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Abstract

The synchronization problem for complex discrete–time systems is revisited from a control perspective and it is argued that the problem may be viewed as an observer problem. It is shown that for several classes of systems a solution for the synchronization (observer) problem exists. Also, by allowing past measurements a dynamic mechanism for state reconstruction is provided.

1.1 Introduction

Since the work of Pecora and Carroll [18], a huge interest in (chaos) synchronization has arisen. Among others, this is illustrated by the appearance of a number of special issues of journals devoted to the subject, cf. [29, 28, 30]. One clear motivation for this widespread interest lies in the fact that Pecora and Carroll indicated that chaos synchronization might be useful in communications. Although by now this claim is not fully justified yet, several interesting applications of (chaos) synchronization are envisioned.

Synchronization as it was introduced by Pecora and Carroll has been studied from various viewpoints. Following [18], often a receiver-transmitter (or master-slave) formalism is taken, where typically the receiver system is an exact copy of the transmitter system, and the aim is to synchronize the receiver response with that of the transmitter, provided the receiver dynamics are driven by a scalar signal from the transmitter, see [18, 4, 26].

More recently, the above method was recast in an *active-passive decomposition*, see [17], where the decomposition idea has to be understood in the way that part of the transmitter state needs to be transmitted, while the "passive" part then will be derived asymptotically.

Another idea to achieve synchronization between (identical) transmitter and receiver dynamics is to include (linear) feedback of the drive signal in the receiver system, see [16] and [11] where a number of sucessfull experimental settings of this type are discussed.

A third way to achieve synchronization between transmitter and receiver was recently put forward in [6] and essentially advertises the idea of system inversion for (state) synchronization.

Notwithstanding the widespread interest in the synchronization problem, sofar the problem leaves quite some ambiguity in how to make an active-passive decomposition, or how to build successfully an (stable) inverse system. Indeed, this ambiguity disappears when the synchronization problem is viewed as the question how to reconstruct the full state trajectory of the transmitter system, given some (scalar) drive signal from the transmitter. This is essentially the observer problem from control theory, and has, following the earlier attempts [5, 19, 13], by now obtained a prominent place within recent synchronization literature, see for instance [14] and various other observer-based synchronization papers.

The purpose of the present chapter is to revisit the synchronization problem for discrete-time systems using (discrete-time) observers. Synchronization of complex/chaotic discrete-time systems has been the subject of various publications, see e.g. [2, 1, 7, 21, 27], but only little attention for an observer-based viewpoint exists (see, however, [24, 25] where this viewpoint *is* taken, and [26], which may be interpreted as a particular application of the observer-based viewpoint (although this is not mentioned explicitly in [26])). One could argue, however, that the synchronization problem for discrete-time systems is as important as the continuous-time counterpart. First, for communications of *binary* signals one can very well base oneself on discrete-time transmitter systems instead of continuous-time transmitters. A second motivation to look at discrete-time synchronization is that many continuous-time models are in the end -for instance for simulation and implementation- discretized or sampled. A third motivation is that discrete-time dynamics are obtained when one considers the Poincaré map at a suitably defined Poincaré section of a chaotic transmitter system.

As stated, we pursue an observer-based view on (discrete-time) synchronization. Although there exist some clear analogies between discrete-time and continuous-time observers, there are various results available in either context which do not admit a proper analogon in the other domain.

This chapter is organized as follows. In Section 1.2, we treat some preliminaries and give our problem statement. Section 1.3 is devoted to nonlinear discrete–time transmitters of a special form, the so called Lur'e form. It is shown that for this kind of systems the construction of an observer is relatively easy. In Section 1.4, we study the question when a given nonlinear discrete–time transmitter is equivalent to a system in Lur'e form by means of a coordinate transformation. In Section 1.5, we introduce a so called extended Lur'e form, indicate how observers for transmitters in this extended Lur'e form may be constructed, and give conditions under which a nonlinear discrete–time transmitter may be transformed into an extended Lur'e form. Section 1.6 treats the observer design for perturbed linear transmitters. Section 1.7 finally, contains some conclusions.

1.2 Preliminaries and Problem Statement

Throughout this chapter, we consider discrete-time nonlinear (transmitter) dynamics of the form

$$x(k+1) = f(x(k)), \quad x(0) = x_0 \in \mathbb{R}^n \tag{1.2.1}$$

where the state transition map f is a smooth mapping from $\mathbb{R}^n$ into itself. Note that direct extensions of (1.2.1) are possible by allowing the state to belong to an open subset of $\mathbb{R}^n$, or to a differentiable manifold. The solution $x(k, x_0)$ of (1.2.1) is not directly available, but only an *output* is measured, say

$$y(k) = h(x(k)) \tag{1.2.2}$$

where $y \in \mathbb{R}^p$ and $h : \mathbb{R}^n \to \mathbb{R}^p$ is the smooth output map. Though in the sequel there is no restriction in assuming the transmitted signal $y(k)$ to be p-dimensional, we will for simplicity -and following most work on synchronization- take $p = 1$.

The observer problem for (1.2.1,1.2.2) now deals with the question how to reconstruct the state trajectory $x(k, x_0)$ on the basis of the measurements $y(k)$. A *full observer* (or briefly *observer*) for the system (1.2.1,1.2.2) is a dynamical system of the form

$$\hat{x}(k+1) = \hat{f}(\hat{x}(k), y(k)), \quad \hat{x}(0) = \hat{x}_0 \in \mathbb{R}^n \tag{1.2.3}$$

where $\hat{x} \in \mathbb{R}^n$, and $\hat{f}$ is a smooth mapping on $\mathbb{R}^n$ parametrized by y, such that the error $e(k) := x(k) - \hat{x}(k)$ asymptotically converges to zero as $k \to \infty$ for all initial conditions x_0 and $\hat{x}_0$. Moreover, we require that if $e(k_0) = 0$ for some k_0, then $e(k) = 0$ for all $k \geq k_0$.

1.3 Systems in Lur'e Form

The problem of observer design in its full generality is a problem that is difficult to solve. Basically, only the observer design problem for linear systems has been solved in its full generality, cf. [20]. Therefore, we start our survey of possible approaches to observer-based synchronization by considering a class of nonlinear systems that is slightly more general than linear systems, namely systems in so called Lur'e form.

Assume that the master dynamics are governed by the following system of difference equations

$$x(k+1) = Ax(k) + \varphi(y(k)), \quad y(k) = Cx(k), \tag{1.3.4}$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^1$ is the scalar output, $\varphi : \mathbb{R}^1 \to \mathbb{R}^n$ is a smooth function and A, C are constant matrices of appropriate dimensions. Dynamics of the form (1.3.4) are referred to as dynamics in *Lur'e form*. The question we now pose is, under what conditions it is possible to design an observer for (1.3.4)? As a possible observer candidate one can build a copy of (1.3.4) augmented with so called *output injection*:

$$\widehat{x}(k+1) = A\widehat{x}(k) + \varphi(y(k)) + L(y(k) - \widehat{y}(k)), \quad \widehat{y}(k) = C\widehat{x}(k), \tag{1.3.5}$$

where $\widehat{x}(k) \in \mathbb{R}^n$ is the estimate of $x(k)$ and L is a $n \times 1$ matrix, see [10].

The solutions of the systems (1.3.4) and (1.3.5) will synchronize if for all initial conditions the error $e(k) := x(k) - \widehat{x}(k)$ tends to zero when k tends to infinity. Substracting (1.3.5) from (1.3.4) one can easily see that the error vector $e(k)$ obeys the following linear difference equation

$$e(k+1) = (A - LC)e(k). \tag{1.3.6}$$

Therefore, if all eigenvalues of $A - LC$ lie in the open unit disc (i.e., the set $\{z \in \mathbb{C} \mid |z| < 1\}$), then (1.3.5) is an observer for (1.3.4). In other words, for the system (1.3.4) the synchronization problem can be reduced to the following question: given A, C, under what conditions does there exist a matrix L such that $A - LC$ has all eigenvalues in the open unit disc? This linear algebraic problem has a simple solution. Namely, a sufficient condition for existence of L is the invertibility of the following linear mapping

$$\mathcal{O}(x) := \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x. \tag{1.3.7}$$

In linear control theory, a pair of matrices (C, A) such that $\mathcal{O}(x)$ in (1.3.7) is invertible, is said to be an *observable pair*. Using this terminology we can formulate the following result.

THEOREM 1.1
Assume that the pair (C, A) is observable. Then the system (1.3.4) admits an observer (1.3.5) with the exponentially stable linear error dynamics (1.3.6).

The proof of this result can be found in any textbook on linear control theory (see e.g. [20]). It is worth mentioning that the proof is constructive. Namely, the linear mapping $\mathcal{O}$ defines a similarity transformation such that the matrix $A - LC$ is similar to the following matrix in Frobenius form

$$\begin{bmatrix} 0 & \cdots & 0 & a_1 - l_1 \\ 1 & \cdots & 0 & a_2 - l_2 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & a_n - l_n \end{bmatrix}$$

where $\mathrm{col}(l_1, l_2, \ldots, l_n) = \mathcal{O}(L)$, and the a_i are the coefficients of the characteristic polynomial of A. Since $a_i - l_i$ are the coefficients of the characteristic polynomial of $A - LC$ it is always possible to locate the eigenvalues of $A - LC$ in the open unit disc by means of an appropriate choice of the matrix L.

It is worth mentioning that the condition of observability is in fact a sufficient, but not a necessary condition allowing to design an observer.

Namely, the system may have $\mathcal{O}(x)$ of rank lower than n, but at the same time it may admit an observer. This situation occurs when the so-called unobservable dynamics are exponentially stable. In the control literature linear systems with exponentially stable unobservable dynamics are referred to as *detectable* (cf. [20]). In practice it often means that such systems can be transformed to an observable system via model reduction.

Example 1 Consider the following discrete-time dynamics in Lur'e form:

$$\begin{aligned} \begin{bmatrix} z_1(k+1) \\ z_2(k+1) \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & -\alpha \\ 1 & 1+\alpha \end{bmatrix}}_{A} \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -\beta \cos y(k) \end{bmatrix}}_{\varphi(y(k))} \\ y(k) &= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} z(k) \end{aligned} \tag{1.3.8}$$

where $\alpha, \beta > 0$. In this case, we obtain

$$\begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1+\alpha \end{bmatrix} \tag{1.3.9}$$

which clearly is an invertible matrix. Thus, one may construct an observer for (1.3.8) of the following form:

$$\begin{aligned} \begin{bmatrix} \hat{z}_1(k+1) \\ \hat{z}_2(k+1) \end{bmatrix} &= \underbrace{\begin{bmatrix} 0 & -\alpha \\ 1 & 1+\alpha \end{bmatrix}}_{A} \begin{bmatrix} \hat{z}_1(k) \\ \hat{z}_2(k) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ -\beta \cos y(k) \end{bmatrix}}_{\varphi(y(k))} + \\ &\quad + L(y(k) - \hat{y}(k)) \\ \hat{y}(k) &= \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \hat{z}(k) \end{aligned} \tag{1.3.10}$$

where $L = \mathrm{col}(l_1, l_2)$, and l_1 and l_2 are chosen such that all eigenvalues of the matrix

$$A - LC = \begin{bmatrix} 0 & -\alpha - l_1 \\ 1 & 1+\alpha - l_2 \end{bmatrix} \tag{1.3.11}$$

lie in the open unit disc. △

1.4 Transformation into Lur'e Form

In the previous section we learned that if the transmitter dynamics are in Lur'e form (1.3.4) and the pair (C, A) is observable, then it is always possible to design a receiver system which synchronizes with (1.3.4).

The result presented in the previous section is very simple. However, the following question remains open: what can we do if the transmitter dynamics are not in the form (1.3.4)? In this section we will present a partial answer to this question.

First of all, notice that the representation (1.3.4) is coordinate dependent. This means that if one rewrites the system (1.3.4) in a new coordinate system via a (nonlinear) coordinate change $z = T(x)$, then a new representation of the *same* dynamical system is not necessarily in the form (1.3.4).

By the same token, however, this may also mean that it is possible to transform a system into Lur'e form by means of a nonlinear coordinate change $z = T(x)$. Hence, we arrive at the following problem.

Let a discrete–time system (1.2.1,1.2.2) with scalar output be given, and assume that $f(0) = 0$, $h(0) = 0$. The problem is to find conditions ensuring existence of an invertible coordinate change $z = T(x)$ such that the system (1.2.1) is locally (or globally) equivalent to the following Lur'e system

$$z(k+1) = Az(k) + \varphi(y(k)), \quad y(k) = Cz(k) \tag{1.4.12}$$

where the pair (C, A) is observable.

As one can see from the problem statement, the coordinate change $z = T(x)$ can be either locally or globally defined (i.e., the inverse mapping T^{-1} can exist on a neighborhood of the origin or everywhere). In the first case the systems (1.2.1,1.2.2) and (1.4.12) are equivalent if for all k one has that $||x(k)||$ is sufficiently small. In the second case there are no restrictions of such kind.

The following result from [12] gives a (local) solution to the problem.

THEOREM 1.2
A discrete–time system (1.2.1,1.2.2) with single output is locally equivalent to a system in Lur'e form (1.4.12) with observable pair (C, A) via a coordinate change $z = T(x)$ if and only if

(i) *the pair $(\partial h(0)/\partial x, \partial f(0)/\partial x)$ is observable,*

(ii) *the Hessian matrix of the function $h \circ f^n \circ \mathcal{O}^{-1}(s)$ is diagonal, where $x = \mathcal{O}^{-1}(s)$ is the inverse map of*

$$\mathcal{O}(x) = \begin{bmatrix} h(x) \\ h \circ f(x) \\ \vdots \\ h \circ f^{n-1}(x) \end{bmatrix}, \tag{1.4.13}$$

with $h \circ f(x) := h(f(x))$, $f^1 := f$, $f^j := f \circ f^{j-1}$.

It is important to notice that condition (i) means that the Jacobian $\partial\mathcal{O}(0)/\partial x$ is invertible. In an equivalent form it can be rewritten in the form

$$\dim\left(\operatorname{span}\left\{\frac{\partial h}{\partial x}(0), \frac{\partial h\circ f}{\partial x}(0), \ldots, \frac{\partial h\circ f^{n-1}}{\partial x}(0)\right\}\right) = n$$

The condition (ii) may be interpreted in the following way. As indicated above, if condition (i) holds, the transformation $s = \mathcal{O}(x)$ is a local diffeomorphism. Thus, s forms a new set of local coordinates for the dynamics (1.2.1) around the origin. It is straightforwardly checked that in these new coordinates the system (1.2.1,1.2.2) takes the form

$$\left\{\begin{array}{rcl} s_1(k+1) & = & s_2(k) \\ & \vdots & \\ s_{n-1}(k+1) & = & s_n(k) \\ s_n(k+1) & = & f_s(s(k)) \\ y(k) & = & s_1(k) \end{array}\right. \tag{1.4.14}$$

where $f_s(s) := h\circ f^n\circ\mathcal{O}^{-1}(s)$. In the literature (cf. [15]), the form (1.4.14) is referred to as the *observable form* of the system (1.2.1,1.2.2). Condition (ii) then is equivalent to the local existence of functions $\varphi_1, \cdots, \varphi_n : \mathbb{R}\to\mathbb{R}$ such that

$$f_s(s) = \varphi_1(s_1) + \varphi_2(s_2) + \cdots + \varphi_n(s_n) \tag{1.4.15}$$

With the functions $\varphi_1, \cdots, \varphi_n$ at hand, the transformation

$$z_i := s_{n+1-i} - \sum_{k=i+1}^{n} \varphi_k(s_{k-i}) \quad (i = 1, \cdots, n) \tag{1.4.16}$$

then transforms the observable form (1.4.14) into the following Lur'e form:

$$\left\{\begin{array}{rcl} z_1(k+1) & = & \varphi_1(y(k)) \\ z_2(k+1) & = & z_1(k) + \varphi_2(y(k)) \\ & \vdots & \\ z_n(k+1) & = & z_{n-1}(k) + \varphi_n(y(k)) \\ y(k) & = & z_n(k) \end{array}\right. \tag{1.4.17}$$

The mapping $\mathcal{O}$ in (1.4.13) and the observable form play an important role in the observer design for nonlinear discrete–time systems. As one can easily see, in the linear case the mapping $\mathcal{O}$ is exactly the linear operator (1.3.7) introduced in the previous section. Since the Jacobian of $\mathcal{O}$ is invertible around $x = 0$ the mapping $\mathcal{O}$ is a local diffeomorphism. If one is interested in finding a coordinate change $z = T(x)$ which is globally defined it is sufficient to check that $\mathcal{O}$ is a global diffeomorphism from $\mathbb{R}^n$ to $\mathbb{R}^n$ *and* the functions $\varphi_1, \cdots, \varphi_n$ satisfying (1.4.15) exist globally.

Example 2 Bouncing ball. Consider the following discrete–time model which describes the bouncing ball system [23, 3]:

$$\begin{cases} x_1(k+1) = x_1(k) + x_2(k) \\ x_2(k+1) = \alpha x_2(k) - \beta \cos(x_1(k) + x_2(k)) \end{cases} \tag{1.4.18}$$

where $x_1(k)$ is the phase of the table at the k-th impact, $x_2(k)$ is proportional to the velocity of the ball at the k-th impact, the parameter α is the coefficient of restitution, and $\beta = 2\omega^2(1+\alpha)A/g$. Here ω is the angular frequency of the table oscillation, A is the corresponding amplitude and g is the gravitational acceleration. For some values of the parameters the system can exhibit very complex behavior. However, we will show that this is not an obstacle for the design of an observer.

Suppose only the first variable x_1 (the phase) is available for measurement. The question is: can we reconstruct the second variable? Clearly the system (1.4.18) is not in Lur'e form. However, using the theory presented in this section, we will show that there exists a coordinate change that transform (1.4.18) into Lur'e form.

So, we assumed that

$$y(k) = h(x(k)) = x_1(k).$$

Let us check the conditions of Theorem 1.2. A simple calculation gives

$$\frac{\partial h(0)}{\partial x} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad \frac{\partial f(0)}{\partial x} = \begin{bmatrix} 1 & 1 \\ 0 & \alpha \end{bmatrix}$$

and this pair is clearly observable. Hence, condition (i) is satisfied.

To check condition (ii), let us find the mapping $\mathcal{O}$. Obviously,

$$\mathcal{O}(x) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x \tag{1.4.19}$$

with $x = \mathrm{col}(x_1, x_2)$. This mapping is linear, it is invertible and therefore it is a global diffeomorphism. Introducing $s = \mathrm{col}(s_1, s_2) := \mathcal{O}(x)$ we see that

$$f_s(s) := h \circ f^2 \circ \mathcal{O}^{-1}(s) = -\alpha s_1 + (1+\alpha)s_2 - \beta \cos s_2$$

and it is clear that the Hessian of this function is diagonal. Thus, condition (ii) is satisfied as well. Note that, in view of (1.4.15), we have that $f_s(s) = \varphi_1(s_1) + \varphi_2(s_2)$, with $\varphi_1(s_1) := -\alpha s_1$, $\varphi_2(s_2) := (1+\alpha)s_2 - \beta \cos s_2$. Therefore there exists a coordinate change which locally transforms the system (1.4.18) into Lur'e form. Moreover, the mapping $\mathcal{O}$ is a global diffeomorphism *and* the functions φ_1, φ_2 are globally defined, which gives that this coordinate change is in fact global.

From (1.4.19),(1.4.16) we obtain the following coordinate change:

$$\begin{cases} z_1 = -\alpha x_1 + x_2 + \beta \cos x_1 \\ z_2 = x_1 \end{cases} \tag{1.4.20}$$

with the output $y = z_2 = x_1$. In the new coordinate system the original system (1.4.18) has the following form

$$\begin{cases} z_1(k+1) = -\alpha z_2(k) \\ z_2(k+1) = z_1(k) + (1+\alpha) z_2(k) - \beta \cos z_2(k). \end{cases} \tag{1.4.21}$$

Note that the dynamics (1.4.21) are identical to the dynamics (1.3.8). Therefore, an observer for (1.4.21) is given by (1.3.10).

The estimates $\widehat{x}_1, \widehat{x}_2$ for x_1, x_2 are given by the following relations, which immediately follow from (1.4.20)

$$\begin{cases} \widehat{x}_1 = \widehat{z}_2 \\ \widehat{x}_2 = \widehat{z}_1 + \alpha \widehat{z}_2 - \beta \cos \widehat{z}_2 \end{cases} \tag{1.4.22}$$

with $\hat{z}_1, \hat{z}_2$ the observer state for (1.4.21). Moreover, by means of an appropriate choice of l_1, l_2 one can achieve arbitrarily fast convergence of $\widehat{x}(k)$ to $x(k)$. △

1.5 Transformation into Extended Lur'e Form

In the previous section we found that if the observability mapping $\mathcal{O}$ is a diffeomorphism and condition (ii) of Theorem 1.2 holds, then there exists a coordinate change transforming the system (1.2.1,1.2.2) into Lur'e form, which makes the observer design a simple linear algebraic problem. Especially condition (ii) of Theorem 1.2 is quite restrictive. Therefore, the question arises whether, and in what way, this condition may be relaxed.

To answer this question, we will assume in this section that at time k besides $y(k)$ also past output measurements $y(k-1), \cdots, y(k-N)$ for some $N > 0$ are available, and first consider nonlinear dynamics of the following form:

$$\begin{cases} x(k+1) &= Ax(k) + \varphi(y(k), y(k-1), \cdots, y(k-N)) \\ y(k) &= Cx(k) \end{cases} \tag{1.5.23}$$

where $x(k) \in \mathbb{R}^n$, $y(k) \in \mathbb{R}^1$, $\varphi : \mathbb{R}^{N+1} \to \mathbb{R}^n$ is a smooth mapping, and A, C are matrices of appropriate dimensions. Note that the dynamics (1.5.23) for $N = 0$ are just the dynamics (1.3.4). Therefore, we refer to dynamics of the form (1.5.23) as dynamics in *extended Lur'e form with buffer N*. Assume that the pair (C, A) is observable. As we have seen in

Section 1.3 there then exists a matrix L such that all eigenvalues of $A-LC$ lie in the open unit disc. Along the same lines as in Section 1.3, it may then be shown that the following dynamics are an observer for (1.5.23):

$$\left\{ \begin{array}{rcl} \widehat{x}(k+1) & = & A\widehat{x}(k) + \varphi(y(k), \cdots, y(k-N)) + L(y(k) - \widehat{y}(k)) \\ \widehat{y}(k) & = & C\widehat{x}(k) \end{array} \right. \tag{1.5.24}$$

As in Section 1.4, we now ask ourselves the question under what conditions the discrete-time system (1.2.1,1.2.2) may be transformed into an extended Lur'e form for some $N \geq 0$. The transformations we are going to use here, are more general than the transformation in Section 1.4, in the sense that we also allow them to depend on the past output measurements $y(k-1), \cdots, y(k-N)$. More specifically, we will be looking at parametrized transformations $z = T(x, \xi_1, \cdots, \xi_N)$, where $z \in \mathbb{R}^n$, with the property that (locally or globally) there exists a mapping $T^{-1}(\cdot, \xi_1, \cdots, \xi_N) : \mathbb{R}^n \to \mathbb{R}^n$ parametrized by $(\xi_1, \cdots, \xi_N)$, such that for all $(\xi_1, \cdots, \xi_N)$ we have

$$T(T^{-1}(z, \xi_1, \cdots, \xi_N), \xi_1, \cdots, \xi_N) = z$$

A mapping having this property will be referred to as an *extended coordinate change*. We will then say that the system (1.2.1,1.2.2) may be transformed into an extended Lur'e form with buffer N if there exists an extended coordinate change $T(\cdot, \xi_1, \cdots, \xi_N) : \mathbb{R}^n \to \mathbb{R}^n$ parametrized by $(\xi_1, \cdots, \xi_N)$ such that the variable

$$z(k) := T(x(k), y(k-1), \cdots, y(k-N)) \tag{1.5.25}$$

satisfies (1.5.23), where the pair (C, A) is observable. As pointed out above, one may then build an observer (1.5.24) for $z(k)$ in (1.5.25). From this observer, one then obtains estimates $\widehat{x}(k)$ for $x(k)$ by inverting the extended coordinate change T:

$$\widehat{x}(k) := T^{-1}(\widehat{z}(k), y(k-1), \cdots, y(k-N)) \tag{1.5.26}$$

The following result from [9] (see also [8]) gives conditions under which a system (1.2.1,1.2.2) may be transformed into an extended Lur'e form with buffer N.

THEOREM 1.3
Consider a discrete–time system (1.2.1,1.2.2), and assume that the mapping $\mathcal{O}$ in (1.4.13) is a local diffeomorphism. Let $N \in \{0, \cdots, n-1\}$ be given. Then (1.2.1,1.2.2) may be locally transformed into an extended Lur'e form with buffer N if and only if there locally exist functions $\varphi_{N+1}, \cdots, \varphi_n : \mathbb{R}^{N+1} \to \mathbb{R}$ such that the function f_s in the observable form (1.4.14) satis-

fies

$$f_s(s_1, \cdots, s_n) = \sum_{i=N+1}^{n} \varphi_i(s_i, \cdots, s_{i-N}) \tag{1.5.27}$$

The proof of the above theorem is constructive. Namely, assume that functions $\varphi_{N+1}, \cdots, \varphi_n$ satisfying (1.5.27) exist, and define an extended coordinate change by

$$z_i := \begin{cases} s_{n-i+1} - \sum_{j=N+1}^{n} \varphi_j(s_{j-i}, \cdots, s_{j-i-N}) & (i = 1, \cdots, N-1) \\ \\ s_{n-i+1} - \sum_{j=i+1}^{n} \varphi_j(s_{j-i}, \cdots, s_{j-i-N}) & (i = N, \cdots, n) \end{cases} \tag{1.5.28}$$

It is then straightforwardly checked that in these new extended coordinates the observable form (1.4.14) takes the following extended Lur'e form:

$$\begin{cases} z_1(k+1) &= 0 \\ z_2(k+1) &= z_1(k) \\ &\vdots \\ z_N(k+1) &= z_{N-1}(k) \\ z_{N+1}(k+1) &= z_N(k) + \varphi_{N+1}(y(k), \cdots, y(k-N)) \\ &\vdots \\ z_n(k+1) &= z_{n-1}(k) + \varphi_n(y(k), \cdots, y(k-N)) \\ y(k) &= z_n(k) \end{cases} \tag{1.5.29}$$

Theorem 1.3 gives necessary and sufficient conditions for the local existence of an extended Lur'e form with buffer N for (1.2.1,1.2.2). For global existence of an extended Lur'e form with buffer N, the mapping $\mathcal{O}$ in (1.4.13) needs to be a global diffeorphism, and the mappings $\varphi_{N+1}, \cdots, \varphi_n$ satisfying (1.5.27) need to exist globally.

It is easily checked that for $N = n-1$, condition (1.5.27) is always satisfied globally. Thus, we have that a system (1.2.1,1.2.2) for which the mapping $\mathcal{O}$ in (1.4.13) is a local (global) diffeomorphism may always be locally (globally) transformed into an extended Lur'e form with buffer $n-1$.

1.6 Observers for Perturbed Linear Systems

So far the design procedure for observers has been based on the assumption that for the discrete–time system under consideration the mapping $\mathcal{O}$ in

(1.4.13) is a (local or global) diffeomorphism. In the sequel, we consider a particular class of systems for which this might not be the case. Namely, we consider systems of the form

$$\left\{ \begin{array}{rcl} x(k+1) & = & Ax(k) + Bf(x(k)) \\ y(k) & = & Cx(k) \end{array} \right. \tag{1.6.30}$$

where $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^1$ is the scalar output, the function $f : \mathbb{R}^n \to \mathbb{R}^1$ is smooth, A, B, C are matrices of appropriate dimensions, and the pair (C, A) is observable. Clearly, depending on the specific structure of f and B, the system (1.6.30) may have a mapping $\mathcal{O}$ that is not a diffeomorphism. Nevertheless, we may derive conditions on (1.6.30) that guarantee the existence of an observer.

Define the rational function $G(s)$ by

$$G(s) := C(sI - A)^{-1}B \tag{1.6.31}$$

Then $G(s)$ has the form $G(s) = \frac{q(s)}{p(s)}$, where q and p are polynomials in s, with $\deg(p) > \deg(q)$. We now assume that $\deg(p) - \deg(q) = 1$. It may be shown that this is equivalent to the fact that $CB \neq 0$. To obtain an observer for (1.6.30), we first define new coordinates in the following way. Since $CB \neq 0$, there exists an $(n-1) \times n$ matrix N such that $NB = 0$ and the matrix $S := \left[\begin{array}{cc} C^T & N^T \end{array} \right]^T$ is invertible. Thus, $(\xi, z) := (Cx, Nx)$ forms a new set of coordinates for (1.6.30). It is straightforwardly checked that in these new coordinates the system (1.6.30) takes the form

$$\left\{ \begin{array}{rcl} \xi(k+1) & = & \bar{f}(\xi(k), z(k)) \\ z(k+1) & = & A_1\xi(k) + A_2 z(k) \\ y(k) & = & \xi(k) \end{array} \right. \tag{1.6.32}$$

where

$$\bar{f}(\xi, z) = C\left[AS^{-1}\left[\begin{array}{c} \xi \\ z \end{array}\right] + CBf\left(S^{-1}\left[\begin{array}{c} \xi \\ z \end{array}\right]\right)\right]$$

and

$$\left[\begin{array}{cc} A_1 & A_2 \end{array}\right] = NAS^{-1}$$

We now assume the following:

A1 The mapping $\bar{f}$ in (1.6.32) is globally Lipschitz with respect to z, i.e., there exists an $L > 0$ such that

$$(\forall \xi \in \mathbb{R})(\forall z, \bar{z} \in \mathbb{R}^{n-1})(|\bar{f}(\xi, \bar{z}) - \bar{f}(\xi, z)| < L||\bar{z} - z||)$$

A2 All zeros of the polynomial $q(s)$ are located in the open unit disc.

As an observer candidate we take the following system:

$$\begin{cases} \widehat{\xi}(k+1) &= \bar{f}(y(k), \widehat{z}(k)) \\ \widehat{z}(k+1) &= A_1 y(k) + A_2 \widehat{z}(k) \end{cases} \tag{1.6.33}$$

We then have the following result.

THEOREM 1.4
Assume that for (1.6.30) we have that the pair (C, A) is observable, that $CB \neq 0$, and that assumptions **A1** *and* **A2** *hold. Then (1.6.33) is an observer for (1.6.32).*

PROOF Defining the error signals $e_\xi(k) := \xi(k) - \widehat{\xi}(k)$, $e_z(k) := z(k) - \widehat{z}(k)$, we obtain the following error equations:

$$\begin{cases} e_\xi(k+1) &= \bar{f}(\xi(k), e_z(k) + \widehat{z}(k)) - \bar{f}(\xi(k), \widehat{z}(k)) \\ e_z(k+1) &= A_2 e_z(k) \end{cases} \tag{1.6.34}$$

It is easily checked that assumption **A2** implies that all eigenvalues of A_2 are in the open unit disc. This implies on its turn that there exist $\gamma > 0, 0 < \lambda < 1$ such that $e_z(k)$ satisfies

$$||e_z(k)|| \leq \gamma \lambda^k ||e_z(0)|| \tag{1.6.35}$$

Using assumption **A1** and (1.6.35), we then obtain

$$|e_\xi(k)| = |\bar{f}(\xi(k-1), e_z(k-1) + \widehat{z}(k-1)) - \bar{f}(\xi(k-1), \widehat{z}(k-1))| <$$

$$L||e_z(k-1)|| \leq L\gamma\lambda^{k-1}||e_z(0)|| \tag{1.6.36}$$

Since $0 < \lambda < 1$, it follows from (1.6.35),(1.6.36) that $e_\xi(k), e_z(k) \to 0$ for $k \to +\infty$, and thus (1.6.33) is an observer for (1.6.32). ▯

REMARK A.
The result in this section may be generalized to systems (1.6.30) for which we have that $\deg(p) - \deg(q) > 1$. This generalization will be given in a forthcoming paper.

1.7 Conclusions

Following a similar line of research as in [14] we develop an observer perspective on the synchronization problem for nonlinear (complex) discrete–time systems. For several classes of discrete–time systems it is shown that a suitable observer can be found. In case such an observer does not exist, or, can not be found analytically we propose to use an extended observer. The latter method follows [8], see also [9], and presents an observer that uses also past measurements, and can be applied under fairly general conditions. Like the continuous–time paper [14] it seems that control theory might be a very valuable tool in the study of synchronization.

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