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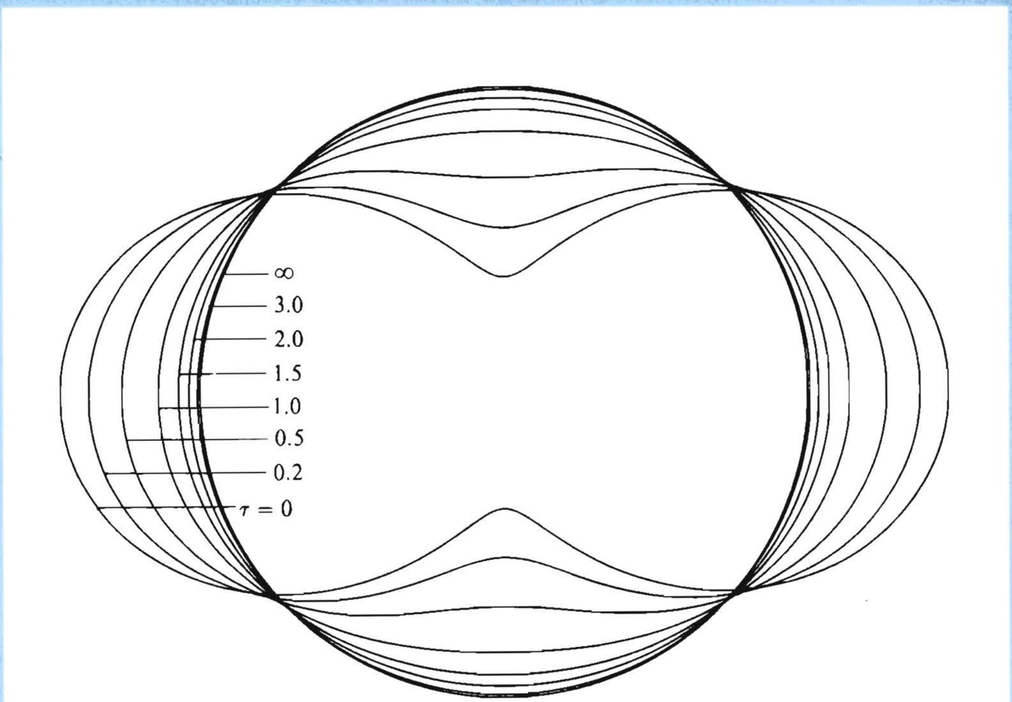

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Chapter 1

Introduction

1.1 Free boundary problems

For the last three decades, the subject of free boundary problems (FBP) has attracted increasing attention because of its theoretical interest and its numerous applications in physics and engineering. Typically, a free boundary problem consists of one or more partial differential equations (PDE) or systems of PDE with corresponding initial and boundary conditions which are supposed to hold on an a priori unknown domain. Hence, solving an FBP consists not only of finding the unknown functions that (in an appropriate sense) solve the given equations but also in determining and characterizing these unknown domains. Quite often this is the most interesting part of the problem but also its main difficulty.

Even this rather unprecise characterization of FBP can provide an idea of how huge the field of applications in modeling various physical phenomena is. Without any attempt to be complete we mention

- deformation of rigid bodies, including fracture,
- liquid and gas flow with free boundaries, including reactive flows,
- phase transition processes.

It is not surprising that to this variety of applications corresponds a variety of theoretical methods for their treatment, and that lots of challenging problems have emerged from them. In many cases, they have even determined the direction of development of the theories and they continue to do so. Obviously, the theory of PDE with all its aspects plays the most prominent role in this, but also tools from other areas as functional analysis or complex function theory have been applied successfully to FBP.

Motivated by the applications and supported by the theory, the treatment of FBP also forms a field of rapidly growing interest in numerical mathematics and scientific computing.

1.2 Existence and uniqueness results in modeling

A main objective of this thesis is to obtain existence and uniqueness results for certain FBP that occur as models for physical processes. Clearly, the main interest from a practical point of view is in the prediction of qualitative properties and behavior of the solution and in efficient calculation of approximations rather than in such propositions. It is not exceptional that one encounters the (more or less explicitly expressed or tacitly assumed) belief that a "reasonable" model of a "real-world" physical problem will always automatically have precisely one solution. This reasoning is erroneous because it neglects the crucial simplifications that have been made by replacing the physical problem by its model. On the contrary, only the knowledge about existence and uniqueness of the solution will tell whether the chosen model can be "reasonable". Thus, existence and uniqueness theorems are not only of theoretical interest but in the context of modeling they give important information on the quality of the model.

To illustrate this, suppose that a certain model yields no solution, as for example in the Stokes paradoxon of two-dimensional hydrodynamics [57]. This means that conflicting assumptions have been made in the process of modeling, even if this is not obvious at all. On the other hand, a model can admit more than one solution. This is the case, for instance, for certain one-dimensional conservation laws which can be formulated mathematically as Cauchy problems for first-order quasilinear hyperbolic PDE (see e.g. [74] for an introduction to this). Such a situation gives rise to the conclusion that the model does not contain enough information to describe the reality and has therefore to be supplemented with additional conditions for the choice of one of the solutions as the "correct" one. On this issue, see also the remarks in the classical work of Lichtenstein [58], especially Section 7.9.[1]

Moreover, in many cases the theoretical effort that has been made in order to give an existence and uniqueness proof will also provide more insight into the structure of the problem and the qualitative behavior of solutions as well as hints on effective numerical methods and their properties for the problem in question.

With regard to the FBPs which are considered here, we will briefly return to this point in Chapter 6.

1.3 The models

This thesis is concerned with two instationary free boundary problems in fluid mechanics, namely the problems of so-called Stokes flow and Hele-Shaw flow. In both cases, the driving mechanism which will be mainly considered is the influence of surface tension, hence the concept of mean curvature of a surface will play an prominent role. Let Γ be a twice differentiable, $(N-1)$-dimensional hypersurface in $\mathbb{R}^N$. Its mean curvature $\tilde{\kappa}$ is usually defined as the sum of the principal curvatures divided by $(N-1)$. For our purposes it will be convenient to call $\kappa = \tilde{\kappa}(N-1)$ the mean curvature (in short: curvature) of Γ. We will always consider κ as a real-valued function on Γ. If Γ bounds a domain in $\mathbb{R}^N$, we will choose the sign of κ such that it is negative where the domain is convex.

1.3.1 Viscous sintering

Before we formulate the models in mathematical terms, let us have a brief look at the technological process of viscous sintering whose theoretical investigation is a main motivation of this thesis. For a more detailed description the reader is referred to [36, 53, 54, 91] and the references given there.

[1] In the preface of [58] there is even a striking reference to FBP: "Gleichfalls ein Desideratum bilden heute Existenzsätze inkompressibler Flüssigkeiten, sobald eine freie Oberfläche vorliegt." ("Today, existence theorems for incompressible fluids are also a desideratum whenever there is a free surface.") This has been written already in 1929!

In the production of high-quality glasses, it is sometimes preferable to work at lower temperatures than usually in glass technology. This renders the possibility to avoid impurities induced by chemical reactions with the container walls and to use components which are too volatile at higher temperatures. The primary product of the viscous sintering technology to be discussed here is a so-called aerogel, a porous glass substance of low density that can be described as a mass of microscopically small droplets which are only loosely connected to each other.[2] At temperatures of 500–700 °C, the glass is in the state of a highly viscous liquid (comparable to heavy oil or syrup) which can be deformed by the forces arising from surface tension. These forces are acting normal on the surface of the glass droplets and their amount is proportional to the curvature of the surface. This leads to the coalescence of adjacent droplets. In this way, larger clusters are formed, which macroscopically results in an increasing density of the glass. Ideally, the process yields eventually a homogeneous glass body without pores and inclusions.

The viscous sintering process will be modeled in two steps: At first, a general model which leads to an FBP for the full Navier-Stokes equations is given, and afterwards some simplifications are made which are possible due to the high viscosity.

1.3.2 The full Navier-Stokes equations

We start with the consideration of the following physical situation: A liquid drop moves freely under the influence of an exterior force and of surface tension. (As usual at this stage of the discussion, we will assume that all occurring derivatives exist.) At time $t \geq 0$ the drop occupies the domain $\Omega(t)$ having the boundary $\Gamma(t)$. The liquid is assumed to be Newtonian and incompressible with constant positive density ρ and viscosity ν. The velocity and pressure fields $v(\cdot,t)$ and $p(\cdot,t)$ are defined on $\Omega(t)$ and satisfy the Navier-Stokes equations

$$\left.\begin{aligned} \rho\left(\frac{\partial v}{\partial t} + (v\cdot\nabla)v\right) - \nu\Delta v + \nabla p &= f \\ \operatorname{div} v &= 0 \end{aligned}\right\} \text{ in } \Omega(t), \tag{1.1}$$

where $f = f(x,t)$ is the density field of the volume forces.

The boundary condition expresses the equilibrium between the normal forces onto the surface of the liquid and the normal component of the stress tensor $\mathcal{T}$ given by

$$\mathcal{T}(v,p) = \nu(\nabla v + (\nabla v)^T) - p\mathcal{I},$$

where ∇v denotes the velocity gradient and $\mathcal{I}$ is the identity tensor. As mentioned earlier, this normal force is proportional to the mean curvature $\kappa(t)$ of $\Gamma(t)$, hence

$$\mathcal{T}(v,p)n(t) = \gamma\kappa(t)n(t), \tag{1.2}$$

where $n(t)$ denotes the outer normal vector on $\Gamma(t)$, and γ is a positive real proportionality factor, the surface tension coefficient, which is a material parameter depending only on the liquid and its environment. For a derivation of (1.2) from physical reasonings see [56]. At initial time $t = 0$ the velocity field inside the liquid is prescribed as

$$v(\cdot,0) = v_0 \qquad \text{in } \Omega(0). \tag{1.3}$$

[2] It is beyond the scope of this thesis to take into account the highly complex topology of the glass body in this process. Instead, we will restrict ourselves to the investigation of simple topologies wherever this will be necessary.

As usual in continuum mechanics, a parametrization of $\Omega(t)$ by Lagrangian coordinates $\xi \in \Omega(0)$ is introduced where the parameter functions $x = x(\cdot, t)$ satisfy the Volterra integral equation

$$x(\xi, t) = \xi + \int_0^t v(x(\xi, s), s)\, ds. \tag{1.4}$$

The equations (1.1), (1.2), (1.3) together with (1.4) and

$$\Omega(t) = x[\Omega(0), t] \tag{1.5}$$

constitute an instationary free boundary problem. Appendix A gives a brief account of the results concerning its solvability, mainly for the purpose of comparison to the results to be obtained for the model of Stokes flow which is introduced next.

1.3.3 Stokes flow

In order to make use of more detailed information of the physical properties of viscous sintering let us make equations (1.1), (1.2) dimensionless. Following [53], we choose a characteristic length x_c resembling the spatial extent of the liquid domain and characteristic velocity, pressure, and time accordingly as

$$v_c = \frac{\gamma}{\nu}, \quad p_c = \frac{\gamma}{x_c}, \quad t_c = \frac{x_c}{v_c} = \frac{x_c \nu}{\gamma}.$$

Furthermore, we assume f to be given by the gravity, i.e. $f = \rho g e_g$ where g is the gravity acceleration and e_g a dimensionless fixed unit vector. Rewriting (1.1), (1.2) in the dimensionless variables

$$\tilde{v} = \frac{v}{v_c}, \quad \tilde{p} = \frac{p}{p_c}, \quad \tilde{x} = \frac{x}{x_c}, \quad \tilde{t} = \frac{t}{t_c}$$

yields

$$\begin{aligned} \mathrm{Re}\left(\frac{\partial \tilde{v}}{\partial \tilde{t}} + (\tilde{v}\cdot\nabla)\tilde{v}\right) - \Delta\tilde{v} + \nabla\tilde{p} &= \mathrm{B} e_g \\ \operatorname{div}\tilde{v} &= 0 \\ \mathcal{T}_1(\tilde{v}, \tilde{p}) n(t) &= \kappa(t) n(t), \end{aligned}$$

where all spatial derivatives have to be taken with respect to $\tilde{x}$,

$$\mathcal{T}_1(\tilde{v}, \tilde{p}) = \left(\nabla\tilde{v} + (\nabla\tilde{v})^T\right) - \tilde{p}\mathcal{I}$$

and

$$\mathrm{Re} = \frac{\rho x_c v_c}{\nu} = \frac{\rho x_c \gamma}{\nu^2}, \qquad \mathrm{B} = \frac{\rho x_c^2 g}{\gamma}.$$

These dimensionless numbers, characterizing the respective influence of convective and gravitational effects compared to viscosity, are the well-known Reynolds number and the Bond number. (The name Suratman number instead of Reynolds number is also used in our special situation where $v_c = \frac{\gamma}{\nu}$ is chosen [43]. Note, moreover, that $\mathrm{B} = \frac{\mathrm{Re}}{\mathrm{Fr}}$ where Fr is the Froude number.)

For a typical viscous sintering problem, the values of these numbers are [53, 91]

$$\mathrm{Re} = 10^{-19} \dots 10^{-21}, \qquad \mathrm{B} = 10^{-6} \dots 10^{-8}.$$

Hence it is justified to neglect both the convective and the gravitational terms and to replace (1.1) by the equations for incompressible "creeping flow":

$$\left.\begin{array}{rcl} -\Delta v + \nabla p &=& 0 \\ \operatorname{div} v &=& 0 \end{array}\right\} \qquad \text{in } \Omega(t), \tag{1.6}$$

$$\mathcal{T}(v,p)n(t) = \kappa(t)n(t) \qquad \text{on } \Gamma(t). \tag{1.7}$$

(Here and in the sequel, the tilde is suppressed, we write $\mathcal{T}$ for $\mathcal{T}_1$, and we will refer to v and p as velocity and pressure field as before.)

Complemented by (1.4) and (1.5), the equations (1.6), (1.7) form another free boundary problem which will be discussed in detail in this thesis. At the moment, we only want to direct the attention to the crucial changes in the character of the problem that results from replacing (1.1) by (1.6). For $\Omega(t)$ known, the latter form an elliptic system in the sense of Agmon-Douglis-Nirenberg [3] with complementing boundary condition (1.7) (see Chapter 3). Therefore $v(\cdot, t)$, $p(\cdot, t)$ depend only on $\Omega(t)$ but not on its evolution in time. In particular, this holds for the initial time $t = 0$, i.e. there is no initial velocity to be prescribed for the creeping flow FBP. Consequently, the evolution of the domain as well as the velocity and pressure fields at any time are essentially[3] determined by $\Omega(0)$.

This fact can be clarified by the following intuitive reasoning: In a nonstationary initial-boundary value problem for the full Navier-Stokes equations, the initial momentum of the liquid is dissipated by inner friction due to the viscosity. For higher viscosities, i.e. smaller Reynolds numbers, the characteristic time of this dissipation process becomes shorter, and "in the limit" the influence of the initial velocity vanishes for all positive times t.

For a discussion of the solvability of the corresponding two-phase problem with $N = 2$ and a special geometry see [10]. The approach used there is based on Fourier analysis and a contraction argument in a scale of Banach spaces.

1.3.4 Hele-Shaw flow

Besides Stokes flow and in many respects parallel to it, so-called Hele-Shaw flow will be considered. We will restrict our attention to the one-phase problem. Back in 1898, this model was introduced to describe the motion of a thin layer of liquid confined in a so-called Hele-Shaw cell, a narrow interstice between two parallel plates [28, 40]. Let $\Omega(t)$ be the domain occupied by the liquid again. (In this original problem, we have $N = 2$.) The liquid is supposed to be incompressible and its velocity v is proportional to the gradient of the pressure p. Thus, we have in dimensionless form

$$\left.\begin{array}{rcl} v &=& -\nabla p \\ \operatorname{div} v &=& 0 \end{array}\right\} \text{ in } \Omega(t). \tag{1.8}$$

These equations are also encountered as the simplest model of groundwater flow when the soil is supposed to be homogeneous [90]. In this case, v is the so-called specific discharge vector, describing the flux through an oriented area element per unit of time, and the first equation is called Darcy's law. It is known [12] that the equations (1.8) are consistent with the Navier-Stokes equations if the inertial effects are disregarded and a certain averaging procedure is applied.

Based on (1.8) and depending on the applications, various driving mechanisms that induce a motion of the liquid can be considered. The most usual ones are gravity and injection or suction

[3] In fact, the velocity fields as well as the evolution of the domain are determined only up to rigid body motions. This will be discussed in Chapter 3.

of liquid at point sources or sinks, at some parts of the boundary of $\Omega(t)$ or, if $\Omega(t)$ is unbounded, at infinity. (If point sources or sinks are included, the equations hold only in the liquid domain with the source/sink points removed.)

Describing the evolution of the domain by (1.4), (1.5) again, it is not hard to derive the expression

$$V_n(t) = v(\cdot,t)|_{\Gamma(t)} \cdot n(t) \quad \text{on } \Gamma(t) \tag{1.9}$$

for the normal velocity V_n of the boundary $\Gamma(t)$. More precisely, $V_n = V_n(t)$ is a function, defined on $\Gamma(t)$, which assigns to each point of $\Gamma(t)$ the component of its velocity that is normal to $\Gamma(t)$ in this point (see Chapter 3).

Writing $u = -p$ we find from (1.8), (1.9) the equations

$$\left.\begin{array}{rcll} \Delta u & = & 0 & \text{in } \Omega(t) \\ V_n & = & \dfrac{\partial u}{\partial n(t)} & \text{on } \Gamma(t) \end{array}\right\} \tag{1.10}$$

(or modifications of them including source terms.) For these equations to constitute a "well-defined" free boundary problem, a boundary condition for u at $\Gamma(t)$ has to be added. This can be done in different ways. The simplest one is the homogeneous Dirichlet condition

$$u = 0 \quad \text{on } \Gamma(t)$$

corresponding to continuity of the pressure across the free boundary $\Gamma(t)$ and constant pressure outside $\Omega(t)$. Other boundary conditions that are encountered in literature are the homogeneous Robin condition

$$u + \varepsilon \frac{\partial u}{\partial n(t)} = 0 \quad \text{on } \Gamma(t) \tag{1.11}$$

and the inhomogeneous Dirichlet condition

$$u = \gamma\kappa(t) \quad \text{on } \Gamma(t), \tag{1.12}$$

where ε and γ are positive constants. Both conditions have first been applied to the closely related famous Stefan problem

$$\begin{array}{rcll} c\dfrac{\partial u}{\partial t} - \Delta u & = & 0 & \text{in } \Omega(t) \\ V_n & = & \dfrac{\partial u}{\partial n(t)} & \text{on } \Gamma(t) \end{array}$$

describing phase change processes where u represents the temperature and c is the specific heat. (Note that we recover (1.10) by setting $c = 0$.) Referring to this context, (1.11) is called kinetic undercooling regularization and accounts for certain nonequilibrium thermodynamic effects. The condition (1.12) arises if a surface energy term is included in the model and is called Gibbs-Thomson relation. In the original Hele-Shaw problem, it is a rough approximation of the influence of surface tension forces on the free surface of the liquid. For more details on the modeling aspects and for results concerning solvability and properties of the various FBP mentioned here we refer to the survey article [46], where special emphasis is laid on the case $N = 2$ and the complex variable methods applicable there.

In this thesis, we will exclusively deal with the boundary condition (1.12). Due to its inhomogeneity, it represents a driving mechanism, and for the main part of this thesis we will not include other ones. As in the case of the Stokes flow FBP described above, the evolution of the liquid domain is then completely determined by $\Omega(0)$.

We remark that the FBP (1.10), (1.12) is also obtained in the description of the motion of phase boundaries by capillarity and volume diffusion in metallurgy [64].

In the sequel, unless stated otherwise, we will refer to the FBP (1.4)–(1.7) by the name "Stokes flow" and to the FBP (1.10), (1.12) by the name "Hele-Shaw flow" without explicitly mentioning surface tension as the (only) driving mechanism considered.

1.4 Basic ideas and properties

Let us outline now some essential points of the following investigations, in a way that is aimed at clarifying the crucial concepts by deliberately skipping technicalities as far as possible.

1.4.1 Quasistationary approximation and surface motion laws

We recall that equations (1.4), (1.5) imply (1.9), hence the latter equation holds both for Stokes flow and for Hele-Shaw flow. Comparing (1.6) and (1.7) on one hand and the inhomogeneous Dirichlet problem

$$\left.\begin{array}{rcll} \Delta u & = & 0 & \text{in } \Omega(t) \\ u & = & \gamma\kappa(t) & \text{on } \Gamma(t) \end{array}\right\} \tag{1.13}$$

on the other, we see that both the vector-valued function v and the scalar function u satisfy elliptic boundary value problems in $\Omega(t)$ with inhomogeneous boundary conditions involving the curvature. We will call them fixed-time problems in the sequel.

Roughly speaking, elliptic BVP typically occur as models for stationary processes. Accordingly, in our case they result from omitting the "nonstationary" inertia terms. The FBPs under consideration are, however, obviously nonstationary. This apparently contradictory approach is used quite often (e.g. in thermodynamics) to model processes where the considered system, within the given precision, can be seen as evolving along a trajectory of equilibrium states. This is called quasistationary (or quasistatic) approximation.

As a consequence of this approach for our problems, we find the following structure of the FBPs: The evolution of the domain is given by

$$V_n = \tilde{\mathcal{F}}(\Omega(t)) = \mathcal{F}(\Gamma(t)), \tag{1.14}$$

where $\tilde{\mathcal{F}}$ involves the solution of the corresponding fixed-time problem. Equations of the form (1.14), with $\mathcal{F}$ a given operator which assigns to any sufficiently smooth surface Γ a real-valued function on it, are called surface motion laws: the motion of $\Gamma(t)$ is completely determined by $\Gamma(t)$ itself. The most extensively studied example of such a surface motion law is the so-called mean curvature flow:

$$\mathcal{F}(\Gamma) = \kappa.$$

A survey on surface motion laws based on the curvature is given in Appendix B. Here we want to point out one important difference between mean curvature flow and the FBPs considered here: they are nonlocal, i.e. the value of $\mathcal{F}(\Gamma)$ at a point $x \in \Gamma$ does not only depend on the behavior of Γ near x but on Γ as a global object. Nevertheless, the identification of our FBPs as surface motion laws does not only help to understand their nature but also provides hints as to what methods should be chosen for their treatment and what results can be expected.

1.4.2 The direct mapping method

The most obvious difficulty in the mathematical treatment of FBPs is the unknown or changing domain. A widely used method to overcome this problem is to choose a fixed reference domain Ω_0 and to introduce an unknown diffeomorphism mapping Ω_0 onto $\Omega(t)$. This approach is called direct mapping method. As we consider moving domains, we will have to work with (sufficiently smooth) time-dependent diffeomorphisms $z = z(\cdot, t)$. The treatment of an FBP by the direct mapping method proceeds then by transforming it to a system of equations (nonlinear at least in z) and boundary and initial conditions from which both the (transformed) solution and the diffeomorphism z have to be determined.

Of course, a major obstacle in this approach is the fact that $\Omega(t)$ does by no means determine $z(\cdot, t)$ uniquely. In fact, if a diffeomorphism z satisfies $z[\Omega_0, t] = \Omega(t)$, then the same holds for $\tilde{z} = z \circ \zeta$ where ζ is any diffeomorphism of Ω_0 onto itself. Therefore the freedom in the choice of z has to be removed which can (for instance) be done by the following means:

- Choose $\Omega_0 = \Omega(0)$ and let z be the parametrization of $\Omega(t)$ given by Lagrangian coordinates. This approach is the most well-known in continuum mechanics. The treatment of the FBP for the full Navier-Stokes equations as described in appendix A is based on it.

- If $N = 2$, let Ω_0 be a suitable standard domain (e.g. the unit disk) and let z be the conformal mapping of Ω_0 onto $\Omega(t)$. This approach will be used in Chapter 2.

- If $\partial\Omega_0 (= \Gamma_0)$ and $\Gamma(t)$ are in a suitable sense close to each other, it is possible to fix in a geometrically determined way a diffeomorphism from Γ_0 onto $\Gamma(t)$ and to extend it uniquely to a diffeomorphism from Ω_0 onto $\Omega(t)$. This will be done in Section 3.1 and applied in the subsequent parts of this thesis.

Once (essentially) uniqueness of z is enforced, the special character of surface motion laws implies a special structure for the problem on Ω_0, namely a (nonlinear) evolution equation

$$\frac{\partial z}{\partial t} = F(z(t))$$

with an initial condition given by $\Omega(0)$. (Note that this is not the case e.g. for the FBP in appendix A.) For technical reasons it will be convenient to consider the evolution equation for z on Γ_0 rather than on Ω_0. The nonlocal character of the surface motion laws yields also nonlocality of the operator F.

Hence, our quasistationary FBPs will be reformulated as nonlinear, nonlocal evolution equations on a (compact) manifold without boundary. The study of these equations is the core of this thesis, and most of the results on the FBPs will be obtained by investigation of these evolution equations.

1.5 Contents of the thesis

There is a great variety of methods in the study of nonlinear evolution equations whose applicability depends on the special situation. In this thesis, we will use the following three methods in order to obtain results on existence, uniqueness, and regularity of solutions:

- abstract Cauchy-Kovalevskaya theorems,

- an abstract approach to fully nonlinear parabolic equations,

- quasilinearization and a priori estimates.

Other methods, such as hard implicit function theorems of Nash-Moser type, which also have been successfully used for FBP, are not considered here.

The properties of analytic functions and conformal mappings are favorable for the description of Stokes flow. From a functional analytic point of view, they lend themselves in a natural way to the construction of scales of Banach spaces. Such scales, together with the concept of quasidifferential operators, form the framework for the abstract Cauchy-Kovalevskaya theorem. It will be applied to an evolution equation arising from the reformulation of the Stokes flow FBP in Chapter 2. Due to the use of conformal mappings, this technique is restricted to the case $N = 2$. On the other hand, short-time existence can be proved (for analytic initial data) even "backward in time", and no detailed knowledge about the type of the evolution equation is needed. Moreover, exponential stability of the equilibrium is proved. (The reason why parallel results for the Hele-Shaw problem cannot be obtained by the same methods will be given in Section 3.5).

The further chapters are devoted to the Stokes and Hele-Shaw flow problems in arbitrary space dimensions. The approach chosen there is by the direct mapping method with Ω_0 near $\Omega(t)$, i.e. we consider small perturbations of Ω_0, represented by real-valued functions r on Γ_0 for which an initial value problem (IVP)

$$\left.\begin{aligned} \frac{\partial r}{\partial t} &= \rho(r), \\ r(0) &= r_0, \end{aligned}\right\} \tag{1.15}$$

r_0 sufficiently small, is derived and its investigation is started. This is done in Chapter 3 by first studying the fixed-time problems and then investigating their dependence on the perturbation r.

A main tool in the analysis is the linearization of (1.15), i.e. the determination and investigation of the operator $\rho'(0)$, the Fréchet derivative of ρ at $r = 0$. Roughly speaking, the "leading term" in this linear operator turns out to be the composition of the Laplace-Beltrami operator on $\Gamma(0)$ with the Neumann-to-Dirichlet operator for the Stokes equations in the Stokes flow FBP and the composition of the Laplace-Beltrami operator with the Dirichlet-to-Neumann operator for the Laplacian in the Hele-Shaw FBP.

The crucial result is that, in appropriate function spaces, $-\rho'(0)$ generates an analytic semigroup, i.e. the IVP (1.15) is (abstract) parabolic. This fact makes it possible to apply general results on nonlinear problems of this type in order to obtain existence and uniqueness results for the solution of (1.15).

In Chapter 5 the investigation is continued by choosing a different approach to (1.15). Instead of working with analytic semigroups, sharper statements concerning the solvability of (1.15) can be derived using the technique of a priori estimates and Galerkin approximations. The basis for this as well as for the proof of additional smoothness properties is a generalized chain rule which is derived from the invariance of the fixed time problems with respect to rotations. For this technique to be applicable, however, an additional restriction on the geometry of Ω_0 has to be imposed.

All results in Chapters 4 and 5 are local in time, i.e. they ensure the existence of a solution to (1.15) on a short time interval $[0, T]$. Section 6.1 is devoted to the analysis of our FBPs near equilibrium states, i.e. where $\Omega(0)$ is a slightly perturbed ball. In such cases it is possible to prove solvability of (1.15) for all positive times and to show that the solution exponentially decays to the equilibrium state.

Finally, some extensions and variations of the FBPs considered so far are indicated together with the necessary modifications in their treatment. Some remarks, results and references concerning the numerical treatment of the Stokes flow problem are also given.

Chapter 2

A complex analysis approach to plane Stokes flow

The Stokes flow problem (1.6), (1.7), (1.9) has been approached by several authors since 1990 for $N = 2$ by means of complex function theory. The strategy which is common in their papers can roughly be described as follows:

- Representation of the fixed-time problem as a BVP for (bi-)analytic functions in $\Omega(t)$
- Transformation by conformal mapping to a BVP for analytic functions on the unit circle (or another fixed standard domain). The crucial ingredients here are the Riemann mapping theorem and the fact that the composition of two analytic mappings is analytic.
- Derivation of an evolution equation for the conformal mapping from (1.9)

Depending on the purpose, this evolution equation can be in implicit or explicit forms. In the former case, it is called Hoppers equation and can be used to construct explicit solutions of the Stokes flow FBP [43, 44, 45, 51, 75]. In explicit form, the evolution equation can be considered as a nonlinear Löwner-Kufarev equation [51]. This form is more suited for numerical treatment as well as for obtaining existence and uniqueness results by means of a Cauchy-Kovalevskaya theorem [5, 6, 8].

Since 1970, such theorems have been applied to instationary free boundary flow problems in various geometries, for various driving mechanisms, and for various governing equations. Without attempting to be complete, we mention potential flow of a free liquid drop [68] and of a liquid layer above a fixed bottom [69], two-phase flow in porous media [25], coupled flow of surface and ground water [80], and Hele-Shaw flow [42, 72, 73].

In the derivation of the evolution equation as well as in the choice of the function spaces we follow [5, 8]. As our main interest is in the existence and uniqueness proof, for some details in the derivation of the evolution equation we will refer to the original papers. The main results of this chapter have been published in [71].

2.1 Preliminaries

The basis for the formulation of abstract Cauchy-Kovalevskaya theorems is the following concept:

Definition (Scale of Banach spaces): Let $\mathcal{I}$ be an open interval of $\mathbb{R}$ and $\{X_\rho,\, \rho \in \mathcal{I}\}$ an indexed family of Banach spaces $(X_\rho, \|\cdot\|_\rho)$. $\{X_\rho\}$ is called a scale of Banach spaces iff, for all $r, \rho \in \mathcal{I}$ with $r < \rho$, X_ρ is continuously embedded in X_r and the corresponding (linear) embedding operator is injective and has an operator norm ≤ 1.

We will use the following special scale: Let G be the unit disk of the complex plane $\mathbb{C}$, $\rho \geq 0$, consider the spaces B_ρ of (complex-valued) functions on ∂G having a Fourier series

$$f(\tau) = \sum_{k\in\mathbb{Z}} f_k \tau^k, \qquad \tau \in \partial G, \tag{2.1}$$

for which the expression

$$\|f\|_\rho = \sum_{k\in\mathbb{Z}} |f_k| e^{|k|\rho} \tag{2.2}$$

is finite. (The Weierstrass criterion ensures that for any $\rho \geq 0$ all such f are continuous.)

Moreover, for $\rho > 0$ we will consider the spaces $\tilde{B}_\rho$ consisting of equivalence classes of functions in B_ρ which differ only by a constant and for which the expression

$$\frac{\partial}{\partial \rho} \|f\|_\rho = \sum_{k\in\mathbb{Z}} |k||f_k| e^{|k|\rho}$$

is finite.

Lemma 1 *(The scale B_ρ)*

(i) The spaces $\{B_\rho, \|\cdot\|_\rho\}$ form a scale of Banach spaces with $\mathcal{I} = (0, +\infty)$.

(ii) The spaces $\{\tilde{B}_\rho, \frac{\partial}{\partial\rho}\|\cdot\|_\rho\}$ form a scale of Banach spaces with $\mathcal{I} = (0, +\infty)$.

(iii) The embedding $B_\rho \hookrightarrow\!\!\hookrightarrow B_r$ is compact for $r < \rho$.

(iv) Each space B_ρ with $\rho \geq 0$ is a Banach algebra, i.e. if $f, g \in B_\rho$, then their product fg, defined by

$$(fg)(\tau) = f(\tau)g(\tau) \qquad \forall \tau \in \partial G,$$

is in B_ρ, and

$$\|fg\|_\rho \leq \|f\|_\rho \|g\|_\rho .$$

(v) If $\rho > 0$, $f \in B_\rho$, then f can be analytically extended into the annulus

$$\mathcal{A}_\rho = \{\zeta \mid e^{-\rho} < |\zeta| < e^\rho\}.$$

On the other hand, if w is an analytic function in $\mathcal{A}_\rho$ then its restriction to ∂G belongs to all B_r with $r < \rho$.

Proof: (i) It is straightforward to check that B_ρ is a Banach space under the norm $\|\cdot\|_\rho$. Naturally, the identity is chosen as embedding operator, hence the scale properties follow from the monotonicity of the mapping $\rho \mapsto \|f\|_\rho$ in its domain of definition.

(ii) It is straightforward to check that $\tilde{B}_\rho$ is a Banach space under the norm $\frac{\partial}{\partial\rho}\|\cdot\|_\rho$. Again one chooses the identity as embedding operator, and the scale property follows from the convexity of the mapping $\rho \mapsto \|f\|_\rho$.

(iii) We approximate the embedding operator $I_{\rho,r}$ by a sequence of finite-rank operators $I_n \in L(B_\rho, B_r)$ defined by truncation of the Fourier series:

$$(I_n f)(\tau) = \sum_{|k|\leq n} f_k \tau^k.$$

For the difference, one gets

$$\|(I_{\rho,r} - I_n)f\|_r = \sum_{|k|>n} |f_k| e^{|k|r} = \sum_{|k|>n} |f_k| e^{|k|\rho} e^{-|k|(\rho-r)} \leq \|f\|_\rho \, e^{-n(\rho-r)},$$

hence $I_n \longrightarrow I_{\rho,r}$ in $\mathcal{L}(B_\rho, B_r)$ and therefore $I_{\rho,r}$ is compact.

(iv) By direct calculation, we find

$$\begin{aligned} \|fg\|_\rho &= \sum_m \left| \sum_l f_l g_{m-l} \right| e^{|m|\rho} \leq \sum_{m,l} |f_l||g_{m-l}| e^{|m|\rho} \\ &= \sum_{k,l} |f_l||g_k| e^{|k+l|\rho} \leq \sum_{k,l} |f_l||g_k| e^{(|k|+|l|)\rho} = \|f\|_\rho \|g\|_\rho , \end{aligned}$$

where all summations have to be carried out over $\mathbb{Z}$.

(v) follows from standard results on the convergence, analyticity, and uniqueness of the Laurent series

$$F(\zeta) = \sum_{k\in\mathbb{Z}} f_k \zeta^k . \qquad \blacksquare$$

If we introduce the arc argument θ by $\tau = e^{i\theta}$ and consider f as a function of θ, we find for all $\rho > 0$ and all $f \in B_\rho$

$$\frac{\partial}{\partial\rho} \|f\|_\rho = \left\| \frac{\partial f}{\partial\theta} \right\|_\rho ,$$

and from this and Lemma 1 (iv) it follows that

$$\frac{\partial}{\partial\rho} \|fg\|_\rho = \frac{\partial}{\partial\rho} \|f\|_\rho \|g\|_\rho + \|f\|_\rho \frac{\partial}{\partial\rho} \|g\|_\rho .$$

In order to apply complex analysis to the Stokes equations in two dimensions we identify as usual $(x, y) \in \mathbb{R}^2$ with $x+iy \in \mathbb{C}$. In the sequel, we will not indicate in the notation the difference between points, domains, and functions that correspond to each other via this identification. Let $\mathcal{U}$ be a domain in $\mathbb{R}^2$. We introduce the Cauchy-Riemann operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

and note that $\dfrac{\partial u}{\partial \bar{z}} = \overline{\dfrac{\partial \bar{u}}{\partial z}}$, where the bar denotes the complex conjugate. We recall that

$$w : \mathcal{U} \longrightarrow \mathbb{C}$$

is analytic in $\mathcal{U}$ iff

$$\frac{\partial w}{\partial \bar{z}} = 0$$

there, and in this case

$$\frac{\partial w}{\partial z} = w' .$$

Hence

$$\frac{\partial u}{\partial z} = 0 \tag{2.3}$$

iff $\overline{u}$ is analytic in $\mathcal{U}$.

A function $u : \mathcal{U} \longrightarrow \mathbb{C}$ is called bianalytic iff

$$\left(\frac{\partial}{\partial \overline{z}}\right)^2 w = \frac{\partial^2 w}{\partial \overline{z}^2} = 0.$$

Lemma 2 *(Representation of homogeneous Stokes flow in 2D,* [5]*)*
Let (v, p) solve the equations

$$\left.\begin{array}{rcl} -\nu\Delta v + \nabla p & = & 0 \\ \operatorname{div} v & = & 0 \end{array}\right\} \tag{2.4}$$

in the domain $\Omega \subset \mathbf{R}^2$.

(i) There are analytic functions w_0, w_1 in Ω such that

$$\begin{array}{rcl} v & = & w_1 - z\overline{w_1'} - \overline{w_0'} \\ p & = & -4\nu \operatorname{Re} w_1' \end{array}$$

The bianalytic stress-stream function

$$w = \overline{z} w_1 + w_0 = \varphi + i\psi$$

(φ, ψ real-valued) satisfies

$$v = i\nabla\psi, \quad p = -\nu\Delta\varphi. \tag{2.5}$$

(ii) Among all bianalytic functions $w : \Omega \longrightarrow \mathbb{C}$, w is determined for given (v, p) by (2.5) *up to a linear function* $\operatorname{Re}(a\overline{z}) + b$, *$a, b \in \mathbb{C}$*

Proof: Using the Cauchy-Riemann operators, we can rewrite (2.4) as

$$2\frac{\partial}{\partial \overline{z}}\left(p - 2\nu\frac{\partial v}{\partial z}\right) = 0, \tag{2.6}$$

$$2 \operatorname{Re} \frac{\partial v}{\partial z} = 0. \tag{2.7}$$

From (2.6) we conclude that $p - 2\nu\frac{\partial v}{\partial z}$ is analytic in Ω, hence

$$p - 2\nu\frac{\partial v}{\partial z} = -4\nu w_1' \tag{2.8}$$

for a certain analytic function w_1. Using now (2.7) and the fact that p is real-valued, we can give a decomposition of (2.8) into its real and imaginary part:

$$\begin{array}{rcl} p & = & -4\nu \operatorname{Re} w_1' \\ \dfrac{\partial v}{\partial z} & = & 2 \operatorname{Im} w_1' = w_1' - \overline{w_1'}. \end{array}$$

This implies, by the above remark on the solvability of (2.3),

$$v = w_1 - z\overline{w_1'} - \overline{w_0'}$$

with some analytic function w_0. Using

$$w_1' = \frac{\partial^2 w}{\partial z \partial \overline{z}}, \quad v = \frac{\partial w}{\partial \overline{z}} - \overline{\frac{\partial w}{\partial z}} = \frac{\partial}{\partial \overline{z}}(w - \overline{w}),$$

the representation formulas (2.5) are obtained straightforwardly.

(ii) We have to find all solution of the system of equations

$$\begin{aligned} \nabla \psi &= 0, \\ \Delta \varphi &= 0, \\ \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} &= 2\frac{\partial^2 \psi}{\partial x \partial y}, \\ \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2} &= -2\frac{\partial^2 \varphi}{\partial x \partial y}, \end{aligned}$$

where the third and fourth equation are consequences (actually, an equivalent formulation) of the bianalyticity of w. From the first equation we see that ψ is a constant, and from this and the other equations one concludes that all second partial derivatives of φ vanish, hence it is a linear function in x and y. This completes the proof. ■

Remarks: If Ω is not simply-connected, w will in general not be single-valued. For a detailed discussion of this see [5]. The above representation method, originally developed for two-dimensional problems in elasticity [52, 66], has been applied to the Stokes equations since the 1960s (e.g. [35, 57], for more references see [8].) □

The following standard results will play an essential role in the derivation of an explicit evolution equation involving the solution of the Stokes equations. For the proof we refer to [33, 81].

Lemma 3 *(Schwarz integral and Hilbert transform)*

Let $f : \partial G \longrightarrow \mathbb{R}$ be Hölder-continuous.

(i) The complex singular integral

$$\mathbf{S}[f](\zeta) = \frac{1}{2\pi} \int_0^{2\pi} S(\lambda, \zeta) f(\lambda)\, d\nu, \quad S(\lambda, \zeta) = \frac{\lambda + \zeta}{\lambda - \zeta}, \quad \lambda = e^{i\nu}$$

is an analytic function in G. Moreover, $\mathbf{S}[f](0) \in \mathbb{R}$.

(ii) For the limit of $\mathbf{S}[f]$ on ∂G from the interior, the Plemelj formula

$$\lim_{\zeta \in G \to \tau} \mathbf{S}[f](\zeta) = f(\tau) + i\mathbf{H}[f](\tau)$$

holds, where

$$\mathbf{H}[f](\tau) = \frac{1}{2\pi} \int_0^{2\pi} H(\tau, \lambda) f(\lambda)\, d\nu, \quad H(\tau, \lambda) = iS(\tau, \lambda) = \cot \frac{\theta - \nu}{2},$$

and the integral is to be understood as Cauchy principal value.

(iii) If f is given by (2.1), then

$$\mathbf{H}[f] = -i \sum_{k \in \mathbb{Z}} \operatorname{sgn}(k) f_k \tau^k.$$

Remarks: $\mathbf{S}[f]$ and $\mathbf{H}[f]$ are called the Schwarz integral of f and the Hilbert transform of f (on the unit circle), respectively. Note that $\mathbf{H}[f]$ is real, hence we have that if F is any analytic function in G which is continuous on $\overline{G}$ and $\operatorname{Re} F = f$ on ∂G, then

$$\operatorname{Im} F = \mathbf{H}[f] + iC \qquad \text{on } \partial G,$$

where iC is an imaginary constant. The definition of $\mathbf{H}$ can be extended to complex-valued functions in an obvious way. If this is done it is not hard to see that $f = i\mathbf{H}[f]$ iff f has an analytic extension into G that vanishes at 0. □

2.2 The evolution equation

We consider the following slightly modified Stokes flow FBP describing the quasistationary thermocapillary motion of a bubble [8]:

Let $\Omega(t)$ be the outer domain bounded by the bounded simple curve $\Gamma(t)$. The velocity and pressure field in it satisfy the Stokes equations

$$\left.\begin{array}{rcl} -\nu\Delta v + \nabla p &=& 0 \\ \operatorname{div} v &=& 0 \end{array}\right\} \text{ in } \Omega(t) \tag{2.9}$$

Near infinity, velocity and pressure are assumed to approach constant values:

$$v \to v_\infty,\ p \to p_\infty \qquad \text{as } |x| \to \infty, \tag{2.10}$$

where $x \in \mathbf{R}^2$ denotes the space variable. These constants are time-dependent and a priori unknown. Moreover, incompressibility of the bubble has to be demanded:

$$\int_{\mathbf{R}^2\setminus\Omega(t)} dx = \text{const} = \pi a^2. \tag{2.11}$$

This is an equivalent formulation for the condition that no fluid is injected or extracted at infinity.

The evolution of $\Omega(t)$ is determined by

$$V_n = v|_{\Gamma(t)} \cdot n(t), \tag{2.12}$$

and for the normal component of the stress tensor we have, by the action of surface tension forces,

$$\mathcal{T}(v,p)n = \frac{d}{ds}\left(\sigma \frac{dx}{ds}\right), \tag{2.13}$$

where s is the arclength parameter along $\Gamma(t)$, taken clockwise as $\Omega(t)$ is an outer domain, and σ is the surface tension coefficient. Note that for constant σ this is identical with (1.2). In order to consider thermocapillary motion, the dependence of σ on the temperature T has to be taken into account. For our purpose it is sufficient to take the simplest case of linear dependence:

$$\sigma = \sigma_* + \gamma T. \tag{2.14}$$

The temperature field is given as the solution of the auxiliary elliptic boundary value problem

$$\begin{array}{rcll} \Delta T &=& 0 & \text{in } \Omega(t) \\ \dfrac{\partial T}{\partial n} &=& 0 & \text{at } \Gamma(t) \end{array} \tag{2.15}$$

with the asymptotic condition

$$T \to C_\infty \cdot x \qquad \text{as } |x| \to \infty, \tag{2.16}$$

where C_∞ can be interpreted as temperature gradient at infinity.

To rewrite the equations (2.9)–(2.13) in terms of the functions φ, ψ, w_0, and w_1 as defined in Lemma 2 we note that, in (x, y)-coordinates,

$$T(v,p) = 2\nu \begin{pmatrix} \varphi_{yy} & -\varphi_{xy} \\ -\varphi_{xy} & \varphi_{xx} \end{pmatrix}$$

and if $\Gamma(t)$ is parameterized by $z = z(s,t)$, then $n = -i\frac{\partial z}{\partial s}$. Using this and (2.5) one straightforwardly calculates on $\Gamma(t)$

$$\begin{aligned} T(v,p)n &= -2\nu i \frac{d}{ds}(\nabla\varphi), \\ V_n &= -\operatorname{Im}\left(\frac{\partial \bar{z}}{\partial s}\frac{\partial z}{\partial t}\right), \\ v \cdot n &= -\frac{\partial \psi}{\partial s}, \end{aligned}$$

hence by (2.12)

$$\operatorname{Im}\left(\frac{\partial \bar{z}}{\partial s}\frac{dz}{dt}\right) = \frac{\partial \psi}{\partial s} \text{ on } \Gamma(t) \tag{2.17}$$

and by (2.13)

$$-2\nu \frac{d}{ds}(\nabla\varphi) = \frac{d}{ds}\left(\sigma \frac{dz}{ds}\right),$$

which yields by integration

$$2\nu\nabla\varphi = -\sigma n$$

and splitting this into two scalar equations corresponding to the tangential and normal components gives, after another integration for the tangential component,

$$\frac{\partial \varphi}{\partial n} = -\sigma \quad \text{on } \Gamma(t), \tag{2.18}$$

$$\varphi = 0 \quad \text{on } \Gamma(t). \tag{2.19}$$

We have omitted the integration constants here because, according to Lemma 2 (ii), φ is determined only up to a linear function, i.e. taking the integration constants to be 0 has no influence on (v, p) but enforces uniqueness of φ. Furthermore we used the fact that $w = \varphi + i\psi$ is single-valued due to (2.10), (2.11). For a proof of this see Theorem 2 in [5] and the corresponding remark in [8]. To eliminate $\varphi = \operatorname{Re}(\bar{z}w_1 + w_0)$ we use that on $\Gamma(t)$

$$\begin{aligned} 2\operatorname{Im}\left(\frac{\partial \bar{z}}{\partial s} w_1\right) &= 2\operatorname{Re}\left(-i\frac{\partial \bar{z}}{\partial s}\frac{\partial w}{\partial \bar{z}}\right) = \operatorname{Re}\left(\overline{i\frac{\partial z}{\partial s}}(\nabla\varphi + i\nabla\psi)\right) \\ &= \operatorname{Re}\left(-\bar{n}\nabla\varphi + \overline{\frac{\partial z}{\partial s}}\nabla\psi\right) = -\frac{\partial\varphi}{\partial n} + \frac{\partial\psi}{\partial s} = \sigma + \frac{\partial \psi}{\partial s} \end{aligned} \tag{2.20}$$

by (2.18). With an appropriate choice of a moving coordinate system we get from (2.10) the asymptotic conditions [8]

$$w_0 = O(1),\ w_1 = -\frac{p_\infty}{4\nu}z + v_\infty + O\left(\frac{1}{z}\right) \text{ as } |z| \to \infty. \tag{2.21}$$

To transform our moving boundary problem to the unit disk G, one introduces now a time-dependent conformal mapping $z(\zeta, t)$ from G onto the flow domain $\Omega(t)$. From the Riemann mapping theorem it follows that such a conformal mapping exists, and it is of the form

$$z(\zeta, t) = \sum_{k=-1}^{\infty} z_k(t)\zeta^k,$$

where $z_{-1} \in \mathbf{R}$, $z_{-1} > 0$ without loss of generality. We assume, moreover, that z' does not vanish on ∂G. By Kellogg's theorem [38], $\Omega(t) \in C^{1,\alpha}$ is sufficient for this. In the following, all variables will be considered as functions of ζ but the same notation as before will be used. Obviously, the functions w_0 and w_1 are analytic in $G\backslash\{0\}$ for any t. We will denote the complex variable along ∂G by τ and its argument by θ.

Note that on ∂G

$$\frac{\partial \bar{z}}{\partial s} = \frac{1}{|z'|}\overline{\frac{\partial z}{\partial \theta}} = \frac{1}{|z'|}\overline{z'\frac{\partial \tau}{\partial \theta}} = \frac{1}{|z'|}\overline{z' i \tau} = -i\frac{\overline{z'}}{\tau z'} = -i\frac{|z'|}{\tau z'},$$

hence

$$\begin{aligned} \operatorname{Im}\left(\frac{\partial \bar{z}}{\partial s}\frac{\partial z}{\partial t}\right) &= |z'| \operatorname{Re}\left(\frac{\partial z/\partial t}{\tau z'}\right), \\ \operatorname{Im}\left(\frac{\partial \bar{z}}{\partial s} w_1\right) &= |z'| \operatorname{Re}\left(\frac{w_1}{\tau z'}\right). \end{aligned}$$

Thus we get from (2.17), (2.19), (2.20)

$$\operatorname{Re}\frac{\partial z/\partial t}{\tau z'} + u = 0 \quad \text{on } \partial G \tag{2.22}$$

$$\operatorname{Re}(w_0 + \bar{z}w_1) = 0 \quad \text{on } \partial G \tag{2.23}$$

$$2\operatorname{Re}\frac{w_1}{\tau z'} + u + A = 0 \quad \text{on } \partial G, \tag{2.24}$$

where

$$A(\tau, t) = \frac{\sigma_* + \gamma T(z(\tau, t))}{2\nu |z'(\tau, t)|},$$

$$u(\tau, t) = \frac{\partial \psi/\partial \theta}{|z'(\tau, t)|^2}. \tag{2.25}$$

From (2.21) we get asymptotic conditions now for $\zeta \to 0$:

$$\begin{aligned} \frac{\partial z/\partial t}{\zeta z'} &= -\frac{d}{dt}\log|z_{-1}| + O(\zeta) \\ w_0 &= O(1) \\ w_1 &= -\frac{p_\infty}{4\nu}(z_{-1}\zeta^{-1} + z_0) + v_\infty + O(\zeta). \end{aligned} \tag{2.26}$$

Moreover, the conformal mapping introduced above enables one to solve the problem (2.15), (2.16) explicitly. For the values of T at the unit circle one gets

$$T(\tau, t) = \mathrm{Re}(\overline{C}_\infty(2z_{-1}(t)\tau^{-1} + z_0(t))).$$

By introducing the scaling factors a for length, $\frac{\sigma_*}{2\nu}$ for velocity, $\frac{2\nu a}{\sigma_*}$ for time, and $\frac{\sigma_*}{a}$ for pressure, all equations can be made dimensionless. One thus obtains

$$A(\tau, t) = \frac{1 + \mathrm{Re}(\overline{c}(2z_{-1}(t)\tau^{-1} + z_0(t)))}{|z'(\tau, t)|}, \tag{2.27}$$

where, again, the same notation as before is used and the dimensionless constant $c = \dfrac{\gamma C_\infty a}{\sigma_*}$ is the so-called crispation number.

Let us have a look at the equations (2.22)–(2.26) (in dimensionless form): Applying Lemma 3 to $\frac{\partial z/\partial t}{\zeta z'}$ we find from (2.22) and the first equation in (2.26) that

$$\frac{\partial z}{\partial t} = \zeta z' \mathbf{S}[u] \qquad \text{in } G. \tag{2.28}$$

For given z, on the other hand, it is possible to determine u from (2.23), (2.24) and the asymptotic conditions on w_0 and w_1. This will be done below. Hence the complex partial differential equation (2.28) is the explicit evolution equation for z that had to be derived. We remark again that it has the form of a (nonlinear) Löwner-Kufarev equation [51]

$$\frac{\partial z}{\partial t} = \zeta z' f$$

where f depends on z.

It remains to describe how to obtain u for given z which is equivalent to the solution of the fixed-time problem. From the asymptotics of w_1 near $\zeta = 0$ it follows that $\frac{w_1}{\zeta z'}$ has a removable singularity at 0 and approaches a real value there. Moreover, we can demand without loss of generality that $w_0(0) \in \mathbf{R}$ because up to now w has been determined only up to an imaginary constant. Hence on ∂G

$$\begin{aligned} w_0 &= -\mathrm{Re}\, w_1\overline{z} - i\mathbf{H}[\mathrm{Re}\, w_1\overline{z}], \\ w_1 &= -\frac{\tau z'}{2}(u + A + i\mathbf{H}[u + A]). \end{aligned}$$

Let us define the functions $\Phi : \partial G \longrightarrow \mathbb{C}$ and $g : \partial G \times \partial G \longrightarrow \mathbb{C}$ by

$$\begin{aligned} \Phi(\tau) &= \tau(u + A + i\mathbf{H}[u + A]), \\ g(\lambda, \tau) &= \lambda(z'(\lambda)\overline{z(\lambda)} - z'(\tau)\overline{z(\tau)}). \end{aligned} \tag{2.29}$$

For later use we note that, according to the remark after Lemma 3, $\Phi = i\mathbf{H}[\Phi]$ and by straightforward calculation in terms of Fourier coefficients of z

$$\int_0^{2\pi} g(\lambda, \tau)\, d\nu \in \mathbf{R} \qquad (\lambda = e^{i\nu}). \tag{2.30}$$

Now we are able to obtain

$$\begin{aligned}\psi &= \operatorname{Im}(w_0+\overline{z}w_1) = \operatorname{Im}\left(-\operatorname{Re} w_1\overline{z} - i\mathbf{H}[\operatorname{Re}(w_1\overline{z})] - \frac{\overline{z}z'}{2}\Phi\right)\\ &= -\operatorname{Re}\mathbf{H}[w_1\overline{z}] - \operatorname{Im}\left(i\frac{\overline{z}z'}{2}\mathbf{H}[\Phi]\right) = \frac{1}{2}\operatorname{Re}(\mathbf{H}[\overline{z}z\Phi] - \overline{z}z'\mathbf{H}[\Phi]) \qquad (2.31)\end{aligned}$$

or

$$\psi(\tau) = \frac{1}{2}\operatorname{Re}\mathbf{H}[(u+A+i\mathbf{H}[u+A])g(\cdot,\tau)](\tau) = \frac{1}{2\pi}\int_0^{2\pi} I_1(\tau,\lambda)(u(\lambda)+A(\lambda))\,d\nu \quad (2.32)$$

with

$$I_1(\tau,\lambda) = \frac{1}{2}\operatorname{Re}\left(H(\tau,\lambda)g(\lambda,\tau) + \frac{1}{2\pi}\int_0^{2\pi} iH(\tau,\zeta)H(\zeta,\lambda)g(\zeta,\tau)\,d\phi\right), \qquad (2.33)$$

$\lambda = e^{i\nu}$, $\zeta = e^{i\phi}$. Using that

$$H(\tau,\zeta)H(\zeta,\lambda) = H(\tau,\lambda)(H(\tau,\zeta) - H(\lambda,\zeta)) - 1$$

and (2.30) one calculates

$$I_1(\tau,\lambda) = H(\tau,\lambda)(Z(\tau) - Z(\lambda)),$$

where

$$Z(\tau) = \frac{1}{2}\operatorname{Re}\left(\tau\overline{z}z' - \frac{1}{2\pi}\int_{\partial G} H(\tau,\zeta)\overline{z(\zeta)}dz(\zeta)\right). \qquad (2.34)$$

Gathering the results, we find that u satisfies the boundary integral equation

$$u = \tilde{K}(u+A) \qquad \text{on } \partial G$$

with

$$\begin{aligned}\tilde{K}(f) &= |z'|^{-2}L(Z,f),\\ L(Z,f) &= \frac{\partial}{\partial\theta}(\mathbf{H}[Zf] - Z\mathbf{H}[f]). \qquad (2.35)\end{aligned}$$

We recall that solving this integral equation is equivalent to the determination of V_n at $\Gamma(t)$ for fixed t. Under the conditions imposed here, existence and uniqueness of the solution of this problem can be shown (see [5]). Hence, it is justified to consider u as a function of z.

It will be convenient to work with a real-valued function h on ∂G instead of z whose relation with z is given by

$$\begin{aligned}h(z,t) &= \operatorname{Re}(\tau z(\tau,t) - 1),\\ z &= \frac{1}{\tau}(1 + h + i\mathbf{H}[h]). \qquad (2.36)\end{aligned}$$

Note that $h = 0$ corresponds to a circular bubble with unit radius. Moreover, the conservation of the (dimensionless) bubble area

$$\pi = -\frac{1}{2}\operatorname{Re}\int_0^{2\pi} \overline{z}z\tau\,d\theta$$

yields a priori

$$h_0(t) = \left(1 + 4\sum_{k=2}^{\infty}(k-1)|h_k(t)|^2\right)^{\frac{1}{2}} - 1, \tag{2.37}$$

where h_k are the Fourier coefficients of h.

From (2.28) we find

$$\frac{\partial h}{\partial t} = u + B(h, u),$$

where

$$B(h, u) = \left(h - \frac{\partial \mathbf{H}[h]}{\partial \theta}\right) u - \left(\frac{\partial h}{\partial \theta} + \mathbf{H}[h]\right) \mathbf{H}[u]. \tag{2.38}$$

2.3 Existence of solutions

Summarizing, we consider the following nonlocal Cauchy problem for a real function

$$h : \partial G \times J \longrightarrow \mathbb{R},$$

where J is a time interval containing 0:

$$\begin{aligned} \frac{\partial h}{\partial t} &= F(h) = U[h] + B(h, U[h]) \\ h(\tau, 0) &= h_*(\tau), \end{aligned} \tag{2.39}$$

where B is defined by (2.38), U is the solution operator of the integral equation

$$\begin{aligned} u &= K[h](u + A), \\ K[h](f) &= |z'|^{-2} L(Z, f), \end{aligned} \tag{2.40}$$

and A, z, L, and Z are given by (2.27), (2.36), (2.35), and (2.34), respectively. h_* is the function corresponding to the initial domain Ω_0. Without loss of generality we demand (2.37) to hold for the solution h. This equation enables us to recover h from an element of $\tilde{B}_\rho$ in an unique way. Therefore, in the sequel we will use the notation h for elements of $\tilde{B}_\rho$ as well, and the Fourier coefficient h_0 will be considered as a function on $\tilde{B}_\rho$.

The crucial step in the existence proof will be an inequality which ensures that F is a quasidifferential operator in the scale of spaces $\tilde{B}_\rho$ (cf. Lemma 1 (ii)) in the sense of Ovsiannikov [68]. As a preparation for this, we introduce the notation

$$\mathcal{U}_\rho(h_*, r) = \left\{ h \in \tilde{B}_\rho \ : \ \frac{\partial}{\partial \rho} \|h - h_*\|_\rho < r \right\}$$

for all ρ for which $h_* \in \tilde{B}_\rho$.

Lemma 4 *Assume $h_* \in \tilde{B}_{\rho_*}$ for a certain $\rho_* > 0$ and let Ω_0 be a $C^{1,\alpha}$-domain with $\alpha > 0$. Then there are constants $\hat{\rho} \in (0, \rho_*]$, $r > 0$ and $C > 0$ such that for all $\rho \in (0, \rho)$ and all $h_1, h_2 \in \mathcal{U}_\rho(h_*, r)$ the inequalities*

$$\begin{aligned} \frac{\partial}{\partial \rho} \|F(h_1) - F(h_2)\|_\rho \ &\le \ C \left(\frac{\partial^2}{\partial \rho^2} \|h_1 - h_2\|_\rho + \right. \\ &+ \left. \left(\frac{\partial^2}{\partial \rho^2} \|h_1\|_\rho + \frac{\partial^2}{\partial \rho^2} \|h_2\|_\rho \right) \frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho \right) \end{aligned} \tag{2.41}$$

and

$$\frac{\partial}{\partial\rho}\|F(h_*)\|_\rho \le C\left(\frac{\partial^2}{\partial\rho^2}\|h_*\|_\rho + 1\right) \tag{2.42}$$

hold.

Proof: The assertions of the lemma will be proved by a sequence of inequalities which are obtained from working with the Fourier coefficients in a manner similar to [8]. Furthermore, perturbation arguments are used to ensure the boundedness of certain expressions. The smoothing property of the operator $K[h](\cdot)$ is used to apply a compactness argument and to ensure the uniformity of some estimates with respect to ρ.

At first, a suitable $\hat{\rho}$ has to be determined. Let z_* denote the initial conformal map corresponding to h_*. By the smoothness presumption on Ω_0 and Kellogg's theorem [38], z'_* may be extended continuously to ∂G, and it is not vanishing there, i.e.

$$|z'_*|^2 > \gamma > 0 \qquad \text{on } \partial G \tag{2.43}$$

due to a compactness argument. An easy calculation analogously to the derivation of inequality (2.49) below shows that $z'_* \in B_{\rho_*}$ and hence $|z'_*|^2 = z'_*\overline{z'_*} \in B_{\rho_*}$ from the Banach algebra property. Hence by Lemma 1 (v) there is a function w that is analytic in $\mathcal{A}_{\rho_*}$ whose restriction to ∂G is $|z'_*|^2$. From (2.43) and continuity and compactness arguments it follows that we can choose a $\hat{\rho} \in (0, \frac{\rho_*}{2}]$ such that $\operatorname{Re} w > \tilde{\gamma} > 0$ in $\mathcal{A}_{2\hat{\rho}}$. In this smaller annulus, the functions $w^{\frac{1}{2}}$ and $w^{-\frac{1}{2}}$ are analytic. (Here and in the sequel, we preserve single-valuedness by choosing the branch of the square root which maps positive real numbers to positive real numbers.) Restriction of these functions to ∂G yields $|z'_*|, |z'_*|^{-1} \in B_{\hat{\rho}}$ by Lemma 1 (v).

Let $\rho \in (0, \hat{\rho})$ be arbitrary, $r > 0$ small. (The upper bounds that are to be imposed on r will become clear from the arguments used within the proof.) Let $h, h_1, h_2 \in \mathcal{U}_\rho(h_*, r)$ be arbitrary, having the Fourier coefficients $h_k, h_k^{(1)}, h_k^{(2)}$, respectively.

It is clear that the functions A, z, and Z have to be considered now as functions of h with values in B_ρ. Throughout the proof, the index 1 or 2 will indicate the values of them at h_1 and h_2, respectively. If no index is used, the value of these functions at h is meant. Furthermore, all occurring constants will be denoted by C if their actual value is of no interest. Without explicit statement in every single case, all inequalities are to be understood in the sense that they hold *with the same constant(s)* C for all $h, h_1, h_2 \in \mathcal{U}_\rho(h_*, r)$ and for all $\rho \in (0, \hat{\rho})$.

It is immediately clear that

$$\frac{\partial}{\partial\rho}\|h\|_\rho \le \frac{\partial}{\partial\rho}\|h_*\|_\rho + r \le C. \tag{2.44}$$

From this and (2.37) one obtains (cf. [8])

$$|h_0| \le \left(\frac{\partial}{\partial\rho}\|h\|_\rho\right)^2 \le C. \tag{2.45}$$

With the notation

$$\eta_j = 4\sum_{k=2}^{\infty}(k-1)|h_k^{(j)}|^2 > 0, \qquad j = 1, 2$$

one can write

$$|h_0^{(1)} - h_0^{(2)}| = |\sqrt{1+\eta_1} - \sqrt{1+\eta_2}| = \left|\frac{\eta_1 - \eta_2}{\sqrt{1+\eta_1} + \sqrt{1+\eta_2}}\right|$$

and thus

$$\begin{aligned}
|h_0^{(1)} - h_0^{(2)}| &\le 2\sum_{k=2}^{\infty}(k-1)\left||h_k^{(1)}|^2 - |h_k^{(2)}|^2\right| \\
&\le 2\sum_{k=2}^{\infty}(k-1)(|h_k^{(1)}| + |h_k^{(2)}|)|h_k^{(1)} - h_k^{(2)}| \\
&\le \frac{\partial}{\partial\rho}(\|h_1\|_\rho + \|h_2\|_\rho)\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho \le C\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho. \qquad (2.46)
\end{aligned}$$

The Fourier coefficients of Z_j are (cf. [8])

$$Z_k^{(j)} = -h_0^{(j)}h_k^{(j)} - h_k^{(j)} + 2\sum_{m=2}^{\infty}(m-1)\overline{h_m^{(j)}}h_{m+k}^{(j)}.$$

Therefore $(n = 1, 2)$,

$$\begin{aligned}
&\frac{\partial^n}{\partial\rho^n}\|Z_1 - Z_2\|_\rho \le \\
\le\ & 4\sum_{k=1}^{\infty}\sum_{m=2}^{\infty}k^n(m-1)|\overline{(h_m^{(1)} - h_m^{(2)})}h_{m+k}^{(1)} - \overline{h_m^{(2)}}(h_{m+k}^{(1)} - h_{m+k}^{(2)})|e^{k\rho} \\
+\ & \frac{\partial^n}{\partial\rho^n}\|h_1 - h_2\|_\rho + 2\sum_{k=1}^{\infty}k^n|(h_0^{(2)} - h_0^{(1)})h_k^{(2)} + h_0^{(1)}(h_k^{(2)} - h_k^{(1)})|e^{k\rho} \\
\le\ & 4\sum_{k=1}^{\infty}\sum_{m=2}^{\infty}(k+m)^n m(|h_m^{(1)} - h_m^{(2)}||h_{k+m}^{(1)}| \\
& + |h_m^{(2)}||h_{m+k}^{(1)} - h_{m+k}^{(2)}|)e^{(k+m)\rho}e^{-m\rho} \\
& + \frac{\partial^n}{\partial\rho^n}\|h_1 - h_2\|_\rho + |h_0^{(1)} - h_0^{(2)}|\frac{\partial^n}{\partial\rho^n}\|h_2\|_\rho + |h_0^{(1)}|\frac{\partial^n}{\partial\rho^n}\|h_1 - h_2\|_\rho \\
\le\ & \frac{\partial}{\partial r}\|h_1 - h_2\|_r\Big|_{r=-\rho}\frac{\partial^n}{\partial\rho^n}\|h_2\|_\rho + \frac{\partial}{\partial r}\|h_1\|_r\Big|_{r=-\rho}\frac{\partial^n}{\partial\rho^n}\|h_1 - h_2\|_\rho \\
& + \frac{\partial^n}{\partial\rho^n}\|h_1 - h_2\|_\rho + |h_0^{(1)} - h_0^{(2)}|\frac{\partial^n}{\partial\rho^n}\|h_2\|_\rho + |h_0^{(1)}|\frac{\partial^n}{\partial\rho^n}\|h_1 - h_2\|_\rho \\
\le\ & C\left(\frac{\partial^n}{\partial\rho^n}\|h_1 - h_2\|_\rho + \frac{\partial^n}{\partial\rho^n}\|h_1\|_\rho\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho\right), \qquad (2.47)
\end{aligned}$$

where (2.45) and (2.46) have been used. For $n = 1$, this may be estimated further by

$$C\frac{\partial}{\partial\rho}\|h_1 - h_2\|_\rho$$

using (2.44) again.

Replacing h_1 by h and h_2 by 0 in the estimate (2.47) yields

$$\frac{\partial^n}{\partial\rho^n}\|Z\|_\rho \le C\frac{\partial^n}{\partial\rho^n}\|h\|_\rho \qquad (2.48)$$

which for $n = 1$ reduces to $\frac{\partial}{\partial \rho} \|Z\|_\rho \leq C$.

Because of (2.36) and

$$z' = -\frac{i}{\tau}\frac{\partial z}{\partial \theta}$$

one finds

$$\begin{aligned} z' &= -\frac{i}{\tau^2}\frac{\partial}{\partial \theta}(h + i\mathbf{H}[h]) - \frac{1}{\tau^2}(1 + h + i\mathbf{H}[h]) \\ \frac{\partial}{\partial \theta} z' &= -\frac{i}{\tau^2}\frac{\partial^2}{\partial \theta^2}(h + i\mathbf{H}[h]) - \frac{3}{\tau^2}\frac{\partial}{\partial \theta}(h + i\mathbf{H}[h]) + \frac{2i}{\tau^2}(1 + h + i\mathbf{H}[h]). \end{aligned}$$

Taking the norms $\|\cdot\|_\rho$ of these expressions and applying the properties introduced above, one gets for $n = 0, 1$

$$\frac{\partial^n}{\partial \rho^n} \|z'\|_\rho \leq C\left(\frac{\partial^{n+1}}{\partial \rho^{n+1}} \|h\|_\rho + 1\right). \tag{2.49}$$

In an analogous way,

$$\frac{\partial^n}{\partial \rho^n} \|z_1' - z_2'\|_\rho \leq C\frac{\partial^{n+1}}{\partial \rho^{n+1}} \|h_1 - h_2\|_\rho \tag{2.50}$$

may be obtained. An immediate consequence is

$$\begin{aligned} \left\| |z_1'|^2 - |z_2'|^2 \right\|_\rho &= \left\| \overline{z_1'} z_1' - \overline{z_2'} z_2' \right\|_\rho \\ &\leq (\|z_1'\|_\rho + \|z_2'\|_\rho) \|z_1' - z_2'\|_\rho \leq C\frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho . \end{aligned} \tag{2.51}$$

A series expansion for the square root gives

$$|z'| - |z_*'| = \sqrt{|z'|^2} - \sqrt{|z_*'|^2} = |z_*'| \sum_{n=1}^{\infty} \frac{a_n}{|z_*'|^{2n}} \left(|z'|^2 - |z_*'|^2\right)^n,$$

where all the coefficients a_n satisfy $|a_n| < 1$. Hence, using (2.51),

$$\begin{aligned} \left\| |z'| - |z_*'| \right\|_\rho &\leq \left\| |z_*'| \right\|_\rho \sum_{n=1}^{\infty} \left(\left\| |z_*'|^{-1} \right\|_\rho^2 \left\| |z'|^2 - |z_*'|^2 \right\|_\rho \right)^n \\ &\leq \frac{C\frac{\partial}{\partial \rho} \|h - h_*\|_\rho}{1 - C\frac{\partial}{\partial \rho} \|h - h_*\|_\rho} \leq C\frac{\partial}{\partial \rho} \|h - h_*\|_\rho \end{aligned} \tag{2.52}$$

if r is small. This yields, moreover, $\||z'|\|_\rho \leq C$, and by repeating the above argument we get

$$\left\| |z_1'| - |z_2'| \right\|_\rho \leq C\frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho . \tag{2.53}$$

Furthermore, using (2.52),

$$\begin{aligned} \left\| |z'|^{-1} - |z_*'|^{-1} \right\|_\rho &\leq \frac{\left\| |z_*'|^{-1} \right\|_\rho^2 \left\| |z'| - |z_*'| \right\|_\rho}{1 - \left\| |z_*'|^{-1} \right\|_\rho \left\| |z'| - |z_*'| \right\|_\rho} \\ &\leq C \left\| |z'| - |z_*'| \right\|_\rho \leq C\frac{\partial}{\partial \rho} \|h - h_*\|_\rho \end{aligned} \tag{2.54}$$

for sufficiently small r, hence

$$\left\| |z'|^{-1} \right\|_\rho \le C. \tag{2.55}$$

With the use of (2.55), one obtains analogously to (2.52)

$$\left\| |z_1'|^{-1} - |z_2'|^{-1} \right\|_\rho \le C \frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho , \tag{2.56}$$

and from this and (2.55)

$$\begin{aligned} \left\| |z_1'|^{-2} - |z_2'|^{-2} \right\|_\rho &\le \left(\left\| |z_1'|^{-1} \right\|_\rho + \left\| |z_2'|^{-1} \right\|_\rho \right) \left\| |z_1'|^{-1} - |z_2'|^{-1} \right\|_\rho \\ &\le C \frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho . \end{aligned} \tag{2.57}$$

As a next step, some derivatives with respect to ρ have to be estimated. We find

$$\begin{aligned} \frac{\partial}{\partial \rho} \left\| |z'|^{-1} \right\|_\rho &= \left\| \frac{\partial}{\partial \theta} |z'|^{-1} \right\|_\rho \le \left\| |z'|^{-2} \right\|_\rho \left\| \frac{\partial}{\partial \theta} |z'| \right\|_\rho \le C \left\| \frac{\partial}{\partial \theta} \sqrt{z' \overline{z'}} \right\|_\rho \\ &\le C \left\| |z'|^{-1} \right\|_\rho \left\| \frac{\partial z'}{\partial \theta} \right\|_\rho \|z'\|_\rho \le C \left(\frac{\partial^2}{\partial \theta^2} \|h\|_\rho + 1 \right) \end{aligned} \tag{2.58}$$

from (2.55) and (2.49), and

$$\begin{aligned} &\frac{\partial}{\partial \rho} \left\| |z_1'| - |z_2'| \right\|_\rho = \\ &= \left\| |z_1'|^{-1} \frac{\partial z_1'}{\partial \theta} z_1' - |z_2'|^{-1} \frac{\partial z_2'}{\partial \theta} z_2' \right\|_\rho \le \left\| |z_1'|^{-1} \right\|_\rho \left\| \frac{\partial z_1'}{\partial \theta} \right\|_\rho \|z_1' - z_2'\|_\rho \\ &\quad + \left\| |z_1'|^{-1} \right\|_\rho \left\| \frac{\partial}{\partial \theta} (z_1' - z_2') \right\|_\rho \|z_2'\|_\rho + \left\| |z_1'|^{-1} - |z_2'|^{-1} \right\|_\rho \left\| \frac{\partial z_2'}{\partial \theta} \right\|_\rho \|z_2'\|_\rho \\ &\le C \left(\frac{\partial^2}{\partial \rho^2} \|h_1 - h_2\|_\rho + \left(\frac{\partial^2}{\partial \rho^2} \|h_1\|_\rho + \frac{\partial^2}{\partial \rho^2} \|h_2\|_\rho \right) \frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho \right), \end{aligned} \tag{2.59}$$

where (2.49), (2.50), (2.55), and (2.56) have been used. Moreover, using (2.49), (2.57), and (2.59),

$$\begin{aligned} &\frac{\partial}{\partial \rho} \left\| |z_1'|^{-1} - |z_2'|^{-1} \right\|_\rho = \left\| |z_1'|^{-2} \frac{\partial}{\partial \theta} |z_1'| - |z_2'|^{-2} \frac{\partial}{\partial \theta} |z_2'| \right\|_\rho \\ &\le \left\| |z_1'|^{-2} \right\|_\rho \frac{\partial}{\partial \rho} \left\| |z_1'| - |z_2'| \right\|_\rho + \frac{\partial}{\partial \rho} \left\| |z_2'| \right\|_\rho \left\| |z_1'|^{-2} - |z_2'|^{-2} \right\|_\rho \\ &\le C \left(\frac{\partial^2}{\partial \rho^2} \|h_1 - h_2\|_\rho \right. \\ &\qquad \left. + \left(\frac{\partial^2}{\partial \rho^2} \|h_1\|_\rho + \frac{\partial^2}{\partial \rho^2} \|h_2\|_\rho \right) \frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho \right). \end{aligned} \tag{2.60}$$

In a similar way,

$$\frac{\partial}{\partial \rho} \left\| |z_1'|^{-2} - |z_2'|^{-2} \right\|_\rho \le \left(\frac{\partial}{\partial \rho} \left\| |z_1'|^{-1} \right\|_\rho + \frac{\partial}{\partial \rho} \left\| |z_2'|^{-1} \right\|_\rho \right) \left\| |z_1'|^{-1} - |z_2'|^{-1} \right\|_\rho$$

$$
\begin{aligned}
&\quad + \left(\left\| |z_1'|^{-1} \right\|_\rho + \left\| |z_2'|^{-1} \right\|_\rho \right) \frac{\partial}{\partial \rho} \left\| |z_1'|^{-1} - |z_2'|^{-1} \right\|_\rho \\
\leq &\ C \left(\frac{\partial^2}{\partial \rho^2} \|h_1 - h_2\|_\rho \right. \\
&\quad \left. + \left(\frac{\partial^2}{\partial \rho^2} \|h_1\|_\rho + \frac{\partial^2}{\partial \rho^2} \|h_2\|_\rho \right) \frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho \right), \quad (2.61)
\end{aligned}
$$

applying (2.55), (2.58), (2.56), and (2.60).

The estimates concerning the operator L have already been given in [8]. We repeat them here only for the sake of completeness. For the Fourier coefficients g_k of $L(Z, f)$ one easily calculates

$$g_k = k \sum_{m \in \mathbb{Z}} (\operatorname{sgn}(k) - \operatorname{sgn}(m)) Z_{k-m} f_m, \qquad g_{-k} = \overline{g_k}$$

and therefore ($n = 0, 1$)

$$
\begin{aligned}
&\frac{\partial^n}{\partial \rho^n} \|L(Z, f)\|_\rho = 2 \sum_{k=1}^{\infty} k^n |g_k| \leq \\
\leq &\ 2 \sum_{k=1}^{\infty} k^{n+1} \left(2 \sum_{m=1}^{\infty} |f_m||Z_{k+m}| + |f_0||Z_k| \right) e^{k\rho} \\
\leq &\ 4 \sum_{m=1}^{\infty} |f_m| e^{-m\rho} \sum_{k=1}^{\infty} k^{n+1} |Z_{k+m}| e^{(k+m)\rho} + |f_0| \frac{\partial^{n+1}}{\partial \rho^{n+1}} \|Z\|_\rho \\
\leq &\ \|f\|_0 \frac{\partial^{n+1}}{\partial \rho^{n+1}} \|Z\|_\rho \leq C \|f\|_0 \frac{\partial^{n+1}}{\partial \rho^{n+1}} \|h\|_\rho \qquad (2.62)
\end{aligned}
$$

because of $\rho > 0$ and (2.48).

Now we are able to investigate the crucial question of dependence of the operator K on h. Applying the linearity of L in the first argument, (2.47), (2.48), (2.57), and (2.55), we find

$$
\begin{aligned}
&\|(K[h_1] - K[h_2])(f)\|_\rho \leq \\
\leq &\ \left\| |z_1'|^{-2} \right\|_\rho \|L(Z_1 - Z_2, f)\|_\rho + \left\| |z_1'|^{-2} - |z'^2|^{-2} \right\|_\rho \|L(Z_2, f)\|_\rho \\
\leq &\ C \frac{\partial}{\partial \rho} \|Z_1 - Z_2\|_\rho \|f\|_0 + C \frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho \frac{\partial}{\partial \rho} \|Z_2\|_\rho \|f\|_0 \\
\leq &\ C \frac{\partial}{\partial \rho} \|h_1 - h_2\|_\rho \|f\|_0. \qquad (2.63)
\end{aligned}
$$

For the derivative one obtains

$$
\begin{aligned}
&\frac{\partial}{\partial \rho} \|(K[h_1] - K[h_2])(f)\|_\rho \leq \\
\leq &\ \frac{\partial}{\partial \rho} \left\| |z_1'|^{-2} L(Z_1 - Z_2, f) \right\|_\rho + \frac{\partial}{\partial \rho} \left\| (|z_1'|^{-2} - |z_2'|^{-2}) L(Z_2, f) \right\|_\rho \\
\leq &\ \frac{\partial}{\partial \rho} \left\| |z_1'|^{-2} \right\|_\rho \|L(Z_1 - Z_2, f)\|_\rho + \left\| |z_1'|^{-2} \right\|_\rho \frac{\partial}{\partial \rho} \|L(Z_1 - Z_2, f)\|_\rho \\
&+ \frac{\partial}{\partial \rho} \left\| |z_1'|^{-2} - |z_2'|^{-2} \right\|_\rho \|L(Z_2, f)\|_\rho \left\| |z_1'|^{-2} - |z_2'|^{-2} \right\|_\rho \frac{\partial}{\partial \rho} \|L(Z_2, f)\|_\rho
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\left(\frac{\partial^2}{\partial\rho^2}\|h_1-h_2\|_\rho \right.\\
&\quad\left.+\left(\frac{\partial^2}{\partial\rho^2}\|h_1\|_\rho+\frac{\partial^2}{\partial\rho^2}\|h_2\|_\rho\right)\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\right)\|f\|_0, \qquad (2.64)
\end{aligned}
$$

where (2.55), (2.58), (2.62), (2.48), (2.47), and (2.57) have been used.

By help of (2.36) we may rewrite (2.27) as

$$A(\tau,t)=\frac{1+2\operatorname{Re}(\bar{c}((h_0(t)+1)\tau^{-1}+h_1(t)))}{|z'(\tau,t)|}$$

and using (2.55) and (2.56) it is straightforward to prove

$$\|A\|_0\leq C \qquad (2.65)$$

and

$$\|A_1-A_2\|_0\leq C\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho. \qquad (2.66)$$

For estimates concerning the solution operator U of (2.40) it is important to remark that

$$
\begin{aligned}
\|K[h](f)\|_\rho &\leq \left\||z'|^{-2}\right\|_\rho\|L(Z,f)\|_\rho\leq C\frac{\partial}{\partial\rho}\|Z\|_\rho\|f\|_0\\
&\leq C\frac{\partial}{\partial\rho}\|h\|_\rho\|f\|_0\leq C\|f\|_0 \qquad (2.67)
\end{aligned}
$$

for all $f\in B_0$ because of (2.55), (2.62), and (2.48), therefore $K[h]$ is continuous from B_0 in B_ρ. Together with the compactness of the embedding $B_\rho\hookrightarrow B_0$ this ensures compactness of $K[h]$ in B_0. Hence, the Fredholm alternative holds for the operator $I-K[h]$ in this space. According to the above remark, the integral equation (2.40), which may be written as

$$(I-K[h])(u)=K[h](A),$$

has a unique solution. Therefore $I-K[h]$ is a homeomorphism of B_0. This means, in particular, $(I-K[h_*])^{-1}\in\mathcal{L}(B_0,B_0)$.

In the following, if $\|\cdot\|_0$ is applied to an operator instead of a function on ∂G, it will denote the usual norm in $\mathcal{L}(B_0,B_0)$. Note that from (2.63) it follows that

$$\|K[h_1]-K[h_2]\|_0\leq C\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho.$$

Thus, applying a standard perturbation result concerning the inverse of regular linear operators,

$$
\begin{aligned}
\left\|(I-K[h])^{-1}-(I-K[h_*])^{-1}\right\|_0 &\leq \frac{\left\|(I-K[h_*])^{-1}\right\|_0^2\|K[h]-K[h_*]\|_0}{1-\|K[h]-K[h_*]\|_0\left\|(I-K[h_*])^{-1}\right\|_0}\\
&\leq Cr
\end{aligned}
$$

and therefore $\left\|(I-K[h])^{-1}\right\|_0\leq C$ (with C independent of h) if r is chosen small enough. Consequently,

$$\left\|(I-K[h_1])^{-1}-(I-K[h_2])^{-1}\right\|_0\leq C\|K[h_1]-K[h_2]\|_0\leq C\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho.$$

Moreover,

$$\begin{aligned}\frac{\partial}{\partial\rho}\|K[h](f)\|_\rho &\le \frac{\partial}{\partial\rho}\left\||z'|^{-2}\right\|_\rho\|L(Z,f)\|_\rho+\left\||z'|^{-1}\right\|_\rho^2\frac{\partial}{\partial\rho}\|L(Z,f)\|_\rho\\ &\le C\left(\frac{\partial^2}{\partial\rho^2}\|h\|_\rho+1\right)\|f\|_0, \end{aligned}\tag{2.68}$$

where (2.58), (2.55), (2.62), and (2.48) have been used.

After these preparations, the necessary estimates for U can be given. Indeed,

$$U[h]=(I-K[h])^{-1}K[h](A)=K[h](I-K[h])^{-1}(A)$$

and therefore

$$\|U[h]\|_\rho\le\left\|(I-K[h])^{-1}(A)\right\|_0\le C\|A\|_0\le C,\tag{2.69}$$

$$\frac{\partial}{\partial\rho}\|U[h]\|_\rho\le C\left(\frac{\partial^2}{\partial\rho^2}\|h\|_\rho+1\right)\left\|(I-K[h])^{-1}(A)\right\|_0\le C\left(\frac{\partial^2}{\partial\rho^2}\|h\|_\rho+1\right),\tag{2.70}$$

$$\begin{aligned}\|U[h_1]-U[h_2]\|_\rho &\le \left\|K[h_1](I-K[h_1])^{-1}(A_1-A_2)\right\|_\rho\\ &\quad+\left\|K[h_1]((I-K[h_1])^{-1}-(I-K[h_2])^{-1})(A_2)\right\|_\rho\\ &\quad+\left\|(K[h_1]-K[h_2])(I-K[h_2])^{-1}(A_2)\right\|_\rho\\ &\le C\Big(\left\|(I-K[h_1])^{-1}(A_1-A_2)\right\|_0\\ &\quad+\left\|((I-K[h_1])^{-1}-(I-K[h_2])^{-1})(A_2)\right\|_0\\ &\quad+\left\|(I-K[h_2])^{-1}(A_2)\right\|_0\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\Big)\\ &\le C\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho, \end{aligned}\tag{2.71}$$

$$\begin{aligned}\frac{\partial}{\partial\rho}\|U[h_1]-U[h_2]\|_\rho &\le \frac{\partial}{\partial\rho}\left\|K[h_1](I-K[h_1])^{-1}(A_1-A_2)\right\|_\rho\\ &\quad+\frac{\partial}{\partial\rho}\left\|K[h_1]((I-K[h_1])^{-1}-(I-K[h_2])^{-1})(A_2)\right\|_\rho\\ &\quad+\frac{\partial}{\partial\rho}\left\|(K[h_1]-K[h_2])(I-K[h_2])^{-1}(A_2)\right\|_\rho\\ &\le C\left(\frac{\partial^2}{\partial\rho^2}\|h_1\|_\rho+1\right)(\left\|(I-K[h_1])^{-1}(A_1-A_2)\right\|_0\\ &\quad+\left\|((I-K[h_1])^{-1}-(I-K[h_2])^{-1})(A_2)\right\|_0)\\ &\quad+C\left(\frac{\partial^2}{\partial\rho^2}\|h_1-h_2\|_\rho+\left(\frac{\partial^2}{\partial\rho^2}\|h_1\|_\rho+\frac{\partial^2}{\partial\rho^2}\|h_2\|_\rho\right)\cdot\right.\\ &\qquad\left.\cdot\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\right)\left\|(I-K[h_2])^{-1}(A_2)\right\|_0\\ &\le C\left(\frac{\partial^2}{\partial\rho^2}\|h_1-h_2\|_\rho\right. \end{aligned}$$

$$+\left(\frac{\partial^2}{\partial\rho^2}\|h_1\|_\rho+\frac{\partial^2}{\partial\rho^2}\|h_2\|_\rho\right)\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\Big), \quad (2.72)$$

using the above estimates concerning K, $(I-K[h])^{-1}$, and A.

For arbitrary $u, \tilde{h} \in B_\rho$, the estimate

$$\begin{aligned}
&\frac{\partial}{\partial\rho}\left\|B(\tilde{h},u)\right\|_\rho \leq \\
\leq\;& \frac{\partial}{\partial\rho}\left\|\frac{\partial}{\partial\theta}\mathbf{H}[\tilde{h}]u\right\|_\rho+\frac{\partial}{\partial\rho}\left\|\tilde{h}u\right\|_\rho+\frac{\partial}{\partial\rho}\left\|\frac{\partial\tilde{h}}{\partial\theta}\mathbf{H}[u]\right\|_\rho+\frac{\partial}{\partial\rho}\left\|\mathbf{H}[\tilde{h}]\mathbf{H}[u]\right\|_\rho \\
\leq\;& 2\left(\frac{\partial^2}{\partial\rho^2}\left\|\tilde{h}\right\|_\rho\|u\|_\rho+\frac{\partial}{\partial\rho}\left\|\tilde{h}\right\|_\rho\frac{\partial}{\partial\rho}\|u\|_\rho\right. \\
&\left.+\frac{\partial}{\partial\rho}\left\|\tilde{h}\right\|_\rho\|u\|_\rho+\left\|\tilde{h}\right\|_\rho\frac{\partial}{\partial\rho}\|u\|_\rho\right) \quad (2.73)
\end{aligned}$$

holds. Using the bilinearity of B, we find

$$\frac{\partial}{\partial\rho}\|B(h_1,u_1)-B(h_2,u_2)\|_\rho\leq\frac{\partial}{\partial\rho}\|B(h_1-h_2,u_1)\|_\rho+\frac{\partial}{\partial\rho}\|B(h_2,u_1-u_2)\|_\rho \quad (2.74)$$

for arbitrary $u_1, u_2 \in B_\rho$. The lemma follows now from the subsequent application of the inequalities (2.69)–(2.74). (Note that

$$\|h\|_\rho \leq C, \qquad \|h_1-h_2\|_\rho \leq C\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho$$

as a consequence of (2.45), (2.46)). ∎

The inequalities (2.41) and(2.42) may be written as

$$\begin{aligned}
\frac{\partial}{\partial\rho}\|F(h_1)-F(h_2)\|_\rho &\leq C\frac{\partial}{\partial\rho}\left[\left(\frac{\partial}{\partial\rho}\|h_1\|_\rho+\frac{\partial}{\partial\rho}\|h_2\|_\rho+1\right)\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho\right] \\
\frac{\partial}{\partial\rho}\|F(h_*)\|_\rho &\leq C\frac{\partial}{\partial\rho}\left[\frac{\partial}{\partial\rho}\|h_*\|_\rho+\rho\right].
\end{aligned}$$

The expressions in square brackets are positive convex functions of the real parameter ρ, hence for arbitrary $\rho, \rho' \in (0,\hat{\rho})$ with $\rho' < \rho$:

$$\begin{aligned}
\frac{\partial}{\partial s}\|F(h_1)-F(h_2)\|_s\Big|_{s=\rho'} &\leq C\frac{\left(\frac{\partial}{\partial\rho}\|h_1\|_\rho+\frac{\partial}{\partial\rho}\|h_2\|_\rho+1\right)\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho}{\rho-\rho'} \\
&\leq \frac{C}{\rho-\rho'}\frac{\partial}{\partial\rho}\|h_1-h_2\|_\rho, \quad (2.75)
\end{aligned}$$

$$\frac{\partial}{\partial\rho}\|F(h_*)\|_\rho\leq C\frac{\frac{\partial}{\partial s}\|h_*\|_s\big|_{s=\hat{\rho}}+\hat{\rho}}{\hat{\rho}-\rho}\leq\frac{C}{\hat{\rho}-\rho} \quad (2.76)$$

because of the usual estimate of the derivative by a difference quotient. According to the abstract Cauchy-Kovalevskaya theorem proved by Nishida [67], the inequalities (2.75) (holding uniformly for all $h_1, h_2 \in \mathcal{U}_\rho(h_*, r)$) and (2.76) ensure the existence of a solution to (2.39) locally in time.

Proposition 1 *For arbitrary $h_* \in \tilde{B}_{\rho_*}$ with $\rho_* > 0$ there is a $\beta > 0$ and a $\hat{\rho} > 0$ such that* (2.39) *has a unique solution $h(t)$ in the time interval $\left(-\frac{\hat{\rho}}{\beta}, \frac{\hat{\rho}}{\beta}\right)$ with $h(t) \in \tilde{B}_{\hat{\rho}-\beta|t|}$. The number β is completely determined by r and C in Lemma 4.*

For the proof, see, for example, [89]. The results for the problem "backward in time", i.e. for $t < 0$, are an immediate consequence of the autonomous character of (2.39). The function

$$h_-(t) = h(-t)$$

is described by the initial value problem

$$\frac{\partial h_-}{\partial t} = -F(h_-), \qquad h_-(0) = h_*$$

for which existence, uniqueness, and smoothness properties of the solution for $t > 0$ can be obtained as in the original problem without any changes.
Remark: For the results obtained in this section, it is not necessary to demand positivity of the surface tension coefficient σ. □

2.4 Near equilibrium

Because of $U[0] = 0$ it is clear that $h = 0$ is a stationary solution of (2.39). It describes the uniform thermocapillary drift of a circular bubble.

To investigate the behavior of solutions near this equilibrium state, we calculate the Fréchet derivative of F at 0 and find

$$\begin{aligned} F'(0)[h] &= -L(h, A_0), \\ A_0(\tau) &= 1 + 2\,\mathrm{Re}(c\tau), \qquad \tau \in \partial G. \end{aligned}$$

If $|c| < \frac{1}{2}$ it can easily be shown that the spectrum of the operator $L(\cdot, A_0)$ in the spaces B_ρ, $\rho > 0$, consists precisely of all nonnegative integers, hence (2.39) is linearly stable at $h = 0$. This is in accordance with the physical expectations as $|c| < \frac{1}{2}$ iff the surface tension coefficient is strictly positive on $\Gamma(t)$. The further considerations are restricted to this case.
Remark: The operator $L(\cdot, A_0)$ generates a semigroup which is smoothing in the scale $\{B_\rho\}$ in the sense that there is an $\alpha > 0$ depending only on $|c| < \frac{1}{2}$ such that

$$||e^{-tL(\cdot, A_0)}h||_{\rho+\alpha t} \leq ||h||_\rho .$$

This result, together with an estimate on the nonlinear remainder term $F(h) + L(h, A_0)$, is the basis of the following proposition. For an abstract setting describing such a situation see [24]. □

In [8] the following a priori estimate is shown:

Proposition 2 (L.K. Antanovskii) *(A priori estimate near equilibrium)*

Let $|c| < \frac{1}{2}$, $\alpha \in (0, 1 - 2|c|)$, $\rho_* > 0$ be given. There is a $q_1 > 0$ such that for all solutions h of (2.39) *with $h_* \in \mathcal{U}_{\rho_*}(0, q_1)$ and for all $t > 0$ for which h exists in $[0, t]$ the estimate*

$$\left\| \frac{\partial h}{\partial \theta}(t) \right\|_{\rho_*+\alpha t} \leq \left\| \frac{\partial h}{\partial \theta}(0) \right\|_{\rho_*} \tag{2.77}$$

holds.

An immediate consequence of (2.77) is

$$\left\| \frac{\partial h}{\partial \theta}(t) \right\|_{\rho_*} \leq e^{-\alpha t} \left\| \frac{\partial h}{\partial \theta}(0) \right\|_{\rho_*} \tag{2.78}$$

and hence (see (2.45)) $||h(t)||_{\rho_*} \leq Ce^{-\alpha t}$, i.e. small perturbations of the equilibrium are "smoothed out" exponentially in the considered norm.

By help of this a priori estimate it is possible to globalize the existence result of the previous section in the case of such small perturbations:

Proposition 3 *(Global existence and exponential stability of solutions near equilibrium)*
Let $|c| < \frac{1}{2}$, $\rho_* > 0$ be given. Then there is a $q > 0$ such that all solutions of (2.39) for which $h_* \in \mathcal{U}_{\rho_*}(0, q)$ exist for all $t > 0$.

Proof: The idea of the proof is to show that for a certain $q \in (0, q_1)$ there is a $T > 0$ such that all solutions of (2.39) with $h_* \in \mathcal{U}_{\rho_*}(0, q)$ exist on the interval $[0, T]$. The estimate (2.78) with arbitrary $\alpha \in (0, 1 - 2|c|)$ ensures then $h(T) \in \mathcal{U}_{\rho_*}(0, q)$, and a simple induction argument will finally prove the existence of h on the interval $[0, nT]$ for all $n \in \mathbf{N}$.

In other words, it is sufficient to find a uniform lower bound for the length of the existence intervals of the solutions of (2.39) with $h_* \in \mathcal{U}_{\rho_*}(0, q)$. This can be done by proving that Lemma 4 holds with the same constants r, C, and $\hat{\rho}$ for all $h_* \in \mathcal{U}_{\rho_*}(0, q)$ if $q > 0$ is chosen small enough.

At first we repeat the arguments for the proofs of (2.52) and (2.54) with z_* in place of z and ζ^{-1} in place of z_*. We can replace $\hat{\rho}$ by ρ_* here because no smoothness is lost when taking the square root of $|\zeta^{-1}|^2 = 1$ or its reciprocal. Hence we find

$$\| |z_*'| - 1 \|_{\rho_*} \leq Cq, \qquad \left\| |z_*'|^{-1} - 1 \right\|_{\rho_*} \leq Cq. \tag{2.79}$$

Hence $|z_*'|$ and $|z_*'|^{-1}$ are in B_{ρ_*} with uniformly bounded norms, and we can choose $\rho = \rho_*$.

A reexamination of the proof of Lemma 4 in this situation shows that for all h_* that satisfy

$$\max \left\{ \left\| \frac{\partial h_*}{\partial \theta} \right\|_{\rho_*}, \| |z_*'| \|_{\rho_*}, \left\| |z_*'|^{-1} \right\|_{\rho_*}, \left\| (I - K[h_*])^{-1} \right\|_0 \right\} \leq M \tag{2.80}$$

the inequality (2.41) holds with C and r only depending on M. It it easily seen that $K[0]$ is the zero operator, and an estimate analogously to (2.63) with h_* in place of h_1 and 0 in place of h_2 shows that $\|K[h_*]\|_0 \leq Cq$, hence $\left\| (I - K[h_*])^{-1} \right\|_0 \leq 2$ for sufficiently small q.

From this and (2.79) it follows now that if q is chosen sufficiently small, then (2.80) holds for all $h_* \in \mathcal{U}_{\rho_*}(0, q)$ with a certain fixed M. This completes the proof. ■

2.5 Bounded flow domains

We will conclude the discussion of the complex analysis approach to the Stokes flow problem by giving a brief account of the necessary changes that have to be made if instead of a bubble one considers a simply-connected, bounded flow domain, i.e. a liquid drop. It is based on [6]. Our main interest is in the short-time existence theorem for general initial domains again, therefore we restrict ourselves to the case of a constant surface tension coefficient $\sigma > 0$.

The solution of (2.9) can be represented by the functions φ, ψ, w_0, and w_1 in the same way as described above. The fixed time problem (2.9), (2.13) determines the velocity field v only up to rigid-body motions. (We postpone a detailed discussion of this in arbitrary space dimension to Section 3.2.) However, it is shown in [6] that if one demands $0 \in \Omega(0)$,

$$\begin{aligned} w_1(0) &= 0, \\ \operatorname{Im} w_1'(0) &= 0, \end{aligned}$$

(which corresponds to the choice of a suitable moving coordinate system) these degrees of freedom are removed and, moreover, $0 \in \Omega(t)$ for all t for which the solution of the FBP exists.

Thus, the unit disk G has to be mapped onto a bounded flow domain containing 0, hence it is natural to normalize

$$z(\zeta, t) = \sum_{k=1}^{\infty} z_k(t)\zeta^k, \qquad z_1 \in \mathbf{R},\ z_1 > 0.$$

Differing from the case of the bubble, w_1 (pulled back to G) is analytic in the whole of G, i.e. $w_1 = i\mathbf{H}[w_1]$ and instead of (2.31) we obtain

$$\begin{aligned} \psi &= -\operatorname{Re}\mathbf{H}[\overline{z}w_1] + \operatorname{Im}(\overline{z}w_1) = -\operatorname{Re}\mathbf{H}[\overline{z}w_1] + \operatorname{Im}(\overline{z}i\mathbf{H}[w_1]) \\ &= \operatorname{Re}(\overline{z}\mathbf{H}[w_1] - H[\overline{z}w_1]) = \frac{1}{2}\operatorname{Re}(\overline{z}\mathbf{H}[z'\Phi] - \mathbf{H}[\overline{z}z'\Phi]) \end{aligned}$$

with Φ still being defined by (2.29). We replace g in the following calculations by

$$\tilde{g}(\lambda, \tau) = \lambda z'(\lambda)(\overline{z(\lambda)} - \overline{z(\tau)})$$

and take into account that also

$$\int_0^{2\pi} \tilde{g}(\lambda, \tau)\, d\nu \in \mathbf{R}$$

and

$$\frac{i}{2\pi}\int_0^{2\pi} H(\tau, \lambda)\lambda z'(\lambda)\, d\nu = \tau z'(\tau). \qquad (\lambda = e^{i\nu})$$

In the same way as in Section 2.3, this leads to the equations (2.32) with g replaced by $\tilde{g}$, (2.33), and (2.34). The latter can be rewritten now as

$$\begin{aligned} Z(\tau) &= \frac{1}{4\pi}\operatorname{Re}\int_{\partial G} H(\tau, \lambda)(\overline{z(\tau)} - \overline{z(\lambda)}\, dz(\lambda) \\ &= \frac{1}{2\pi}\int_G \frac{1 - |\zeta|^2}{|\tau - \zeta|^2}|z'(\zeta)|^2\, d\xi d\eta \end{aligned}$$

with $\zeta = \xi + i\eta$, $\xi, \eta \in \mathbf{R}$.

Finally, we introduce h by

$$h(\tau) = \operatorname{Re}\left(\frac{z(\tau)}{\tau} - 1\right), \qquad \tau \in \partial G$$

which leads to

$$z(\tau) = \tau(1 + h(\tau) + i\mathbf{H}[h](\tau))$$

and the evolution equation

$$\frac{\partial h}{\partial t} = -U[h] - \left(h + \frac{\partial \mathbf{H}[h]}{\partial \theta}\right) U[h] - \left(\frac{\partial h}{\partial \theta} - \mathbf{H}[h]\right) \mathbf{H}[U[h]].$$

The condition on the volume of the drop is now

$$h_0(t) = \left(1 - 4\sum_{k=1}^{\infty}(k+1)|h_k(t)|^2\right)^{\frac{1}{2}} - 1.$$

Based on these equations, analogous results as in Sections 2.3 and 2.4 can be obtained without significant changes in the proofs.

Chapter 3

Derivation of the evolution equations

This chapter is devoted to the reformulation of the Stokes flow FBP (1.4)–(1.7) and the Hele-Shaw flow FBP (1.10), (1.12) as nonlocal evolution equations for a real-valued function on a smooth, compact reference manifold without boundary. Due to the special structure of the considered FBPs (cf. Section 1.4) this can be done along the following steps:

- Construction of a correspondence between perturbations of a fixed reference domain and real-valued functions defined on its boundary; these functions will be called perturbation functions

- Representation of the fixed-time problems on the reference domain as operator equations and investigation of existence, uniqueness, and regularity of their solutions

- Investigation of the dependence of these operator equations on the perturbation functions

- Reformulation of the surface motion law as an evolution equation for the perturbation function

This approach is rather straightforward and essentially of geometric nature. Basically, the technique of perturbation functions is a widely used tool in the analysis of FBP and other surface motion laws, both instationary (e.g. [17, 27, 30]) and stationary (e.g. [1, 2, 13]). (In the case of stationary problems, one of course obtains a time-independent equation determining the perturbation function instead of an evolution equation.) It has to be pointed out that the choice of this method already limits the scope of the results which can be expected: By perturbation functions it is in general only possible to describe domains that are close to the reference domain, hence "global" results concerning domain evolutions over "large" distances cannot be obtained.

Moreover, the investigation of (weak formulations of) the fixed-time problems will immediately yield some results on stationary solutions and the global behavior of the corresponding moving boundary problems.

3.1 Preliminaries

Let $\Omega_0 \in \mathbb{R}^N$, $N \geq 2$, be a simply-connected bounded domain with C^∞-boundary Γ_0 and outer normal vector field n. On Ω_0 we define the Sobolev spaces $H^s_{\mathbb{C}}(\Omega_0)$, $s \geq 0$, in the usual way [59]: For $m \in \mathbb{N}$ one defines

$$\|u\|^2_{H^m_{\mathbb{C}}(\Omega_0)} = \sum_{|\alpha| \leq m} \|\partial^\alpha u\|^2_{L^2_{\mathbb{C}}}, \qquad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1, \ldots, \partial^{\alpha_N} x_N},$$

where u is, in general, complex-valued, and the partial derivatives ∂^α are taken in distributional sense. For noninteger s we define

$$H^s_{\mathbb{C}}(\Omega_0) = \left[H^0_{\mathbb{C}}(\Omega_0), H^m_{\mathbb{C}}(\Omega_0)\right]_\theta, \qquad m = [s] + 1,\ \theta = \frac{s}{m},$$

where $[X, Y]_\theta$ denotes the complex interpolation space between the Hilbert spaces X and $Y \hookrightarrow X$ corresponding to the parameter $\theta \in (0, 1)$, and $[s]$ denotes the largest integer not larger than s.

It will be convenient to work with spaces of real-valued functions in the sequel, so we define

$$H^s(\Omega_0) = \{u \in H^s_{\mathbb{C}}(\Omega_0) \,|\, u \text{ real}\}, \tag{3.1}$$

which obviously is a real Banach space and for integer s also a real Hilbert space.

On Γ_0 we introduce the Sobolev spaces $H^s_{\mathbb{C}}(\Gamma_0)$, $s \in \mathbb{R}$, by defining their scalar products $(\cdot, \cdot)$ with the help of the operator $(I - \Delta_{\Gamma_0})$, where I and Δ_{Γ_0} denote the identity and the Laplace-Beltrami operator of Γ_0, respectively. We recall that in arbitrary regular local coordinates we have

$$\Delta_{\Gamma_0}\phi = \frac{1}{\sqrt{g}} \frac{\partial}{\partial w_i}\left(\sqrt{g} g^{ij} \frac{\partial \phi}{\partial w_j}\right), \tag{3.2}$$

where g^{ij} are the matrix elements of $G^{-1} = \left(\left(\frac{\partial \xi}{\partial w}\right)^T \frac{\partial \xi}{\partial w}\right)^{-1}$ and $g = \det G$. Here and in the sequel, unless stated otherwise, summation has to be performed over indices that occur twice in the same term. An alternative way of defining Δ_{Γ_0} is given in equation (3.12) below.

The operator $(I - \Delta_{\Gamma_0})$ is $L^2_{\mathbb{C}}$-self-adjoint and has a purely discrete spectrum consisting of positive eigenvalues, hence it has a complete orthonormal system $\{\psi_k\}$ of eigenfunctions in $L^2_{\mathbb{C}}(\Gamma_0) = H^0_{\mathbb{C}}(\Gamma_0)$. For any $s \in \mathbb{R}$ we set

$$\begin{aligned} (u, v)_0 &= \int_{\Gamma_0} u\bar{v}\, d\Gamma, \\ \Lambda^s u &= \sum_k \lambda_k^{\frac{s}{2}} (u, \psi_k)_0 \psi_k, \\ (u, v)_s &= (\Lambda^s u, \Lambda^s v)_0, \end{aligned} \tag{3.3}$$

where λ_k is the eigenvalue corresponding to ψ_k. It is well-known [59, 87] that the norms generated by the scalar products $(\cdot, \cdot)$ are equivalent to the ones obtained using local charts, partitions of unity, and the standard norms of the spaces $H^s_{\mathbb{C}}(\mathbb{R}^{N-1})$. As Δ_{Γ_0} maps real-valued functions to real-valued functions it is not hard to check that the ψ_k can be chosen to be real. Therefore Λ^s

maps real-valued functions to real-valued functions, hence we can define the real Sobolev spaces $H^s(\Gamma_0)$ analogously to (3.1), retaining the same scalar products.

Let us fix some notation for the remaining part of this thesis. We will use the letters C and c for "large" and "small" positive constants, respectively, if their actual value is of no interest. Sometimes an index is used to indicate their dependence on parameters.

A function that is given on Ω_0 and its restriction or trace at the boundary Γ_0 are sometimes denoted by the same symbol.

Let M be an arbitrary metric space. For $x \in M$, $r > 0$, we denote by $B_x(\rho, M)$ the open ball of radius ρ centered at x in M.

If X is a normed space, we denote by X' its dual and by X^k the product space of k copies of X, equipped with the usual norm (and the usual scalar product if X is a Hilbert space).

The norms in $H^s(\Omega_0)$ and $H^s(\Gamma_0)$ will be denoted by $\|\cdot\|_s^{\Omega_0}$ and $\|\cdot\|_s^{\Gamma_0}$, respectively, and the same notation will be used for the norms of the product spaces $(H^s(\Omega_0))^k$, $(H^s(\Gamma_0))^k$. Analogous notation will be used for the corresponding spaces on other domains and manifolds.

We denote the kernel of a linear map A by $\mathcal{N}(A)$ and its range by $\mathcal{R}(A)$.

For bounded domains Ω and bounded surfaces Γ we will denote by $|\Omega|$ and $|\Gamma|$ their volume and area, respectively.

To be able to describe perturbed domains, we choose a fixed C^∞-function $\zeta : \Gamma_0 \longrightarrow \mathbb{R}^N$ satisfying

$$\zeta(\xi) \cdot n(\xi) > 0 \qquad \forall \xi \in \Gamma_0. \tag{3.4}$$

Lemma 5 *(Perturbed domains and perturbation functions)*

Let $s > 2 + \frac{N}{2}$. There is a $\delta_0 > 0$ depending only on Ω_0, ζ, and s such that the following holds:

(i) For all $r \in B_0(\delta_0, H^s(\Gamma_0))$, the set

$$\Gamma_r = \{\xi + \zeta(\xi) r(\xi) \mid \xi \in \Gamma_0\}$$

is homeomorphic to Γ_0.

(ii) There is a mapping $z : \; B_0(\delta_0, H^s(\Gamma_0)) \longrightarrow (C^2(\Omega_0))^N$ such that $z(r)$ is a global diffeomorphism of Ω_0 onto $\Omega_r = z[\Omega_0]$ and

$$\|z - I\|_{(C^2(\Omega_0))^N} \leq C \|r\|_s^{\Gamma_0}$$

with C independent of r, and $\Gamma_r = z[\Gamma_0]$.

Proof: Let $\mathcal{I}$ be a (small) open interval containing 0 and consider the C^∞-mapping

$$\phi : \; \Gamma_0 \times \mathcal{I} \longrightarrow \mathbb{R}^N$$

defined by

$$\phi(\xi, \eta) = \xi + \zeta(\xi)\eta. \tag{3.5}$$

Pick a fixed $\xi_0 \in \Gamma_0$ and let $\xi = \xi(u)$ be a regular local parametrization of Γ_0 near ξ_0. The differential map of ϕ in $(\xi_0, 0)$

$$D\phi(\xi_0, 0) : \; T_{\xi_0}(\Gamma_0) \times \mathbb{R} \longrightarrow \mathbb{R}^N$$

is given by

$$D\phi(\xi_0, 0)[t, \tau] = \frac{\partial \xi}{\partial u_i} t^i + \tau \zeta(\xi_0),$$

where $T_{\xi_0}(\Gamma_0)$ denotes the tangential space of Γ_0 in ξ_0 and $t = t^i \dfrac{\partial}{\partial u_i}$ its elements. Due to (3.4), $D\phi(\xi_0, 0)$ is surjective, hence by the local diffeomorphism theorem there is a neighborhood $\mathcal{U}$ of $(\xi_0, 0)$ in $\Gamma_0 \times \mathcal{I}$ such that ϕ acts as a C^∞-diffeomorphism of $\mathcal{U}$ into $\mathbf{R}^N$.

By a compactness argument we can conclude that there is a $\delta_1 > 0$ such that ϕ acts as a (global) C^∞-diffeomorphism of $\Gamma_0 \times (-\delta_1, \delta_1)$ onto some neighborhood of $\Gamma_0 \subset \mathbf{R}^N$ in $\mathbf{R}^N$. Taking into account the continuous embedding $H^s(\Gamma_0) \hookrightarrow C^2(\Omega_0)$ and the fact that the submanifold $\tilde{\Gamma}_r \subset \Gamma_0 \times (-\delta_1, \delta_1)$ consisting of the points $(\xi, r(\xi))$ is homeomorphic to Γ_0 if $r \in B_0(\delta_1, C(\Gamma_0))$ we can conclude that (i) holds.

To show (ii) we need some preparations. It is easily seen from the assumptions of compactness and smoothness of Γ_0 that there is a $\delta_2 > 0$ depending only on Ω_0 and having the property that if for any $x, y \in \Omega_0$

$$\max\{\operatorname{dist}(x, \Gamma_0), \operatorname{dist}(y, \Gamma_0), |x - y|\} < \delta_2,$$

then there are an open set $\mathcal{U} \subset \Omega_0$ and a diffeomorphism $\Phi_{\mathcal{U}} : \mathcal{U} \longrightarrow \mathbf{R}^N$ such that $x, y \in \mathcal{U}$, $\Phi_{\mathcal{U}}[\mathcal{U}]$ convex, and $\Phi_{\mathcal{U}}$ and $\Phi_{\mathcal{U}}^{-1}$ are Lipschitz-continuous with constants that do not depend on $\mathcal{U}$. Using this and the mean value theorem, the following estimate can be shown: Let g be a Fréchet-differentiable mapping from Ω_0 into some normed space E. Then for $|x - y| < \delta_2$

$$\|g(y) - g(x)\|_E \le C \sup_{w \in \Omega_0 \cap B_x(C|y-x|, E)} \|g'(w)\|\,|y - x| \tag{3.6}$$

holds, which in particular implies Lipschitz-continuity of g if g and g' are bounded.

We construct the mapping z by choosing an arbitrary but fixed linear continuous right inverse T^{-1} of the trace operator $T: H^{s+\frac{1}{2}}(\Omega_0) \longrightarrow H^s(\Gamma_0)$ and setting

$$z(r) = T^{-1}(r\zeta) + I,$$

where T^{-1} has to be applied separately to the components of $r\zeta$. This yields immediately $\Gamma_r = z[\Gamma_0]$, and the estimate for $\|z(r) - \mathrm{id}\|_{(C^2(\Omega_0))^N}$ is a consequence of the continuity of the embedding $H^{s+\frac{1}{2}}(\Omega_0) \hookrightarrow C^2(\Omega)$. This estimate ensures for sufficiently small δ_0 and all $x \in \Omega_0$ that $\dfrac{\partial z(r)}{\partial x}(x)$ is near the identity and hence nonsingular. Therefore $z(r)$ is a local diffeomorphism, it remains to show that it is globally injective, i.e. that $z(x_1) = z(x_0)$ implies $x_1 = x_0$ for all $x_0, x_1 \in \Omega_0$. For this purpose, the equation $z(x) = z(x_0)$ is rewritten equivalently as

$$x = S(x) := x - z'(x_0)^{-1}(z(x) - z(x_0)) \tag{3.7}$$

The mapping S is differentiable in all $x \in \Omega_0$ and has the derivative

$$S'(x) = I - z'(x_0)^{-1} z'(x) = z'(x_0)^{-1}(z'(x_0) - z'(x)).$$

According to the above remark, $z \in \left(C^2(\Omega_0)\right)^N$ implies Lipschitz-continuity of z', hence

$$\|S'(x)\| \le C\|z'(x_0) - z'(x)\| \le C|x - x_0|,$$

where $\|\cdot\|$ denotes an arbitrary operator norm on $\mathcal{L}\left(\mathbf{R}^N, \mathbf{R}^N\right)$. Consequently,

$$\begin{aligned} \sup_{x \in \Omega_0 \cap B_{x_0}(C|x_1 - x_0|, \mathbf{R}^N)} \|S'(x)\| &\le C \sup_{x \in \Omega_0 \cap B_{x_0}(C|x_1 - x_0|, \mathbf{R}^N)} |x - x_0| \\ &\le C|x_1 - x_0|. \end{aligned}$$

Moreover, assuming $z(x_1) = z(x_0)$,

$$|x_1 - x_0| \le |x_1 - z(x_1)| + |z(x_0) - x_0| \le C\delta_0, \tag{3.8}$$

hence, for sufficiently small δ_0, (3.6) may be applied to S and this yields

$$|x_1 - x_0| = |S(x_1) - S(x_0)| \le C|x_1 - x_0|^2,$$

i.e. $x_0 = x_1$ or $|x_1 - x_0| \ge C^{-1}$, but the latter of these two possibilities is in contradiction with (3.8) for small δ_0. ■

Remark: It is clear from the proof that Γ_r is actually "as smooth as r", i.e. if $r \in C^{k,\alpha}(\Gamma_0)$, then Γ_r belongs to the same smoothness class. □

In the following investigations the concept of (locally) analytic operators will be useful.

Definition (Analytic operators in Banach spaces): Let X and Y be Banach spaces. A mapping F defined on $B_{x_0}(\varepsilon, X)$ with positive ε and values in Y is called analytic near x_0 (from X to Y) iff F has a series representation

$$F(x) = \sum_{k=0}^{\infty} F_k(x - x_0, \dots, x - x_0) \qquad \forall x \in B_{x_0}(\varepsilon, X), \tag{3.9}$$

where the F_k are bounded k-linear symmetric operators from X^k to Y for which the majorant series

$$\sum_{k=0}^{\infty} \|F_k\|_{\mathcal{L}(X^k,Y)} \varepsilon^k \tag{3.10}$$

converges, which implies the absolute convergence of the series in (3.9) and the boundedness of F. As usual, we will call F analytic in the open set $U \subset X$ iff F is analytic near any point of U.

Obviously, the series (3.9) represents a generalization of the concept of a power series. The rules for calculations with them can also be generalized to our situation, more precisely, we will use the following results concerning local analyticity, differentiability, derivatives, "comparison of coefficients", composition, pointwise products, and the Implicit Function theorem:

Lemma 6 *(Properties of analytic operators)*

Let F be given by (3.9) *and suppose* (3.10) *converges. Then:*

(i) F is analytic in $B_{x_0}(\varepsilon, X)$.

(ii) For any $m \in \mathbb{N}$, the m-th Fréchet derivative of F exists and is an analytic function in $B_{x_0}(\varepsilon, X)$ valued in $\mathcal{L}(X^m, Y)$ given by

$$F^{(m)}(x)[h_1, \dots, h_m] = \sum_{k=0}^{\infty} \frac{(m+k)!}{k!} F_{m+k}(x - x_0, \dots, x - x_0, h_1, \dots, h_m).$$

Furthermore:

(iii) If $\tilde{F}$ is an analytic function near x_0 valued in Y, then $F \equiv \tilde{F}$ near x_0 iff $F_k = \tilde{F}_k$ for all $k \in \mathbb{N}$.

(iv) If G is analytic near $F(x_0)$ from Y to the Banach space Z, then the composition $G \circ F$ is analytic near x_0 from X to Z.

(v) If Y is a Banach algebra and H is analytic near x_0 from X to Y, then the pointwise product FH is analytic near x_0 from X to Y.

(vi) Suppose the mapping $\mathcal{F} : X \times Y \longrightarrow Z$ is analytic near (x_0, y_0), $\mathcal{F}(x_0, y_0) = 0$,

$$\mathcal{F}_y(x_0, y_0) = \mathcal{F}'(x_0, y_0)[0, \cdot]$$

is bijective from Y to Z, and let $\mathcal{U}$ be a sufficiently small neighborhood of x_0 in X. Then the uniquely defined continuous function $y : \mathcal{U} \longrightarrow Y$ satisfying $y(x_0) = y_0$ and $\mathcal{F}(x, y(x)) = 0$ for all $x \in \mathcal{U}$ is analytic near x_0.

Proof: (i)–(iv) are straightforward generalizations of the corresponding standard results for power series (see e.g. [22] Ch. IX). To indicate the proof of (v) it is sufficient to remark that

$$(FH)_k[h_1, \ldots, h_k] = \frac{1}{k!} \sum_{\sigma \in S_k} \sum_{l=0}^{k} F_l(h_{\sigma(1)}, \ldots, h_{\sigma(l)}) H_{k-l}(h_{\sigma(l+1)}, \ldots, h_{\sigma(k)}),$$

where S_k denotes the set of all permutations of $\{1, \ldots, k\}$. By the Banach algebra property of Y this implies

$$\|(FH)_k\|_{\mathcal{L}(X^k, Y)} \leq C \sum_{l=0}^{k} \|F_l\|_{\mathcal{L}(X^l, Y)} \|H_{k-l}\|_{\mathcal{L}(X^{k-l}, Y)}$$

with C independent of k. For the proof of (vi) we refer to [93], Ch. 8.2. ■

In particular, the following result will be applied:

Lemma 7 *("Square roots" and inverses as analytic operations)*

Let X be a Banach algebra and let $x_0, u_0 \in X$ be such that $x_0^2 = u_0$ and x_0 and u_0 are invertible. There exist functions $u \mapsto \sqrt{u}$ and $u \mapsto u^{-1}$ defined in a neighborhood of u_0 which are analytic near u_0 with values in X such that $(\sqrt{u})^2 = u$ and $u^{-1}u = e$, where e denotes the unit element of X.

Remark: Note that the symbol "$\sqrt{\ }$" is used here in a sense depending on x_0. □

Proof: The lemma follows from applying Lemma 6 (v) and (vi) to the equations

$$\mathcal{F}(u, x) = x^2 - u = 0, \quad \mathcal{G}(u, x) = xu - e = 0,$$

respectively. ■

Let us conclude this preliminary section with a well-known result concerning a natural representation of the curvature of a surface in terms of its Laplace-Beltrami operator. Its importance for the analysis of the FBPs considered here can hardly be overestimated.

Lemma 8 *(Curvature and the Laplace-Beltrami operator)*

Let Γ_0 be as before and $x \in (C^\infty(\Gamma_0))^N$ be the mapping that assigns to each point of Γ_0 its (cartesian) coordinates in $\mathbf{R}^N$. Then

$$\Delta_{\Gamma_0} x = \kappa n,$$

where Δ_{Γ_0} has to be applied to every component of x separately.

Proof: For $N = 3$ the lemma is proved in [21] (Section 2.5, Theorem 1). We prefer to give an alternative proof based on some remarks in [37], Section 15.1, that can also provide a more intuitive idea about the Laplace-Beltrami operator.

Let ϕ be a smooth function on Γ_0 and Φ a smooth extension of it into a neighborhood of $\Gamma_0 \subset \mathbb{R}^N$. The vector-valued operator $\phi \mapsto \delta\Phi$ defined on Γ_0 by

$$\delta\phi = \nabla\Phi - (\nabla\Phi \cdot n)n$$

is called the surface gradient of ϕ and is easily seen to be independent of the values of Φ outside Γ_0. Writing $\delta_i\phi = \delta\phi \cdot e_i, n_i = n \cdot e_i$, with $i = 1, \dots, N$ and e_i denoting the i-th unit vector we have by definition

$$\delta\phi \cdot n = \delta_i\phi n_i = 0. \tag{3.11}$$

The components δ_i of δ are first-order differential operators on Γ_0 for which we have ([37], Section 15.1)

$$\begin{aligned} \delta_i\delta_i\phi &= \Delta_{\Gamma_0}\phi, \\ \delta_i n_i &= -\kappa. \end{aligned} \tag{3.12}$$

Using this and (3.11) and writing as usual δ_{ij} for the Kronecker symbol we calculate

$$\begin{aligned} \delta_i x_j &= \delta_{ij} - n_i n_j, \\ \delta_i\delta_i x_j &= -(\delta_i n_j)n_i - (\delta_i n_i)n_j = \kappa n_j, \end{aligned}$$

which proves the lemma. ■

3.2 The fixed-time problems

A thorough understanding of the properties of the fixed-time problems (cf. (1.6), (1.7), (1.10), (1.12))

$$\left.\begin{aligned} -\Delta u + \nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned}\right\} \text{ in } \Omega_0 \qquad T(u,p)n = \kappa n \quad \text{on } \Gamma_0 \tag{3.13}$$

and

$$\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega_0 \\ u &= \kappa \quad \text{on } \Gamma_0 \end{aligned} \tag{3.14}$$

is a necessary prerequisite for the study of the corresponding moving boundary problems. (κ and n denote the curvature and the outer normal vector of Γ_0). Essentially, this understanding is provided by the theory of elliptic boundary value problems. For the Stokes equations, however, some care has to be taken of certain technical details.

3.2.1 The fixed-time problem for Stokes flow

We will discuss the fixed-time problem for the Stokes flow problem in three steps:

- investigation of the weak formulation, based on a Green formula for the Stokes operator,
- obtaining of regularity results in Sobolev spaces, based on the theory of hydrodynamic Lorentz-Ladyshenskaya potentials ([55], [60]),
- proof of a commutator property for the Neumann-to-Dirichlet operator of the Stokes equations.

The main idea in this is to use the strong resemblance of the properties of the Stokes operator to the Laplacian. In the Stokes equations, the normal component of the stress tensor $\mathcal{T}(u,p)n$ plays the same role that is played by the normal derivative $\frac{\partial u}{\partial n}$ for the Laplace equation. Hence, (3.13) is a Neumann boundary value problem, and we will recover a nontrivial space of solutions for the homogeneous problem and necessary solvability conditions in strict analogy to the second BVP for the Laplacian. Moreover, the hydrodynamic potentials as well as the singular integral operators arising from them have properties that correspond to the well-known ones in potential theory.

Weak formulation

To conveniently deal with the N-dimensional problem, we generalize some notions of vector algebra and analysis to $\mathbf{R}^N$. Let K be an arbitrary but fixed bijection from the set

$$\{(i,j) \mid 1 \leq i < j \leq N\}$$

to the set $\left\{1,\ldots,\binom{N}{2}\right\}$. We define the bilinear mappings

$$\begin{aligned} \wedge &: \mathbf{R}^N \times \mathbf{R}^N \longrightarrow \mathbf{R}^{\binom{N}{2}} \\ \rfloor &: \mathbf{R}^{\binom{N}{2}} \times \mathbf{R}^N \longrightarrow \mathbf{R}^N \end{aligned}$$

by

$$(a \wedge b)_{K(i,j)} = a_i b_j - a_j b_i \qquad (1 \leq i < j \leq N)$$

and

$$(c \rfloor a)_i = \sum_{j=1}^{i-1} c_{K(j,i)} a_j - \sum_{j=i+1}^{N} c_{K(i,j)} a_j \qquad (i = 1,\ldots,N).$$

It is easy to check that

$$c \cdot (a \wedge b) = b \cdot (c \rfloor a) \qquad \forall a, b \in \mathbf{R}^N, c \in \mathbf{R}^{\binom{N}{2}}. \tag{3.15}$$

We define, moreover, the differential operator

$$\mathrm{rot}: \left(H^1(\Omega_0)\right)^N \longrightarrow \left(L^2(\Omega_0)\right)^{\binom{N}{2}}$$

by

$$(\mathrm{rot}\, v)_{K(i,j)} = \frac{\partial v_j}{\partial x_i} - \frac{\partial v_i}{\partial x_j} \qquad (1 \leq i < j \leq N),$$

which yields the integral theorem

$$\int_{\Omega_0} \mathrm{rot}\, v \, dx = \int_{\Gamma_0} n \wedge v \, d\Gamma. \tag{3.16}$$

Note that if $N = 3$, then the usual definitions of the outer product and the curl (rotation) of a vector field can be obtained, up to the sign of the second component, by choosing the suitable bijection K.

The basis for the weak formulation is the following integral identity:

Lemma 9 *(First Stokes-Green formula* [55]*)*
For all $u, v \in \left(C^2(\Omega_0)\right)^N$, $p \in C^1(\Omega_0)$ *we have*

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_0}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)dx-\int_{\Omega_0} p \operatorname{div} v\, dx \\ =\ &\int_{\Omega_0}(-\Delta u+\nabla p-\nabla(\operatorname{div} u))\cdot v\, dx+\int_{\Gamma_0}\mathcal{T}(u,p)n\cdot v\, d\Gamma. \end{aligned}$$

Proof: The identity follows easily from applying the Gauss integral theorem to the vector-valued function $\mathcal{T}(u,p)v$. ■

We introduce the space

$$V_0=\left\{v\in\left(H^1(\Omega_0)\right)^N \mid v_i(x)=s_{ij}x_j+c_i,\ \ s_{ij},c_i\in\mathbf{R},\ \ s_{ij}=-s_{ji}\right\}$$

of the velocity fields corresponding to rigid body motions in $\mathbf{R}^N$. Clearly $\Delta v=0$, $\operatorname{div} v=0$, $\mathcal{T}(v,0)=0$ for all $v\in V_0$, and thus $(u,p)=(v,0)$ is a solution of the homogeneous problem

$$\begin{aligned} \left.\begin{aligned} -\Delta u+\nabla p &= 0 \\ \operatorname{div} u &= 0 \end{aligned}\right\} &\quad \text{in } \Omega_0 \\ \mathcal{T}(u,p)n = 0 \qquad &\quad \text{on } \Gamma_0. \end{aligned}$$

Hence, the velocity component in a solution of (3.13) is defined only up to an element of V_0. To enforce uniqueness of the solution it is natural to demand

$$\int_{\Omega_0} u\, dx=0, \qquad \int_{\Omega_0}\operatorname{rot} u\, dx=0. \tag{3.17}$$

We proceed now by discussing a variational problem which is a weak formulation of a generalization of (3.13), (3.17). For this purpose we introduce the Hilbert spaces

$$\begin{aligned} X &= (H^1(\Omega_0))^N\times L^2(\Omega_0)\times\left(\mathbf{R}^N\times\mathbf{R}^{\binom{N}{2}}\right), \\ Y &= ((H^1(\Omega_0))^N)'\times\left(L^2(\Omega_0)\times\mathbf{R}^N\times\mathbf{R}^{\binom{N}{2}}\right) \end{aligned}$$

and the (bi-)linear operators

$$\begin{aligned} L:&\ \ X\longrightarrow Y \\ A:&\ \ \left(H^1(\Omega_0)\right)^N\longrightarrow\left(\left(H^1(\Omega_0)\right)^N\right)', \\ B:&\ \ \left(H^1(\Omega_0)\right)^N\longrightarrow L^2(\Omega_0)\times\left(\mathbf{R}^N\times\mathbf{R}^{\binom{N}{2}}\right), \\ a:&\ \ \left(H^1(\Omega_0)\right)^N\times\left(H^1(\Omega_0)\right)^N\longrightarrow\mathbf{R}, \\ \varphi_1:&\ \ \left(H^1(\Omega_0)\right)^N\longrightarrow\mathbf{R}^N, \\ \varphi_2:&\ \ \left(H^1(\Omega_0)\right)^N\longrightarrow\mathbf{R}^{\binom{N}{2}}, \end{aligned}$$

defined by

$$L\begin{bmatrix} u \\ p \\ \lambda \end{bmatrix} = \begin{bmatrix} Au+B'\begin{bmatrix} p \\ \lambda \end{bmatrix} \\ Bu \end{bmatrix},$$

$$\begin{aligned}
(Au)v &= a(u,v), \\
Bu &= \begin{bmatrix} -\operatorname{div} u \\ \varphi_1(u) \\ \varphi_2(u) \end{bmatrix}, \\
a(u,v) &= \frac{1}{2}\int_{\Omega_0} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) dx, \\
\varphi_1(u) &= \int_{\Omega_0} u\, dx, \\
\varphi_2(u) &= \int_{\Omega_0} \operatorname{rot} u\, dx,
\end{aligned}$$

where $B' : L^2(\Omega_0) \times \left(\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}\right) \longrightarrow ((H^1(\Omega_0))^N)'$ is the operator dual to B.

Lemma 10 *(Weak formulation)*
(i) The operator L is a homeomorphism between X and Y.
(ii) Suppose $L[u\ p\ \lambda]^T = [f\ 0]^T$ with

$$f(v) = \int_{\Gamma_0} \kappa n \cdot v\, d\Gamma \qquad \forall v \in \left(H^1(\Omega_0)\right)^N.$$

Then $\lambda = 0$ and (u,p) is a weak solution of (3.13), (3.17).

Proof: (i) The equation

$$L[u\ p\ \lambda]^T = F \tag{3.18}$$

is a variational problem with linear restrictions to which the usual existence results apply (see e.g. [15]). In order to establish (i) it is therefore sufficient to show that a is elliptic on $(\ker B, \|\cdot\|_1^{\Omega_0})$ and B is surjective.

The first statement follows from Poincarés inequality (e.g. [93] Vol. II/B, Appendix, (53a))

$$\int_{\Omega_0} |\nabla w|^2\, dx + \left(\int_{\Omega_0} w\, dx\right)^2 \geq c \|w\|_1^{\Omega_0\,2} \qquad \forall w \in H^1(\Omega_0)$$

and Korns second inequality (e.g. [93] Vol. IV Theorem 62.F)

$$a(v,v) \geq c \sum_{i,j=1}^{N} \int_{\Omega_0} \left(\frac{\partial v_i}{\partial x_j}\right)^2 dx \qquad \forall v \in \left\{ v \in \left(H^1(\Omega_0)\right)^N \,\middle|\, \varphi_2(v) = 0 \right\}.$$

To show the surjectivity of B we note that $\varphi = [\varphi_1\ \ \varphi_2]^T$ is surjective from V_0 onto $\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}$. The equation

$$Bu = \begin{bmatrix} q \\ \mu \end{bmatrix} \in L^2(\Omega_0) \times \left(\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}\right)$$

has a solution $u = u_0 + u_1$ with $u_0 \in V_0$, $\varphi(u_0) = \mu$, $u_1 = \nabla\Phi$, where $\Phi \in H^2(\Omega_0) \cap H_0^1(\Omega_0)$ solves $-\Delta\Phi = q$ in Ω_0.

(ii) The equation (3.18) with $F = [f\ 0]^T$ can be written as

$$\left.\begin{aligned} a(u,v) - \int_{\Omega_0} p \operatorname{div} v\, dx + \lambda_1 \cdot \varphi_1(v) + \lambda_2 \cdot \varphi_2(v) &= f(v)\ \forall v \in \left(H^1(\Omega_0)\right)^N \\ \operatorname{div} u &= 0, \\ \varphi(u) &= 0. \end{aligned}\right\} \quad (3.19)$$

Using Lemma 8 and the Green formula for closed surfaces we find

$$\int_{\Gamma_0} \kappa n \cdot v\ d\Gamma = \int_{\Gamma_0} \Delta_{\Gamma_0} x \cdot v\ d\Gamma = -\int_{\Gamma_0} \delta x_i \cdot \delta v_i\ d\Gamma$$

and from this it is easy to see that f vanishes on V_0. Choosing $v \in V_0$ in (3.19) yields

$$\lambda_1 \cdot \varphi_1(v) + \lambda_2 \cdot \varphi_2(v) = 0 \qquad \forall v \in V_0.$$

We use again the surjectivity of φ from V_0 onto $\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}$ to conclude that $\lambda = 0$. From this we see by Lemma 9 that (3.19) is a weak formulation of (3.13), (3.17). ■

Regularity

For later use we introduce on V_0 the basis $\{v_{ij}\}$ by

$$\begin{aligned} v_{1,j} &= \frac{1}{|\Omega_0|} e_j \qquad (j = 1, \ldots, N), \\ v_{2,K(i,j)} &= \frac{1}{2|\Omega_0|}(x_j e_i - x_i e_j) - \frac{1}{2|\Omega_0|^2} \int_{\Omega_0} (x_j e_i - x_i e_j)\, dx\ (1 \le i < j \le N) \end{aligned}$$

which is dual to the functionals $\{\varphi_{ij}\}$, i.e.

$$\varphi_{ij}(v_{kl}) = \delta_{ik}\delta_{jl}. \qquad (3.20)$$

For fixed $s \ge 2$ we introduce the spaces

$$\begin{aligned} X_s &= (H^s(\Omega_0))^N \times H^{s-1}(\Omega_0) \times \left(\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}\right) \\ Y_s &= \left(H^{s-2}(\Omega_0)\right)^N \times H^{s-1}(\Omega_0) \times \left(H^{s-\frac{3}{2}}(\Gamma_0)\right)^N \times \mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}} \end{aligned}$$

and the operator

$$\tilde{L} : X_s \longrightarrow Y_s$$

defined by

$$\tilde{L} \begin{bmatrix} u \\ p \\ \lambda \end{bmatrix} = \begin{bmatrix} -\Delta u + \nabla p + \lambda_1 \\ -\operatorname{div} u \\ \mathcal{T}(u,p)n + \lambda_2 \rfloor n \\ \varphi_1(u) \\ \varphi_2(u) \end{bmatrix}.$$

Lemma 11 *(Regularity of the Stokes fixed-time problem)*
(i) The operator $\tilde{L}$ is a homeomorphism between the spaces X_s and Y_s.

(ii) Suppose $\tilde{L}[u\ p\ \lambda]^T = [0\ 0\ F_B\ 0\ 0]^T$. *Then*

$$\|\lambda\|_{\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}} \leq C_\sigma \|F_B\|_\sigma^{\Gamma_0} \tag{3.21}$$

for all $\sigma \in \mathbb{R}$ *and*

$$\|u\|_0^{\Omega_0} \leq C \|F_B\|_{-\frac{3}{2}}^{\Gamma_0}. \tag{3.22}$$

Proof: Note that, according to (3.15), (3.16),

$$\int_{\Gamma_0} v \cdot (\lambda_2 \rfloor n)\, d\Gamma = \int_{\Gamma_0} \lambda_2 \cdot (n \wedge v)\, d\Gamma = \lambda_2 \cdot \int_{\Omega_0} \operatorname{rot} v\, dx = \lambda_2^T \varphi_2(v). \tag{3.23}$$

Using this and Lemma 9 we find from

$$\tilde{L}[u\ p\ \lambda]^T = [F_I\ g\ F_B\ h_1\ h_2]^T$$

the variational formulation

$$\begin{aligned} & a(u,v) - \int_{\Omega_0} p \operatorname{div} v\, dx + \lambda_1^T \varphi_1(v) + \lambda_2^T \varphi_2(v) \\ = \; & \int_{\Omega_0} (F_I + \nabla g) \cdot v\, dx + \int_{\Gamma_0} F_B \cdot v\, d\Gamma \qquad \forall v \in \left(H^1(\Omega_0)\right)^N, \\ -\operatorname{div} u \; = \; & g, \\ \varphi_1(u) \; = \; & h_1, \\ \varphi_2(u) \; = \; & h_2. \end{aligned} \tag{3.24}$$

Lemma 10 yields that this problem has a unique solution $[u\ p\ \lambda]^T \in X$, and from the fact that $a(u,\cdot)$ and div vanish on V_0 we find

$$\lambda_{ij} = \int_{\Omega_0} (F_I + \nabla g) \cdot v_{ij}\, dx + \int_{\Gamma_0} F_B \cdot v_{ij}\, d\Gamma.$$

All v_{ij} are smooth, hence

$$\|\lambda\|_{\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}} \leq C_s \left(\|F_I\|_{s-2}^{\Omega_0} + \|g\|_{s-1}^{\Omega_0} + \|F_B\|_{s-\frac{3}{2}}^{\Gamma_0} \right)$$

and (3.21) follow. We will determine u and p by setting

$$\begin{aligned} u &= u_0 + u_1 + u_2, \\ p &= p_1 + p_2, \end{aligned}$$

where

$$\left. \begin{aligned} -\Delta u_1 + \nabla p_1 &= F_I - \lambda_1 \\ -\operatorname{div} u_1 &= g \end{aligned} \right\} \text{ in } \Omega_0, \qquad u_1 = -\frac{1}{|\Gamma_0|} \int_{\Omega_0} g\, dx \cdot n \text{ on } \Gamma_0, \qquad \int_{\Omega_0} p_1\, dx = 0, \tag{3.25}$$

$u_0 \in V_0$ such that $\varphi_i(u_0) = -\varphi_i(u_1) + h_i$, $i = 1, 2$, and

$$\begin{aligned} \left. \begin{aligned} -\Delta u_2 + \nabla p_2 &= 0 \\ -\operatorname{div} u_2 &= 0 \end{aligned} \right\} & \text{ in } \Omega_0, \\ T(u_2, p_2) n = -T(u_1, p_1) n + F_B - \lambda_2 \rfloor n = \Phi \quad & \text{ on } \Gamma_0, \\ \varphi_i(u_2) = 0 \quad & (i = 1, 2). \end{aligned} \tag{3.26}$$

The regularity results for the Dirichlet problem of the Stokes equations ([34], Theorem IV.6.1) yield that (3.25) has precisely one solution $(u_1, p_1) \in (H^s(\Omega_0))^N \times H^{s-1}(\Omega_0)$ and an estimate

$$||u_1||_s^{\Omega_0} + ||p_1||_{s-1}^{\Omega_0} \leq C_s \left(||F_I||_{s-2}^{\Omega_0} + ||\lambda_1||_{\mathbb{R}^N} + ||g||_{s-1}^{\Omega_0} \right)$$

holds. Thus we have $\Phi \in \left(H^{s-\frac{3}{2}}(\Gamma_0) \right)^N$ and

$$\begin{aligned} ||\Phi||_{s-\frac{3}{2}}^{\Gamma_0} &\leq C_s \left(||u_1||_s^{\Omega_0} + ||p_1||_{s-1}^{\Omega_0} + ||\lambda_2||_{\mathbb{R}\binom{N}{2}} + ||F_B||_{s-\frac{3}{2}}^{\Gamma_0} \right) \\ &\leq C_s \left(||F_I||_{s-2}^{\Omega_0} + ||g||_{s-1}^{\Omega_0} + ||F_B||_{s-\frac{3}{2}}^{\Gamma_0} + ||\lambda||_{\mathbb{R}^N \times \mathbb{R}\binom{N}{2}} \right). \end{aligned}$$

It remains to show that (3.26) has a unique solution $(u_2, p_2) \in (H^s(\Omega_0))^N \times H^{s-1}(\Omega_0)$ satisfying an estimate

$$||u_2||_s^{\Omega_0} + ||p_2||_{s-1}^{\Omega_0} \leq C_s ||\Phi||_{s-\frac{3}{2}}^{\Gamma_0}. \tag{3.27}$$

Note that due to Lemma 9 and (3.23)

$$\int_{\Gamma_0} \Phi \cdot v = 0 \qquad \forall v \in V_0. \tag{3.28}$$

From the discussion of the weak formulation as in Lemma 10 with κn replaced by Φ we find that this condition is necessary and sufficient for the existence of a unique weak solution $(u_2, p_2) \in \left(H^1(\Omega_0)\right)^N \times L^2(\Omega_0)$. A density argument shows that we can assume $\Phi \in (C(\Gamma))^N$.

To show (3.27) we will apply representation formulas and weakly singular integral equations from the theory of hydrodynamic potentials. For $N = 3$ this theory can be found in [55], Ch. III, the generalization to arbitrary N is straightforward. For the sake of brevity, we will describe the details here only for $N \notin \{2, 4\}$ because of the logarithmic terms entering the representation formulas in these cases.

For $x \in \Omega$ we use the ansatz

$$\begin{aligned} u_2(x) &= V(x, \psi) \\ V(x, \psi) &= \frac{1}{2\omega_N} \int_{\Gamma_0} \left(\frac{I}{(N-2)|x-y|^{N-2}} + \frac{(x-y)(x-y)^T}{|x-y|^N} \right) \psi(y)\, d\Gamma_y, \\ p_2(x) &= \frac{1}{\omega_N} \int_{\Gamma_0} \frac{x-y}{|x-y|^N} \psi(y)\, d\Gamma_y, \end{aligned}$$

where ψ is a $\mathbb{R}^N$-valued (measurable) function on Γ_0 and

$$\omega_N = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)} \tag{3.29}$$

denotes the $N-1$-dimensional surface area of the unit sphere in $\mathbb{R}^N$.

It can be shown as in [55] that (u_2, p_2) satisfies the first three equations in (3.26) if ψ is continuous and satisfies

$$\left(\frac{1}{2} I + K \right) \psi = \Phi \tag{3.30}$$

with

$$(K\psi)(x) = -\frac{N}{\omega_N}\int_{\Gamma_0}((x-y)\cdot n(x))\frac{(x-y)(x-y)^T}{|x-y|^{N+2}}\psi(y)\,d\Gamma_y \qquad (x \in \Gamma_0).$$

The operator K is a weakly singular integral operator, hence it is compact on $\left(H^0(\Gamma_0)\right)^N$ and continuity of Φ implies continuity for all $\psi \in \left(H^0(\Gamma_0)\right)^N$ that satisfy (3.30) (see e.g. [88], theorems 12.1., 12.7., 12.8.) Moreover, K is a pseudodifferential operator [78], hence it is compact on $\left(H^{s-\frac{3}{2}}(\Gamma_0)\right)^N$ and therefore $(\frac{1}{2}I + K)$ is a Fredholm operator of index 0 on this space. Taking into account that $\mathcal{N}(\frac{1}{2}I+K)$ consists of continuous functions, one can conclude, using the results about the weak formulation, that $V(\cdot,\psi) \in V_0$ for all $\psi \in \mathcal{N}(\frac{1}{2}I+K)$. $\psi \in \mathcal{N}(\frac{1}{2}I+K)$ and $V(\cdot,\psi) = 0$ implies $\psi = 0$ [55], hence $\dim \mathcal{N}(\frac{1}{2}I+K) \le N + \binom{N}{2}$. The necessary solvability conditions (3.28) imply $\operatorname{codim} \mathcal{R}(\frac{1}{2}I+K) \ge N + \binom{N}{2}$, hence

$$\dim \mathcal{N}\left(\frac{1}{2}I + K\right) = \operatorname{codim} \mathcal{R}\left(\frac{1}{2}I + K\right) = N + \binom{N}{2},$$

i.e. the solvability conditions (3.28) are also sufficient and the mapping $\psi \mapsto V(\cdot,\psi)$ maps $\mathcal{N}(\frac{1}{2}I + K)$ onto V_0. Thus we can conclude that (3.30) has precisely one solution such that $\varphi_i(V(\cdot,\psi)) = 0$, $i = 1,2$, satisfying an estimate

$$||\psi||^{\Gamma_0}_{s-\frac{3}{2}} \le C_s||\Phi||^{\Gamma_0}_{s-\frac{3}{2}}.$$

Finally, we use the fact that the singular integral operator that maps ψ to $V(\cdot,\psi)|_{\Gamma_0}$ is a pseudodifferential operator of order -1 [78], hence we find that the trace of u_2 on Γ is in $\left(H^{s-\frac{1}{2}}(\Gamma_0)\right)^N$ and

$$||u_2||^{\Gamma_0}_{s-\frac{1}{2}} \le C||\psi||^{\Gamma_0}_{s-\frac{3}{2}} \le C||\Phi||^{\Gamma_0}_{s-\frac{3}{2}}.$$

The proof of (3.27) is completed now by another application of the regularity result on the Dirichlet problem.

To show (3.22), consider the "adjoint" problem

$$\left.\begin{array}{rcl} -\Delta v + \nabla q &=& u \\ \operatorname{div} v &=& 0 \end{array}\right\} \text{ in } \Omega_0$$
$$T(v,q)n = -\mu \rfloor n \quad \text{on } \Gamma_0,$$

with $\mu \in \mathbb{R}^{\binom{N}{2}}$ given by

$$\mu_j = \int_{\Omega_0} u \cdot v_{2,j}\,dx, \qquad (1 \le j \le \binom{N}{2})$$

which implies $||\mu||_{\mathbb{R}^{\binom{N}{2}}} \le C||u||^{\Omega_0}_0$. By examining the variational formulation of this problem in the same way as in Lemma 10 we find the existence of a weak solution of it that satisfies

$$\varphi_1(v) = 0, \quad \varphi_2(v) = 0.$$

By the above regularity results we get

$$||v||^{\Omega_0}_2 \le C\left(||u||^{\Omega_0}_0 + ||\mu \rfloor n||^{\Gamma_0}_{\frac{1}{2}}\right) \le C||u||^{\Omega_0}_0.$$

With this, we find by the second Green formula for the Stokes equations

$$\begin{aligned} \|u\|_0^{\Omega_0 2} &= (u, -\Delta v + \nabla q)_0 + \int_{\Gamma_0} (\mathcal{T}(u,p)n \cdot v - \mathcal{T}(v,q)n \cdot u)\, d\Gamma \\ &= \int_{\Gamma_0} (F_B \cdot v + \mu \rfloor n)\, d\Gamma = \int_{\Gamma_0} F_B \cdot v\, d\Gamma - \mu \cdot \varphi_2(u) \\ &\leq C\|F_B\|_{-\frac{3}{2}}^{\Gamma_0} \|v\|_{\frac{3}{2}}^{\Gamma_0} \leq C\|F_B\|_{-\frac{3}{2}}^{\Gamma_0} \|v\|_2^{\Omega_0} \leq C\|F_B\|_{-\frac{3}{2}}^{\Gamma_0} \|u\|_0^{\Omega_0}, \end{aligned}$$

which proves (3.22). ■

The Neumann-to-Dirichlet operator

We will conclude the discussion of the fixed-time problem with the proof of a commutator property for the operator that describes the correspondence between the Neumann and Dirichlet boundary data for the Stokes equations. More precisely, we consider for $f \in (H^s(\Gamma_0))^N$, $s \geq -\frac{1}{2}$, the problem

$$L[u\ p\ \lambda]^T = [\tilde{f}\ 0] \tag{3.31}$$

with

$$\tilde{f}(v) = \int_{\Gamma_0} fv\, d\Gamma$$

and define

$$Sf = u|_{\Gamma_0}. \tag{3.32}$$

Combining the results about the weak and strong formulations for the fixed-time problem with an interpolation argument we get that S is a well-defined operator in

$$\mathcal{L}\left((H^s(\Gamma_0))^N, (H^{s+1}(\Gamma_0))^N\right)$$

for all $s \geq -\frac{1}{2}$.

We recall the definition of the operators Λ^σ from (3.3) and note the fact that they are pseudodifferential operators of order σ [87].

Lemma 12 *(Commutator property)*
For any $s, \sigma \in \mathbf{R}$ such that $s + \sigma \geq -\frac{1}{2}$ the operator $S\Lambda^\sigma - \Lambda^\sigma S$ is in

$$\mathcal{L}\left((H^s(\Gamma_0))^N, (H^{s-\sigma+2}(\Gamma_0))^N\right).$$

Proof: Even for $s < \frac{1}{2}$ it is enough to show the inequality

$$\|(S\Lambda^\sigma - \Lambda^\sigma S)f\|_{s-\sigma+2}^{\Gamma_0} \leq C_{s,\sigma}\|f\|_s^{\Gamma_0}$$

for all $f \in \left(H^{\frac{1}{2}}(\Gamma_0)\right)^N$, hence we are allowed to work with the strong formulation of the problem (3.31).

Based on the above discussion of the regularity properties of our boundary value problem we find that

$$Sf = \left(u_2 - \lambda_{1k}(f)u^{(k)}\right)\Big|_{\Gamma_0},$$

where $(u^{(k)}, p^{(k)})$ are the solution of the Dirichlet problems

$$\left.\begin{array}{rcl} -\Delta u^{(k)} + \nabla p^{(k)} & = & e_k \\ \operatorname{div} u^{(k)} & = & 0 \end{array}\right\} \quad \text{in } \Omega_0,$$
$$u^{(k)} = 0 \qquad \text{on } \Gamma_0,$$
$$\textstyle\int_{\Omega_0} p^{(k)}\, dx = 0,$$

(u_2, p_2) is the solution of

$$\begin{array}{rcll} -\Delta u_2 + \nabla p_2 & = & 0 & \multirow{2}{*}{$\Big\}$ in Ω_0,} \\ \operatorname{div} u_2 & = & 0 & \\ \mathcal{T}(u_2, p_2)n = \lambda_{1k}(f)\mathcal{T}(u^{(k)}, p^{(k)})n - \lambda_2(f) \rfloor\, n + f & = & Pf & \text{on } \Gamma_0, \\ \varphi_i(u) = \lambda_{1k}(f)\varphi_i(u^{(k)}) & = & h_i(f) & (i = 1, 2), \end{array}$$

and

$$\lambda_{ij}(f) = \int_{\Gamma_0} f \cdot v_{ij}\, d\Gamma. \qquad (i = 1, 2).$$

Note that for all $\theta \in \mathbb{R}$

$$\begin{aligned} \|h_1(f)\|_{\mathbb{R}^N} + \|h_2(f)\|_{\mathbb{R}^{\binom{N}{2}}} &\le C_\theta \|f\|_\theta^{\Gamma_0}, \\ \|(\Lambda^\sigma P - P\Lambda^\sigma) f\|_s^{\Gamma_0} &\le C_{\theta,\sigma,s} \|f\|_\theta^{\Gamma_0}. \end{aligned} \tag{3.33}$$

Using hydrodynamic potentials as above and writing $(V\psi)(x) = V(x, \psi)$ for $x \in \Gamma_0$ one finds

$$Sf = V\psi - \lambda_{1k}(f)u^{(k)},$$

where ψ is the (unique) solution of

$$\begin{aligned} \left(\frac{1}{2}I + K\right)\psi &= Pf \\ \varphi_i(V\psi) &= h_i(f). \end{aligned} \tag{3.34}$$

From the above discussion of this problem we recall

$$\|\psi\|_s^{\Gamma_0} \le C\|f\|_s^{\Gamma_0}. \tag{3.35}$$

In the same way we get

$$S\Lambda^\sigma f = V\zeta - \sum_k \lambda_{1k}(\Lambda^\sigma f)u^{(k)},$$

where ζ is the (unique) solution of

$$\begin{aligned} \left(\frac{1}{2}I + K\right)\zeta &= P\Lambda^\sigma f \\ \varphi_i(V\zeta) &= h_i(\Lambda^\sigma f). \end{aligned} \tag{3.36}$$

Applying Λ^σ to (3.34) yields

$$\left(\frac{1}{2}I + K\right)\Lambda^\sigma\psi = \Lambda^\sigma Pf + (K\Lambda^\sigma - \Lambda^\sigma K)\psi$$

and after subtracting (3.36)

$$\left(\frac{1}{2}I+K\right)(\Lambda^\sigma\psi-\zeta)=(\Lambda^\sigma P-P\Lambda^\sigma)f+(K\Lambda^\sigma-\Lambda^\sigma K)\psi,$$

hence, using (3.33), (3.35), and the fact that K is a pseudodifferential operator of order 0,

$$||\Lambda^\sigma\psi-\zeta||_{s+1-\sigma}^{\Gamma_0}\leq C||f||_s^{\Gamma_0}$$

and from this, using that V is a pseudodifferential operator of order -1,

$$\begin{aligned}
&||\Lambda^\sigma Sf-S\Lambda^\sigma f||_{s+2-\sigma}^{\Gamma_0}\\
=\ &\left\|\Lambda^\sigma V\psi-\lambda_{1k}(f)\Lambda^\sigma u^{(k)}-V\zeta+\lambda_{1k}(\Lambda^\sigma f)u^{(k)}\right\|_{s+2-\sigma}^{\Gamma_0}\\
=\ &\left\|V(\Lambda^\sigma\psi-\zeta)+\left(\lambda_{1k}(\Lambda^\sigma f)u^{(k)}-\lambda_{1k}(f)\Lambda^\sigma u^{(k)}\right)+(\Lambda^\sigma V-V\Lambda^\sigma)\psi\right\|_{s+2-\sigma}^{\Gamma_0}\\
\leq\ &C_s||f||_s^{\Gamma_0}. \quad \blacksquare
\end{aligned}$$

Remark: It is clear from the proof that the same commutation relation holds with Λ^σ replaced by an arbitrary pseudodifferential operator of order σ. □

3.2.2 The fixed-time problem for Hele-Shaw flow

For the discussion of the fixed-time problem (3.14) it is sufficient to invoke standard results from the theory of elliptic boundary value problems. Moreover, there is no need for an explicit discussion of the weak formulation.

Lemma 13 *(The fixed-time problem for Hele-Shaw flow)*
(i) (Existence, uniqueness, and regularity)
For all $s\geq 2$, the mapping

$$L:\ H^s(\Omega_0)\longrightarrow H^{s-2}(\Omega_0)\times H^{s-\frac{1}{2}}(\Gamma_0)$$

defined by

$$Lu=\begin{bmatrix}\Delta u\\ u|_{\Gamma_0}\end{bmatrix}$$

is a homeomorphism.
(ii) (The Dirichlet-to-Neumann operator)
The operator $S:\ H^{s-\frac{1}{2}}(\Gamma_0)\longrightarrow H^{s-\frac{3}{2}}(\Gamma_0)$ defined by

$$S\phi=\frac{\partial}{\partial n}\left(L^{-1}\begin{bmatrix}0\\ \phi\end{bmatrix}\right) \tag{3.37}$$

is a pseudodifferential operator of order 1 in the manifold Γ_0.

Proofs of (i) and (ii) can be found e.g. in [59] and [87], respectively.
Remark: It follows from (ii) that S can be extended to $H^{\frac{1}{2}}(\Gamma_0)$ with values in $H^{-\frac{1}{2}}(\Gamma_0)$. □

3.3 Some elementary consequences

Even in this stadium of the discussion it is possible to give some qualitative properties of the evolution problems we are interested in. This will be done under the assumption of existence of (sufficiently smooth) solutions of the FBPs under consideration. In particular, we will find that the surface motions corresponding to them conserve the enclosed volume and diminish the area of the surface, and that all stationary solutions are given by those domains whose boundaries have constant curvature, i.e. by the circles for $N = 2$ and the balls for $N = 3$ (see [14] § 100).

As we are dealing in this section with domains that are not necessarily infinitely smooth, we modify the definitions of the norms in the spaces $H^{\frac{1}{2}}(\Gamma(t))$ and $H^{-\frac{1}{2}}(\Gamma(t))$. Given a norm on $H^{\frac{1}{2}}(\Gamma(t))$, we choose on $H^{-\frac{1}{2}}(\Gamma(t))$ the dual norm given (for sufficiently smooth v) by

$$||v||_{-\frac{1}{2}}^{\Gamma(t)} = \sup_{||\Phi||_{\frac{1}{2}}^{\Gamma(t)}=1} \int_{\Gamma(t)} v\Phi \, d\Gamma(t).$$

3.3.1 Stokes flow

Let us assume in this subsection that $u(\cdot,t), p(\cdot,t), \Omega(t), t \in [0,T)$ where $T = +\infty$ is allowed, are parametrized families of sufficiently smooth functions and domains such that (1.4)-(1.7) are satisfied,

$$\int_{\Omega(t)} u \, dx = 0, \quad \int_{\Omega(t)} \operatorname{rot} u \, dx = 0,$$

and the functions

$$\left.\begin{array}{rcl} A(t) & = & \int_{\Gamma(t)} d\Gamma(t), \\ V(t) & = & \int_{\Omega(t)} dx, \\ M(t) & = & \int_{\Omega(t)} dx \end{array}\right\} \tag{3.38}$$

representing the surface area, volume, and center of gravity of $\Omega(t)$, respectively, are differentiable with respect to t. (Here and in the following, we suppress the variable t in the notation of u.) Let the bilinear form a_t be defined as a in Section 3.2 with Ω_0 replaced by $\Omega(t)$. We recall that

$$a_t(u,u) \geq c_t ||u||_1^{\Omega(t)^2}, \tag{3.39}$$

where c_t depends on the domain $\Omega(t)$ only. Moreover, a standard result from the theory of elliptic boundary value problems ensures that if $g \in H^{\frac{1}{2}}(\Gamma(t))$ and $\int_{\Gamma(t)} g \, dx = 0$, then the Neumann problem

$$\begin{array}{rcl} \Delta\Phi & = & 0 \text{ in } \Omega(t), \\ \dfrac{\partial\Phi}{\partial n} & = & g \text{ on } \Gamma(t), \\ \int_{\Omega(t)} \Phi \, dx & = & 0 \end{array} \tag{3.40}$$

has a solution $\Phi \in H^2(\Omega(t))$ for which an estimate

$$||\Phi||_2^{\Omega(t)} \leq C_t ||g||_{\frac{1}{2}}^{\Gamma(t)} \tag{3.41}$$

holds with C_t depending only on $\Omega(t)$.

Lemma 14 *(Global properties of Stokes flow)*

(i) (Conserved quantities and surface diminishing)

The functions V and M are constant. The function A is nonincreasing in t.

(ii) (Asymptotic behavior)

Suppose $T = +\infty$, $\dot{A} = \frac{dA}{dt}$ is uniformly Lipschitz-continuous and

$$\max\left\{\frac{1}{c_t}, C_t, \|\mathbf{1}\|_{\frac{1}{2}}^{\Gamma(t)}, \|\mathbf{1}\|_{-\frac{1}{2}}^{\Gamma(t)}\right\} \leq K \quad \forall t \geq 0, \tag{3.42}$$

where c_t and C_t are the constants from (3.39) and (3.41), respectively, and $\mathbf{1}$ denotes the constant function mapping $\Gamma(t)$ to 1. Then $\|\kappa(t) - \overline{\kappa}(t)\|_{-\frac{1}{2}}^{\Gamma(t)} \to 0$ as $t \to +\infty$ with

$$\overline{\kappa}(t) = \frac{1}{|\Gamma(t)|}\int_{\Gamma(t)} \kappa(t)\, d\Gamma(t). \tag{3.43}$$

(iii) (Stationary solutions)

The function A is constant in time only at stationary solutions, and if a domain Ω yields a stationary solution, then $\partial\Omega$ has constant curvature.

Proof: Let $t \in [0, T)$ be fixed. We find from the fixed-time problems

$$\begin{aligned}
\frac{dV}{dt} &= \int_{\Omega(t)} \operatorname{div} u\, dx = 0,\\
\frac{dA}{dt} &= \int_{\Gamma(t)} \kappa(t) n(t)\cdot u\, d\Gamma(t) = -a_t(u,u) \leq 0,\\
\frac{dM}{dt} &= \int_{\Omega(t)} u\, dx + \int_{\Omega(t)} x \operatorname{div} u\, dx = 0.
\end{aligned}$$

This implies (i) and, for $T = +\infty$ and $\dot{A}$ uniformly Lipschitz-continuous,

$$\dot{A}(t) \to 0 \quad \text{as } t \to +\infty \tag{3.44}$$

as A is obviously bounded from below.

Due to the duality between the spaces $H^{-\frac{1}{2}}(\Omega(t))$ and $H^{\frac{1}{2}}(\Omega(t))$ there is a $\phi \in H^{\frac{1}{2}}(\Omega(t))$ such that $\|\phi\|_{\frac{1}{2}}^{\Gamma(t)} = 1$ and

$$\|\kappa(t) - \overline{\kappa}(t)\|_{-\frac{1}{2}}^{\Gamma(t)} = \int_{\Gamma(t)} (\kappa(t) - \overline{\kappa}(t))\phi\, d\Gamma(t).$$

Let $\overline{\phi} = \frac{1}{|\Gamma(t)|}\int_{\Gamma(t)} \phi\, d\Gamma(t)$ and set $v = \nabla\Phi$, where Φ is the solution of (3.40) with $g = \phi - \overline{\phi}$. From (3.41) and (3.42) we get

$$\begin{aligned}
\|\overline{\phi}\|_{\frac{1}{2}}^{\Gamma(t)} &= \overline{\phi}\|\mathbf{1}\|_{\frac{1}{2}}^{\Gamma(t)} = \frac{\|\mathbf{1}\|_{\frac{1}{2}}^{\Gamma(t)}}{A(t)}\int_{\Gamma(t)} \phi\, d\Gamma(t) \leq C\|\mathbf{1}\|_{\frac{1}{2}}^{\Gamma(t)}\|\mathbf{1}\|_{-\frac{1}{2}}^{\Gamma(t)}\|\phi\|_{\frac{1}{2}}^{\Gamma(t)} \leq CK^2,\\
\|v\|_1^{\Omega(t)} &\leq K\|\phi - \overline{\phi}\|_{\frac{1}{2}}^{\Gamma(t)} \leq CK^3
\end{aligned}$$

with C independent of t.

Using this we find from (3.39) and (3.42)

$$\begin{aligned}||\kappa(t)-\overline{\kappa}(t)||_{-\frac{1}{2}}^{\Gamma(t)} &= \int_{\Gamma(t)} \kappa(t)n(t)\cdot v\, d\Gamma(t) - \overline{\kappa}(t)\int_{\Gamma(t)} \frac{\partial\Phi}{\partial n}\, d\Gamma(t) = a_t(u,v) \\ &\leq CK^3||u||_1^{\Omega(t)} \leq CK^{\frac{7}{2}}a_t(u,u)^{\frac{1}{2}} = CK^{\frac{7}{2}}\left(-\dot{A}(t)\right)^{\frac{1}{2}}.\end{aligned}$$

This implies (ii) because of (3.44) and (iii) because for stationary solutions A is obviously constant. (On the other hand, if $\kappa(t) = \overline{\kappa}$ is constant on $\Gamma(t)$, then $u = 0$, $p = -\overline{\kappa}$, hence we have a stationary solution.) ■

Remarks: The mathematical content of this lemma is limited because of the strong and hardly verifiable assumptions in (ii). However, it provides a mathematical formulation for the consequences of the energy balance considerations familiar in physics. In particular, the equation

$$a_t(u,u) + \dot{A}(t) = 0$$

has an explicit interpretation in physical terms: At any instant of time, the amount of energy "produced" by area diminishing is dissipated by inner friction. This observation is the basis for the viscous sintering models that are used in material science [31, 77]. It is noteworthy that in these models neither the Stokes equations nor the curvature of the boundary occur explicitly. □

3.3.2 Hele-Shaw flow

By parallel reasonings we can obtain very similar results on Hele-Shaw flow. Let $u(\cdot,t)$ and $\Omega(t)$ be families of sufficiently smooth functions and domains parametrized by $t \in [0,T)$ and satisfying

$$\begin{aligned} \Delta u &= 0 && \text{in } \Omega(t), \\ u &= \kappa(t) && \text{on } \Gamma(t), \\ V_n &= \frac{\partial u}{\partial n(t)} && \text{on } \Gamma(t), \end{aligned}$$

recall that the velocity field inside the liquid domain is given by $v = \nabla u$ and define A, V, M, and $\overline{\kappa}(t)$ as in (3.38) and in (3.43), respectively. As in the case of Stokes flow, we assume differentiability of V, A, and M with respect to t. Recall that the trace theorem yields

$$||w||_{\frac{1}{2}}^{\Gamma(t)} \leq C_{1,t}||w||_1^{\Omega(t)} \quad \forall w \in H^1(\Omega(t)) \tag{3.45}$$

and we have, as a consequence of the Bramble-Hilbert lemma, the Poincaré-type inequality ([93] Vol II/B, Appendix, (53a))

$$||w||_1^{\Omega(t)} \leq C_{2,t}\left(||\nabla w||_0^{\Omega(t)} + \left|\int_{\Gamma(t)} u\, d\Gamma(t)\right|\right) \tag{3.46}$$

Lemma 15 *(Global properties of Hele-Shaw flow)*

(i) (Conserved quantities and surface diminishing)
The functions V and M are constant. The function A is nonincreasing in t.
(ii) (Asymptotic behavior)
Suppose $T = +\infty$, $\dot{A} = \frac{dA}{dt}$ is uniformly Lipschitz-continuous and

$$\max\{C_{1,t}, C_{2,t}\} \leq K \qquad \forall t \geq 0 \tag{3.47}$$

for the constants in the inequalities (3.45) *and* (3.46). *Then* $\|\kappa(t)-\overline{\kappa}(t)\|_{\frac{1}{2}}^{\Gamma(t)} \to 0$ *as* $t \to +\infty$.

(iii) (Stationary solutions)

The function A is constant in time only at stationary solutions, and if a domain Ω yields a stationary solution, then $\partial\Omega$ has constant curvature.

Proof: (i) can be shown directly by calculating, using integration by parts,

$$
\begin{aligned}
\frac{dV}{dt} &= \int_{\Omega(t)} \operatorname{div} v \, dx = 0, \\
\frac{dM}{dt} &= \int_{\Omega(t)} v \, dx + \int_{\Omega(t)} x \operatorname{div} v \, dx = \int_{\Omega(t)} \nabla u \, dx \\
&= \int_{\Gamma(t)} \kappa n(t) \, d\Gamma(t) = \int_{\Gamma(t)} \Delta_{\Gamma(t)} x \, d\Gamma(t) = 0, \\
\frac{dA(t)}{dt} &= -\int_{\Gamma(t)} \kappa(t) n(t) \cdot v \, d\Gamma(t) = -\int_{\Gamma(t)} u \frac{\partial u}{\partial n(t)} \, d\Gamma(t) \\
&= -\int_{\Omega(t)} |\nabla u|^2 \, dx \le 0.
\end{aligned} \tag{3.48}
$$

As in the Stokes flow case this implies (3.44). Furthermore, using (3.45), (3.46), and (3.47),

$$
\begin{aligned}
\|\kappa(t)-\overline{\kappa}(t)\|_{\frac{1}{2}}^{\Gamma(t)} &\le K\|u-\overline{\kappa}(t)\|_1^{\Omega(t)} \\
&\le K^2\left(\|\nabla(u-\overline{\kappa}(t))\|_0^{\Omega(t)} + \int_{\Gamma(t)} (\kappa(t)-\overline{\kappa}(t)) \, d\Gamma(t)\right) \\
&= K^2\left(\int_{\Omega(t)} |\nabla u|^2 \, dx\right)^{\frac{1}{2}} = K^2\left(-\dot{A}(t))^{\frac{1}{2}}\right),
\end{aligned}
$$

and (ii) and (iii) follow from this as in the proof of Lemma 14. ■

3.4 Dependence on perturbations

In our case, the application of the direct mapping method proceeds as follows: A given perturbation $r \in H^s(\Gamma_0)$, $s > 2 + \frac{N}{2}$, $\|r\|_s^{\Gamma_0}$ small, defines by Lemma 5 a domain Ω_r on which we consider the fixed-time problems. Using the diffeomorphism $z(r)$, these problems can be "pulled back" to the fixed domain Ω_0 and written as operator equations in which r occurs as a parameter. For $r = 0$ we recover the fixed-time problems on Ω_0 that have been discussed in Section 3.2. Near $r = 0$, the investigation of the dependence of all occurring operators and functions on r will yield the necessary information about the solutions of the fixed-time problems as functions of r, i.e. "of the domain".

Suppose $s > 2 + \frac{N}{2}$ and let Ω_r and $z = z(r)$ be defined by Lemma 5. We denote by $\tilde{\kappa}(r)$ and $\tilde{\nu}(r)$ the curvature and the outer normal vector of Ω_r, respectively, considered as functions on Γ_r. On Γ_0 we define

$$
\begin{aligned}
\kappa(r) &= \tilde{\kappa}(r) \circ z, \\
\nu(r) &= \tilde{\nu}(r) \circ z.
\end{aligned}
$$

Lemma 16 *(Perturbation of outer normal and curvature)*
(i) The mapping $r \mapsto \nu(r)$ is analytic near 0 from $H^s(\Gamma_0)$ to $\left(H^{s-1}(\Gamma_0)\right)^N$.
(ii)The mapping $r \mapsto \kappa(r)$ is analytic near 0 from $H^s(\Gamma_0)$ to $H^{s-2}(\Gamma_0)$.

Proof: (i) Let $\Gamma_0 = \bigcup_m \Gamma_0^{(m)}$ be a finite covering of Γ_0 by coordinate patches $\Gamma_0^{(m)}$ and $\{\chi_m\}$ a smooth partition of unity subordinate to it. Let $\xi^{(m)} = \xi^{(m)}(w)$, $w \in \tilde{W}_m$, be smooth regular parametrizations of $\Gamma_0^{(m)}$. Let $W_m \subset \tilde{W}_m$ be a domain such that

$$\operatorname{supp} \chi_m \circ \xi^{(m)} \subset W_m \subset \overline{W_m} \subset \tilde{W}_m.$$

Without loss of generality one can assume that the W_m are bounded and have smooth boundary. Moreover, for all m we choose functions $\psi_m \in C_0^\infty\left(\Gamma_0^{(m)}\right)$ such that $\psi_m \equiv 1$ in $\operatorname{supp} \chi_m$. Note that

$$\nu(r) = \sum_m \chi_m \left(\nu^{(m)}(r) \circ \left(\xi^{(m)}\right)^{-1}\right),$$

where $\nu = \nu^{(m)}(r)$ is the solution of

$$F^{(m)}(r,\nu) = \begin{bmatrix} \left(\dfrac{\partial\left(\xi^{(m)} + (\psi_m r\zeta)\circ\xi^{(m)}\right)}{\partial w}\right)^T \\ \nu^T \end{bmatrix} \nu - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = 0 \tag{3.49}$$

and $\nu^{(m)}(0) = n \circ \xi^{(m)} = n^{(m)}$.

Due to the well-known results concerning equivalence of norms for Sobolev spaces on manifolds it is sufficient to show that the mapping $r \mapsto \nu^{(m)}(r)$ is analytic near 0 from $H^s(\Gamma_0)$ to $\left(H^{s-1}(W_m)\right)^N$ for any m. Let m be fixed and note that $F^{(m)}$ maps

$$(r,\nu) \in H^s(\Gamma_0) \times \left(H^{s-1}(W_m)\right)^N$$

analytic near $(0, n^{(m)})$ into $\left(H^{s-1}(W_m)\right)^N$ because $H^{s-1}(W_m)$ is a Banach algebra. Furthermore, for the Fréchet derivative of $F^{(m)}$ at $(0, n^{(m)})$ with respect to the second argument we get

$$D_2 F^{(m)}(0, n^{(m)})h = \begin{bmatrix} \left(\dfrac{\partial \xi^{(m)}}{\partial w}\right)^T \\ 2n^{(m)T} \end{bmatrix} h.$$

Due to the regularity of the parametrization $\xi^{(m)}$, the matrix defining $D_2F^{(m)}(0, n^{(m)})$ is nonsingular on $\tilde{W}_m$ and all its elements are smooth functions, hence $D_2F^{(m)}(0, n^{(m)})$ is an homeomorphism of $\left(H^{s-1}(W_m)\right)^N$. By Lemma 6 this implies the analyticity of $\nu^{(m)}(r)$ as a function of r near 0.

(ii) Suppressing for the sake of brevity the composition with $\xi^{(m)}$ and the dependence of G, g, and g^{ij} on m in the notation we have, for any fixed m, on W_m (cf. (3.2))

$$\chi_m \kappa(r) = \frac{\chi_m}{\sqrt{g(\psi_m r)}} \frac{\partial}{\partial w_i}\left(\sqrt{g(\psi_m r)} g^{ij}(\psi_m r) \frac{\partial(\xi^{(m)} + \psi_m r \zeta)}{\partial w_j}\right) \cdot \nu^{(m)}(\psi_m r)$$

with

$$\begin{aligned}
g(R) &= \det G(R), \\
g^{ij}(R) &= [G(R)^{-1}]_{ij}, \\
G(R) &= \left(\frac{\partial(\xi^{(m)} + R\zeta)}{\partial w}\right)^T \left(\frac{\partial(\xi^{(m)} + R\zeta)}{\partial w}\right).
\end{aligned}$$

It is clear that the mapping $r \mapsto \psi_m r$ is analytic from $H^s(\Gamma_0)$ to $H^s(W_m)$ and the mappings $R \mapsto g(R)$ and $R \mapsto G(R)$ are analytic near 0 from $H^s(W_m)$ to the Banach algebras $H^{s-1}(W_m)$ and $\left(H^{s-1}(W_m)\right)^{N\times N}$. The regularity of the parametrization implies that $G(0)$ is smooth and invertible on $\tilde{W}_m$, hence its restriction to W_m is invertible in $\left(H^{s-1}(W_m)\right)^{N\times N}$. It follows from Lemmas 6 and 7 together with (i) and the facts that both $H^{s-1}(W_m)$ and $H^{s-2}(W_m)$ are Banach algebras that $(\chi_m \kappa(r)) \circ \xi^{(m)}$ depends analytically on $r \in H^s(\Gamma_0)$ near 0 with values in $H^{s-2}(W_m)$, and the assertion follows from this. ■

We recall the definitions of the spaces X, X_s, Y_s from Section 3.2. As announced, we have to investigate the solution $(\tilde{u}, \tilde{p}, \tilde{\lambda}) \in X$ of the fixed-time problem

$$\begin{array}{rl}
\left.\begin{array}{rcl} -\Delta\tilde{u} + \nabla\tilde{p} + \tilde{\lambda}_1 &=& 0 \\ -\operatorname{div}\tilde{u} &=& 0 \end{array}\right\} & \text{in } \Omega_r, \\
\mathcal{T}(\tilde{u},\tilde{p})\tilde{\nu}(r) + \tilde{\lambda}_2 \rfloor \tilde{\nu}(r) = \tilde{\kappa}(r)\tilde{\nu}(r) & \text{on } \Gamma_r, \\
\int_{\Omega_r} \tilde{u}\, dx = 0, & \\
\int_{\Omega_r} \operatorname{rot}\tilde{u}\, dx = 0. &
\end{array}$$

Transformation to Ω_0 using the diffeomorphism z yields

$$\tilde{L}(r)\begin{bmatrix} u(r) \\ p(r) \\ \lambda(r) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \kappa(r)\nu(r) \\ 0 \\ 0 \end{bmatrix}, \tag{3.50}$$

where

$$u(r) = \tilde{u} \circ z, \quad p(r) = \tilde{p} \circ z, \quad \lambda(r) = \tilde{\lambda} \circ z,$$

$$\begin{aligned}
\tilde{L}(r)\begin{bmatrix} u \\ p \\ \lambda \end{bmatrix} &= \begin{bmatrix} -\Delta_r u + \nabla_r p + \lambda_1 \\ -\operatorname{div}_r u \\ \mathcal{T}_r(u,p)\nu(r) + \lambda_2 \rfloor \nu(r) \\ \int_{\Omega_0} u \det \mathcal{A}\, dx \\ \int_{\Omega_0} \operatorname{rot}_r u \det \mathcal{A}\, dx \end{bmatrix}, \\
(\nabla_r p)_i &= a^{ji}\frac{\partial p}{\partial x_j}, \\
\operatorname{div}_r u &= a^{ji}\frac{\partial u_i}{\partial x_j}, \\
(\operatorname{rot}_r u)_{K(i,j)} &= a^{li}\frac{\partial u_j}{\partial x_l} - a^{lj}\frac{\partial u_i}{\partial x_l}, \\
(\Delta_r u)_i &= a^{jl}\frac{\partial}{\partial x_j}\left(a^{kl}\frac{\partial u_i}{\partial x_k}\right),
\end{aligned}$$

$$\begin{aligned}
(\nabla_r u)_{ij} &= a^{kj}\frac{\partial u_i}{\partial x_k}, \\
\mathcal{T}_r(u,p) &= (\nabla_r u) + (\nabla_r u)^T - pI, \\
\mathcal{A} &= \frac{\partial z}{\partial x}, \\
a^{ij} &= \left[\mathcal{A}^{-1}\right]_{ij}.
\end{aligned}$$

Lemma 17 *(Perturbation of the Stokes equations)*

(i) The mapping $r \mapsto \tilde{L}(r)$ is analytic near 0 from $H^{s-\frac{1}{2}}(\Gamma_0)$ to $\mathcal{L}(X_s, Y_s)$.

(ii) For sufficiently small $r \in H^{s+\frac{1}{2}}(\Gamma_0)$, $u(r) \in (H^s(\Omega_0))^N$ is well-defined. The mapping $r \mapsto u(r)$ is analytic near 0 from $H^{s+\frac{1}{2}}(\Gamma_0)$ to $(H^s(\Omega_0))^N$.

Proof: (i) The mapping $r \mapsto z(r)$ is linear and continuous, hence analytic from $H^{s-\frac{1}{2}}(\Gamma_0)$ to $(H^s(\Omega_0))^N$. Consequently, $r \mapsto \mathcal{A}(r)$ is analytic from $H^{s-\frac{1}{2}}(\Gamma_0)$ to $\left(H^{s-1}(\Omega_0)\right)^{N\times N}$ which is a Banach algebra. Note that $\mathcal{A}(0)$ is the identity, hence by Lemma 7 the mappings $r \mapsto \mathcal{A}(r)^{-1}$ and $r \mapsto a^{ij}(r)$ are analytic near 0 from $H^{s-\frac{1}{2}}(\Gamma_0)$ to $\left(H^{s-1}(\Omega_0)\right)^{N\times N}$ and $H^{s-1}(\Omega_0)$, respectively. The assertion follows now straightforwardly from the Banach algebra properties of $H^{s-2}(\Omega_0)$, $H^{s-1}(\Omega_0)$, and $H^{s-\frac{3}{2}}(\Gamma_0)$.

(ii) For any sufficiently small $r \in H^{s+\frac{1}{2}}(\Gamma_0)$, $u(r)$ is the first component in the solution $v \in X_s$ of the equation

$$F(r,v) = \tilde{L}(r)v - \begin{bmatrix} 0 \\ 0 \\ \kappa(r)\nu(r) \\ 0 \\ 0 \end{bmatrix}.$$

According to Lemma 16 and (i) together with the Banach algebra property of $H^{s-\frac{3}{2}}(\Gamma_0)$, F is analytic near $\left(0, \begin{bmatrix} u_0 \\ p_0 \\ 0 \end{bmatrix}\right)$ into Y_s, where u_0 and p_0 denote the velocity and pressure components of the solution of the fixed-time problem on Ω_0. Clearly the Fréchet derivative of F with respect to v at this point is $\tilde{L}(0)$ which is a homeomorphism by Lemma 11. Hence by Lemma 6 (vi), v depends analytically near 0 on $r \in H^{s+\frac{1}{2}}(\Gamma_0)$ with values in X_s which implies the assertion. ■

In an analogous way, for the Hele-Shaw flow problem we have to discuss the Dirichlet boundary value problem for the Laplacian

$$\begin{aligned}
\Delta\tilde{u} &= 0 && \text{in } \Omega_r \\
\tilde{u} &= \tilde{\kappa}(r) && \text{on } \Gamma_r
\end{aligned}$$

which is, after transformation by z to Ω_0,

$$L(r)u(r) = \begin{bmatrix} 0 \\ \kappa(r) \end{bmatrix}$$

with

$$\begin{aligned}
u(r) &= \tilde{u} \circ z, \\
L(r)u &= \begin{bmatrix} \Delta_r u \\ u|_{\Gamma_0} \end{bmatrix},
\end{aligned}$$

$$\Delta_r = a^{jl} \frac{\partial}{\partial x_j} \left(a^{kl} \frac{\partial u}{\partial x_k} \right),$$

where the a^{ij} are defined as in the Stokes flow problem.

Lemma 18 *(Perturbation of the Laplace equation)*
(i) The mapping $r \mapsto L(r)$ is analytic near 0 from $H^{s-\frac{1}{2}}(\Gamma_0)$ to

$$\mathcal{L}\left(H^s(\Omega_0),\ H^{s-2}(\Omega_0) \times H^{s-\frac{1}{2}}(\Gamma_0)\right).$$

(ii) For sufficiently small $r \in H^{s+\frac{3}{2}}(\Gamma_0)$, $u(r) \in H^s(\Omega_0)$ is well-defined. The mapping $r \mapsto u(r)$ is analytic near 0 from $H^{s+\frac{3}{2}}(\Gamma_0)$ to $H^s(\Omega_0)$.

Proof: (i) can be proved in complete analogy to the proof of Lemma 17 (i). The assertion of (ii) follows from Lemma 16 (ii) together with the application of the Implicit Function theorem (Lemma 6 (vi)) to the equation

$$F(r, v) = L(r)v - \begin{bmatrix} 0 \\ \kappa(r) \end{bmatrix}$$

near $(0, u_0)$, where u_0 is the solution of the fixed-time problem on Ω_0. The nondegeneracy condition is the bijectivity of $L(0)$. It is satisfied due to Lemma 13 (i). ■

3.5 Evolution of the perturbation function

We are able now to formulate the moving boundary problem in terms of an evolution equation for the perturbation functions r by allowing it to depend on time. With a slight change of notation, from now on we consider r as a function from a time interval $[0, T)$ into $B_0(\delta_0, H^s(\Gamma_0))$, where δ_0 is given by Lemma 5. For $t \in [0, T)$, $\xi \in \Gamma_0$, we will write $r(t, \xi)$ instead of $(r(t))(\xi)$. Let $\Omega(t)$, $t \in [0, T)$ be a given family of domains evolving according to (1.4), (1.5), such that for all t there is an $r(t) \in B_0(\delta_0, H^s(\Gamma_0))$ with

$$\Omega(t) = \Omega_{r(t)}, \tag{3.51}$$

where the right side is defined as in Lemma 5. Our aim is to express the time derivative of r in terms of v and geometric quantities. (We will assume the existence of all occurring derivatives.)

Introducing Lagrangian coordinates $p \in \Gamma(0)$ for the parametrization of $\Gamma(t)$ we have

$$\Gamma(t) = \{x(p, t) \,|\, p \in \Gamma(0)\}$$

with

$$\begin{aligned} x(0, p) &= p, \\ \frac{\partial x}{\partial t}(p, t) &= v(x(p, t), t). \end{aligned} \tag{3.52}$$

On the other hand, we get from Lemma 5 and (3.51) another parametrization, namely

$$\Gamma(t) = \{\xi + r(t, \xi) \,|\, \xi \in \Gamma_0\}.$$

Due to the globality of both parametrizations and the differentiability assumptions there is a differentiable function $q(\cdot,t)$ from Γ_0 to $\Gamma(0)$ such that

$$x(q(\xi,t),t) = \xi + r(t,\xi)\zeta(\xi).$$

Differentiating this equation with respect to t and using (3.52) yields

$$\frac{\partial x}{\partial p}\frac{\partial q}{\partial t} + v(\xi + r(t,\xi)\zeta(\xi)) = \frac{\partial r}{\partial t}(t,\xi)\zeta(\xi).$$

The first term on the left side represents a vector tangential to $\Gamma(t)$ at $\xi + r(t,\xi)\zeta(\xi)$, hence multiplication with $\nu(r(t))(\xi)$, the outer normal of $\Gamma(t)$ in this point, yields (on Γ_0)

$$\frac{\partial r}{\partial t} = \frac{(v \circ z(r(t)))|_{\Gamma_0} \cdot \nu(r(t))}{\zeta \cdot \nu(r(t))} = \rho(r(t)), \tag{3.53}$$

where the argument $\xi \in \Gamma_0$ has been suppressed.

This evolution equation for the perturbation function r has been derived exclusively from (1.4), (1.5), and (3.51), i.e. it is a purely kinematic relation. It will yield the evolution equations both for Stokes flow and for Hele-Shaw flow by making the appropriate choices for v. In both cases, the study of the FBPs is reduced to the thorough investigation of the operator ρ.

Remark: If we set, in particular, $\zeta = n$ and $r = 0$, we have $\dfrac{\partial r}{\partial t} = V_n$ and recover the kinematic boundary condition

$$V_n = v \cdot n$$

that has been announced earlier (cf. 1.9)). □

Stokes flow

For Stokes flow we have to set $v \circ z(r(t)) = u(r(t))$, where u is defined by Lemma 17 (ii). Hence we get, suppressing the time argument,

$$\frac{\partial r}{\partial t} = \rho(r) = \frac{(u(r))|_{\Gamma_0} \cdot \nu(r)}{\zeta \cdot \nu(r)}. \tag{3.54}$$

Lemma 19 *(Analyticity of the Stokes evolution operator)*
The operator ρ defined by (3.54) *is analytic near* 0 *from $H^{s+\frac{1}{2}}(\Gamma_0)$ to $H^{s-\frac{1}{2}}(\Gamma_0)$.*

Proof: The result follows, by the Banach algebra property of $H^{s-\frac{1}{2}}(\Gamma_0)$ and the fact that, due to (3.4), $\zeta \cdot \nu(0)$ is a strictly positive C^∞-function on Γ_0, hence an invertible element of $H^{s-\frac{1}{2}}(\Gamma_0)$, from Lemmas 6, 7, 16 (i), and 17 (ii). ■

Hele-Shaw flow

For Hele-Shaw flow we have to set $v \circ z(r(t)) = \nabla_{r(t)} u(r(t))$ with u defined by Lemma 18 (ii) and

$$(\nabla_r u)_i = a^{ji}(r)\frac{\partial u}{\partial x_j}. \tag{3.55}$$

Hence we get

$$\frac{\partial r}{\partial t} = \rho(r) = \frac{(\nabla_r u(r))|_{\Gamma_0} \cdot \nu(r)}{\zeta \cdot \nu(r)}. \tag{3.56}$$

Lemma 20 *(Analyticity of the Hele-Shaw evolution operator)*
The operator ρ *defined by* (3.56) *is analytic near* 0 *from* $H^{s+\frac{3}{2}}(\Gamma_0)$ *to* $H^{s-\frac{3}{2}}(\Gamma_0)$.

Proof: Note that the mappings $r \longmapsto (\nabla_r u(r))|_{\Gamma_0}$ and $r \longmapsto \nu(r)$ are analytic from $H^{s+\frac{3}{2}}(\Gamma_0)$ to $H^{s-\frac{3}{2}}(\Gamma_0)$, respectively. The assertion follows from this by the same arguments as in the proof of Lemma 19. ∎

Remark: From the point of view of the mapping properties in the scale of Sobolev spaces $\{H^\theta(\Gamma_0)\}$, the operators on the right side of the evolution equations for Stokes flow and Hele-Shaw flow behave as differential operators of first and of third order, respectively. This explains why the methods that have been used in Chapter 2 for the treatment of the Stokes flow problem in 2D cannot be used for plane Hele-Shaw flow: Operators of higher than first order are in general not quasidifferential in scales of Banach spaces like $\{B_r\}$, so the abstract Cauchy-Kovalevskaya theorem is not applicable. □

Chapter 4

Existence results via linearization

The most important tool in the analysis of the equations (3.54), (3.56) is the determination and investigation of the linear operator ρ_1, representing the Fréchet derivative of ρ at 0. More precisely, the procedure for obtaining short-time existence and uniqueness results for the evolution equations will be as follows:

- Identification of ρ_1 in terms of the corresponding fixed-time problem and differential operators on Γ_0
- Proof of coercivity estimates for $-\rho_1$
- Proof that ρ_1 generates an analytic semigroup
- Application of a general theorem on fully nonlinear parabolic equations to (3.54), (3.56)

In order to clarify the concepts and ideas of this approach before giving the technical details let us start with the following outline:

In a very informal way, the problems we consider can be characterized by the scheme in Table 4.1. Taking into account, moreover, that the linearization of the mapping $r \longmapsto \kappa(r)$ is in highest order essentially given by the Laplace-Beltrami operator Δ_{Γ_0} we find, for the leading-order terms of ρ_1, a structure that strictly resembles the scheme in Table 4.1. Namely, for the Stokes flow problem, we get the composition of Δ_{Γ_0} with the (normal component of) the Neumann-to-Dirichlet operator of the Stokes equations. On the other hand, for Hele-Shaw flow, we get the composition of Δ_{Γ_0} with the Dirichlet-to-Neumann operator of the Laplacian. The general theory of elliptic boundary value problems shows that in these problems the operators on the boundary manifold that assign to each other the Dirichlet and Neumann boundary data are elliptic pseudodifferential operators of the "expected" order [87].

Thus, in both cases, the operator ρ_1 is (at least in highest order) an elliptic pseudodifferential operator of order 1 or 3, respectively. It should be noted that for our purposes there is no need to explicitly calculate their principal symbols because the coercivity estimates can be obtained in a straightforward way from the coercivity of the underlying fixed time problems.

This discussion makes also clear in which (abstract) sense the equations (3.54) and (3.56) are parabolic (although they are not even differential equations): The linearization of the right-hand side generates an analytic semigroup (on suitable spaces). The equations are fully nonlinear (as opposed to semilinear, cf. [61]) in the sense that the nonlinear remainder term $\rho - \rho_1$ has the same

	Stokes flow	Hele-Shaw flow
governing elliptic operator	$\left[\begin{array}{ccc|c} -\Delta & & & \\ & \ddots & & \nabla \\ & & -\Delta & \\ \hline & \text{div} & & 0 \end{array}\right]$	Δ
Type of boundary condition in the fixed-time problem	Neumann ($\mathcal{T}(u,p)n$)	Dirichlet
Type of boundary data prescribing V_n	Dirichlet	Neumann $\left(\dfrac{\partial u}{\partial n}\right)$

Table 4.1: schematic comparison of Stokes flow and Hele-Shaw flow

properties as ρ with respect to "differentiation order", i.e. concerning continuity properties in the scale $\{H^{\theta}(\Gamma_0)\}$.

4.1 Calculation of the linearizations

In the sequel we will use the term "first-order differential operator on Γ_0" for linear differential operators l that are of the form $l = l^{(1)}h + \mu h$, where $\mu \in C^{\infty}(\Gamma_0)$ and $l^{(1)}$ corresponds to a C^{∞}-smooth tangential vector field on Γ_0, i.e. differential operators having smooth coefficients in any smooth local coordinate system. A linear operator mapping $C^{\infty}(\Gamma_0)$ to $(C^{\infty}(\Gamma_0))^N$ will be called vector-valued first-order differential operator on Γ_0 iff all its components are first-order differential operators on Γ_0.

We recall from Lemmas 6 (ii) and 16 that

$$\nu(r) = \sum_{k=0}^{\infty} \nu_k(r,\ldots,r), \quad \kappa(r) = \sum_{k=0}^{\infty} \kappa_k(r,\ldots,r),$$

and the Fréchet derivatives at of ν and κ at $r = 0$ are

$$\nu_1 \in \mathcal{L}\left(H^s(\Gamma_0), \left(H^{s-1}(\Gamma_0)\right)^N\right), \quad \kappa_1 \in \mathcal{L}\left(H^s(\Gamma_0), H^{s-2}(\Gamma_0)\right),$$

respectively.

Lemma 21 *(Linearization of geometric quantities)*
(i) The operator ν_1 is a vector-valued first-order differential operator on Γ_0.
(ii) We have

$$\kappa_1(r) = \gamma\Delta_{\Gamma_0} r + l(r),$$

where l is a first-order differential operator on Γ_0.

Proof: (i) We use the same notation as in the proof of Lemma 16 and note that it is sufficient to show that the Fréchet derivative of the mapping $r \mapsto \nu^{(m)}(r)$ at $r = 0$ is a vector-valued differential operator with smooth coefficients on W_m. Differentiation of (3.49) with respect to r at $r = 0$ yields, with obvious notation,

$$\nu_1^{(m)}[h] = -\begin{bmatrix} \left(\dfrac{\partial \xi^{(m)}}{\partial w}\right)^T \\ 2n^{(m)} \end{bmatrix}^{-1} \begin{bmatrix} \left(\dfrac{\partial((\psi_m h\zeta)\circ \xi^{(m)})}{\partial w}\right)^T \\ 0 \end{bmatrix} n^{(m)}$$

which proves the assertion due to the invertibility of

$$\begin{bmatrix} \left(\dfrac{\partial \xi^{(m)}}{\partial w}\right)^T \\ 2n^{(m)} \end{bmatrix}$$

and the smoothness of $\xi^{(m)}$, $\zeta \circ \xi^{(m)}$, and $n^{(m)}$.

(ii) Without loss of generality we can assume that r is smooth and small in $C(\Gamma_0)$. It will be convenient to work with perturbations of Γ_0 in normal direction, therefore we introduce on a small neighborhood of Γ_0 in $\mathbf{R}^N$ the functions B and Ξ by

$$B(x) = \pm\,\mathrm{dist}(x, \Gamma_0)$$

with positive sign for $x \notin \Omega_0$ and negative sign for $x \in \Omega_0$, and $\Xi(x) \in \Gamma_0$ as the (unique) solution of the minimization problem

$$|\xi - x| \to \min, \qquad \xi \in \Gamma_0.$$

Then clearly

$$x = \Xi(x) + B(x)n(\Xi(x))$$

and for sufficiently small $r \in C(\Gamma_0)$ we have a parametrization of Γ_r by

$$x = \xi + b_r(\xi)n(\xi), \quad \xi \in \Gamma_0,$$

where

$$b_r(\Xi(\xi + r(\xi)\zeta(\xi))) = B(\xi + r(\xi)\zeta(\xi)). \tag{4.1}$$

Note that (4.1) defines b_r at all points of Γ_0, b_r is smooth and small in $C(\Gamma_0)$.

On the other hand, replacing the vector field ζ by n we can describe domains near Γ_0 as

$$\tilde{\Gamma}_b : \quad x = \xi + b(\xi)n(\xi)$$

(cf. Lemma 5) and define $\tilde{\kappa}(b)(\xi)$ for all $\xi \in \Gamma_0$ as the curvature of $\tilde{\Gamma}_b$ in the point $\xi + b(\xi)n(\xi)$. It is clear that the mapping $b \mapsto \tilde{\kappa}(b)$ enjoys the same local analyticity properties as the mapping $r \mapsto \kappa(r)$. Denoting for the moment the Fréchet derivatives with respect to r and b at 0 with D_r and D_b, respectively, we find from

$$\kappa(r) = \tilde{\kappa}(b_r)$$

by the chain rule

$$\kappa_1[h] = D_r\kappa[h] = D_b\tilde{\kappa}[D_r b_r[h]]. \tag{4.2}$$

In order to calculate the "inner" derivative we introduce the smooth function

$$\Xi_r : \Gamma_0 \longrightarrow \Gamma_0$$

given by $\Xi_r = \Xi(\xi + r(\xi)\zeta(\xi))$. Using that $\Xi = \Xi_0$ is the identity on Γ_0 and $b_0 \equiv 0$ we find

$$D_r b_r = D_r(b_r \circ \Xi_0) = D_r(b_r \circ \Xi_r) - D_r(b_0 \circ \Xi_r) = D_r(b_r \circ \Xi_r)$$

and from this with $\nabla B = n$ on Γ_0 and (4.1)

$$D_r b_r[h] = \gamma h.$$

It remains to calculate $D_b\tilde{\kappa}$. To simplify the notation we set for the rest of the proof $\zeta = n$ which implies $r = b$, and the tilde can be omitted. Working with local parametrizations as in the proof of Lemma 16 and taking into account that, in the notation used there,

$$\begin{aligned}
D_R G[h] &= \left(\frac{\partial \xi^{(m)}}{\partial w}\right)^T \left(\frac{\partial (hn)}{\partial w}\right) + \left(\frac{\partial (hn)}{\partial w}\right)^T \left(\frac{\partial \xi^{(m)}}{\partial w}\right) \\
&= \left(\frac{\partial \xi^{(m)}}{\partial w}\right)^T \left(n\frac{\partial h}{\partial w} + h\frac{\partial n}{\partial w}\right) + \left(n\frac{\partial h}{\partial w} + h\frac{\partial n}{\partial w}\right)^T \left(\frac{\partial \xi^{(m)}}{\partial w}\right) \\
&= \left(\frac{\partial \xi^{(m)}}{\partial w}\right)^T \left(h\frac{\partial n}{\partial w}\right) + \left(h\frac{\partial n}{\partial w}\right)^T \left(\frac{\partial \xi^{(m)}}{\partial w}\right), \\
D_R(G^{-1})[h] &= -G(0)^{-1} D_R G[h] G(0)^{-1},
\end{aligned}$$

we find that $D_r g[h]$ and $D_r g^{ij}[h]$ are computed by pointwise multiplication of h with a fixed smooth function.

The assertion follows from this by straightforward calculation, using (4.2) and the facts that

$$h \longmapsto \Delta_{\Gamma_0}(hn) - (\Delta_{\Gamma_0} h)n$$

is a vector-valued first-order differential operator on Γ_0 and

$$h \longmapsto \Delta_{\Gamma_0}(\gamma h) - \gamma \Delta_{\Gamma_0} h$$

is a first-order differential operator on Γ_0. ■

After this preparation we can describe the structure of the linearization both for Stokes flow and for Hele-Shaw flow.

Stokes flow

In the notation of Lemma 17 we find from (3.54)

$$\begin{aligned}
\rho_1(r) &= \left(-\frac{u(0)\cdot n}{\gamma^2}\zeta + \frac{1}{\gamma}u(0)\right)\cdot \nu_1(r) + \frac{1}{\gamma} n \cdot u_1(r), \\
u_1(r) &= \operatorname*{Tr}_{\Gamma_0} \Pi_1 \tilde{L}(0)^{-1} \left(\begin{bmatrix} 0 \\ 0 \\ \kappa_1(r)n + \kappa(0)\nu_1(r) \\ 0 \\ 0 \end{bmatrix} - \tilde{L}_1(r) \begin{bmatrix} u(0) \\ p(0) \\ 0 \end{bmatrix} \right),
\end{aligned}$$

writing Π_1 for the canonical projection of X_s onto its first component $(H^s(\Omega_0))^N$ and using the fact that λ vanishes for the solution of the fixed-time problem. Hence, using Lemma 21 (ii) and the operator S as defined in (3.31), (3.32)

$$\rho_1 = \rho_1^* + l_1 + l_0, \tag{4.3}$$

where

$$\begin{aligned}
\rho_1^*(r) &= n \cdot S(\Delta_{\Gamma_0} r\, n), \\
l_1(r) &= \left(-\frac{u(0) \cdot n}{\gamma^2} \zeta + \frac{1}{\gamma} u(0) \right) \cdot \nu_1(r), \\
l_0(r) &= \frac{1}{\gamma} n \cdot \left(\operatorname*{Tr}_{\Gamma_0} \Pi_1 \tilde{L}(0)^{-1} \left(\begin{bmatrix} 0 \\ 0 \\ \kappa(0)\nu_1(r) \\ 0 \\ 0 \end{bmatrix} - L_1(r) \begin{bmatrix} u(0) \\ p(0) \\ 0 \end{bmatrix} \right) \right) \\
&\quad + \frac{1}{\gamma} n \cdot S(\gamma \Delta_{\Gamma_0} r\, n) - n \cdot S(\Delta_{\Gamma_0} r\, n).
\end{aligned}$$

From the regularity results on the fixed time problem (Lemma 11) and the smoothness of Γ_0 it follows that $u(0)|_{\Gamma_0}$ is smooth, hence we find from Lemma 21 (i) that l_1 is a first-order differential operator on Γ_0. Furthermore, the commutator property of S (Lemma 12 and the remark after its proof) together with Lemma 21 (i), Lemma 17 (i) and the fact that $p(0)$ is also smooth yields

$$l_0 \in \mathcal{L}\left(H^{s-\frac{1}{2}}(\Gamma_0), H^{s-\frac{1}{2}}(\Gamma_0) \right) \tag{4.4}$$

for all $s > 2 + \frac{N}{2}$.

Hele-Shaw flow

In the notation of Lemma 18, we find from (3.56)

$$\rho_1(r) = \left(-\frac{1}{\gamma^2} \frac{\partial u(0)}{\partial n} \zeta + \frac{1}{\gamma} \nabla u(0) \right) \cdot \nu_1(r) + \frac{1}{\gamma} \left(a_1^{ji}(r) \frac{\partial u(0)}{\partial x_j} \right) \Big|_{\Gamma_0} n_i + \frac{1}{\gamma} \frac{\partial}{\partial n} u_1(r) \tag{4.5}$$

with

$$a_1^{ji}(r) = -\frac{\partial \left(T^{-1}(r\zeta_j) \right)}{\partial x_i},$$

where $T^{-1} \in \mathcal{L}\left((H^\sigma(\Gamma_0))^N, \left(H^{\sigma+\frac{1}{2}}(\Omega_0) \right)^N \right)$ $(\sigma > 0)$ is the fixed inverse of the trace operator that has been introduced in the proof of Lemma 5, and

$$u_1(r) = L(0)^{-1} \left(\begin{bmatrix} 0 \\ \kappa_1(r) \end{bmatrix} - L_1(r) u(0) \right).$$

Summarizing and using the operator S defined in (3.37) we get from the Lemmas 13, 18 (i), and 21 that

$$\rho_1 = S\Delta_{\Gamma_0} + l_2 \tag{4.6}$$

with $l_2 \in \mathcal{L}\left(H^{s+\frac{1}{2}}(\Gamma_0), H^{s-\frac{3}{2}}(\Gamma_0)\right)$ for all $s > 2 + \frac{N}{2}$. Here we have used the fact that the operator

$$h \mapsto \frac{1}{\gamma} S \gamma \Delta_{\Gamma_0} h - S \Delta_{\Gamma_0} h$$

is a pseudodifferential operator of order 2 because of the commutator properties of S.

4.2 Coercivity and generation results

As we are exclusively concerned with linear operators in this section, we will follow the usual notational conventions and omit the brackets around the arguments of linear operators like ρ_1, l_1, l_0 etc.

Lemma 22 *(Coercivity of $-\rho_1$ for Stokes flow)*
Let ρ_1 be given by (4.3). *For all $s > \frac{3}{2} + \frac{N}{2}$ there are positive constants c and C_s such that*

$$(-\rho_1 r, r)_s \geq c\|r\|_{s+\frac{1}{2}}^{\Gamma_0\,2} - C_s\|r\|_s^{\Gamma_0 2} \qquad \forall r \in H^{s+1}(\Gamma_0).$$

Proof: Using the decomposition in (4.3) we will give the proof by showing the inequalities

$$\begin{array}{lrcl} \text{(i)} & |(l_0 r, r)_s| & \leq & C_s\|r\|_s^{\Gamma_0 2} \\ \text{(ii)} & |(l_1 r, r)_s| & \leq & C_s\|r\|_s^{\Gamma_0 2} \\ \text{(iii)} & (-\rho_1^* r, r)_s & \geq & c\|r\|_{s+\frac{1}{2}}^{\Gamma_0\,2} - C_s\|r\|_s^{\Gamma_0 2}. \end{array}$$

(i) is immediate from (4.4) and the Schwarz inequality. To show (ii) we recall that l_1 is a linear first-order differential operator on Γ_0 due to Lemma 21 (i) and denote its adjoint in $H^0(\Gamma_0)$ by l_1^*. Partial integration on Γ_0 shows that the operator $l_1 + l_1^*$ is given by multiplication with a smooth function. Using this, from

$$(l_1 r, r)_0 = -(l_1 r, r)_0 + ((l_1 + l_1^*)r, r)_0$$

we can conclude

$$|(l_1 r, r)_0| = \frac{1}{2}\,|((l_1 + l_1^*)r, r)_0| \leq C\|r\|_0^{\Gamma_0 2}$$

and further, using that $\Lambda^s l_1 - l_1 \Lambda^s \in \mathcal{L}\left(H^0(\Gamma_0), H^s(\Gamma_0)\right)$

$$|(l_1 r, r)_s| = |((\Lambda^s l_1 - l_1\Lambda^s)r, \Lambda^s r)_0| + |(l_1\Lambda^s r, \Lambda^s r)_0| \leq C_s\|r\|_s^{\Gamma_0 2}.$$

To show (iii) we recall that $\rho^\star r = \dot{u}\cdot n$, where $(\dot{u}, \dot{p}, \dot{\lambda})$ is the solution of the variational problem (see Section 3.2)

$$\begin{array}{rcl} a(\dot{u}, v) - \int_{\Omega_0} \dot{p} \operatorname{div} v \, dx + \dot{\lambda}_1^T \varphi_1(v) & & \\ +\dot{\lambda}_2^T \varphi_2(v) & = & \int_{\Gamma_0} \Delta_{\Gamma_0} r\, n \cdot v \, d\Gamma \quad \forall v \in (H^1(\Omega))^N \\ \operatorname{div} \dot{u} & = & 0 \\ \varphi_1(\dot{u}) & = & 0 \\ \varphi_2(\dot{u}) & = & 0. \end{array} \tag{4.7}$$

From (3.21) we find

$$\|\dot{\lambda}\|_{\mathbb{R}^N \times \mathbb{R}^{\binom{N}{2}}} \leq C\|\Delta_{\Gamma_0} r\, n\|_{-\frac{3}{2}}^{\Gamma_0} \leq C\|r\|_{\frac{1}{2}}^{\Gamma_0}. \tag{4.8}$$

Setting $v = \dot{u}$ in the first equation of (4.7) and applying the ellipticity of a we find

$$\begin{aligned}(-\rho_1^\star r, r)_1 &= (\rho_1^\star r, \Delta_{\Gamma_0} r)_0 - (\rho_1^\star r, r)_0 \geq \int_{\Gamma_0} \dot{u}\cdot n\, \Delta_{\Gamma_0} r\, d\Gamma - C\|r\|_1^{\Gamma_0\,2} \\ &= a(\dot{u},\dot{u}) - C\|r\|_1^{\Gamma_0\,2} \geq c\|\dot{u}\|_1^{\Omega_0\,2} - C\|r\|_1^{\Gamma_0\,2}. \end{aligned} \tag{4.9}$$

On the other hand,

$$\begin{aligned}\|r\|_{\frac{3}{2}}^{\Gamma_0} &\leq C\left(\|\Delta_{\Gamma_0} r\|_{-\frac{1}{2}}^{\Gamma_0} + \|r\|_{\frac{1}{2}}^{\Gamma_0}\right) \\ &\leq C \sup_{\substack{\varphi\in H^{\frac{1}{2}}(\Gamma_0)\\ \|\varphi\|_{\frac{1}{2}}^{\Gamma_0}=1}} \left(\int_{\Gamma_0} \Delta_{\Gamma_0} r\, \varphi\, d\Gamma + \|r\|_{\frac{1}{2}}^{\Gamma_0}\right)\end{aligned} \tag{4.10}$$

and we proceed by estimating the integral in the last expression for arbitrary $\varphi \in H^{\frac{1}{2}}(\Gamma_0)$ with $\|\varphi\|_{\frac{1}{2}}^{\Gamma_0} = 1$. At first we define the mean value

$$\overline{\varphi} = \frac{1}{|\Gamma_0|}\int_{\Gamma_0} \varphi\, d\Gamma$$

and note that

$$\|\overline{\varphi}\|_{\frac{1}{2}}^{\Gamma_0} \leq C|\overline{\varphi}| \leq C\|\varphi\|_0^{\Gamma_0} \leq C.$$

We consider the Neumann problem

$$\left.\begin{aligned} \Delta\Phi &= 0 && \text{in } \Omega_0 \\ \frac{\partial\Phi}{\partial n} &= \varphi - \overline{\varphi} && \text{on } \Gamma_0 \\ \int_{\Omega_0} \Phi\, dx &= 0 \end{aligned}\right\} \tag{4.11}$$

and set $v_\varphi = \nabla\Phi$. We get $v_\varphi \in \left(H^1(\Omega_0)\right)^N$ and

$$\|v_\varphi\|_1^{\Omega_0} \leq \|\Phi\|_2^{\Omega_0} \leq C\|\varphi - \overline{\varphi}\|_{\frac{1}{2}}^{\Gamma_0} \leq C.$$

Setting now $v = v_\varphi$ in the first equation of (4.7) yields

$$\begin{aligned}\int_{\Gamma_0} \Delta_{\Gamma_0} r\varphi\, d\Gamma &= \int_{\Gamma_0} \Delta_{\Gamma_0} r(\varphi - \overline{\varphi})\, d\Gamma = \int_{\Gamma_0} \Delta_{\Gamma_0} r n\cdot v_\varphi\, d\Gamma \\ &= a(\dot{u}, v_\varphi) + \dot{\lambda}_1^T\varphi_1(v_\varphi) + \dot{\lambda}_2^T\varphi_2(v_\varphi) \\ &\leq C\left(\|\dot{u}\|_1^{\Omega_0} + \|\dot{\lambda}\|_{\mathbf{R}^N\times\mathbf{R}^{\binom{N}{2}}}\right) \leq C\left(\|\dot{u}\|_1^{\Omega_0} + \|r\|_{\frac{1}{2}}^{\Gamma_0}\right),\end{aligned} \tag{4.12}$$

where (4.8) has been used. Hence, together with (4.10),

$$\|r\|_{\frac{3}{2}}^{\Gamma_0\,2} \leq C\left(\|\dot{u}\|_1^{\Omega_0\,2} + \|r\|_{\frac{1}{2}}^{\Gamma_0\,2}\right)$$

and with (4.9)

$$(\rho_1^\star r, r)_1 \geq c\|r\|_{\frac{3}{2}}^{\Gamma_0\,2} - C\|r\|_{\frac{1}{2}}^{\Gamma_0\,2}.$$

The inequality (iii) follows from this, using the commutator property

$$\rho_1^\star \Lambda^{s-1} - \Lambda^{s-1}\rho_1^\star \in \mathcal{L}\left(H^1(\Gamma_0), H^s(\Gamma_0)\right)$$

as in the proof of (ii). ■

Lemma 23 *(Coercivity of $-\rho_1$ for Hele-Shaw flow)*
Let ρ_1 be given by (4.5)*, $s > \frac{3}{2} + \frac{N}{2}$. There are positive constants c and C_s such that*

$$(-\rho_1 r, r)_s \geq c\|r\|_{s+\frac{3}{2}}^{\Gamma_0\,2} - C_s\|r\|_{s+1}^{\Gamma_0\,2} \qquad \forall r \in H^{s+3}(\Gamma_0)$$

Proof: Because of (4.6) and

$$|(l_2 r, r)_s| \leq \|l_2 r\|_{s-1}^{\Gamma_0}\|r\|_{s+1}^{\Gamma_0} \leq C_s\|r\|_{s+1}^{\Gamma_0\,2}$$

it is sufficient to show the above coercivity estimate for $-S\Delta_{\Gamma_0}$. As a first step, we give an elementary $H^0(\Gamma_0)$-coercivity estimate for S. For any $\varphi \in H^{\frac{1}{2}}(\Gamma_0)$, let $\tilde{u}$ be the solution of

$$\begin{aligned} \Delta\tilde{u} &= 0 \text{ in } \Omega_0, \\ \tilde{u}|_{\Gamma_0} &= \varphi \end{aligned}$$

and set

$$u = \tilde{u} - \frac{1}{|\Omega_0|}\int_{\Omega_0} \tilde{u}\, dx.$$

Then $S\varphi = \dfrac{\partial\tilde{u}}{\partial n}$ and by the well-known "dual estimate"

$$\|\tilde{u}\|_0^{\Omega_0} \leq C\|\varphi\|_{-\frac{1}{2}}^{\Gamma_0}$$

(see e.g. [59]) we have

$$\begin{aligned} \|\tilde{u}\|_1^{\Omega_0\,2} &\leq C\left(\|u\|_1^{\Omega_0\,2} + \|\tilde{u} - u\|_1^{\Omega_0\,2}\right) \leq C\left(\|u\|_1^{\Omega_0\,2} + \|\tilde{u}\|_0^{\Omega_0\,2}\right) \\ &\leq C\left(\|u\|_1^{\Omega_0\,2} + \|\varphi\|_{-\frac{1}{2}}^{\Gamma_0\,2}\right). \end{aligned}$$

Hence, using the trace theorem,

$$\|u\|_1^{\Omega_0\,2} \geq c\|\tilde{u}\|_1^{\Omega_0\,2} - \|\varphi\|_{-\frac{1}{2}}^{\Gamma_0\,2} \geq c\|\varphi\|_{\frac{1}{2}}^{\Gamma_0\,2} - \|\varphi\|_{-\frac{1}{2}}^{\Gamma_0\,2}$$

and by the Green formula and the Poincaré inequality

$$(S\varphi, \varphi)_0 = \|\nabla\tilde{u}\|_0^{\Omega_0\,2} = \|\nabla u\|_0^{\Omega_0\,2} \geq c\|u\|_1^{\Omega_0\,2} \geq c\|\varphi\|_{\frac{1}{2}}^{\Gamma_0\,2} - C\|\varphi\|_{-\frac{1}{2}}^{\Gamma_0\,2}.$$

Using this, we can conclude

$$\begin{aligned} (-S\Delta_{\Gamma_0} r, r)_0 &= -(Sr, r)_0 + (S\Lambda^2 r, r)_0 \\ &= -(Sr, r)_0 + ((S\Lambda - \Lambda S)\Lambda r, r)_0 + (S\Lambda r, \Lambda r)_0 \\ &\geq c\|\Lambda r\|_{\frac{1}{2}}^{\Gamma_0\,2} - C\|r\|_1^{\Gamma_0\,2} \end{aligned}$$

and from this we get the assertion, using the fact that S is a pseudodifferential operator and the same arguments as in the proof of Lemma 22. ■

The assertions on the generator properties for the operator ρ_1 follow from the coercivity estimates by standard arguments.

Lemma 24 *(Parabolic character of Stokes flow)*
Let $s > 2 + \frac{N}{2}$ and consider ρ_1 given by (4.3) as an unbounded operator on $H^s(\Gamma_0)$.
(i) $\mathcal{D}(\rho_1) = H^{s+1}(\Gamma_0)$, and the graph norm on $\mathcal{D}(\rho_1)$ is equivalent to $||\cdot||_{s+1}^{\Gamma_0}$.
(ii) The operator ρ_1 generates an analytic semigroup on $H^s(\Gamma_0)$.

Proof: We observe that due to the identity

$$(u, v)_{s+1} = (\Lambda^{\frac{1}{2}} u, \Lambda^{\frac{1}{2}} v)_{s+\frac{1}{2}} = (\Lambda^{-\frac{1}{2}}(\Lambda u), \Lambda^{\frac{1}{2}} v)_{s+\frac{1}{2}} \quad \forall u, v \in H^{s+1}(\Gamma_0)$$

the spaces $H^{s+1}(\Gamma_0)$ and $H^s(\Gamma_0)$ are in duality with respect to the bilinear form

$$\langle \cdot, \cdot \rangle : H^s(\Gamma_0) \times H^{s+1}(\Gamma_0) \longrightarrow \mathbb{R}$$

defined by

$$\langle u, v \rangle = (\Lambda^{-\frac{1}{2}} u, \Lambda^{\frac{1}{2}} v)_{s+\frac{1}{2}}.$$

Note that

$$\langle u, v \rangle = (u, v)_{s+\frac{1}{2}} \qquad \forall u \in H^{s+\frac{1}{2}}(\Gamma_0). \tag{4.13}$$

From Lemma 22 we have that for all sufficiently large real ω

$$(-\rho_1 + \omega r, r)_{s+\frac{1}{2}} \geq c||r||_{s+1}^{\Gamma_0\,2} \qquad \forall r \in H^{s+\frac{3}{2}}(\Gamma_0) \tag{4.14}$$

and from this and (4.13) we get by a continuity and density argument

$$\langle -\rho_1 + \omega r, r \rangle \geq c||r||_{s+1}^{\Gamma_0\,2} \qquad \forall r \in H^{s+1}(\Gamma_0).$$

Hence, by the Lax-Milgram lemma, $-\rho_1 + \omega I$ is an homeomorphism from $H^{s+1}(\Gamma_0)$ to $H^s(\Gamma_0)$.

If we suppose now $r \in H^s(\Gamma_0)$ and $\rho_1 r \in H^s(\Gamma_0)$, then $(-\rho_1 + \omega I)r \in H^s(\Gamma_0)$ and thus $r \in H^{s+1}(\Gamma_0)$ and

$$||r||_{s+1}^{\Gamma_0} \leq C_1 \left(||\rho_1 r||_s^{\Gamma_0} + ||r||_s^{\Gamma_0} \right) \leq C_2 ||r||_{s+1}^{\Gamma_0}.$$

This proves (i).

To show (ii) we complexify the space $H^s(\Gamma_0)$ and the operator ρ_1 in the usual way, with the scalar product in the complexified space $H^s(\Gamma_0)_{\mathbb{C}}$ given by

$$[u_1 + iu_2, v_1 + iv_2]_s = (u_1, v_1)_s + (u_2, v_2)_s + i\left((u_2, v_1)_s - (u_1, v_2)_s\right)$$

$(u_j, v_j \in H^s(\Gamma_0))$. It is sufficient to establish that for a certain real $\omega > 0$ the operator

$$A = \rho_1 - \omega I$$

generates an analytic semigroup on $H^s(\Gamma_0)$, the result for $\rho_1 = A + \omega I$ follows from this by a well-known perturbation result (e.g. [74] Theorem 11.37). We choose ω to be large enough to ensure (4.14) with s replaced by $s - \frac{1}{2}$. Then

$$(Ar, r)_s \leq -c||r||_{s+\frac{1}{2}}^{\Gamma_0\,2} \qquad \forall r \in H^{s+1}(\Gamma_0)$$

and, by arguments analogous to the ones in the proof of (i), the positive real axis belongs to the resolvent set of A. Furthermore, we find for all $z = u + iv \in H^s(\Gamma_0)_{\mathbb{C}}$, $u, v \in H^s(\Gamma_0)$,

$$\begin{aligned}
\mathrm{Re}[Az, z]_s &= (Au, u)_s + (Av, v)_s \le -c\left(\|u\|_{s+\frac{1}{2}}^{\Gamma_0\,2} + \|v\|_{s+\frac{1}{2}}^{\Gamma_0\,2}\right) \le 0, \\
|\mathrm{Im}[Az, z]_s| &\le |(Au, v)_s| + |(Av, u)_s| \le \|Au\|_{s-\frac{1}{2}}^{\Gamma_0}\|v\|_{s+\frac{1}{2}}^{\Gamma_0} + \|Av\|_{s-\frac{1}{2}}^{\Gamma_0}\|u\|_{s+\frac{1}{2}}^{\Gamma_0} \\
&\le C\|u\|_{s+\frac{1}{2}}^{\Gamma_0}\|v\|_{s+\frac{1}{2}}^{\Gamma_0} \le C\left(\|u\|_{s+\frac{1}{2}}^{\Gamma_0\,2} + \|v\|_{s+\frac{1}{2}}^{\Gamma_0\,2}\right) \le -C\,\mathrm{Re}[Az, z]_s,
\end{aligned}$$

i.e. the numerical range of A

$$\left\{[Az, z]_s \;\middle|\; z \in H^{s+1}(\Gamma_0)_{\mathbb{C}}\right\}$$

is contained in a sector of the left half complex plane whose opening angle is smaller than π. By Proposition VII.3.2 in [62], this implies the generation result for A. ■

Lemma 25 *(Parabolic character of Hele-Shaw flow)*
Let $s > 2 + \frac{N}{2}$ and consider ρ_1 given by (4.5) *as an unbounded operator on $H^s(\Gamma_0)$.*
(i) $\mathcal{D}(\rho_1) = H^{s+3}(\Gamma_0)$, and the graph norm on $\mathcal{D}(\rho_1)$ is equivalent to $\|\cdot\|_{s+3}^{\Gamma_0}$.
(ii) The operator ρ_1 generates an analytic semigroup on $H^s(\Gamma_0)$.

Proof: The proof can essentially be given in analogy to the proof of Lemma 24, so we restrict ourselves to the indication of the necessary changes.

Due to the interpolation inequality

$$\|r\|_{s+1}^{\Gamma_0\,2} \le \varepsilon\|r\|_{s+\frac{3}{2}}^{\Gamma_0\,2} + C_{s,\varepsilon}\|r\|_s^{\Gamma_0\,2}$$

holding for all positive ε we can infer from Lemma 23

$$(-\rho_1 r, r)_s \ge c\|r\|_{s+\frac{3}{2}}^{\Gamma_0\,2} - C_s\|r\|_s^{\Gamma_0\,2} \qquad \forall r \in H^{s+3}(\Gamma_0). \tag{4.15}$$

(i) can be proved now using the duality between $H^{s+3}(\Gamma_0)$ and $H^s(\Gamma_0)$ given by the bilinear form

$$\langle\cdot,\cdot\rangle : \; H^s(\Gamma_0) \times H^{s+3}(\Gamma_0) \longrightarrow \mathbb{R}$$

defined by

$$\langle u, v\rangle = (\Lambda^{-\frac{3}{2}}u, \Lambda^{\frac{3}{2}}v)_{s+\frac{3}{2}}.$$

The estimate (4.15) with s replaced by $s + \frac{3}{2}$ yields $\mathcal{D}(\rho_1) = H^{s+3}(\Gamma_0)$ in the same way as in Lemma 24.

To show (ii) we consider again an operator $A = \rho_1 - \omega I$ with sufficiently large $\omega \in \mathbb{R}$ and estimate, based on (4.15),

$$\begin{aligned}
\mathrm{Re}[Az, z]_s &\le -c\left(\|u\|_{s+\frac{3}{2}}^{\Gamma_0\,2} + \|v\|_{s+\frac{3}{2}}^{\Gamma_0\,2}\right), \\
|\,\mathrm{Im}[Az, z]_s| &\le \|Au\|_{s-\frac{3}{2}}^{\Gamma_0}\|v\|_{s+\frac{3}{2}}^{\Gamma_0} + \|Av\|_{s-\frac{3}{2}}^{\Gamma_0}\|u\|_{s+\frac{3}{2}}^{\Gamma_0} \le C\|u\|_{s+\frac{3}{2}}^{\Gamma_0}\|v\|_{s+\frac{3}{2}}^{\Gamma_0}
\end{aligned}$$

and the generator property of A follows as above. ■

4.3 Existence results I

The favorable properties of the linearization ρ_1 obtained in the previous section enable us to obtain (short-time) existence and uniqueness results on the initial value problem

$$\left.\begin{aligned}\frac{\partial r}{\partial t} &= \rho(r)\\ r(0) &= r_0,\end{aligned}\right\} \tag{4.16}$$

both for Stokes flow and for Hele-Shaw flow, by invoking a corresponding theorem on abstract fully nonlinear parabolic equations [61].

We start with a preparatory lemma that generalizes the previously obtained results on $\rho_1 = \rho'(0)$ to $\rho'(r)$ for all sufficiently small r.

Lemma 26 *(Generation properties of $\rho'(r)$ for small r)*
Let $s > 2 + \frac{N}{2}$.
(i) (Stokes flow)
Let ρ be given by (3.54), $D = H^{s+1}(\Gamma_0)$. There is a $\delta > 0$ such that for all $r \in B_0(\delta, D)$ the operator $\rho'(r) \in \mathcal{L}(D, H^s(\Gamma_0))$, considered as an unbounded operator on $H^s(\Gamma_0)$, satisfies $\mathcal{D}(\rho'(r)) = D$, the graph norm on $\mathcal{D}(\rho'(r))$ is equivalent to $\|\cdot\|_D$, and $\rho'(r)$ generates an analytic semigroup on $H^s(\Gamma_0)$.
(ii) (Hele-Shaw flow)
Let ρ be given by (3.56), $D = H^{s+3}(\Gamma_0)$. Then the same assertions as in (i) hold.

Proof: (i) We recall that $\rho'(0) = \rho_1$ and that the mapping $r \mapsto \rho'(r)$ is continuous near 0 from $D = H^{s+1}(\Gamma_0)$ to $\mathcal{L}\left(H^{s+1}(\Gamma_0), H^s(\Gamma_0)\right)$. Hence, for any fixed $r \in B_0(\delta, H^{s+1}(\Gamma_0))$ we have, using the same notation as in the proof of Lemma 24,

$$\begin{aligned}\langle -\rho'(r)[h], h\rangle &= \langle -\rho_1 h, h\rangle + \langle -(\rho'(r) - \rho_1)h, h\rangle\\ &\geq c\|h\|_{s+1}^{\Gamma_0\,2} - C_s\|h\|_s^{\Gamma_0\,2} - \|\rho'(r) - \rho_1\|_{\mathcal{L}(H^{s+1}(\Gamma_0),H^s(\Gamma_0))}\|h\|_{s+1}^{\Gamma_0\,2}\\ &\geq \frac{c}{2}\|h\|_{s+1}^{\Gamma_0\,2} - C_s\|h\|_s^{\Gamma_0\,2}\end{aligned} \tag{4.17}$$

for sufficiently small δ, where Lemma 22 has been used. Based on this estimate the assertions on the domain of definition and the generation of an analytic semigroup can be shown as for ρ_1 in Lemma 24.

The proof of (ii) proceeds analogously, based on Lemma 23. ■

Following [61], we introduce weighted Hölder spaces of functions of a real variable with values in a Banach space X. These spaces are designed to handle the singular behavior of the function $t \mapsto e^{tA}x$ mapping an interval $(0, T]$ into $\mathcal{D}(A)$ for $t \downarrow 0$, $x \notin \mathcal{D}(A)$, where $\{e^{tA}\}$ is an analytic semigroup.

For $\alpha \in (0, 1)$, $a, b \in \mathbb{R}$, $a < b$, let $C_\alpha^\alpha((a, b], X)$ be the linear space of all bounded functions $f : (a, b] \longrightarrow X$ for which

$$[f]_{C_\alpha^\alpha((a,b],X)} = \sup_{0<\varepsilon<b-a} \varepsilon^\alpha [f]_{C^\alpha([a+\varepsilon,b],X)} < \infty,$$

where

$$[f]_{C^\alpha([c,b],X)} = \sup_{c\leq s<t\leq b} \frac{\|f(t) - f(s)\|_X}{(t-s)^\alpha} \qquad (a < c < b).$$

$C_\alpha^\alpha((a,b],X)$ is a Banach space under the norm given by

$$\|f\|_{C_\alpha^\alpha((a,b],X)} = \sup_{t\in(a,b]} \|f(t)\|_X + [f]_{C_\alpha^\alpha((a,b],X)}.$$

Proposition 4 *(Existence theorem I)*

Let $s > 2 + \frac{N}{2}$.

(A) (Stokes flow)

Let ρ *be given by* (3.54), $D = H^{s+1}(\Gamma_0)$.

(i) (Existence)

There are positive real constants δ, T *depending only on* s *and* Γ_0 *such that for all* $r_0 \in B_0(\delta, D)$ *the problem* (4.16) *has a solution*

$$r \in C([0,T],D) \cap C^1([0,T],H^s(\Gamma_0)) \cap \bigcap_{0<\alpha<1} C_\alpha^\alpha((0,T],D).$$

(ii) (Uniqueness)

This solution is unique in

$$\bigcup_{0<\alpha<1} C_\alpha^\alpha((0,T],D) \cap C([0,T],D).$$

(iii) (Time regularity)

The mapping $t \mapsto r(t)$ *is analytic from* $(0,T)$ *into* D.

(iv) (Dependence on the initial value)

For all $\alpha \in (0,1)$ *the mapping* $r_0 \mapsto r$ *is analytic from* $B_0(\delta, D)$ *into* $C_\alpha^\alpha((0,T],D)$.

(B) (Hele-Shaw flow)

Let ρ *be given by* (3.56), $D = H^{s+3}(\Gamma_0)$. *Then (i)–(iv) also hold.*

Proof: The assertions (i) and (ii) follow from Theorem 8.1.1. in [61]; the assertions (iii) and (iv) follow from Theorem 8.3.9. in [61]. To see this, we have to set $X = H^s(\Gamma_0)$, $\mathcal{O}$ a sufficiently small neighborhood of 0 in D, $\overline{u} = 0$, $\overline{t} = 0$, $F(u,t) = \rho(u)$ which yields $A = \rho_1$. The smoothness of F in u follows from Lemma 19 in case (A) and from Lemma 20 in case (B). Our evolution problems are autonomous, hence the assumptions on Hölder continuity of F and its Fréchet derivative with respect to u in t are trivially satisfied with any $\alpha \in (0,1)$. The compatibility condition $F(\overline{u},\overline{t}) \in \overline{D}$ is clearly valid because in our cases D is dense in X. The crucial assumption on the parabolic character ((8.0.3) in [61]) is satisfied due to Lemma 26.

We will confine ourselves here to a sketch of the basic approach used for the proof: Any solution of (4.16) is a fixed point of the operator Γ defined by

$$(\Gamma r)(t) = e^{t\rho_1} r_0 + \int_0^t e^{(t-s)\rho_1}(\rho(r(s)) - \rho_1(r(s)))\, ds.$$

Based on a detailed study of the operator

$$f \mapsto \int_0^t e^{(t-s)A} f(s)\, ds$$

in the weighted Hölder spaces defined above it is possible to show that Γ is a contraction on the metric space

$$Y = \{u \in B_0(\varepsilon, C_\alpha^\alpha((0,T],D)) \cap C([0,T],D) \mid u(0) = r_0\}$$

if ε and T are chosen small enough, and the existence follows from the Banach fixed point theorem. Very loosely speaking, the assertions (ii)–(iv) are proved analogously to the corresponding results for ordinary differential equations, based on the analyticity of ρ. ■

Chapter 5

Further results via quasilinearization

The results on the solution of (4.16) obtained in the previous chapter appear to be not quite satisfactory in a number of respects. The analytical tools are relatively complicated, uniform estimates for $||r(t)||_s^{\Gamma_0}$ are not easily available, and no results on higher space regularity could be obtained although we expect $r(t) \in C^\infty(\Gamma_0)$ for all $t > 0$ due to the parabolic character of the evolution equations under consideration, even if r_0 has finite smoothness only.

These drawbacks can be overcome for evolution equations

$$\frac{\partial u}{\partial t} = F(u)$$

where F is a quasilinear operator. (The restriction to autonomous problems in this informal discussion is just for simplicity of notation.) In an abstract setting, the quasilinearity of F can be described in the following way (cf. e.g. [30, 49]): Let $X_2 \hookrightarrow X_1 \hookrightarrow X$ be three Banach spaces with continuous and dense embeddings, $X_1 \not\hookrightarrow X_2$. $F: \; X_2 \longrightarrow X$ is called quasilinear iff there are continuous operators

$$\begin{aligned} A: &\;\; X_1 \longrightarrow \mathcal{L}(X_2, X) \\ G: &\;\; X_1 \longrightarrow X \end{aligned}$$

such that

$$F(u) = A(u)u + G(u). \tag{5.1}$$

(Depending on the situation, stronger smoothness assumptions on A and G have to be made in many cases. Moreover, A and G may be defined only on some open set in X_1.) The parabolic character can be characterized here by the demand that $A(u)$ generates an analytic semigroup on X, at least for all u in some open set of X_1.

Our approach does not provide a decomposition of the form (5.1) for the operator ρ.[1] To take advantage of a quasilinear structure in spite of this, we will use a quasilinearization technique, i.e. we will replace the original equation by a system of equations for r and spatial derivatives of r up to a certain order. This system is quasilinear, and a priori estimates, solvability, and regularity

[1] It has to be mentioned here, however, that for Hele-Shaw flow the very similar approach in [30] actually leads to a quasilinear evolution equation. This is due to the quasilinearity of the operator $r \mapsto \kappa(r)$.

of the solution can be shown in the usual way. This method was introduced by Eidel'man ([26], III.4).

In the actual considerations below, the quasilinear system will not occur explicitly. In order to clarify the basic idea as well as an important difficulty that we encounter let us first look in an informal way at the following model problem: Consider the fully nonlinear evolution equation

$$u_t = f(x, u_{xx}) \qquad \text{on } \mathbf{R}$$

with smooth $f : \mathbf{R}^2 \longrightarrow \mathbf{R}$. Differentiating this equation with respect to the space variable x and writing $v = u_x$ we get the equation

$$v_t = \partial_1 f(x, v_x) + \partial_2 f(x, v_x) v_{xx} \tag{5.2}$$

which is quasilinear (in appropriate function spaces) because v_{xx} occurs only linear on the right side.

Needless to say that the chain rule has been used to find (5.2). Note, however, that its applicability depends on the special structure of the operator $u \mapsto f(\cdot, u_{xx}(\cdot))$ while in the case of general (nonlocal) operators $u \mapsto F(u)$ there is no obvious link between $(F(u))_x$ and u_x. Such a link can be established, however, if F is Fréchet-differentiable and satisfies the assumption

$$T_h \circ F = F \circ T_h \qquad \forall h \in \mathbf{R}, \tag{5.3}$$

where T_h is the translation operator on $\mathbf{R}$ defined by

$$T_h u(x) = u(x + h),$$

i.e. F is equivariant with respect to translations. Differentiating the equation

$$T_h F(u) = F(T_h u)$$

with respect to h at $h = 0$ we find

$$(F(u))_x = F'(u)[u_x], \tag{5.4}$$

i.e. a "generalized chain rule". This name is justified by the fact that (5.3) is a weakening of the assumption made for the usual chain rule, namely that $F(u)(x) = f(u(x))$, and (5.4) specializes to $(F(u))_x(x) = f'(u(x))u_x(x)$ in this case.

The crucial idea for the application of this approach to the operator ρ is to use the rotational invariance of the fixed time problems for the derivation of an equivariance relation analogous to (5.3). Consequently, the corresponding generalized chain rule holds for the differential operators generated by "infinitesimal rotations". The estimates that are made possible by the quasilinearization rely on the fact that these differential operators can be used to define equivalent norms on the spaces $H^s(\Gamma_0)$.

Hence, the program to be carried out in this chapter is the following:

- Proof of a generalized chain rule in an abstract context
- Application of this to the operator ρ; note that here an additional assumption (Assumption 1) on the geometry of Γ_0 has to be made
- A priori estimates in the norms of $H^s(\Gamma_0)$
- Existence proof based on Galerkin approximations
- Proof of the smoothing property, based on a bootstrapping argument.

5.1 A chain rule for equivariant operators

As we are concerned here with invariance and equivariance with respect to a group of motions, it is natural to formulate the abstract result in the context of Lie groups and their representations on Banach spaces.

Let G be a finite-dimensional Lie group, $\mathcal{G}$ its Lie algebra, $a_1, \ldots, a_d$ a basis of $\mathcal{G}$ and for $i = 1, \ldots, d$ let $t \mapsto e^{-ta_i}$ be the one-parameter subgroup of G generated by a_i. Let X and Y be Banach spaces and let

$$\begin{aligned} U: \quad & G \longrightarrow \mathcal{L}(X) \\ V: \quad & G \longrightarrow \mathcal{L}(Y) \end{aligned}$$

be strongly continuous representations of G on X and Y, respectively. We denote by $D_i^{(X)}$ and $D_i^{(Y)}$ the generators of the strongly continuous semigroups of operators $t \mapsto U(e^{-ta_i})$ and $t \mapsto V(e^{-ta_i})$ on X and Y, respectively. For the sake of brevity we will suppress the indication of the spaces X and Y in the notation for the generators.

For any multiindex $\alpha = (\alpha_1 \ldots \alpha_d) \in \mathbf{N}^d$ we define $|\alpha| = \alpha_1 + \ldots + \alpha_d$ and

$$D^\alpha = D_1^{\alpha_1} \ldots D_d^{\alpha_d}.$$

Note that due to the structure equations of $\mathcal{G}$ we have

$$D_i D_j - D_j D_i = c_{ij}^k D_k \qquad i, j = 1, \ldots, d,$$

and this implies

$$D^\alpha D^\beta = D^{\alpha+\beta} + \sum_{|\gamma| < |\alpha+\beta|} C_{\alpha\beta\gamma} D^\gamma \tag{5.5}$$

for arbitrary multiindices α, β.

By the Hille-Yosida theorem, the operators D_i are closed, hence for all $n \in \mathbf{N}$ the spaces

$$X^{(n)} = \bigcap_{|\alpha| \leq n} \mathcal{D}(D^\alpha)$$

normed by

$$||u||_{X^{(n)}} = \sum_{|\alpha| \leq n} ||D^\alpha u||_X$$

are Banach spaces, and Banach spaces $Y^{(n)}$ are defined analogously. It is a routine task to check that the spaces $X^{(n)}, Y^{(n)}$ are, up to equivalence of norms, independent of the basis choice in $\mathcal{G}$. (See [76], Section I.1.) From (5.5) it follows that if $|\alpha| \leq n$, then D^α maps $X^{(n)}$ continuously into $X^{(n-|\alpha|)}$, corresponding results hold for the $Y^{(n)}$.

We will consider a situation in which X and Y are spaces of real-valued functions on a manifold on which the Lie group G is acting as a group of diffeomorphisms. In this case, the spaces $X^{(n)}$ and $Y^{(n)}$ can be seen as subspaces of X and Y containing the functions which are "n times differentiable" with respect to the differential operators that constitute $\mathcal{G}$. The following lemma will make this idea more precise in the case where we will need it. We will work with Sobolev spaces $H^s(S^{N-1})$ whose norms are defined as described in Section 3.1.

Lemma 27 *(A characterization for $H^s(S^{N-1})$)*
Let the Lie group $G = SO(N)$ be represented by the rotations of $\mathbf{R}^N$ around the origin. For arbitrary $s \in \mathbf{R}$ set $X = H^s(S^{N-1})$, $U(g)u = u \circ g$. Then $X^{(n)} = H^{s+n}(S^{N-1})$ with equivalent norms.

Proof: Let $\mathcal{H} = L^2(S^{N-1})$ and note that U acts on $\mathcal{H}$ in a natural way as a unitary representation. If we choose the basis in $\mathcal{G}$ such that (formally)

$$D_{K(i,j)} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i},$$

then we have ([92] p.13)

$$\sum_{k=1}^{\binom{N}{2}} D_k^2 \varphi = \Delta_{S^{N-1}} \varphi \qquad \forall \varphi \in C^\infty(S^{N-1}). \tag{5.6}$$

Set

$$\tilde{\Delta} = \sum_{k=1}^{\binom{N}{2}} D_k^2, \qquad \mathcal{D}(\tilde{\Delta}) = \mathcal{H}^{(2)}.$$

Theorem I.6.1 in [76] shows that the operator $\tilde{\Delta}$ is self-adjoint. On the other hand, the restriction of $\Delta_{S^{N-1}}$ to the dense subspace $C^\infty(S^{N-1})$ is essentially self-adjoint (cf. theorem 31.1 of [88]). Denoting this restriction by Δ_0 we find from (5.6) that $\Delta_0 \subset \tilde{\Delta}$. This implies $\Delta_{S^{N-1}} = \tilde{\Delta}$ with equality of the domains, and from Theorem I.6.1. in [76] we find

$$\mathcal{H}^{(n)} = \mathcal{D}\left(\tilde{\Delta}^{\frac{n}{2}}\right) = \mathcal{D}\left(\Delta_{S^{N-1}}^{\frac{n}{2}}\right)$$

with equivalent norms. The lemma follows from this by Theorem 3.17 in [29]. ■

Lemma 28 *(Regularity and a chain rule for equivariant operators)*
(i) Let $\mathcal{U} \subset X$ be open, $F : \mathcal{U} \longrightarrow Y$ K times Fréchet-differentiable, $n \leq K$. If the equivariance relation

$$V(g)F(u) = F(U(g)u) \tag{5.7}$$

holds for all $u \in \mathcal{U}$ and all g near the unit element in G, then the restriction of F to $\mathcal{U} \cap X^{(n)}$ maps $\mathcal{U} \cap X^{(n)}$ $K - n$ times Fréchet-differentiable into $Y^{(n)}$, and for all $\alpha \in \mathbf{N}^d$ and $u \in \mathcal{U} \cap X^{(|\alpha|)}$ one has the chain rule

$$D^\alpha F(u) = \sum_{k=1}^{|\alpha|} \sum_{\beta_1 + \ldots + \beta_k = \alpha} C_{\beta_1 \ldots \beta_k} F^{(k)}(u)[D^{\beta_1} u, \ldots, D^{\beta_k} u], \tag{5.8}$$

where only $\beta_l \neq 0$ occur and $C_\alpha = 1$.
(ii) If, in particular, F is analytic near $u_0 \in \mathcal{U}$, then there is an $\varepsilon > 0$ such that the restriction of F to $B_{u_0}(\varepsilon, X) \cap X^{(n)}$ is analytic and bounded into $Y^{(n)}$ for all $n \in \mathbf{N}$.

Proof: (i) The proof of (i) will be given by induction over $|\alpha|$. Suppose $u \in \mathcal{U} \cap X^{(1)}$. By assumption, we have for sufficiently small $|t|$ and all $i = 1, \ldots, d$

$$V(e^{-ta_i})F(u) = F(U(e^{-ta_i})u).$$

The right side is differentiable with respect to t at $t = 0$, hence the same holds for the expression on the left, therefore $F(u) \in Y^{(1)}$. Carrying out the differentiation yields

$$D_i F(u) = F'(u)[D_i u], \qquad i = 1, \ldots, d. \tag{5.9}$$

The expression on the right is a $K-1$ times differentiable function from $\mathcal{U} \cap X^{(1)}$ into Y, hence all assertions are proved for $|\alpha| = 1$. In particular, if $k \le K-1$ and $h_1, \ldots, h_k \in X^{(1)}$, then $F^{(k)}(u)[h_1, \ldots, h_k] \in Y^{(1)}$, and calculating the k-th-order Fréchet derivative on both sides of (5.9) yields

$$\begin{aligned} & (D_j F)^{(k)}(u)[h_1, \ldots, h_k] = D_j(F^{(k)}(u)[h_1, \ldots, h_k]) \\ = & \sum_{l=1}^{k} F^{(k)}(u)[h_1, \ldots, h_{l-1}, D_j h_l, h_{l+1}, \ldots, h_k] \\ & + F^{(k+1)}(u)[D_j u, h_1, \ldots, h_k] \end{aligned} \tag{5.10}$$

for all $h_1, \ldots, h_k \in X^{(1)}$ which is easily proved by induction.

Suppose the assertions hold for all α' with $|\alpha'| \le m \le K-1$, consider α with $|\alpha| = m+1$, $u \in \mathcal{U} \cap X^{(m+1)}$. We can write $D^\alpha = D_j D^{\alpha'}$ and apply the induction assumption to $D^{\alpha'} F(u)$. Due to $D^\beta u \in X^{(1)}$ for all β with $|\beta| \le |\alpha'|$ we have $D^{\alpha'} F(u) \in Y^{(1)}$ and (5.10) may be applied. Rearranging the terms according to the order of the Fréchet derivatives and noting that the expressions on the right are $K - m - 1$ times Fréchet-differentiable from $\mathcal{U} \cap X^{(m+1)}$ to $Y^{(m+1)}$ completes the proof of (5.8).

(ii) In view of the definition of the space $Y^{(n)}$ it is sufficient to show that the mappings $u \mapsto D^\alpha F(u)$ are analytic and bounded from $B_{u_0}(\varepsilon, X^{(n)}) \cap X^{(n)}$ to Y. The analyticity follows immediately from the above remark on the analyticity of the Fréchet derivatives. The boundedness of the $F^{(k)}$ implies

$$\|F^{(k)}(u)\|_{\mathcal{L}(X^k, Y)} \le C_k \qquad \forall u \in B_{u_0}(\varepsilon, X)$$

and if we demand $\|u\|_{X^{(n)}} \le M$, then

$$\|D^\beta u\|_X \le C_{\beta,n} M \qquad \forall \beta : \ |\beta| \le n$$

and hence by (5.8)

$$\|D^\alpha F(u)\|_Y \le C_k (C_n M)^k \le C_n M^n. \quad \blacksquare$$

To apply Lemma 28 to our operators ρ, we have to show an equivariance relation (5.7). As announced above, this will be based on the invariance of the fixed time problems under rotations, i.e. we will choose $G = SO(N)$. Here, however, we encounter the following difficulty: Our problem can be put in the framework considered above only if it can be formulated in terms of functions on the unit sphere. Hence we have to make the following assumption:

Assumption 1 *The reference domain Ω_0 is strictly star-shaped, i.e. there is a constant $x_0 \in \mathbf{R}^N$ and a strictly positive function $R_0 \in C^\infty(S^{N-1})$ such that*

$$\Gamma_0 = \{R_0(\theta)\theta + x_0 \,|\, \theta \in S^{N-1}\}.$$

We will assume that $x_0 = 0$. In this chapter, this can be done without loss of generality.

In view of the rotational symmetry it is natural to choose

$$\zeta(\xi) = \frac{\xi}{|\xi|}. \tag{5.11}$$

This obviously meets the demands on ζ that have been made in Section 3.1.

Remark: It is enlightening to consider the meaning of the Assumptions 1 and (5.11) from a geometrical point of view: From the proof of lemma 5 we recall that, in the general situation considered there, there is a small $\delta > 0$ and a C^∞-diffeomorphism ϕ given by (3.5) that maps $\Gamma_0 \times (-\delta, \delta)$ onto some neighborhood $\mathcal{V}$ of Γ_0 in $\mathbf{R}^N$. We introduce the smooth mapping

$$\Xi : \mathcal{V} \longrightarrow \Gamma_0$$

by $\Xi = \Pi_1 \circ \phi^{-1}$, where Π_1 is the canonical projection of $\Gamma_0 \times (-\delta, \delta)$ onto Γ_0. Pick now an arbitrary $x \in \mathcal{V}$ and an arbitrary skew-symmetric matrix Q. For t ranging in a sufficiently small interval around 0, we have $e^{tQ} \in \mathcal{V}$. It is easily seen that the choice (5.11) ensures for all possible x, Q, and t that $\Xi(e^{tQ}x)$ depends only on $\Xi(x)$ but not on the second component of $\phi^{-1}(x)$. Consequently, a (small) rotation of Γ_r around 0 generates a (local) flow on Γ_0 which does not depend on r. In the general case, this flow and the vector field generating it are depending on r, and so the corresponding chain rule will involve nonlinear differential operators. Therefore we will not investigate this general case but accept the geometrical restriction given by Assumption 1. □

The mapping $\Phi : S^{N-1} \longrightarrow \Gamma_0$ defined by $\Phi(\theta) = \zeta R_0(\theta)$ is a C^∞- diffeomorphism between S^{N-1} and Γ_0, hence the direct image map Φ^* defined by $(\Phi^*\varphi)(\theta) = \varphi(\Phi(\theta))$ is an homeomorphism from $C^\infty(\Gamma_0)$ to $C^\infty(S^{N-1})$ and from $H^\sigma(\Gamma_0)$ to $H^\sigma(S^{N-1})$ for all $\sigma \in \mathbf{R}$. Due to (5.11), we have for any sufficiently small $r \in H^s(\Gamma_0)$

$$\Gamma_r = \{\theta R(\theta) \mid \theta \in S^{N-1}\} = \tilde{\Gamma}_R,$$

where $R = \Phi^* r + R_0$. Moreover, we define differential operators $\tilde{D}_j$ by transferring the D_j to Γ_0:

$$\tilde{D}_j = \Phi^{*-1} D_j \Phi^*, \qquad j = 1 \ldots \tbinom{N}{2}.$$

Their compositions $\tilde{D}^\alpha$ are defined analogously to D^α.

We are ready now to show some important properties of the nonlinear operator ρ.

Lemma 29 *(Smoothness and a chain rule for ρ)*

Suppose Assumption 1 *and* (5.11) *hold, let n be a positive integer.*

(A) (Stokes flow)

Let ρ be defined by (3.54), $d = 1, s > 2 + \frac{N}{2} - \frac{d}{2}$. *Then there is an $\varepsilon > 0$ such that the following holds:*

(i) ρ is analytic and bounded from

$$B_0(\varepsilon, H^{s+d}(\Gamma_0)) \cap H^{s+n+d}(\Gamma_0)$$

to $H^{s+n}(\Gamma_0)$ for all $n \in \mathbf{N}$.

(ii) ρ is weakly sequentially continuous from $B_0(\varepsilon, H^{s+d}(\Gamma_0)) \cap H^{s+n+d}(\Gamma_0)$ to $H^{s+n}(\Gamma_0)$ for all integer $n \geq 1$.

(iii) For all $n \in \mathbf{N}$, $r \in B_0(\varepsilon, H^{s+d}(\Gamma_0)) \cap H^{s+n+d}(\Gamma_0)$, and $\alpha \in \mathbf{N}^{\binom{N}{2}}$ with $|\alpha| \leq n$ we have

$$\tilde{D}^\alpha \rho(r) = \sum_{k=1}^{|\alpha|} \sum_{\beta_1+\ldots+\beta_k=\alpha} C_{\beta_1,\ldots,\beta_k} \rho^{(k)}(r)[\tilde{D}^{\beta_1}(r+\mathcal{R}_0), \ldots, \tilde{D}^{\beta_k}(r+\mathcal{R}_0)] \tag{5.12}$$

with the same constants $C_{\beta_1,\dots,\beta_k}$ as in Lemma 28 and $\mathcal{R}_0 = {\Phi^*}^{-1} R_0$.
(B) (Hele-Shaw flow)
Let ρ be defined by (3.56), $d = 3$. Then the same assertions as in (A) hold.

Proof: On a ball $B_{R_0}(\delta, H^{s+d}(S^{N-1}))$ with sufficiently small $\delta > 0$ we define the operators $\tilde\rho$ and $\tilde\nu$ by

$$\begin{aligned}\tilde\rho(R) &= \Phi^*\rho(r) = \Phi^*\rho \circ {\Phi^*}^{-1}(R - R_0) \\ \tilde\nu(R) &= \Phi^*\nu(r) = \Phi^*\nu \circ {\Phi^*}^{-1}(R - R_0).\end{aligned} \tag{5.13}$$

These operators are obviously analytic near R_0 from $H^{s+d}(S^{N-1})$ to $H^s(S^{N-1})$ and $\left(H^{s+d-1}(S^{N-1})\right)^N$, respectively. Taking the k-th Frechet derivative of (5.13) yields

$$\tilde\rho^{(k)}(R)[h_1, \dots, h_k] = \Phi^*\rho^{(k)}(r)[{\Phi^*}^{-1}h_1, \dots, {\Phi^*}^{-1}h_k] \tag{5.14}$$

for all $h_1, \dots, h_k \in H^{s+d}(S^{N-1})$.

As in Lemma 27, we consider the Lie group G and the same representation U of it on $H^s(S^{N-1})$ as introduced there. The restriction of U to $H^{s+d}(S^{N-1})$ is strongly continuous due to Lemma 27 and theorem 3.17 in [29]. Consequently, for any fixed $R \in B_{R_0}(\delta, H^{s+d}(S^{N-1}))$ there is a neighborhood $\mathcal{V}$ of the identity in $SO(N)$ (represented as group of orthogonal matrices) such that $R \circ Q \in B_{R_0}(\delta, H^{s+d}(S^{N-1}))$ for all $Q \in \mathcal{V}$. Picking an arbitrary $Q \in \mathcal{V}$, we find $\tilde\Gamma_{R\circ Q} = Q^{-1}[\tilde\Gamma_R]$ and hence

$$\tilde\nu(R \circ Q) = Q^{-1}\tilde\nu(R) \circ Q.$$

We write

$$r_Q = {\Phi^*}^{-1}(R \circ Q - R_0)$$

and from the rotational invariance of the Stokes equations, together with the boundary conditions and auxiliary conditions, we have in case (A)

$$\Phi^*(u(r_Q)|_{\Gamma_0}) = Q^{-1}\Phi^*(u(r)|_{\Gamma_0}) \circ Q \tag{5.15}$$

with u defined by Lemma 17 (ii). Using this and (5.11), we get

$$\begin{aligned}\tilde\rho(R \circ Q) &= \Phi^*\rho(r_Q) = \frac{\Phi^*(u(r_Q)|_{\Gamma_0}) \cdot \Phi^*\nu(r_Q)}{\Phi^*\zeta \cdot \Phi^*\nu(r_Q)} \\ &= \frac{Q^{-1}\Phi^*(u(r)|_{\Gamma_0}) \circ Q \cdot \tilde\nu(R \circ Q)}{\Phi^*\zeta \cdot \tilde\nu(R \circ Q)} \\ &= \frac{Q^{-1}\Phi^*(u(r)|_{\Gamma_0}) \circ Q \cdot Q^{-1}\tilde\nu(R) \circ Q}{Q^{-1}\Phi^*\zeta \circ Q \cdot Q^{-1}\tilde\nu(R) \circ Q} \\ &= \Phi^*\left(\frac{u(r)|_{\Gamma_0} \cdot \nu(r)}{\zeta \cdot \nu(r)}\right) \circ Q \\ &= \Phi^*\rho({\Phi^*}^{-1}(R - R_0)) \circ Q = \tilde\rho(R) \circ Q.\end{aligned} \tag{5.16}$$

The equality between the first and the last term of (5.16) holds also in case (B), and the proof is analogous; we have to use

$$\Phi^*\left((\nabla_{r_Q}u(r_Q))|_{\Gamma_0}\right) = Q^{-1}\Phi^*\left((\nabla_r u(r))|_{\Gamma_0}\right) \circ Q$$

instead of (5.15), with u and ∇_r defined by by Lemma 18 (ii) and (3.55), respectively. Setting $X = H^{s+d}(S^{N-1})$, $Y = H^s(S^{N-1})$, $U(g)u = V(g)u = u \circ g$, $F = \tilde{\rho}$ and applying Lemmas 27 and 28 (ii) we find analyticity and boundedness of $\tilde{\rho}$ from

$$B_{R_0}(\delta, H^{s+d}(S^{N-1})) \cap H^{s+d+n}(S^{N-1})$$

to $H^{s+n}(S^{N-1})$, and (i) follows from this because of $\rho(r) = \Phi^{*-1}\tilde{\rho}(\Phi^* r + R_0)$.

Lemma 28 (i) yields

$$D^\alpha \tilde{\rho}(R)) = \sum_{k=1}^{|\alpha|} \sum_{\beta_1+\ldots+\beta_k=\alpha} C_{\beta_1\ldots\beta_k} \tilde{\rho}^{(k)}(R)[D^{\beta_1}R, \ldots, D^{\beta_k}R]$$

for all $R \in B_{R_0}(\delta, H^{s+d}(\Gamma_0)) \cap H^{s+d+n}(\Gamma_0)$. This implies (iii) as can be seen by setting $R = \Phi^* r + R_0$, applying Φ^{*-1} on both sides and using (5.14) and $\Phi^{*-1}D^\beta = \tilde{D}^\beta \Phi^{*-1}$ for arbitrary multiindices β.

In the sequel, we will use the notations $x_n \xrightarrow{X} x$ for norm convergence and $x_n \xrightharpoonup{X} x$ for weak convergence in the (Banach) space X.

In order to prove (ii), consider an arbitrary sequence

$$\{r_n\} \subset B_0(\varepsilon, H^{s+d}(\Gamma_0)) \cap H^{s+n+d}(\Gamma_0)$$

such that $r_n \xrightharpoonup{H^{s+n+d}(\Gamma_0)} r^*$. We choose a $\sigma \in \left(2 + \frac{N}{2} - \frac{d}{2}, s\right)$. Due to the compactness of the embedding

$$H^{s+n+d}(\Gamma_0) \hookrightarrow\hookrightarrow H^{\sigma+n+d}(\Gamma_0)$$

this implies $r_n \xrightarrow{H^{\sigma+n+d}(\Gamma_0)} r^*$ and thus by (i)

$$\rho(r_n) \xrightarrow{H^{\sigma+n}(\Gamma_0)} \rho(r^*). \tag{5.17}$$

On the other hand, $\{r_n\}$ is bounded in $H^{s+n+d}(\Gamma_0)$ and thus by (i) $\{\rho(r_n)\}$ is bounded in $H^{s+n}(\Gamma_0)$. Consider now an arbitrary subsequence $\{\rho(r_{n'})\}$ of $\{\rho(r_n)\}$ such that

$$\rho(r_{n'}) \xrightharpoonup{H^{s+n}(\Gamma_0)} \rho^*.$$

This implies, by compactness of the embedding $H^{s+n}(\Gamma_0) \hookrightarrow\hookrightarrow H^{\sigma+n}(\Gamma_0)$,

$$\rho(r_{n'}) \xrightarrow{H^{\sigma+n}(\Gamma_0)} \rho^*$$

and thus by (5.17) $\rho^* = \rho(r^*)$. Hence we can conclude ([93], Proposition 10.13 (4))

$$\rho(r_n) \xrightharpoonup{H^{s+n}(\Gamma_0)} \rho(r^*). \quad \blacksquare$$

5.2 Existence results II

In view of the chain rule which has been proved in the previous section, it will be advantageous to work with norms which are generated by the differential operators $\tilde{D}_j$. As we are dealing only with the manifold Γ_0 in the sequel, the tilde will be omitted. In order to avail ourselves of an Hilbert space structure, for fixed $\sigma \in \mathbf{R}$ we define for $n \in \mathbf{N}$ Hilbert spaces $H^{\sigma,n}(\Gamma_0)$ by the scalar product

$$(u,v)_{\sigma,n} = \sum_{|\alpha|\le n} (D^\alpha u, D^\alpha v)_{H^\sigma(\Gamma_0)}.$$

By Lemma 27, we clearly have $H^{s,n}(\Gamma_0) = (H^s(\Gamma_0))^{(n)} = H^{s+n}(\Gamma_0)$ with equivalence of norms. Note that

$$(u,v)_{\sigma,n} = (S_{\sigma,n}u, v)_0,$$

where

$$S_{\sigma,n} = \sum_{|\alpha|\le n} (D^\alpha)^* \Lambda^{2\sigma} D^\alpha \tag{5.18}$$

is an elliptic pseudodifferential operator of order $2(\sigma+n)$.

In the following lemma we take advantage of the chain rule: Although our approach has not provided a decomposition of ρ as a quasilinear operator, we obtain one for the operators $D^\alpha \circ \rho$.

Lemma 30 *(Quasilinearity of $D^\alpha \circ \rho$)*

Suppose Assumption 1 *and* (5.11) *hold, let α be a multiindex with $0 < |\alpha| \le n$.*

(A) (Stokes flow)

Let ρ be given by (3.54), *$d = 1$. Suppose $s > 2 + \frac{N}{2} - \frac{d}{2}$.*

(i) We have the decomposition

$$D^\alpha \rho(r) = \rho'(r)[D^\alpha r] + G_\alpha(r), \qquad r \in B_0(\varepsilon, H^{s+d}(\Gamma_0)) \cap H^{s+d+n}(\Gamma_0)$$

for sufficiently small $\varepsilon > 0$, where G_α maps $B_0(\varepsilon, H^{s+d}(\Gamma_0)) \cap H^{s+d+n-1}(\Gamma_0)$ analytically into $H^s(\Gamma_0)$ and

$$\|G_\alpha(r)\|_s^{\Gamma_0} \le C_{s,\alpha}(\|r\|_{s+|\alpha|+d-1}^{\Gamma_0}{}^2 + 1).$$

(ii) If, furthermore, $\|r\|_{s+d+\frac{1}{2}}^{\Gamma_0} < K$, then for all $\delta > 0$ there is a constant $C_{s,\alpha,K,\delta}$ such that

$$\|G_\alpha(r)\|_s^{\Gamma_0} \le \delta \|r\|_{s+|\alpha|+d}^{\Gamma_0} + C_{s,\alpha,K,\delta}.$$

(iii) If $r, w \in B_0(\varepsilon, H^{s+d}(\Gamma_0)) \cap B_0(K, H^{s+d+|\alpha|-1}(\Gamma_0))$, then there is a constant $C_{s,|\alpha|,K}$ such that

$$\|G_\alpha(r) - G_\alpha(w)\|_s^{\Gamma_0} \le C_{s,\alpha,K}\|r-w\|_{s+d+|\alpha|-1}^{\Gamma_0}.$$

(B) (Hele-Shaw flow)

Let ρ be given by (3.56), *$d = 3$. Then the same assertions as in (A) hold.*

Proof: The chain rule in Lemma 29 (iii) yields

$$\begin{aligned} G_\alpha(r) &= \rho'(r)[D^\alpha \mathcal{R}_0] \\ &\quad + \sum_{k=2}^{|\alpha|} \sum_{\beta_1+\ldots+\beta_k=\alpha} C_{\beta_1,\ldots,\beta_k} \rho^{(k)}(r)[D^{\beta_1}(r+\mathcal{R}_0), \ldots, D^{\beta_k}(r+\mathcal{R}_0)]. \end{aligned}$$

We recall from Lemma 6 (ii) that the mappings $r \longmapsto \rho^{(k)}(r)$ are analytic from $B_0(\varepsilon, H^{s+d}(\Gamma_0))$ into $\mathcal{L}\left((H^{s+d}(\Gamma_0))^k, H^s(\Gamma_0)\right)$, hence G_α maps $B_0(r, H^{s+d}(\Gamma_0)) \cap H^{s+d+n-1}(\Gamma_0)$ analytically into $H^s(\Gamma_0)$, and

$$\begin{aligned}\|G_\alpha(r)\|_s^{\Gamma_0} &\le C_{s,\alpha}\left(1+\sum_{k=2}^{|\alpha|}\sum_{\beta_1+\ldots+\beta_k=\alpha}\left\|D^{\beta_1}(r+\mathcal{R}_0)\right\|_{s+d}^{\Gamma_0}\cdots\right.\\ &\qquad\left.\cdots\left\|D^{\beta_k}(r+\mathcal{R}_0)\right\|_{s+d}^{\Gamma_0}\right). \end{aligned} \tag{5.19}$$

To obtain (i) we estimate

$$\left\|D^{\beta_1}(r+\mathcal{R}_0)\right\|_{s+d}^{\Gamma_0} \le C_{s,\alpha}\|r+\mathcal{R}_0\|_{s+d+|\alpha|-1}^{\Gamma_0},$$

where we used that $|\beta_1| \le |\alpha| - 1$. On the other hand, we also have

$$h = \sum_{j=2}^{k} |\beta_j| \le |\alpha| - 1.$$

Taking into account that

$$s+d+|\beta_j| = (s+d+h)\frac{|\beta_j|}{h} + (s+d)\left(1-\frac{|\beta_j|}{h}\right), \qquad j = 2,\ldots,k,$$

we find by an interpolation inequality

$$\begin{aligned}\left\|D^{\beta_1}(r+\mathcal{R}_0)\right\|_{s+d}^{\Gamma_0} &\le C_{s,\alpha}\|r+\mathcal{R}_0\|_{s+d+|\beta_j|}^{\Gamma_0}\\ &\le C_{s,\alpha}\|r+\mathcal{R}_0\|_{s+d+h}^{\Gamma_0}{}^{\frac{|\beta_j|}{h}}\|r+\mathcal{R}_0\|_{s+d}^{\Gamma_0}{}^{1-\frac{|\beta_j|}{h}}\end{aligned}$$

and hence from (5.19)

$$\begin{aligned}\|G_\alpha(r)\|_s^{\Gamma_0} &\le C_{s,\alpha}\left(1+\|r+\mathcal{R}_0\|_{s+d+|\alpha|-1}^{\Gamma_0}{}^{2}\|r+\mathcal{R}_0\|_{s+d}^{\Gamma_0}{}^{k-2}\right)\\ &\le C_{s,\alpha}\left(\|r\|_{s+d+|\alpha|-1}^{\Gamma_0}{}^{2}+1\right),\end{aligned}$$

where we have used that $\mathcal{R}_0$ is a fixed smooth function and $\|r\|_{s+d}^{\Gamma_0} < \varepsilon$.

To obtain (ii) we note that $|\beta_j| < |\alpha|$ for $j = 1\ldots,k$ and hence

$$s+d+|\beta_j| < (s+d+|\alpha|)\frac{|\beta_j|}{|\alpha|} + \left(s+d+\frac{1}{2}\right)\left(1-\frac{|\beta_j|}{|\alpha|}\right),$$

from which we can conclude that there is a $\sigma > 0$ depending only on α such that

$$s+d+|\beta_j| \le (s+d+|\alpha|-\sigma)\frac{|\beta_j|}{|\alpha|} + \left(s+d+\frac{1}{2}\right)\left(1-\frac{|\beta_j|}{|\alpha|}\right).$$

Hence, by an interpolation inequality as in the proof of (i),

$$\begin{aligned}\|G_\alpha(r)\|_s^{\Gamma_0} &\le C_{s,\alpha}\left(1+\|r+\mathcal{R}_0\|_{s+d+|\alpha|-\sigma}^{\Gamma_0}\|r+\mathcal{R}_0\|_{s+d+\frac{1}{2}}^{\Gamma_0}{}^{k-1}\right)\\ &\le C_{s,\alpha,K}\left(\|r+\mathcal{R}_0\|_{s+d+|\alpha|-\sigma}^{\Gamma_0}+1\right)\\ &\le C_{s,\alpha,K}\left(\|r\|_{s+d+|\alpha|-\sigma}^{\Gamma_0}+1\right) \le \delta\|r\|_{s+d+\alpha}^{\Gamma_0}+C_{s,\alpha,K,\delta}\end{aligned}$$

for arbitrary $\delta > 0$, where in the last step another interpolation inequality has been used.

Assertion (iii) follows in a straightforward manner from

$$\begin{aligned} G_\alpha(r) - G_\alpha(w) &= (\rho'(r) - \rho'(w))[D^\alpha \mathcal{R}_0] \\ &+ \sum_{k=2}^{|\alpha|} \sum_{\beta_1+\ldots+\beta_k=\alpha} C_{\beta_1,\ldots,\beta_k}(\rho^{(k)}(r) - \rho^{(k)}(w))[D^{\beta_1}(w+\mathcal{R}_0), \ldots, D^{\beta_k}(w+\mathcal{R}_0)] \\ &+ \sum_{k=2}^{|\alpha|} \sum_{\beta_1+\ldots+\beta_k=\alpha} C_{\beta_1,\ldots,\beta_k} \sum_{j=1}^{k} \rho^{(k)}[D^{\beta_1}(r+\mathcal{R}_0), \ldots, D^{\beta_{j-1}}(r+\mathcal{R}_0), \\ &\quad D^{\beta_j}(r-w), D^{\beta_{j+1}}(w+\mathcal{R}_0), \ldots, D^{\beta_k}(w+\mathcal{R}_0)] \end{aligned}$$

and the Lipschitz continuity of the mappings $\rho^{(k)}$ from $B_0(\varepsilon, H^{s+d}(\Gamma_0))$ to

$$\mathcal{L}\left((H^{s+d}(\Gamma_0))^k, H^s(\Gamma_0)\right). \quad \blacksquare$$

The quasilinearity of the $D^\alpha \rho$ enables us to give energy estimates for solutions of (4.16) in the usual way.

Lemma 31 *(Local a priori estimate)*
Suppose Assumption 1 *and* (5.11) *hold,* $s > 2 + \frac{N}{2}$.
(A) (Stokes flow)
Let ρ *be defined by* (3.54), $d = 1$. *There is an* $\varepsilon > 0$ *such that*
(i) for all integer $n \geq d$ *we have an estimate*

$$(\rho(r), r)_{s,n} \leq C_{s,n}\left(\|r\|_{s,n}^{\Gamma_0\,2} + 1\right) \qquad \forall r \in B_0(\varepsilon, H^{s+\frac{d}{2}+\frac{1}{2}}(\Gamma_0)) \cap H^{s+n+d}(\Gamma_0),$$

(ii) for all $r, w \in B_0(\varepsilon, H^{s+\frac{d}{2}}(\Gamma_0)) \cap B_0(K, H^{s+\frac{3}{2}d}(\Gamma_0))$ *we have an estimate*

$$(\rho(r) - \rho(w), r - w)_{s,d} \leq C_{s,K}\|r - w\|_{s,d}^{\Gamma_0\,2}.$$

(B) (Hele-Shaw flow)
Let ρ *be defined by* (3.56), $d = 3$. *Then the same assertions as in (A) hold.*

Remark: In view of the application given later, we emphasize that ε can be chosen independently of n. □

Proof: (i) By Lemma 30, we have

$$\begin{aligned} (\rho(r), r)_{s,n} &= \sum_{|\alpha|\leq n} (D^\alpha \rho(r), D^\alpha r)_s \\ &= (\rho(r), r)_s + \sum_{1\leq|\alpha|\leq n} \left((\rho'(r)[D^\alpha r], D^\alpha r)_s + (G_\alpha(r), D^\alpha r)_s\right) \end{aligned}$$

and estimate the terms on the right hand side separately. Due to the analyticity of ρ near 0 on $H^{s+\frac{d}{2}}(\Gamma_0)$, we immediately have

$$(\rho(r), r)_s \leq \|\rho(r)\|_{s-\frac{d}{2}}^{\Gamma_0}\|r\|_{s+\frac{d}{2}}^{\Gamma_0} \leq C_s\|r\|_{s+\frac{d}{2}}^{\Gamma_0\,2}.$$

From the proof of Lemma 26 we recall (cf. (4.17))

$$(\rho'(r)[D^\alpha r], D^\alpha r)_s \leq -c_s \|D^\alpha r\|^{\Gamma_0\;2}_{s+\frac{d}{2}} + C_s \|D^\alpha r\|^{\Gamma_0\;2}_{s+\frac{d}{2}-\frac{1}{2}}$$

and from Lemma 30 (ii) with s replaced by $s - \frac{d}{2}$, using that $\|r\|^{\Gamma_0}_{s+\frac{d}{2}+\frac{1}{2}} < \varepsilon$,

$$(G_\alpha(r), D^\alpha r)_s \leq \|G_\alpha(r)\|^{\Gamma_0}_{s-\frac{d}{2}} \|D^\alpha r\|^{\Gamma_0}_{s+\frac{d}{2}} \leq \left(\delta \|r\|^{\Gamma_0}_{s+\frac{d}{2}+|\alpha|} + C_{s,\alpha,\delta}\right) \|r\|^{\Gamma_0}_{s+\frac{d}{2}+|\alpha|}$$

for any positive δ. Carrying out the summation over α, we obtain

$$(\rho(r), r)_{s,n} \leq -c\|r\|^{\Gamma_0\;2}_{s+\frac{d}{2},n} + C_s \|r\|^{\Gamma_0\;2}_{s+\frac{d}{2}+n-\frac{1}{2}} + C_n \delta \|r\|^{\Gamma_0\;2}_{s+n+\frac{d}{2}} + C_{s,n,\delta} \|r\|^{\Gamma_0}_{s+\frac{d}{2}+n}$$

and after choosing a sufficiently small δ

$$\begin{aligned} (\rho(r), r)_{s,n} &\leq -c\|r\|^{\Gamma_0\;2}_{s+\frac{d}{2},n} + C_s \|r\|^{\Gamma_0\;2}_{s+\frac{d}{2}+n-\frac{1}{2}} + C_{s,n,\delta} \|r\|^{\Gamma_0}_{s+\frac{d}{2}+n} \\ &\leq -c\|r\|^{\Gamma_0\;2}_{s+\frac{d}{2}+n} + C_s \|r\|^{\Gamma_0\;2}_{s+\frac{d}{2}+n-\frac{1}{2}} + C_{s,n}. \end{aligned}$$

In case (A) the assertion is immediate now, in case (B) it follows from the interpolation inequality

$$\|r\|^{\Gamma_0\;2}_{s+\frac{d}{2}+n-\frac{1}{2}} \leq \delta \|r\|^{\Gamma_0\;2}_{s+\frac{d}{2}+n} + C_{s,n,\delta} \|r\|^{\Gamma_0\;2}_{s}$$

with sufficiently small δ.

(ii) We have, using Lemma 30 again,

$$\begin{aligned} &(\rho(r) - \rho(w), r - w)_{s,d} = (\rho(r) - \rho(w), r - w)_s \\ &+ \sum_{1 \leq |\alpha| \leq n} \{((\rho'(r) - \rho'(w))[D^\alpha w], D^\alpha (r - w))_s \\ &\qquad + (\rho'(r)[D^\alpha (r - w)], D^\alpha (r - w))_s + (G_\alpha(r) - G_\alpha(w), D^\alpha (r - w))_s\} \end{aligned}$$

and estimate the terms on the right separately. Using the analyticity of ρ and ρ' near 0 on $B_0(\varepsilon, H^{s+\frac{d}{2}}(\Gamma_0))$ and $\|w\|^{\Gamma_0}_{s+\frac{3}{2}d} < K$ we get

$$\begin{aligned} (\rho(r) - \rho(w), r - w)_s &\leq \|\rho(r) - \rho(w)\|^{\Gamma_0}_{s-\frac{d}{2}} \|r - w\|^{\Gamma_0}_{s+\frac{d}{2}} \\ &\leq C_s \|r - w\|^{\Gamma_0\;2}_{s+\frac{d}{2}}, \\ ((\rho'(r) - \rho'(w))[D^\alpha w], D^\alpha (r - w))_s &\leq C_s \|r - w\|^{\Gamma_0}_{s+\frac{d}{2}} \|D^\alpha w\|^{\Gamma_0}_{s+\frac{d}{2}} \times \\ &\qquad \times \|D^\alpha (r - w)\|^{\Gamma_0}_{s+\frac{d}{2}} \\ &\leq C_s K \|r - w\|^{\Gamma_0}_{s+\frac{d}{2}} \|r - w\|^{\Gamma_0}_{s+\frac{3}{2}d}, \end{aligned}$$

and using the coercivity of $\rho'(r)$ and Lemma 30 (iii) (with s replaced by $s - \frac{d}{2} + \frac{1}{2}$)

$$\begin{aligned} (\rho'(r)[D^\alpha (r - w)], D^\alpha (r - w))_s &\leq -c\|r - w\|^{\Gamma_0\;2}_{s+\frac{d}{2}} + C_s \|r - w\|^{\Gamma_0\;2}_{s+\frac{3}{2}d-\frac{1}{2}}, \\ (G_\alpha(r) - G_\alpha(w), D^\alpha (r - w))_s &\leq \|G_\alpha(r) - G_\alpha(w)\|^{\Gamma_0}_{s-\frac{d}{2}+\frac{1}{2}} \|D^\alpha (r - w)\|^{\Gamma_0}_{s+\frac{d}{2}-\frac{1}{2}} \\ &\leq C_{s,K} \|r - w\|^{\Gamma_0\;2}_{s+\frac{3}{2}d-\frac{1}{2}}. \end{aligned}$$

Summing up and using an interpolation inequality in case (B), we get

$$\begin{aligned}(\rho(r)-\rho(w),r-w)_{s,d} &\le -c\|r-w\|^{\Gamma_0}_{s+\frac{d}{2},d}{}^2 + C_{s,K}\|r-w\|^{\Gamma_0}_{s+\frac{3}{2}d-\frac{1}{2}}{}^2 \\ &\le C_{s,K}\|r-w\|^{\Gamma_0}_{s,d}{}^2 . \quad \blacksquare\end{aligned}$$

Remark: In the Stokes flow case (A), the fact that ρ is a first-order operator has an interesting consequence: Using Lemma 30 (i), we can estimate (under the same assumptions as in Lemma 31)

$$\begin{aligned}(G_\alpha(r),D^\alpha r)_s &\le \|G_\alpha(r)\|^{\Gamma_0}_s\|D^\alpha r\|^{\Gamma_0}_s \le C_{s,\alpha}\left(\|r\|^{\Gamma_0}_{s+|\alpha|-\frac{1}{2}}{}^2+1\right)\|r\|^{\Gamma_0}_{s+|\alpha|} \\ &\le C_{s,\alpha}\left(\|r\|^{\Gamma_0}_{s+|\alpha|}{}^3+1\right).\end{aligned}$$

Consequently, we get an estimate

$$(\rho(r),r)_{s,n} \le C_{s,n}\left(\|r\|^{\Gamma_0}_{s,n}{}^3+1)\right)$$

just by using

$$(\rho'(r)[h],h)_s \le C_s\|h\|^{\Gamma_0}_s{}^2,$$

i.e. without using the coercivity of the linearization. This indicates the possibility of applying our method to related nonlinear first-order hyperbolic evolution problems. In case (B), however, the coercivity of the linearization is needed to keep the nonlinearity under control. □

The a priori estimates given in Lemma 31 strongly suggest the application of Galerkin approximations for the existence proof, proceeding essentially as in the proof of the of the abstract Theorem A in [50]. However, due to the local character of all our considerations, the operator ρ is defined and has the necessary properties only for arguments which are small in a fixed Hilbert space. Hence, we will have to control the growth of the Galerkin approximations not only in one fixed space but in two different spaces of the scale $\{H^\sigma(\Gamma_0)\}$. The following lemma which generalizes the idea of diagonalizing the Gram matrix provides a preparation for this.

Lemma 32 *(Orthogonal basis for a pair of Sobolev spaces)*
Let $\sigma_1,\sigma_2 \in \mathbf{R}$, n_1,n_2 nonnegative integers such that $n_2+\sigma_2 > n_1+\sigma_1$. There is an orthonormal basis of $H^{\sigma_1,n_1}(\Gamma_0)$, consisting of smooth functions, which is an orthogonal basis for $H^{\sigma_2,n_2}(\Gamma_0)$.

Proof: We recall the definition of the operators $S_{\sigma,n}$ as given in (5.18). The unbounded linear operator S on $H^{\sigma_1,n_1}(\Gamma_0)$, defined by

$$\begin{aligned}D(S) &= H^{2(\sigma_2+n_2)-\sigma_1-n_1}(\Gamma_0), \\ S &= S^{-1}_{\sigma_1,n_1}S_{\sigma_2,n_2}\end{aligned}$$

is easily seen to be symmetric on $H^{\sigma_1,n_1}(\Gamma_0)$ and it is an elliptic pseudodifferential operator of order $2(\sigma_2+n_2-\sigma_1-n_1)$. Hence $R(S)=H^{\sigma_1,n_1}(\Gamma_0)$ and thus (cf. e.g. [88], Satz 17.6(b)) S is self-adjoint. By Rellich's theorem, the compactness of the embedding

$$H^{\sigma_2,n_2}(\Gamma_0) \hookrightarrow\hookrightarrow H^{\sigma_1,n_1}(\Gamma_0)$$

implies that S has a purely discrete spectrum and thus a complete orthonormal system of eigenfunctions $\{e_j\}$ which is obviously an orthogonal basis of $H^{\sigma_2,n_2}(\Gamma_0)$. The smoothness of the e_j follows from elliptic regularity theory. ■

As in [50], we will use the notations IT for the interval $[0,T]$ and $C_w(IT,X)$, $C^1_w(IT,X)$ for the space of weakly continuous and weakly (continuously) differentiable functions from IT into the Banach space X, i.e. $g \in C^1_w(IT,X)$ iff the mapping $t \mapsto \langle\varphi, g(t)\rangle$ is in $C^1(IT)$ for all $\varphi \in X'$.

Proposition 5 *(Existence theorem II)*

Suppose Assumption 1 *and* (5.11) *hold, let* $s > 2 + \frac{N}{2}$.

(A) (Stokes flow)

Let ρ *be defined by* (3.54), $d = 1$. *There are positive constants* ε *and* T *depending only on* s *and* Γ_0 *such that:*

(i) For any integer $n \geq d$ *and any* $r_0 \in B_0(\varepsilon, H^{s,d}(\Gamma_0)) \cap H^{s+n}(\Gamma_0)$ *the initial value problem* (4.16) *has a solution*

$$r \in C_w(IT, H^{s+n}(\Gamma_0)) \cap C^1_w(IT, H^{s+n-d}(\Gamma_0))$$

with $r(t) \overset{H^{s+n}(\Gamma_0)}{\longrightarrow} r_0$ *as* $t \to 0$.

(ii) For any $r_0 \in B_0(\varepsilon, H^{s,d}(\Gamma_0))$, (4.16) *has at most one solution in*

$$C^1(IT, H^{s+d}(\Gamma_0)) \cap L^\infty(IT, H^{s+\frac{3}{2}d}(\Gamma_0)).$$

(B) (Hele-Shaw flow)

Let ρ *be given by* (3.56), $d = 3$. *Then the same assertions hold.*

Proof: (i) The proof of (i) can be given as a modification of the proof of theorem A in [50]. The correspondence with the notation that is used there is as follows:

$$\begin{aligned} A(\cdot,t) &= -\rho, \\ \{V,H,X\} &= \{H^{s+n+d}(\Gamma_0), H^{s,n}(\Gamma_0), H^{s+n-d}(\Gamma_0)\} \\ \langle u,v\rangle &= \left(S_{s,n}^{\frac{s+n+d}{2(s+n)}} u, S_{s,n}^{\frac{s+n-d}{2(s+n)}} v\right)_0 . \end{aligned}$$

Hence $\langle u,v\rangle = (u,v)_{s,n}$ for all $u,v \in H^{s,n}(\Gamma_0)$, i.e. the triplet $\{V,H,X\}$ is admissible. To avoid ambiguity, we will keep the notation as it has been used previously.

By Lemmas 31 and 29 (ii) we can choose ε small enough to ensure that

$$(\rho(r), r)_{s,d} \leq C_s^*\left(1 + \|r\|_{s,d}^{\Gamma_0\,2}\right), \tag{5.20}$$

$$(\rho(r), r)_{s,n} \leq C_{s,n}\left(1 + \|r\|_{s,n}^{\Gamma_0\,2}\right), \tag{5.21}$$

$$(\rho(r) - \rho(w), r - w)_{s,d} \leq C_{s,\|r\|^{\Gamma_0}_{s+\frac{3}{2}d},\|w\|^{\Gamma_0}_{s+\frac{3}{2}d}} \|r-w\|_{s,d}^{\Gamma_0\,2} \tag{5.22}$$

for all $r, w \in B_0(2\varepsilon, H^{s,d}(\Gamma_0)) \cap H^{s+n}(\Gamma_0)$ and ρ is weakly sequentially continuous on $B_0(2\varepsilon, H^{s,d}(\Gamma_0)) \cap H^{s+n}(\Gamma_0)$ into $H^{s+n-d}(\Gamma_0)$.

By Lemma 32 there is an orthonormal basis of $H^{s,d}(\Gamma_0)$ which is also an orthogonal basis $\{e_j\}$ of $H^{s,n}(\Gamma_0)$. Let P_j be the orthogonal projection in $H^{s,d}(\Gamma_0)$ onto $M_j = \mathrm{span}\{e_1,\ldots,e_j\}$.

Clearly the restriction of P_j to $H^{s,n}(\Gamma_0)$ is the orthogonal projection onto M_j in $H^{s,n}(\Gamma_0)$. For all positive $j \in \mathbb{N}$ we define the Galerkin approximations r_j as usual by

$$\frac{\partial r_j}{\partial t} = P_j \rho(r_j), \qquad r_j(0) = P_j r_0. \tag{5.23}$$

We have to prove now that there is a $T > 0$ independent of n and a constant K such that

$$\begin{aligned} \|r_j(t)\|_{s,d}^{\Gamma_0} &\leq 2\varepsilon \qquad \forall t \in IT\ \forall j \in \mathbb{N}, & (5.24)\\ \|r_j(t)\|_{s,n}^{\Gamma_0} &\leq K \qquad \forall t \in IT\ \forall j \in \mathbb{N}, & (5.25)\end{aligned}$$

the assertion (i) will follow then by the arguments given in [50]. Consider the unique solution m of the initial value problem

$$\frac{\partial m}{\partial t} = 2C_s^*(1+m), \qquad m(0) = \varepsilon^2,$$

where C_s^* is the constant from (5.20) and choose T to be the (uniquely defined) positive number for which $m(T) = 4\varepsilon^2$. At first we will show (5.24). Suppose the opposite: This would imply that for some j there is a $T^* \in (0, T)$ such that

$$\|r(T^*)\|_{s,d}^{\Gamma_0} = 2\varepsilon, \quad \|r(t)\|_{s,d}^{\Gamma_0} < 2\varepsilon\ \forall t \in [0, T).$$

On IT^* we get from (5.20) the differential inequality

$$\frac{d}{dt}\left(\|r_j(t)\|_{s,d}^{\Gamma_0\,2}\right) \leq C_s^*\left(1 + \|r_j(t)\|_{s,d}^{\Gamma_0\,2}\right)$$

and integrating it and using the strict monotonicity of m we find

$$\|r_j(T^*)\|_{s,d}^{\Gamma_0\,2} \leq m(T^*) < 4\varepsilon^2$$

in contradiction to the definition of T^*. Hence (5.24) holds, and on the basis of this (5.25) can be proved analogously to (5.24), using (5.21) instead of (5.20).

(ii) Suppose $r_1, r_2 \in C^1(IT, H^{s+d}(\Gamma_0)) \cap L^\infty(IT, H^{s+\frac{3}{2}d}(\Gamma_0))$ are two solutions of (4.16). From (5.20) one concludes $\|r_1(t)\|_{s,d}^{\Gamma_0}, \|r_2(t)\|_{s,d}^{\Gamma_0} < 2\varepsilon$ for all $t \in IT$ with a certain $T > 0$ in the same way as (5.24) was proved above. Thus, (5.22) together with $r_1, r_2 \in L^\infty(IT, H^{s+\frac{3}{2}d}(\Gamma_0))$ yield

$$\frac{d}{dt}\left(\|r_1(t) - r_2(t)\|_{s,d}^{\Gamma_0\,2}\right) \leq C_{r_1,r_2,s}\|r_1(t) - r_2(t)\|_{s,d}^{\Gamma_0\,2}$$

for almost all $t \in IT$, and from the Gronwall inequality follows $r(t) = v(t)$ for all $t \in IT$. ■

Remarks:

1. We want to emphasize that the initial value r_0 is not assumed to be small in $H^{s,n}(\Gamma_0)$ and that no smoothness is lost on the whole existence interval IT. Such results are not easily obtained by the methods for fully nonlinear equations that have been used in Section 4.3.

2. Taking into account that $C_w^1(IT, H^s(\Gamma_0)) \subset C^1(IT, H^\sigma(\Gamma_0))$ for $s > \sigma$ due to the compactness of the embedding $H^s(\Gamma_0) \hookrightarrow\hookrightarrow H^\sigma(\Gamma_0)$ and $C_w(IT, X) \subset L^\infty(IT, X)$ for any Banach space X due to the Uniform Boundedness principle we find uniqueness for the solution of (4.16) obtained in the proof of (i) if $r_0 \in H^{s+\frac{3}{2}d}(\Gamma_0)$.

3. If the initial values are supposed to be bounded in a slightly higher Sobolev norm, then, for fixed $t \in IT$, Lipschitz-continuous dependence of $r(t)$ on the initial value r_0 in a lower norm can be shown in a similar way as in the proof of (ii). □

5.3 Smoothness of the boundary

The quasilinear structure of the evolution problem (4.16) and its parabolic character enable us to give a proof for the smoothing property of the surface motion laws we consider. The basis for this is the following abstract result:

Proposition 6 *(Spatial smoothness for nonautonomous linear parabolic evolution equations)*
Let D and X be Banach spaces, let D be densely and continuously embedded in X. For $t_0, t_1 \in \mathbf{R}$, $t_0 < t_1$, consider mappings

$$\begin{aligned} A: &\quad [t_0, t_1] \longrightarrow \mathcal{L}(D, X), \\ f: &\quad [t_0, t_1] \longrightarrow X \end{aligned}$$

which are Hölder-continuous with exponent $\gamma \in (0,1)$. Assume that for all $t \in [t_0, t_1]$ the norm $\|\cdot\|_{A(t)}$ defined by

$$\|x\|_{A(t)} = \|A(t)x\|_X + \|x\|_X \qquad \forall x \in D$$

is an equivalent norm on D, and $A(t)$ generates an analytic semigroup on X.
Then, for any $u_0 \in X$, the initial value problem

$$\left.\begin{aligned} \frac{du}{dt} &= A(t)u + f(t) \\ u(t_0) &= u_0 \end{aligned}\right\}$$

has a unique solution $u \in C^\gamma([t_0, t_1], X)$ which for any $\delta \in (t_0, t_1)$ satisfies

$$u \in C^1([\delta, t_1], X) \cap C^\gamma([\delta, T], D).$$

Proof: The proposition is an easy consequence of Theorem 6.1.4. in [61]. ■
Remark: Proposition 6 is by no means an optimal result. For our purposes, however, it will be sufficient. □

Using the chain rule and a bootstrapping argument, we can conclude from this that for positive times the solution of our initial value problems are C^∞-smooth:

Proposition 7 *(Smoothing property of the evolution)*
Suppose Assumption 1 and (5.11) hold, let $s > 3 + \frac{N}{2}$.
(A) (Stokes flow)
Let ρ be defined by (3.54), $d = 1$. There are positive numbers ε and T, such that for any solution of (4.16) which satisfies

$$r_0 \in B_0\left(\varepsilon, H^{s+d}(\Gamma_0)\right) \cap B_0\left(\varepsilon, H^{s+d-1}(\Gamma_0)\right)$$

and

$$r \in C^1(IT, H^{s+d}(\Gamma_0)), \tag{5.26}$$

we have

$$r \in C^1([\delta, T], C^\infty(\Gamma_0))$$

for any $\delta > 0$.
(B) (Hele-Shaw flow)
Let ρ be given by (3.56), $d = 3$. Then the same assertion holds.

Remarks: The assertion is to be understood in the sense that the time derivative of r, taken in the fixed space $H^{s+d}(\Gamma_0)$, maps $[\delta, T]$ continuously into $C^\infty(\Gamma_0)$.

From Proposition 5 and the remark after it it is clear that assumption (5.26) is satisfied if a slightly higher smoothness for r_0 is demanded. □

Proof: By the assumptions together with Lemmas 26 and 29, we can choose ε and T such that the following properties hold:

$$\|r(t)\|_{s+d}^{\Gamma_0} < 2\varepsilon, \ \|r(t)\|_{s+d-1}^{\Gamma_0} < 2\varepsilon \qquad \forall t \in IT,$$

the mappings

$$\begin{array}{ll} \rho: & B_0(2\varepsilon, H^{s+d-1}(\Gamma_0)) \longrightarrow H^{s-1}(\Gamma_0), \\ \rho: & B_0(2\varepsilon, H^{s+d}(\Gamma_0)) \cap H^{s+d+n}(\Gamma_0) \longrightarrow H^{s+n}(\Gamma_0) \end{array}$$

are analytic for all $n \in \mathbf{N}$, and for all $r \in B_0(2\varepsilon, H^{s+d-1}(\Gamma_0)) \cap B_0(2\varepsilon, H^{s+d}(\Gamma_0))$ we have that $\rho'(r)$ generates analytic semigroups both on $H^{s-1}(\Gamma_0)$ and on $H^s(\Gamma_0)$. Moreover, for $j = 0, 1$, $\mathcal{D}(\rho'(r)) = H^{s+d-j}(\Gamma_0)$ if $\rho'(r)$ is considered as unbounded operator on $H^{s-j}(\Gamma_0)$ with equivalence of the graph norm and the original norm on $H^{s-j}(\Gamma_0)$.

We arbitrarily choose $\gamma \in (0,1)$ and a strictly increasing sequence $\{\delta_k\} \subset (0, \delta)$. Due to the Sobolev embedding theorems and Lemma 27, we have

$$C^\infty(\Gamma_0) = \bigcap_{k=0}^{\infty} C^k(\Gamma_0) = \bigcap_{k=0}^{\infty} H^{s,k}(\Gamma_0)$$

with equivalent topologies, where the intersections are endowed with the projective limit topologies. Hence it is sufficient to show

$$D^\beta r \in C^1\left([\delta_k, T], H^s(\Gamma_0)\right) \cap C^\gamma\left([\delta_k, T], H^{s+d}(\Gamma_0)\right) \qquad \forall \beta : \ |\beta| \le k \tag{5.27}$$

for all $k \in \mathbf{N}$. This will be done inductively. For $k = 0$, (5.27) is ensured by (5.26). Suppose now (5.27) holds for $k = n$. This implies

$$r \in C^1\left([\delta_n, T], H^{s+n}(\Gamma_0)\right) \cap C^\gamma\left([\delta_n, T], H^{s+n+d}(\Gamma_0)\right).$$

We pick an arbitrary α with $|\alpha| = n + 1$ and set $u = D^\alpha r$. By Lemma 30 we get that $u \in C^1([\delta_n, T], H^{s-1}(\Gamma_0))$ satisfies

$$\left.\begin{array}{rcl} \dfrac{du}{dt} & = & A(t)u + f(t) \\ u(\delta_n) & = & D^\alpha r(\delta_n) \end{array}\right\}, \tag{5.28}$$

where, by our assumptions,

$$\begin{array}{rcl} A(t) & = & \rho'(r(t)) \in C^\gamma\left([\delta_n, T], \mathcal{L}\left(H^{s+d-1}(\Gamma_0), H^{s-1}(\Gamma_0)\right)\right) \\ & & \cap\, C^\gamma\left([\delta_n, T], \mathcal{L}\left(H^{s+d}(\Gamma_0), H^s(\Gamma_0)\right)\right) \\ f(t) & = & G_\alpha(r(t)) \in C^\gamma\left([\delta_n, T], H^s(\Gamma_0)\right). \end{array}$$

Applying Proposition 6 with $t_0 = \delta_n$, $t_1 = T$, $X = H^{s-1}(\Gamma_0)$, $D = H^{s+d-1}(\Gamma_0)$ to (5.28) yields $D^\alpha r\left(\frac{\delta_n + \delta_{n+1}}{2}\right) \in H^s(\Gamma_0)$. Hence, applying now Proposition 6 with $t_0 = \frac{\delta_n + \delta_{n+1}}{2}$,

$t_1 = T$, $X = H^s(\Gamma_0)$, $D = H^{s+d}(\Gamma_0)$ to

$$\left.\begin{array}{rcl} \dfrac{du}{dt} & = & A(t)u + f(t) \\ u\left(\frac{\delta_n+\delta_{n+1}}{2}\right) & = & D^\alpha r\left(\frac{\delta_n+\delta_{n+1}}{2}\right) \end{array}\right\}$$

yields

$$D^\alpha r \in C^1\left([\delta_{n+1}, T], H^s(\Gamma_0)\right) \cap C^\gamma\left([\delta_{n+1}, T], H^{s+d}(\Gamma_0)\right)$$

which completes the induction proof. ■

Chapter 6

Extensions and remarks

6.1 Near equilibrium

We recall from Lemmas 14 and 15 that both for Stokes flow and for Hele-Shaw flow the stationary solutions are given by balls of liquid at rest. These equilibrium states are, moreover, expected to be stable and to occur as limit states of the evolution. This has a physical reason: they correspond to the global minimum of the free energy. The aim of this section is to give a strict verification of these expectations.

More precisely, we will show that if (in an appropriate sense) $\Omega(0)$ is near a ball, then the solutions of our FBPs exist for all positive times and decay exponentially fast to the corresponding equilibrium state. Results of this type have been obtained previously for Hele-Shaw flow with $N = 2$ in [20]. The approach via complex function theory and spaces of analytic functions which is used in that paper is comparable to the one in Chapter 2 of this thesis.

The key idea in our discussion is to split the space $H^s(\Gamma_0)$ into eigenspaces of the operator ρ_1 that correspond to its negative and nonnegative eigenvalues, respectively. Working with a semi-norm adapted to this decomposition and taking advantage of the fact that we linearize around a stationary point will lead to estimates in which no lower order terms occur. On the other hand, the a priori valid conservation of volume and of the center of gravity will be used to keep control over the full norm in $H^s(\Gamma_0)$.

We recall the definition of ω_N from (3.29). Lemma 14 and 15 yield that the quantities $V = V(t)$ and $M = M(t)$ as defined by (3.38) are constant in time for any solution of the Stokes flow or the Hele-Shaw flow FBP. By appropriate shifting and scaling we can assume without loss of generality $V = \frac{\omega_N}{N}$ and $M = 0$, i.e. the volume and the center of gravity of $\Omega(t)$ are the same as for the unit ball. To consider small perturbations of the unit ball, it is natural to set

$$\Gamma_0 = S^{N-1}, \quad \zeta = n, \tag{6.1}$$

i.e. we have $\gamma \equiv 1$, Assumption 1 holds with $R_0 \equiv 1$, and, with the notations of Section 5.1, we have that Φ and Φ^* are the identity, and $\mathcal{R}_0 \equiv 0$. Therefore, taking the Fréchet derivative with respect to r at $r = 0$ on both sides of (5.12) yields the exact commutation relation

$$D^\alpha \rho_1 r = \rho_1 D^\alpha r \qquad \forall r \in H^{s+d+|\alpha|}(\Gamma_0). \tag{6.2}$$

In the special situation given by (6.1), ρ_1 has a simpler structure than in the general case:

Lemma 33 *(Linearization around the equilibrium)*
Suppose (6.1) *holds.*
(A) (Stokes flow)
Let ρ and S be defined by (3.54) *and* (3.31), (3.32), *respectively. Then*

$$\rho_1 r = n \cdot S((\Delta_{\Gamma_0} + (N-1)I)r\, n).$$

(B) (Hele-Shaw flow)
Let ρ and S be defined by (3.56) *and* (3.37), *respectively. Then*

$$\rho_1 r = S((\Delta_{\Gamma_0} + (N-1)I)r.$$

Proof: (A) It is easy to see that adding a fixed multiple of the normal vector to the right hand side of the boundary condition in the fixed-time problem will only change the pressure but not the velocity field. Hence we have, using the same notation as in Section 4.1,

$$\begin{aligned} u(r) &= \operatorname*{Tr}_{\Gamma_0} \Pi_1 \tilde{L}(r)^{-1} \begin{bmatrix} 0 \\ 0 \\ \kappa(r)\nu(r) \\ 0 \\ 0 \end{bmatrix} \\ &= \operatorname*{Tr}_{\Gamma_0} \Pi_1 \tilde{L}(r)^{-1} \begin{bmatrix} 0 \\ 0 \\ (\kappa(r) + N - 1)\nu(r) \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{6.3}$$

From $\kappa(0) \equiv -(N-1)$, $\gamma \equiv 1$, and $u(0) \equiv 0$ we find by calculating the Fréchet derivative of (6.3)

$$\rho_1 r = n \cdot S(\kappa_1 r n).$$

It remains to calculate κ_1. We proceed as in the proof of Lemma 21 (ii) and denote by DF the derivative of a quantity F with respect to r at $r = 0$. Due to $n = \xi = \zeta$ we find

$$\begin{aligned} DG[h] &= 2G(0)h, \\ D(G^{-1})[h] &= -2G(0)^{-1}h. \end{aligned}$$

Moreover, we will use the facts that $\kappa(0)$ is constant, $D\nu[h]$ is a vector field tangential to Γ_0 for all h and, as before,

$$D(\sqrt{g})[h] = -\kappa(0)\sqrt{g(0)}h.$$

Working with an arbitrary local parametrization, we obtain

$$\begin{aligned} D\kappa[h] &= D\left(\frac{1}{\sqrt{g}}\right)[h]\frac{\partial}{\partial w_i}\left(\sqrt{g(0)}g^{ij}(0)\frac{\partial n}{\partial w_j}\right) \cdot n \\ &+ \frac{1}{\sqrt{g(0)}}\frac{\partial}{\partial w_i}\left(D\left(\sqrt{g}\right)[h]g^{ij}(0)\frac{\partial n}{\partial w_j}\right) \cdot n \\ &+ \frac{1}{\sqrt{g(0)}}\frac{\partial}{\partial w_i}\left(\sqrt{g(0)}Dg^{ij}[h]\frac{\partial n}{\partial w_j}\right) \cdot n \\ &+ \frac{1}{\sqrt{g(0)}}\frac{\partial}{\partial w_i}\left(\sqrt{g(0)}g^{ij}(0)\frac{\partial hn}{\partial w_j}\right) \cdot n + \Delta_{\Gamma_0} n \cdot D\nu[h]. \end{aligned}$$

Hence, writing $\nabla_{\Gamma_0}(h,n) = g^{ij}(0)\dfrac{\partial h}{\partial w_i}\dfrac{\partial n}{\partial w_j}$ and using that this is a vector field tangential to Γ_0,

$$\begin{aligned} D\kappa[h] &= \kappa(0)h\Delta_{\Gamma_0}n\cdot n - \frac{\kappa(0)}{\sqrt{g(0)}}\frac{\partial}{\partial w_i}\left(\sqrt{g(0)}hg^{ij}(0)\frac{\partial n}{\partial w_j}\right)\cdot n \\ &\quad -\frac{2}{\sqrt{g(0)}}\frac{\partial}{\partial w_i}\left(\sqrt{g(0)}g^{ij}(0)h\frac{\partial n}{\partial w_j}\right)\cdot n \\ &\quad +\Delta_{\Gamma_0}h + 2\nabla_{\Gamma_0}(h,n)\cdot n + \Delta_{\Gamma_0}n\cdot hn + \Delta_{\Gamma_0}n\cdot D\nu[h] \\ &= \kappa(0)^2h - \kappa(0)^2h - \kappa_0\nabla_{\Gamma_0}(h,n)\cdot n - 2\kappa(0)h \\ &\quad -2\nabla_{\Gamma_0}(h,n)\cdot n + \Delta_{\Gamma_0}h + \kappa(0)h = \Delta_{\Gamma_0}h + (N-1)h. \end{aligned}$$

(B) Using analogous arguments, the assertion follows straightforwardly from the calculation of κ_1 and the fact that $u(0) = 0$. ∎

Let $\{Y_{kl}\,|\,l \in \mathbf{N},\ 1 \le k \le K(l,N)\}$ be an L^2-orthonormal basis of the spherical harmonics such that Y_{kl} is an eigenfunction of Δ_{Γ_0} belonging to the eigenvalue $-l(l+N-2)$. Using the expansion coefficients

$$u_{kl} = (u, Y_{kl})_0$$

we will work with the scalar product

$$(u,v)_s = u_{1\,0}v_{1\,0} + u_{k1}v_{k1} + \sum_{l=2}^{\infty}(l(l+N-2)-(N-1))^s u_{kl}v_{kl}$$

for all $s \in \mathbf{R}$. These scalar products are obviously equivalent to the usual ones. Note that $K(0,N) = 1$, $K(1,N) = N$, $Y_{1\,0}$ is constant, and

$$\operatorname{span}\{Y_{k1}\,|\,k = 1\ldots N\} = \operatorname{span}\{x_1\ldots,x_N\}.$$

Both for Stokes flow and for Hele-Shaw flow, the linearizations ρ_1 vanish on the subspace $\{Y_{kl}\,|\,l \le 1\}$. This follows from $\Delta_{\Gamma_0}Y_{k1} = -(N-1)Y_{k1}$ and the facts that the operator S defined by (3.31), (3.32) maps the normal vector field to 0 while the operator S defined by (3.37) vanishes on constants. Hence, it is natural to introduce the projection operator $\mathcal{P}$ by

$$\mathcal{P}u = \sum_{l\ge 2}u_{kl}Y_{kl}$$

which is orthogonal in all spaces $H^s(\Gamma_0)$ and commutes with ρ_1.

Using spherical coordinates, one straightforwardly obtains

$$\begin{aligned} V(r) &= \frac{1}{N}\int_{\Gamma_0}(1+r)^N\,d\Gamma, \\ M(r) &= \frac{1}{N+1}\int_{\Gamma_0}(1+r)^{N+1}n\,d\Gamma \end{aligned}$$

for the volume and the center of gravity of the domain Ω_r, respectively. On $H^s(\Gamma_0)$, $s > \frac{N-1}{2}$, we define the analytic function

$$F:\ H^s(\Gamma_0) \longrightarrow \mathbf{R}\times\mathbf{R}^N$$

by

$$F(r) = \left[\begin{array}{c} V(r) - \frac{\omega_N}{N} \\ M(r) \end{array} \right]$$

and the submanifold

$$\mathcal{M}_s = \{ r \in H^s(\Gamma_0) \mid F(r) = 0 \}.$$

Lemma 34 *(Norms and seminorms on $H^s(\Gamma_0)$ and $\mathcal{M}_s$)*
Let $s > \frac{N-1}{2}$. There are positive constants C and ε depending only on s such that

$$\begin{aligned} \|r\|_s^{\Gamma_0} &\leq C(\|\mathcal{P}r\|_s^{\Gamma_0} + \|F(r)\|_{\mathbf{R}\times\mathbf{R}^N}) \quad \forall r \in B_0(\varepsilon, H^s(\Gamma_0)), &(6.4)\\ \|r\|_s^{\Gamma_0} &\leq (1 + C\|\mathcal{P}r\|_s^{\Gamma_0})\|\mathcal{P}r\|_s^{\Gamma_0} \quad \forall r \in B_0(\varepsilon, H^s(\Gamma_0)) \cap \mathcal{M}_s. &(6.5)\end{aligned}$$

Proof: Consider the mapping

$$\Phi : H^s(\Gamma_0) \longrightarrow \mathcal{P}[H^s(\Gamma_0)] \times (\mathbf{R} \times \mathbf{R}^N)$$

defined by

$$\Phi(r) = \left[\begin{array}{c} \mathcal{P}r \\ F(r) \end{array} \right].$$

Note that $\Phi(0) = 0$,

$$\Phi'(0)[h] = \left[\begin{array}{c} \mathcal{P}h \\ \left[\begin{array}{c} \int_{\Gamma_0} h \, d\Gamma \\ \int_{\Gamma_0} hn \, d\Gamma \end{array} \right] \end{array} \right].$$

$\Phi'(0)$ is a bijection from $H^s(\Gamma_0)$ onto $\mathcal{P}[H^s(\Gamma_0)] \times (\mathbf{R} \times \mathbf{R}^N)$, hence (6.4) follows as a consequence of the Local Diffeomorphism theorem applied to Φ in the neighborhood of 0.

Furthermore, we define the function

$$\tilde{F} : \mathcal{P}[H^s(\Gamma_0)] \times (I - \mathcal{P})[H^s(\Gamma_0)] \longrightarrow \mathbf{R} \times \mathbf{R}^N$$

by

$$\tilde{F}(r_1, r_2) = F(r_1 + r_2).$$

For the Fréchet derivatives $D_1\tilde{F}$ and $D_2\tilde{F}$ of $\tilde{F}$ at $(0,0)$ with respect to the first and second argument, respectively, we find the formally identical expressions

$$D_i\tilde{F}(0,0) = \left[\begin{array}{c} \int_{\Gamma_0} h \, d\Gamma \\ \int_{\Gamma_0} hn \, d\Gamma \end{array} \right] \qquad (i = 1, 2).$$

Note that $D_1\tilde{F}(0,0)$ is the zero operator while $D_2\tilde{F}(0,0)$ is invertible. Due to the orthogonality of $\mathcal{P}$ we have

$${\|r\|_s^{\Gamma_0}}^2 = {\|\mathcal{P}r\|_s^{\Gamma_0}}^2 + {\|\bar{r}\|_s^{\Gamma_0}}^2$$

with $\bar{r} = (I - \mathcal{P})r$. If we assume $r \in \mathcal{M}_s$, $\bar{r}$ satisfies the equation

$$\tilde{F}(\mathcal{P}r, \bar{r}) = F(r) = 0.$$

By the Implicit Function theorem, this implies that $\bar{r}$ can be interpreted as a function of $\mathcal{P}r$ and

$$\|\bar{r}\|_s^{\Gamma_0} \leq C{\|\mathcal{P}r\|_s^{\Gamma_0}}^2$$

if $||\mathcal{P}r||_s^{\Gamma_0}$ is sufficiently small. Assertion (6.5) follows easily from this. ■

On $H^s(\Gamma_0)$ we introduce the degenerate bilinear forms

$$[u, v]_s = (\mathcal{P}u, \mathcal{P}v)_s$$

and, for any $n \in \mathbf{N}$,

$$[u, v]_{s,n} = \sum_{|\alpha| \le n} [D^\alpha u, D^\alpha v]_s.$$

Moreover, we define corresponding seminorms $|\cdot|_{s,n}$ by

$$|u|_{s,n}^2 = [u, u]_{s,n}.$$

Using these notations, the necessary estimates for the linearization can be given:

Lemma 35 *(Coercivity estimate for $-\rho_1$ near equilibrium)*
Suppose (6.1) holds.

*(A) **(Stokes flow)***
Let ρ be defined by (3.54). *There is a constant $c > 0$ such that*

$$-[\rho_1 r, r]_1 \ge c|r|_{\frac{3}{2}}^2 \qquad \forall r \in H^2(\Gamma_0).$$

*(B) **(Hele-Shaw flow)***
Let ρ be defined by (3.56). *There is a constant $c > 0$ such that*

$$-[\rho_1 r, r]_s \ge c|r|_{s+\frac{3}{2}}^2 \qquad \forall r \in H^{s+3}(\Gamma_0).$$

Proof: (A) We introduce the notation

$$\begin{bmatrix} u^* \\ p^* \\ \lambda^* \end{bmatrix} = \tilde{L}(0)^{-1} \begin{bmatrix} 0 \\ 0 \\ (\Delta_{\Gamma_0} r + (N-1)r)n \\ 0 \\ 0 \end{bmatrix}$$

which implies $\rho_1 r = u^* \cdot n$ and (cf. the proof of Lemma 11)

$$\begin{aligned} (\lambda_1^*)_i &= \frac{N}{\omega_N} \int_{\Gamma_0} (\Delta_{\Gamma_0} r + (N-1)r)n \cdot e_j \, d\Gamma = \frac{N}{\omega_N} \int_{\Gamma_0} (\Delta_{\Gamma_0} r + (N-1)r) x_j \, d\Gamma \\ &= \frac{N}{\omega_N} \int_{\Gamma_0} (\Delta_{\Gamma_0} x_j + (N-1)x_j) \, d\Gamma = 0, \\ (\lambda_2^*)_k &= \int_{\Gamma_0} (\Delta_{\Gamma_0} r + (N-1)r)n \cdot v_{2,k} \, d\Gamma = 0, \end{aligned}$$

where we used that $\Delta_{\Gamma_0} x_j = -(N-1)x_j$ and that the vector fields $v_{2,k}$ describing rigid body rotations around the origin are tangential to the unit sphere. Taking into account that

$$(\rho_1 r)_{00} = \omega_N^{-\frac{1}{2}} \int_{\Gamma_0} u^* \cdot n \, d\Gamma = \omega_N^{-\frac{1}{2}} \int_{\Omega_0} \operatorname{div} u^* \, dx = 0$$

we find

$$\begin{aligned}
-[\rho_1 r, r]_1 &= -\sum_{l=2}^{\infty}\sum_k (l(l+1)-(N-1))(\rho_1 r)_{kl} r_{kl} \\
&= -\sum_{l=0}^{\infty}\sum_k (l(l+1)-(N-1))(\rho_1 r)_{kl} r_{kl} \\
&= -\sum_{l=0}^{\infty}\sum_k (\rho_1 r)_{kl}(-\Delta_{\Gamma_0} r-(N-1)r)_{kl} \\
&= \int_{\Gamma_0} \rho_1 r(\Delta_{\Gamma_0} r+(N-1)r)\, d\Gamma \\
&= \int_{\Gamma_0} u^*(\Delta_{\Gamma_0} r+(N-1)r)\, d\Gamma = a(u^*,u^*) \geq c\|u^*\|_1^{\Omega_0\,2}.
\end{aligned}$$

On the other hand,

$$|r|_{\frac{3}{2}}^2 = (\Delta_{\Gamma_0} r + (N-1)r, \varphi)_0$$

with

$$\begin{aligned}
\varphi &= \sum_{l=0}^{\infty}\sum_k \varphi_{kl} Y_{kl}, \\
\varphi_{kl} &= \begin{cases} (l(l+N-2)-(N-1))^{-\frac{1}{2}} r_{kl} & (l \geq 2) \\ 0 & (l < 2). \end{cases}
\end{aligned}$$

Note that $\varphi \in H^{\frac{1}{2}}(\Gamma_0)$, $\|\varphi\|_{\frac{1}{2}}^{\Gamma_0} = |r|_{\frac{3}{2}}$.

Considering the Neumann problem (4.11) with this φ and taking into account that $\overline{\varphi} = 0$ due to $\varphi_{1\,0} = 0$ we can show

$$|r|_{\frac{3}{2}}^2 \leq C\left(\|u^*\|_1^{\Omega_0} + \|\lambda^*\|_{\mathbf{R}^N\times\mathbf{R}^{\binom{N}{2}}}\right)\|\varphi\|_{\frac{1}{2}}^{\Gamma_0} \leq C\|u^*\|_1^{\Omega_0}|r|_{\frac{3}{2}}$$

in the same way as in (4.12). This implies the assertion.

(B) The basis $\{Y_{kl}\}$ of the spherical harmonics is a complete system of eigenfunctions also for the operator S defined by (3.37); the corresponding eigenvalue is l. Hence we can prove the assumption by straightforward calculation with

$$c = \inf_{l\geq 2} \frac{l}{(l(l+N-2)-(N-1))^{\frac{1}{2}}}. \qquad \blacksquare$$

Based on this estimate for the linearization, the following new a priori estimate can be given:

Lemma 36 *(A priori estimate near equilibrium)*

Suppose (6.1) *holds.*

(A) (Stokes flow)

Let ρ be given by (3.54). *For any integer $n > \frac{N}{2}+1$ there are positive numbers c_n, C_n, and ε_n such that*

$$[\rho(r), r]_{1,n} \leq -c_n|r|_{1,n}^2 + C_n\|F(r)\|_{\mathbf{R}\times\mathbf{R}^N}^2 \tag{6.6}$$

for all $r \in B_0\left(\varepsilon_n, H^{n+1}(\Gamma_0)\right) \cap H^{n+2}(\Gamma_0)$.

(B) (Hele-Shaw flow)

Let ρ be given by (3.56). *For any $s > 2 + \frac{N}{2}$ there are positive numbers c_s, C_s, and ε_s such that*

$$[\rho(r), r]_{s,3} \leq -c_s |r|^2_{s,3} + C_s \|F(r)\|^2_{\mathbf{R}\times\mathbf{R}^N} \tag{6.7}$$

for all $r \in B_0\left(\varepsilon_s, H^{s+3}(\Gamma_0)\right) \cap H^{s+6}(\Gamma_0)$.

Proof: (A) Calculating the Fréchet derivatives of $D^\alpha \rho$ at $r = 0$ using (5.12) and $\mathcal{R}_0 \equiv 1$ yields

$$D^\alpha \rho_k(r, \ldots, r) = \sum_{\beta_1 + \ldots + \beta_k = \alpha} C_{\beta_1, \ldots, \beta_k} \rho_k(D^{\beta_1} r, \ldots, D^{\beta_k} r), \tag{6.8}$$

where, in contrast to the original chain rule, also multiindices $\beta_j = 0$ are allowed. Together with the estimates for the ρ_k, this yields

$$\|D^\alpha \rho_k(r, \ldots, r)\|^{\Gamma_0}_{\frac{1}{2}} \leq C_n \|r\|^{\Gamma_0\,k-1}_{n+1} \|r\|^{\Gamma_0}_{n+\frac{3}{2}}.$$

Using this, together with (6.2) and $\rho(0) = 0$ we obtain

$$\begin{aligned} [D^\alpha \rho(r), D^\alpha r]_1 &= [D^\alpha \rho_1 r, D^\alpha r]_1 + \sum_{k=2}^\infty [D^\alpha \rho_k(r, \ldots, r), D^\alpha r]_1 \\ &\leq -c|D^\alpha \rho|^2_{\frac{3}{2}} + C_\alpha \sum_{k=2}^\infty M^k \|r\|^{\Gamma_0\,k-1}_{n+1} \|r\|^{\Gamma_0\,2}_{n+\frac{3}{2}}. \end{aligned}$$

Carrying out the summation over all α with $|\alpha| \leq n$ we find from (6.4) and the fact that $\mathcal{P}$ and D^α commute

$$[\rho(r), r]_{1,n} \leq -c_n |r|^2_{n+\frac{3}{2}} + C_n \sum_{k=2}^\infty \|r\|^{\Gamma_0\,k-1}_{n+1} \left(|r|^2_{n+\frac{3}{2}} + \|F(r)\|^2_{\mathbf{R}\times\mathbf{R}^N}\right),$$

and the assertion follows by choosing ε_n sufficiently small.

(B) The assertion can be shown in an essentially analogous way, using the estimate

$$[u, v]_s \leq C_s \|u\|^{\Gamma_0}_{s-\frac{3}{2}} \|v\|^{\Gamma_0}_{s+\frac{3}{2}}$$

and (6.8) for all α with $|\alpha| > 0$. ■

Using this a priori estimate, we can show now the following result which gives a justification for the expectations from physical reasonings that were mentioned above. By $\mathbf{R}_+$ we denote the infinite time interval $[0, +\infty)$.

Proposition 8 *(Global existence of solutions and exponential stability near equilibrium)*

Suppose (6.1) *holds.*

(A) (Stokes flow)

Let ρ be given by (3.54), *$n > \frac{N}{2} + 2$. There are positive constants ε_n and c_n such that for any*

$$r_0 \in \mathcal{M}_{n+1} \cap B_0(\varepsilon_n, H^{1,n}(\Gamma_0))$$

the initial value problem (4.16) *has a solution*

$$r \in C_w(\mathbf{R}_+, H^{n+1}(\Gamma_0)) \cap C^1_w(\mathbf{R}_+, H^n(\Gamma_0))$$

for which an estimate

$$\|r(t)\|_{n,1}^{\Gamma_0} \le e^{-c_n t}\|r_0\|_{n,1}^{\Gamma_0} \tag{6.9}$$

holds for all sufficiently large t.

(B) (Hele-Shaw flow)

Let ρ be given by (3.56), $s > \frac{N}{2}+2$. *There are positive constants ε_s and c_s such that for any*

$$r_0 \in \mathcal{M}_{s+3} \cap B_0(\varepsilon_s, H^{s,3}(\Gamma_0))$$

the initial value problem (4.16) *has a solution*

$$r \in C_w(\mathbf{R}_+, H^{s+3}(\Gamma_0)) \cap C_w^1(\mathbf{R}_+, H^s(\Gamma_0))$$

for which an estimate

$$\|r(t)\|_{s,3}^{\Gamma_0} \le e^{-c_s t}\|r_0\|_{s,3}^{\Gamma_0} \tag{6.10}$$

holds for all sufficiently large t.

Proof: (A) By Proposition 5 (i) we find that for sufficiently small ε_n there is a solution

$$r \in C_w(IT, H^{n+1}(\Gamma_0)) \cap C_w^1(IT, H^n(\Gamma_0))$$

of (4.16) for small T. From the conservation of volume and center of gravity we conclude $r(t) \in \mathcal{M}_{n+1}$ for all $t \in IT$. According to the proof of Proposition 5 (i), $r(t)$ is given by

$$r(t) = \operatorname*{w-lim}_{j\to\infty} r_j(t) \qquad \forall t \in IT,$$

where w-lim denotes the weak limit in $H^{n+1}(\Gamma_0)$, the $r_j \in C^1(IT, H^{s+2}(\Gamma_0))$ are the solutions of the Galerkin equations (5.23), and the convergence is uniform in t. Hence $r_j(t) \overset{H^{\frac{N}{2}+1}(\Gamma_0)}{\longrightarrow} r(t)$ uniformly in t and thus

$$\|F(r_j(t))\|_{\mathbf{R}\times\mathbf{R}^N} \to 0 \quad \text{uniformly in } t \in IT \tag{6.11}$$

because, as remarked above, $r(t) \in \mathcal{M}_{s+1}$. We choose the finite-dimensional subspaces M_j in such a way that $\mathcal{P}$ and P_j commute for all j. Thus we have for all $t \in IT$

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(|r_j(t)|_{1,n}^2\right) &= [P_j\rho(r_j(t)), r_j(t)]_{1,n} \\
&= (\mathcal{P}P_j\rho(r_j(t)), \mathcal{P}r_j(t))_{1,n} = (\mathcal{P}\rho(r_j(t)), \mathcal{P}P_j r_j(t))_{1,n} \\
&= [\rho(r_j(t)), r_j(t)]_{1,n} \le -c|r_j(t)|_{1,n}^2 + C\|F(r_j(t))\|_{\mathbb{R}\times\mathbb{R}^N}^2
\end{aligned}$$

because of (6.6), if ε_n is chosen sufficiently small. Hence

$$|r_j(t)|_{1,n}^2 \le e^{-ct}|r_0|_{1,n}^2 + C\int_0^t e^{c(\tau-t)}\|F(r_j(\tau))\|_{\mathbb{R}\times\mathbb{R}^N}^2\,d\tau,$$

and thus, using (6.11),

$$\begin{aligned}
|r(t)|_{1,n}^2 &= \|\mathcal{P}r(t)\|_{1,n}^{\Gamma_0} = \left\|\mathcal{P}\operatorname*{w-lim}_{j\to\infty} r_j(t)\right\|_{1,n}^{\Gamma_0} = \left\|\operatorname*{w-lim}_{j\to\infty}\mathcal{P}r_j(t)\right\|_{1,n}^{\Gamma_0} \\
&\le \varliminf_{j\to\infty}\|\mathcal{P}r_j(t)\|_{1,n}^{\Gamma_0} = \varliminf_{j\to\infty}|r_j(t)|_{1,n} \le e^{-ct}|r_0|_{1,n}.
\end{aligned}$$

Finally, $r(t) \in \mathcal{M}_{1,n} \cap B_0(2\varepsilon_n, H^{1,n}(\Gamma_0))$ implies for small ε_n by (6.4)

$$\begin{aligned} \|r(T)\|_{1,n}^{\Gamma_0} &\leq (1 + C|r(T)|_{1,n})|r(T)|_{1,n} \leq (1 + C\|r_0\|_{1,n}^{\Gamma_0})e^{-cT}\|r_0\|_{1,n}^{\Gamma_0} \\ &\leq e^{-\frac{c}{2}T}\|r_0\|_{1,n}^{\Gamma_0} \leq \varepsilon. \end{aligned} \quad (6.12)$$

Therefore we can continue the solution to $[T, 2T]$ and by induction to $[mT, (m+1)T]$ for all $m \in \mathbf{N}$. The estimate (6.9) can be shown for all $t > T$ in the same way as for $t = T$ in (6.12).

(B) The assertion can be proved analogously, using the norm $\|\cdot\|_{s,3}^{\Gamma_0}$ and the a priori estimate (6.7). ■

6.2 Point sources as additional driving force

In this section, we will briefly describe the necessary generalizations that occur if, additionally to the surface tension force, the flow is driven by a point source or sink of prescribed strength in the interior of the liquid domain. We will give the explicit calculations only for the case of Stokes flow and one point source; the generalization to more than one source and the parallel treatment of Hele-Shaw flow are straightforward. As far as short-time existence is concerned, the results remain essentially unchanged. (Of course, in order to generalize the considerations involving rotational symmetry of the equations we cannot have more than one point source, and we have to assume star-shapedness with respect to the source point.) This is due to the fact that the inclusion of a point source in the interior of the liquid domain preserves the analytic dependence of all occurring functions and operators on r, contributes only a lower-order term to the modified fixed time problem (6.13) below, and does not essentially change the structure of the evolution equation for r.

It is clear that one cannot expect a generalization of the global existence result near equilibrium in such an easy way.

The fixed-time problem

Let us adopt the notation of Section 3.2 and assume without loss of generality $0 \in \Omega_0$. If we include a source or sink of strength Q at 0, the governing equations of the flow, described by its velocity field $\overline{u}$ and pressure field $\overline{p}$, become

$$\left.\begin{aligned} -\Delta\overline{u} + \nabla\overline{p} &= 0 \\ \operatorname{div}\overline{u} &= Q\delta \end{aligned}\right\} \quad \text{in } \Omega_0$$
$$T(\overline{u}, \overline{p})n = \kappa n \qquad \text{on } \Gamma_0,$$

where δ denotes the Dirac distribution in 0.

We split the velocity field $\overline{u}$ in a singular and a regular part by setting

$$\begin{aligned} \overline{u} &= u + u_S, \\ u_S(x) &= \frac{Q}{\omega_N}\frac{x}{|x|^N} \qquad (x \neq 0). \end{aligned}$$

Note that, in the sense of distributions,

$$\begin{aligned} \operatorname{div} u_S &= Q\delta, \\ \Delta u_S &= Q\nabla\delta. \end{aligned}$$

Thus, setting $p = \overline{p} - Q\delta$ we obtain the modified fixed time problem

$$\left.\begin{array}{rcl} \left.\begin{array}{rcl} -\Delta u + \nabla p &=& 0 \\ \operatorname{div} u &=& 0 \end{array}\right\} \text{ in } \Omega_0 & & \\ \mathcal{T}(u,p)n &=& \kappa n - \mathcal{T}(u_S, 0)n \\ &=& \kappa n - \dfrac{2Q}{\omega_N}\left(\dfrac{\delta_{ij}}{|x|^N} - N\dfrac{x_i x_j}{|x|^{N+2}}\right) n_j e_i \quad \text{on } \Gamma_0, \end{array}\right\}, \tag{6.13}$$

where δ_{ij} is the Kronecker symbol and e_i denotes the i-th unit vector.

In order to establish the existence of solutions of (6.13) one has to show (additionally to the considerations in Section 3.2) that

$$\int_{\Gamma_0} \mathcal{T}(u_S, 0)n \cdot v \, d\Gamma = 0$$

for arbitrary $v \in V_0$. This can be done by applying the first Stokes-Green formula (Lemma 9) on the domain $\Omega_\varepsilon = \Omega_0 \backslash B_0(\varepsilon, \mathbb{R}^N)$ with ε small enough to ensure $B_0(2\varepsilon, \mathbb{R}^N) \subset \Omega_0$, $u = u_s$, and $p = 0$. Using that

$$\begin{array}{rcl} \dfrac{\partial v_i}{\partial x_j} + \dfrac{\partial v_j}{\partial x_i} &=& 0, \\ \Delta u_S &=& 0, \\ \operatorname{div} u_S &=& 0 \end{array}$$

in Ω_ε one finds

$$\begin{array}{rcl} \displaystyle\int_{\Gamma_0} \mathcal{T}(u_S, 0)n \cdot v \, d\Gamma &=& -\displaystyle\int_{\partial B_0(\varepsilon, \mathbb{R}^N)} \mathcal{T}(u_S, 0)n \cdot v \, d\Gamma \\ &=& \dfrac{2Q}{\omega_N} \displaystyle\int_{\partial B_0(\varepsilon, \mathbb{R}^N)} \left(\dfrac{\delta_{ij}}{|x|^N} - N\dfrac{x_i x_j}{|x|^{N+2}}\right) \dfrac{x_j}{|x|} v_i \, d\Gamma \\ &=& \dfrac{2Q}{\omega_N}\dfrac{1-N}{\varepsilon^{N+1}} \displaystyle\int_{\partial B_0(\varepsilon, \mathbb{R}^N)} x \cdot v \, d\Gamma = 0. \end{array}$$

Perturbations, evolution equation, and linearization

We adopt the notation of Section 3.4. Starting from (6.13) and carrying out the perturbations we find, instead of (3.50),

$$\begin{array}{rcl} \tilde{L}(r)\begin{bmatrix} u(r) \\ p(r) \\ \lambda(r) \end{bmatrix} &=& \begin{bmatrix} 0 \\ 0 \\ \phi(r) \\ 0 \\ 0 \end{bmatrix}, \\ \phi(r) &=& \kappa(r)\nu(r) \\ & & -\dfrac{2Q}{\omega_N}\left(\dfrac{\delta_{ij}}{|\xi + r\zeta|^N} - N\dfrac{(\xi_i + r\zeta_i)(\xi_j + r\zeta_j)}{|\xi + r\zeta|^{N+2}}\right)(\nu(r))_j e_i. \end{array}$$

To establish the assertions of Lemma 17 in this more general situation, it is sufficient to remark that the mappings $x \mapsto |x|^n$ are analytic near x_0 for all $x_0 \neq 0$ and all integer n.

The evolution equation for r reads now

$$\begin{aligned} \frac{\partial r}{\partial t} &= \rho(r) = \frac{(u(r) + u_S(\xi + r\zeta)) \cdot \nu(r)}{\zeta \cdot \nu(r)} \\ &= \left(u(r) + \frac{Q}{\omega_N} \frac{\xi + r\zeta}{|\xi + r\zeta|^N} \right) \cdot \frac{\nu(r)}{\zeta \cdot \nu(r)} \end{aligned} \tag{6.14}$$

and the assertions of Lemma 19 can be proved as above.

Linearization of ρ as given in (6.14) yields, in the notation of Section 4.1,

$$\begin{aligned} \rho_1(r) &= \left(-\frac{1}{\gamma^2} \left(\left(u(0) + \frac{Q}{\omega_N} \frac{\xi}{|\xi|^N} \right) \cdot n \right) \zeta + \frac{1}{\gamma} \left(u(0) + \frac{Q}{\omega_N} \frac{\xi}{|\xi|^N} \right) \right) \cdot \nu_1(r) \\ &\quad + \frac{1}{\gamma} n \cdot \left(u_1(r) + \frac{Q}{\omega_N} \left(\frac{\zeta}{|\xi|^N} - N \frac{\zeta \cdot \xi}{|\xi|^{N+2}} \xi \right) r \right), \\ u_1(r) &= \operatorname*{Tr}_{\Gamma_0} \Pi_1 \tilde{L}(0)^{-1} \left(\begin{bmatrix} 0 \\ 0 \\ \phi_1(r) \\ 0 \\ 0 \end{bmatrix} - \tilde{L}_1(r) \begin{bmatrix} u(0) \\ p(0) \\ 0 \end{bmatrix} \right), \\ \phi_1(r) &= \kappa_1(r) n + \kappa(0) \nu_1(r) - \frac{2Q}{\omega_N} \left(\frac{\delta_{ij} \zeta \cdot \xi}{|\xi|^{N+2}} \right) (\nu_1(r))_j e_i \\ &\quad + \frac{2QN}{\omega_N} \left(\frac{\zeta \cdot \xi}{|\xi|^{N+2}} + \frac{\zeta_i \xi_j + \xi_i \zeta_j}{|\xi|^{N+2}} - (N+2) \frac{\xi_i \xi_j (\xi \cdot \zeta)}{|\xi|^{N+4}} \right) n_j e_i r. \end{aligned}$$

The terms in ρ_1 that contain Q represent a differential operator ("with smooth coefficients") and the same is true for the terms in ϕ_1. Hence, also for ρ given by (6.14) we get a decomposition like (4.3) and find identical results on coercivity and generation of analytic semigroups.

Results

In this way, we arrive at the following result:

Proposition 9 *(Existence results for Stokes flow with a point source I)*
Let ρ be given by (6.14)*. Then the same assertions as in Proposition* 4 *(A) hold.*

To generalize the results of Chapter 5, we have to demand the validity of Assumption 1 again. In our new context, however, after choosing the source point to be 0, the choice $x_0 = 0$ is no longer without loss of generality. As announced above, this means that we have to demand now strict star-shapedness of the liquid domain with respect to the source point. Taking into account that u_S is an invariant vector field with respect to rotations around the origin, the equivariance property (5.16) can be shown also for ρ given by (6.14).

This implies the following result:

Proposition 10 *(Existence results for Stokes flow with a point source II)*
Let ρ be given by (6.14)*, suppose Assumption* 1 *with $x_0 = 0$ and* (5.11) *hold. Then the same assertions as in Propositions* 5 *(A) and* 7 *(A) hold.*

Remark: If we consider a time-dependent source strength $Q = Q(t)$, the operator ρ will depend explicitly on t, i.e. the evolution problem is not longer autonomous. Even in this case, if Q depends continuously differentiable on t, the results on existence, uniqueness, and spatial smoothness of the solution are still valid. It is clear, however, that the solution r will only be "as smooth as Q" as a function of t. □

6.3 Further remarks

6.3.1 Other geometries

The approaches which are chosen in this thesis are not essentially restricted to the geometry of a bounded, simply-connected domain. The results of Chapters 3 and 4 are valid, for example, also for multiply-connected domains with smooth compact boundary. The assumption of boundedness, however, is essential because analytic difficulties occur if Γ_0 is not compact: Lemma 5 is not valid in this case, and the embeddings in the scale $H^s(\Gamma_0)$ are not compact. Qualitatively, one expects that holes in the liquid domain shrink and eventually vanish in finite time (cf. [18] for plane Stokes flow). Of course, the actual closing of a hole cannot be described by our methods. For a generalization of the approach in Chapter 2 to multiply-connected domains see [5] and also [7], where Dirichlet boundary conditions on one connected component of the boundary are considered.

Other geometric settings that are considered both for theoretical and for practical purposes are (infinite) liquid layers above a fixed bottom and domains given by

$$\Omega = \{x \mid x_N < \varphi(x_1, \ldots, x_{N-1}),\ (x_1, \ldots, x_{N-1}) \in \mathbf{R}^{N-1}\}. \tag{6.15}$$

Such geometries have been treated, for example, in [23] and [69]. It is expected that, by imposing suitable asymptotic conditions at infinity and boundary conditions at the fixed bottom, an analogous treatment of our moving boundary problems can be carried out if periodicity assumptions are made, i.e. if we can reformulate the problem on the compact manifold T^{N-1} (or a topologically equivalent one) rather than on the noncompact manifold $\mathbf{R}^{N-1}$. In this geometry, it is natural to use for the quasilinearization the invariance of the governing equations with respect to translations, and the assumption of strict star-shapedness has to be replaced by the demand that all lines $x_N = \text{const}$ have precisely one point in common with the free boundary. (If Ω is given by (6.15), this assumption is obviously satisfied.)

6.3.2 Numerical aspects of the Stokes flow FBP

Three different approaches for the numerical treatment of the Stokes flow FBP are listed in Table 6.1, together with their analytic background and some references to the literature. (We do not attempt to give an exhaustive literature review, more references can be found e.g. in [91].) The so-called level set method [79] which has been successfully applied to e.g. mean curvature flow does not seem to be promising due to the nonlocal character of the surface motion law considered here.

The aim of this subsection is to make some remarks on the three methods mentioned above that at the same time illustrate the interplay between analysis and numerical mathematics.

The method based on conformal mappings can briefly be described in the following way: An ansatz with a finite number of free parameters is chosen for the conformal mapping, e.g. one assumes that $z(\cdot, t)$ is a polynomial of fixed degree or a rational function of prescribed form in

Analytic approach	*Numerical method*	*References*
Evolution equation for a conformal map	Solving ODEs for coefficients	[8, 9]
Weak formulation of the fixed-time problem	FEM + time integration	[41, 48]
Hydrodynamic potentials for the fixed-time problem	BEM +time integration	[53, 91]

Table 6.1: Numerical methods for Stokes flow driven by surface tension

the first argument. The evolution equation (in explicit or implicit form) is used to derive a system of ordinary differential equations for the free parameters which can be solved numerically. This method is advantageous if the initial shapes $\Omega(0)$ can be described exactly by a conformal map of the type used in the ansatz. Otherwise, the necessary approximation procedure for the initial domain will demand a considerable computational effort; moreover, the stability properties of the method with respect to the approximation error are in general unclear.

It is beyond our aims to give a complete discussion of all advantages and disadvantages of FEM versus BEM in the case of the simulation of Stokes flow. For this, we refer to the cited literature. We only remark that the use of the boundary element method is natural in the sense that only the velocity field at the boundary is computed. This is sufficient for the determination of its motion.

In [91], Ch. 5.1., the behavior of a semi-discretized problem (Eqn. (5.1) there) is investigated by means of an eigenvalue analysis of its linearization. The result obtained there is that, in general, the system of ordinary differential equations describing the motion of material points on the boundary is stiff. From an analytic point of view, this can be straightforwardly explained by the fact that solving the discretized integral equations that arise in the numerical treatment of the fixed-time problem is actually an approximate computation of a pseudodifferential operator of order 1. The unboundedness of this operator in the continuous problem corresponds to the occurrence of eigenvalues with large absolute values in the discretized problem. From this point of view, the situation is in strict analogy with the numerical analysis of parabolic differential equations.

In [91], the difficulties of solving a stiff system of ODEs are overcome by using an implicit backward difference method. However, if the Euler-forward method is used for the time integration as in [41], a stability bound for the timestep Δt depending on the spatial discretization parameter Δx of the form

$$\frac{\Delta t}{\Delta x} \leq C \tag{6.16}$$

is expected, in accordance with the order 1 of the pseudodifferential operator in the linearized continuous problem. This bound has also been found in computational experiments [82] for a closely related problem [83].

The bound (6.16) is less restrictive than analogous bounds on $\frac{\Delta t}{(\Delta x)^2}$ that occur in the stability analysis of discretizations of second-order parabolic differential equations like the heat equation.

This explains why the Euler-forward method could be used in [41] without stability problems.

6.3.3 Open problems

As pointed out earlier, for general initial domains the methods that have been used in Chapters 3–5 can only yield local existence results. The length of the time interval on which existence of the solution is ensured depends on the reference domain, hence no answers can be found to the interesting questions about (non-) development of singularities (e.g. cusps or corners) in the boundary of the liquid domain. As we have shown above, the influence of surface tension leads in general to a smoothing of the boundary. However, in special examples the development of a cusp for a single moment of time has been found by Howison e.a. as a limiting case between a smooth evolution and an evolution in which the connectivity of the liquid domain changes.

As a first step towards global existence results, it seems reasonable to look for global geometric properties of the liquid domain that are preserved by the evolution under a given surface motion law. Apparently, the only cases for which such questions have been studied thoroughly by now are mean curvature flow and related problems [27, 32, 47, 79]. For instance, it can be shown that convexity of the domain enclosed by the moving surface is preserved by mean curvature flow [47]. An analogous result has been obtained in the case $N = 2$ for so-called area-preserving mean curvature flow [32]. On the basis of this, global existence of a solution for convex initial domains can be proved in both cases. A crucial tool in this, however, is the application of maximum principles for elliptic second-order differential operators. Therefore, it is not possible to prove corresponding results along the same lines for evolution problems that involve the solutions of elliptic BVP, i.e. where elliptic pseudodifferential operators occur instead of second-order elliptic operators.

For Stokes flow, Hopper [44] conjectures that convexity and star-shapedness are preserved by the evolution, to prove this, however, seems to be a rather difficult problem. Apparently, the only strict result in this direction is the following theorem by Plotnikov [70]: For Stokes flow with $N = 2$ and a connected inital domain, the liquid domain will be connected for all times for which the solution exists.

Finally, we want to remark that the analysis given in the preceding chapters does not rely too strongly on special properties of the Stokes operator or the Laplacian: We only use coercivity and regularity of standard boundary value problems associated with them and, in Chapter 5, their rotational invariance. Hence, it is expected that an analogous approach can be used for a much wider class of free boundary problems, even in cases where the governing equations are nonlinear. In this respect, see [1, 2].

Appendix A

A free boundary problem for the Navier-Stokes equations

We assume without loss of generality $\rho \equiv 1$ and restrict our attention to the case $N = 3$. In order to transform the equations (1.1), (1.2) to the fixed domain $\Omega(0)$ we introduce the functions

$$\begin{aligned} u(\xi, t) &= v(x(\xi, t), t), \\ q(\xi, t) &= p(x(\xi, t), t) \end{aligned}$$

representing the velocity and pressure fields in Lagrangian coordinates. They satisfy the nonlinear initial-boundary value problem

$$\left.\begin{aligned} \frac{\partial u}{\partial t} - \nu \Delta_u u - \nabla_u q &= f(x_u, t) \\ \operatorname{div}_u u &= 0 \\ u(\cdot, 0) &= v_0 \end{aligned}\right\} \quad \text{in } \Omega(0)$$
$$\mathcal{T}_u(u, q) n_{u,t} = \gamma \kappa_{u,t} n_{u,t} \qquad \text{on } \Gamma(0), \tag{A.1}$$

where Δ_u, ∇_u, $\mathcal{T}_u$ denote the differential operators Δ, ∇, and $\mathcal{T}$ with respect to the ξ-coordinates,

$$x_u(\xi, t) = \xi + \int_0^t u(\xi, s)\, ds$$

and $n_{u,t}$ and $\kappa_{u,t}$ are the outer normal vector and the mean curvature of the surface

$$\Gamma_{u,t} = x_u[\Gamma(0), t],$$

respectively. For a given domain Ω in $\mathbf{R}^3$ and $r, T > 0$, $r \neq \mathbf{N}$ we introduce the notation $Q_T = \Omega \times (0, T)$ and the Sobolev spaces of noninteger order $W_2^r(\Omega)$, $W_2^{\frac{r}{2}}(0, T)$, $W_2^{r, \frac{r}{2}}(Q_T)$ by Hilbert norms whose squares are

$$\begin{aligned} \|u\|^2_{W_2^r(\Omega)} &= \sum_{|\alpha| \le [r]} \|\partial^\alpha u\|^2_{L^2(\Omega)} + \sum_{|\alpha| = [r]} \int_{\Omega \times \Omega} \frac{|\partial^\alpha u(x) - \partial^\alpha u(y)|^2}{|x - y|^{3 + 2(r - [r])}}\, dx dy, \\ \|u\|^2_{W_2^{\frac{r}{2}}(0,T)} &= \sum_{j=0}^{[\frac{r}{2}]} \|\partial^j u\|^2_{L^2(0,T)} + \int_0^T \int_0^T \frac{\left|\partial^{[\frac{r}{2}]} u(t) - \partial^{[\frac{r}{2}]} u(\tau)\right|^2}{|t - \tau|^{1 + 2(\frac{r}{2} - [\frac{r}{2}])}}\, dt d\tau, \end{aligned}$$

$$\|u\|^2_{W_2^{r,\frac{r}{2}}(Q_T)} = \int_0^T \|u(\cdot,t)\|^2_{W_2^r(\Omega)}\,dt + \int_\Omega \|u(x,\cdot)\|^2_{W_2^{\frac{r}{2}}(0,T)}\,dx,$$

where the usual multiindex notation is applied, $[s]$ denotes the largest integer not larger than s, and all differentiations are to be understood in generalized sense. Using local charts and partitions of unity subordinate to them, one can define analogous function spaces on manifolds. For r large enough, the spaces W_2^r can also be used to characterize the smoothness of surfaces. In the following theorem we will write S^2 for the unit ball in $\mathbf{R}^3$, $\Omega(0) = \Omega$, $\Gamma(0) = \Gamma$, $G_T = \Gamma \times (0,T)$, $R^* = \sqrt[3]{\frac{3|\Omega|}{4\pi}}$, and $p^* = \frac{2\gamma}{R^*}$. Note that R^* is the radius of a ball of volume $|\Omega|$ and p^* is the constant pressure inside a ball of this radius, consisting of resting liquid governed by (1.1), (1.2).

Proposition 11 (V.A. Solonnikov [84, 85, 86])

(i) (Short-time existence and uniqueness)

Suppose Ω is bounded, $l \in (\frac{1}{2}, 1)$, Γ is of class $W_2^{\frac{5}{2}+l}$, $v_0 \in \left(W_2^{1+l}(\Omega)\right)^3$ satisfies the compatibility conditions

$$\begin{array}{rcll} \operatorname{div} v_0 &=& 0 & \text{in } \Omega, \\ \left(\nabla v_0 + (\nabla v_0)^T\right) n(0) &\|& n(0) & \text{on } \Gamma, \end{array}$$

and f has Lipschitz-continuous first derivatives in the space variables and is Hölder-continuous with exponent $\frac{1}{2}$ in time. Then there is a constant $T_1 > 0$ such that (A.1) *has a unique solution (u,q) such that*

$$u \in \left(W_2^{2+l,1+\frac{l}{2}}(Q_{T_1})\right)^3, \quad \nabla q \in \left(W_2^{l,\frac{l}{2}}(Q_{T_1})\right)^3, \quad q|_{G_{T_1}} \in W_2^{\frac{1}{2}+l,\frac{1}{4}+\frac{l}{2}}(G_{T_1}).$$

(ii) (Global existence near equilibrium)

Additionally to the assumptions of (i), suppose $f \equiv 0$ and there is a function R_0 such that

$$\Gamma = \{R_0(\omega)\omega \mid \omega \in S^2\},$$

and $\|R_0 - R^*\|_{W_2^{\frac{5}{2}+l}}$ and $\|v_0\|_{(W_2^{1+l}(\Omega))^3}$ are sufficiently small. Then the following statements hold:

- *The assertion of (i) holds with any $T_1 > 0$,*
- *For any $t \geq 0$, $\Omega(t)$ is such that there is a function $R_t \in W_2^{\frac{5}{2}+l}$ with*

$$\Gamma(t) = \{R_t(\omega)\omega \mid \omega \in S^2\}.$$

- *Let $(v(\cdot,t), p(\cdot,t))$ be the solution of the original problem* (1.1), (1.2), (1.3). *The norms*

$$\left\|\frac{\partial v}{\partial t}(\cdot,t)\right\|_{(W_2^l(\Omega(t)))^3}, \quad \|v(\cdot,t)\|_{(W_2^{l+2}(\Omega(t)))^3},$$

$$\|p(\cdot,t) - p^*\|_{W_2^{l+1}(\Omega(t))}, \quad \|R_t - R^*\|_{W_2^{\frac{5}{2}+l}(\Omega(t))}$$

are uniformly bounded with respect to $t \geq t_0 > 0$.

The crucial part in the proof of (i) is the investigation of a linear problem

$$\left.\begin{array}{rcl}\dfrac{\partial u}{\partial t} - \nu\Delta_w u - \nabla_w q & = & f(\xi,t) \\ \operatorname{div}_w u & = & 0 \\ u(\cdot,0) & = & v_0\end{array}\right\} \quad \text{in } \Omega(0)$$
$$T_w(u,q)n_{u,t} = \gamma\kappa_{w,t}n_{w,t} \qquad \text{on } \Gamma(0),$$

for which a priori estimates depending on $w \in \left(W_2^{l+2}(\Omega(t))\right)^3$ are derived which can be used to prove the short-time solvability of (A.1) by a fixed point argument.

For estimates of u and q in the appropriate norms as well as an "intermediate" result concerning the unlimited growth of T_1 if the data approach the situation of (ii) the reader is referred to the original articles. Furthermore, a generalization of the FBP discussed here is treated in [63] using Hölder function spaces instead of Sobolev spaces. Similar result can be obtained for $N = 2$.

Corresponding results concerning a layer of viscous liquid above a fixed bottom under the influence of gravity and surface tension are obtained in [4] and [11].

Appendix B

Surface motion by curvature: an overview

In this appendix, a brief survey on surface motion laws governed by curvature is given. In table B.1 the surface motion laws that are discussed in the literature are listed together with the fixed-time problems and/or the normal velocities that define them. The notation is essentially as in the introduction, $\left[\frac{\partial u}{\partial n(t)}\right]_{\Gamma(t)}$ denotes the jump of the normal derivative of u across $\Gamma(t)$ and Ξ is a domain containing $\Gamma(t)$. (If Ξ is unbounded then the boundary condition at $\partial\Xi$ has to be supplemented or replaced by an appropriate asymptotic condition on u.) No attempt is made to give a complete list of literature references, instead we restrict ourselves to some references that deal with existence and uniqueness results.

It is a typical property of surface motion laws by curvature that they occur, at least formally, as limiting cases of other well-known FBP or nonlinear PDE that describe phase changes. The corresponding problems are listed in the last column of table B.1. Note, however, that in order to obtain Stokes flow or one-phase Hele-Shaw flow from the FBP for the Navier-Stokes equations or from the Stefan problem one has to impose boundary conditions which already involve the curvature while in the other cases the surface $\Gamma(t)$ is the zero-level set of the solution of the corresponding equations. For details of this we refer to the original articles, see also [27].

Finally, we remark that all surface motion laws considered here, except for the mean curvature flow $V_n = \kappa(t)$, are surface-diminishing and volume-preserving, i.e. if $\Gamma(t)$ is a closed surface (curve) evolving according to one of these laws, having area (length) $A(t)$ and enclosing the volume (area) $V(t)$ then

$$\begin{aligned} \frac{dA}{dt} &\leq 0, \\ \frac{dV}{dt} &= 0. \end{aligned}$$

Name	*Fixed-time problem*	V_n	*(formal) limit of:*
Stokes flow [6, 8, 43, 51]	$-\Delta v + \nabla p = 0$, $\operatorname{div} v = 0$ in $\Omega(t)$; $T(v,p)n(t) = \kappa(t)n(t)$ on $\Gamma(t)$	$v \cdot n(t)$	Navier-Stokes eq. (Re=0)
(one-phase) Hele-Shaw flow [19, 20, 23, 30]	$\Delta u = 0$ in $\Omega(t)$; $u = \kappa(t)$ on $\Gamma(t)$	$\dfrac{\partial u}{\partial n(t)}$	Stefan problem ($c = 0$)
Mullins-Sekerka or two-phase Hele-Shaw flow [17, 30]	$\Delta u = 0$ in $\Xi \backslash \Gamma(t)$; $u = \kappa(t)$ on $\Gamma(t)$; $\dfrac{\partial u}{\partial n} = 0$ on $\partial\Xi$	$\left[\dfrac{\partial u}{\partial n(t)}\right]_{\Gamma(t)}$	Cahn-Hilliard eq. ($\varepsilon \downarrow 0$)
Mean curvature and related flows	–	$\kappa(t)$ [47]	Allen-Cahn eq. ($\varepsilon \downarrow 0$)
		$\kappa(t) - \overline{\kappa(t)}$ [32]	
		$\Delta_{\Gamma(t)}\kappa(t)$ [16, 65]	Cahn-Hilliard eq. ($\varepsilon \downarrow 0$)
		$\Delta_{\Gamma(t)}A^{-1}\kappa(t)$, $A = \frac{1}{M}\Delta_{\Gamma(t)} - \frac{1}{D}I$	(see [27])

Table B.1: Laws of surface motion by curvature

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Index

Summary

This thesis is devoted to the mathematical investigation of certain free boundary value problems that arise from the description of liquid flows for which inertia effects are negligible, i.e. where a quasistastionary approximation can be applied. Two problems of this type are studied here which describe the change of shape of a freely moving drop or blob of liquid. The underlying models are known as Stokes flow and Hele-Shaw flow, respectively. In both cases, the governing equations are elliptic (Stokes equations and Laplace equation, respectively), and the forces arising from surface tension are considered as driving mechanism. This leads to inhomogeneous boundary conditions in which the curvature of the boundary occurs.

The most interesting aspect of these free boundary problems is the evolution of the domain in which the equations are defined. The direct mapping method is applied for its description, i.e. the liquid domain is represented as the image of a time-dependent diffeomorphism which is defined on a fixed reference domain. Such diffeomorphisms can be constructed by conformal mapping or from small perturbations of the boundary of the reference domain. The free boundary problems are reformulated as nonlinear, nonlocal evolution equations for these time-dependent diffeomorphisms.

The investigation of the evolution equations obtained in this way forms the core of the thesis. Existence, uniqueness, and smoothness properties of solutions to them are established using various methods from functional analysis and the theory of partial differential equations. In the case of Stokes flow in two dimensions, short-time existence and uniqueness results (both forward and backward in time) can be derived for domains with analytic boundary using an abstract Cauchy-Kovalevskaya theorem in a scale of Banach spaces of analytic functions.

Both for Stokes flow and for Hele-Shaw flow in arbitrary space dimensions, the underlying boundary value problems are discussed, and their dependence on small perturbations of the domain in appropriate function spaces is considered. It is shown by the investigation of a linearized problem that the discussed evolution equations are parabolic. Accordingly, short-time existence and uniqueness of solutions (under appropriate assumptions on the smoothness of the initial condition) can be obtained using results from the theory of fully nonlinear parabolic equations. Under an additional assumption on the geometry of the liquid domain, stronger results including C^∞-smoothness of the boundary of the liquid domain can be shown, using Galerkin approximations and a chain rule which is derived from the invariance of the problem with respect to rigid body rotations.

Moreover, it is proved that both for Stokes flow and Hele-Shaw flow driven by surface tension in arbitrary dimension the balls of liquid at rest are exponentially stable equilibria. Finally, the essential results are extended to the case where sources or sinks are present in the liquid domain, and some remarks are made on numerical aspects and open theoretical questions.

Samenvatting

Dit proefschrift is gewijd aan wiskundig onderzoek aan zekere vrije rand problemen die optreden bij de beschrijving van vloeistofstromingen waar traagheidseffecten verwaarloosbaar zijn, d.w.z. waar een quasistationaire benadering toegepast kan worden. De twee problemen van deze aard die hier centraal staan beschrijven de gedaanteverandering van een vrij bewegende vloeistofdruppel en staan bekend onder de namen Stokes flow en Hele-Shaw flow. In beide gevallen worden de stromingen bepaald door elliptische vergelijkingen (Stokes-vergelijkingen of Laplace-vergelijking), de capillaire krachten vormen het aandrijvende mechanisme. Dit leidt tot inhomogene randvoorwaarden waarin de kromming van de rand optreedt.

Het meest interessante aspect van deze vrije rand problemen is de evolutie van het gebied waarop de vergelijkingen gedefinieerd zijn. Voor de beschrijving hiervan wordt de directe afbeeldingsmethode toegepast, d.w.z. het vloeistofgebied wordt voorgesteld als beeld van een tijdsafhankelijk diffeomorfisme dat op een vast referentiegebied gedefinieerd is. Zulke diffeomorfismen kunnen geconstrueerd worden door conforme afbeelding of via kleine verstoringen van de rand van het referentiegebied. De vrije rand problemen worden geherformuleerd als niet-lineaire, niet-lokale evolutievergelijkingen voor deze tijdsafhankelijke diffeomorfismen.

De analyse van de op deze manier verkregen evolutievergelijkingen vormt de kern van dit proefschrift. Existentie, eenduidigheid en gladheidseigenschappen van hun oplossingen worden met behulp van verschillende methoden uit de functionaalanalyse en de theorie van partiële differentiaalvergelijkingen aangetoond. In het geval van twee-dimensionale Stokes flow met analytisch begingebied kunnen, door de toepassing van de abstracte stelling van Cauchy-Kovalevskaya in een schaal van Banachruimten van analytische functies, existentie- en eenduidigheidsresultaten voor korte tijd (voorwaarts en achterwaarts) afgeleid worden.

Voor Stokes flow en Hele-Shaw flow, in willekeurige ruimtelijke dimensies, worden de onderliggende randwaardeproblemen beschouwd. Hun afhankelijkheid van kleine verstoringen van het gebied wordt onderzocht in geschikte functieruimten. Het parabolische karakter van de evolutievergelijkingen wordt aangetoond door de analyse van een gelineariseerd probleem. Op basis daarvan worden existentie- en eenduidigheidsresultaten (onder geschikte condities aan de gladheid van de beginvoorwaarde) uit algemene stellingen uit de theorie van volledig niet-lineaire parabolische vergelijkingen verkregen. Onder een verdere voorwaarde aan de geometrie van het vloeistofgebied kunnen sterkere resultaten bewezen worden, waaronder C^∞-gladheid van de rand. De belangrijkste methoden hierbij zijn Galerkin approximaties en de toepassing van een kettingregel, die met de invariantie van het probleem onder starre lichaamsrotaties samenhangt.

Verder wordt aangetoond dat, in willekeurige dimensies, voor Stokes flow en Hele-Shaw flow aangedreven door oppervlaktespanning de bollen van vloeistof in rust exponentiëel stabiele evenwichtstoestanden zijn. Tenslotte worden de belangrijkste resultaten uitgebreid tot het geval van bronnen en putten in het vloeistofgebied en er worden enkele opmerkingen over numerieke as-

Curriculum Vitae

De schrijver van dit proefschrift werd op 17 april 1969 in Dresden (toen DDR) geboren. Na het behalen van het einddiploma aan de Spezialschule physikalisch-technischer Richtung "M.A. Nexö" in Dresden is hij in 1988 wiskunde gaan studeren aan de Technische Universiteit Dresden. In het kader van het ECMI-studentenuitwisselingsprogramma verbleef hij in 1992 voor drie maanden aan de Technische Universiteit Eindhoven. Onder begeleiding van prof.dr. H.-G. Roos en dr. A. Felgenhauer studeerde hij in 1993 af in de specialisering numerieke wiskunde op een scriptie met de titel "Analytical and numerical models for the motion of a viscous capillary liquid drop."

Sinds november 1993 is hij als assistent in opleiding werkzaam bij de vakgroep analyse van de faculteit wiskunde en informatica van de Technische Universiteit Eindhoven. In deze functie heeft hij onder leiding van prof.dr. J. de Graaf het onderzoek verricht dat geleid heeft tot dit proefschrift.

STELLINGEN

behorende bij het proefschrift

Parabolic evolution equations for quasistationary free boundary problems in capillary fluid mechanics

door G. Prokert

1. De vrije-rand problemen voor Stokes-stromingen en Hele-Shaw-stromingen aangedreven door oppervlaktespanning kunnen geherformuleerd worden als niet-lokale, niet-lineaire parabolische evolutievergelijkingen op een vaste referentievariëteit. Onder geschikte voorwaarden hebben deze vergelijkingen voor korte tijd precies één oplossing.

   Hoofdstukken 3–5 van dit proefschrift

2. De geldigheid van de kettingregel is niet beperkt tot functies die verkregen worden door compositie. In een algemener kader kunnen kettingregels opgevat worden als infinitesimale formuleringen van equivariantie-eigenschappen met betrekking tot fluxen.

   cf. Hoofdstuk 5.1 van dit proefschrift

3. Het karakter van Hoppers vergelijking [6] als impliciete evolutievergelijking komt duidelijker naar voren als zij met behulp van de Hilberttranformatie H op de eenheidscirkel in het complexe vlak op de volgende manier geschreven wordt:

$$(I - iH)\left[\frac{d}{dt}(\overline{\Omega}\Omega') - \frac{d}{d\zeta}\left(\zeta\overline{\Omega}\Omega'(I + iH)\left[\frac{1}{2|\Omega'|}\right]\right)\right] = 0.$$

4. De bewijzen van de existentiestellingen voor Stokes flow met vrije rand aangedreven door oppervlaktespanning in de omgeving van het evenwicht in [1] en [2] zijn onvolledig omdat de theorie van semilineaire vergelijkingen op dit probleem niet van toepassing is.

5. De beweringen in [4] over Hele-Shaw flow aangedreven door oppervlaktespanning voor algemene begingebieden worden in dit artikel niet bewezen.

6. De in [8] geïntroduceerde vierde-orde oppervlaktebewegingswet is geen geschikt model voor viskeus sinteren omdat de niet-lokaliteit van het probleem verwaarloosd wordt.

7. Gegeven een rooster $0 = x_0 < x_1 < \ldots < x_n = 1$ en een functie $f \in C^4[0,1]$. Beschouw de differentiaaloperator S gedefinieerd door

$$Sv = v'' - \omega^2 v, \qquad \omega > 0.$$

Dan geldt:

(i) Er is precies één interpolerende exponentiële spline $u \in C^2[0,1]$ zodanig dat

$$\begin{array}{rcll} S^2u & = & 0 & \text{in } (x_{i-1}, x_i) \quad (i = 1, \ldots, n), \\ u(x_i) & = & f(x_i) & \qquad\qquad\quad (i = 0, \ldots, n), \\ Su(0) & = & Sf(0), & \\ Su(1) & = & Sf(1), & \end{array}$$

(ii) voor deze u geldt

$$||u - f||_{C^j[0,1]} \leq K_j h^{4-j} \qquad (j = 0, 1, 2)$$

met $h = \max\{x_i - x_{i-1} \mid i = 1, \ldots, n\}$. De constanten K_j zijn onafhankelijk van het rooster. [7]

8. Bij de numerieke oplossing van het niet-lineaire slecht gestelde parameteridentificatieprobleem voor het randwaardeprobleem

$$\begin{array}{rcl} \partial_x(a(x)\partial_x u) & = & f \text{ in } (0,1), \\ u(0) = u(1) & = & 0, \end{array}$$

waarbij a moet worden bepaald uit f en u levert regularisatie met behulp van de discrete H^1-norm vaak duidelijk betere resultaten op dan regularisatie met behulp van de discrete L^2-norm.

Zie [5].

9. De vooruitgang op het gebied van hard- en software voor numerieke simulatie en visualisatie is van het grootste belang voor de toepasbaarheid en de uitstraling van de wiskunde. Deze vooruitgang dient echter niet verward te worden met vooruitgang bij het begrijpen en oplossen van de onderliggende wiskundige problemen.

10. "In mathematics as elsewhere close attention to immediately useful ends is not always the most effective way of being practical."

"Curiosity may be idle if allowed its own way too long; but without it, little of even the lowest practical value has been achieved."

E. T. Bell [3]

11. De onjuiste bewering dat communisme en fascisme min of meer als twee vormen van hetzelfde verschijnsel gezien moeten worden geeft blijk van een gevaarlijke ignorantie ten opzichte van de wortels en drijfveren van beide bewegingen.

12. "Het gelijk van rechts" berust op optisch bedrog.

13. Om het leggen van een onbedoeld verband te voorkomen dienen op spoorwegstations plakkaten met het opschrift "Vloek niet !" niet te dicht naast de dienstregeling te worden opgehangen.

14. Onze maatschappij moet zich onthaasten, en wel zo spoedig mogelijk.

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