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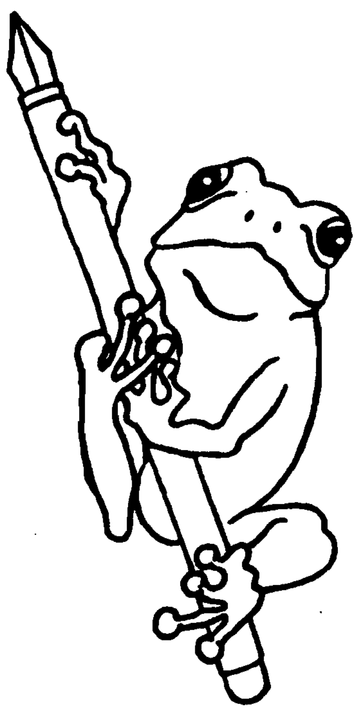

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# On linear subsystems of nonlinear control systems *

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**Abstract**

In this paper we consider the problem of simultaneous (partial) feedback linearization and input-output linearization for SISO nonlinear control systems. It is shown that the problem of existence of a linear subsystem of a certain dimension may be reduced to a well-known problem from real algebraic geometry.

## 1 Introduction and problem statement

In this paper we consider a smooth SISO nonlinear control system $\Sigma$ of the form

$$\Sigma \left\{ \begin{array}{lcll} \dot{x} & = & f(x) + g(x)u & ,x \in \mathbb{R}^n,\ u \in \mathbb{R} \\ y & = & h(x) & ,y \in \mathbb{R} \end{array} \right. \tag{1}$$

Further, consider a linear SISO system $\bar{\Sigma}$ of the form

$$\bar{\Sigma} \left\{ \begin{array}{lcll} \dot{\xi} & = & \bar{A}\xi + \bar{B}\bar{u} & ,\xi \in \mathbb{R}^{\bar{n}},\ \bar{u} \in \mathbb{R} \\ \eta & = & \bar{C}\xi & ,\eta \in \mathbb{R} \end{array} \right. \tag{2}$$

where $\bar{n} \leq n$. We will call $\bar{\Sigma}$ a (linear) *subsystem* of $\Sigma$ if for $\Sigma$ there exist a regular static state feedback $Q_s:\ u = \alpha(x) + \beta(x)v$ and new coordinates $\bar{x}(x) = (\bar{x}_1(x), \bar{x}_2(x))$ such that in the new coordinates $\bar{x}(x)$ the system $\Sigma \circ Q_s$ takes the form

$$\Sigma \circ Q_s \left\{ \begin{array}{lcl} \dot{\bar{x}}_1 & = & \bar{A}\bar{x}_1 + \bar{B}v \\ \dot{\bar{x}}_2 & = & \bar{a}(\bar{x}) + \bar{b}(\bar{x})v \\ y & = & \bar{C}\bar{x}_1 \end{array} \right. \tag{3}$$

In this paper we answer the question whether, given $\bar{n} \in \{1, \cdots, n\}$, the system $\Sigma$ has a controllable linear subsystem of dimension $\bar{n}$. Note that if $\Sigma$ has a linear subsystem, one may partially feedback linearize the system by means of regular static state feedback and coordinate transformation, while at the same time achieving a linear input-output behavior.

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In this respect the problem considered in this paper may be seen as a combined (partial) feedback linearization problem and input-output linearization problem. For an overview of the literature on (partial) feedback linearization we refer to [9],[10],[11] and the references therein, while for an overview of the literature on input-output linearization we refer to [8] and the references therein. Further, note that the question whether a system has a linear subsystem of dimension $n$ has been answered in [3].

The organization of the paper is as follows. In the next section we will introduce some notation, concepts and results that will be used in the rest of the paper. In Section 3 preliminary necessary and sufficient conditions for the existence of a controllable linear subsystem of a given dimension will be derived. Starting from these conditions, it will be shown in Section 4 that the problem under consideration may be reduced to a well known problem from real algebraic geometry. In Section 5, we give an example, and in Section 6 some conclusions are drawn.

# 2 Preliminaries

## 2.1 Relative degree of one-forms

In this subsection we give a differential-geometric treatment of the relative degree of one-forms. The concept of relative degree of a one-form was introduced in [2] in an algebraic framework. Define the manifold $M_0 := \mathbb{R}^n$ with local coordinates $x$, and the manifolds $M_k := M_{k-1} \times \mathbb{R}$ with local coordinates $(x, u, \cdots, u^{(k-1)})$ $(k = 1, \cdots, 2n+1)$. Clearly, $M_k$ is an immersed submanifold of $M_\ell$ $(k = 0, \cdots, 2n;\ \ell = k+1, \cdots, 2n+1)$, with the natural immersion $i_{k\ell} : M_k \to M_\ell$ defined by

$$i_{k\ell}(x, u, \cdots, u^{(k-1)}) = (x, u, \cdots, u^{(k-1)}, 0, \cdots, 0) \tag{4}$$

Let $\Xi_k$ denote the codistribution $\mathrm{span}\{dx\}$ on $M_k$ $(k = 0, \cdots, 2n+1)$. On $M_{2n+1}$, we define the extended vector field

$$f^e := (f + gu)\frac{\partial}{\partial x} + \sum_{i=0}^{2n} u^{(i+1)} \frac{\partial}{\partial u^{(i)}} \tag{5}$$

For a one-form $\omega$ on $M_k$ $(k = 0, \cdots, n+1)$, we define $\omega^{(\ell)}$ on $M_{2n+1}$ by

$$\begin{aligned} &\omega^{(\ell)} := \mathcal{L}^{\ell}_{f^e}((i_{k2n+1})_* \omega) \\ &\qquad (\omega \in M_k; k = 0, \cdots, n+1; \ell = 0, \cdots, 2n+1-k) \end{aligned} \tag{6}$$

Then $\omega^{(\ell)}$ may be interpreted as a one-form on on $M_{k+\ell}$, in the sense that

$$\begin{aligned} &(i_{k+\ell 2n+1})_*(i_{k+\ell 2n+1})^* \omega^{(\ell)} = \omega^{(\ell)} \\ &\qquad (\omega \in M_k; k = 0, \cdots, n+1; \ell = 0, \cdots, 2n+1-k) \end{aligned} \tag{7}$$

Let $\omega \in \Xi_k$ $(k = 0, \cdots, n)$, and assume that there exists an $\ell \in \{1, \cdots, n\}$ such that $\omega^{(\ell)} \notin \Xi_{2n+1}$. Then the smallest such $\ell$ is called the *relative degree* of $\omega$, to be denoted by $r_\omega$. If for all $\ell \in \{1, \cdots, n\}$ we have that $\omega^{(\ell)} \in \Xi_{2n+1}$, we define $r_\omega := +\infty$. For a function $\phi$

satisfying $d\phi \in \Xi_k$, we define its relative degree by $r_\phi := r_{d\phi}$. Define the codistributions $\mathcal{H}_k^\ell$ $(k = 1, \cdots, n;\ \ell = k-1, \cdots, 2n+1-k)$ by

$$\mathcal{H}_k^\ell := \{\omega \in \Xi_\ell \mid r_\omega \geq k\} \tag{8}$$

Using (7), it may then be shown that $\mathcal{H}_k^\ell$ may be identified with $\mathcal{H}_k^{k-1}$, in the sense that

$$(i_{k-1\ell})_*(i_{k-1\ell})^*\mathcal{H}_k^\ell = (i_{k-1\ell})_*\mathcal{H}_k^{k-1}$$
$$(k = 1, \cdots, n; \ell = k-1, \cdots, 2n+1-k) \tag{9}$$

We further define the codistribution $\mathcal{H}_\infty^n$ on $M_n$ by

$$\mathcal{H}_\infty^n := \{\omega \in \Xi_n \mid r_\omega = +\infty\} \tag{10}$$

Next, define

$$\mathcal{H}_k := (i_{k-1 2n+1})_*\mathcal{H}_k^{k-1} \quad (k = 1, \cdots, n) \tag{11}$$

$$\mathcal{H}_\infty := (i_{n 2n+1})_*\mathcal{H}_\infty^n \tag{12}$$

We then have the following properties (for a proof, see (mutatis mutandis) [2]).

**Lemma 2.1** *(i)* $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots \supset \mathcal{H}_n \supset \mathcal{H}_\infty$.

*(ii)* $\mathcal{H}_\infty$ *is integrable.*

*(iii)* $\Sigma$ *is strongly accessible if and only if* $\mathcal{H}_\infty = \{0\}$.

*(iv)* $\mathcal{H}_k = \{\omega \in \mathcal{H}_{k-1} \mid ((i_{k-2 2n+1})^*\omega)^{(1)} \in \mathcal{H}_k\}$ $(k = 1, \cdots, n)$.

*(v)* $\mathcal{H}_\infty = \{\omega \in \mathcal{H}_n \mid ((i_{n-1 2n+1})^*\omega)^{(1)} \in \mathcal{H}_n\}$.

*(vi) Define*

$$\sigma := n + 1 - \dim(\mathcal{H}_\infty) \tag{13}$$

*Then*

$$\dim(\mathcal{H}_k) = n + 1 - k \quad (k = 1, \cdots, \sigma) \tag{14}$$

*and*

$$\mathcal{H}_k = \mathcal{H}_\infty \quad (k = \sigma, \cdots, n) \tag{15}$$

*(vii) Let* $\lambda \in \mathcal{H}_{\sigma-1} \backslash \mathcal{H}_\infty$. *Then we have for* $k \in \{1, \cdots, \sigma - 1\}$:

$$\mathcal{H}_k = \operatorname{span}\{((i_{n-2 2n+1})^*\lambda)^{(\ell)} \mid \ell = 0, \cdots, \sigma - 1 - k\} \oplus \mathcal{H}_\infty \tag{16}$$

■

## 2.2 Parametrized post compensated system

In the sequel, the notion of a parametrized post compensated system will be of key importance. In this subsection we introduce this notion, and give some properties. Consider a smooth SISO system $\Sigma$ of the form (1), and let $d \in \mathbb{N}$ be given. Let $s_1, \cdots, s_d$ be parameters that take their values in $\mathbb{R}$. We then define a *parametrized post compensated system* $\Sigma^p\langle s_1, \cdots, s_d\rangle$ by

$$\Sigma^p\langle s_1,\cdots,s_d\rangle \begin{cases} \dot{x} &= f(x)+g(x)u \\ \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{d-1} &= z_d \\ \dot{z}_d &= h(x) - \sum\limits_{k=1}^{d} s_k z_k \end{cases} \tag{17}$$

Similarly to what has been done in the previous subsection, one may define a sequence of parametrized codistributions $\mathcal{H}_k\langle s_1, \cdots, s_d\rangle$ for $\Sigma^p\langle s_1, \cdots, s_d\rangle$. Define $M := M_{2n+1}$, where $M_{2n+1}$ has been defined in the previous subsection, and define $M^p := \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{2(n+d)+1}$ with local coordinates $(x, z, u, \cdots, u^{(2(n+d))})$. Define the immersion $i : M \rightarrow M^p$ by

$$i(x, u, \cdots, u^{(2n)}) := (x, 0, u, \cdots, u^{(2n)}, 0, \cdots, 0)$$

Further, let $\Xi$, $\Xi^p$ denote the codistribution $\mathrm{span}\{dx\}$ on $M$ and $M^p$ respectively. For $\Sigma^p\langle s_1, \cdots, s_d\rangle$, we define the codistributions

$$\mathcal{H}_k^e := i_*\mathcal{H}_k \quad (k = 1, \cdots, n) \tag{18}$$

$$\mathcal{H}_\infty^e := i_*\mathcal{H}_\infty \tag{19}$$

It then follows from the form of $\Sigma^p\langle s_1, \cdots, s_d\rangle$ that

$$\forall_{s_1,\cdots,s_d\in\mathbb{R}}\ \forall_{k\in\{1,\cdots,n\}}\ \mathcal{H}_k^e \subset \mathcal{H}_k^p\langle s_1, \cdots, s_d\rangle \tag{20}$$

$$\forall_{s_1,\cdots,s_d\in\mathbb{R}}\ \forall_{k\in\{n+1,\cdots,n+d,\infty\}}\ \mathcal{H}_\infty^e \subset \mathcal{H}_k^p\langle s_1, \cdots, s_d\rangle \tag{21}$$

$$\forall_{s_1,\cdots,s_d\in\mathbb{R}}\ \forall_{k\in\{1,\cdots,n\}}\ \mathcal{H}_k^p\langle s_1, \cdots, s_d\rangle \cap \Xi^p = \mathcal{H}_k^e \tag{22}$$

$$\forall_{s_1,\cdots,s_d\in\mathbb{R}}\ \forall_{k\in\{n+1,\cdots,n+d,\infty\}}\ \mathcal{H}_k^p\langle s_1, \cdots, s_d\rangle \cap \Xi^p = \mathcal{H}_\infty^e \tag{23}$$

We now show that the codistributions $\mathcal{H}_k^p\langle s_1, \cdots, s_d\rangle$ $(k = 1, \cdots, \sigma)$ may be parametrized in a polynomial way. Let $\mathcal{S}$ denote the ring of smooth functions of $(x, u, \cdots, u^{(2n)})$, and define the polynomial ring $\mathcal{R} := \mathcal{S}[s_1, \cdots, s_d]$.

**Lemma 2.2** *Consider the parametrized post compensated system $\Sigma^p\langle s_1, \cdots, s_d\rangle$ and the sequence of parametrized codistributions $\mathcal{H}_k^p\langle s_1, \cdots s_d\rangle$ $(k = 1, \cdots, \sigma)$. Let $\lambda \in \mathcal{H}_n \backslash \mathcal{H}_\infty$ satisfy*

$$(i_{n-12n+1})_*(i_{n-12n+1})^*\lambda = \lambda \tag{24}$$

*Define $r := r_h$. Then*

$$\dim(\mathcal{H}_k^p\langle s_1, \cdots, s_d\rangle) = \dim(\mathcal{H}_k^e) + d \quad (k = 1, \cdots, \sigma) \tag{25}$$

*and there exist* $\phi_{k\ell} \in \mathcal{R}$ $(k = 1, \cdots, d;\ \ell = 0, \cdots, \sigma - r - d - 2 + k)$ *such that*

$$\mathcal{H}_k^p\langle s_1, \cdots, s_d\rangle = \operatorname{span}\{i_*\omega_k\langle s_1, \cdots, s_d\rangle - dz_k \mid k = 1, \cdots, d\} \oplus \mathcal{H}_\infty^e \qquad (k = 1, \cdots, \sigma) \tag{26}$$

*where*

$$\omega_k := \sum_{\ell=0}^{\sigma-r-d-2+k} \phi_{k\ell}\lambda^{(\ell)} \tag{27}$$

**Proof** Equality (25) follows straightforwardly from Lemma 2.1 and (20),$\cdots$,(23). It then follows from (21),(23),(25) that there exist parametrized one-forms $\tilde{\omega}_k\langle s_1, \cdots, s_d\rangle \in \Xi^p$ $(k = 1, \cdots, d)$ such that

$$\mathcal{H}_\sigma^p\langle s_1, \cdots, s_d\rangle = \operatorname{span}\{\tilde{\omega}_k\langle s_1, \cdots, s_d\rangle - dz_k \mid k = 1, \cdots, d\} \oplus \mathcal{H}_\infty^e \tag{28}$$

From Lemma 2.1.*(i)* and (20),(22),(28) it then follows that

$$\mathcal{H}_\ell^p\langle s_1, \cdots, s_d\rangle = \operatorname{span}\{\tilde{\omega}_k\langle s_1, \cdots, s_d\rangle - dz_k \mid k = 1, \cdots, d\} \oplus \mathcal{H}_\ell^e \qquad (\ell = 1, \cdots, \sigma) \tag{29}$$

What remains to be shown is that $\tilde{\omega}_k = i_*\omega_k$ $(k = 1, \cdots, d)$, where the $\omega_k$ are of the form (27). We give the proof for $d = 2$. The proof for $d > 2$ is analogous. Since $r_h = r$, there exist $\alpha_0, \cdots, \alpha_{\sigma-1-r} \in \mathcal{S}$ such that $\alpha_{\sigma-1-r} \neq 0$, and

$$dh = \sum_{\ell=0}^{\sigma-1-r} \alpha_\ell \lambda^{(\ell)} \tag{30}$$

From Lemma 2.1.*(iv)* and (29) it follows that

$$\dot{\tilde{\omega}}_1 - d\dot{z}_1 = \dot{\tilde{\omega}}_1 - \tilde{\omega}_2 + (\tilde{\omega}_2 - dz_2) \in \mathcal{H}_{\sigma-1}^p\langle s_1, s_2\rangle \tag{31}$$

and

$$\dot{\tilde{\omega}}_2 - d\dot{z}_2 = \dot{\tilde{\omega}}_2 + s_1\tilde{\omega}_1 + s_2\tilde{\omega}_2 - dh - s_1(\tilde{\omega}_1 - dz_1) - s_2(\tilde{\omega}_2 - dz_2) \in \mathcal{H}_{\sigma-1}^p\langle s_1, s_2\rangle \tag{32}$$

Let $\mathcal{S}^p$ denote the ring of smooth functions of $(x, z, u, \cdots, u^{(2(n+d))})$. With Lemma 2.1.*(vii)* it follows from (31),(32) that there exist parametrized functions $\beta_1\langle s_1, s_2\rangle, \beta_2\langle s_1, s_2\rangle$ satisfying $\beta_1\langle s_1, s_2\rangle, \beta_2\langle s_1, s_2\rangle \in \mathcal{S}^p$, $(\forall s_1, s_2 \in \mathbb{R})$ and parametrized one-forms $\pi_1\langle s_1, s_2\rangle, \pi_2\langle s_1, s_2\rangle$ satisfying $\pi_1\langle s_1, s_2\rangle, \pi_2\langle s_1, s_2\rangle \in \mathcal{H}_\infty^e$, $(\forall s_1, s_2 \in \mathbb{R})$ such that

$$\dot{\tilde{\omega}}_1 = \tilde{\omega}_2 + \beta_1(i_*\lambda) + \pi_1 \tag{33}$$

$$\dot{\tilde{\omega}}_2 = dh - s_1\tilde{\omega}_1 - s_2\tilde{\omega}_2 + \beta_2(i_*\lambda) + \pi_2 \tag{34}$$

From (33),(34) it follows in particular that $r_{\tilde{\omega}_1} = r + 2$, $r_{\tilde{\omega}_2} = r + 1$, and hence there exist parametrized functions $\tilde{\phi}_{k\ell}\langle s_1, s_2\rangle$ $(k = 1, 2;\ \ell = 0, \cdots, \sigma - 4 - r + k)$ and parametrized one-forms $\eta_1\langle s_1, s_2\rangle, \eta_2\langle s_1, s_2\rangle$ such that

$$\forall_{s_1, s_2 \in \mathbb{R}}\ \eta_1\langle s_1, s_2\rangle, \eta_2\langle s_1, s_2\rangle \in \mathcal{H}_\infty^e \tag{35}$$

$$\forall_{s_1,s_2\in\boldsymbol{R}}\ \forall_{k\in\{1,2\}}\ \forall_{\ell\in\{0,\cdots,\sigma-4-r+k\}}\ \tilde{\phi}_{k\ell}\in\mathcal{S}^p \tag{36}$$

$$\tilde{\omega}_k=\sum_{\ell=0}^{\sigma-4-r+k}\tilde{\phi}_{k\ell}(i_*\lambda)^{(\ell)}+\eta_k\ \ (k=1,2) \tag{37}$$

Comparing (30),(33),(34),(37) we then obtain:

$$\dot{\tilde{\phi}}_{10}-\tilde{\phi}_{20}=\beta_1 \tag{38}$$

$$\dot{\tilde{\phi}}_{1\ell}+\tilde{\phi}_{1\ell-1}-\tilde{\phi}_{2\ell}=0\ \ (\ell=1,\cdots,\sigma-3-r) \tag{39}$$

$$\tilde{\phi}_{1\sigma-3-r}-\tilde{\phi}_{2\sigma-2-r}=0 \tag{40}$$

$$\dot{\tilde{\phi}}_{20}-s_1\tilde{\phi}_{10}-s_2\tilde{\phi}_{20}=\alpha_0+\beta_2 \tag{41}$$

$$\dot{\tilde{\phi}}_{2\ell}+\tilde{\phi}_{2\ell-1}-s_1\tilde{\phi}_{1\ell}-s_2\tilde{\phi}_{2\ell}=\alpha_\ell\ \ (\ell=1,\cdots,\sigma-3-r) \tag{42}$$

$$\dot{\tilde{\phi}}_{2\sigma-2-r}+\tilde{\phi}_{2\sigma-3-r}-s_2\tilde{\phi}_{2\sigma-2-r}=\alpha_{\sigma-2-r} \tag{43}$$

$$\tilde{\phi}_{2\sigma-2-r}=\alpha_{\sigma-1-r} \tag{44}$$

From (40),(44) it follows that

$$\tilde{\phi}_{1\sigma-3-r}=\tilde{\phi}_{2\sigma-2-r}=\alpha_{\sigma-1-r}\in\mathcal{S}\subset\mathcal{R} \tag{45}$$

Equalities (43),(45) then give

$$\tilde{\phi}_{2\sigma-3-r}=\alpha_{\sigma-2-r}-\dot{\tilde{\phi}}_{2\sigma-2-r}+s_2\tilde{\phi}_{2\sigma-2-r}\in\mathcal{R} \tag{46}$$

Using an induction argument, it then follows from (39),(42),(45),(46) that

$$\tilde{\phi}_{k\ell}\in\mathcal{R}\ \ (k=1,2;\ell=1,\cdots,\sigma-4-r+k) \tag{47}$$

It further follows from (38),(41) that $\tilde{\phi}_{10}$, $\tilde{\phi}_{20}$ are arbitrary. Together with (47), this establishes our claim. ∎

## 3 Necessary and sufficient conditions

In this section we derive necessary and sufficient conditions for the existence of a linear subsystem of dimension $\bar{n}\in\{1,\cdots,n\}$ for a *strongly accessible* SISO system $\Sigma$. We start with some (rather trivial) observations.

**Lemma 3.1** *Consider a SISO system $\Sigma$ of the form (1), and define $r:=r_h$. Let $\bar{n}\in\{1,\cdots,n\}$ be given. Then $\Sigma$ has a linear subsystem of dimension $\bar{n}$* only if *$\bar{n}\geq r$.*

**Proof** Follows immediately from (3) and the fact that the relative degree of $h$ is invariant under regular static state feedback and coordinate transformations. ∎

**Lemma 3.2** *Consider a SISO system $\Sigma$ of the form (1), and define $r := r_h$. Then $\Sigma$ has a linear controllable subsystem of dimension $r$.*

**Proof** As is well known (see e.g. [9],[11]), the differentials $dy^{(k)}$ $(k = 0, \cdots, r-1)$ are linearly independent, and $y^{(r)} = a(x) + b(x)u$, where $b(x) \neq 0$. The result then follows by defining $\bar{x}_{1k} = y^{(k-1)}$ $(k = 1, \cdots, r)$ and $v := a(x) + b(x)u$. ∎

Our main result is as follows.

**Theorem 3.3** *Consider a strongly accessible SISO system $\Sigma$ of the form (1), and define $r := r_h$. Let $\bar{n} \in \{r+1, \cdots, n\}$ be given, and define $d := \bar{n} - r$. Consider the parametrized post compensated system $\Sigma^p\langle s_1, \cdots, s_d\rangle$, and the sequence of parametrized codistributions $\mathcal{H}_k^p\langle s_1, \cdots, s_d\rangle$. Then $\Sigma$ has a linear controllable subsystem of dimension $\bar{n}$ if and only if there exist $a_1, \cdots, a_d \in \mathbb{R}$ such that*

$$\mathcal{H}_\infty^p\langle a_1, \cdots, a_d\rangle = \mathcal{H}_{n+1}\langle a_1, \cdots, a_d\rangle \tag{48}$$

**Proof** *(necessity)* Assume that $\Sigma$ has a linear controllable subsystem $\bar{\Sigma}$ of dimension $\bar{n}$. Since $\bar{\Sigma}$ is controllable, one may assume without loss of generality that the matrices $\bar{A}$, $\bar{B}$ in (2) are in Brunovsky canonical form. Let $\bar{c}_i$ $(i = 1, \cdots, n)$ denote the entries of $\bar{C}$ in (2). Since the relative degree is invariant under state space transformations and regular static state feedback, we then have that $\bar{c}_{d+1} \neq 0$ and $\bar{c}_{d+2} = \cdots = \bar{c}_n = 0$. Consider the post compensated system $\Sigma^p\langle \frac{\bar{c}_1}{\bar{c}_{d+1}}, \cdots, \frac{\bar{c}_d}{\bar{c}_{d+1}}\rangle$, and define new coordinates $(x, \xi)$ for this system, with $\xi_i := z_i - \bar{c}_{d+1}\bar{x}_{1i}$ $(i = 1, \cdots, d)$. In these new coordinates we have

$$\dot{\xi}_i = z_{i+1} - \bar{c}_{d+1}\bar{x}_{1i+1} = \xi_{i+1} \quad (i = 1, \cdots, d-1) \tag{49}$$

and

$$\dot{\xi}_d = \sum_{k=1}^{d+1} \bar{c}_k \bar{x}_{1k} - \sum_{k=1}^{d} \frac{\bar{c}_k}{\bar{c}_{d+1}} z_k - \bar{c}_{d+1}\bar{x}_{id+1} = -\sum_{k=1}^{d} \frac{\bar{c}_k}{\bar{c}_{d+1}} \xi_k \tag{50}$$

From (49),(50) it follows that

$$\mathcal{H}_\infty^p\langle \frac{\bar{c}_1}{\bar{c}_{d+1}}, \cdots, \frac{\bar{c}_d}{\bar{c}_{d+1}}\rangle = \operatorname{span}\{d\xi_1, \cdots, d\xi_d\} \tag{51}$$

From Lemma 2.2 and the fact that $\mathcal{H}_\infty^p\langle \frac{\bar{c}_1}{\bar{c}_{d+1}}, \cdots, \frac{\bar{c}_d}{\bar{c}_{d+1}}\rangle \subset \mathcal{H}_{n+1}\langle \frac{\bar{c}_1}{\bar{c}_{d+1}}, \cdots, \frac{\bar{c}_d}{\bar{c}_{d+1}}\rangle$ it then follows that there exist $a_1, \cdots, a_d \in \mathbb{R}$ such that (48) holds.

*(sufficiency)* Assume that there exist $a_1, \cdots, a_d \in \mathbb{R}$ such that (48) holds. It then follows from Lemma 2.2 that there exist one-forms $\omega_1, \cdots, \omega_d \in \operatorname{span}\{dx\}$ such that

$$\mathcal{H}_\infty^p\langle a_1, \cdots, a_d\rangle = \operatorname{span}\{\omega_1 - dz_1, \cdots, \omega_d - dz_d\} \tag{52}$$

and

$$\mathrm{d}\omega_i \in \operatorname{span}\{\pi \wedge \rho \mid \pi, \rho \in \operatorname{span}\{dx, du, \cdots, du^{(2n)}\}\} \tag{53}$$

From (52) and the form of $\Sigma^p\langle a_1,\cdots,a_d\rangle$ it follows that

$$\dot{\omega}_i = \omega_{i+1} \quad (i = 1,\cdots,d) \tag{54}$$

and

$$dh = \dot{\omega}_d + \sum_{k=1}^{d} a_k\omega_k \tag{55}$$

Combining (54) and (55), we obtain

$$dh = \omega_1^{(d)} + \sum_{k=1}^{d} a_k\omega_1^{(d-1)} \tag{56}$$

We next show that $\omega_1$ is exact. The fact that $\mathcal{H}^p_\infty\langle s_1,\cdots,s_d\rangle$ is integrable implies that we have

$$\begin{aligned} 0 = \quad & \mathrm{d}(\omega_1 - dz_1)\wedge(\omega_1 - dz_1)\wedge\cdots\wedge(\omega_d - dz_d) = \\ & \mathrm{d}\omega_1 \wedge(\omega_1 - dz_1)\wedge\cdots\wedge(\omega_d - dz_d) \end{aligned} \tag{57}$$

Together with (53) this gives that $\mathrm{d}\omega_1 = 0$, and hence $\omega_1$ is (locally) exact. Let $\bar{x}_{11}$ be such that $\omega_1 = d\bar{x}_{11}$. It follows from Lemma 2.2 that $r_{\bar{x}_{11}} = r + d$. Defining $\bar{x}_{1k} := \mathcal{L}_f^{k-1}\bar{x}_{11}$ $(k = 2,\cdots,r+d)$, this then gives that the differentials $d\bar{x}_{11},\cdots,d\bar{x}_{1r+d}$ are linearly independent, and that $\dot{\bar{x}}_{1r+d} = a(x) + b(x)u$, where $b(x) \neq 0$. Further, it follows from (56) that $y = \sum_{k=1}^{d} a_k\bar{x}_{1k} + \bar{x}_{1d+1}$. Defining $v := a(x) + b(x)u$, it is then established that $\Sigma$ has a linear subsystem of dimension $r + d = \bar{n}$. ∎

**Remark 3.4** Let $d \in I\!N$ be given. Checking the proof of Theorem 3.3, one sees that $\Sigma$ has a linear subsystem of dimension $r + d$ if and only if there exist a function $\phi : I\!R^n \to I\!R$ and $a_1,\cdots,a_d \in I\!R$ such that

$$r_\phi = r + d \tag{58}$$

and

$$h = \mathcal{L}_f^d\phi + \sum_{k=1}^{d} a_k\mathcal{L}_f^{d-1}\phi \tag{59}$$

Rewriting (58) as

$$\mathcal{L}_\tau\phi = 0 \quad (\forall\tau \in \mathcal{H}^\perp_{r+d}) \tag{60}$$

one obtains a set of nonlinear PDE's for $\phi$. The integrability conditions for this set of PDE's are given by (48). Further, note that it follows from the sufficiency-part of the proof of Theorem 3.3 that the zeros of the linear subsystem are given by the zeros of the polynomial $p(s) := s^d + \sum_{k=1}^{d} a_k s^{k-1}$.

# 4 Reduction to an algebro-geometric problem

In this section we show that the question whether there exists a linear subsystem of dimension $\bar{n} > r$ is equivalent to a well-known problem from real algebraic geometry. For reasons of clarity of exposition and space limitations, we first restrict to the case $\bar{n} = r + 1$. At the end of the section we make some remarks about the case $\bar{n} > r + 1$. Assume that $\Sigma$ is strongly accessible. Let $\lambda \in \mathcal{H}_n - \{0\}$ be such that (16),(24) hold. Define $r := r_h$. Then there exist $\alpha_0, \cdots, \alpha_{n-r} \in \mathcal{S}$ such that $\alpha_{n-r} \neq 0$ and

$$dh = \sum_{\ell=0}^{n-r} \alpha_\ell \lambda^{(\ell)} \tag{61}$$

Consider the parametrized post compensated system $\Sigma^p\langle s\rangle$. It then follows from Lemma 2.2 that there exist $\phi_\ell \in \mathcal{R}$ $(\ell = 0, \cdots, n - r - 1)$ such that

$$\mathcal{H}^p_{n+1}\langle s\rangle = \mathrm{span}\{ \sum_{\ell=0}^{n-r-1} \phi_\ell\langle s\rangle\lambda^{(\ell)} - dz\} \tag{62}$$

Define $\psi_0, \cdots, \psi_{n-r} \in \mathcal{R}$ by

$$\psi_0 := \dot{\phi}_0 + s\phi_0 - \alpha_0 \tag{63}$$

$$\psi_\ell := \dot{\phi}_\ell + \phi_{\ell-1} + s\phi_\ell - \alpha_\ell \quad (\ell = 1, \cdots, n - r - 1) \tag{64}$$

$$\psi_{n-r} := \phi_{n-r-1} - \alpha_{n-r} \tag{65}$$

Let $0_\mathcal{S}$ denote the zero-function. We now have the following result.

**Theorem 4.1** *Consider a strongly accessible SISO system $\Sigma$ of the form (1), and define $r := r_h$. Let $\psi_0, \cdots, \psi_{n-r}$ be defined by (63),(64),(65). Then $\Sigma$ has a linear subsystem of dimension $r + 1$ if and only if $\psi_0, \cdots, \psi_{n-r}$ have a common real zero, i.e.,*

$$\exists_{a\in \mathbb{R}} \forall_{\ell\in\{0,\cdots,n-r\}} \ \psi_\ell\langle a\rangle = 0_\mathcal{S} \tag{66}$$

**Proof** From Theorem 3.3 it follows that $\Sigma$ has a linear subsystem of dimension $r + 1$ if and only if there exists an $a \in \mathbb{R}$ such that $\mathcal{H}_{n+1}\langle a\rangle = \mathcal{H}_\infty\langle a\rangle$. It is straightforwardly shown that this is equivalent to the existence of an $a \in \mathbb{R}$ such that

$$\frac{d}{dt}( \sum_{\ell=0}^{n-r-1} \phi_\ell\langle a\rangle\lambda^{(\ell)}) + a( \sum_{\ell=0}^{n-r-1} \phi_\ell\langle a\rangle\lambda^{(\ell)}) = dh \tag{67}$$

It then easily follows that this is equivalent to (66). ■

We next indicate how (66) may be checked by reducing the question to the question whether a set of polynomials in $\mathbb{R}[s]$ has a common real zero. Define $\xi := \mathrm{col}(x, u, \cdots, u^{(2n)}) \in \mathbb{R}^{3n+1}$, and let $\nu$ denote the maximal degree in $s$ of the polynomials $\psi_0, \cdots, \psi_{n-r}$. Then there exist functions $\psi^k_\ell \in \mathcal{S}$ such that

$$\psi_\ell\langle s\rangle(\xi) = \sum_{k=0}^{\nu} \psi^k_\ell(\xi)s^k \quad (\ell = 0, \cdots, n - r) \tag{68}$$

Define the $(n-r+1, \nu+1)$-matrix $P(\xi)$ with entries $P_{ij}(\xi) := \psi_i^j(\xi)$ $(i = 0, \cdots, n-r;\ j = 0, \cdots, \nu)$. Further, define for $s \in \mathbb{R}$ the vector $v_s := \mathrm{col}(1, s, \cdots, s^\nu)$. Then the question to be considered is whether there exists a *real* solution to the equation $P(\xi)v_s \equiv 0$. Obviously, there exists a real solution to this equation *only if* there exists a $v \in \mathbb{R}^{\nu+1}$ satisfying the equation $P(\xi)v \equiv 0$. Note that this equation may be extended by the equations $(\partial/\partial\xi_i(P(\xi)))v \equiv 0$ $(i = 1, \cdots, 2n)$ and equations obtained by taking higher-order partial derivatives. One may now come up with an algorithm that performs this extension in a controlled way ([13]). The algorithm is reminiscent of the Structure Algorithm ([9],[11]). The final result of the algorithm will be a constant right-invertible $(q, \nu+1)$-matrix $P$ (for some $q \in \mathbb{N}$ with the property that $\{v \in \mathbb{R}^{\nu+1} \mid P(\xi)v \equiv 0\} = \mathrm{Ker}P$. It then follows that $a \in \mathbb{R}$ satisfies (66) if and only if $Pv_a = 0$, i.e, if and only if $a$ is a common zero of the polynomials $p_i(s) := \sum_{j=1}^{\nu+1} P_{ij}s^{j-1}$ $(i = 1, \cdots, q)$. Let $\langle p_1, \cdots, p_q\rangle$ denote the polynomial ideal in $\mathbb{R}[s]$ spanned by $p_1, \cdots, p_q$. Since $\mathbb{R}[s]$ is a principal ideal domain, there exists a polynomial $\hat{p} \in \mathbb{R}[s]$ with the property that $\langle p_1, \cdots, p_q\rangle = \langle\hat{p}\rangle$. Hence $a \in \mathbb{R}$ satisfies (66) if and only if $\hat{p}(a) = 0$. Thus, we have reduced our problem to the problem whether a monovariable polynomial has a real root. This is a well-known problem from real algebraic geometry, that has received attention since the time of Newton and Descartes. Obviously, there exists a real root when the polynomial $\hat{p}$ is of odd degree. When $\hat{p}$ is of even degree, one can check whether $\hat{p}$ has a real zero (in fact one can even determine the number of real zeros) using the so called Newton sums and Hankel forms associated with the polynomial. We refer to [6] for details on this topic.

In case one is trying to answer the question whether $\Sigma$ has a real subsystem of dimension $\bar{n} > r+1$, one can proceed roughly in the same way as above. In this case, it may be shown that there exists a linear subsystem of dimension $\bar{n}$ if and only if a set of polynomials $\psi_0, \cdots, \psi_\gamma \in \mathcal{S}[s_1, \cdots, s_d]$, where $d := \bar{n} - r$, has a common real zero. Applying the same kind of algorithm as indicated above, the problem may then reduced to the problem whether a set of polynomials $p_1, \cdots, p_q \in \mathbb{R}[s_1, \cdots, s_d]$ has a common real zero. This problem has first been solved by Tarski ([12]). Later on, the problem has been considered by Collins ([4], see also [1],[5]) by using the concept of Cylindrical Algebraic Decomposition (CAD) of $\mathbb{R}^n$. Further, with the method of CAD one can also tackle problems in which polynomial equalities as well as polynomial inequalities play a role. By using polynomial inequalities obtained from the Routh-Hurwitz test, this also allows to check whether there exist linear subsystems with stable zero dynamics. By now, MAPLE-implementations of the algorithm for Cylindrical Algebraic Decomposition are available. A drawback, however, is that the complexity of the algorithm is doubly exponential.

# 5 Example

Consider on $\{x \in \mathbb{R}^3 \mid x_2 \geq 0\}$ the nonlinear SISO system $\Sigma$ given by

$$\Sigma \left\{ \begin{array}{rcl} \dot{x}_1 &=& x_1^2x_2 + x_1u \\ \dot{x}_2 &=& x_2 - \frac{1}{2}x_1 \\ \dot{x}_3 &=& -x_2 + x_3 - x_1x_2x_3 - x_3u \\ y &=& x_1x_2 \end{array} \right. \tag{69}$$

We have $r := r_h = 1$, and hence $\Sigma$ has a linear subsystem of dimension 1. We next check whether $\Sigma$ has a linear subsystem of dimension 2. To this end, we consider the post compensated system $\Sigma^p\langle s\rangle$. Define the one-forms $\omega_1, \omega_2, \omega_3$ by

$$\begin{array}{rcl}\omega_1 & := & dx_2^2 \\ \omega_2 & := & d(x_1x_3) \\ \omega_3 & := & d(x_1x_2)\end{array} \tag{70}$$

The one-forms $\omega_1$ and $\omega_2$ satisfy

$$\begin{array}{rcl}\dot\omega_1 & = & 2\omega_1 - \omega_3 \\ \dot\omega_2 & = & \omega_2 - \omega_3\end{array} \tag{71}$$

For $\Sigma^p\langle s\rangle$ we find

$$\mathcal{H}_4^p\langle s\rangle = \text{span}\{(s+1)\omega_1 - (s+2)\omega_2 - dz\} \tag{72}$$

From (70),(71),(72) it follows that $a \in I\!R$ satisfies $\mathcal{H}_\infty^p\langle a\rangle = \mathcal{H}_4^p\langle a\rangle$ if and only if it satisfies $a^2 + 3a + 2 = 0$, and hence $a = -1$ or $a = -2$. We have

$$\mathcal{H}_4^p\langle -2\rangle = \text{span}\{\omega_1 - dz\} \tag{73}$$

Defining new coordinates $\bar x_1 := x_2^2$, $\bar x_2 := \frac{d}{dt}(x_2^2) = 2x_2^2 - x_1x_2$, $\bar x_3 := x_3$, and choosing $u$ in an appropriate way, we then obtain the form (3) for $\Sigma$. We further have

$$\mathcal{H}_4^p\langle -1\rangle = \text{span}\{-\omega_2 - dz\} \tag{74}$$

If we now define new coordinates $\bar x_1 := x_1x_3$, $\bar x_2 := \frac{d}{dt}(x_1x_3) = -x_1x_2 + x_1x_3$, $\bar x_3 := x_2$, and choose $u$ in an appropriate way, we also obtain the form (3) for $\Sigma$.

We next check whether $\Sigma$ has a linear subsystem of dimension 3. Considering the post compensated system $\Sigma^p\langle s_1, s_2\rangle$, we obtain

$$\mathcal{H}_4^p\langle s_1, s_2\rangle = \text{span}\{\omega_2 - \omega_1 - dz_1, (s_2 - 2)(\omega_2 - \omega_1) - \omega_1 - dz_1\} \tag{75}$$

It then follows from (70),(71),(75) that $\mathcal{H}_4^p\langle a_1, a_2\rangle = \mathcal{H}_\infty^p\langle a_1, a_2\rangle$ if and only if

$$\begin{array}{rcl}a_2 & = & 3 \\ a_2^2 + a_2 + a_1 - 2 & = & 0 \\ a_2^2 - a_2 - 2 & = & 0\end{array} \tag{76}$$

Clearly, the first and last equation in (76) are contradictory. Hence $\Sigma$ does not have a linear subsystem of dimension 3. Note, however, that by choosing new coordinates $\bar x_1 := x_2^2 - x_1x_2$, $\bar x_2 := 2x_2^2 - x_1x_3$, $\bar x_3 := 4x_2^2 - x_1x_3 - x_1x_2$, and by choosing $u$ in an appropriate way, we may feedback linearize the state equations of $\Sigma$.

# 6 Conclusions

In this paper we have characterized the linear subsystems of a nonlinear SISO system. Further, it has been shown that the existence of a linear subsystem of a given dimension can be checked by reducing the problem to a well known problem from real algebraic geometry, that can be tackled by means of the so called Cylindrical Algebraic Decomposition (CAD). A drawback of using CAD is that the complexity of existing algorithms is doubly exponential. This brings up the question whether the use of CAD could be circumvented. One way to do this might be to investigate whether or not the polynomial equations obtained have some special (preferably triangular) structure that can be employed. This remains a topic for future research. A more practically oriented way is to come up with an "educated guess" of the possible zeros of a linear subsystem by using the linearization of the system around an equilibrium point. This will be the topic of a forthcoming paper ([7]). In this paper, we have restricted ourselves on the one hand to SISO systems, and on the other hand to regular static state feedback. We expect that an extension of the results in the paper to MIMO systems (using regular static state feedback) is possible. Also an extension to the regular dynamic feedback case (at least for square systems having an invertible decoupling matrix) seems possible. This last extension would be useful in the solution of the model matching problem by means of minimal order dynamic state feedback. These remain topics for future research.

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