same reasoning can be applied to obtain Gaussian bounds on the derivatives $A^\alpha K$ of the kernel but this requires crossnorm bounds on

$$U_\rho A^\alpha S U_\rho^{-1} = (U_\rho A^\alpha U_\rho^{-1})S^\rho = \sum_{\beta;|\beta|\leq|\alpha|} c_\beta(\rho) A^\beta S^\rho \quad .$$

Now if $|\alpha| > 1$ it no longer suffices to have crossnorm bounds on the $A^\beta S^\rho$ uniform for $\sum_{i=1}^{d'} |(A_i\psi)|^2 \leq 1$ because the coefficients $c_\beta(\rho)$ depend on higher derivatives of ψ. This would present no essential problem if the distance $d'_n(\,\cdot\,;\,\cdot\,)$ defined by

$$d'_n(g\,;h) = \sup\{\, |\psi(g) - \psi(h)| \,:\, \psi \in C_c^\infty(G)\;,\; \sup_{1\leq|\alpha|\leq n} |(A^\alpha\psi)| \leq 1\,\}$$

were equivalent to the distance $d'(\,\cdot\,;\,\cdot\,)$. This is indeed the situation if $a_1, \ldots, a_{d'}$ is a vector space basis of $\mathfrak{g}$, i.e., if one is in the strongly elliptic case, by the discussion on pages 201–202 of [Rob], but one does not have equivalence in the strictly subelliptic framework. The equivalence breaks down at short distances. An example is provided by the Heisenberg group with vector space basis $a_1, a_2, a_3 = [a_1, a_2]$ and algebraic basis a_1, a_2. If $d_1(\cdot\,;\,\cdot)$ denotes the distance with respect to the vector space basis then $d'_2(g\,;h) \leq d_1(g\,;h) \leq d'_1(g\,;h)$ for all $g, h \in G$ and the equivalence of $d'_2(\cdot\,;\,\cdot)$ and $d'_1(\cdot\,;\,\cdot)$ would imply that $d'_1(\cdot\,;\,\cdot)$ and $d_1(\cdot\,;\,\cdot)$ are equivalent. But this is a clear contradiction. Therefore the perturbation approach is not directly applicable to the discussion of higher order smoothness properties.

The alternative method we introduce is a simple form of interpolation. The input data for the interpolation consists of the Gaussian bounds on the kernel provided by Theorem 1.1 and uniform bounds on the derivatives of the kernel stemming from the crossnorm bounds on S and S^*.

Proposition 4.1 *Let H be a subelliptic operator in divergence form or non-divergence form. Fix $N, N^* \in \mathbf{N}_0$ and $\nu, \nu^* \in \langle 0,1\rangle$. Let $a, b > 0$ and $\omega \geq 0$. Suppose $S_t C_c^\infty(G) \subset C^{N+\nu\,\prime}$, $S_t^* C_c^\infty(G) \subset C^{N^*+\nu^*\,\prime}$ and*

$$\|S_t\varphi\|_{C^{N+\nu\,\prime}} \leq a\, t^{-D'/4} t^{-(N+\nu)/2} e^{\omega t} \|\varphi\|_{\hat{2}}$$

$$\|S_t^*\varphi\|_{C^{N^*+\nu^*\,\prime}} \leq a\, t^{-D'/4} t^{-(N^*+\nu^*)/2} e^{\omega t} \|\varphi\|_{\hat{2}}$$

for all $t > 0$ and $\varphi \in C_c^\infty(G)$. Moreover, suppose

$$|K_t(g\,;h)| \leq a\, t^{-D'/2} e^{\omega t} e^{-b(|gh^{-1}|')^2 t^{-1}} \tag{11}$$

uniformly for all $g, h \in G$ and $t > 0$.

Then the kernel of the semigroup S generated by H is N-times differentiable in the first variable, the derivatives with respect to the first are N^*-times differentiable with respect to the second and the derivatives are continuous. Moreover, for all $\kappa > 0$, $\tau \in \langle 0,1\rangle$, $\gamma, \gamma^* \in \langle 0,1\rangle$ and $\alpha, \beta \in J(d')$ with $|\alpha| + \gamma < N + \nu$ and $|\beta| + \gamma^* < N^* + \nu^*$ there exist $a', b' > 0$ and $\omega' \geq 0$ such that

$$|(A^\alpha B^\beta K_t)(g\,;h)| \leq a'\, t^{-(D'+|\alpha|+|\beta|)/2} e^{\omega' t} e^{-b'(|gh^{-1}|')^2 t^{-1}} \tag{12}$$

and

$$\begin{aligned}&|(A^\alpha B^\beta K_t)(k^{-1}g\,;l^{-1}h) - (A^\alpha B^\beta K_t)(g\,;h)|\\ &\quad\leq a'\, t^{-(D'+|\alpha|+|\beta|)/2} e^{\omega' t} \left(\left(\frac{|k|'}{t^{1/2} + |gh^{-1}|'} \right)^\gamma + \left(\frac{|l|'}{t^{1/2} + |gh^{-1}|'} \right)^{\gamma^*} \right) e^{-b'(|gh^{-1}|')^2 t^{-1}}\end{aligned} \tag{13}$$

uniformly for all $t > 0$, $g, h \in G$ and $k, l \in G$ such that $|k|' + |l|' \leq \kappa t^{1/2} + \tau |gh^{-1}|'$.

The constants a', b' and ω' depend only on N, N^*, ν, ν^*, a, b, ω, κ, τ, α, β, γ and γ^*.

Proof It follows as in the proof of Proposition 3.5 in [ElR6] that the kernel K is N-times differentiable in the first variable, the derivatives with respect to the first are N^*-times differentiable with respect to the second and the derivatives are continuous. Moreover, one has the following uniform bounds. For all $\gamma, \gamma^* \in \langle 0,1\rangle$ and $\alpha, \beta \in J(d')$ with $|\alpha| + \gamma \leq N + \nu$ and $|\beta| + \gamma^* \leq N^* + \nu^*$ there exist $a', b' > 0$ and $\omega' \geq 0$ such that

$$\begin{aligned}&\Big|(A^\alpha B^\beta K_t)(k^{-1}g\,;l^{-1}h) - (A^\alpha B^\beta K_t)(g\,;h)\Big|\\ &\quad\leq a' \Big((|k|' t^{-1/2})^\gamma (|l|' t^{-1/2})^{\gamma^*} + (|k|' t^{-1/2})^\gamma + (|l|' t^{-1/2})^{\gamma^*} \Big) t^{-(D'+|\alpha|+|\beta|)/2} e^{\omega' t}\end{aligned} \tag{14}$$

uniformly for all $g, h, k, l \in G$ and $t > 0$.

Now we come to the interpolation argument. It consists of two basic steps. The first step is to interpolate between Gaussian bounds on the kernel and Hölder bounds which are uniform over $G \times G$ to obtain Gaussian bounds on the Hölder derivatives.

The next lemma gives an abstract version of the simplest result of this nature. It interpolates between Gaussian bounds and uniform Hölder bounds, to give Gaussian Hölder bounds.

Lemma 4.2 *Let $\nu, \nu^* \in \langle 0,1\rangle$, $b > 0$, $\gamma \in \langle 0, \nu\rangle$, $\gamma^* \in \langle 0, \nu^*\rangle$, $\kappa > 0$ and $\tau \in \langle 0,1\rangle$. Then there exist $b' > 0$ and $M > 0$ such that for every $t > 0$, $a > 0$ and every function $\Phi: G \times G \to \mathbf{C}$ satisfying*

$$|\Phi(g\,;h)| \le a\, e^{-b(|gh^{-1}|')^2 t^{-1}}$$

and

$$|\Phi(k^{-1}g\,;l^{-1}h) - \Phi(g\,;h)| \le a\left((|k|'t^{-1/2})^{\nu}(|l|'t^{-1/2})^{\nu^*} + (|k|'t^{-1/2})^{\nu} + (|l|'t^{-1/2})^{\nu^*}\right)$$

uniformly for all $g, h, k, l \in G$ one has the intermediate bounds

$$|\Phi(k^{-1}g\,;l^{-1}h) - \Phi(g\,;h)| \le a\,M\left(\left(\frac{|k|'}{t^{1/2} + |gh^{-1}|'}\right)^{\gamma} + \left(\frac{|l|'}{t^{1/2} + |gh^{-1}|'}\right)^{\gamma^*}\right) e^{-b'(|gh^{-1}|')^2 t^{-1}}$$

uniformly for all $g, h \in G$ and $k, l \in G$ such that $|k|' + |l|' \le \kappa\, t^{1/2} + \tau |gh^{-1}|'$.

Proof First, with $\kappa > 0$ and $\tau \in \langle 0,1\rangle$ fixed we argue that the assumed Gaussian bounds give bounds

$$|\Phi(k^{-1}g\,;l^{-1}h) - \Phi(g\,;h)| \le a\,M_1\, e^{-b'(|gh^{-1}|')^2 t^{-1}} \tag{15}$$

uniformly for all $g, h \in G$ and $k, l \in G$ such that $|k|' + |l|' \le \kappa\, t^{1/2} + \tau |gh^{-1}|'$. Secondly, we establish that for each $\delta > 0$ the uniform Hölder bounds yield estimates

$$\begin{aligned} &|\Phi(k^{-1}g\,;l^{-1}h) - \Phi(g\,;h)| \\ &\qquad \le a\,M_\delta\left(\left(\frac{|k|'}{t^{1/2} + |gh^{-1}|'}\right)^{\gamma} + \left(\frac{|l|'}{t^{1/2} + |gh^{-1}|'}\right)^{\gamma^*}\right) e^{2\delta(|gh^{-1}|')^2 t^{-1}} \end{aligned} \tag{16}$$

Thirdly, setting $\theta = \max(\gamma\nu^{-1}, \gamma^*(\nu^*)^{-1})$ and choosing $\delta = 4^{-1}\theta^{-1}(1-\theta)b'$ one has

$$\begin{aligned} &|\Phi(k^{-1}g\,;l^{-1}h) - \Phi(g\,;h)| \\ &\quad = |\Phi(k^{-1}g\,;l^{-1}h) - \Phi(g\,;h)|^{1-\theta}\, |\Phi(k^{-1}g\,;l^{-1}h) - \Phi(g\,;h)|^{\theta} \\ &\quad \le a\, M_1^{1-\theta}\, M_\delta^{\theta}\left(\left(\frac{|k|'}{t^{1/2} + |gh^{-1}|'}\right)^{\theta\nu} + \left(\frac{|l|'}{t^{1/2} + |gh^{-1}|'}\right)^{\theta\nu^*}\right) e^{-b''(|gh^{-1}|')^2 t^{-1}} \end{aligned}$$

where $b'' = b'(1-\theta) - 2\delta\theta > 0$. Now the lemma follows easily. Thus it remains to prove (15) and (16).

If $g, h, k, l \in G$ and $|k|' + |l|' \le \kappa\, t^{1/2} + \tau |gh^{-1}|'$ then

$$|\Phi(k^{-1}g\,;l^{-1}h)| \le a\, e^{-b(|k^{-1}gh^{-1}l|')^2 t^{-1}} \quad .$$

But

$$(|gh^{-1}|')^2 \le (|k^{-1}gh^{-1}l|' + |k|' + |l|')^2 \le (|k^{-1}gh^{-1}l|' + \kappa t^{1/2} + \tau |gh^{-1}|')^2$$
$$\le \tau^2(1+\varepsilon)(|gh^{-1}|')^2 + 2(1+\varepsilon^{-1})((|k^{-1}gh^{-1}l|')^2 + \kappa^2 t)$$

for all $\varepsilon > 0$. Now set $\varepsilon = (1-\tau^2)(1+\tau^2)^{-1}$. Then it follows that

$$-(|k^{-1}gh^{-1}l|')^2 \le -2^{-1}\varepsilon(1+\varepsilon^{-1})^{-1}(|gh^{-1}|')^2 + \kappa^2 t \quad .$$

So

$$|\Phi(k^{-1}g\,;l^{-1}h) - \Phi(g\,;h)| \le a\,M_1\,e^{-b'(|gh^{-1}|')^2 t^{-1}} \quad ,$$

where $M_1 = 1 + e^{b\kappa^2}$ and $b' = 2^{-1}\varepsilon(1+\varepsilon^{-1})^{-1}b$. Thus (15) is valid.

Next, for all $\delta > 0$ one has

$$|gh^{-1}|'\,t^{-1/2} \le a_\delta\, e^{\delta(|gh^{-1}|')^2 t^{-1}}$$

where $a_\delta = 2^{-1}\delta^{-1/2}$. Therefore

$$t^{-1/2}(t^{1/2} + |gh^{-1}|') = 1 + |gh^{-1}|' t^{-1/2} \le (1+a_\delta)e^{\delta(|gh^{-1}|')^2 t^{-1}}$$

and hence

$$t^{-1/2} \le (1+a_\delta)\frac{1}{t^{1/2} + |gh^{-1}|'}\,e^{\delta(|gh^{-1}|')^2 t^{-1}} \quad .$$

Then

$$(|k|'t^{-1/2})^\nu + (|l|'t^{-1/2})^{\nu^*} \le (1+a_\delta)\left(\left(\frac{|k|'}{t^{1/2}+|gh^{-1}|'}\right)^\nu + \left(\frac{|l|'}{t^{1/2}+|gh^{-1}|'}\right)^{\nu^*}\right)e^{\delta(|gh^{-1}|')^2 t^{-1}}$$

and

$$(|k|'t^{-1/2})^\nu(|l|'t^{-1/2})^{\nu^*} \le (1+a_\delta)^2\left(\frac{|k|'}{t^{1/2}+|gh^{-1}|'}\right)^\nu\left(\frac{|l|'}{t^{1/2}+|gh^{-1}|'}\right)^{\nu^*} e^{2\delta(|gh^{-1}|')^2 t^{-1}}$$
$$\le (1+a_\delta)^2(\kappa+\tau)^\nu\left(\frac{|l|'}{t^{1/2}+|gh^{-1}|'}\right)^{\nu^*} e^{2\delta(|gh^{-1}|')^2 t^{-1}} \quad .$$

Combination of these estimates with the assumed uniform Hölder bound immediately yields (16) with $M_\delta = (1+a_\delta) + (1+a_\delta)^2(\kappa+\tau)^\nu$. □

The second lemma interpolates between Gaussian Hölder bounds and uniform Hölder bounds on a derivative to give Gaussian bounds on the derivative. The interpolation is based on the Taylor series of first-order.

Lemma 4.3 *Let $\nu,\gamma \in \langle 0,1\rangle$, $b > 0$ and $i \in \{1,\ldots,d'\}$. Then there exists $b' > 0$ such that for every $t > 0$, $a > 0$ and every function $\Phi\colon G\times G \to \mathbf{C}$ which is pointwise partially differentiable in the first variable in the direction a_i and satisfies bounds*

$$|\Phi(k^{-1}g\,;h) - \Phi(g\,;h)| \le a\,(|k|'t^{-1/2})^\gamma\, e^{-b(|gh^{-1}|')^2 t^{-1}}$$

and

$$\Big|(A_i\Phi)(k^{-1}g\,;h) - (A_i\Phi)(g\,;h)\Big| \le a\,t^{-1/2}(|k|'t^{-1/2})^\nu$$

uniformly for all $g, h, k \in G$ with $|k|' \leq t^{1/2}$ one has the intermediate bounds

$$|(A_i\Phi)(g\,;h)| \leq 2a\,t^{-1/2}e^{-b'(|gh^{-1}|')^2t^{-1}}$$

uniformly for all $g, h \in G$.

A similar statement is valid for derivatives in the second variable.

Proof The Duhamel formula,

$$\Big((I - L(\exp(sa_i))\varphi\Big)(g) = -\int_0^s du\,\Big(L(\exp(ua_i)\,A_i\varphi)\Big)(g)$$
$$= -s\,(A_i\varphi)(g) + \int_0^s du\,\Big((I - L(\exp(ua_i))\,A_i\varphi)\Big)(g) \quad ,$$

gives a general form of the first-order Taylor expansion valid for all $s > 0$ and any function φ continuously differentiable in the direction a_i. Consequently

$$(A_i\varphi)(g) = -s^{-1}\Big((I - L(\exp(sa_i))\varphi\Big)(g) + s^{-1}\int_0^s du\,\Big((I - L(\exp(ua_i))\,A_i\varphi)\Big)(g) \quad .$$

Therefore setting $\varphi(g) = \Phi(g\,;h)$ and using the estimate $|\exp(ua_i)|' \leq u$ one finds

$$|(A_i\Phi)(g\,;h)| \leq s^{-1}\,a\,(s\,t^{-1/2})^{\gamma}\,e^{-b(|gh^{-1}|')^2t^{-1}} + s^{-1}\int_0^s du\,a\,t^{-1/2}(u\,t^{-1/2})^{\nu}$$
$$= a\,t^{-1/2}\Big((s\,t^{-1/2})^{-1+\gamma}e^{-b(|gh^{-1}|')^2t^{-1}} + (1+\nu)^{-1}(s\,t^{-1/2})^{\nu}\Big)$$

for all $s \in \langle 0, t^{1/2}]$. Now set $s = t^{1/2}e^{-b(1+\nu-\gamma)^{-1}(|gh^{-1}|')^2t^{-1}}$. Then

$$|(A_i\Phi)(g\,;h)| \leq a\,(1 + (1+\nu)^{-1})\,t^{-1/2}e^{-b'(|gh^{-1}|')^2t^{-1}}$$

where $b' = b\nu(1+\nu-\gamma)^{-1}$ and the proof of the lemma is complete. □

Proposition 4.1 is now follows by iteration of the conclusions of these lemmas.

End of the proof of Proposition 4.1. First, using the bounds (11) and (14) with $|\alpha| = |\beta| = 0$, $\gamma = \nu$ and $\gamma^* = \nu^*$ one deduces from Lemma 4.2 that the bounds (13) are valid for $|\alpha| = |\beta| = 0$ and all $\gamma \in \langle 0, \nu\rangle$ and $\gamma^* \in \langle 0, \nu^*\rangle$. Next, for all $n \in \mathbf{N}_0$ let $P(n)$ be the assertion

> The bounds (12) and (13) are valid for all $\alpha \in J_n(d')$, $|\beta| = 0$, $\gamma \in \langle 0, \nu\rangle$ and $\gamma^* \in \langle 0, \nu^*\rangle$.

Then $P(0)$ is valid. Now let $n \in \{0, \ldots, N-1\}$ and suppose $P(n)$ is valid. If $\alpha \in J_n(d')$ with $|\alpha| = n$, $i \in \{1, \ldots, d'\}$ and $|\beta| = 0$ then one can apply Lemma 4.3 with $\Phi = A^{\alpha}K_t$ and it follows that the bounds (12) are valid for all $\alpha \in J_{n+1}(d')$ and $|\beta| = 0$. Then the assertion $P(n+1)$ follows by applying Lemma 4.2 on $\Phi = A^{\alpha}K_t$ with $|\alpha| = n+1$.

By induction it follows that the assertion $P(N)$ is valid. But then the bounds (13) for $|\alpha| \leq N-1$, $|\beta| = 0$, $\gamma \in \langle 0, 1\rangle$ and $\gamma^* \in \langle 0, \nu^*\rangle$ follow from Lemma 4.2 applied to $A^{\alpha}K_t$, the bounds (12) and the bounds (14).

Finally, for all $n \in \mathbf{N}_0$ let $Q(n)$ be the assertion

The bounds (12) and (13) are valid for all $\alpha \in J_N(d')$, $\beta \in J_n(d')$, $\gamma \in \langle 0,1\rangle$ and $\gamma^* \in \langle 0,\nu^*\rangle$ with $|\alpha|+\gamma < N+\nu$.

Then we have just proved that $Q(0)$ is valid and one proves by induction, as above, that $Q(N^*)$ is valid. Then the proposition follows readily. □

Proof of Theorem 1.2. This follows immediately from Propositions 2.6 and 4.1. □

Proof of Theorem 1.3. This follows immediately from Propositions 3.5, 2.6 and 4.1, except the very last statement, which can be established separately by a similar argument. □

5 Miscellany

In this section we discuss two related topics. First we state the optimal regularity for the fractional powers of H on the regular L_p-spaces and secondly we give an improvement of Theorem 1.3 for strongly elliptic operators.

5.1 Regularity

The aim of this subsection is to characterize the domain of the fractional power of a subelliptic operators on L_p, with $1<p<\infty$. If $H_L = -\sum_{i=1}^{d'} A_i^2$ denotes the sublaplacian then $L'_{p;n} = D((I+H_L)^{n/2})$ for all $n \in \mathbf{N}_0$ and $p \in \langle 1,\infty\rangle$, with equivalent norms (see [BER]). Therefore we define $L'_{p;\gamma} = D((I+H_L)^{\gamma/2})$ for all $\gamma \in \langle 0,\infty\rangle\backslash\mathbf{N}$, equipped with the graph norm.

We first consider operators in divergence form and then in non-divergence form. The best results are for $p=2$.

Theorem 5.1 *Let H be a subelliptic operator in divergence form* (1). *If $\nu \in \langle 0,1\rangle$, $\gamma \in [0,1+\nu\rangle$, $c_{ij}, c'_i \in C^{\nu\,\prime}$, and $c_i, c_0 \in L_\infty$ then $D((\lambda I+H)^{\gamma/2}) = L'_{2;\gamma}$ for large λ.*

Moreover, for all $\nu \in \langle 0,1\rangle$, $M,\mu>0$ there exists a $\lambda_0>0$ such that for all $\gamma \in [0,1+\nu\rangle$ and $\lambda \geq \lambda_0$ there exists an $a>0$ such that $D((\lambda I+H)^{\gamma}) = L'_{2;\gamma}$ and

$$a^{-1}\|(\lambda I+H)^{\gamma}\varphi\|_2 \leq \|\varphi\|'_{2;\gamma} \leq a\,\|(\lambda I+H)^{\gamma}\varphi\|_2$$

uniformly for all $H \in \mathcal{E}^{\mathrm{div}}(\nu,0,\nu,0,\mu,M)$ and $\varphi \in L'_{2;\gamma}$.

Similar conclusions are valid on the L_2-spaces.

Proof The proof is precisely the same as in [ElR6] Corollary 4.5, since we now have the various kernel bounds for subelliptic operators. □

Using interpolation one can then extend this theorem to other L_p-spaces for $\gamma \leq 1$ as in Corollary 4.8 in [ElR6]

Theorem 5.2 *Let H be a subelliptic operator in divergence form* (1). *If $\nu \in \langle 0,1\rangle$, $c_{ij}, c_i, c'_i \in C^{\nu\,\prime}$, and $c_0 \in L_\infty$ then $D((\lambda I+H)^{\gamma/2}) = L'_{p;\gamma}$ for all large λ, $\gamma \in [0,1]$ and $p \in \langle 1,\infty\rangle$.*

Moreover, for all $\nu \in \langle 0,1\rangle$, $M,\mu > 0$ there exists a $\lambda_0 > 0$ such that for all $p \in \langle 1,\infty\rangle$, $\lambda \geq \lambda_0$ and $\gamma \in [0,1]$ there exists an $a > 0$ such that $D((\lambda I + H)^{\gamma/2}) = L'_{p;\gamma}$ and

$$a^{-1}\|(\lambda I + H)^{\gamma}\varphi\|_p \leq \|\varphi\|'_{p;\gamma} \leq a\,\|(\lambda I + H)^{\gamma}\varphi\|_p$$

uniformly for all $H \in \mathcal{E}^{\mathrm{div}}(\nu,\nu,\nu,0,\mu,M)$ and $\varphi \in L'_{p;\gamma}$.

Similar conclusions are valid on the $L_{\hat{p}}$-spaces.

Next we consider non-divergence form operators on L_2.

Proposition 5.3 *Let H be a second-order subelliptic operator in non-divergence form* (4) *with $c_{ij}, c_i, c_0 \in \mathcal{C}^{n+\nu\,\prime}(G)$ for some $n \in \mathbf{N}$ and $\nu \in \langle 0,1\rangle$. If $\gamma \in [0, n+2+\nu\rangle$ then $D((\lambda I + H)^{\gamma/2}) \subseteq L'_{2;\gamma}$ for all large λ, and the embedding is continuous.*

Moreover, for all $M,\mu > 0$ there exists a $\lambda_0 > 0$ such that for all $\gamma \in [0, n+2+\nu\rangle$ there exists an $a > 0$ such that

$$\|\varphi\|'_{2;\gamma} \leq a\,\|(\lambda I + H)^{\gamma/2}\varphi\|_2$$

uniformly for all $H \in \mathcal{E}^{\mathrm{nondiv}}(n+\nu, n+\nu, n+\nu, \mu, M)$, $\lambda \geq \lambda_0$ and $\varphi \in D((\lambda I + H)^{\gamma/2})$.

Similar conclusions are valid on $L_{\hat{2}}$.

Proof The proof of this proposition is almost the same as the proof of Theorem 4.4 in [ElR6]. □

The converse inclusion is valid for all regular L_p-spaces.

Proposition 5.4 *Let H be a second-order subelliptic operator in non-divergence form* (4) *with $c_{ij}, c_i, c_0 \in \mathcal{C}^{n+\nu\,\prime}(G)$ for some $n \in \mathbf{N}$ and $\nu \in \langle 0,1\rangle$. If $\gamma \in [0, n+2+\nu\rangle$ then $L'_{p;\gamma} \subseteq D((\lambda I + H)^{\gamma/2})$ for all $p \in \langle 1,\infty\rangle$ and all large λ, and the embedding is continuous.*

Moreover, for all $M,\mu > 0$ there exists a $\lambda_0 > 0$ such that for all $\gamma \in [0, n+2+\nu\rangle$, $\lambda \geq \lambda_0$ and $p \in \langle 1,\infty\rangle$ there exists an $a > 0$ such that

$$\|(\lambda I + H)^{\gamma/2}\varphi\|_p \leq a\,\|\varphi\|'_{p;\gamma}$$

uniformly for all $H \in \mathcal{E}^{\mathrm{nondiv}}(n+\nu, n+\nu, n+\nu, \mu, M)$ and $\varphi \in L'_{p;\gamma}$.

Similar conclusions are valid on $L_{\hat{p}}$.

Proof If n is even then obviously $L'_{p;n+2} \subset D(H^{(n+2)/2})$ and if n is odd then $L'_{p;n+1} \subset D(H^{(n+1)/2})$ for all $p \in [1,\infty]$. So by interpolation one has $L'_{p;n+\delta} \subset D((\lambda I + H)^{(n+\delta)/2})$ if λ is large enough, $p \in \langle 1,\infty\rangle$ and $\delta \in \langle 0,1\rangle$. Now the proposition can be proved by the argument at the end of the proof of Theorem 6.4 in [ElR6]. □

Corollary 5.5 *Let H be a second-order subelliptic operator in non-divergence form* (4) *with $c_{ij}, c_i, c_0 \in \mathcal{C}^{n+\nu\,\prime}(G)$ for some $n \in \mathbf{N}$ and $\nu \in \langle 0,1\rangle$. If $\gamma \in [0, n+2+\nu\rangle$ then $L'_{2;\gamma} = D((\lambda I + H)^{\gamma/2})$ for all $p \in \langle 1,\infty\rangle$ and all large λ, with equivalent norms.*

Theorem 5.6 *Let H be a second-order subelliptic operator in non-divergence form (4) with $c_{ij}, c_i, c_0 \in C^{n+\nu\,\prime}(G)$ for some $n \in \mathbf{N}$ and $\nu \in \langle 0,1\rangle$. If $\gamma \in [0, n+2]$ then $L'_{p;\gamma} = D((\lambda I + H)^{\gamma/2})$ for all $p \in \langle 1, \infty\rangle$ and all large λ, with equivalent norms.*

Moreover, for all $M, \mu > 0$ there exists a $\lambda_0 > 0$ such that for all $\gamma \in [0, n+2]$, $\lambda \geq \lambda_0$ and $p \in \langle 1, \infty\rangle$ there exists an $a > 0$ such that

$$a^{-1} \|(\lambda I + H)^{\gamma/2}\varphi\|_p \leq \|\varphi\|'_{p;\gamma} \leq a\,\|(\lambda I + H)^{\gamma/2}\varphi\|_p$$

uniformly for all $H \in \mathcal{E}^{\mathrm{nondiv}}(n+\nu, n+\nu, n+\nu, \mu, M)$ and $\varphi \in L'_{p;\gamma}$.

Similar conclusions are valid on $L_{\hat{p}}$.

Proof The inclusion $D((\lambda I + H)^{(n+2)/2}) \subseteq L'_{p;n+2}$ follows as in [BER], since we have the appropriate kernel bounds (see also the proof of Theorem 4.6 in [ElR6]). Then the inclusions $D((\lambda I + H)^{\gamma/2}) \subseteq L'_{p;\gamma}$ for $\gamma \in [0, n+2]$ follow by interpolation. The bounds are a consequence of the proof. □

For strongly elliptic operators one has $D((\lambda I + H)^{(n+2)/2}) = L_{p;n+2}$ whenever $c_{ij}, c_i, c_0 \in L_{\infty;n}$ (see [ElR6] Corollary 5.2). Moreover, in the strongly elliptic case the condition $\gamma \in [0, n+2]$ in Theorem 5.6 can be weakened to $\gamma \in [0, n+2+\nu\rangle$. It is unclear whether similar statements are valid in the subelliptic case.

5.2 Strongly elliptic non-divergence operators

For divergence form operators with $C^{\nu\,\prime}$-coefficients we have proved in Theorem 1.2 that the kernel satisfies $C^{1+\nu-\varepsilon\,\prime}$-Gaussian bounds for all $\varepsilon > 0$ and for non-divergence form operators with $C^{n+\nu\,\prime}$-coefficients the kernel satisfies $C^{n+2+\nu-\varepsilon\,\prime}$-Gaussian bounds in the first variable and $C^{n+\nu-\varepsilon\,\prime}$-Gaussian bounds in the second variable. If the operator is strongly elliptic and in divergence form with C^{ν}-coefficients one can take $\varepsilon = 0$, i.e., the kernel satisfies $C^{1+\nu}$-Gaussian bounds (see Theorem 1.1 in [ElR6]). We next show that one can also take $\varepsilon = 0$ in the strongly elliptic non-divergence form case.

For $\psi \in C_c^\infty(G)$, $\rho \in \mathbf{R}$ and elliptic operator H let S^ρ be the perturbed semigroup generated by the perturbed operator H_ρ by Davies' technique (see Section 4). The main step in the proof is a perturbed version of Lemma 3.4.

Lemma 5.7 *Suppose $a_1, \ldots, a_{d'}$ is a vector space basis of $\mathfrak{g}$. Let $n \in \mathbf{N}_0$, $\nu \in \langle 0,1\rangle$, $M, \mu > 0$ and $a, \omega > 0$. Suppose for all $H \in \mathcal{E}^{\mathrm{nondiv}}((n\vee 1)+\nu, (n\vee 1)+\nu, (n\vee 1)+\nu, \mu, M)$ one has*

$$\|S_t^\rho \varphi\|_{C^{n+1+\nu}} \leq a\, t^{-d/4} t^{-(n+1+\nu)/2} e^{\omega(1+\rho^2)t} \|\varphi\|_{\hat{2}}$$

for all $t > 0$, $\varphi \in C_c^\infty(G)$, $\rho \in \mathbf{R}$ and $\psi \in C_c^\infty(G)$ with $\|A^\gamma \psi\|_\infty \leq 1$ for all $\gamma \in J(d)$ with $1 \leq |\gamma| \leq n+2$.

Then there exist $a', \omega' > 0$ such that for all $H \in \mathcal{E}^{\mathrm{nondiv}}((n\vee 1)+\nu, (n\vee 1)+\nu, (n\vee 1)+\nu, \mu, M)$ one has

$$\|S_t^\rho \varphi\|_{C^{n+2+\nu}} \leq a'\, t^{-d/4} t^{-(n+2+\nu)/2} e^{\omega'(1+\rho^2)t} \|\varphi\|_{\hat{2}}$$

for all $t > 0$, $\varphi \in C_c^\infty(G)$, $\rho \in \mathbf{R}$ and $\psi \in C_c^\infty(G)$ with $\|A^\gamma \psi\|_\infty \leq 1$ for all $\gamma \in J(d)$ with $1 \leq |\gamma| \leq n+3$.

Proof We follow the proof of Lemma 3.4. If $\psi \in C_c^\infty(G)$, $\rho \in \mathbf{R}$, $t > 0$ and $\varphi \in C_c^\infty(G)$ then one has as in (9)

$$H_R^w \eta_{\sigma R} S_t^\rho \varphi = H \eta_{\sigma R} S_t^\rho \varphi = \eta_{\sigma R} H S_t^\rho \varphi + [H, \eta_{\sigma R}] S_t^\rho \varphi$$

$$= \eta_{\sigma R} S_{t/2}^\rho H_\rho S_{t/2}^\rho \varphi + \sum_{\substack{|\beta|+|\gamma|\leq 2 \\ |\gamma|\leq 1}} c_{\beta,\gamma} (A^\beta \eta_{\sigma R}) A^\gamma S_t^\rho \varphi$$

$$+ \rho\, \eta_{\sigma R} \sum_{i,j=1}^d c_{ij}\, A_i\, \psi_j\, S_t^\rho \varphi + \rho\, \eta_{\sigma R} \sum_{i,j=1}^d c_{ij}\, \psi_i\, A_j\, S_t^\rho \varphi$$

$$+ \rho^2\, \eta_{\sigma R} \sum_{i,j=1}^d c_{ij}\, \psi_i\, \psi_j\, S_t^\rho \varphi - \rho\, \eta_{\sigma R} \sum_{i=1}^d c_i\, \psi_i\, S_t^\rho \varphi$$

strongly, and hence weakly on $L_{2;1}$, where $\psi_i = A_i \psi$. Then we apply Lemma 3.3. The contribution of the first two terms is as before, one only has to replace the factor $e^{\omega t}$ by $e^{\omega(1+\rho^2)t}$. Next,

$$\|\rho\, \eta_{\sigma R} \sum_{i,j=1}^d c_{ij}\, A_i\, \psi_j\, S_t^\rho \varphi\|_{C^{n+\nu}} \leq c'\, |\rho| \sum_{i,j=1}^d \|\psi_j\|_{\infty;n+2} \|S_t^\rho \varphi\|_{C^{n+1+\nu}}$$

$$\leq c''\, t^{-1/2} e^{\omega'(1+\rho^2)t} t^{-d/4} t^{-(n+1+\nu)/2} e^{\omega'(1+\rho^2)t} \|\varphi\|_{\hat{2}}$$

$$= c''\, t^{-d/4} t^{-(n+2+\nu)/2} e^{2\omega'(1+\rho^2)t} \|\varphi\|_{\hat{2}}$$

for suitable c', c'' and ω', depending only on n, ν, M and μ. The other three terms can be estimated similarly. Hence it follows from Lemma 3.3 that

$$\|\eta_{\sigma R} S_t^\rho \varphi\|_{C^{n+2+\nu}} \leq a'\, t^{-d/4} t^{-(n+2+\nu)/2} e^{\omega' t} \|\varphi\|_{\hat{2}} \quad .$$

Arguing as before in the proof of Lemma 3.4 it follows that there exist $a, \omega > 0$ such that for all $H \in \mathcal{E}^{\mathrm{nondiv}}((n \vee 1) + \nu, (n \vee 1) + \nu, (n \vee 1) + \nu, \mu, M)$, $t > 0$ and $\varphi \in L_{\hat{2}}$ one has

$$|(A^\alpha e^{-\rho\psi} S_t e^{\rho\psi} \varphi)(e)| \leq a\, t^{-d/4} t^{-(n+2)/2} e^{\omega(1+\rho^2)t} \|\varphi\|_{\hat{2}}$$

and

$$\Big|\Big((I - L(g))(A^\alpha e^{-\rho\psi} S_t e^{\rho\psi} \varphi)\Big)(e)\Big| \leq a\, |g|^\nu t^{-d/4} t^{-(n+2+\nu)/2} e^{\omega(1+\rho^2)t} \|\varphi\|_{\hat{2}}$$

uniformly for all $g \in B(\sigma^2 R)$, $\alpha \in J_{n+2}(d)$, $\rho \in \mathbf{R}$ and $\psi \in C_c^\infty(G)$ with $\|A^\gamma \psi\|_\infty \leq 1$ for all $\gamma \in J(d)$ with $1 \leq |\gamma| \leq n+3$, where we have written the function ψ explicitly in the perturbed semigroup. Next, let $H \in \mathcal{E}^{\mathrm{nondiv}}((n \vee 1) + \nu, (n \vee 1) + \nu, (n \vee 1) + \nu, \mu, M)$, $t > 0$, $h \in G$, $g \in B(\sigma^2 R)$, $\alpha \in J_{n+2}(d)$, $\varphi \in L_{\hat{2}}$, $\rho \in \mathbf{R}$ and $\psi \in C_c^\infty(G)$ with $\|A^\gamma \psi\|_\infty \leq 1$ for all $\gamma \in J(d)$ with $1 \leq |\gamma| \leq n+3$. Then $\|A^\gamma R(h)\psi\|_\infty \leq 1$ for all $\gamma \in J(d)$ with $1 \leq |\gamma| \leq n+3$. If $H^{(h)}$ is the non-divergence form operator with coefficients $R(h)c_{ij}$, $R(h)c_i$ and $R(h)c_0$ obtained by right translation of the coefficients of H, then $H^{(h)} \in \mathcal{E}^{\mathrm{nondiv}}((n \vee 1) + \nu, (n \vee 1) + \nu, (n \vee 1) + \nu, \mu, M)$. So with $S^{(h)}$ the semigroup generated by $H^{(h)}$ one deduces that

$$\Big|\Big((I - L(g))(A^\alpha e^{-\rho\psi} S_t e^{\rho\psi} \varphi)\Big)(h)\Big| = \Big|\Big((I - L(g))(A^\alpha e^{-\rho R(h)\psi} S_t^{(h)} e^{\rho R(h)\psi} R(h)\varphi)\Big)(e)\Big|$$

$$\leq a\, |g|^\nu t^{-d/4} t^{-(n+2+\nu)/2} e^{\omega(1+\rho^2)t} \|R(h)\varphi\|_{\hat{2}}$$

$$= a\, |g|^\nu t^{-d/4} t^{-(n+2+\nu)/2} e^{\omega(1+\rho^2)t} \|\varphi\|_{\hat{2}} \quad .$$

Similarly,

$$|(A^\alpha e^{-\rho\psi} S_t e^{\rho\psi}\varphi)(h)| \leq a\, t^{-d/4} t^{-(n+2)/2} e^{\omega(1+\rho^2)t} \|\varphi\|_2$$

and one can extend the condition $g \in B(\sigma^2 R)$ to $g \in B(1)$. □

For sake of completeness and reference we quote the next proposition.

Proposition 5.8 *Suppose $a_1, \ldots, a_{d'}$ is a vector space basis of $\mathfrak{g}$. Let $n \in \mathbf{N}$ and $\nu \in \langle 0,1 \rangle$. Then for all $M, \mu > 0$ there exist $a, \omega > 0$ such that for all $H \in \mathcal{E}^{\rm nondiv}(n+\nu, n+\nu, n+\nu, \mu, M)$ one has*

$$\|S_t^\rho \varphi\|_{C^{n+2+\nu}} \leq a\, t^{-d/4} t^{-(n+2+\nu)/2} e^{\omega(1+\rho^2)t} \|\varphi\|_2$$

for all $t > 0$, $\varphi \in C_c^\infty(G)$, $\rho \in \mathbf{R}$ and $\psi \in C_c^\infty(G)$ with $\|A^\gamma \psi\|_\infty \leq 1$ for all $\gamma \in J(d)$ with $1 \leq |\gamma| \leq n+3$.

Proof This follows as in the proof of Proposition 3.5 with Lemma 3.4 replaced by Lemma 5.7. The first step of the proof is now given by [ElR6], Proposition 3.3. □

Theorem 5.9 *Suppose $a_1, \ldots, a_{d'}$ is a vector space basis of $\mathfrak{g}$. Let $n \in \mathbf{N}$, $\nu \in \langle 0,1 \rangle$ and let H be a strongly elliptic operator in non-divergence form* (4) *with complex coefficients $c_{ij}, c_i, c_0 \in C^{n+\nu}$. Then for all $\kappa > 0$ and $\tau \in \langle 0,1 \rangle$ there exist $a, b > 0$ and $\omega \geq 0$, such that*

$$|(A^\alpha B^\beta K_t)(k^{-1}g\,;l^{-1}h) - (A^\alpha B^\beta K_t)(g\,;h)| \leq a\, t^{-(d+|\alpha|+|\beta|)/2} e^{\omega t} \Big(\frac{|k|+|l|}{t^{1/2}+|gh^{-1}|} \Big)^\nu e^{-b|gh^{-1}|^2 t^{-1}}$$

uniformly for all $\alpha \in J_{n+2}(d)$, $\beta \in J_n(d)$, $t > 0$, $g, h \in G$ and $k, l \in G$ such that $|k|+|l| \leq \kappa\, t^{1/2} + \tau\, |gh^{-1}|$.

Moreover, for all n, ν, κ, τ, M and μ the constants a, b and ω can be chosen uniformly for all $H \in \mathcal{E}^{\rm nondiv}(n+\nu, n+\nu, n+\nu, \mu, M)$.

Proof This follows from Proposition 5.8 and [ElR6] Propositions 3.3 and 5.6. □

It is unclear whether a subelliptic version of the above theorem is valid. If, however, G is stratified and $a_1, \ldots, a_{d'}$ is a basis for the generating subspace of the stratification of $\mathfrak{g}$ then a subelliptic version is valid, since the same proof works. The higher derivatives on the perturbation function ψ play no role because of the scaling mechanism used in [ElR6].

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