Moments of consecutive sums of ARMA(1,1) processes

Department of Operations Planning and Control -- Working Paper Series

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Research Report TUE/BDK/LBS/93-06
May, 1993

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Abstract

In this paper we develop expressions for the evaluation of the two central moments of sums of consecutive terms of ARMA(1,1) stochastic processes. We also present an expression for the evaluation of the correlation parameters between two consecutive sums of consecutive terms of such processes. These expressions subsequently form the basis for evaluating the corresponding parameters of AR(1) and MA(1) processes, which can be viewed as special cases of the ARMA(1,1) process.

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1. Introduction

Problems in several areas of Management Science (queuing theory, inventory control, quality control and forecasting) necessitate the consideration of stochastic processes. As a result, models often involve sums of random variables (representing successive observations of a stochastic process) whose parameters need to be evaluated or estimated. Good examples are the models developed for the evaluation of the cumulative lead time demand parameters in inventory systems (see Eppen and Martin 1988, An et al. 1989 and Lagodimos et al. 1993) and for the determination of the action limits of quality control charts (see Wardell et al. 1992).

One simplifying assumption usually underlying the analysis of such models is that successive terms of the stochastic processes involved are independent. This assumption, however, is not always realistic (see Wagner 1980), and so processes with serially correlated terms need to be considered. A general class of such processes is the Autoregressive Moving Average (ARMA) process, popularised by Box and Jenkins (1976). It is with the determination of the statistical parameters of consecutive sums of ARMA(1,1) processes that we are concerned here.

The remainder of this paper is organised as follows. In section 2 we introduce the ARMA(1,1) model and deal with the problem of determining the model coefficients so as to obtain some desired mean, variance and autocorrelation characteristics. In section 3 we derive expressions for the first central moments (mean and variance) of consecutive terms of ARMA(1,1) processes. In section 4 we derive an expression for the autocorrelation coefficient between consecutive sums of consecutive terms of such processes. Finally, in section 5 we present some concluding remarks.

2. The ARMA(1,1) process

Let $\{d_t\}$ be a discrete-time stationary stochastic process. In the following we use the notation μ, σ^2 and θ_k to represent the (time invariant) mean, variance and autocorrelation coefficient between any two terms d_t and d_{t+k} of the process respectively.

The ARMA(1,1) process is a particular member of the ARMA class of stochastic processes consisting of one autoregressive and one moving average term (see Box and Jenkins 1976). If $\{d_t\}$ is an ARMA(1,1) process, then:

$$d_t - \alpha d_{t-1} = \beta + \gamma \varepsilon_{t-1} + \varepsilon_t \qquad (1)$$

where α, β and γ are real coefficients and $\{\epsilon_t\}$ is a stochastic error process representing a stream of independent identically distributed random variables having zero mean and constant variance V. In general, we need not make any assumption concerning the distribution of $\{\epsilon_t\}$. However, when $\{\epsilon_t\}$ are normally distributed each term of the ARMA(1,1) process will be normally distributed as well.

In order for $\{d_t\}$ to be stationary we need that $|\alpha| < 1$ and $|\gamma| < 1$. There are also two particular cases: for $\alpha=0$, $\{d_t\}$ becomes a Moving Average MA(1) process, while for $\gamma=0$ it becomes an Autoregressive AR(1) process.

It is useful to have an expression for representing d_{t+k} as a function of d_t. Using (1) we have:

$$\begin{aligned} d_{t+k} &= \beta + \alpha d_{t+k-1} + \gamma \varepsilon_{t+k-1} + \varepsilon_{t+k} \\ &= \beta + \varepsilon_{t+k} + \gamma \varepsilon_{t+k-1} + \alpha\{\beta + \alpha d_{t+k-2} + \gamma \varepsilon_{t+k-2} + \varepsilon_{t+k-1}\} \\ &= \beta(1+\alpha) + (\varepsilon_{t+k} + \alpha \varepsilon_{t+k-1}) + \gamma(\varepsilon_{t+k-1} + \alpha \varepsilon_{t+k-2}) + \alpha^2 d_{t+k-2} \end{aligned}$$

After a number of iterative steps we obtain the following expression for d_{t+k}:

$$d_{t+k} = A_1 + A_2 + A_3 + \alpha^k d_t \tag{2}$$

$$\text{with} \quad A_1 := \beta \sum_{i=0}^{k-1} \alpha^i$$

$$A_2 := \sum_{i=1}^{k} \alpha^{k-i} \varepsilon_{t+i}$$

$$A_3 := \gamma \sum_{i=0}^{k-1} \alpha^{k-1-i} \varepsilon_{t+i}$$

We can now evaluate the statistical parameters of d_t in terms of α, β and γ. First consider the mean μ. Taking expectations of both terms of (1) we obtain:

$$\begin{aligned} E[d_t - ad_{t-1}] &= E[\beta + \gamma\varepsilon_{t-1} + \varepsilon_t] \\ \mu - a\mu &= \beta \\ \mu &= \beta \,/\, (1-\alpha) \end{aligned} \tag{3}$$

We can also evaluate the covariance of d_t and d_{t+k}. By definition:

$$\begin{aligned} Cov(d_t, d_{t+k}) &= E[(d_t - \mu)(d_{t+k} - \mu)] \\ &= E[(d_t - \mu)(\alpha^k d_t + A_1 + A_2 + A_3 - \mu)] \\ &= E[\alpha^k d_t^2 + d_t A_1 + d_t A_2 + d_t A_3 - \mu d_t - \mu\alpha^k d_t - \mu A_1 - \mu A_2 - \mu A_3 + \mu^2] \\ &= \alpha^k (E[d_t^2] - \mu^2) + E[d_t A_3] \\ &= \alpha^k \sigma^2 + \gamma\alpha^{k-1} E[d_t \varepsilon_t] \\ &= \alpha^k \sigma^2 + \gamma\alpha^{k-1} V \end{aligned} \tag{4}$$

$$\begin{aligned} \text{where} \quad E[d_t \varepsilon_t] &= E[\ \beta\varepsilon_t + ad_{t-1}\varepsilon_t + \gamma\varepsilon_{t-1}\varepsilon_t + \varepsilon_t^2\ ] \\ &= E[\varepsilon_t^2] \\ &= V \end{aligned}$$

Using the relation $\theta_k = Cov(d_t, d_{t+k})/\sigma^2$, the correlation coefficient θ_k can be obtained directly.

We finally evaluate $Var(d_t)$. Using (1) we have:

$$\begin{aligned} Var(d_t - ad_{t-1}) &= Var(\beta + \gamma\varepsilon_{t-1} + \varepsilon_t) \\ Var(d_t) + Var(ad_{t-1}) - 2Cov(d_t, ad_{t-1}) &= Var(\gamma\varepsilon_{t-1}) + Var(\varepsilon_t) \\ \sigma^2(1+\alpha^2) - 2\alpha Cov(d_t, d_{t-1}) &= (\gamma^2 + 1)V \end{aligned}$$

Finally, replacing $Cov(d_t, d_{t-1})$ above by expression (4) with k=1, we obtain:

$$\sigma^2 = \frac{(\gamma^2 + 2\alpha\gamma + 1)V}{1 - \alpha^2} \tag{5}$$

which is a function of α and γ and which can also be found in Box and Jenkins (1976).

One problem often encountered in practice is the determination of the process coefficients α, β and γ and of the variance V of the error process $\{\epsilon_t\}$ in order to obtain some desired μ, σ^2 and θ_k. To deal with this problem, we simply need to solve the following algebraic system of equations:

$$\begin{aligned} \beta &= \mu(1-\alpha) \\ \sigma^2 &= \frac{(\gamma^2+2\alpha\gamma+1)V}{1-\alpha^2} \\ \theta_k &= \alpha^k+\gamma\alpha^{k-1}\ \frac{V}{\sigma^2} \end{aligned}$$

Since we are interested in stationary ARMA(1,1) processes we need only to consider values of α and γ such that $|\alpha|<1$ and $|\gamma|<1$. In order to solve this system for α, β, γ and V, we need to know the values of the statistical parameters μ, σ^2, θ_1 and θ_2 of $\{d_t\}$. Notice that for the two special cases (the AR(1) and MA(1) process) we only need to know the values of μ, σ^2 and θ_1, since we have only three unknown parameters. That is: α, β and V for the AR(1) process and β, γ and V for the MA(1) process respectively.

3. Sums of consecutive terms

In this section we develop expressions for the mean and variance of sums consisting of consecutive ARMA(1,1) terms of the form:

$$X := \sum_{k=1}^{L} d_{t+k} \tag{6}$$

We start with E[X]. Since $E[d_t]=\mu$, it is straightforward that:

$$E[X] = L * \mu \tag{7}$$

The derivation of $Var(d_t)$ is more involving. It is convenient for this purpose to rewrite X in a more concise form. Introducing (2) in expression (6) we have:

$$\begin{aligned} X &= \sum_{k=1}^{L} d_{t+k} \\ &= \sum_{k=1}^{L} \beta \sum_{i=0}^{k-1} \alpha^i + \sum_{k=1}^{L} \sum_{i=1}^{k} \alpha^{k-i}\ \varepsilon_{t+i} + \gamma \sum_{k=1}^{L} \sum_{i=0}^{k-1} \alpha^{k-1-i}\ \varepsilon_{t+i} + \sum_{k=1}^{L} \alpha^k d_t \\ &= X_1 + X_2 + \gamma X_3 + X_4 d_t \end{aligned} \tag{8}$$

We can evaluate the sums in X_1, X_2, X_3 and X_4 as follows:

$$\begin{aligned}
X_1 &= \sum_{k=1}^{L} \beta \sum_{i=0}^{k-1} \alpha^i \\
&= \beta \sum_{k=1}^{L} \frac{1-\alpha^k}{1-\alpha} \\
&= \frac{\beta}{1-\alpha}\left[L-\sum_{k=1}^{L}\alpha^k\right] \\
&= \frac{\beta}{1-\alpha}\left[L-\alpha\frac{1-\alpha^L}{1-\alpha}\right]
\end{aligned}$$

$$\begin{aligned}
X_2 &= \sum_{k=1}^{L}\sum_{i=1}^{k}\alpha^{k-i}\,\varepsilon_{t+i} \\
&= \sum_{k=1}^{L}\left(\alpha^{k-1}\varepsilon_{t+1}+\alpha^{k-2}\varepsilon_{t+2}+\ldots+\alpha\varepsilon_{t+k-1}+\varepsilon_{t+k}\right) \\
&= (\varepsilon_{t+1})+(\alpha\varepsilon_{t+1}+\varepsilon_{t+2})+\ldots+(\alpha^{L-1}\varepsilon_{t+1}+\alpha^{L-2}\varepsilon_{t+2}+\ldots+\varepsilon_{t+L}) \\
&= \sum_{k=1}^{L}\varepsilon_{t+k}\left[\sum_{i=0}^{L-k}\alpha^i\right] \\
&= \sum_{k=1}^{L}\varepsilon_{t+k}\,\frac{1-\alpha^{L-k+1}}{1-\alpha} \\
&= \frac{1}{1-\alpha}\sum_{j=1}^{L}(1-\alpha^j)\varepsilon_{t+L+1-j}
\end{aligned}$$

$$\begin{aligned}
X_3 &= \sum_{k=1}^{L}\sum_{i=0}^{k-1}\alpha^{k-1-i}\varepsilon_{t+i} \\
&= \sum_{k=1}^{L}\left(\alpha^{k-1}\varepsilon_t+\alpha^{k-2}\varepsilon_{t+1}+\ldots+\alpha\varepsilon_{t+k-2}+\varepsilon_{t+k-1}\right) \\
&= (\varepsilon_t)+(\alpha\varepsilon_t+\varepsilon_{t+1})+\ldots+(\alpha^{L-1}\varepsilon_t+\alpha^{L-2}\varepsilon_{t+1}+\ldots+\alpha\varepsilon_{t+L-2}+\varepsilon_{t+L-1}) \\
&= \varepsilon_t\sum_{i=0}^{L-1}\alpha^i+\sum_{k=1}^{L-1}\varepsilon_{t+k}\left[\sum_{i=0}^{L-k-1}\alpha^i\right] \\
&= \frac{1-\alpha^L}{1-\alpha}\varepsilon_t+\frac{1}{1-\alpha}\sum_{k=1}^{L-1}(1-\alpha^{L-k})\varepsilon_{t+k} \\
&= \frac{1}{1-\alpha}\left[(1-\alpha^L)\varepsilon_t+\sum_{j=1}^{L-1}(1-\alpha^j)\varepsilon_{t+L-j}\right]
\end{aligned}$$

$$\begin{aligned}
X_4 &= \sum_{k=1}^{L}\alpha^k \\
&= \alpha\,\frac{1-\alpha^L}{1-\alpha}
\end{aligned}$$

Introducing these expressions in (8) we obtain:

$$
\begin{aligned}
X &= X_1+X_2+\gamma X_3+X_4 d_t \\
&= \frac{\beta}{1-\alpha}\left[L-\alpha\,\frac{1-\alpha^L}{1-\alpha}\right]+\frac{1}{1-\alpha}\sum_{j=1}^{L}(1-\alpha^j)\varepsilon_{t+L+1-j} \\
&\quad +\gamma\,\frac{1}{1-\alpha}\left[(1-\alpha^L)\varepsilon_t+\sum_{j=1}^{L-1}(1-\alpha^j)\varepsilon_{t+L-j}\right]+\alpha\,\frac{1-\alpha^L}{1-\alpha}\,d_t
\end{aligned}
$$

Taking the variance of the above we have:

$$
Var(X)=Var(X_2)+\gamma^2 Var(X_3)+Var(X_4 d_t)+2\gamma\, Cov(X_2,X_3)+2\gamma\, Cov(X_3,X_4 d_t) \tag{9}
$$

which is obtained by noticing that the random variables X_2 and d_t are independent. Each of the terms in the above expression can be evaluated as follows:

$$
\begin{aligned}
Cov(X_3,X_4 d_t) &= Cov\left[\frac{1-\alpha^L}{1-\alpha}\varepsilon_t,\alpha\frac{1-\alpha^L}{1-\alpha}d_t\right] \\
&= \alpha\left[\frac{1-\alpha^L}{1-\alpha}\right]^2 Cov(\varepsilon_t,d_t) \\
&= \alpha\left[\frac{1-\alpha^L}{1-\alpha}\right]^2 V
\end{aligned}
$$

$$
\begin{aligned}
Cov(X_2,X_3) &= Cov\left[\frac{1}{1-\alpha}\sum_{j=1}^{L}(1-\alpha^j)\varepsilon_{t+L+1-j},\frac{1-\alpha^L}{1-\alpha}\varepsilon_t+\frac{1}{1-\alpha}\sum_{j=1}^{L-1}(1-\alpha^j)\varepsilon_{t+L-j}\right] \\
&= \frac{1}{(1-\alpha)^2}\Big\{Cov\big((1-\alpha^L)\varepsilon_{t+1},(1-\alpha^{L-1})\varepsilon_{t+1}\big)+Cov\big((1-\alpha^{L-1})\varepsilon_{t+2},(1-\alpha^{L-2})\varepsilon_{t+2}\big)+\ldots \\
&\quad +Cov\big((1-\alpha^2)\varepsilon_{t+L-1},(1-\alpha)\varepsilon_{t+L-1}\big)\Big\} \\
&= \frac{V}{(1-\alpha)^2}\sum_{j=1}^{L-1}(1-\alpha^{j+1})(1-\alpha^j) \\
&= \frac{V}{(1-\alpha)^2}\sum_{j=1}^{L-1}(1-\alpha^j-\alpha^{j+1}+\alpha^{2j+1}) \\
&= \frac{V}{(1-\alpha)^2}\left(L-1+\alpha\sum_{j=0}^{L-2}\alpha^j-\alpha^2\sum_{j=0}^{L-2}\alpha^j+\alpha^3\sum_{j=0}^{L-2}(\alpha^2)^j\right. \\
&= \frac{V}{(1-\alpha)^2}\left[L-1+\frac{\alpha(1-\alpha^{L-1})}{1-\alpha}-\frac{\alpha^2(1-\alpha^{L-1})}{1-\alpha}+\frac{\alpha^3(1-\alpha^{2(L-1)})}{1-\alpha^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
Var(X_2) &= Var\left[\frac{1}{1-\alpha}\sum_{j=1}^{L}(1-\alpha^j)\varepsilon_{t+L+1-j}\right]\\
&= \frac{V}{(1-\alpha)^2}\sum_{j=1}^{L}(1-\alpha^j)^2\\
&= \frac{V}{(1-\alpha)^2}\left[L-2\sum_{j=1}^{L}\alpha^j+\sum_{j=1}^{L}(\alpha^2)^j\right]\\
&= \frac{V}{(1-\alpha)^2}\left[L-2\alpha\frac{1-\alpha^L}{1-\alpha}+\alpha^2\frac{1-\alpha^{2L}}{1-\alpha^2}\right]\\
Var(X_3) &= Var\left[\frac{1-\alpha^L}{1-\alpha}\varepsilon_t+\frac{1}{1-\alpha}\sum_{j=1}^{L-1}(1-\alpha^j)\varepsilon_{t+L-j}\right]\\
&= \left\{\left[\frac{1-\alpha^L}{1-\alpha}\right]^2 V+\frac{V}{(1-\alpha)^2}\left[L-1-2\alpha\frac{1-\alpha^{L-1}}{1-\alpha}+\alpha^2\frac{1-(\alpha^2)^{L-1}}{1-\alpha^2}\right]\right\}\\
&= \frac{V}{(1-\alpha)^2}\left[(1-\alpha^L)^2+L-1-2\alpha\frac{1-\alpha^{L-1}}{1-\alpha}+\alpha^2\frac{1-\alpha^{2(L-1)}}{1-\alpha^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
Var(X_4 d_t) &= Var\left[\sum_{k=1}^{L}\alpha^k d_t\right]\\
&= Var\left[\alpha\frac{1-\alpha^L}{1-\alpha}d_t\right]\\
&= \alpha^2\left[\frac{1-\alpha^L}{1-\alpha}\right]^2\sigma^2\\
&= \alpha^2\left[\frac{1-\alpha^L}{1-\alpha}\right]^2\frac{V(\gamma^2+1)}{1+\alpha^2-2\alpha\theta_1}
\end{aligned}
$$

Introducing the above in (9) we finally obtain:

$$
\begin{aligned}
Var(X) = \frac{V}{(1-\alpha)^2}\Bigg\{ & L-2\alpha\frac{1-\alpha^L}{1-\alpha}+\alpha^2\frac{1-\alpha^{2L}}{1-\alpha^2}+\gamma^2\left[(1-\alpha^L)^2+L-1-2\alpha\frac{1-\alpha^{L-1}}{1-\alpha}+\alpha^2\frac{1-\alpha^{2(L-1)}}{1-\alpha^2}\right]\\
&+\frac{(\gamma^2+1)\alpha^2(1-\alpha^L)^2}{1+\alpha^2-2\alpha\theta_1}+2\alpha\gamma(1-\alpha^L)^2\\
&+2\gamma\left[L-1+\alpha\frac{1-\alpha^{L-1}}{1-\alpha}-\alpha^2\frac{1-\alpha^{L-1}}{1-\alpha}+\alpha^3\frac{1-\alpha^{2(L-1)}}{1-\alpha^2}\right]\Bigg\} \qquad (10)
\end{aligned}
$$

For the two special cases of the ARMA(1,1) process we have the following expressions:

MA(1) process (α=0):

$$\begin{aligned} Var(X) &= V\{L+\gamma^2(1+L-1)+2\gamma(L-1)\} \\ &= V\{L+L\gamma^2+2\gamma L-2\gamma\} \end{aligned}$$

AR(1) process (γ=0, θ_1=α):

$$Var(X) = \frac{V}{(1-\alpha)^2}\left\{L-2\alpha\frac{1-\alpha^L}{1-\alpha}+\alpha^2\frac{1-\alpha^{2L}}{1-\alpha^2}+\alpha^2\frac{(1-\alpha^L)^2}{1-\alpha^2}\right\}$$

$$= \frac{V}{(1-\alpha)^2(1-\alpha^2)}\{\, 2\alpha^{L+1}-L\alpha^2-2\alpha+L \,\}$$

4. Consecutive sums of consecutive terms

In this section we evaluate the correlation between consecutive sums of consecutive ARMA(1,1) terms. Let X and Z be random variables defined as follows:

$$X:=\sum_{k=1}^{L} d_{t+k} \qquad and \qquad Z:=\sum_{k=L+1}^{L+l} d_{t+k}$$

where L and l are integers representing the number of ARMA(1,1) terms included in X and Z. We are interested in evaluating *Cov*(X,Z) and the resulting correlation coefficient ρ.

Define a new variable Y=X+Z. We can generally represent *Cov*(X,Z) as:

$$Cov(X,Z)=Cov(X,Y-X)=Cov(X,Y)-Var(X) \tag{11}$$

Since *Var*(X) has already been evaluated in section 3, we only need *Cov*(X,Y). By definition we can write:

$$Cov(X,Y)=E[XY]-E[X]E[Y] \tag{12}$$

and so we need to evaluate E[XY] and E[X]E[Y]. Using the results of section 3, it is convenient to rewrite X and Y as:

$$X := \frac{1}{1-\alpha}\left[B_1(L)+B_2(L)+B_3(L)+B_4(L)+B_5(L)d_t \right]$$
$$Y := \frac{1}{1-\alpha}\left[B_1(\lambda)+B_2(\lambda)+B_3(\lambda)+B_4(\lambda)+B_5(\lambda)d_t \right]$$

$$\begin{aligned} \textit{where} \qquad \lambda &:= L+l \\ B_1(p) &:= \beta \left[p-\alpha\frac{1-\alpha^p}{1-\alpha} \right] \\ B_2(p) &:= \sum_{j=1}^{p}(1-\alpha^j)\varepsilon_{t+p+1-j} \\ B_3(p) &:= \gamma(1-\alpha^p)\varepsilon_t \\ B_4(p) &:= \gamma\sum_{j=1}^{p-1}(1-\alpha^j)\varepsilon_{t+p-j} \\ B_5(p) &:= \alpha(1-\alpha^p) \end{aligned}$$

Hence:

$$\begin{aligned} E[XY] = & \frac{1}{(1-\alpha)^2}\Big\{ B_1(L)B_1(\lambda)+B_1(L)B_5(\lambda)\mu+E[B_2(L)B_2(\lambda)]+E[B_2(L)B_4(\lambda)]+E[B_3(L)B_3(\lambda)] \\ & +E[B_3(L)B_5(\lambda)d_t]+E[B_4(L)B_2(\lambda)]+E[B_4(L)B_4(\lambda)]+B_5(L)B_1(\lambda)\mu \\ & +E[B_5(L)B_3(\lambda)d_t]+E[B_5(L)B_5(\lambda)d_t^2] \Big\} \\ E[X]E[Y] = & \frac{1}{(1-\alpha)^2}\Big\{ B_1(L)B_1(\lambda)+B_1(L)B_5(\lambda)\mu+B_5(L)B_1(\lambda)\mu+B_5(L)B_5(\lambda)\mu^2 \Big\} \end{aligned}$$

Introducing the above in (12) we have:

$$\begin{aligned} Cov(X,Y) &= E[XY]-E[X]E[Y] \\ &= \frac{1}{(1-\alpha)^2}\Big\{ E[B_2(L)B_2(\lambda)]+E[B_2(L)B_4(\lambda)]+E[B_3(L)B_3(\lambda)] \\ &\quad +E[B_3(L)B_5(\lambda)d_t]+E[B_4(L)B_2(\lambda)]+E[B_4(L)B_4(\lambda)] \\ &\quad +E[B_5(L)B_3(\lambda)d_t]+B_5(L)B_5(\lambda)\left\{E[d_t^2]-\mu^2\right\} \Big\} \end{aligned} \tag{13}$$

We need to evaluate each of the terms in the above expression. We have:

$$
\begin{aligned}
E[B_2(L)B_2(\lambda)] &= E\left[\sum_{j=1}^{L}(1-\alpha^j)\varepsilon_{t+L+1-j} * \sum_{j=1}^{L+l}(1-\alpha^j)\varepsilon_{t+L+l+1-j}\right] \\
&= V\left[\sum_{j=1}^{L}(1-\alpha^j)(1-\alpha^{j+l})\right] \\
&= V\left[\sum_{j=1}^{L}(1-\alpha^{j+l}-\alpha^j+\alpha^{2j+l})\right] \\
&= V\left[L-\alpha^{l+1}\sum_{j=0}^{L-1}\alpha^j-\alpha\sum_{j=0}^{L-1}\alpha^j+\alpha^{l+2}\sum_{j=0}^{L-1}(\alpha^2)^j\right] \\
&= V\left[L-\alpha^{l+1}\frac{1-\alpha^L}{1-\alpha}-\alpha\frac{1-\alpha^L}{1-\alpha}+\alpha^{l+2}\frac{1-\alpha^{2L}}{1-\alpha^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
E[B_2(L)B_4(\lambda)] &= E\left[\sum_{j=1}^{L}(1-\alpha^j)\varepsilon_{t+L+1-j} * \gamma\sum_{j=1}^{L+l-1}(1-\alpha^j)\varepsilon_{t+L+l-j}\right] \\
&= \gamma V\sum_{j=1}^{L}(1-\alpha^j)(1-\alpha^{j+l-1}) \\
&= \gamma V\sum_{j=1}^{L}(1-\alpha^{j+l-1}-\alpha^j+\alpha^{2j+l-1}) \\
&= \gamma V\left[L-\alpha^{l}\frac{1-\alpha^L}{1-\alpha}-\alpha\frac{1-\alpha^L}{1-\alpha}+\alpha^{l+1}\frac{1-\alpha^{2L}}{1-\alpha^2}\right]
\end{aligned}
$$

$$
E[B_3(L)B_3(\lambda)] = \gamma^2 V\,(1-\alpha^L)(1-\alpha^{L+l})
$$

$$
\begin{aligned}
E[B_3(L)B_5(\lambda)d_t] &= E\left[\gamma\ (1-\alpha^L)\varepsilon_t\ \alpha\ (1-\alpha^{L+l})d_t\right] \\
&= \gamma\ \alpha\ (1-\alpha^L)(1-\alpha^{L+l})E[\varepsilon_t d_t] \\
&= \gamma\ \alpha\ V\ (1-\alpha^L)(1-\alpha^{L+l})
\end{aligned}
$$

$$
\begin{aligned}
E[B_4(L)B_2(\lambda)] &= E\left[\gamma \sum_{j=1}^{L-1} (1-\alpha^j)\varepsilon_{t+L-j} * \sum_{j=1}^{L+l} (1-\alpha^j)\varepsilon_{t+L+l+1-j}\right] \\
&= \gamma\ V \sum_{j=1}^{L-1} (1-\alpha^j)(1-\alpha^{j+l+1}) \\
&= \gamma\ V \left[L-1-\sum_{j=1}^{L-1} \alpha^j - \sum_{j=1}^{L-1} \alpha^{j+l+1} + \sum_{j=1}^{L-1} \alpha^{2j+l+1}\right] \\
&= \gamma\ V \left[L-1-\alpha\frac{1-\alpha^{L-1}}{1-\alpha} - \alpha^{l+2}\frac{1-\alpha^{L-1}}{1-\alpha} + \alpha^{l+3}\frac{1-\alpha^{2(L-1)}}{1-\alpha^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
E[B_4(L)B_4(\lambda)] &= E\left[\gamma \sum_{j=1}^{L-1} (1-\alpha^j)\varepsilon_{t+L-j} * \gamma \sum_{j=1}^{L+l-1} (1-\alpha^j)\varepsilon_{t+L+l-j}\right] \\
&= \gamma^2\ V \sum_{j=1}^{L-1} (1-\alpha^j)(1-\alpha^{j+l}) \\
&= \gamma^2\ V \left[L-1-\sum_{j=1}^{L-1} \alpha^j - \sum_{j=1}^{L-1} \alpha^{j+l} + \sum_{j=1}^{L-1} \alpha^{2j+l}\right] \\
&= \gamma^2\ V \left[L-1-\alpha\frac{1-\alpha^{L-1}}{1-\alpha} - \alpha^{l+1}\frac{1-\alpha^{L-1}}{1-\alpha} + \alpha^{l+2}\frac{1-\alpha^{2(L-1)}}{1-\alpha^2}\right]
\end{aligned}
$$

$$
\begin{aligned}
E[B_5(L)B_3(\lambda)d_t] &= E\left[\ \alpha(1-\alpha^L)d_t\ \gamma(1-\alpha^{L+l})\varepsilon_t\ \right] \\
&= \gamma\ \alpha\ (1-\alpha^L)(1-\alpha^{L+l})E[d_t\varepsilon_t] \\
&= \gamma\ \alpha\ V\ (1-\alpha^L)(1-\alpha^{L+l})
\end{aligned}
$$

$$
\begin{aligned}
B_5(L)B_5(\lambda)\left\{\ E[d_t^2]-\mu^2\ \right\} &= B_5(L)B_5(\lambda)\ \sigma^2 \\
&= \alpha^2\ (1-\alpha^L)(1-\alpha^{L+l})\ \frac{V}{1-\alpha^2}
\end{aligned}
$$

Replacing the above in expression (13) for *Cov*(X,Y) and using expressions (10) and (11), after some algebra, we finally obtain:

$$
\begin{aligned}
Cov(X,Z) &= Cov(X,Y) - Var(X) \\
&= \frac{V}{(1-\alpha)^2} \Bigg\{ \left[L - \alpha^{l+1}\frac{1-\alpha^L}{1-\alpha} - \alpha\frac{1-\alpha^L}{1-\alpha} + \alpha^{l+2}\frac{1-\alpha^{2L}}{1-\alpha^2} \right] \\
&\quad + \gamma \left[L - \alpha^{l}\frac{1-\alpha^L}{1-\alpha} - \alpha\frac{1-\alpha^L}{1-\alpha} + \alpha^{l+1}\frac{1-\alpha^{2L}}{1-\alpha^2} \right] \\
&\quad + \gamma^2 \ (1-\alpha^L)(1-\alpha^{L+l}) \\
&\quad + \gamma \ \ \alpha\,(1-\alpha^L)(1-\alpha^{L+l}) \\
&\quad + \gamma \left[L-1-\alpha\frac{1-\alpha^{L-1}}{1-\alpha} - \alpha^{l+2}\frac{1-\alpha^{L-1}}{1-\alpha} + \alpha^{l+3}\frac{1-\alpha^{2(L-1)}}{1-\alpha^2} \right] \\
&\quad + \gamma^2 \left[L-1-\alpha\frac{1-\alpha^{L-1}}{1-\alpha} - \alpha^{l+1}\frac{1-\alpha^{L-1}}{1-\alpha} + \alpha^{l+2}\frac{1-\alpha^{2(L-1)}}{1-\alpha^2} \right] \\
&\quad + \gamma \ \ \alpha\,(1-\alpha^L)(1-\alpha^{L+l}) \\
&\quad + \frac{\alpha^2}{1-\alpha^2}\,(1-\alpha^L)(1-\alpha^{L+l}) \\
&\quad - \left[L-2\alpha\frac{1-\alpha^L}{1-\alpha} + \alpha^2\frac{1-\alpha^{2L}}{1-\alpha^2} \right] \\
&\quad - \gamma^2 \left[(1-\alpha^L)^2 + L-1-2\alpha\frac{1-\alpha^{L-1}}{1-\alpha} + \alpha^2\frac{1-\alpha^{2(L-1)}}{1-\alpha^2} \right] \\
&\quad - \left[\frac{(\gamma^2+1)\alpha^2(1-\alpha^L)^2}{1+\alpha^2-2\alpha\theta_1} + 2\alpha\gamma(1-\alpha^L)^2 \right] \\
&\quad - 2\gamma \left[L-1+\alpha\frac{1-\alpha^{L-1}}{1-\alpha} - \alpha^2\frac{1-\alpha^{L-1}}{1-\alpha} + \alpha^3\frac{1-\alpha^{2(L-1)}}{1-\alpha^2} \right] \Bigg\}
\end{aligned}
$$

Since

$$
\rho = \frac{Cov(X,Z)}{\sqrt{Var(X)\ Var(Z)}}
$$

the above can be used directly to evaluate ρ.

It is interesting to consider the two special cases where the ARMA(1,1) process reduces to an AR(1) or an MA(1) process.

MA(1)-process ($\alpha=0$):

$$\begin{aligned} Cov(X,Y) &= V\,[\ L+L\gamma^2+2\gamma L-\gamma\ ] \\ Var(X) &= V\,[\ L+L\gamma^2+2\gamma L-2\gamma\ ] \\ Var(Z) &= V\,[\ l+l\gamma^2+2\gamma l-2\gamma\ ] \\ Cov(X,Y)-Var(X) &= \gamma\ V \end{aligned}$$

Hence:

$$\begin{aligned} \rho &= \frac{Cov(X,Y)-Var(X)}{\sqrt{Var(X)}\ *\ \sqrt{Var(Z)}} \\ &= \frac{\gamma}{\sqrt{L+L\gamma^2+2\gamma L-2\gamma}\ *\ \sqrt{l+l\gamma^2+2\gamma l-2\gamma}} \end{aligned}$$

AR(1)-process ($\gamma=0$):

$$\begin{aligned} Cov(X,Y) &= \frac{V}{(1-\alpha)^2}\left[L-\alpha^{l+1}\frac{1-\alpha^L}{1-\alpha}-\alpha\frac{1-\alpha^L}{1-\alpha}+\alpha^{l+2}\frac{1-\alpha^{2L}}{1-\alpha^2}+\frac{\alpha^2}{1-\alpha^2}(1-\alpha^L)(1-\alpha^{L+l})\right] \\ Var(X) &= \frac{V}{(1-\alpha)^2(1-\alpha^2)}\,[\ 2\alpha^{L+1}-L\alpha^2-2\alpha+L\ ] \\ Var(Z) &= \frac{V}{(1-\alpha)^2(1-\alpha^2)}\,[\ 2\alpha^{l+1}-l\alpha^2-2\alpha+l\ ] \\ Cov(X,Y)-Var(X) &= \frac{V}{(1-\alpha)^2(1-\alpha^2)}\,[\ L(1-\alpha^2)-\alpha^{l+1}(1-\alpha^L)(1+\alpha)-\alpha(1-\alpha^L)(1+\alpha)+\alpha^{l+2}(1-\alpha^{2L}) \\ &\qquad +\alpha^2(1-\alpha^L)(1-\alpha^{L+l})-2\alpha^{L+1}+L\alpha^2+2\alpha-L)\ ] \\ &= \frac{\alpha\ V}{(1-\alpha)^2(1-\alpha^2)}\,[\ (\alpha^L-1)(\alpha^l-1)\ ] \end{aligned}$$

Hence:

$$\begin{aligned} \rho &= \frac{Cov(X,Z)-Var(X)}{\sqrt{Var(X)}\ *\ \sqrt{Var(Z)}} \\ &= \frac{\alpha(\alpha^L-1)(\alpha^l-1)}{\sqrt{2\alpha^{L+1}-L\alpha^2-2\alpha+L}\ *\ \sqrt{2\alpha^{l+1}-l\alpha^2-2\alpha+l}} \end{aligned}$$

5. Concluding remarks

In this paper we derived expressions for the central moments (mean and variance) of sums of consecutive terms of ARMA(1,1) stochastic processes. We also derived an expression for the autocorrelation coefficient between consecutive sums of consecutive terms of ARMA(1,1) processes. The derivation of these expressions involves some lengthy and cumbersome algebra. Using a similar methodology, extension of these expressions to more general ARMA(p,q) processes (with p autoregressive and q moving average terms) is straightforward, although the algebra involved is much more involving.

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