

Optimal control of a divergent N-echelon inventory system

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Optimal control of a divergent N -echelon inventory system

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Abstract

Consider a divergent multi-echelon inventory system, e.g. a distribution system or a production system. At every facility in the system orders are placed (or production is initiated) periodically. The order arrives after a fixed lead time. At the end of each period linear costs are incurred at each facility for holding inventory. Also, linear penalty costs are incurred at the most downstream facilities for backorders. The objective is to minimize the expected holding and penalty costs per period. We prove that under the balance assumption it is cost-optimal to control every facility by an order-up-to-policy. The optimal replenishment policy, i.e., the order-up-to-level and the rationing functions at each facility, can be determined by decomposition of the system. This decomposition result reduces complex multi-dimensional control problems to 'simple' one-dimensional problems, which closely resemble the classical newsboy problem achieving a proper trade-off between the consequences of too much stock and too little inventory.

Keywords: multi-echelon, optimal control, inventory, allocation, rationing, divergent

1 Introduction

The research of multi-echelon models has gained importance over the last decade because integrated control of supply chains, consisting of a number of processing and distribution stages, has become feasible through modern information technology. Multi-echelon inventory systems provide a means of modeling such supply chains, thereby enabling quantitative analysis and characterization of optimal control policies (cf. Clark & Scarf [1960], Federgruen & Zipkin [1984], Rosling [1989] and Langenhoff & Zijm [1990]).

The start of research on multi-echelon inventory models is in generally allotted to Clark & Scarf [1960], who study an N -echelon serial system without lot sizing. They introduced the concept of echelon stock for a given stockpoint to prove that the optimal control policies for the N -echelon serial system with discounted penalty and holding costs, are characterized by N so-called echelon order-up-to-levels. The echelon stock of a stockpoint equals all stock at this stockpoint plus in transit to or on hand at any of its downstream stockpoints minus the backorders at its downstream stockpoints. Like Van Houtum & Zijm [1991a] and Zijm & Van Houtum [1994] we also like to define the echelon inventory position of a stockpoint as its echelon stock plus all material in transfer to that stockpoint.

Eppen & Schrage [1981] analyzed a divergent two-echelon system, where the proportional costs of holding and backordering at the end-stockpoints are identical. They derive approximately optimal policies and costs of: (1) An order-up-to-policy at the upstream stockpoint, assuming no fixed ordering cost, and (2) An (m, y) policy at the depot, assuming fixed ordering costs at the upstream stockpoint. An (m, y) policy is a policy in which every m periods the echelon inventory position is raised to an order-up-to-level y . A paper of Federgruen & Zipkin [1984] extends the model and the results considered by Eppen & Schrage. The holding and penalty costs do not have to be identical across the end-stockpoints. The demands at these stockpoints do not have to be normally distributed. They allow for a larger class of distributions (e.g. Erlang and gamma distribution). As already mentioned

before, Eppen & Schrage restrict themselves to an order-up-to-policy, or a (m, y) policy to analyze the system. While Federgruen & Zipkin use an approximate dynamic program to determine the control policy, which is optimal given the *balance assumption*. Under this assumption the rationing rule always allocates non-negative stock quantities. In Eppen & Schrage [1981], Langenhoff & Zijm [1990] and De Kok, Lagodimos & Seidel [1994] similar assumptions are made.

A more 'service related' approach to determine the control parameters is introduced by De Kok [1990] and Lagodimos [1992]. In this approach the target service levels of the different end-stockpoints plays an important role in the analysis, instead of the minimization of a cost-function. In De Kok [1990] a planning procedure has been determined for a divergent two-echelon system with a stockless depot. Later this model is extended in Seidel & De Kok [1990] and De Kok, Lagodimos & Seidel [1994] by allowing the depot to hold stock.

Although much attention has been given to these divergent two-echelon systems, one seldom finds extensions to more general divergent N -echelon systems (e.g. Verrijdt & De Kok [1995]). In practice, however, large production and distribution networks are frequently encountered and therefore generalization of two-echelon policies is needed. In this paper we analyze a divergent N -echelon inventory system in which every stockpoint is allowed to hold stock. Every stockpoint places replenishment orders periodically. The order arrives after a fixed lead time, and then it is decided how much and in what way the stock is allocated among its successors. Only the unfilled demand at the end-stockpoints are backordered. Penalty costs proportional to the amount short at every end-stockpoint are incurred at the end of each period. Also holding costs proportional to the inventory on hand are incurred at the end of each period. The objective is to minimize the average costs per period on the long run.

This model can be regarded as an extension of Langenhoff & Zijm [1990] and Van Houtum & Zijm [1991b]. Langenhoff & Zijm [1990] prove exact decomposition results for a two-echelon assembly system, a two-echelon serial system and a divergent two-echelon system. The analysis of the latter system is more thoroughly analyzed in Van Houtum & Zijm [1991b]. We extend their analysis by relaxing the following constraints: (1) all lead times to the end-stockpoints are identical, (2) the penalty costs at all end-stockpoints are identical, and (3) considered model is a two-echelon model.

The paper is organized as follows. In Section 2 we describe the considered model. In Section 3 we present an average cost analysis for the divergent N -echelon system. Given the balance assumption we derive necessary conditions and important properties of an optimal control policy. This balance assumption is *not* required if immediately after taking a rationing decision there is a sufficiently large 'demandless' period (e.g. week-end). Since such a period enables to transship products from the stockpoints with negative allocation quantities to those with positive allocation quantities. In Section 4 we prove exact decomposition results for divergent N -echelon systems given the balance assumption. From the optimality of the decomposition we know that there exists an optimal policy in which every stockpoint is controlled by an order-up-to-policy. This decomposition result reduces the complex multi-dimensional control problem to simple one-dimensional problems closely resembling the classical newsboy problem. Finally in Section 5 we give a few concluding remarks.

2 Model description

Consider a single-item discrete-time multi-echelon inventory system where every stockpoint is allowed to hold stock. The system has an *arborescent* structure, i.e., each location has a unique supplier. We refer to these kind of systems as divergent multi-echelon systems. Notice that a divergent multi-echelon system can be described by a directed graph (see for example Figure 1). The most upstream stockpoint can place orders at an external supplier which has an infinite capacity, which means that this supplier can always meet the demand.

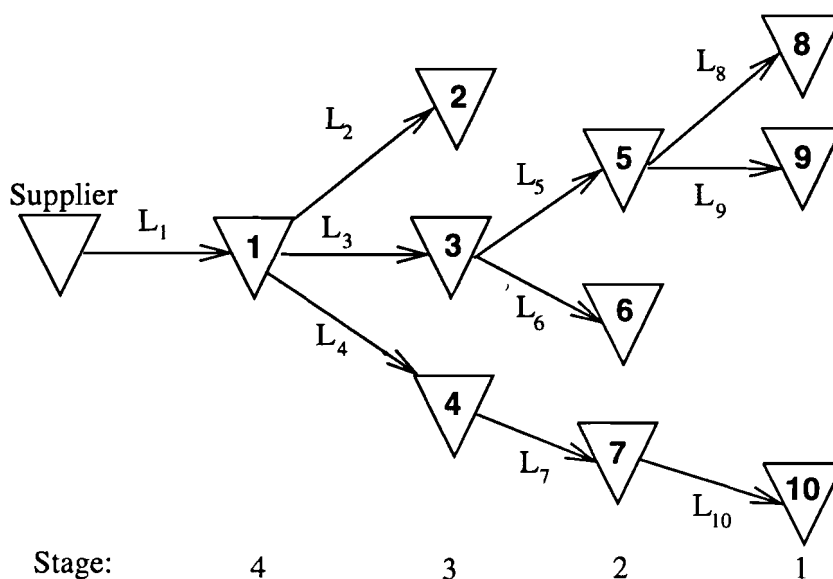


Figure 1: Schematic representation of a divergent 4-echelon inventory system.

The inventory in this multi-echelon system is controlled by periodic review policies. That is, every R periods the most upstream stockpoint, i say, issues a replenishment order. The replenishment order arrives after L_i periods, where L_i is a fixed, non-negative integer. Then the physical stock at stockpoint i is allocated immediately to its successors. There are two possibilities:

- (i). The physical stock is sufficient to raise the echelon inventory position of each successor to its order-up-to-level. Then the required amounts are sent to the successors and excess stock is kept at stockpoint i to be allocated in the next occasion.
- (ii). The physical stock is *not* sufficient to reach the order-up-to-levels. Then material rationing is required to allocate the available physical stock over its successors appropriately. For this purpose we introduce rationing functions in the next section.

A similar allocation procedure is applied at the other intermediate stockpoints when a replenishment order arrives.

Without loss of generality we assume that only the end-stockpoints face external customer demand. In case an intermediate stockpoint i faces external demand, we redirect this demand to a new successor j with lead time $L_j := 0$. By definition this successor j is an end-stockpoint. During one period the demand between end-stockpoints may be correlated, however, the demand in subsequent periods are i.i.d.. With respect to the customer demand process, we assume that all demand which cannot be satisfied immediately is backordered.

At the end of each period both penalty and holding costs are incurred. The penalty costs equals p_i for each backlogged product at end-stockpoint i . For a product at stockpoint i or in transfer to one of its successors the holding costs equals $h_i + \sum_{k \in U_i} h_k$, where U_i represents all stockpoints on the path from the supplier to i . Notice that h_i can be regarded as an additional holding cost due to value added in stockpoint i . No fixed ordering costs are assumed. Note that because all excess customer demand is backordered, linear variable ordering costs do not influence any control policy and can therefore be omitted. The objective of the analysis is to determine a cost-optimal replenishment policy. That is a policy, which minimizes the expected total costs on the long run.

For sake of clarity in the remainder of this paper we refer to the length of a review period as one

period ($R := 1$). Furthermore, we introduce the following notation:

- $ech(i) :=$ Set of stockpoints that constitute the echelon of stockpoint i (e.g. $ech(5) = \{5, 8, 9\}$),
- $pre(i) :=$ Preceding stockpoint of stockpoint i (e.g. $pre(8) = 5$),
- $L_i :=$ Lead time from stockpoint $pre(i)$ to i .
- $U_i :=$ Set of all stockpoints on path from supplier to stockpoint i (e.g. $U_1 = \emptyset$ and $U_6 = \{1, 3\}$),
- $V_i :=$ All stockpoints which are supplied by i (e.g. $V_1 = \{2, 3, 4\}$),
- $E :=$ Set of all end-stockpoints (e.g. $E = \{2, 6, 8, 9, 10\}$),
- $I :=$ Set of all intermediate stockpoints (e.g. $I = \{1, 3, 4, 5, 7\}$),
- $N :=$ Number of stages in inventory system (e.g. $N = 4$).

The examples between the brackets refer to the situation of Figure 1.

3 Properties optimal replenishment policy

In this section we present an average cost analysis for the divergent N -echelon system, which is strongly based on the work of Langenhoff & Zijm [1990] and Van Houtum & Zijm [1991a]. In this section we assume that decomposition of the system, as in Langenhoff & Zijm [1990] and Van Houtum & Zijm [1991b], is a reasonable approach to determine the control parameters of a cost-optimal policy. Indeed, in the next section we proof that decomposition is exact. This decomposition enables us to derive necessary conditions for a cost-optimal replenishment policy, i.e., the order-up-to-level at every stockpoint and the rationing functions to its successors. The optimal order-up-to-policy can be determined by solving a problem similar to the classical newsboy problem. For the optimal rationing functions we derive necessary conditions and useful properties.

For our convenience we introduce some additional notation:

- $z_i^j[x] :=$ The echelon inventory position of stockpoint $j \in V_i$ just after the allocation, if the echelon stock of supplier i just before allocation equals x .
- $y_i :=$ Echelon order-up-to-level of stockpoint i .
- $\Delta_i :=$ Maximum physical stock at stockpoint i , $y_i - \sum_{j \in V_i} y_j$.
- $\Psi_i :=$ All control parameters downstream of stockpoint i ,

$$\begin{cases} \emptyset & i \in E \\ \bigcup_{j \in V_i} (z_i^j, y_j, \Psi_j) & i \in I. \end{cases}$$
- $D_i(y_i, \Psi_i) :=$ The expected total costs in $ech(i)$ at the end of an arbitrary period given the echelon order-up-to-level y_i and Ψ_i (If $\Psi_i = \emptyset$ we suppress Ψ_i).
- $x_i^* := \inf\{x | \partial^2 D_i(x, \Psi_i) / \partial x^2 > 0\}$.
- $F_L^{ech(i)} / f_L^{ech(i)} :=$ cdf/pdf of demand at all end-stockpoints in $ech(i)$ during L periods (if $L = 1$ we suppress the index).
- $\left(F_L^{ech(i)}(x)\right)^\Delta := \begin{cases} 0 & x < 0 \\ F_L^{ech(i)}(x + \Delta) & x \geq 0. \end{cases}$
- $p_i^* :=$ Minimum penalty costs per backlogged product in $ech(i)$, i.e., $\min_{j \in ech(i) \cap E} p_j$.
- $h_i^* :=$ Minimum added value of a successor of stockpoint i , i.e., $\min_{j \in V_i} h_j$.
- $\underline{V}_i :=$ Set of successors of i , which have the same minimal penalty costs as i , i.e., $\{j \in V_i | p_i^* = p_j^*\}$.
- $\bar{V}_i :=$ Set of successors of i which add the minimal amount to a product, i.e., $\{j \in V_i | h_i^* = h_j\}$.

First we derive the cost-function which need to be minimized. Theorem 3.1 constitutes the basis for this derivation.

Theorem 3.1. *When at the end of an arbitrary period the echelon stock of a stockpoint i equals x_i , the total costs incurred at the end of this period equal*

$$\sum_{i \in E} \left(h_i x_i + (h_i + \sum_{j \in U_i} h_j + p_i)(-x_i)^+ \right) + \sum_{i \in I} h_i x_i.$$

Proof. In order to proof this theorem we use that if $i \in I$ then

$$(h_i + \sum_{j \in U_i} h_j) \sum_{k \in V_i} x_k = \sum_{k \in V_i} (h_i + \sum_{j \in U_i} h_j) x_k = \sum_{k \in V_i} \sum_{j \in U_k} h_j x_k. \quad (1)$$

From (1) and the property that for the most upstream stockpoint i we have $U_i = \emptyset$, it follows

$$\sum_{i \in I} (h_i + \sum_{j \in U_i} h_j) \sum_{j \in V_i} x_j = \sum_{i \in I \cup E} \sum_{j \in U_i} h_j x_i. \quad (2)$$

From the definition of the inventory costs h_i and penalty costs p_i it easily follows that the expected total costs equals

$$\begin{aligned} & \sum_{i \in E} \left((h_i + \sum_{j \in U_i} h_j) x_i^+ + p_i (-x_i)^+ \right) + \sum_{i \in I} (h_i + \sum_{j \in U_i} h_j) (x_i - \sum_{j \in V_i} x_j) = \\ & \sum_{i \in E} \left((h_i + \sum_{j \in U_i} h_j) x_i + (h_i + \sum_{j \in U_i} h_j + p_i) (-x_i)^+ \right) + \sum_{i \in I} h_i x_i + \\ & \sum_{i \in I} \sum_{j \in U_i} h_j x_i - \sum_{i \in I} (h_i + \sum_{j \in U_i} h_j) \sum_{j \in V_i} x_j \stackrel{(2)}{=} \\ & \sum_{i \in E} \left((h_i + \sum_{j \in U_i} h_j) x_i + (h_i + \sum_{j \in U_i} h_j + p_i) (-x_i)^+ \right) + \sum_{i \in I} h_i x_i + \sum_{i \in I} \sum_{j \in U_i} h_j x_i - \sum_{i \in I \cup E} \sum_{j \in U_i} h_j x_i. \end{aligned}$$

Rewriting the expression above completes the proof. \square

Theorem 3.1 implies that costs $h_i x_i$ are incurred to each stockpoint i , independent of the sign of x_i , whereas at an end-stockpoint extra costs $-(h_i + \sum_{j \in U_i} h_j + p_i) x_i$ are incurred when this stockpoint is in a backlog position.

In order to relate the inventory of a stockpoint at the beginning of a period with the costs incurred at the end of this period we define the one-period cost-function $L_i(x)$.

$L_i(x) :=$ The expected costs incurred at the end of a period, when at the beginning of this period the echelon stock of stockpoint i is increased to x .

To evaluate $L_i(x)$ we assume that the total demand of all-end-stockpoints in $ech(i)$ during L periods is distributed with cdf $F_L^{ech(i)}$ (if $L = 1$, we suppress the index). From Theorem 3.1 it follows that

$$L_i(x) := \begin{cases} \int_0^\infty h_i(x-u) dF^{ech(i)}(u) + \int_x^\infty (h_i + \sum_{j \in U_i} h_j + p_i)(u-x) dF^{ech(i)}(u) & i \in E \\ \int_0^\infty h_i(x-u) dF^{ech(i)}(u) & i \in I. \end{cases}$$

Before we are able to compute the expected costs of echelon i at the end of an arbitrary period, denoted by $D_i(y_i, \Psi_i)$, we have to decide how to ration the available stock when a stockpoint has insufficient stock to meet all the demand. Suppose that at the beginning of an arbitrary period (just before rationing), stockpoint i has an echelon stock of x products. All its successors $j \in V_i$ want to raise their echelon inventory position to y_j . Hence, if $x \geq \sum_{j \in V_i} y_j$ the echelon inventory position of stockpoint j just after rationing yields y_j , and the remainder $x - \sum_{j \in V_i} y_j$ is retained at stockpoint i . However, if $x < \sum_{j \in V_i} y_j$ we have to deal with one of the main difficulties of divergent multi-echelon systems: how should we ration the available stock over these stockpoints j ? To overcome this problem we define a rationing function $z_i^j[x]$. This means that the rationing policy allocates $z_i^j[x]$ to stockpoint j , and no products are retained at stockpoint i . Thus,

$$\sum_{j \in V_i} z_i^j[x] = x. \quad (3)$$

We assume that when a stockpoint i rations the available echelon stock x over the stockpoints $j \in V_i$, the echelon inventory position of every stockpoint j just after rationing is at least as large as it was just before rationing. In Langenhoff & Zijm [1990] this assumption is referred to as the *balance assumption*.

The next theorem enables us to determine the expected costs of echelon i , given replenishment policy (y_i, Ψ_i) .

Theorem 3.2. *For replenishment policy (y_i, Ψ_i) the average costs of echelon i equals*

$$D_i(y_i, \Psi_i) = \int_0^\infty L_i(y_i - u) dF_{L_i}^{ech(i)}(u) + \sum_{j \in V_i} \left[\int_0^{\Delta_i} D_j(y_j, \Psi_j) dF_{L_i}^{ech(i)}(u) + \int_{\Delta_i}^\infty D_j(z_i^j[y_i - u], \Psi_j) dF_{L_i}^{ech(i)}(u) \right]. \quad (4)$$

Proof. Consider a replenishment policy (y_i, Ψ_i) . Suppose that at the beginning of an arbitrary review period t the echelon inventory position of echelon i is raised to y_i and the total demand in the periods t up to and including period $t + L_i - 1$ equals u . Hence at the end of period $t + L_i - 1$ (just after arrival order) the echelon stock of echelon i equals $y_i - u$. Therefore at time $t + L_i$ the costs incurred for echelon i equals $L_i(y_i - u)$.

If at the beginning of period $t + L_i$ holds $y_i - u > \sum_{j \in V_i} y_j$, every stockpoint $j \in V_i$ raises his inventory position to y_j . Therefore at time $t + L_i + L_j$ the expected costs incurred with echelon j equals $D_j(y_j, \Psi_j)$.

However, if at the beginning of period $t + L_i$ we have $y_i - u < \sum_{j \in V_i} y_j$, every stockpoint $j \in V_i$ raises its inventory position to $z_i^j[y_i - u]$. Therefore at time $t + L_i + L_j$ the expected costs incurred for echelon j equals $D_j(z_i^j[y_i - u], \Psi_j)$.

By conditioning on u and using the above mentioned relations the theorem follows. \square

For simplicity we assume that $F^{ech(i)}(x)$ is a differentiable function for $x \geq 0$. Then from Theorem 3.2 it follows that $D_i(x, \Psi_i)$ is also a differentiable function for x . In order to characterize the properties of the optimal policy we have to analyze the behavior of this function $D_i(x, \Psi_i)$. When i is an end-stockpoint the most important properties of $D_i(x)$ directly results from (4).

Lemma 3.1. *The cost-function $D_i(x)$ of an end-stockpoint i is*

- (i). *linear decreasing with slope $-(\sum_{j \in U_i} h_j + p_i)$ for $x < 0$.*
- (ii). *convex in x . Specifically, if $F^{ech(i)}(x)$ is strictly increasing in $x \geq 0$, then $D_i(x)$ is a strict convex function for $x \geq 0$.*

(iii). $\partial D_i(x)/\partial x$ tends to h_i when x goes to infinity.

Proof. Let $i \in E$. Subsequently substituting the definition of $L_i(x)$ in (4), and differentiating the result to x yields

$$\frac{\partial D_i(x)}{\partial x} = h_i - (h_i + \sum_{j \in U_i} h_j + p_i)(1 - F_{L_i+1}^{ech(i)}(x)). \quad (5)$$

Notice that $F_{L_i+1}^{ech(i)}(x) = 0$ for $x \leq 0$. This proves (i). Result (ii) follows from

$$\frac{\partial^2 D_i(x)}{\partial x^2} = (h_i + \sum_{j \in U_i} h_j + p_i) f_{L_i+1}^{ech(i)}(x) \geq 0.$$

Finally, we notice that when x goes to infinity, $F_{L_i+1}^{ech(i)}(x)$ increases to 1, which proves (iii). \square

From Lemma 3.1 it follows that if h_i is positive, a minimum of the function $D_i(x)$ exists for $x > 0$, and if $F^{ech(i)}(x)$ is strictly increasing in x the unicity of this minimum is guaranteed. However, if h_i equals 0 the minimum is attained in infinity. This can be explained by considering an increase of the order-up-to-level at end-stockpoint i . This results in an increasing ability to meet the customer demands, which yields lower penalty costs. However, no additional holding costs are incurred since $h_i = 0$.

From (4) it follows that the rationing functions $\{z_i^j\}_{j \in V_i}$ directly affects $D_i(y_i, \Psi_i)$. The rationing functions of stockpoint i which minimize $D_i(y_i, \Psi_i)$ are denoted by $\{\hat{z}_i^j\}_{j \in V_i}$. Lemma 3.2 yields the necessary condition for these functions.

Lemma 3.2. *A necessary condition for the rationing functions $\{\hat{z}_i^j\}_{j \in V_i}$ minimizing $D_i(y_i, \Psi_i)$ is*

$$\left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\hat{z}_i^j[x]} = \lambda_i[x] \quad \text{for every } j \in V_i. \quad (6)$$

Proof. When the echelon stock of stockpoint i , say x , is not sufficient to raise the echelon inventory positions of the stockpoints $j \in V_i$ to their echelon order-up-to-level y_j , the echelon stock x is rationed over the stockpoints $j \in V_i$. Theorem 3.2 implies that in order to do this optimally we have to solve

$$\min \sum_{j \in V_i} D_j(z_i^j[x], \Psi_j) \quad \text{s.t.} \quad \sum_{j \in V_i} z_i^j[x] = x.$$

Using the Lagrange-multiplier technique yields that the rationing functions $\{z_i^j\}_{j \in V_i}$ can only be optimal when (6) holds, which completes the proof. \square

Lemma 3.2 proves that the derivative of $D_j(y, \Psi_j)$ to y in $y = \hat{z}_i^j[x]$ is *independent* of stockpoint j . This is very important in order to characterize the optimal replenishment policy. With Lemma 3.2 we are able to derive several properties of $\{\hat{z}_i^j\}_{j \in V_i}$. In Lemma 3.3 we prove that there exists an optimal set of rationing functions $\{\hat{z}_i^j\}$ for which every function \hat{z}_i^j is non-decreasing.

Lemma 3.3. *If for every stockpoint $j \in V_i$ the cost-function $D_j(x, \Psi_j)$ is convex in x , then there exists an optimal set of rationing functions $\{\hat{z}_i^j\}$ such that*

$$\frac{d\hat{z}_i^j[x]}{dx} \geq 0.$$

Specifically, if for every stockpoint $j \in V_i$ the cost-function $D_j(x, \Psi_j)$ is linear decreasing for $x < x_j^$ and strict convex for $x \geq x_j^*$, then for the optimal set of rationing functions holds*

$$\frac{d\hat{z}_i^j[x]}{dx} > 0 \quad \text{for } x \geq x_i^* := \min\{x | \hat{z}_i^j[x] \geq x_j^* \text{ for } j \in V_i\}.$$

Proof. Let $D_j(x, \Psi_j)$ be convex in x for every $j \in V_i$. Assume for every set of optimal rationing functions $\{\hat{z}_i^j\}$ there exists a $m \in V_i$ such that $d\hat{z}_i^m[x]/dx < 0$ for some x . Then there exists an $\epsilon > 0$ such that $\hat{z}_i^m[x + \epsilon] < \hat{z}_i^m[x]$. Let us distinguish between two cases:

- There exists a stockpoint $k \in V_i$ such that

$$\left. \frac{\partial D_k(y, \Psi_k)}{\partial y} \right|_{y=\hat{z}_i^k[x+\epsilon]} > \left. \frac{\partial D_k(y, \Psi_k)}{\partial y} \right|_{y=\hat{z}_i^k[x]}.$$

From Lemma 3.2 it follows that

$$\left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\hat{z}_i^j[x+\epsilon]} > \left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\hat{z}_i^j[x]} \quad \text{for every } j \in V_i,$$

Since $D_j(y, \Psi_j)$ is convex in y we know that $\hat{z}_i^j[x + \epsilon] > \hat{z}_i^j[x]$ for every $j \in V_i$. This contradicts our earlier made assumption.

- For every successor $j \in V_i$ holds

$$\left. \frac{\partial D_j(y, \Psi_k)}{\partial y} \right|_{y=\hat{z}_i^j[x+\epsilon]} \leq \left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\hat{z}_i^j[x]} \quad (7)$$

Suppose there exists a stockpoint $k \in V_i$

$$\left. \frac{\partial D_k(y, \Psi_k)}{\partial y} \right|_{y=\hat{z}_i^k[x+\epsilon]} < \left. \frac{\partial D_k(y, \Psi_k)}{\partial y} \right|_{y=\hat{z}_i^k[x]}.$$

Since $D_j(y, \Psi_j)$ is convex in y we know that $\hat{z}_i^j[x + \epsilon] < \hat{z}_i^j[x]$ for every $j \in V_i$. So, $\sum_{j \in V_i} \hat{z}_i^j[x + \epsilon] < \sum_{j \in V_i} \hat{z}_i^j[x]$. This contradicts (3), which states $\sum_{j \in V_i} \hat{z}_i^j[x + \epsilon] = x + \epsilon > x = \sum_{j \in V_i} \hat{z}_i^j[x]$. Hence, equation (7) reduces to

$$\left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\hat{z}_i^j[x+\epsilon]} = \left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\hat{z}_i^j[x]}.$$

Let $A_i := \{j \in V_i | \hat{z}_i^j[x + \epsilon] \leq \hat{z}_i^j[x]\}$ and $B_i := \{j \in V_i | \hat{z}_i^j[x + \epsilon] > \hat{z}_i^j[x]\}$. Let us consider a set of rationing-functions $\{\tilde{z}_i^j\}$ which is identical to $\{\hat{z}_i^j\}$, except in $x + \epsilon$:

$$\tilde{z}_i^j[x + \epsilon] := \begin{cases} \hat{z}_i^j[x] & j \in A_i \\ \hat{z}_i^j[x] + q_i^j \epsilon & j \in B_i \end{cases}$$

where

$$q_i^j := \frac{\hat{z}_i^j[x + \epsilon] - \hat{z}_i^j[x]}{\sum_{k \in B_i} (\hat{z}_i^k[x + \epsilon] - \hat{z}_i^k[x])}.$$

We proof that $\{\tilde{z}_i^j\}$ is an optimal set of rationing functions in $x + \epsilon$. First, it can be shown that $\sum_{j \in V_i} \tilde{z}_i^j[x + \epsilon] = x + \epsilon$. Furthermore, eq. (6) is satisfied, since

- (1) For $j \in A_i$ we have

$$\left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\tilde{z}_i^j[x+\epsilon]} = \left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\hat{z}_i^j[x]}.$$

- (2) For $j \in B_i$ we have

$$\left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\tilde{z}_i^j[x+\epsilon]} = \left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\hat{z}_i^j[x] + q_i^j \epsilon}.$$

From (3) and $\hat{z}_i^m[x + \epsilon] < \hat{z}_i^m[x]$ we conclude that both A_i and B_i are non-empty sets. Since $D_j(y, \Psi_j)$ is convex in y , and $\hat{z}_i^j[x] < \hat{z}_i^j[x + \epsilon] < \hat{z}_i^j[x + \epsilon]$ we have

$$\left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\hat{z}_i^j[x+\epsilon]} = \left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\hat{z}_i^j[x]+q_i^j \epsilon}.$$

So, $\{\hat{z}_i^j\}$ is an optimal set of rationing functions. However, $\hat{z}_i^j[x + \epsilon] \geq \hat{z}_i^j[x]$ contradicts our earlier made assumption.

Assume for every stockpoint $j \in V_i$ the cost-function $D_j(x, \Psi_j)$ is linear decreasing for $x < x_j^*$ and strict convex for $x \geq x_j^*$. Let $x_i^* := \min\{x | \hat{z}_i^j[x] \geq x_j^* \text{ for } j \in V_i\}$. Now we proof that $\hat{z}_i^j[x]$ is strictly increasing for $x \geq x_i^*$. Assume for a stockpoint $m \in V_i$ holds $d\hat{z}_i^m[x]/dx \leq 0$ for a $x \geq x_i^*$. Then there exists an $\epsilon > 0$ for which $\hat{z}_i^m[x + \epsilon] \leq \hat{z}_i^m[x]$. Since $D_m(y, \Psi_m)$ is a convex function, we obtain

$$\left. \frac{\partial D_m(y, \Psi_m)}{\partial y} \right|_{y=\hat{z}_i^m[x+\epsilon]} \leq \left. \frac{\partial D_m(y, \Psi_m)}{\partial y} \right|_{y=\hat{z}_i^m[x]} \quad (8)$$

Using Lemma 3.2 and (8) yields,

$$\left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\hat{z}_i^j[x+\epsilon]} \leq \left. \frac{\partial D_j(y, \Psi_j)}{\partial y} \right|_{y=\hat{z}_i^j[x]} \quad \text{for every } j \in V_i.$$

Since $x \geq x_i^*$ and $\hat{z}_i^j[x]$ is non-decreasing in x it follows $\hat{z}_i^j[x] \geq x_j^*$. The cost-function $D_j(y, \Psi_j)$ is strict convex in $y = \hat{z}_i^j[x] \geq x_j^*$. Hence, $\hat{z}_i^j[x + \epsilon] \leq \hat{z}_i^j[x]$. So, $\sum_{j \in V_i} \hat{z}_i^j[x + \epsilon] \leq \sum_{j \in V_i} \hat{z}_i^j[x]$. This contradicts (3), which states $\sum_{j \in V_i} \hat{z}_i^j[x + \epsilon] = x + \epsilon > x = \sum_{j \in V_i} \hat{z}_i^j[x]$. Hence $d\hat{z}_i^m[x]/dx > 0$ for $x \geq x_i^*$. \square

Before proving another property of $\{\hat{z}_i^j\}$ we introduce the following definitions:

Definition 3.1. $\Psi_i = \bigcup_{j \in V_i} (\hat{z}_i^j, \hat{y}_j, \Psi_j)$ is *locally optimal* when

(i) Rationing functions $\{\hat{z}_i^j\}_{j \in V_i}$ satisfy (6).

(ii) For every $j \in V_i$ the order-up-to-level equals $\hat{y}_j := \operatorname{argmin} \{y | \partial D_j(y, \Psi_j) / \partial y = 0\}$. \square

Definition 3.2. $\hat{\Psi}_i$ is *optimal* if Ψ_j is locally optimal for every $j \in \operatorname{ech}(i)$. \square

In the next lemma we state an important property of $\{\hat{z}_i^j\}_{j \in V_i}$. It directly follows from Lemma 3.2 and the definition of local optimality. This property guarantees that when stockpoint i has to ration $\sum_{j \in V_i} \hat{y}_j$ products over its successors, it ensures the continuity of the amount allocated to a successor $j \in V_i$.

Lemma 3.4. *If for every stockpoint $j \in V_i$ the cost-function $D_j(y, \Psi_j)$ is convex in y , then for a local optimal $\Psi_i = \bigcup_{j \in V_i} (\hat{z}_i^j, \hat{y}_j, \Psi_j)$ holds $\hat{z}_i^j[\sum_{j \in V_i} \hat{y}_j] = \hat{y}_j$.*

Proof. This proof is by contradiction. Assume $D_j(y, \Psi_j)$ is convex in y for every $j \in V_i$. Suppose there exists a stockpoint $k \in V_i$ for which $\hat{z}_i^k[\sum_{j \in V_i} \hat{y}_j] < \hat{y}_k$. From the definition of \hat{y}_k and the convexity of $D_k(x, \Psi_k)$, we conclude

$$\left. \frac{\partial D_k(y, \Psi_k)}{\partial y} \right|_{y=\hat{z}_i^k[\sum_{j \in V_i} \hat{y}_j]} < \left. \frac{\partial D_k(y, \Psi_k)}{\partial y} \right|_{y=\hat{y}_k} = 0.$$

Hence, from Lemma 3.2 it follows that $\lambda_i[\sum_{j \in V_i} \hat{y}_j] < 0$. Since $\sum_{j \in V_i} \hat{z}_i^j[\sum_{n \in V_i} \hat{y}_n] = \sum_{j \in V_i} \hat{y}_j$, there has to be a stockpoint, m say, for which $\hat{z}_i^m[\sum_{j \in V_i} \hat{y}_j] > \hat{y}_m$. Using similar arguments as above it can be shown that $\lambda_i[\sum_{j \in V_i} \hat{y}_j] \geq 0$. This leads to a contradiction. Hence, $\hat{z}_i^k[\sum_{j \in V_i} \hat{y}_j] \geq \hat{y}_k$.

Analogous we can prove $\hat{z}_i^k[\sum_{j \in V_i} \hat{y}_j] \leq \hat{y}_k$. Thus $\hat{z}_i^k[\sum_{j \in V_i} \hat{y}_j] = \hat{y}_k$. This concludes the proof. \square

Lemma 3.4 yields a necessary condition for every rationing function in order to be optimal. Besides this lemma, we need the definition of $L_i(x)$, Theorem 3.2, Lemma 3.2 and Definition 3.1 to proof Theorem 3.3, giving an explicit expression of the derivative of the cost-function $D_i(x, \Psi_i)$ to x for local optimal Ψ_i .

Theorem 3.3. *For every intermediate stockpoint $i \in I$ with a locally optimal Ψ_i holds if $\Delta_i < 0$,*

$$\frac{\partial D_i(y_i, \Psi_i)}{\partial y_i} = h_i + \int_0^\infty \lambda_i[y_i - u] dF_{L_i}^{ech(i)}(u),$$

if $\Delta_i \geq 0$,

$$\frac{\partial D_i(y_i, \Psi_i)}{\partial y_i} = h_i + \int_0^\infty \lambda_i[y_i - u - \Delta_i] d\left(F_{L_i}^{ech(i)}\right)^{\Delta_i}(u),$$

with $\lambda_i[x]$ as defined in (6).

Proof. Substitution of the definition of $L_i(x)$ in (4) and rewriting the result yields

$$\begin{aligned} \frac{\partial D_i(y_i, \Psi_i)}{\partial y_i} &= \frac{\partial}{\partial y_i} \left\{ h_i(y_i - (L_i + 1)\mu_{ech(i)}) + \sum_{j \in V_i} D_j(y_j, \Psi_j) + \right. \\ &\quad \left. \int_{\Delta_i}^\infty D_j(\hat{z}_i^j[y_i - u], \Psi_j) - D_j(y_j, \Psi_j) dF_{L_i}^{ech(i)}(u) \right\} \\ &= h_i + \sum_{j \in V_i} \int_{\Delta_i}^\infty \frac{d\hat{z}_i^j[x]}{dx} \Big|_{x=y_i-u} \frac{\partial D_j(x, \Psi_j)}{\partial x} \Big|_{x=\hat{z}_i^j[y_i-u]} dF_{L_i}^{ech(i)}(u). \end{aligned}$$

Since Ψ_i is locally optimal, $\{\hat{z}_i^j\}_{j \in V_i}$ satisfy (6). Applying Lemma 3.2 yields

$$\frac{\partial D_i(y_i, \Psi_i)}{\partial y_i} = h_i + \int_{\Delta_i}^\infty \lambda_i[y_i - u] \sum_{j \in V_i} \frac{d\hat{z}_i^j[x]}{dx} \Big|_{x=y_i-u} dF_{L_i}^{ech(i)}(u).$$

Since equality (3) holds for every rationing function, using this property yields

$$\frac{\partial D_i(y_i, \Psi_i)}{\partial y_i} = h_i + \int_{\Delta_i}^\infty \lambda_i[y_i - u] dF_{L_i}^{ech(i)}(u). \quad (9)$$

When $\Delta_i < 0$ the theorem is trivial from (9), since $F_{L_i}^{ech(i)}(u) = 0$ when $u < 0$.

For the case $\Delta_i \geq 0$ we introduce $u^* := u - \Delta_i$. Substitution of u^* in (9) yields

$$\begin{aligned} \frac{\partial D_i(y_i, \Psi_i)}{\partial y_i} &= h_i + \int_0^\infty \lambda_i[y_i - u^* - \Delta_i] dF_{L_i}^{ech(i)}(u^* + \Delta_i) \\ &= h_i + \int_{0+}^\infty \lambda_i[y_i - u^* - \Delta_i] d\left(F_{L_i}^{ech(i)}\right)^{\Delta_i}(u^*). \end{aligned}$$

Since Ψ_i is locally optimal we use Lemma 3.4 to show that $\lambda_i[y_i - \Delta_i] = 0$. This completes the proof. \square

Similar properties as in Lemma 3.1 are derived in the next theorem for an arbitrary stockpoint. On the one hand these properties yield insight in the cost-function $D_i(x, \hat{\Psi}_i)$, while on the other hand it is required for the proof in Section 4 that the decomposition is exact.

Theorem 3.4. *The cost-function $D_i(x, \hat{\Psi}_i)$ with optimal $\hat{\Psi}_i$ is*

- (i). *linear decreasing with slope $-(\sum_{j \in U_i} h_j + p_i^*)$ for $x < x_i^*$.*
- (ii). *convex in x . Specifically, if for every end-stockpoint k in ech(i) the demand function $F^{ech(k)}(x)$ is strictly increasing for $x \geq 0$, then $D_i(x, \hat{\Psi}_i)$ is a strict convex function of $x \geq x_i^*$.*
- (iii). *$\partial D_i(x, \hat{\Psi}_i)/\partial x$ tends to h_i when x goes to infinity.*

Proof. This proof is by induction on i . By defining $x_i^* := 0$ for every end-stockpoint i , Lemma 3.1 proofs the theorem. Next, consider a stockpoint $i \in I$. Assume the theorem holds for every successor $j \in V_i$ (induction assumption). First we prove (i). Let $x < x_i^*$. Then, there exists a stockpoint $m \in V_i$ for which $\hat{z}_i^m[x] \leq x_m^*$. Using the induction assumption and Lemma 3.2 yields $\lambda_i[x] = -(\sum_{j \in U_m} h_j + p_m^*)$. From the definition of $\lambda_i[x]$ we have

$$\left. \frac{\partial D_j(y, \hat{\Psi}_j)}{\partial y} \right|_{y=\hat{z}_i^j[x]} = -\left(\sum_{k \in U_m} h_k + p_m^* \right) \quad \text{for every } j \in V_i. \quad (10)$$

Substitution of $\lambda_i[x]$ in the formulas of Theorem 3.3 yields

$$\frac{\partial D_i(x, \hat{\Psi}_i)}{\partial x} = h_i - \left(\sum_{j \in U_m} h_j + p_m^* \right) = -\left(\sum_{j \in U_i} h_j + p_m^* \right).$$

Next, we prove $p_m^* = p_i^*$. Suppose there exists a $j \in V_i$ for which $p_m^* > p_j^*$. Using $D_j(y, \hat{\Psi}_j)$ is convex in y and the induction assumption yields

$$\frac{\partial D_j(y, \hat{\Psi}_j)}{\partial y} \geq -\left(\sum_{k \in U_j} h_k + p_j^* \right).$$

Since $p_m^* > p_j^*$ and $U_m = U_j$ we know

$$\frac{\partial D_j(y, \hat{\Psi}_j)}{\partial y} > -\left(\sum_{k \in U_m} h_k + p_m^* \right).$$

This contradicts (10). Thus, a necessary condition for m is

$$p_m^* \leq p_j^* \quad \text{for every } j \in V_i.$$

Hence, $p_m^* = p_i^*$, which proves (i).

Next, we prove that $D_i(x, \hat{\Psi}_i)$ is convex in x . From the induction assumption and Lemma 3.4 it can be shown that $\lambda_i[\sum_{j \in V_i} \hat{y}_j] = 0$. This property and (9) yields

$$\frac{\partial^2 D_i(x, \hat{\Psi}_i)}{\partial x^2} = \int_{x - \sum_{j \in V_i} \hat{y}_j}^{\infty} \frac{d\hat{z}_i^j[x-u]}{dx} \left. \frac{\partial^2 D_j(y, \hat{\Psi}_j)}{\partial y^2} \right|_{y=\hat{z}_i^j[x-u]} dF_{L_i}^{ech(i)}(u) \quad \text{for every } j \in V_i.$$

From Lemma 3.3 (using the induction assumption) it immediately follows that $\partial^2 D_i(x, \hat{\Psi}_i)/\partial x^2 \geq 0$. Specifically, let $x \geq x_i^*$. It can be shown that $\sum_{j \in V_i} \hat{y}_j \geq x_i^*$. Thus, there exists an $u \geq 0$ such that $x - \sum_{j \in V_i} \hat{y}_j \leq u \leq x - x_i^*$. Then from Lemma 3.3 and the induction assumption it follows that $D_i(x, \hat{\Psi}_i)$ is strict convex for $x \geq x_i^*$. Finally, we proof that when x goes to infinity $\partial D_i(x, \hat{\Psi}_i)/\partial x$ tends to h_i . From the induction assumption it follows

$$-\left(\sum_{j \in U_i} h_j + p_i^* \right) \leq \lambda_i[x] \leq h_i^*. \quad (11)$$

From (9) and (11) it follows

$$h_i - \left(\sum_{j \in U_i} h_j + p_i^* \right) \int_{\Delta_i}^{\infty} dF_{L_i}^{ech(i)}(u) \leq \frac{\partial D_i(x, \Psi_i)}{\partial x} \leq h_i + h_i^* \int_{\Delta_i}^{\infty} dF_{L_i}^{ech(i)}(u).$$

When y_i tends to infinity both the lower and upper bound converge to h_i . \square

The next corollary is of vital importance for the results in the next section, and immediately follows from Theorem 3.4.

Corollary 3.1. *For every stockpoint $i \in I$ with $\hat{\Psi}_i$ optimal there exists a \hat{y}_i for which*

$$\frac{\partial D_i(x, \hat{\Psi}_i)}{\partial x} < 0, \quad x < \hat{y}_i \quad \text{and} \quad \frac{\partial D_i(x, \hat{\Psi}_i)}{\partial x} \geq 0, \quad x > \hat{y}_i.$$

\square

From Lemma 3.2 and Theorem 3.4 some interesting properties of $\{\hat{z}_i^j\}_{j \in V_i}$ can be derived. These properties are formalized in the Theorem 3.5. This theorem states that if the echelon stock of stockpoint i just before rationing, x say, is less than or equal to x_i^* the optimal rationing rule ensures that only the echelons j with the lowest penalty cost (p_i^*) get a small (or even negative) echelon inventory position after rationing. If x is very large the optimal rationing rule ensures that the major part of x is allocated to the successors j with the lowest added value, i.e., h_i^* . After this theorem we elaborate on the behavior of an optimal rationing function, by giving some examples of the behavior of such a function.

Theorem 3.5. *For every optimal set of rationing functions holds*

(i)

$$\hat{z}_i^j[x] \in [\underline{x}_j^l, \underline{x}_j^u] \quad \text{for} \quad x \leq x_i^* = \sum_{k \in \underline{V}_i} x_k^* + \sum_{k \in V_i \setminus \underline{V}_i} \underline{x}_k^u \quad \text{and} \quad j \in V_i \setminus \underline{V}_i,$$

(ii)

$$\hat{z}_i^j[x] < \bar{x}_j \quad \text{for every} \quad x \quad \text{and} \quad j \in V_i \setminus \bar{V}_i,$$

where

$$\begin{aligned} \underline{x}_j^l &:= \operatorname{argmin}_x \left\{ \frac{\partial D_j(x, \hat{\Psi}_j)}{\partial x} = -\left(\sum_{k \in U_j} h_k + p_i^* \right) \right\} \quad \text{for} \quad j \in V_i \setminus \underline{V}_i, \\ \underline{x}_j^u &:= \operatorname{argmax}_x \left\{ \frac{\partial D_j(x, \hat{\Psi}_j)}{\partial x} = -\left(\sum_{k \in U_j} h_k + p_i^* \right) \right\} \quad \text{for} \quad j \in V_i \setminus \underline{V}_i, \\ \bar{x}_j &:= \operatorname{argmin}_x \left\{ \frac{\partial D_j(x, \hat{\Psi}_j)}{\partial x} = h_i^* \right\} \quad \text{for} \quad j \in V_i \setminus \bar{V}_i. \end{aligned}$$

Proof. (i): Suppose a stockpoint i has to ration its echelon stock $x \leq x_i^*$ over its successors. Consider a successor, m say, for which $p_m^* > p_i^*$. So, $m \in V_i \setminus \underline{V}_i$. We prove (i) by showing that $\hat{z}_i^m[x] \geq \underline{x}_m^l$ and $\hat{z}_i^m[x] \leq \underline{x}_m^u$, respectively. For both cases the proofs are given by contradiction.

• Assume $\hat{z}_i^m[x] < \underline{x}_m^l$. From the convexity of $D_m(y, \hat{\Psi}_m)$ in y , and the definition of \underline{x}_m^l we obtain

$$\left. \frac{\partial D_m(y, \hat{\Psi}_m)}{\partial y} \right|_{y=\hat{z}_i^m[x]} < \left. \frac{\partial D_m(y, \hat{\Psi}_m)}{\partial y} \right|_{y=\underline{x}_m^l} = -\left(\sum_{k \in U_m} h_k + p_i^* \right).$$

Using Lemma 3.2 yields

$$\lambda_i[x] < -(h_i + \sum_{j \in U_i} h_j + p_i^*). \quad (12)$$

Next, we consider a stockpoint, n say, for which $p_n^* = p_i^*$. So, $n \in \underline{V}_i$. From the convexity of $D_n(y, \hat{\Psi}_n)$ and Theorem 3.4 it follows

$$\frac{\partial D_n(y, \hat{\Psi}_n)}{\partial y} \geq -\left(\sum_{k \in U_n} h_k + p_n^*\right) \stackrel{n \in \underline{V}_i}{=} -\left(\sum_{k \in U_n} h_k + p_i^*\right).$$

Again using Lemma 3.2 yields

$$\lambda_i[x] \geq -(h_i + \sum_{k \in U_i} h_k + p_i^*). \quad (13)$$

Equation (12) and (13) lead to a contradiction. Hence $\hat{z}_i^m[x] \geq \underline{x}_m^l$.

• Assume $\hat{z}_i^m[x] > \underline{x}_m^u$. Similar to (12) it can be shown that

$$\lambda_i[x] > -(h_i + \sum_{k \in U_i} h_k + p_i^*). \quad (14)$$

Rewriting (14) shows that for every stockpoint $j \in V_i \setminus \underline{V}_i$ we have

$$\left. \frac{\partial D_j(y, \hat{\Psi}_j)}{\partial y} \right|_{y=\hat{z}_i^j[x]} > -\left(\sum_{k \in U_j} h_k + p_i^*\right) = \left. \frac{\partial D_j(y, \hat{\Psi}_j)}{\partial y} \right|_{y=\underline{x}_j^u}.$$

Since $D_j(y, \hat{\Psi}_j)$ is convex in y we know that $\hat{z}_i^j[x] > \underline{x}_j^u$. Using this and equality (3) yields

$$\sum_{j \in \underline{V}_i} \hat{z}_i^j[x] = x - \sum_{j \in V_i \setminus \underline{V}_i} \hat{z}_i^j[x] < x - \sum_{j \in V_i \setminus \underline{V}_i} \underline{x}_j^u = x - x_i^* + \sum_{j \in \underline{V}_i} x_j^* \leq \sum_{j \in \underline{V}_i} x_j^*.$$

So, there exists a stockpoint $n \in \underline{V}_i$ for which $\hat{z}_i^n[x] < x_n^*$.

$$\left. \frac{\partial D_n(y, \hat{\Psi}_n)}{\partial y} \right|_{y=\hat{z}_i^n[x]} = -\left(\sum_{k \in U_n} h_k + p_n^*\right) \stackrel{n \in \underline{V}_i}{=} -\left(\sum_{k \in U_n} h_k + p_i^*\right).$$

Again using Lemma 3.2 yields

$$\lambda_i[x] = -(h_i + \sum_{k \in U_i} h_k + p_i^*). \quad (15)$$

Equation (14) and (15) lead to a contradiction. Hence $\hat{z}_i^m[x] \leq \underline{x}_m^u$. So $\hat{z}_i^m[x] \in [\underline{x}_m^l, \underline{x}_m^u]$.

(ii): Suppose a stockpoint i has to ration its echelon stock $x \leq x_i^*$ over its successors. Consider such a successor, m say, for which $h_m > h_i^*$. So, $m \in V_i \setminus \bar{V}_i$. Assume that $\hat{z}_i^m[x] \geq \bar{x}_m$. Since $D_m(y, \hat{\Psi}_m)$ is convex in y we have

$$\left. \frac{\partial D_m(y, \hat{\Psi}_m)}{\partial y} \right|_{y=\hat{z}_i^m[x]} \geq \left. \frac{\partial D_m(y, \hat{\Psi}_m)}{\partial y} \right|_{y=\bar{x}_m} = h_i^*.$$

So from Lemma 3.2 it follows that $\lambda_i[x] \geq h_i^*$. From Theorem 3.4 it is clear that for a stockpoint $j \in \bar{V}_i$ holds $\partial D_j(y, \hat{\Psi}_j)/\partial y < h_j = h_i^*$. Then Lemma 3.2 leads to a contradiction, since we obtain $\lambda_i[x] < h_i^*$. This proves $\hat{z}_i^m[x] < \bar{x}_m$. \square

In case the cdf of the customer demand is strictly increasing for every end-stockpoint, it can be shown that $\underline{x}_j^l = \underline{x}_j^u$. So in that case $\hat{z}_i^j[x]$ equals \underline{x}_j^u for $j \notin \underline{V}_i$ and $x \leq x_i^*$. In the more general case where strict monotonicity of the customer demand cdf is not required it can be shown that there exists an optimal set of rationing functions such that $\hat{z}_i^j[x]$ equals \underline{x}_j^u for $j \notin \underline{V}_i$ and $x \leq x_i^*$. In the remainder of this paper we consider this specific set.

From Theorem 3.5 it is possible to distinguish between four classes of rationing functions. Figure 2 depicts an example of a rationing function for each class. Notice that in Figure 2 (a) and (b) the amount of products allocated to successor j tends to a limit when x goes to infinity, although, the rationing functions in Figure 2 (c) and (d) do not have an upperbound. This is intuitively clear. Suppose stockpoint i has to allocate many products, x say. Since high penalty costs are incurred for every backlogged product, we would like to allocate as much as possible to successor j . However, also holding costs are incurred which assures that not too much stock is kept at the various stockpoints. It is clear that there is a trade off between the penalty costs and the holding costs. If the amount x which need to be allocated is large then the allocation decisions are based on the holding costs, since the penalty costs are very small. Specifically, if x goes to infinity the reduction of the penalty costs of echelon j by allocating an extra product to j equals 0, while the additional holding costs are h_j . Therefore, an extra product will be allocated to those successors with minimal added value.

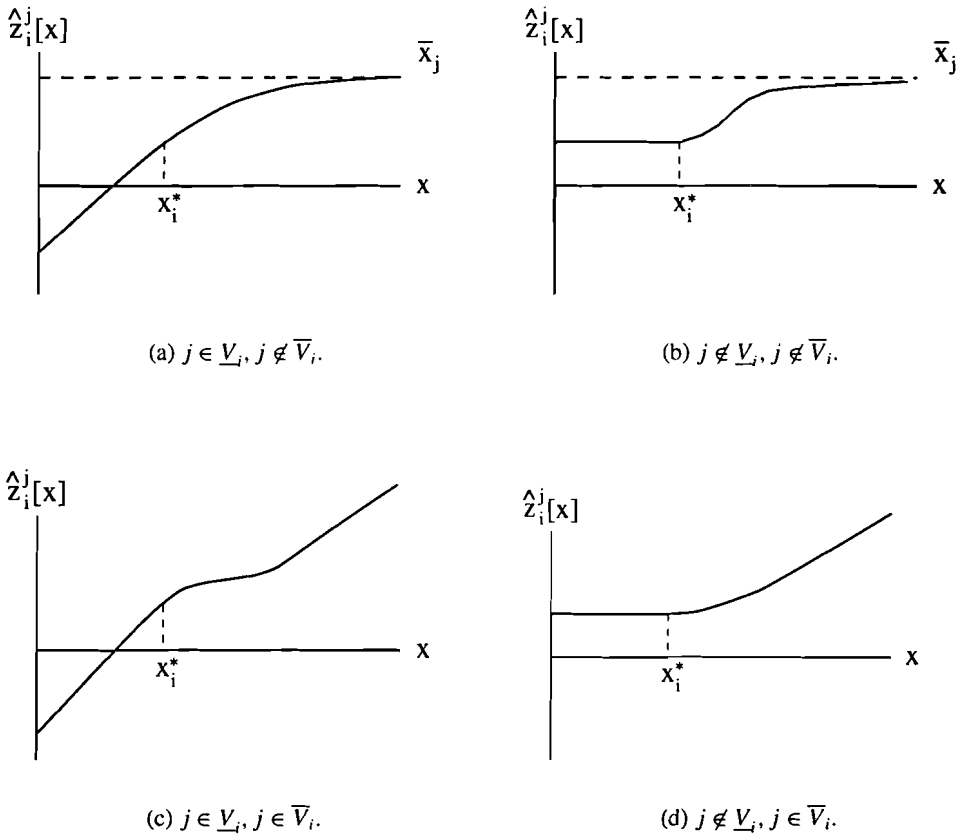


Figure 2: The behavior of a rationing function for the four different classes.

Also notice that in Figure 2 (b) and (d) the amount of products allocated to successor j is fixed when x is sufficiently small, although, this is not true in Figure 2 (a) and (c). This can be explained as follows. If x is very small (or even negative) there is an incentive to allocate the major part to the successors j which (indirectly) supply end-stockpoints with high penalty costs ($j \notin \underline{V}_i$). The actual amount allocated to a successor $j \notin \underline{V}_i$ is such that the marginal costs in $ech(j)$ equals the marginal costs of a successor $j \in \underline{V}_i$.

Next, we address Theorem 3.6 enabling a simplification of Theorem 3.3.

Theorem 3.6. Let $\alpha_k^i(y)$ denotes the non-stock out probability of an end-stockpoint k in a divergent echelon system, in which the most upstream stockpoint i uses an order-up-to-policy with order-up-to-level y . If for every rationing function in stockpoint i holds $z_i^j[\sum_{j \in V_i} y_j] = y_j$, then

$$\alpha_k^i(y) = \begin{cases} F_{L_k+1}^{ech(k)}(y) & i \in E, \\ \int_0^\infty \alpha_k^j(z_i^j[y-u]) dF_{L_i}^{ech(i)}(u) & i \in I, \Delta_i < 0 \\ \int_0^\infty \alpha_k^j(z_i^j[y-u-\Delta_i]) d\left(F_{L_i}^{ech(i)}\right)^{\Delta_i}(u) & i \in I, \Delta_i \geq 0, \end{cases} \quad (16)$$

for every $j \in V_i$ and $k \in E \cap ech(j)$.

Proof. We proof this by using induction on i . When $i \in E$ it is trivial.

Suppose equality (16) holds for a divergent system with most upstream stockpoint j (induction assumption). Then the non-stock out probability of an end-stockpoint k equals $\alpha_k^j(y_j)$, where y_j equals the order-up-to-level of this stockpoint j . Next, consider a divergent system with most upstream stockpoint i , and $j \in V_i$. In order to determine the non-stock out probability of stockpoint k in this system we introduce some additional notation. Let $\alpha_{k,t}^i$ denotes the non-stock out probability of this stockpoint k as a result of the rationing decision at the beginning of period t , and $D_{t-L_i,t}$ equals the total demand of all end-stockpoints during $[t-L_i, t)$.

When at the beginning of an arbitrary period t the echelon stock of stockpoint i is less than the sum of all order-up-to-levels of the stockpoints in V_i , rationing is necessary. This means that every stockpoint $j \in V_i$ gets his appropriate share $z_i^j[y - D_{t-L_i,t}]$ instead of order-up-to-level y_j . From the induction assumption we obtain that for $j \in V_i$ and $k \in ech(j) \cap E$

$$y - D_{t-L_i,t} < \sum_{j \in V_i} y_j \implies \alpha_{k,t}^i(y) = \alpha_k^j(z_i^j[y - D_{t-L_i,t}]). \quad (17)$$

However, if at time t the echelon stock at i is sufficient, all stockpoints $j \in V_i$ raise their inventory positions to their order-up-to-levels. From the induction assumption we obtain that for $j \in V_i$ and $k \in ech(j) \cap E$

$$y - D_{t-L_i,t} \geq \sum_{j \in V_i} y_j \implies \alpha_{k,t}^i(y) = \alpha_k^j(y_j). \quad (18)$$

Since the demand process is stationary we may suppress the index t in $\alpha_{k,t}^i(y)$. From (17) and (18) it can be verified that for $j \in V_i$ and $k \in ech(j) \cap E$,

$$\Delta_i < 0 \implies \alpha_k^i(y) = \int_0^\infty \alpha_k^j(z_i^j[y-u]) dF_{L_i}^{ech(i)}(u), \quad (19)$$

$$\Delta_i \geq 0 \implies \alpha_k^i(y) = \int_0^{\Delta_i} \alpha_k^j(y_j) dF_{L_i}^{ech(i)}(u) + \int_{\Delta_i}^\infty \alpha_k^j(z_i^j[y-u]) dF_{L_i}^{ech(i)}(u). \quad (20)$$

Rewriting (20), using the assumption that $z_i^j[\sum_{j \in V_i} y_j] = y_j$, yields that for $j \in V_i$ and $k \in ech(j) \cap E$

$$\Delta_i > 0 \implies \alpha_k^i(y) = \int_0^\infty \alpha_k^j(z_i^j[y - \Delta_i - u]) d\left(F_{L_i}^{ech(i)}\right)^{\Delta_i}(u). \quad (21)$$

From (19) and (21) it follows that equality (16) also holds for i . Induction proofs the theorem. \square

From Theorem 3.3 and 3.6 we are able to derive an explicit expression for $\partial D_i(x, \hat{\Psi}_i)/\partial x$ with optimal $\hat{\Psi}_i$.

Theorem 3.7. For every $D_i(x, \hat{\Psi}_i)$ with optimal $\hat{\Psi}_i$ holds

$$\frac{\partial D_i(y, \hat{\Psi}_i)}{\partial y} = -\left(\sum_{j \in U_i} h_j + p_k\right) + (h_k + \sum_{j \in U_k} h_j + p_k) \alpha_k^i(y) \quad \text{for every } k \in \text{ech}(i) \cap E.$$

Proof. This theorem can be proved by induction on i . For $i \in E$ it immediately follows from (5). Next, consider an intermediate stockpoint $i \in I$. Assume the theorem holds for every successor $j \in V_i$ (induction assumption). Let $y < \sum_{j \in V_i} \hat{y}_j$. From Theorem 3.3 it follows

$$\frac{\partial D_i(y, \hat{\Psi}_i)}{\partial y} = h_i + \int_0^\infty \lambda_i[y - u] dF_{L_i}^{\text{ech}(i)}(u).$$

Substitution of the definition of $\lambda_i[y - u]$ yields

$$\frac{\partial D_i(y, \hat{\Psi}_i)}{\partial y} = h_i + \int_0^\infty \left. \frac{\partial D_j(x, \hat{\Psi}_j)}{\partial x} \right|_{x=\hat{z}_i^j[y-u]} dF_{L_i}^{\text{ech}(i)}(u) \quad \text{for every } j \in V_i.$$

Using the induction assumption yields

$$\frac{\partial D_i(y, \hat{\Psi}_i)}{\partial y} = h_i + \int_0^\infty -\left(\sum_{n \in U_j} h_n + p_k\right) + (h_k + \sum_{n \in U_k} h_n + p_k) \alpha_k^j(\hat{z}_i^j[y - u]) dF_{L_i}^{\text{ech}(i)}(u),$$

with $k \in E \cap \text{ech}(j)$.

Theorem 3.6 directly completes the proof for $y < \sum_{j \in V_i} \hat{y}_j$. The proof is completely analogous for $y \geq \sum_{j \in V_i} \hat{y}_j$. \square

From Theorem 3.7 it follows that the optimal order-up-to-level in stockpoint i , \hat{y}_i say, has to satisfy

$$\alpha_k^i(\hat{y}_i) = \frac{\sum_{j \in U_i} h_j + p_k}{h_k + \sum_{j \in U_k} h_j + p_k} \quad \text{for every } k \in \text{ech}(i) \cap E. \quad (22)$$

Note that condition (22) resembles the classical newsboy result. This classical newsboy result prescribes the optimal critical ratio for a single location inventory system to be $p/(p + h)$. Similar newsboy style results are derived in Rogers & Tsubakitani [1991]. They considered the optimization problem of minimizing the penalty costs in a divergent two-echelon system, subjected to a budget constraint on the total holding costs. Necessary conditions are derived for optimality for the case where every end-stockpoint faces normal distributed demand.

4 Optimality of decomposition approach

Now we apply the results from the previous section to develop an optimization scheme for $(\hat{y}_i, \hat{\Psi}_i)$ where stockpoint i is the most upstream stockpoint of the N -echelon system. The problem of finding this optimal $(\hat{y}_i, \hat{\Psi}_i)$ can be decomposed into solving one-dimensional problems subsequently. For our convenience we assign a *low level code* (LLC) to every stockpoint. By definition the low level code of an end-stockpoint i equals 1, i.e., $\text{LLC}(i) := 1$. For an intermediate stockpoint i we have $\text{LLC}(i) := 1 + \max_{j \in V_i} \text{LLC}(j)$. Let W_n denote the set of stockpoints with low level code n .

Decomposition approach:

- (i). $n := 1$.
- (ii). \bullet Set $\hat{\Psi}_i := \emptyset$ for every $i \in E$.

- Determine the order-up-to-level \hat{y}_i , which minimizes $D_i(y_i)$, for every stockpoint $i \in E$.
- (iii). $n := n + 1$.
- (iv). Determine for every stockpoint $i \in W_n$:
 - The rationing functions $\{\hat{z}_i^j\}_{j \in V_i}$, which minimizes $\sum_{j \in V_i} D_j(z_i^j[x], \hat{\Psi}_j)$ for every x .
 - $\hat{\Psi}_i := \bigcup_{j \in V_i} (\hat{z}_i^j, \hat{y}_j, \hat{\Psi}_j)$ for $i \in W_n$.
 - The order-up-to-level \hat{y}_i , which minimizes $D_i(y_i, \hat{\Psi}_i)$.
- (v). If $n=N$ then return to step (iv).

In order to prove that the approach above yields the optimal policy we introduce some definitions which are needed for Lemma 4.1.

Definition 4.1. Consider an arbitrary policy $s_0 = y_i \cup \bigcup_{j \in V_i} (z_i^j, y_j, \Psi_j)$ with $i \in W_N$. Let s_n denotes the replenishment policy in which for every $i \in W_{n+1}$

- (i). The order-up-to-levels and the rationing functions of a *stockpoint upstream of i* are identically defined as in the policy s_0 .
- (ii). The order-up-to-level and the rationing functions of i are identically defined as in the policy s_0 and the optimal policy $\hat{\Psi}_i$, respectively.
- (iii). The order-up-to-level and the rationing functions of a *stockpoint downstream of i* are identically defined as in the optimal policy $\hat{\Psi}_i$.

□

Lemma 4.1. Let $g_n(s_n)$ denote the expected total costs of the stockpoints in $\bigcup_{j=1}^n W_j$ when the multi-echelon system is controlled by policy s_n . For $n \in \{1, \dots, N\}$ holds

$$g_n(s_n) \leq g_n(s_{n-1}).$$

Proof. Suppose s_0 is an arbitrary replenishment policy. Let us consider a stockpoint $i \in W_{n+1}$. For our convenience we introduce some additional notation. Let $g_t^i(s_n)$ denote the expected total costs in $\bigcup_{j \in V_i} ech(j)$, due to the replenishment orders placed by policy s_n at the beginning of period t . The rationing function from stockpoint i to successor j when applying policy s_n and s_{n-1} is denoted by \hat{z}_i^j and z_i^j , respectively. The order-up-to-level of j when applying policy s_n and s_{n-1} are denoted by \hat{y}_j and y_j . Let us distinguish between the case where $\sum_{j \in V_i} y_j < \sum_{j \in V_i} \hat{y}_j$ and $\sum_{j \in V_i} y_j > \sum_{j \in V_i} \hat{y}_j$.

- $\sum_{j \in V_i} y_j < \sum_{j \in V_i} \hat{y}_j$.

If at the beginning of period t the echelon stock of stockpoint i , $I_{i,t}$ say, is sufficient to raise the echelon inventory positions of all stockpoints $j \in V_i$ to their order-up-to-level, the expected costs of echelon j equals $D_j(\hat{y}_j, \hat{\Psi}_j)$ and $D_j(y_j, \hat{\Psi}_j)$ for replenishment policy s_n and s_{n-1} , respectively. Hence, Corollary 3.1 yields

$$I_{i,t} \geq \sum_{j \in V_i} \hat{y}_j \implies g_t^i(s_n) = \sum_{j \in V_i} D_j(\hat{y}_j, \hat{\Psi}_j) \leq \sum_{j \in V_i} D_j(y_j, \hat{\Psi}_j) = g_t^i(s_{n-1}).$$

However, if at the beginning of period t the echelon stock of i is not sufficient, all this echelon stock is allocated over the echelons $j \in V_i$.

$$I_{i,t} \leq \sum_{j \in V_i} y_j \implies g_t^i(s_n) = \sum_{j \in V_i} D_j(\hat{z}_i^j[I_{i,t}], \hat{\Psi}_j) \leq \sum_{j \in V_i} D_j(z_i^j[I_{i,t}], \hat{\Psi}_j) = g_t^i(s_{n-1}).$$

Finally, we analyze the situation where $\sum_{j \in V_i} y_j < I_{i,t} < \sum_{j \in V_i} \hat{y}_j$. Then policy s_n rations all the echelon stock $I_{i,t}$ over the stockpoints $j \in V_i$, while policy s_{n-1} raises the echelon inventory positions of stockpoints $j \in V_i$ to their order-up-to-levels y_j , and the remainder ($I_{i,t} - \sum_{j \in V_i} y_j$) is retained at i .

The rationing function \tilde{z}_i^j , defined by

$$\tilde{z}_i^j[x] := y_j - q_i^j \left(\sum_{j \in V_i} y_j - x \right) \quad \text{with} \quad q_i^j := \frac{\hat{y}_j - y_j}{\sum_{k \in V_i} (\hat{y}_k - y_k)},$$

has the following property $y_j \leq \tilde{z}_i^j[I_{i,t}] < \hat{y}_j$ if $y_j \leq \hat{y}_j$, and $\hat{y}_j < \tilde{z}_i^j[I_{i,t}] < y_j$ if $y_j > \hat{y}_j$. From this property and Corollary 3.1 it follows

$$\sum_{j \in V_i} D_j(\tilde{z}_i^j[I_{i,t}], \hat{\Psi}_j) \leq \sum_{j \in V_i} D_j(y_j, \hat{\Psi}_j). \quad (23)$$

Since \tilde{z}_i^j is at least as good as \hat{z}_i^j , then from Corollary 3.1 and (23) it follows that

$$\begin{aligned} \sum_{j \in V_i} y_j < I_{i,t} < \sum_{j \in V_i} \hat{y}_j &\implies g_t^i(s_n) = \sum_{j \in V_i} D_j(\tilde{z}_i^j[I_{i,t}], \hat{\Psi}_j) \leq \sum_{j \in V_i} D_j(\hat{z}_i^j[I_{i,t}], \hat{\Psi}_j) \\ &\leq \sum_{j \in V_i} D_j(y_j, \hat{\Psi}_j) = g_t^i(s_{n-1}). \end{aligned}$$

- $\sum_{j \in V_i} y_j > \sum_{j \in V_i} \hat{y}_j$.

Analogously, we can proof that

$$\begin{aligned} I_{i,t} \geq \sum_{j \in V_i} y_j &\implies g_t^i(s_n) = \sum_{j \in V_i} D_j(\hat{y}_j, \hat{\Psi}_j) \leq \sum_{j \in V_i} D_j(y_j, \hat{\Psi}_j) = g_t^i(s_{n-1}). \\ I_{i,t} \leq \sum_{j \in V_i} \hat{y}_j &\implies g_t^i(s_n) = \sum_{j \in V_i} D_j(\hat{z}_i^j[I_{i,t}], \hat{\Psi}_j) \leq \sum_{j \in V_i} D_j(\hat{z}_i^j[I_{i,t}], \hat{\Psi}_j) = g_t^i(s_{n-1}). \\ \sum_{j \in V_i} \hat{y}_j < I_{i,t} < \sum_{j \in V_i} y_j &\implies g_t^i(s_n) = \sum_{j \in V_i} D_j(\hat{y}_j, \hat{\Psi}_j) \leq \sum_{j \in V_i} D_j(\hat{z}_i^j[I_{i,t}], \hat{\Psi}_j) = g_t^i(s_{n-1}). \end{aligned}$$

Notice that when $\sum_{j \in V_i} y_j = \sum_{j \in V_i} \hat{y}_j$ from Corollary 3.1 we know that $g_t^i(s_n) \leq g_t^i(s_{n-1})$. So for an arbitrary period t and an arbitrary stockpoint $i \in W_{n+1}$ holds

$$g_t^i(s_n) \leq g_t^i(s_{n-1}),$$

which proofs the lemma. \square

In Lemma 4.1 we assumed s_0 to be an arbitrary policy for which the order-up-to-levels are independent of time t . Due to the stationarity of the demand this lemma still holds for a dynamic policy s_0 .

Theorem 4.1. *If the balance assumption holds, then the decomposition approach yields the optimal replenishment policy, i.e., minimizing the long run average costs.*

Proof. This proof is very similar to the proofs in Van Houtum [1990] and Van Houtum & Zijm [1991b]. Consider an arbitrary replenishment policy s_0 . For both s_n and s_{n-1} the behavior of the stock at all stockpoints in $\cup_{j=n+1}^N W_j$ are identical. Hence, for $n = 1, \dots, N$ holds

$$g_N(s_n) - g_n(s_n) = g_N(s_{n-1}) - g_n(s_{n-1}). \quad (24)$$

Using (24) and Lemma 4.1 yields

$$g_N(s_N) \leq g_N(s_{N-1}) \leq \dots \leq g_N(s_0).$$

Because replenishment policy s_N does not have costs larger than an arbitrary policy, specifically the optimal policy, we conclude that s_N is optimal. \square

5 Conclusions

In this paper we have reviewed our theoretical analysis of divergent multi-echelon inventory systems. The objective of the analysis was to determine a cost-optimal replenishment policy, i.e., a policy which minimizes the expected total holding and penalty costs on the long run. It was proved that the decomposition approach as in Langenhoff & Zijm [1990] can be extended to divergent N -echelon systems given the balance assumption. Hence the complex multi-dimensional problem of determining the cost-optimal policy reduces to the problem of determining (1) the optimal order-up-to-level at every stockpoint, and (2) the optimal rationing functions to its successors. From the analysis we can easily determine the optimal order-up-to-level by solving a one dimensional problem closely resembling the classical newsboy problem (cf. eq. (22)). For the rationing functions we derived several properties which lead to a classification into four classes (see Figure 2). It is cumbersome and time-consuming to determine the set of optimal rationing functions. Therefore there is a need for a more approximate approach to determine a replenishment policy which is almost cost-optimal but easy and fast to determine. In an on-going paper we developed such an approach using linear rationing functions. It turns out that this approach yields a fast way to determine a near optimal replenishment policy.

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