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Abstract

In this paper we analyze stock allocation policies in general N-echelon distribution systems, where it is allowed to hold stock at all levels in the network. The goal is to achieve differentiated target customer service levels (fill rates). Various allocation rules and accompanying numerical methods that have already been developed for smaller networks are extended and compared in an extensive numerical experiment. We conclude that the extension of Balanced Stock rationing (see Van der Heijden [1996]) is the most accurate method, in particular in cases of relatively high imbalance. If the imbalance is not too high, the extension of Consistent Appropriate Share rationing (see [De Kok, Lagodimos & Seidel, 1994; Verrijt & De Kok, 1996]) performs good as well.

Keywords: Multi-echelon, inventory, allocation, rationing, divergent

1 Introduction

The last decade many companies have implemented DRP systems as the front-end of their integrated logistics control systems. DRP, Distribution Resource Planning (cf. Martin [1990]), is the equivalent of MRP, Manufacturing Resource Planning (cf. Vollmann, Berry & Whybark [1992]), for the distribution chain: The planning logic of DRP consolidates demand forecasts at different stockpoints into time-phased dependent demand at intermediate stockpoints and ultimately into time-phased demand at the manufacturing location. This top-down logic does not explicitly take into account possible (future) shortages at stockpoints. To circumvent this problem so-called re-scheduling messages are generated to inform a planner that the logic identified a shortage and the planner is supposed to solve this shortage. However, if the planner solves this problem, inevitably his solution impacts a number of decisions already taken by the DRP system at downstream stockpoints of the stockpoint, where the shortage occurred and most likely also at upstream stockpoints of this stockpoint. Hence the planner is forced to overrule the decisions of the planning system, since these decisions are not consistent. This manual replanning process can be quite time-consuming and intricate. This phenomenon has been identified by DRP system software suppliers. So most state-of-the-art DRP systems offer so-called fair shares allocation rules. The idea behind these rules is to share shortages among all downstream successors of a stockpoint where a shortage occurs. The logic of these allocation rules is usually straightforward, e.g., based on the (planned) demand ratios of the successors. An important drawback of application of these rules is that it is not clear whether the decisions, that result from applying them, are consistent with operational objectives regarding customer service at downstream successors.

In this paper we compare stock allocation rules for situations described above. We consider arbitrary N-echelon divergent distribution systems, i.e., distribution systems where each stockpoint has exactly one preceding supplying stockpoint and has itself an arbitrary number of successors. At the most downstream stockpoints of the system, the end-stockpoints, external customer demand occurs.

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We assume that customer demands at a particular end-stockpoint in subsequent review periods are independent and identically distributed. Demands at different stockpoints during a review period may be correlated. Instead of using DRP planning logic, we apply so-called echelon-stock policies. The *echelon stock* of a stockpoint is the sum of its physical stock plus the amount in transit to or on hand at its downstream stockpoints minus backorders at its end-stockpoints. Furthermore, we define the *echelon inventory position* of a stockpoint as its echelon stock plus the amount in transit to this stockpoint. The control policies used are periodic echelon order-up-to policies, i.e., each review period the echelon inventory position is raised to a fixed level by ordering a lot at its predecessor. We assume no lot sizing restrictions. In case the predecessor has not sufficient stock available the available stock is rationed among all successors, including the stockpoint under consideration. The allocation rule should be such that customer service considerations at all most downstream stockpoints of the predecessor are taken into account. We assume that each stockpoint has a fill rate target. The fill rate is defined as the fraction of demand satisfied directly from stock on hand.

The objective of the paper is to compare a number of practically applicable allocation rules. The comparison is based on the difference between target fill rates and actual fill rates, where the actual fill rates are computed by discrete event simulation. We incorporated the allocation rules into algorithms that compute the order-up-to-levels for arbitrary divergent N-echelon systems under periodic demand.

In the literature allocation rules have received considerable attention. Eppen & Schrage [1981] introduced a fair share allocation rule for a two-echelon system without intermediate stocks. The allocation rule ensures that at the end-stockpoints stockout probability are equalized. Extensions of the results of Eppen & Schrage are given by Federgruen & Zipkin [1984] and Van Donselaar & Wijngaard [1987]. An excellent overview on this line of research is given by Federgruen [1993]. The focus of this line of research is to determine allocation policies that minimize holding and shortage costs. Federgruen [1993] shows that with identical holding and penalty costs this implies that the allocation rule should yield equal stockout probabilities. Furthermore, most papers reviewed in Federgruen [1993] discuss two-echelon systems and it is not clear whether the results derived can be easily extended to arbitrary N-echelon systems, when taking into account computational considerations.

As a consequence of the cost structure chosen the allocation rules derived in Federgruen [1993] cannot be applied to the situation discussed in this paper, where we focus on target fill rates at end-stockpoints, which are not necessarily identical. In De Kok [1990] a generalization of the allocation rule proposed by Eppen & Schrage [1981] is presented that enables to compute the order-up-to-level in a two-echelon system with stockless depot, taking into account fill rate targets. De Kok, Lagodimos & Seidel [1994] generalized the results of De Kok [1990] to a two-echelon system where the depot is allowed to hold stock. They introduced the concept of Consistent Appropriate Share (CAS) rationing. Verrijdt & De Kok [1996] present a modification of the heuristic approach in De Kok [1990] to cope with significantly differing fill rate targets. Verrijdt & De Kok [1995] show that the results in De Kok [1990] can be generalized to arbitrary divergent N-echelon systems where only end-stockpoints are allowed to hold stocks. A generalization of the CAS rationing policy is the Balanced Stock (BS) rationing policy introduced by Van der Heijden [1996]. However, these allocation rules and inventory policies have not been extended yet to general N-echelon distribution systems where all upstream, downstream and intermediate stockpoints are allowed to hold stock. In this paper, we make such extensions of the analysis and we carry out an extensive numerical comparison of the different allocation rules. In view of the practical importance of allocation rules in DRP systems such a comparison is needed. The more so as there is hardly any theoretical insight into the way DRP systems should be parameterized such that operational costumer targets are achieved.

The paper is organized as follows. In Section 2 we present the divergent N-echelon system under consideration. The system dynamics of this system are investigated in Section 3. These still depend

on the rationing policy used at every stockpoint. In Section 4 two rationing policies and its variants are investigated by considering a two-echelon system. The application of both policies is extended to an N-echelon system in Section 5. An extensive numerical study has been undertaken to get insight in the performance of both policies. In Section 6 and 7 we consider many instances of a two-echelon system and a three-echelon system, respectively. Finally, we give our conclusions in Section 8.

2   Model description

Consider a single-item multi-echelon inventory system where every stockpoint is allowed to hold stock. The system has an *arborescent* structure, i.e., each location has a unique supplier. We refer to these kind of systems as divergent multi-echelon systems. The most upstream stockpoint (in Figure 1: stockpoint 1) can place orders at an external supplier having an infinite capacity, which means that this supplier can always meet the demand.
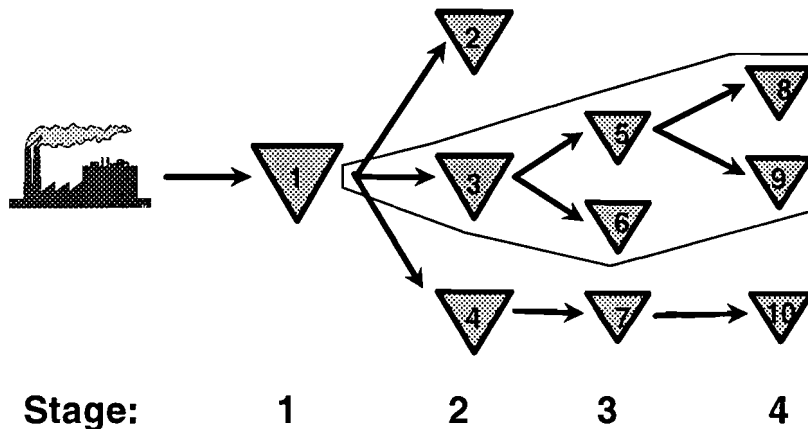
Figure 1: Schematic representation of a divergent 4-echelon inventory system.

The inventory in this system is controlled by a periodic review mechanism. That is, every R periods the most upstream stockpoint, i say, issues a replenishment order that raises the echelon inventory position to its order-up-to level S_i. This replenishment order arrives after a fixed lead time L_i. Then the physical stock at this most upstream stockpoint is allocated immediately to its successors using an allocation rule with two parameters $(S_j, \ p_j)$ for each successor j. When allocating stock, there are two possibilities:

(*i*). The physical stock is sufficient to raise the echelon inventory position of each successor to its maximum allowed level S_j. Then the required amounts are sent to the successors and excess stock is kept at stockpoint i to be allocated at the next occasion.

(*ii*). The physical stock is not sufficient to reach the levels S_j. Then a fraction p_j of the difference is subtracted from the amount that is sent to successor j with $\sum_j p_j = 1$.

A similar allocation procedure is applied at the intermediate stockpoints when a replenishment order arrives.

Without loss of generality, we assume that only the end-stockpoints face external customer demand. If an intermediate stockpoint faces external demand, we redirect this demand to a new successor k with lead time $L_k = 0$. This successor is an end-stockpoint. With respect to the demand process, we assume that all demand which cannot be satisfied immediately is backordered.

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The objective of the analysis is to determine the allocation parameters (S_i, p_i) at each intermediate and upstream stockpoint, such that every end-stockpoint attains its specific target service level. We will use the fill rate as service measure, defined as the fraction of demand satisfied immediately from the stock on hand. This service measure is widely used in practice, see [Silver & Peterson, 1985; Lagodimos, 1992; De Kok, 1990].

Several methods to obtain the allocation parameters are considered in this paper, based on CAS rationing on one hand and BS rationing on the other. We refer to Section 3 for the mathematical details. We introduce the following notation:

$ech(i)$	:=	Set of stockpoints that constitute the echelon of SP_i (e.g. $ech(5) = \{5, 8, 9\}$),
$pre(i)$	:=	Preceding stockpoint of stockpoint i (e.g. $pre(8) = 5$),
U_i	:=	Set of all stockpoints on path from supplier to stockpoint i (e.g. $U_1 = \varnothing$ and $U_6 = \{1, 3\}$),
V_i	:=	All stockpoints which are supplied by i (e.g. $V_1 = \{2, 3, 4\}$),
E	:=	Set of all end-stockpoints (e.g. $E = \{2, 6, 8, 9, 10\}$),
$E(i)$	:=	Set of all end-stockpoints in $ech(i)$ (e.g. $E(3) = \{6, 8, 9\}$),
M	:=	Set of all intermediate stockpoints (e.g. $M = \{1, 3, 4, 5, 7\}$),
N	:=	Number of stages in inventory system (e.g. $N = 4$).

The examples between the brackets refer to the situation of Figure 1.

3 System dynamics of an N-echelon system

In this paper we investigate several control policies, which all use the same kind of allocation rule. When applying this allocation rule we are able to determine the behavior of the stock level in every stockpoint of the N-echelon system. From this behavior a mathematical expression is derived which enables to compute the fill rate at an end-stockpoint *given* the control parameters of the system. For our convenience we use the following notation:

μ_i	:=	Mean of one-period demand at end-stockpoint i,
σ_i^2	:=	Variance of one-period demand at end-stockpoint i,
R	:=	Duration of a review period (in periods),
D_t^i	:=	The demand at the end-stockpoints in $ech(i)$ during t periods, a random variable with mean $t\mu_i$ and standard deviation $\sigma_i\sqrt{t}$.
D_{t_1,t_2}^i	:=	The demand at the end-stockpoints in $ech(i)$ during $(t_1, t_2]$, a random variable with mean $(t_2 - t_1)\mu_i$ and standard deviation $\sigma_i\sqrt{t_2 - t_1}$.
I_t^i	:=	The inventory position of stockpoint i just after rationing,
β_i	:=	Target fill rate at end-stockpoint i,
S_i	:=	Order-up-to-level of stockpoint i,
$z_j[x]$	:=	The echelon inventory position of stockpoint j just after allocation if the echelon inventory position of its supplier just before allocation equals x,
p_j	:=	Allocation-fraction from stockpoint $pre(j)$ to stockpoint j,
Δ_i	:=	Maximum physical stock at stockpoint i, $\Delta_i = S_i - \sum_{j \in V_i} S_j$,
$\mu_{ech(i)}$	:=	The expected demand at end-stockpoint k between the placement of an order by i and the earliest possible arrival time of products from this order at k, aggregated over all end-stockpoints k in $ech(i)$ (e.g. $\mu_{ech(3)}$ in Figure 1 equals $(L_3 + L_6 + R)\mu_6 + (L_3 + L_5 + L_8 + R)\mu_8 + (L_3 + L_5 + L_9 + R)\mu_9$). So, $\mu_{ech(i)} = \sum_{k \in E \cap ech(i)} ((\sum_{j \in U_K \cap ech(i)} L_j) + L_k + R)\mu_k.$
x^+	:=	$\max(0, x)$ for any expression x.

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Note that if the one period demand of all end-stockpoints are independent then $E[D_t^i]$ and $\sigma^2[D_t^i]$ are simply calculated from $E[D_t^i] = t\sum_{j\in E(i)} \mu_j$ and $\sigma^2[D_t^i] = t\sum_{j\in E(i)} \sigma_j^2$. A similar expression applies for D_{t_1,t_2}^i. These expressions can easily be modified to include correlations between end-stockpoints. First, the expressions for $E[D_t^i]$ remain the same. Second, defining ρ_{jk} as the correlation between the one period demand of two stockpoints j and k, we have the following modified expression for the variance: $\sigma^2[D_t^i] = t\sum_{j\in E(i)}\sum_{k\in E(i)} \rho_{jk}\sigma_j\sigma_k$ (where $\rho_{jj} = 1$ of course). Then the analysis in the sequel still applies. However, the introduction of correlations between demand in subsequent periods is not straightforward.

Now we turn to the computation of the fill rates, given the control parameters S_i and p_j. Consider the most upstream stockpoint i say. At the beginning of period $t - L_i$ it raises the echelon inventory position to S_i. Since the lead time equals L_i, this order arrives at the beginning of period t. So the echelon stock of stockpoint i just after the arrival of this order equals

$$S_i - D_{t-L_i,t}^i. \tag{1}$$

If this amount (1) exceeds the sum of the order-up-to-level of its successors, i.e., $\sum_{j\in V_i} S_j$, then every stockpoint $j \in V_i$ is able to raise its echelon inventory position to its order-up-to-level. Thus,

$$D_{t-L_i,t} \le \Delta_i \quad \Longrightarrow \quad I_t^j = S_j \quad \text{for} \quad j \in V_i. \tag{2}$$

However, if (1) is less than $\sum_{j\in V_i} S_j$, then the *complete* echelon stock of echelon i is rationed over its successors $j \in V_i$ by using some rationing functions. Let $z_j[x]$ be the amount allocated to echelon j when $pre(j)$ needs to ration x products. Thus,

$$\Delta_i < D_{t-L_i,t}^i \quad \Longrightarrow \quad I_t^j = z_j\left[S_i - D_{t-L_i,t}^i\right] \quad \text{for} \quad j \in V_i. \tag{3}$$

Both the Consistent Appropriate Share (CAS) rationing policy of De Kok, Lagodimos & Seidel [1994] and the Balanced Stock (BS) rationing policy of Van der Heijden [1996] define this rationing function z_j as follows

$$z_j[x] := S_j - p_j\left(\sum_{n\in V_i} S_n - x\right) \quad \text{for} \quad j \in V_i \quad \text{for} \quad x \le \sum_{n\in V_i} S_n. \tag{4}$$

Clearly, we need that $\sum_{j\in V_i} z_j[x] = x$, which implies that $\sum_{j\in V_i} p_j = 1$. The $\{p_j\}_{j\in V_i}$ are referred to as the allocation-fractions of stockpoint i. From (2)–(4) it follows

$$I_t^j = S_j - p_j\left(D_{t-L_i,t}^i - \Delta_i\right)^+ \quad \text{for} \quad j \in V_i. \tag{5}$$

Next, we consider an arbitrary successor of stockpoint i, say j. At the beginning of time t this stockpoint places an order at i to raise its echelon inventory position to S_j. However, since stockpoint j is supplied by a stockpoint with a finite capacity, it is possible that this order can only be satisfied partially. This (partial) order arrives at stockpoint j at the beginning of period $t + L_j$. Hence, the echelon stock of stockpoint j at the beginning of period $t + L_j$ equals

$$I_t^j - D_{t,t+L_j}^j. \tag{6}$$

If this amount (6) exceeds the sum of the order-up-to-level of its successors, i.e., $\sum_{k\in V_j} S_k$, then every stockpoint $k \in V_j$ is able to raise its echelon inventory position to its order-up-to-level. Thus,

$$I_t^j - \sum_{k\in V_j} S_k \ge D_{t,t+L_j}^j \quad \Longrightarrow \quad I_{t+L_j}^k = S_k \quad \text{for} \quad k \in V_j. \tag{7}$$

However, if (6) is less than $\sum_{k\in V_j} S_k$, then the echelon stock of echelon j is rationed over its successors $k \in V_j$ by using the rationing functions $\{z_k\}_{k\in V_j}$. Thus,

$$I_t^j - \sum_{k\in V_j} S_k < D_{t,t+L_j}^j \quad \Longrightarrow \quad I_{t+L_j}^k = z_k\left[I_t^j - D_{t,t+L_j}^j\right] \quad \text{for} \quad j \in V_j. \tag{8}$$

Now we use a similar allocation rule as (4). Substitution of the definition (4) in (8) and using (7) yields

$$I_{t+L_j}^k = S_k - p_k \left(D_{t,t+L_j}^j - \left(I_t^j - \sum_{k \in V_j} S_k \right) \right)^+ \quad \text{for} \quad k \in V_j. \tag{9}$$

Substitution of (5) in (9) yields

$$I_{t+L_j}^k = S_k - p_k \left(D_{t,t+L_j}^j - \Delta_j + p_j (D_{t-L_i,t}^i - \Delta_i)^+ \right)^+ \quad \text{for} \quad k \in V_j. \tag{10}$$

For sake of clarity let us restrict to stationary demand. Then, by defining $X_i := D_{L_i}^i - \Delta_i$ we are able to simplify (5) and (10)

$$I_t^j \overset{d}{=} S_j - p_j X_i^+ \quad \text{for} \quad j \in V_i,$$

$$I_t^k \overset{d}{=} S_k - p_k \left(X_j + p_j X_i^+ \right)^+ \quad \text{for} \quad k \in V_j,$$

where $X \overset{d}{=} Y$ means that X and Y are identically distributed. Using similar arguments as above it is possible to derive an expression for the echelon inventory position of any stockpoint. Suppose that a stockpoint j is supplied by i_1, and i_n by i_{n+1} for $n = 1, \dots, (r-1)$, with i_r denoting the most upstream stockpoint. Then, it can be shown that

$$I_t^j \overset{d}{=} S_j - p_j \left(X_{i_1} + p_{i_1} \left(\dots + p_{i_{r-1}} \left(X_{i_{r-1}} + p_{i_{r-1}} X_{i_r}^+ \right)^+ \right)^+ \right)^+. \tag{11}$$

In order to satisfy the service-constraint in every end-stockpoint we use the following equation (cf. Hadley & Whitin [1963] and Silver & Peterson [1985]):

$$\beta_j = 1 - \frac{E[(D_{L_j+R}^j - I_t^j)^+ - (D_{L_j}^j - I_t^j)^+]}{R\mu_j} \quad \text{for} \quad j \in E. \tag{12}$$

In the next section we discuss the calculation of all parameters (S_i, p_i) in the system under various allocation policies.

4 Controlling a two-echelon system

In the previous section we derived how to compute the fill rate at an end-stockpoint *given* the control parameters. In the literature several heuristics have been developed to determine the control parameters such that every end-stockpoint attains his pre-determined target service level. In this section we concentrate on the heuristics developed for 2-echelon systems, i.e., one upstream stockpoint, i say, supplying $|V_i|$ end-stockpoints.

In Section 4.1 we describe several heuristics for the CAS rationing policy of De Kok, Lagodimos & Seidel [1994]. In Section 4.2 we describe two heuristics for the BS rationing policy of Van der Heijden [1996]. Finally, in Section 4.3 we address the adaptation of the CAS policy, which was suggested by Diks & De Kok [1996]. This adaptation differs from the CAS and BS rationing policy, since it does not uses a modification of the allocation rule as defined in (4).

4.1 Consistent Appropriate Share rationing

In the CAS-allocation rule of De Kok, Lagodimos & Seidel [1994] it assumed that

$$S_j := \mu_{ech(j)} + p_j \sum_{k \in V_i} (S_k - \mu_{ech(k)}). \tag{13}$$

Substitution of (13) in (5), and next substituting the result in (12) yields

$$\beta_j = 1 - \frac{E[(D_{L_j+R}^j - \mu_{ech(j)} - p_j U_i)^+ - (D_{L_j}^j - \mu_{ech(j)} - p_j U_i)^+]}{R\mu_j}, \tag{14}$$

where $U_i = S_i - \Delta_i - X_i^+ - \sum_{j \in V_i} \mu_{ech(j)} = \sum_{j \in V_i}(S_j - \mu_{ech(j)}) - X_i^+$.

The problem of determining stocknorms which ensure individual fill rate targets at all end-stockpoints, corresponds to the solution of the following system

$$f(p_j, S_i, \Delta_i) = \beta_j, \qquad j \in V_i,$$

$$\sum_{j \in V_i} p_j = 1, \tag{15}$$

where $f(p_j, S_i, \Delta_i)$ equals the right-hand side of (14).

Notice that there are $|V_i| + 2$ decision variables ($\{p_j\}$, S_i, Δ_i), however, only $|V_i| + 1$ equations. Therefore in the remainder of this section we solve this system for a *given* Δ_i. This means that the *maximum* upstream and intermediate stock levels are chosen on before hand. If $\Delta_i \le 0$ the depot will not hold any stock, i.e., when a product arrives at the depot it is immediately allocated to the end-stockpoints. If $\Delta_i = \infty$, the system decomposes into $|V_i|$ single location systems working in parallel.

In the literature several heuristics have been developed to solve (15) for a given Δ_i. Below we discuss four heuristics, respectively indicated by CAS1, CAS2, CAS3 and CAS4. The first two heuristics were proposed by De Kok, Lagodimos & Seidel [1994] based on earlier work by De Kok [1990]. The latter two heuristics are discussed in Verrijdt & De Kok [1996]. We will address these heuristics successively.

CAS1:

(*i*). Initialize S_i.

(*ii*). Use (14) to determine for every end-stockpoint j the allocation-fraction p_j.

(*iii*). If $\sum_{j \in V_i} p_j < 1 - \epsilon$ then decrease S_i and return to step *(ii)*,
   If $\sum_{j \in V_i} p_j > 1 - \epsilon$ then increase S_i and return to step *(ii)*.

The computational burden of this algorithm is related to step *(ii)* where we have to solve $|V_i|$ equations. De Kok, Lagodimos & Seidel [1994] solve each equation by using bisection, since they assumed that $f(p_j, S_i, \Delta_i)$ is an increasing function of p_j. In Diks & De Kok [1996] it is argued that this is only true for high β_j. Therefore Diks & De Kok proposed a minor adaptation of the CAS allocation rule, such that an increase of p_j guarantees an increase of the attained fill rate at stockpoint j. We will address this adaptation extensively in Section 4.3.

CAS2:

(*i*). Determine for every end-stockpoint j the order-up-to-level S'_j such that the fill rate at this stockpoint j equals β_j, *assuming* Δ_i would be infinity. This order-up-to-level can be determined from (12) after substituting $I_t^j = S'_j$.

(*ii*). In correspondence with (13) we define

$$p_j := \frac{S'_j - \mu_{ech(j)}}{\sum_{k \in V_i}(S'_k - \mu_{ech(k)})} \quad \text{for} \quad j \in V_i.$$

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(iii). Use (14) to determine for every end-stockpoint j the required order-up-to-level at most upstream stockpoint i, denoted by $S_i[j]$, such that stockpoint j attains fill rate β_j.

(iv). Define

$$S_i := \frac{\sum_{j \in V_i} S_i[j]}{|V_i|}.$$

Since the allocation-fractions are *defined* in step *(ii)*, we only have one decision variable left (S_i) to satisfy the remaining $|V_i|$ service equations. So, unlike CAS1, the CAS2 heuristic approximates the control parameters satisfying system (15). Therefore, it is reasonable to expect that CAS1 outperforms CAS2 if ϵ is sufficiently small.

As argued in Verrijdt & De Kok [1996] the CAS2 heuristic is justifiable when the differences between the values of $S_i'[j]$ in step *(iii)* for the different end-stockpoints are small. However, when we are dealing with different target fill rates the values of $S_i'[j]$ may differ more than desirable. By averaging over these values in step *(iv)* this leads to end-stockpoints j for which the attained fill rate is too small $(S_i'[j] > S_i')$, and to end-stockpoints j for which the attained fill rate is too high $(S_i'[j] < S_i')$. It was felt that by adjusting the allocation-fractions the performance of CAS2 could be improved. Verrijdt & De Kok [1996] developed two methods for adjusting these allocation-fractions, namely 'the extreme case' method and 'the group' method. In this paper we refer to these two methods as CAS3 and CAS4, respectively. Both methods are an *extension* of CAS2. Now we describe these two methods successively.

CAS3:

(v). Determine a stockpoint $m \in V_i$ for which

$$|S_i'[m] - S_i'| \geq |S_i'[j] - S_i'| \quad \text{for} \quad j \in V_i.$$

(vi). If $S_i'[k] < S_i'$, then the adapted allocation-fractions are defined by

$$\tilde{p}_j := \begin{cases} p_j - \delta & j = m \\ p_j + \delta \dfrac{p_j}{1 - p_m} & j \neq m. \end{cases}$$

If $S_i'[k] > S_i'$, then the adapted allocation-fractions are defined by

$$\tilde{p}_j := \begin{cases} p_j + \delta & j = m \\ p_j - \delta \dfrac{p_j}{1 - p_m} & j \neq m. \end{cases}$$

(vii). Return to step *(iii)* of heuristic CAS2 (after adapting δ) until δ minimizes

$$\frac{S_{i,max}' - S_{i,min}'}{S_i' - \mu_{ech(i)}},$$

where $S_{i,max}' := \max\{S_i'[j] | j \in V_i\}$ and $S_{i,min}' := \min\{S_i'[j] | j \in V_i\}$.

Observe that the adjusted allocation-fractions $\{\tilde{p}_j\}_{j \in V_i}$ sum up to one. The parameter δ determines to what extent the allocation-fractions are increased or decreased.

CAS4:

(v). Divide the successors of stockpoint i into two groups A and B

$$A := \{j \in V_i | S_i'[j] < S_i'\} \quad \text{and} \quad B := \{j \in V_i | S_i'[j] \geq S_i'\}.$$

(vi). Define the adapted allocation-fractions by

$$\tilde{p}_j := \begin{cases} \dfrac{(1-\delta)p_j}{1+\delta-2\delta\sum_{k\in A}p_k} & j \in A \\ \dfrac{(1+\delta)p_j}{1+\delta-2\delta\sum_{k\in A}p_k} & j \in B. \end{cases}$$

(vii). Return to step *(iii)* of heuristic CAS2 (after adapting δ) until δ minimizes

$$\frac{S'_{i,max} - S'_{i,min}}{S'_i - \mu_{ech(i)}},$$

where $S'_{i,max} := \max\{S'_i[j] | j \in V_i\}$ and $S'_{i,min} := \min\{S'_i[j] | j \in V_i\}$.
Again $\{\tilde{p}_j\}_{j\in V_i}$ sum up to one.

4.2 Balanced Stock rationing

In Van der Heijden [1996] it is argued that by not defining the order-up-to-levels $\{S_j\}_{j\in V_i}$ as in (13) we obtain more degrees of freedom, which can be used to better tune the control parameters. Van der Heijden first determines the allocation-fractions $\{p_j\}_{j\in V_i}$ such that an approximate expression for the expected amount of imbalance is minimized as much as possible. Next, the order-up-to-levels $\{S_j\}_{j\in V_i}$ are determined so as to guarantee the target fill rates at the end-stockpoints.

The amount of imbalance caused by stockpoint j at time t is measured as

$$\Omega_j(t) := (-Q_j(t))^+ \quad \text{for} \quad j \in V_i, \tag{16}$$

where $Q_j(t)$ is the amount allocated to stockpoint j at time t. In order to get a tractable expression for $\Omega_j(t)$ it is common [Verrijdt & De Kok, 1996; Van der Heijden, 1996] to assume that stockpoint i did not face any imbalance at the previous allocation. Under this assumption we obtain

$$Q_j(t) = I_t^j - (I_{t-R}^j - D_{t-R,t}^j). \tag{17}$$

In order to determine the allocation-fractions independently of the order-up-to-levels Van der Heijden proposes to determine the allocation-fractions $\{p_j\}_{j\in V_i}$ based on the system with $\Delta_i = 0$. This is reasonable since in practice the amount of stock in intermediate stockpoints usually is small. Now, Van der Heijden [1996] showed by using a normal approximation that

$$E[\Omega_j] \approx \sigma_{\Omega_j} \phi\left(\frac{\mu_{\Omega_j}}{\sigma_{\Omega_j}}\right) + \mu_{\Omega_j} \Phi\left(\frac{\mu_{\Omega_j}}{\sigma_{\Omega_j}}\right), \tag{18}$$

where

$$\mu_{\Omega_j} = -R\mu_j, \qquad \sigma_{\Omega_j}^2 = 2p_j^2 T \sum_{k\in V_i} \sigma_k^2 + (R - 2p_j T)\sigma_j^2 \quad \text{and} \quad T := \min\{R, L_i\}.$$

The purpose is to choose the allocation-fractions $\{p_j\}_{j\in V_i}$ such that the mean imbalance at stockpoint i, i.e., $E[\sum_{j\in V_i} \Omega_j]$, is minimized. Since μ_{Ω_j} does not depend on $\{p_j\}_{j\in V_i}$ we consider the effect of σ_{Ω_j} on this mean imbalance at stockpoint i. Differentiation of (18) to σ_{Ω_j} proofs that the mean imbalance is strictly increasing σ_{Ω_j}, so we have to minimize $\sigma_{\Omega_j}^2$. If we would choose the allocation-fractions such that the mean imbalance at stockpoint i is minimized we obtain

$$\hat{p}_j = \frac{\sigma_j^2}{2 \sum_{k\in V_i} \sigma_k^2}. \tag{19}$$

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Unfortunately, these $\{\hat{p}_j\}_{j \in V_i}$ does not sum up to 1, but to $\frac{1}{2}$. In order to get allocation-fractions which minimize the mean imbalance as much as possible and sum up to one, Van der Heijden [1996] determined $\{p_j\}_{j \in V_i}$ such that for every stockpoint $j \in V_i$ holds

$$\frac{dE(\Omega_j)}{dp_j} = \frac{\phi\left(\frac{\mu_{\Omega_j}}{\sigma_{\Omega_j}}\right)}{\sigma_{\Omega_j}} T \left(2p_j \sum_{k \in V_i} \sigma_k^2 - \sigma_j^2\right) = c_i. \tag{20}$$

The c_i is determined such that the allocation-fractions sum up to one.

In the paper of Van der Heijden [1996] a heuristic is developed to determine all the control parameters. We refer to this heuristic as BS1. An adaptation of this heuristic is proposed by Van Donselaar [1996], which is referred to as the BS2 heuristic. We will address these heuristics successively.

BS1:

(*i*). Compute lower bounds $\hat{p}_j$ for p_j for $j \in V_i$ using (19).

(*ii*). Use bisection to find c_i of (20) such that the allocation-fractions $\{p_j\}$ sum up to one. In each step of the bisection, the corresponding values for $\{p_j\}_{j \in V_i}$ are found by another bisection, where p_j should be in the interval $[\hat{p}_j, 1]$.

(*iii*). Determine for every end-stockpoint j the order-up-to-level S_j such that the fill rate at this stockpoint j equals β_j. This order-up-to-level can be determined from (12) after substituting $I_t^j = S_j - p_j(D_{L_i} - \Delta_i)^+$.

(*iv*). The order-up-to-level S_i follows from

$$S_i = \sum_{j \in V_i} S_j + \Delta_i.$$

BS2:

Instead of minimizing the mean imbalance as much as possible, we could also chose to minimize $\sum_{j \in V_i} \sigma_{\Omega_j}^2$ subject to $\sum_{j \in V_i} p_j = 1$. The Lagrange multiplier technique yields

$$p_j := \frac{\sigma_j^2}{2 \sum_{k \in V_i} \sigma_k^2} + \frac{1}{2|V_i|}. \tag{21}$$

Van Donselaar [1996] suggested to define the allocation-fractions as in (21), since it to simplifies step *(i)* and *(ii)* of the BS1 heuristic considerably. We refer to this variant as the BS2 heuristic. Both the BS1 and BS2 heuristics are tested in Section 6 and 7.

4.3 Adapted Consistent Appropriate Share rationing

When using the CAS policy we know after substituting (13) in (11) that

$$I_t^j := \mu_{ech(j)} + p_j U_i \quad \text{for} \quad j \in E, \tag{22}$$

where U_i is defined as in (14). This U_i is the so-called projected systemwide net inventory introduced by De Kok, Lagodimos & Seidel [1994]. It represents the amount of products which have to be divided over the end-stockpoints *after* allocating $\mu_{ech(j)}$ to every end-stockpoint j. CAS always allocates a fixed fraction p_j of this amount U_i to stockpoint j. Since U_i may be negative an increase of p_j does not necessarily cause an increase of β_j. This depends on how frequent U_i is negative. When the systemwide projected inventory at time t is negative an increase of p_j means that the amount of stock allocated to end-stockpoint j decreases. While when at time t the projected net inventory is positive an

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increase of p_j results in an increase of I_t^j . In order to get a consistent rationing policy Diks & De Kok [1996] suggested to adapt the CAS rationing policy slightly, such that an increase of p_j results in an increase of β_j. This done by rationing such that

$$I_t^j = \mu_{ech(j)} + p_j U_i^+ - q_j(-U_i)^+ \quad \text{for} \quad j \in E, \tag{23}$$

where q_j is a monotonously decreasing function in p_j. Clearly, for $\{q_j\}_{j \in V_i}$ we require $\sum_{j \in V_i} q_j = 1$. In the numerical study of Sections 6 we defined

$$q_j := \frac{1 - p_j}{|V_i| - 1}, \quad \text{for} \quad j \in E.$$

After subsequently substituting this definition of q_j in (23), and substituting the result in (12) we obtain

$$\beta_j = 1 - \frac{E[(D_{L_j+R}^j - \mu_{ech(j)} - p_j U_i - r_j(-U_i)^+)^+ - (D_{L_j}^j - \mu_{ech(j)} - p_j U_i - r_j(-U_i)^+)^+]}{R\mu_j}, \tag{24}$$

with $r_j := p_j - q_j = \frac{|V_i|p_j - 1}{|V_i| - 1}$.

The right-hand side of (24) is denoted by $f'(p_j, S_i, \Delta_i)$. Notice that f' very much resembles the f introduced in Section 4.1. In practice the end-stockpoints usually require high service levels. Therefore most periods U_i is non-negative, which implies $f \approx f'$. In such a case the impact of the adaptation of the CAS rationing policy probably has minor effects on the performance.

In order to determine the allocation-fractions $\{p_j\}_{j \in V_i}$ and S_i (*given* Δ_i) we use a similar heuristic as CAS1. We refer to this heuristic as ACAS.

ACAS:

(*i*). Initialize S_i.

(*ii*). Use (24) to determine for every end-stockpoint j the allocation-fraction p_j.

(*iii*). If $\sum_{j \in V_i} p_j < 1 - \epsilon$ then decrease S_i and return to step *(ii)*,

If $\sum_{j \in V_i} p_j > 1 - \epsilon$ then increase S_i and return to step *(ii)*.

In step *(ii)* of ACAS we use a bisection procedure to determine p_j. Unlike step *(ii)* of CAS1, this always yields a unique solution.

5 Controlling an N-echelon system

In Section 4 we concentrated on heuristics for 2-echelon systems. In theory one seldom finds extensions to more general N-echelon systems, although, in practice large production and distribution networks are frequently encountered. Therefore, generalization of the heuristics of the previous section is needed. In this section we address the extension of each heuristic of the previous section, if to our knowledge there exists such an extension or the extension is straightforward. Section 5.1 describes the generalization of the CAS2 heuristic (as well as the CAS3 and CAS4 heuristics). This generalization is introduced by De Kok [1994]. Section 5.2 describes the generalization of both the BS1 and BS2 heuristic.

For our convenience we assign a *low level code* (LLC) to every stockpoint. By definition the low level code of an end-stockpoint i equals 1, i.e., LLC(i):=1. For an intermediate stockpoint i we have LLC(i):=1+max$_{j \in V_i}$ LLC(j). The set of all stockpoints with low level code n is denoted by W_n.

5.1 Consistent Appropriate Share rationing

The generalization of the CAS2 heuristic and its adaptations (CAS3 and CAS4) is rather straightforward if we use a decomposition approach. We start with the determination of the control parameters at the downstream stockpoints, and then work our way up through the network. When using this decomposition approach the control parameters of a stockpoint, i say, are determined *given* the control parameters of stockpoints downstream of stockpoint i. Thus we do not alter already determined control parameters. So the generalization of the CAS2 heuristic consists of the following steps:

CAS2 (and CAS3/CAS4):

(*i*). n:=1.

(*ii*). Determine for every end-stockpoint j the order-up-to-level S'_j such that the fill rate at this stockpoint j equals β_j, *assuming* Δ_i would be infinity for every $i \in M$. This S'_j can be determined from (12) after substituting $I^j_i = S'_j$.

(*iii*). n:=n+1.

(*iv*). Consider a stockpoint $i \in W_n$. Define for every $j \in V_i$,

$$p_j := \frac{S'_j - \mu_{ech(j)}}{\sum_{k \in V_i}(S'_k - \mu_{ech(k)})}.$$

(*v*). Determine for every end-stockpoint $k \in ech(j)$ with $j \in V_i$ the required order-up-to-level at stockpoint i, denoted by $S'_i[j, k]$, such that end-stockpoint k attains fill rate β_k.

(*vi*). Define

$$S'_i[j] := \frac{\sum_{k \in E \cap ech(j)} S'_i[j, k]}{|E \cap ech(j)|}.$$

(*vii*). Define

$$S'_i := \frac{\sum_{j \in V_i} S'_i[j]}{|V_i|}.$$

(*viii*). In case of CAS3 or CAS4 we adapt the allocation-fraction as suggested in Section 4.1 (cf. step *(v)* and *(vi)*). Next, we return to step *(v)* (after adapting δ) until δ minimizes

$$\frac{S'_{i,max} - S'_{i,min}}{S'_i - \mu_{ech(i)}},$$

where $S'_{i,max} := \max\{S'_i[j] | j \in V_i\}$ and $S'_{i,min} := \min\{S'_i[j] | j \in V_i\}$.

(*ix*). Execute steps *(iv)-(viii)* for every stockpoint $i \in W_n$.

(*x*). if $n < N$ then return to step *(iii)*. Otherwise, the order-up-to-level of the most upstream stockpoint S_i is defined as S^{prime}_i. From S_i and the allocation-fractions determined in step *(iv)* we can determine all the downstream order-up-to-levels.

In Verrijdt & De Kok [1995] the CAS2 heuristic was developed for the case where $\Delta_i = 0$ for all intermediate stockpoints i. De Kok [1994] extended these results to the case where also the intermediate stockpoints may keep stock on hand.

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5.2 Balanced Stock rationing

Expressions for the allocation-fractions become more complicated for these N-echelon systems, since it is cumbersome to determine Ω_j for a stockpoint $j \in V_i$. As a simple approximation Van der Heijden [1996] proposes to assume that the variation in the inventory position of stockpoint j just after rationing has only minor effect on the allocation-fractions. In that case we can determine the allocation-fractions $\{p_j\}_{j \in V_i}$ as we did in Section 4.2, after making the following substitutions in (18)–(21):

$$\mu_j \to \sum_{k \in E \cap ech(j)} \mu_k,$$

$$\sigma_j^2 \to \sum_{k \in E \cap ech(j)} \sigma_k^2. \tag{25}$$

So the BS1 heuristic is as follows

BS1:

(*i*). Determine for every stockpoint $i \in M$ the lower bounds $\{\hat{p}_j\}_{j \in V_i}$ for $\{p_j\}_{j \in V_i}$ by substitution of (25) in (19).

(*ii*). Determine for every stockpoint $i \in M$ the value c_i of (20) (using substitution (25)) such that the allocation-fraction sum up to one. In each step of the bisection, the corresponding values for $\{p_j\}_{j \in V_i}$ are found by another bisection, where p_j should be in the interval $[\hat{p}_j, 1]$.

(*iii*). n:=1.

(*iv*). Determine for every end-stockpoint j the order-up-to-level S_j such that the fill rate at this stockpoint j equals β_j. This order-up-to-level can be determined from (12) after substitution of (11).

(*v*). n:=n+1;

(*vi*). Determine for every stockpoint $i \in W_n$ the order-up-to-level S_i by

$$S_i = \sum_{j \in V_i} S_j + \Delta_i.$$

(*vii*). If $n < N$ then return to step *(v)*.

Again the BS2 heuristic is identical to the BS1 heuristic, except for step *(ii)*. The BS2 heuristic defines the allocation-fractions by (21) after substitution of (25).

6 Numerical experiment for two-echelon models

We extensively tested all rationing policies as described in Section 4 by comparing analytical results to simulation results. That is, we analyze the performance of five variants of CAS rationing and two variants of BS rationing. We use the difference between target fill rate and actual fill rate achieved by a particular rationing policy as a performance measure. One policy is considered to be more accurate than the other if the mean absolute deviation from the target fill rate is smaller over all test runs. Also we consider the maximum deviation between actual and target fill rate as a measure of robustness. The experimental design for two-echelon models is described in the next subsection. The numerical results are presented and discussed in Section 6.2.

6.1 Experimental design for two echelon models

In our experiment we test two-echelon models, in which a central warehouse supplies products to two so-called service groups. A service group consists of a number of local stockpoints with the same

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service, demand and lead time characteristics. The number of local stockpoints in both service groups is the same. To normalize time and quantities, we made the following choices for all test runs:

- the review period equals $R=1$.

- the mean demand per time unit for each local stockpoint in service group A equals $E[D_A] = 10$.

Furthermore, the one period demands of all stockpoints are independent. Since the downstream lead times are usually small, we take $L_i = 1$ as lead time between central warehouse (denoted by index 0) and each local stockpoint i in all test runs. Eight other parameters are varied in our experiment. We chose two different values for each parameter (see Table 1), except for the central stock level. As discussed in Section 4, the amount of central stock is a result of the choice of the parameter Δ_0. From equation (5) it can be shown that the amount of central stock equals $E[\Delta_0 - D_{L_0}^0]^+$, so it is convenient to express Δ_0 in the mean system demand during the lead time L_0, say $\Delta_0 = cL_0 E[D^0]$ for some constant c. We have relatively much central stock if $c > 1$ (say $c = 1.2$), relatively little central stock if $c < 1$ (say $c = 0.8$) and no central stock if $c = 0$. Using these three values of the constant c, we determine the appropriate value of Δ_0 for each case.

parameter	description	values in test runs
n	Number of local stockpoints per service group	1, 3
$E[D_A]$	The mean demand per period at a local stockpoint in service group A	10
$E[D_B]$	The mean demand per period at a local stockpoint in service group B	10, 30
$c[D_A]$	Coefficient of variation of demand per period at a local stockpoint in service group A	0.4, 0.8
$c[D_B]$	Coefficient of variation of demand per period at a local stockpoint in service group B	0.4, 0.8
β_A	Target fill rate at a local stockpoint in service group A	90%, 99%
β_B	Target fill rate at a local stockpoint in service group B	90%, 99%
L_0	Lead time from external supplier to the central warehouse	1, 3
c	Constant, describing the level of stock at the central warehouse $\Delta_0 := cL_0 E[D_{L_0}^0]$	0, 0.8, 1.2

Table 1: Parameter values in the experiment with two-echelon models.

We tested all possible parameter combinations, yielding 3*27=384 cases per rationing policy. The performance of the rationing policies for each case is tested by an extensive simulation of 200,000 time periods to ensure high simulation accuracy. This requires a run time of several minutes up to about 20 minutes CPU time for specific cases on a Pentium-75 PC. The time required to calculate the rationing parameters for one test run usually equals less than 1 second.

6.2 Results for two echelon models

The performance of each rationing policy, the variants of Consistent Appropriate Share (CAS) and Balanced Stock (BS) rationing, is shown in Figure 2–5. Because a deviation from the target service level has usually more serious consequences in the case of a high target service level, we separately give the rationing policy performance for each fill rate level (see Figure 2 and 3). Further, Figure 4 and 5 show the performance of each rationing policy depending on the central stock level. Note that rationing policy CAS1 did not converge in two cases. These cases are removed from the figures for CAS1 only.

The overall results show that BS rationing performs better than CAS rationing with respect to both average performance and worst case performance. The original BS rationing performs best, but the
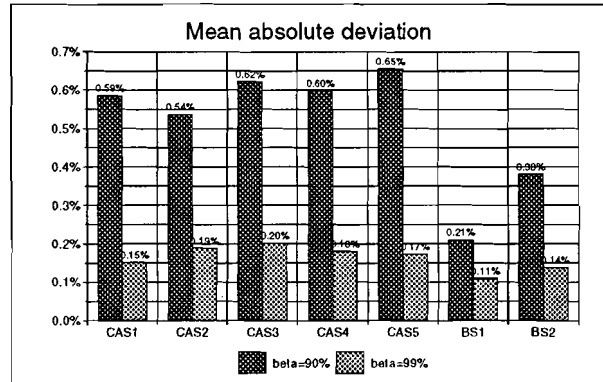
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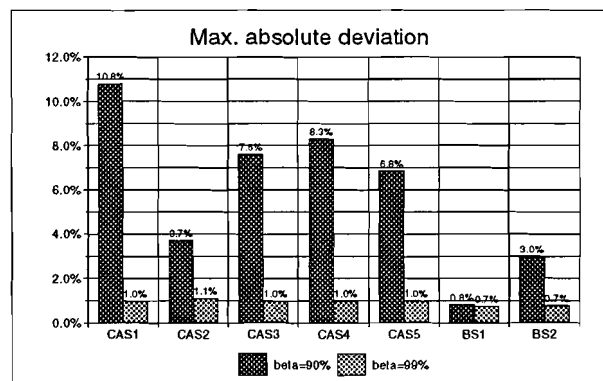Figure 2: Mean absolute deviation of the target fill rate per target fill rate.

Figure 3: Maximum absolute deviation of the target fill rate per target fill rate.

simple variant as suggested by Van Donselaar [1996] is also better than all variants of CAS rationing. Because BS rationing aims to reduce imbalance, the deviation from target fill rate is less than for CAS rationing. Note that the mean physical stock in the system is approximately equal for all rationing policies. Over all cases, the mean physical stock varies between 3.37 weeks (BS2 rationing) and 3.43 weeks (CAS1 rationing).
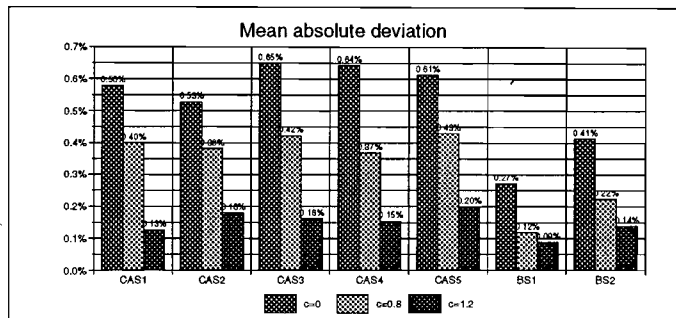

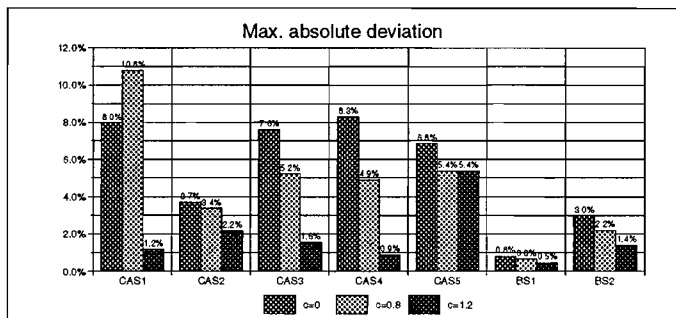Figure 4: Mean absolute deviation of the target fill rate per central stock level.

Figure 5: Maximum absolute deviation of the target fill rate per central stock level.

It is remarkable that the so-called improved variants of CAS rationing do not perform better than the basic CAS allocation rule by De Kok [1990]. In some cases improvement is obtained indeed as is shown in Verrijdt & De Kok [1996] and De Kok, Lagodimos & Seidel [1994]. However, this extensive test shows that worsening occurs as well in some other cases. As an example, consider the following case. A stockless central warehouse ($\Delta_0 = 0$) supplies two service group consisting of one local stock-point each. The supply lead time to the local warehouse equals $L_0 = 3$. The characteristics per service group are shown in Table 2.

service group	$E[D]$	$c[D]$	β_{target}	β_{CAS1}	β_{CAS2}	β_{CAS3}	β_{CAS4}	β_{ACAS}	β_{BS1}	β_{BS2}
A	10	0.8	99%	100.0%	99.3%	99.9%	100.0%	99.9%	98.9%	99.4%
B	30	0.8	90%	82.0%	90.7%	82.6%	81.7%	83.2%	89.8%	88.8%

Table 2: An example where 'improved' CAS rationing is worse than basic CAS rationing.

Table 2 shows that all 'improved' CAS rationing policies yield highly imbalanced results. As a consequence of a poor choice for the rationing parameters, the actual fill rate is too high for service group A and far too low for service group B. Apparently the approximate solution of the system of nonlinear equations (15) deviates strongly from the real solution or the service level is very sensitive to the value of the rationing parameters, but the rationing parameters are accidentally better. Note that also here BS rationing is better than basic CAS rationing.

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Finally we consider the performance of the allocation rule depending on target fill rate and central stock level. Firstly, Figure 2 and 3 show that fortunately all rationing policies perform better for high service levels than for low service levels. Extreme deviations from target occur mainly for $\beta = 90\%$ and for some rationing policies only. In the second place, Figure 4 and 5 show that all rationing policies perform better in the presence of much central stock. This is not surprising, because central stock diminishes imbalance.

7 Numerical experiment for three-echelon models

In this section we discuss the design and results of an experiment with three-echelon models. We analyze only three variants of CAS rationing for the following reasons:

- Extension of rationing policy ACAS (Diks & De Kok) to a three echelon context is not straightforward. In principle it is possible, but the numerical results of the experiment with two-echelon models shows that this is not worth while.

- The CAS1 allocation rule is similar to CAS3 and CAS4, because all these rules try to find an exact solution of the nonlinear system of equations (15). Because CAS1 rationing does not perform better than CAS3 and CAS4 in the two-echelon experiment, it does not seem to be worth while to extend this approach to a three-echelon setting as well.

Hence we analyze both variants of BS rationing and only three variants of CAS rationing: CAS2, CAS3 and CAS4. The experimental design for three-echelon models is described in the next subsection. The numerical results are presented and discussed in Section 7.2.

7.1 Experimental design for three-echelon models

In our experiment we test three-echelon models in which a central warehouse supplies products to two so-called *echelon groups* (see Figure 6). An echelon group consists of a number of intermediate stockpoints that each deliver products to two *service groups*. Each service group consists of an equal number of identical local stockpoints, but two service groups may be different.

When designing the experiment, some attention should be given to the values of Δ_m, defining the intermediate stock levels. From equation (10) it can be shown that the mean amount of physical stock of an intermediate stockpoint k in echelon group m equals $E[\Delta_m - D_{L_k}^k - p_k(D_{L_0}^0 - \Delta_0)^+]^+$, so it is convenient to define Δ_m as

$$\Delta_m := a_m \left(E[D_{L_k}^k] + p_k E[D_{L_0}^0 - \Delta_0]^+ \right). \tag{26}$$

Because the rationing fractions $\{p_k\}$ are not known on input for a specific rationing policy, we plug in the approximation (21) of Van Donselaar [1996]. Now we obtain reasonable values for Δ_m in our experiment using appropriate choices for a_m, see below.

We take the following parameters fixed for all test runs:

- the review period equals $R = 1$.

- the mean demand per period for each local stockpoint in service group A within echelon group I equals 10.

- the lead time between each combination of intermediate stockpoint k and local stockpoint i equals $L_{ki} = 1$.

Further we impose the following restrictions within the experiment to keep the number of test runs within reasonable limits:

- the number of intermediate stockpoints is the same for both echelon groups in a single test run.
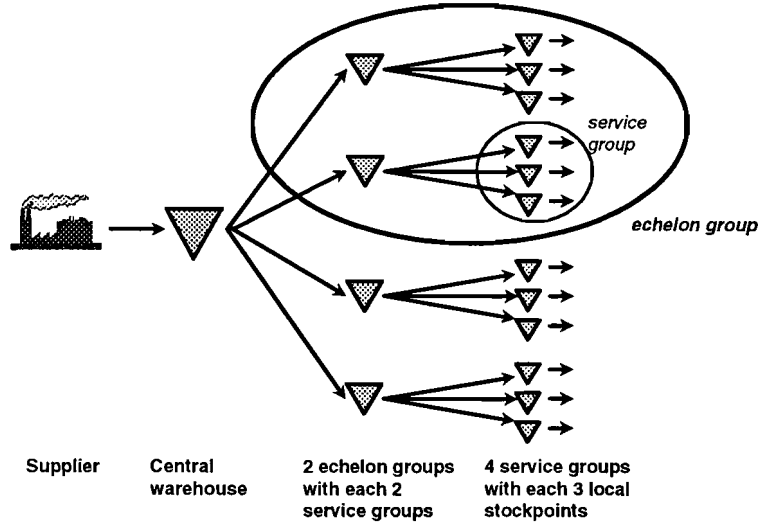
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Figure 6: Echelon groups and service groups.

- the number of local stockpoints per service group is identical for all service groups in a single test run.

- the lead time between central warehouse (denoted by index 0) and each intermediate stockpoint (denoted by index k) is the same.

- The values of am in (26) are the same for all intermediate stockpoints k in echelon group m. Using these values a_m, the values Δ_m are computed from (26).

For the demand and service characteristics at each local stockpoint we take the following values:

- the mean demand at a stockpoint in service group j of echelon group m equals $E[D_{mj}] = 10$ or 30 (except $E[D_{11}]$ which equals 10, see above).

- the coefficient of variation of the demand at a stockpoint in service group j of echelon group m equals $c[D_{mj}]$=0.4 or 0.8.

- the target fill rate at a stockpoint in service group j of echelon group m equals $\beta_{mj} = 90\%$ or 99%.

Like in Section 6 we assume that the one period demands of all stockpoints are independent. When the set of experimental runs is carefully chosen, we need only 87 parameter combinations to analyze the 11 demand- and service parameters (see Appendix A). For the remaining parameters we make the following choices:

- two values for Δ_0, defined by $\Delta_0 := cL_0 E[D_{L_0}^0]$ for $c = 0$ and $c = 1.2$.

- two values for Δ_m, defined by $\Delta_m := a_m (E[D_{L_k}^k] + p_k E[D_{L_0}^0 - \Delta_0]^+)$ for $a_m = 0$ and $a_m = 1.2$. The index k denotes an intermediate stockpoint from echelon group m.

- number of stockpoints:

    (*i*). one local stockpoint per service group and one intermediate stockpoint per echelon group.

    (*ii*). three local stockpoints per service group and one intermediate stockpoint per echelon group.

    (*iii*). three local stockpoints per service group and two intermediate stockpoints per echelon group.

- lead times:

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(*i*). $L_0 = 1$ and $L_k = 1$ for all intermediate stockpoints k.

(*ii*). $L_0 = 3$ and $L_k = 1$ for all intermediate stockpoints k.

(*iii*). $L_0 = 3$ and $L_k = 4$ for all intermediate stockpoints k.

In total we now have 87*2*2*3*3 = 3132 test runs for each rationing policy. This is still a large amount of numerical effort, but it is acceptable. The performance of the rationing policies for each case is tested by a simulation of 100,000 time periods.

7.2 Results for three echelon models

The performance of each rationing policy, the three variants of CAS rationing and the two variants of BS rationing, is shown in Figure 7–10 below. Again we give separate results per target fill rate (Figure 7 and 8) and per upstream stock level (Figure 9 and 10).
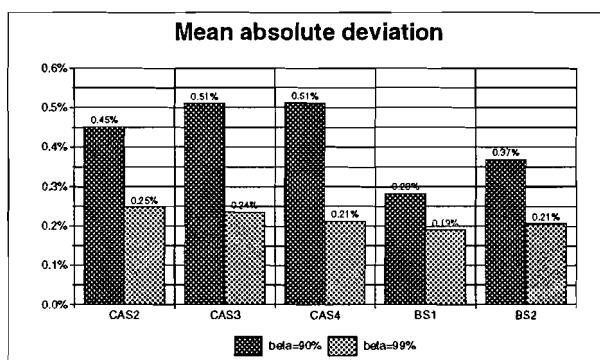

Figure 7: Mean absolute deviation of the target fill rate per target fill rate.
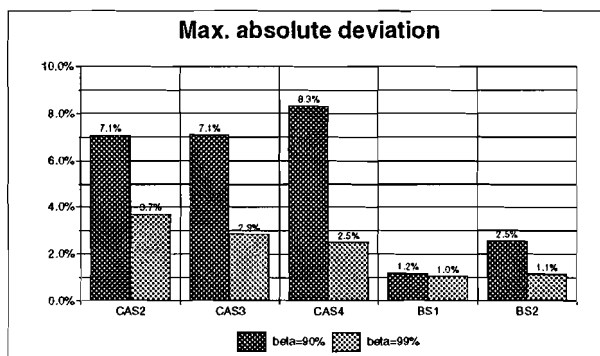

Figure 8: Maximum absolute deviation of the target fill rate per target fill rate.

The results of the three-echelon experiment are a logical extension of the results of the two-echelon experiment. Again, BS rationing performs better than CAS rationing and the original BS rationing performs best. It is remarkable that the performance of the various rationing policies is not worse than for two echelon models. Apparently there is no accumulation of approximation errors. For CAS rationing, the errors seem even to compensate each other slightly. The performance of BS rationing is however slightly worse than for two-echelon models, probably because of the fact that an additional approximation is made when establishing the rationing parameters: The effect of central and intermediate stocks is neglected and only taken into account when calculating the order-up-to levels. Also it is remarkable that the so-called improved variants of CAS rationing do not perform better than the basic CAS allocation rule by De Kok [1990]. Note that also here the mean physical stock in the system is
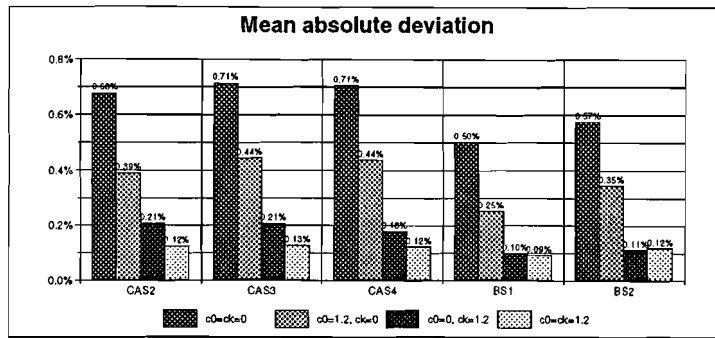
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Figure 9: Mean absolute deviation of the target fill rate per central stock level.
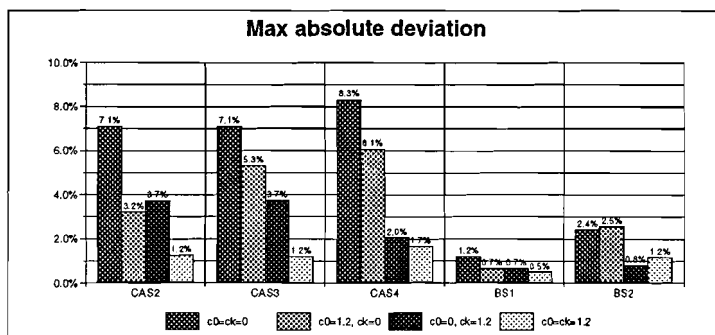

Figure 10: Maximum absolute deviation of the target fill rate per central stock level

approximately equal for all rationing policies. Over all case, the mean physical stock varies between 5.24 weeks (CAS2 rationing) and 5.36 weeks (BS1 rationing).

Finally we consider the performance of the allocation rule depending on target fill rate and central stock level. First, Figure 7 and 8 show that fortunately all rationing policies perform better for high service levels than for low service levels. Extreme deviations from target occur mainly for $\beta=90\%$, although significant deviations may now occur for $\beta=99\%$ as well. In the second place, Figure 9 and 10 show that all rationing policies perform better in the presence of much upstream stock, because imbalance is reduced.

8 Conclusions

In this paper we have extensively examined various rationing policies for integral inventory control in divergent N- echelon systems with periodic review and no lot sizing. Both models with and without intermediate stock are analyzed. Where necessary, the policies as available in the literature are extended to a general N-echelon context with intermediate stocks.

The most important result is that BS rationing performs better than CAS rationing, both on average and worst case. Within the class of CAS rationing policies, it is remarkable that the most simple approach of De Kok [1990] performs best. The variants of CAS rationing which have been derived in the past years do not appear to be better than the simple, original approach by De Kok [1990]. The most serious errors occur in cases of high imbalance, especially

- for relatively low service levels
- in situations with little or no central or intermediate stocks

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Overall, the original BS rationing is the best rationing policy in the test. From a practical point of view it is worth to notice that the simple Van Donselaar variant is a "good-value-for-money" second best. Although the original BS rationing policy is not very difficult to implement, the Van Donselaar variant is even more simple and can easily be used in spreadsheet applications. Another advantage of BS rationing is the fact that the determination of the rationing parameters is decoupled from the determination of the order-up-to levels. Because of this, BS rationing can probably be used more easily for model extensions, such as the introduction of stochastic lead times, order points or lot sizing. These are subjects for further research.

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A Details on the experiment with three echelon models

In this appendix we show how the set of $2^{11} = 2048$ demand- and service parameter combinations for the three-echelon experiment can be reduced to 87 combinations only. First we define the following parameter levels:

- category $E[D_{mj}]$: L(ow) = 10 and H(igh) = 30
- category $c[D_{mj}]$: L(ow) = 0.4 and H(igh) = 0.8
- category β_{mj}: L(ow) = 90% and H(igh) = 99%

We will define the set of parameter combinations to be used in the three-echelon experiment in two stages. First we state which target fill rate combinations are useful, given the demand characteristics within service groups and within echelon groups. Then we select combinations of demand characteristics and combine these with all target fill rate combinations as defined in the first step.

Step 1: selection of the combinations of the four target fill rate parameters β_{mj}.
We distinguish between the following situations:

- both echelon groups are the same.
- the service groups within an echelon group are the same.

If the echelon groups are the same, we can skip those models that are obtained by interchanging the target service levels between echelon groups. If the service groups within an echelon group are the same, we can skip those models that are obtained by interchanging the target service levels between service groups within the same echelon group. Therefore we need only the set of target fill rate combinations as shown in Table 3.

fill rate combinations (β_{11}, β_{12}) and (β_{21}, β_{22})	echelon groups are the same	echelon groups are different
service groups within an echelon group are the same	(L, L) and (L, L) (L, L) and (H, H) (H, H) and (H, H) (L, H) and (L, H)	(L, L) and (L, L) (L, L) and (H, H) (H, H) and (L, L) (H, H) and (H, H) (L, H) and (L, H)
service groups within an echelon group are different	(L, L) and (L, L) (L, L) and (H, H) (H, H) and (H, H) (L, H) and (L, H) (H, L) and (H, L) (L, H) and (H, L)	(L, L) and (L, L) (L, L) and (H, H) (H, H) and (L, L) (H, H) and (H, H) (L, H) and (L, H) (H, L) and (H, L) (L, H) and (H, L) (H, L) and (L, H)

Table 3: Combinations of fill rates given the demand characteristics.

Step 2: selection of the combinations of the seven demand characteristics $E[D_{mj}]$ ($m \neq 1$ or $j \neq 1$) and $c[D_{mj}]$.
For the mean demand per local stockpoint, we select the following combinations, see Table 4:

(*i*). all local stockpoints have low mean demand (all stockpoints with high mean demand give similar results, this is a matter of scaling).

(*ii*). the local stockpoints in echelon group I all face low mean demand, while the local stockpoints in echelon group II all face high mean demand (interchange of echelon groups yields similar models then).

(*iii*). the mean demand levels in both echelon groups are the same, but service group A faces high mean demand and service group B faces low mean demand (interchange of service groups within the same echelon group yields similar models then).

For each set of values for the mean demand, we choose all combinations of demand coefficients of variation, but we eliminate those models that can be transformed to each other by interchanging local stockpoints within and/or between service groups. As a result, we obtain the following parameter sets, see Table 4: Table 4 shows 15 combinations of demand parameters and the relevant number of

(μ_{11}, μ_{12}) and (μ_{21}, μ_{22})	(c_{11}, c_{12}) and (c_{21}, c_{22})	echelon groups	service groups within echelon groups	number of fill rate combinations
(L, L) and (L, L)	(L, L) and (L, L)	same	same	4
	(L, L) and (H, H)	different	same	5
	(H, H) and (H, H)	same	same	4
	(L, H) and (L, H)	same	different	6
(L, L) and (H, H)	(L, L) and (L, L)	different	same	5
	(L, L) and (H, H)	different	same	5
	(H, H) and (L, L)	different	same	5
	(H, H) and (H, H)	different	same	5
	(L, H) and (L, H)	different	different	8
(L, H) and (L, H)	(L, L) and (L, L)	same	different	6
	(L, L) and (H, H)	different	different	8
	(H, H) and (H, H)	same	different	6
	(L, H) and (L, H)	same	different	6
	(H, L) and (H, L)	same	different	6
	(L, H) and (H, L)	different	different	8

Table 4: Combinations of $E[D_{mj}]$ and $c[D_{mj}]$.

fill rate combinations (see Table 4). In this way, we end up with 87 service and demand parameter combinations (sum up the numbers in the last column) instead of $2^{11} = 2048$.