

Why no additive hazards models?

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Why No Additive Hazards Models?

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Key Words - Proportional hazards model, accelerated failure time model, additive hazards model, competing risks model, identifiability

Reader Aids —

General purpose: Widen state-of-art, explain model Special math needed for explanations: Probability, statistics Special math needed to use results: Same Results useful to: Reliability analysts, statisticians

Summary & Conclusions - The intuitively attractive additive hazards model is compared with the proportional hazards and accelerated failure time models. The lack of identifiability limits the use of the model and prevents the application of regression versions using covariates. Fortunately, data analysis based on nonhomogeneous Poisson processes or on proportional hazards is likely to yield most of the information available in the data, even though they: 1) do not necessarily represent the underlying process, and 2) even seem unlikely in certain situations. In particular, proportional hazards modeling appears very robust and requires few assumptions.

1. INTRODUCTION

This paper briefly discusses 3 well-known models:

- proportional hazards,
- · accelerated failure time,
- competing risks.

A fourth model, additive hazards, is reasonable & attractive and is prompted by the 3 others. Although the additive hazards model is intuitively attractive, its applications are limited by an identifiability problem. Because the model is not identifiable, the observation of explanatory variables adds nothing to the knowledge obtained from the event data. Thus if an additive hazards model is hypothesized, then the money & time spent on measuring explanatory covariates is wasted and no decisions can be based on the assumption that different operating conditions lead to differences in performance.¹

The models imply that the time reverts to zero after each event (usually a failure). At the event, the item is restored (eg, repaired, adjusted, replaced) to some known condition and time is set back to zero. Thus, the distribution of times between events is an ordinary Cdf, as opposed to a type of process wherein there are many events, and time is measured from the beginning of the process.

¹Acronyms, nomenclature, and notation are given at the end of the Introduction.

It is easiest to build the models in terms of the hazard rate for the current interval [1, 2]. Aspects of aging can be reflected in:

• the behavior of the hazard rate between failures

• values of the hazard rate at the beginning of each interval.

Since a hazard rate can be interpreted as an aging rate, a change of time scale, or a simple scaling up of the hazard rate, can capture the effects of explanatory factors between failures. The effects of repair can be captured in the initial value of the hazard rate. A hazard rate that is zero at time zero could indicate that a repair has removed all age effects.

This paper discusses the behavior of the hazard rate, both within an interval and at a repair. The observations apply correspondingly to the analysis of data from non-repairable systems where a common underlying model is assumed to be modified by field or experimental conditions.

Acronyms² & Nomenclature

AFT	accelerated failure-time model
AH	additive-hazards model
PH	proportional-hazards model
CAM	competing-risks model
series	implies a system that fails iff at least

X	interval between events, or lifetime of a subject; a r.v.
X ₀	the X under baseline operating conditions
Τ	time of events in a process; a r.v.
z	covariate, to describe the operating conditions
$\psi(z)$	relative-risk factor (dependence of the model on z):

1 component fails.

- $\psi(z)$ $\psi(z) \geq 0, \ \psi(z_{\text{baseline}}) \equiv 1,$
- number of risks/hazards in the system n
- X_i, z_i [X, z] for risk/hazard i; i=0 is the baseline value
- coordinate j of vector z_i for risk/hazard iζi $f_i, F_i,$
- [pdf, Cdf, Sf] for risk/hazard i; i=0 is the baseline R_i value
- hazard [rate, function] for risk/hazard i; i=0 is the h_i, H_i baseline value
- implies the system value sys
- Å parameter of baseline distribution
- β parameter of ψ .

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

²The singular & plural of an acronym are always spelled the same.

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Notation

2. ACCELERATED FAILURE-TIME MODELS

AFT is perhaps the most intuitively attractive model. In AFT, the effect of the covariates is seen as changes in the time scale for the system. The duration is a r.v. and the covariates are summarized in a vector. The basic assumption is:

$$X = \psi(z) \cdot X_0 \tag{1}$$

The life is stochastically increased or decreased according to whether $\psi < 1$ or $\psi > 1$. The behavior of the model is most easily understood by considering the effect of the transformation (1) on a 1-parameter family of distributions.

Let X_0 be described by a 1-parameter Cdf, $F_0(x;\theta)$; the system with covariate $z_j = (\zeta_{j1}, \zeta_{j2}, ..., \zeta_{jk})$ has:

$$F_j(x;z_j, \theta) = F_0\left(\frac{x}{\psi(z_j)}; \theta\right)$$

A detailed treatment of estimation-interpretation for such models is in [12-14].

3. PROPORTIONAL HAZARDS

PHM is widely used for studying the effects of covariates. The basis of PHM is the simple assumption that the hazard rate is affected in a multiplicative way by a relative-risk factor. Since the hazard rate is a measure of *aging*, an increase in relative risk indicates a more rapid *aging*. The model is expressed in terms of $h_0(x) \& \psi(z)$. The system hazard rate is:

$$h_i(x) = \psi(z_i;\beta) \cdot h_0(x)$$

The most common choice for $\psi(z)$ is the Cox-Model:

 $\psi(z;\beta) = \exp(\beta' \cdot z).$

The basic properties of the model are easy to deduce. The most valuable property is: If the baseline hazard rate is unspecified, then the analysis of the effects of covariates can proceed on the basis of a *partial likelihood* that is independent of h_0 . PHM details are readily accessible in the literature [1, 2, 4] and in statistical packages such as BMDP.

4. COMPETING RISKS

CRM [7, 8] has two interpretations:

- the life of a system is subject to several competing risks
- the life of a system of several components ends as soon as one of the components fails.

The failure times can be regarded as a vector of r.v.'s $\{X_i\}$ so that the failure time is the minimum of the X_i . Let the $n X_i$ be *s*-independent; then —

$$R_{sys} = \prod_{i=1}^{n} R_i(x),$$
$$h_{sys}(x) = \sum_{i=1}^{n} h_i(x).$$

CRM are often used in reliability problems, in particular in the analysis of *series* systems and from *weakest link* arguments for systems of *s*-independent components. The β -factor method for dealing with *s*-dependency is also a version of CRM since it, in effect, splits the system into two parts: 1) the *s*-independent components, and 2) a common-cause component in *series* [11: chapter 7].

5. ADDITIVE HAZARDS

Considering the AFT, PH, CAM in chapters 2 - 4, it is reasonable to develop an additive analogue of the PHM in which h_0 is modified in an additive way by covariates [15, 16]:

$$h_{j}(x) = \psi(z_{j}) + h_{0}(x)$$

$$H_{j}(x) = \psi(z_{j}) \cdot x + H_{0}(x)$$

$$R_{j}(x) = \exp[-H_{j}(x)] = \exp[-\psi(z_{j}) \cdot x - H_{0}(x)]$$

$$= \exp[-\psi(z_{j}) \cdot x] \cdot R_{0}(x)$$

$$f_{j}(x) = [\psi(z_{j}) - h_{0}(x)] \cdot \exp[-\psi(z_{j}) \cdot x - H_{0}(x)]$$

The distribution of X is the distribution of the minimum of 2 s-independent r.v.,

$$X = \min(X_1, X_2),$$
$$R_{X_1} = \exp[-\psi(z_j) \cdot x].$$

Thus the AHM is a form of CRM [8] with 2, possibly fictional, components in *series*. However, the failing component cannot be determined. This point is useful for simulating data from an AHM, since X is simply the minimum of an exponential r.v. X_1 sampled from the distribution with parameter $\psi(z_j)$ and a r.v. X_0 .

The moments are easily obtained by Laplace transforms. The transforms are:

$$R_{j}^{*}(s) = \int_{0}^{\infty} R_{j}(x) \exp(-\psi_{j}x) \exp(-sx) dx = R_{0}^{*}(s+\psi_{j})$$
$$R_{j}^{*}(s) = R_{0}^{*}(s+\psi_{j}) = \frac{1}{s+\psi_{j}} \left\{ 1 - f_{0}^{*}(s+\psi_{j}) \right\}$$

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$$F_{0}^{*}(s - \psi_{j} = \frac{1}{s + \psi_{j}} f_{0}^{*}(s + \psi_{j})$$

$$f_{j}^{*}(s) = \psi_{j} R_{0}^{*}(s + \psi_{j}) + f_{0}^{*}(s + \psi_{j})$$

$$f_{0}^{*}(s + \psi_{j}) = \sum_{n=0}^{\infty} (-1)^{n} \frac{(s + \psi_{j})^{n}}{n!} \nu_{n}$$

1

Notation

 $\begin{array}{ll} \nu_n & \text{non-central moments} \\ c_n & \nu_{n+1}/(n+1) \\ m_p & \text{moment } p. \end{array}$

Setting s=0 in these 5 equations gives the first moments; higher moments can be derived by differentiating under the (implied) integral.

$$m_{1}(\psi_{j}) = R_{0}^{*}(\psi_{j}) = \frac{1}{\psi_{j}} [1 - f_{0}^{*}]$$

$$= \sum_{n=0}^{\infty} (-1)^{n} \frac{\psi_{j}^{n}}{n!} c_{n}$$
(2)

By repeatedly differentiating (2) the higher moments are:

$$m_p(\psi_j) = (-1)^{p-1} \frac{d^{p-1}}{d\psi_j^{p-1}} m_1$$
$$= p \sum_{n=0}^{\infty} (-1)^n \frac{\psi_j^n}{n!} c_{n+p-1}$$

Observations are made at *n* values of the covariate, $z_1, z_2, ..., z_n$.

Notation

 $\begin{array}{ll} x_{j,i} & \text{observation } i \text{ (censored or actual) at level } z_j \\ \beta, \ \theta & \text{parameters: } \beta \in \mathbf{R}^p, \ \theta \in \mathbf{R}^q \\ \theta & \text{parameter vector for } f_0 \\ \psi_j & \psi(z_j;\beta), \\ L^{(j)}(\beta,\theta) & \text{contribution to the log-likelihood for observations} \end{array}$

(censored or actual) at z_j ,

$$\psi_{j;k} = -\frac{\partial}{\partial \beta_k} \psi(z_j,\beta)$$

 h_{0r} , H_{0r} derivatives of $[h_0, H_0]$ w.r.t θ_r .

The likelihood equations are:

$$L^{(j)}(\beta,\theta) = \sum_{i \in D_j} \ln[f_j(x_{ji}; \psi_j] + \sum_{i \in C_j \cup D_j} \ln[R_j(x_{ji}; \psi_j)]$$

$$= \sum_{i \in D_j} \ln[\psi_j + h_0(x_{ji})] - \psi_j \sum_{i \in C_j \cup D_j} x_{ji} - \sum_{i \in C_j \cup D_j} H_0(x_{ji})$$
$$\mathcal{L} = \sum_{i=1}^p L^{(j)}(\beta, \theta)$$

The partial derivatives are:

$$\frac{\partial L^{(j)}}{\partial \beta_k} = \psi_{j;k} \left\{ \sum_{i \in D_j} \frac{1}{\psi_j + h_0(x_{ji})} - \sum_{i \in C_j \cup D_j} x_{ji} \right\}$$
$$= \psi_{j;k}(\beta) u_j(\beta, \theta)$$

This equation has defined $u_i(\beta, \theta)$.

$$\frac{\partial L^{(j)}}{\partial \beta_k} = \sum_{i \in D_j} \frac{h_{0r}(x_{ji})}{\psi_j + h_0(x_{ji})} - \sum_{i \in C_j \cup D_j} H_{0r}(x_{ji})$$

The maximum likelihood equations are:

$$\frac{\partial \mathcal{L}}{\partial \theta_r} = \sum_j \frac{\partial L^{(j)}}{\partial \theta_r}$$
$$= \sum_j \left\{ \sum_{i \in D_j} \frac{h_{0r}(x_{ji})}{\psi_j + h_0(x_{ji})} - \sum_{i \in C_j \cup D_j} H_{0r}(x_{ji}) \right\} = 0$$

This can be written as a matrix equation by setting $\psi = (\psi_{j,k})^T$.

$$\frac{\partial \mathcal{L}}{\partial \beta_k} = \sum_j \frac{\partial L^{(j)}}{\delta \beta_k} = \sum_j \psi_{j;k}(\beta) u_j(\beta,\theta) = 0$$
$$\Psi(z;\beta) u(\theta,\beta) = 0$$
(3)

Eq (3) shows that for a fixed θ there is always the solution:

$$u(\beta|\theta) = 0, \text{ or}$$

$$\sum_{i \in D_j} \frac{1}{\psi_j + h_0(x_{ji})} - \sum_{i \in C_j \cup D_j} x_{ji} = 0, \quad j = 1, \dots, p \quad (4)$$

and that the existence of other solutions depends on the rank of Ψ . In particular, if the ψ_j are not functionally related, but each is simply a parameter of distribution *j*, then $(\psi_{j;k})^T$ is an identity matrix and (4) gives the unique solutions. When p < n the rank of ψ is at most *p* and there are solutions determined by the null-space of ψ as well as those from *u*. Further, hazard rates satisfy (3) in the sense that [19]:

$$E\{1/h(x)\} = E\{x\}.$$
 (5)

Thus if $\psi_j + h_0(t)$ is a hazard rate, then it should satisfy (5): and the empirical version of (5) is just (4). This means that solutions determined by the null-space of Ψ that do not also satisfy (4) do not yield hazard rates. Conversely solutions of (4) which are not in the null-space of Ψ do not allow the values of ψ to be estimated. In short there is an identifiability problem in which the values of ψ_i can be estimated, but not the parametric form.

6. REPAIRABLE SYSTEMS

The ideas of a renewal process can be extended to a system that after repair -

- is returned to the working state, but in a condition between the new state and the failed state.
- can age more rapidly than before.

All of the models (AFTM, PHM, CRM, AHM) can incorporate these aspects. The 2 questions about such a system are:

- Can the state of the system through time be predicted?
- Can the effects of various operating conditions be incorporated in the model, *eg*, can the effects of modification or a different maintenance regime be modelled?

The easiest assumption is that the behavior of the interval lengths is modelled by changes in the covariates. The techniques outlined for samples can be extended following Cox [6] to a modulated renewal process by assuming that in place of the ordinary renewal process the assumption of i.i.d. intervals is dropped and replaced by that of interval length depending only on the state at the beginning of the interval. The modulated renewal process is a Markov renewal process [4: chapter 3] in which the sequence of intervals is deterministic and the distributions of interval length are modulated by another process; the assumptions are strengthened by taking the covariates z_i as deterministic. The assumption is that X_i has Cdf $F_{X_i}(x)$. The aim of the models is to reflect in the $F_{X_i}(x)$ the changes in the system as its history unfolds. Through such models we hope to discover whether the system improves or deteriorates through time, and eg, determine optimal repair policies.

The simplifying assumption that X_j has Cdf $F_{X_j}(x)$ means that the durations of X_j can be treated simply as a sample, and used to construct a log-likelihood $L^{(j)}$ for that interval alone. The log-likelihood is:

$$\mathfrak{L} = \sum_{j=1}^{n} L^{(j)}$$

This log-likelihood also provides log-likelihoods for the parameters of $F_{X_j}(x)$ as functions of explanatory variables. The estimators are, in principle, obtained as solutions of the likelihood equations obtained setting the appropriate derivatives of \mathcal{L} to zero.

Techniques for the analysis of renewal processes can also be carried over, in principle, to the modulated renewal process, although explicit closed formulas for measures of interest are mostly not available. The strong assumption that the explanatory variables are deterministic (eg, interval number), allows a direct imitation of the renewal process argument: the time of event n, t_n is simply the sum of the n s-independently distributed interval lengths x_i :

$$t_n = \sum_{i=1}^n x_i.$$

Thus the Laplace transform of $g_n(t) = pdf\{t_n\}$ is:

$$g_n^*(s) = \prod_{i=1}^n f_{X_i}^*(s)$$

The renewal function V(t) has Laplace transform:

$$V^*(s) = (1/s) \sum_{i=1}^{\infty} g_i(s).$$

PHM & AFTM can only model the rate of *aging* between events. However, if these models are based on $h_0(x)$ with the property $h_0(0)=0$, then every interval hazard rate h_j also satisfies $h_j(0)=0$. If h(0)=0, then the system is instantaneously as good as new after a repair, even though thereafter it might *age* faster. The additive hazards approach provides a means of modelling situations in which $h_j(0) > 0$. Hardly any of the common distributions fulfil the requirements of a hazard rate (and thus pdf) that is non-zero at time zero and has an increasing hazard rate. Apart from the exponential, all the common distributions have a zero or an infinite hazard rate at time zero.

Additive hazards arise reasonably from the desire to model a system that: after a repair is better than it was just before the repair, but not as good as new. Thus if a new system has interval hazard rate $h_0(x)$ after failure j-1, then the interval hazard rate is $\psi_j + h_0(x)$. The model is a simple additive analogue of PHM in which each failure contributes something to the *age* of the system. Moreover, AHM offers hazard rates that are not zero at time zero.

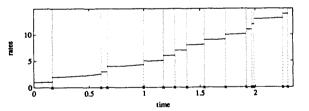


Figure 1. Simulated Additive-Hazards Process

Figure 1 shows a simulated AHM with imperfect repair. The stippled lines indicate the times of the random events, and the solid lines show how the hazard develops through time. The rate of *aging* between events repeats itself, while *aging* is seen in the rising initial value for the hazard rate. However, in view of the difficulties of estimation described above, it can only be used in a phenomenological way to measure the magnitude of the jumps ψ_j in the hazard rate. Models for ψ_j which use explanatory variables yield unsatisfactory estimators of the parameters [15, 16]. The statistical problems are unfortunate since in this case the Laplace transforms required for a renewal

process approach are likely to be more readily obtained. The Laplace transforms of $g_n(t) \& V(t)$ are:

$$g_n(t) \& V(t)$$
 are:

$$g_n^*(s) = \prod_{i=1}^n f_{X_i}^*(s) = \prod_{i=1}^n f_0^*(s + \psi_i)$$
$$V^*(s) = (1/s) \sum_{i=0}^\infty g_i^*(s).$$

In a study of failure data [9, 10], Pijnenburg [15] showed that on the basis of a graphical analysis, an AHM seemed plausible with the sequence number of the failure as an explanatory variable, but failed to find estimators for the AHM:

$$H_{i}(x) = (\beta_{1} + \beta_{2} \cdot j) + h_{0}(x).$$

The reasons for the lack of estimators are now clear from this paper.

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Martin Newby: For biography, see *IEEE Trans. Reliability*, vol 43, 1994 June, p ???.

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Failure Distributions for Local vs Global and Replicate vs Standby Redundancies

(Continued from page 483)

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