Using wavelets in the dual reciprocity method

Using Wavelets in the Dual Reciprocity Method

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Abstract

Radial basis functions, in especial thin plate splines or conic splines seem to be very useful in the Dual Reciprocity Method (DRM) for solving partial differential equations by means of BEM. In DRM the radial basis function have been used for computing a particular solution for an inhomogenuous partial differential. Two main steps are important in applying DRM. First, a function f must be approximated by shifted radial basis functions. Second, a particular solution must be found in an easy way for the radial basis function itself. Despite the fact that radial basis functions may have good approximation properties, the approximation scheme's involved are far from local. The consequence is that many coefficients are needed to represent functions by series of shifted radial basis functions, which causes a lot of computational efforts. Wavelets have the potential to overcome this problem. However, for wavelets the problem of finding a particular solution looks more complicated. In this chapter, it will be shown that for the so-called hexagonal wavelets in two-dimensions and the Poisson equation, it is still possible to derive a particular solution by doing elementary function evaluations.

1 Introduction

Consider the Poisson equation in two dimensions:

$$\triangle u(x) = f(x) \qquad (x \in \Omega).$$

Here $x = (x_1, x_2) \in \mathbb{R}^2$, $\Delta = \frac{\partial^2}{\partial^2_{x_1}} + \frac{\partial^2}{\partial^2_{x_2}}$ is the 2D Laplacian, f is a presribed bivariate function and $\Omega \subset \mathbb{R}^2$ is a bounded simply connected domain in $\mathbb{R}^2$. To find the function $u(x, y)$, which satisfies the Poisson equation on Ω subject to the Dirichlet condition

$$u(x) = g(x) \qquad (x \in \Gamma).$$

where Γ is the closed piecewise smooth boundary of Ω and g a prescribed continuous function, by means of BEM methods, one first convert this problem into an integral as follows:

Let $G(x)$ be the fundamental function corresponding to the Laplacian $\triangle$. So, $\triangle G(x) = \delta(x)$, where $\delta(x)$ is the bivariate Dirac delta function. One has

$$G(x) = \frac{1}{2\,\pi} \log |x|, \tag{1}$$

where $|x| = \sqrt{x_1^2 + x_2^2}$. It follows from Green's second formula that for points x in the interior of Ω the following holds.

$$u(x) - \int_\Omega G_x(y)\, f(y)\, d\,\Omega = \int_\Gamma u(y) \frac{\partial G_x(y)}{\partial n}\, d\,\Gamma - \int_\Gamma G_x(y)\, \frac{\partial u(y)}{\partial n}\, d\,\Gamma. \tag{2}$$

where $G_x(y)$ stands for $G(x-y)$ and where n is the outer normal of Ω.

When x belongs to the boundary Γ, then depending on the local smoothness of Γ at x, the value $u(x)$ in the foregoing equation must be replaced by $c(x)\,u(x)$ for some factor $c(x)$. For instance if Γ is C^1 at x, then $c(x) = 1/2$. So if x is a boundary point, we have

$$c(x)\,u(x) - \int_\Omega G_x(y)\, f(y)\, d\,\Omega = \int_\Gamma u(y) \frac{\partial G_x(y)}{\partial n}\, d\Gamma - \int_\Gamma G_x(y)\, \frac{\partial u(y)}{\partial n}\, d\Gamma,$$

which in case of our Dirichlet condition yields an integral equation for the normal derivative of u on the boundary. If this equation has been solved, then we may apply formula (2) to obtain the solution u in the interior of Ω. Evidently, the same strategy can be used for the Neumann problem (the Dirichlet boundary condition $u(x) = g(x)$ is replaced by the Neumann boundary condition $\frac{\partial u(x)}{\partial n} = g(x)$) and even for mixed versions of the Dirichlet and Neumann conditions involving different parts of the boundary (cf. [5]).

The boundary integral equation (3) contains a domain integral, this is the integral over Ω. If a function u_p can be found such that $\triangle u_p = f$, then before applying numerical methods to solve the integral equation, we first reduces the domain integral to boundary integrals by again using Green's second formula. This is, what P.W. Partridge e.a. call in their book ([5]) the Dual Reciprocity Method (DRM). The DRM avoids computing domain integrals. For our Dirichlet problem this leads to:

$$\int_\Omega G_x(y)\, f(y)\, d\Omega = \int_\Omega G_x(y)\, \triangle u_p(y)\, d\Omega =$$
$$c(x)\, u_p(x) + \int_\Gamma G_x(y)\, \frac{\partial u_p(y)}{\partial n}\, d\Gamma - \int_\Gamma u_p(y)\, \frac{\partial G_x(y)}{\partial n}\, d\Gamma.$$

So, the integral equation that must be solved has the form:

$$\begin{aligned} c(x)\, g(x) \quad &= \quad c(x)\, u_p(x) + \int_\Gamma G_x(y)\, \frac{\partial u_p(y)}{\partial n}\, d\Gamma - \int_\Gamma u_p(y)\, \frac{\partial G_x(y)}{\partial n}\, d\Gamma \\ &\quad + \int_\Gamma g(y) \frac{\partial G_x(y)}{\partial n}\, d\Gamma - \int_\Gamma G_x(y)\, \frac{\partial u(y)}{\partial n}\, d\Gamma. \end{aligned} \tag{3}$$

To solve this boundary integral equation by means of finite boundary elements, one normally needs the values of $u_p(x)$ and the gradient $\nabla u_p(x)$ at a certain set of boundary points. If elementary function evaluations for $u_p(x)$ and $\nabla u_p(x)$ are available, then these values can be computed in a direct way.

In practice the function f will be approximated by a superpostion $f = \sum_k f_k$ of functions f_k such that for each function f_k an analytic expression for a corresponding particular solution can be found straightforwardly. Popular choices are radial basis functions $f_k(x) = \phi(|x - x_k|)$, where $\phi(r)$ is a function defined on $\mathbb{R}^+$ and (x_k) a collection of points, which can be chosen in some way. Since the Laplacian is translation invariant, we only have to look for a particular solution of the equation $\Delta\, u = \phi(|x|)$, which may also be considered as a function of $|x|$ only. Finding a particular solution is then solving an ordinary 1D differential equation. In 1995, M.A. Golberg ([4]) gave a survey of the post-1990 literature on the numerical evaluation of particular solutions in the BEM until that time. With respect to the Poisson equation radial basis functions, in especial thin plate splines or conic splines (cf. [6]), seem to be very useful. The disadvantage is the complexity of the associated approximation problem, since the radial basis functions are in general not compactly supported.

In this chapter we replace the radial basis functions by compactly supported 2D-wavelets, called hexagonal wavelets, which are radial symmetric in some sense. Then we may profit from the fact that in a wavelet expansion of a function f, a relatively small number of the wavelet coefficients are needed to represent the function f. For our applications this means, that by clipping the wavelet coefficients of f with are small compared to a certain treshold value, a vast amount of coefficients in the wavelet expansion of f can be set to zero, which reduces the computational effort for setting up the boundary integral equation.

In this chapter, we are concerned with the problem of finding a particular solution of the inhomogenous Poisson equation when these hexagonal wavelets are used. It turns out that a particular solution can be expressed in terms of third order differences of a function at the vertices of a cube in $\mathbb{R}^3$. Section 2 contains an overwiev of the theory of wavelets in two dimensions, which is needed to understand the hexagonal wavelets presented in Section 3. We also give in Section 3 some examples of the consequences of tresholding wavelet coefficients in the wavelet expansion of some functions. Finally at Section 4, particular solutions will be derived corresponding to the hexagonal wavelets and a numerical test will be carried out.

2 Wavelets preliminaries

One of the applications of wavelet decompositions in numerical analysis is the sparsification of full matrices which may occur in the discretisation of integral equations. It is based on the property that a relative high percentage (depending on the choice of the wavelets) of the wavelet coefficients in the wavelet expansion of sufficiently smooth functions $f \in L^2(\mathbb{R}^2)$ is closed to zero. In order to understand this property of wavelets, which is important for our contribution to the dual reciprocity method, this section will contain an introductory overview of the theory of 2D-wavelets. For (a lot) more information the reader is referred to text books on wavelets. We mention a few of them here: Daubechies ([3]), Chui ([1]), Sidney Burrus e.a ([7]) and Strang ([8]).

For bivariate functions a wavelet expansion is normally based on three so-called mother wavelets ψ^1, ψ^2 and ψ^3, from which by translations and dyadic scaling the wavelets are derived. The corresponding wavelet expansion usually has the form:

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_{l=1}^{3} \sum_{n \in \mathbb{Z}^2} d^l_{n,j} \psi^l_{n,j}(x), \tag{4}$$

where the wavelet coefficients $d^l_{n,j}$ have finite l^2-norm, i.e.,

$$\sum_{l=1}^{3} \sum_{j=-\infty}^{\infty} \sum_{n \in \mathbb{Z}^2} |d^l_{n,j}|^2 < \infty.$$

In this expansion the functions $\psi^l_{n,j}$, called wavelets, are defined as follows:

$$\psi^l_{n,j}(x) = 2^j \psi^l(2^j x - n_1 e_1 - n_2 e_2) \quad (x \in \mathbb{R}^2). \tag{5}$$

Here $n = (n_1, n_2) \in \mathbb{Z}^2$ and e_1 and e_2 are two independent vectors in $\mathbb{R}^2$, which may differ from the unit vectors $(1,0)$ and $(0,1)$. Note that $\|\psi^l_{n,j}\| = \|\psi^l\|$, where $\|\cdot\|$ denotes the L^2-norm in the function space $L^2(\mathbb{R}^2)$.

Apparently, the function f is written as a superposition of translated and scaled versions of the three so-called mother wavelets. It is required that

$$\iint_{\mathbb{R}^2} \psi^l(x)\, dx = 0 \qquad (l = 1, 2, 3). \tag{6}$$

So, the mean value of all the wavelets $\psi^l_{n,j}$ are equal to zero.

The wavelet expansion of f can also be seen as a decomposition of f into functions f_j corresponding to different space scales. To be more precise, we may write

$$f = \sum_{j=-\infty}^{\infty} f_j,$$

where $f_j(x) = \sum_{i=1}^{3} \sum_{n \in \mathbb{Z}^2} d_{n,j}^l \psi_{n,j}^l(x)$ "represents" f in the scale space $W_j \subset \mathbb{R}^2$ spanned by the wavelets $\{\psi_{n,j}^l \,|\, l = 1,2,3;\ n \in \mathbb{Z}^2\}$, where $j \in \mathbb{Z}$ is fixed. The spaces W_j are closed linear subspaces of $\mathrm{L}^2(\mathbb{R}^2)$ and, in fact, dilated versions of the space W_0:

$$f(x) \in W_0 \text{ if and only if } f(2^j\, x) \in W_j.$$

Moreover one has:

(i) $W_j \bigcap W_j = \{0\}$ $(j \neq i)$,
(ii) $L^2(\mathbb{R}^2) = \cdots + W_{-2} + W_{-1} + W_0 + W_1 + W_2 + \cdots$.

Here the $+$-sign refers to direct summation of spaces. In case the basis $\{\psi_{n,j}^l \,|\, l = 1,2,3;\ j \in Z\,, n \in \mathbb{Z}^2\}$ is an orthogonal basis in $L^2(\mathbb{R}^2)$, the wavelets are called orthogonal wavelets. The well-known Daubechies family of wavelets delivers examples of orthogonal wavelets having finite supports. However, orthogonal wavelets having compact supports lack symmetry properties. A way to overcome this problem is to leave the orthonormality property and to consider the more general class of the so-called bi-orthogonal wavelets. In the context of bi-orthogonal wavelets, one is faced with two wavelet bases of $L^2(\mathbb{R}^2)$, the ψ-basis and the $\tilde{\psi}$ basis, which are mutually related in the following way:

$$(\psi_{n,j}^l, \tilde{\psi}_{m,i}^k) = \delta_{l,k}\, \delta_{n,m}\, \delta_{j,i}.$$

For orthonormal wavelet bases, these two bases coincide. It follows straight forward from the definition of bi-orthogonal wavelet bases that the wavelet coefficients $d_{n,j}^l$ in (4) are given by the innerproducts:

$$d_{n,j}^l = (f, \tilde{\psi}_{n,j}^l).$$

For stability purposes it is important that the wavelet bases is a so-called Riesz-basis. This means that positive constants A and B exist such that for the wavelet coefficients in (4) the following inequalities hold:

$$A \sum_{j=-\infty}^{\infty} \sum_{l=1}^{3} \sum_{n \in \mathbb{Z}^2} |d_{n,j}^l|^2 \leq \|f\|^2 \leq B \sum_{j=-\infty}^{\infty} \sum_{l=1}^{3} \sum_{n \in \mathbb{Z}^2} |d_{n,j}^l|^2$$

If the ψ-basis is a Riesz basis then also the $\tilde{\psi}$ basis is a Riesz- basis. The constants A and B are then replaced by $1/B$ and $1/A$ respectively. We now return to our scale spaces W_j.

Note that W_j is a direct sum $W_j = W_j^1 + W_j^2 + W_j^3$ of the spaces W_j^1, W_j^2 and W_j^3 which are dilated versions of W_0^1, W_0^2 and W_0^3 respectively. The space W_0^l is spanned by the wavelets $\{\psi^l(x - n_1\, e_1 - n_2\, e_2) \,|\, n \in \mathbb{Z}^2\}$. If we add (by a direct sum) all the scaled spaces W_j up to the scale $j = k - 1$, then one obtains a sequence of spaces (V_k) $(k \in \mathbb{Z})$, given by

$$V_k = \sum_{j=-\infty}^{k-1} W_j.$$

These spaces satisfy the properties:

(i) $\quad \cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots,$

(ii) $\quad f(x) \in V_k$ if and only if $f(2x) \in V_{k+1}$,

(iii) $\quad \bigcap_{k \in \mathbb{Z}} V_k = \{o\},$

(iv) $\quad \overline{\bigcup_{k \in \mathbb{Z}} V_k} = L^2(\mathbb{R}^2).$

For the most common wavelets the following property holds:

(v) A function ϕ exists such that
$\{\phi(x - n_1 e_1 - n_2 e_2) \,|\, n = (n_1, n_2) \in \mathbb{Z}^2\}$
is a Riesz-basis of V_0.

The sequence of spaces (V_k) is called a multiresolution analysis (MRA) of $L^2(\mathbb{R}^2)$ and ϕ its scaling function. Often, the scaling function is normalized by the condition

$$\iint_{\mathbb{R}^2} \phi(x)\, dx = 1. \tag{7}$$

However, we prefer to have scaling functions, like the B-splines in the approximation theory, being a partition of unity. This means:

$$\sum_{n_1, n_2} \phi(x - n_1 e_1 - n_2 e_2) = 1 \qquad (x \in \mathbb{R}^2). \tag{8}$$

The wavelets, we will apply in this chapter have a nonnegative scaling function, so one has

$$\iint_{\mathbb{R}^2} \phi(x)\, dx > 0. \tag{9}$$

In general, the construction of wavelets initiates by setting up a multiresolution analysis to which an appropriate scaling function ϕ is associated. The dual wavelets stem from the dual MRA, which is described by a sequence of spaces $\tilde{V}_k$ and associated scaling function $\tilde{\phi}$, which has the property:

$$(\phi_{n,0}, \tilde{\phi}_{m,0}) = \delta_{n,m}.$$

We summarize some properties in the following list.

- $W_i \perp \tilde{W}_j \; (i \neq j)$,
- $V_k = V_{k-1} + W^1_{k-1} + W^2_{k-1} + W^3_{k-1}$,
- $\tilde{V}_k = \tilde{V}_{k-1} + \tilde{W}^1_{k-1} + \tilde{W}^2_{k-1} + W^3_{k-1}$,
- W^l_k is spanned by $\{\psi^l_{n,k} | \, n \in \mathbb{Z}^2\}$,
- $\tilde{W}^l_k$ is spanned by $\{\tilde{\psi}^l_{n,k} | \, n \in \mathbb{Z}^2\}$,
- V_k is spanned by $\{\phi_{n,k} | \, n \in \mathbb{Z}^2\}$,
- $\tilde{V}^l_k$ is spanned by $\{\tilde{\phi}_{n,k} | \, n \in \mathbb{Z}^2\}$,

¿From this list a lot of mutual relations between the wavelets and the scaling functions can be obtained. These relations will show how close the wavelets are related to the theory of filter banks.

Since $\phi \in V_0 \subset V_1$, there exists a sequence of numbers (q^0_n) such that

$$\phi(x) = \sum_{n \in \mathbb{Z}^2} q^0_n \, \phi_{n,1}(x). \tag{10}$$

It can be shown that for scaling functions having finite supports, only a finite number of elements in the sequence (q^0_n) differs from zero. In such a situation a sequence is said to be finite. We like to have finitely supported scaling functions and wavelets, so we have to assume that the sequence (q^0_n) is finite.

Applying the 2D Fourier-transform ($f(x) \rightarrow \hat{f}(\omega) = \iint_{\mathbb{R}^2} f(x) \, e^{-i\,\omega \cdot x} \, dx$) to the foregoing relation, one has:

$$\hat{\phi}(\omega) = \tfrac{1}{2} \sum_{n \in \mathbb{Z}^2} q^0_n \, e^{-i\,(n_1\,\omega \cdot e_1 + n_2\,\omega \cdot e_2)/2} \, \hat{\phi}(\tfrac{\omega}{2}). \tag{11}$$

Here the dot $\cdot$ stands for the inner-product in $\mathbb{R}^2$. Similarly, since ψ^l has finite support and $\psi^l \in W^l_0 \subset V_1$, there exist three finite sequences (q^l_n) $(l = 1, 2, 3)$ such that

$$\psi^l(x) = \sum_{n \in \mathbb{Z}^2} q^l_n \, \phi_{n,1}(x). \tag{12}$$

Again using the 2D Fourier-transform this yields

$$\hat{\psi}^l(\omega) = \tfrac{1}{2} \sum_{n \in \mathbb{Z}^2} q^l_n \, e^{-i\,(n_1\,\omega \cdot e_1 + n_2\,\omega \cdot e_2)/2} \, \hat{\phi}^l(\tfrac{\omega}{2}).$$

We introduce now the functions $M_l(\omega)$ as follows:

$$M_l(\omega) = \frac{1}{2} \sum_{n \in \mathbb{Z}^2} q_n^l \, e^{-i\,(n_1\,\omega\cdot e_1 + n_2\,\omega\cdot e_2)} \qquad (l = 0, 1, 2, 3). \tag{13}$$

The functions $M_l(\omega)$ are $2\,\pi$-periodic with respect to the dual pair of vectors $\tilde{e}_1$ and $\tilde{e}_2$ defined by

$$\begin{aligned} e_1 \cdot e_1 = 1, \tilde{e}_1 \cdot e_2 = 0, \\ \tilde{e}_2, e_1 = 0, \tilde{e}_2 \cdot e_2 = 1. \end{aligned} \tag{14}$$

So $M_l(\omega + 2\,\pi\,\tilde{e}_1) = M_l(\omega + 2\,\pi\,\tilde{e}_2) = M_l(\omega)\,(\omega \in \mathbb{R}^2)$. Moreover, one has

$$\hat{\phi}(\omega) = M_0(\frac{\omega}{2})\,\hat{\phi}(\frac{\omega}{2}), \tag{15}$$

$$\hat{\psi}^l(\omega) = M_l(\frac{\omega}{2})\,\hat{\phi}(\frac{\omega}{2}) \qquad (l = 1, 2, 3). \tag{16}$$

Similar relations hold for the dual wavelets and scaling function. To be complete we list them here:

$$\tilde{\phi}(x) = \sum_{n \in \mathbb{Z}^2} \tilde{q}_n^0\,\tilde{\phi}_{1,n}(x), \tag{17}$$

$$\tilde{\psi}^l(x) = \sum_{n \in \mathbb{Z}^2} \tilde{q}_n^l\,\tilde{\phi}_{1,n}(x) \qquad (l = 1, 2, 3), \tag{18}$$

$$\hat{\tilde{\phi}}(\omega) = \tilde{M}_0(\frac{\omega}{2})\,\hat{\tilde{\phi}}(\frac{\omega}{2}), \tag{19}$$

$$\hat{\tilde{\psi}}^l(\omega) = \tilde{M}_l(\frac{\omega}{2})\,\hat{\tilde{\phi}}(\frac{\omega}{2}) \qquad (l = 1, 2, 3). \tag{20}$$

Also the sequences $(\tilde{q}_n^l)$ $(l = 0, 1, 2, 3)$ are assumed to be finite.
It follows form (6) and (9) that

$$\hat{\psi}^l(0) = 0, \qquad (l = 1, 2, 3),$$
$$\hat{\phi}(0) \neq 0.$$

It is also required that

$$\hat{\tilde{\psi}}^l(0) = 0, \qquad (l = 1, 2, 3),$$
$$\hat{\tilde{\phi}}(0) \neq 0.$$

So, one has

$$M_l(0) = \tilde{M}_l(0) = 0 \qquad (l = 1, 2, 3),$$
$$M_0(0) = \tilde{M}_0(0) = 1.$$

The duality relations may also be expressed by means of Fourier-transforms. By setting $\pi_0 = (0,0), \pi_1 = \pi\,\tilde{e}_1, \pi_2 = \pi\,\tilde{e}_2$ (cf. (14)) and $\pi_3 = \pi(\tilde{e}_1 + \tilde{e}_2)$, it can be shown that

$$\sum_{l=0}^{3} M_j(\omega + \pi_l)\,\overline{\tilde{M}_k(\omega + \pi_l)} = \delta_{j,k} \qquad (\omega \in \mathbb{R}^2, j \neq k). \tag{21}$$

The functions $M_l(\omega)$ and $\tilde{M}_l(\omega)$ play an important role in the description of wavelets with help of filter banks. The filter banks are used to decompose a function in wavelets and also to reconstruct a function from its decomposition.

Decomposition and reconstruction

In practice, the wavelet coefficients are seldom computed by taking the inner products with the dual wavelets. If a function f is given, then first f will be approximated by a function $f_0 \in V_0$ say, by means of an approximation scheme; for instance by interpolation. The space V_0 must have the property that it contains a "good" set of candidates to approximate the given function f. In the next section, the space V_0 will consist of functions which are continuous and piecewise linear on a regular mesh with small mesh size. By taking into account the approximation error, we replace f by f_0. So, we have

$$\begin{aligned} f(x) \approx f_0(x) &= \sum_{n\in\mathbb{Z}^2} a_{n,0}\,\phi_{n,0}(x) \\ &= \sum_{j=-\infty}^{-1} \sum_{l=1}^{3} \sum_{n\in\mathbb{Z}^2} d^l_{j,n}\,\psi^l_{n,j}(x) \qquad (x \in \mathbb{R}^2). \end{aligned}$$

Since, $V_0 = V_{-1} + W^1_{-1} + W^2_{-1} + W^3_{-1}$, we may write

$$f_0(x) = \sum_{n\in\mathbb{Z}^2} a_{n,-1}\phi_{n,-1}(x) + \sum_{l=1}^{3} \sum_{n\in\mathbb{Z}^2} d^l_{n,-1}\psi^l_{n,-1}(x). \tag{22}$$

How the new approximation coefficients $(a_{n,-1})$ and the wavelet coefficients $(d^l_{n,-1})$ (also called detail coeficients in this context) at level -1 depend on the approximation coefficients (a^0_n) at level 0, is shown by the formulas:

$$
\begin{aligned}
a_{n,-1} &= \sum_{k\in\mathbb{Z}^2} \tilde{q}^0_{k-2\,n}\, a_{k,0}, \\
d^l_{n,-1} &= \sum_{k\in\mathbb{Z}^2} \tilde{q}^l_{k-2\,n}\, a_{k,0} \qquad\qquad (l = 1, 2, 3).
\end{aligned}
$$

Note the occurence of the coefficients $(\tilde{q}^l_n)$ (cf. 18) in these formulas. We also observe that the coefficients at level -1 are of the form $\sum_{k\in\mathbb{Z}^2} \tilde{q}^l_{k-m}\, a_{k,0}$, where $m = 2\,n$. So, first one has to compute the convolution product of the sequences $(a_{n,0})$ and $(\tilde{q}^l_{-n})$, which can be considered as a linear filter proces (convolution) with frequency response

$$
\sum_{n\in\mathbb{Z}^2} \tilde{q}^l_{-n}\, e^{-i\,(n_1 \omega\cdot e_1 + n_2\, \omega\cdot e_2)} = 2\,\overline{M_l(\omega)}.
$$

Then the "output sequence ", (b_n) say, is downsampled by restricting to the elements $(b_{2\,n})$. The filter proces + downsampling is schematically presented in Figure 1

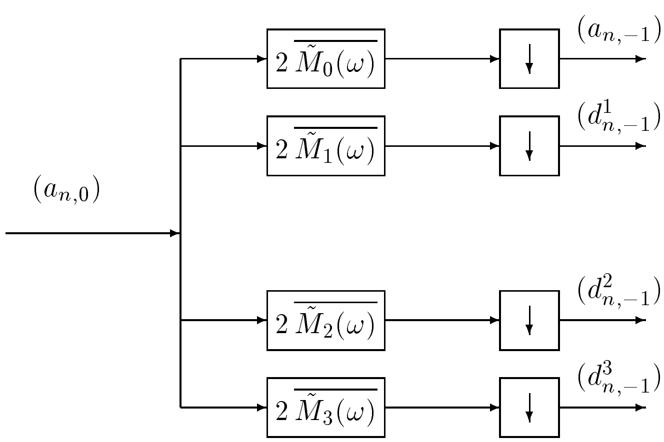

Figure 1: decomposition

On the other hand, if the coefficients $(a_{n,-1})$, $(d^l_{n,-1})$, are given then we may reconstruct the approximation coefficients $(a_{n,0})$ by means of the formulas:

$$
\begin{aligned}
c^0_{n,0} &= \sum_{k \in \mathbb{Z}^2} q^0_{n-2\,k}\, a_{k,-1}, \\
c^1_{n,0} &= \sum_{k \in \mathbb{Z}^2} q^1_{n-2\,k}\, d^1_{k,-1}, \\
c^2_{n,0} &= \sum_{k \in \mathbb{Z}^2} q^2_{n-2\,k}\, d^2_{k,-1}, \\
c^3_{n,0} &= \sum_{k \in \mathbb{Z}^2} q^3_{n-2\,k}\, d^3_{k,-1}, \\
\\
a_{n,0} &= c^0_{n.0} + c^1_{n,0} + c^2_{n,0} + c^3_{n,0}
\end{aligned}
$$

Also, this can be considered as a linear filter proces, which is followed by summation. However, before the filter process is applied, an upsampling is needed by inserting zeros. Upsampling of a sequence (α_n) gives a new sequence (β_n) such that $\beta_{2\,n} = \alpha_n$ for all $n \in \mathbb{Z}^2$. The reconstruction is schematically shown in Figure 2. The frequency responses of the filters are given by $2\,M_0(\omega)$, $2\,M_1(\omega)$, $2\,M_2(\omega)$, $2\,M_3(\omega)$ respectively.

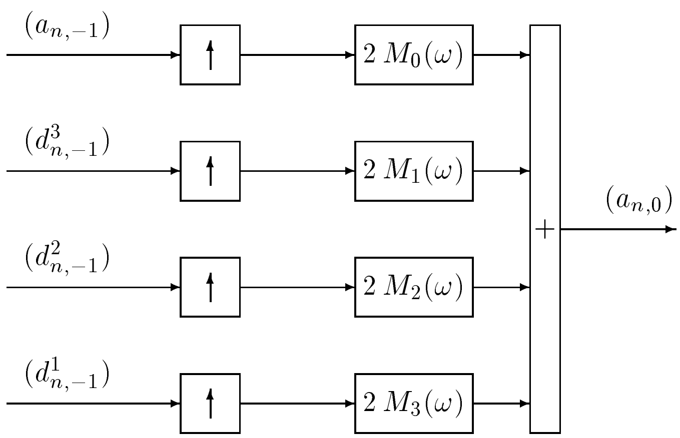

Figure 2: reconstruction

The decomposition filters decompose the function f_0 in four functions; the new approximation function f_{-1} and at level -1 the three detail functions g^1_{-1}, g^2_{-1} and g^3_{-1}. So, $f_0 = f_{-1} + g^1_{-1} + g^2_{-1} + g^3_{-1}$. We may continu, using the same decomposition filters, to decompose the function f_{-1} into the functions f_{-2}, g^1_{-2}, g^2_{-2}, and g^3_{-2}. Then we have: $f_0 = f_{-2} + g^1_{-2} + g^2_{-2} + g^3_{-2} + g^1_{-1} + g^2_{-1} + g^3_{-1}$. Of course, we still may go on and decompose the function f_{-2} etc. Since, g^l_{-1}, ($l = 1,2,3$) contains "information" of f_0 on the finest available scale, a lot of detail coefficients in $(d^l_{n,-1})$ are expected to be close to zero. By choosing an appropriate treshold value these coefficients can be replaced by zero. This also may happen for the functions g^l_{-2}, etc. In image processing, this proces is applied as an effective method to compress images. In numerical analysis, this method can be used to reduce the number of parameters for the representation of a function, which generally will lead to less computations. An indication for choosing an appropriate can be the number of vanishing moments of the dual wavelets $\tilde{\psi}^l$ ($l = 1,2,3$). This is the maximal number m for which

$$\int_{\mathbb{R}^2} p(x)\tilde{\psi}^l(x)\,dx = 0 \quad (l = 1,2,3) \tag{23}$$

for all polynomials $p(x)$ of (total) degree at most m.

Since

$$d^l_{n,-1} = \int_{\mathbb{R}^2} f(x)\,\psi^l_{n,-1}(x)\,dx,$$

it follows from Taylors formula and the definition of m that for functions $f(x)$ wich are $m+1$ times continuous differentiable the following inequality holds

$$|d^l_{n,-1}| \le C \int_{\mathbb{R}^2} |x-a|^{m+1}\,|\psi^l_{n,-1}(x)|\,dx, \tag{24}$$

where a is any point in the support of $\psi^l_{n,-1}$.

The functions $\tilde{\psi}^l$ we shall apply, have finite support, so it follows from (24) that $|d^l_{n,-1}|$ is of the order of $(\mathrm{diam}(\mathrm{support}(\psi^l_{n,-1}))^{m+3}$, where diam is the diameter of the support of ψ^l. So if the support of $\tilde{\psi}^l$ is small, we may expect small values for the numbers $d^l_{n,-1}$.

In the next section, we will focuss on specific wavelets, called hexagonal wavelets. Moreover, an one level decomposition will be applied to the Gaussian function $e^{-x_1^2+x_2^2}$.

3 Hexagonal wavelets

The most frequently used bivariate wavelets have been constructed by means of tensorproduct of univariate wavelets. In this case, the starting point is a biorthogonal system of wavelets generated by a univariate motherwavelet $\psi(x_1)$ (we allow ourselves to use the same function symbols as in the two-dimensional case, only the number of arguments differ) its dual $\tilde{\psi}(x_1)$ and the univariate scaling functions $\phi(x_1)$ and $\tilde{\phi}(x_1)$. Then, from these functions the bivariate wavelets and scaling functions can be obtained as follows:

$$\begin{aligned}
\phi(x) &= \phi(x_1)\,\phi(x_2),\ \tilde{\phi}(x) = \tilde{\phi}(x_1)\,\tilde{\phi}(x_2),\\
\psi^1(x) &= \phi(x_1)\,\psi(x_2),\ \tilde{\psi}^1(x) = \tilde{\phi}(x_1)\,\tilde{\psi}(x_2),\\
\psi^2(x) &= \psi(x_1)\,\phi(x_2),\ \tilde{\psi}^2(x) = \tilde{\psi}(x_1)\,\tilde{\phi}(x_2),\\
\psi^3(x) &= \psi(x_1)\,\psi(x_2),\ \tilde{\psi}^1(x) = \tilde{\psi}(x_1)\,\tilde{\psi}(x_2).
\end{aligned}$$

Here $x = (x_1, x_2)$.

In one dimension, one has two reconstruction filters $\sqrt{2}\,M_0(\omega_1)$ and $\sqrt{2}\,M_1(\omega_1)$ with two decomposition filters $\sqrt{2}\,\tilde{M}_0(\omega_1)$ and $\sqrt{2}\,\tilde{M}_1(\omega_1)$.

These filters generate the two dimensional filters using the formulas:

$$\begin{aligned}
M_0(\omega) &= M_0(\omega_1)\,M_0(\omega_2),\ \tilde{M}_0(\omega) = \tilde{M}_0(\omega_1)\,\tilde{M}_0(\omega_2),\\
M_1(\omega) &= M_0(\omega_1)\,M_1(\omega_2),\ \tilde{M}_1(\omega) = \tilde{M}_0(\omega_1)\,\tilde{M}_1(\omega_2),\\
M_2(\omega) &= M_1(\omega_1)\,M_0(\omega_2),\ \tilde{M}_2(\omega) = \tilde{M}_1(\omega_1)\,\tilde{M}_0(\omega_2),\\
M_3(\omega) &= M_1(\omega_1)\,M_0(\omega_2),\ \tilde{M}_3(\omega) = \tilde{M}_1(\omega_1)\,\tilde{M}_2(\omega_2)
\end{aligned}$$

Here $\omega = (\omega_1, \omega_2)$.

Apparently, in the tensor case a two dimesional filter is a cascade of two one dimensional filters. In one dimension, the filters $M_0(\omega_1)$ and $\tilde{M}_0(\omega_1)$ are examples of low-pass filters (integration filters), whereas $M_1(\omega_1)$ and $\tilde{M}_1(\omega_1)$ are examples of high-pass filters (differentiation filters). So, we may interpret a two dimensional filter in our 2D filter bank, as a cascade of integration or differentiation in the x_1-direction followed by integration or differentiation in the x_2-direction. However, it seems to be unnatural to apply tensor product wavelets in problems where the Laplacian is involved. With respect to the Laplacian, scaling functions and wavelets which are radial symmetric should be preferred. But, it can be shown that in the filter bank setting these scaling functions and wavelets do not exist. A compromise could be the scaling functions and wavelets introduced by Cohen and Schenkler in ([2]). Cohen and Schenkler consider scaling functions and wavelets on an hexagonal grid, $\mathcal{T}$ say (cf. Figure 3), which is generated by the vectors $e_1 = (1, 0)$, $e_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. and $e_3 = -e_1 - e_2 = (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$ (cf Figure 4).

The space V_0 consists of all those continuous functions in $L^2(\mathbb{R}^2)$, from which the restrictions to the triangles of $\mathcal{T}$ coincide with linear functions. The underlying scaling function $\phi(x)$ is the function, which vanishes outside the hexagon $\mathcal{H}$ (cf Figure 4). This function ϕ is also known as the Courant-Hilbert function. Evidently, the translations are also adapted to the hexagonal grid, so we define

$$\phi_{n,j}(x_1, x_2) = 2^j\,\phi(2^j x_1 - n_1\,e_1, 2^j\,x_2 - n_2\,e_2) \qquad (n = (n_1, n_2) \in \mathbb{Z}^2).$$

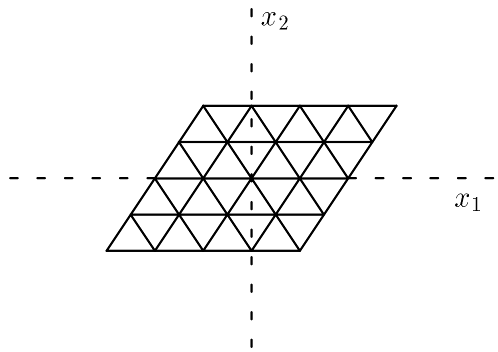

Figure 3: Hexagonaal grid $\mathcal{T}$

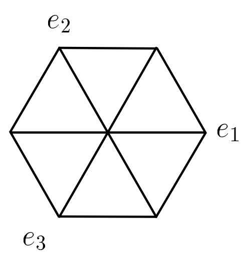

Figure 4: The hexagon $\mathcal{H}$

An elegant representation of the scaling function ϕ can be given in terms of convolution integrals. In order to obtain such a representation, let $\mathcal{U}$ be the parallelogram

$$\mathcal{U} = \{x = \lambda e_1 + \mu e_2 \,|\, 0 < \lambda < 1,\, 0 < \mu < 1\},$$

and let $U(x)$ be the characteristic function of $\mathcal{U}$, i.e.

$$U(x) = \begin{cases} 1 & (x \in \mathcal{U}) \\ 0 & (x \notin \mathcal{U}). \end{cases}$$

Then, $\phi(x)$ may be represented by means of the following convolution type integral:

$$\phi(x) = \int_0^1 U(x - t\, e_3)\, dt. \qquad (25)$$

As a consequence, the Fourier-transform of $\phi(x)$ equals

$$\hat{\phi}(\omega) = \frac{1-e^{-i\,\omega\cdot e_1}}{i\,\omega\cdot e_1}\,\frac{1-e^{-i\,\omega\cdot e_2}}{i\,\omega\cdot e_2}\,\frac{1-e^{-i\,\omega\cdot e_3}}{i\,\omega\cdot e_3} \qquad (\omega \in \mathbb{R}^2).$$

¿From this, it easily follows that

$$\hat{\phi}(\omega) = \tfrac{1}{8}(1+e^{-i\,\omega\cdot e_1/2})\,(1+e^{-i\,\omega\cdot e_2/2})\,(1+e^{-i\,\omega\cdot e_3/2})\,\hat{\phi}(\omega/2).$$

Hence (cf (15))

$$M_0(\omega) = \tfrac{1}{8}(1+e^{-i\,\omega\cdot e_1})\,(1+e^{-i\,\omega\cdot e_2})\,(1+e^{-i\,\omega\cdot e_3}). \tag{26}$$

The scaling function ϕ is invariant with respect to rotation over an angle of $2\pi/3$, i.e.

$$\phi(Rx) = \phi(x) \qquad (x \in \mathbb{R}^2),$$

where R is the rotation operator given by the matrix

$$R = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}.$$

Cohen and Schrenkler (cf [Cohen]) succeeded to compute wavelets ψ^l corresponding to ϕ, which are global rotational invariant with respect to R in the sense that

$$\psi^2(x) = \psi^1(Rx),\ \psi^3(x) = \psi^2(Rx) \qquad (x \in \mathbb{R}^2). \tag{27}$$

They presented different examples for such functions $\psi^1(x)$. In this chapter we will use the function $\psi^1(x)$ which is associated with the following function $M_1(\omega)$.

$$\begin{aligned} M_1(\omega) = \big(&12 + 6\,e^{-i\omega\cdot e_1} - 24\,e^{i\omega\cdot e_1} + 4\,e^{2\,i\omega\cdot e_1} + 2\,e^{3\,i\omega\cdot e_1} + 5\,e^{-i\omega\cdot e_2} + \\ &5\,e^{i\omega\cdot e_2} - 2\,e^{-2\,i\omega\cdot e_2} - 2\,e^{2\,i\omega\cdot e_1} - e^{-3\,i\omega\cdot e_2} - e^{3\,i\omega\cdot e_2} - 24\,e^{i\omega\cdot(e_1+e_2)} + \\ &4\,e^{2\,i\omega\cdot(e_1+e_2)} + 2\,e^{3\,i\omega\cdot(e_1+e_2)} + 6\,e^{-i\omega\cdot(e_1+e_2)} - e^{-i\omega\cdot(e_1+2\,e_2)} - \\ &e^{-i\omega\cdot(e_1+3\,e_2)} - e^{-i\omega\cdot(e_1-e_2)} - e^{-i\omega\cdot(e_1-2\,e_2)} + e^{i\omega\cdot(e_1-e_2)} - e^{i\omega\cdot(e_1-2\,e_2)} + \\ &2\,e^{i\omega\cdot(2\,e_1-e_2)} + e^{i\omega\cdot(e_1+2\,e_2)} - e^{i\omega\cdot(e_1+3\,e_2)} + 2\,e^{i\omega\cdot(2\,e_1+3\,e_2)} + 2\,e^{i\omega\cdot(3\,e_1+2\,e_2)} + \\ &2\,e^{i\omega\cdot(3\,e_1+e_2)} + 4\,e^{i\omega\cdot(2\,e_1+e_2)}\big)/64. \end{aligned} \tag{28}$$

Because of (27), the function $M_2(\omega)$ is obtained from $M_1(\omega)$ by replacing e_1 by e_2, e_2 by $-e_1-e_2$. Subsequently, the function $M_3(\omega)$ is obtained from $M_1(\omega)$ by replacing e_1 by $-e_1-e_2$, and e_2 by e_1. So, the filters at the reconstruction part of the filterbank are known for this moment. For the decomposition part, we need the duals $\tilde{M}_l(\omega)\,(l=0,1,2,3)$. Again, Cohen and Schenkler presented in [Cohen] the following formula:

$$\begin{aligned}32\,\tilde{M}_0(\omega) = 14 + 6\,e^{-i\,\omega\cdot e_1} + 6\,e^{i\,\omega\cdot e_1} - e^{-2\,i\,\omega\cdot e_1} - e^{2\,i\,\omega\cdot e_1} + 6\,e^{-i\,\omega\cdot e_2} +\\ 6\,e^{i\,\omega\cdot e_2} - e^{-2\,i\,\omega\cdot e_2} - e^{2\,i\,\omega\cdot e_2} + 6\,e^{-i\,\omega\cdot(e_1+e_2)} + 6\,e^{i\,\omega\cdot(e_1+e_2)} -\\ e^{-2\,i\,\omega\cdot(e_1+e_2)} - e^{2\,i\,\omega\cdot(e_1+e_2)} - 2\,e^{-i\,\omega\cdot(e_1-e_2)} - 2\,e^{i\,\omega\cdot(e_1-e_2)}\\ - 2\,e^{-i\,\omega\cdot(e_1+2\,e_2)} - 2\,e^{i\,\omega\cdot(e_1+2\,e_2)} - 2\,e^{-i\,\omega\cdot(2\,e_1+e_2)} - 2\,e^{i\,\omega\cdot(2\,e_1+e_2)}\end{aligned} \tag{29}$$

Finally, the decomposition filter $\tilde{M}_1(\omega)$ is given by

$$4\,\tilde{M}_1(\omega) = 1 - e^{i\,\omega\cdot e_1} - e^{i\,\omega\cdot(e_1+e_2)} + e^{i\,\omega\cdot(2\,e_1+e_2)} \tag{30}$$

Evidently, the functions $\tilde{M}_l(\omega)\,(l=2,3)$ can be derived from $\tilde{M}_1(\omega)$ by replacing the vectors e_j in an appropriate way.

Now, our filter bank is completed, since the filter coefficients are known.

Example

As an application we consider the smooth Gaussian function $f(x_1,x_2) = e^{-(x_1^2+x_2^2)}$, which is sampled at the points

$h\,(n_1\,e_1 + n_2\,e_2)$, $(n_1, n_2 = -N, -N+1, \dots, N)$,

with $N = 20$ and $h = 0.05$. In fact the space V_0 now consists of the continuous piecewise linear functions on the hexagonal grid $h\,\mathcal{T}$. However, sampling the function $f(x)$ at the grid $h\,\mathcal{T}$ coincides with sampling the function $f(h\,x)$ at the grid $\mathcal{T}$. So, we may apply our filter bank to the matrix $a_{n,0} = f(h\,(n_1\,e_1 + n_2\,e_2))$. At the grid points of $h\,\mathcal{T}$, where a sampled value is not available, it is assumed that $a_{n,0} = 0$. Therefore, we have extended $a_{n,0}$ to the whole grid by zero padding. This zero padding will introduce boundary errors in the reconstructed values as will be clear by the next tables. The input of the decompostion part of the filter bank is a 41×41 matrix of $a_{n,0}$ values. The output consists of four 20×20 matrices representing the values of $a_{n,-1}$, $d^1_{n,-1}$, $d^2_{n,-1}$, $d^3_{n,-1}$ respectively. In the four output matrices, we replace all the elements for which the absolute value is less then a certain treshold value by zeros. Then the reconstruction part of the filter bank is applied. For an exact reconstruction, the output of the reconstruction part (the reconstruction matrix) should be equal to the input of the decomposition part. Because of tresholding and zeropadding this will not be the case. In the third column of the tables below we have plot the error ε_1, which is the maximum absolute error in the reconstruction matrix, compared to the original matrix. As will be clear from the tables, this error is mainly caused by zeropadding. The error due to zeropadding will occur at the "boundary" of the matrix. Therefore, the fourth column contains the absolute error (ε_2), without taking into account the first three colums and rows and the last three columns and rows of the reconstruction matrix. In ε_2, the error due to zeropadding is eliminated, so this error is mainly caused by tresholding. It can be shown that it is comparable to the treshold value. In the first column the different applied treshold values are listed. The second column gives the ratio of the total number of nonzero elements of the four output matrices (after tresholding) and the size of the input matrix ($a_{n,0}$. This ratio is the compression factor.

treshold	compr. factor	ε_1	ε_2
0.01	0.26	$3.8\ 10^{-1}$	$4.5\ 10^{-3}$
0.001	0.42	$3.8\ 10^{-1}$	$3.2\ 10^{-3}$
0.0001	0.90	$3.8\ 10^{-1}$	$1.8\ 10^{-4}$
0	0.95	$3.8\ 10^{-1}$	0

$f(x_1, x_2) = \exp(-x_1^2 + x_2^2)$, $h = 0.05$, $N = 20$

treshold	compr. factor	ε_1	ε_2
0.01	0.32	$8.7\ 10^{-1}$	$6.9\ 10^{-3}$
0.001	0.77	$8.7\ 10^{-1}$	$2.2\ 10^{-3}$
0.0001	0.91	$8.7\ 10^{-1}$	0
0	0.95	$8.7\ 10^{-1}$	0

$f(x_1, x_2) = \sin(|x_1 - 0.1| + |x_2 + 0.1|)$, $h = 0.05$, $N = 20$

It is clear that small treshold values have negative influence on the compression factors. Since the approximation error by linear interpolation has order h^2 and the error due to tresholding is comparable to the treshold value, to use a treshold value of order h^2 seems to be a good choice. In the next table, we used the function $f(x_1, x_2) = e^{-x_1^2 - x_2^2}$ again, but $h = 0.05$ is replaced by $h = 0.1$ and $N = 20$ is replaced by $N = 10$.

treshold	compr. factor	ε_1	ε_2
0.01	0.27	$3.8\ 10^{-1}$	$1.9\ 10^{-2}$
0.001	0.80	$3.8\ 10^{-1}$	$2.3\ 10^{-3}$
0.0001	0.91	$3.8\ 10^{-1}$	$4.7\ 10^{-5}$
0	091	$3.8\ 10^{-1}$	0

$f(x_1, x_2) = \exp(-x_1^2 + x_2^2)$, $h = 0.1$, $N = 10$

4 The computation of a particular solution

In this section, we will focus on the problem, how to compute a particular solution of the differential equation $\Delta u(x) = f(x)$ $(x \in \mathcal{D})$, for a given function $f(x)$ and a bounded domain $\mathcal{D}$. First, the function f will be approximated by piecewise linear interpolation on a part of the hexagonal grid $h\,\mathcal{T}$, which contains the given domain $\mathcal{D}$. Let $f_0(x)$ be the approximating function, then

$$f(x) \approx f_0(x) = \sum_{n_1,n_2} f(h\,n_1\,e_1 + h\,n_2\,e_2)\,\phi(x/h - n_1\,e_1 - n_2\,e_2),$$

where $\phi(x)$ is our scaling function for the hexagonal wavelets.

Due to linear interpolation, the approximation error on $\mathcal{D}$ is of order h^2, under the assumption that f is a continuous function. In order to apply compression, as demonstrated in the previous section, the decomposition part of the filter bank (cf Figure 1) associated to the hexagonal wavelets must be used. The four output matrices of the decomposition part contain the coefficients for the decompostion of f_0 as given by (22). In this decomposition the function f_0 is written as a linear combination of dilated and translated versions of the function $\phi(x)$, and the three mother wavelets $\psi^1(x)$, $\psi^2(x)$ and $\psi^3(x)$. Tresholding the coefficients in this combination which are small compared to a given treshold value will introduce a second error, which can be controlled by computing the reconstruction error. This error is comparable to the treshold value. Because of (12), to find a particular solution corresponding to the wavelets, it is sufficient to find a particular solution correponding to functions of the type $\phi(\frac{2^{-l}}{h}\,x - n_1\,e_1 - n_2\,e_2)$. If $p(x)$ is a particular solution corresponding to the scaling function $\phi(x)$, then $4^l\,h^2\,p(2^{-l}\,x/h - n_1\,e_1 - n_2\,e_2)$ is a particular solution for $\phi(2^{-l}\,x/h - n_1\,e_1 - n_2\,e_2)$. Therefore we only have to compute $p(x)$.

Evidently, a particular solution $p(x)$ can be expressed by the convolution integral:

$$p(x) = \iint\limits_{\mathbb{R}^2} \phi(\xi)\,G(x-\xi)\,d\,\xi. \qquad (31)$$

where $G(x) = \frac{1}{2\,\pi}\log|x|$ is the fundamental solution correponding to the Laplacian.

Our main task is to evaluate the integral (31).

By substituting expression (25) into (31), we get

$$p(x) = \int_0^1 \iint\limits_{\mathbb{R}^2} U(\xi - t\,e_3)\,G(x-\xi)\,d\xi\,dt = \int_0^1 \iint\limits_{\mathbb{R}^2} U(\xi)\,G(x-\xi-t\,e_3)\,d\xi\,dt$$

$$= \int_0^1 \iint\limits_{\mathcal{U}} G(x-\xi-t\,e_3)\,d\xi\,dt.$$

Now, we replace $\xi = (\xi_1, \xi_2)$ by $\xi' = (\xi_1', \xi_2')$ such that $\xi = \xi_1'\,e_1 + \xi_2'\,e_2$. Hence, $\xi_1 = \xi_1' - \frac{1}{2}\xi_2'$ and $\xi_2 = \frac{1}{2}\sqrt{3}\,\xi_2'$.

By setting $\xi_3' = t$, we arrive at the representation

$$p(x) = \tfrac{1}{2}\sqrt{3}\,\int_0^1\int_0^1\int_0^1 G(x - \xi_1'\,e_1 - \xi_2'\,e_2 - \xi_3'\,e_3)\,d\xi_1'\,d\xi_2'\,d\xi_3'. \tag{32}$$

The next natural step is to represent $x \in \mathbb{R}^2$ in terms of e_1, e_2 and e_3 as follows:

$$x = \mu_1\,e_1 + \mu_2\,e_2 + \mu_3\,e_3,$$
$$\mu_1 + \mu_2 + \mu_3 = 1.$$

The numbers μ_1, μ_2 and μ_3 are called barycentric coordinates of x with respect to e_1, e_2 and e_3.

By substituting $\xi_1' = \mu_1 - u\,\xi_2' = \mu_2 - v$ and $\xi_3' = \mu_3 - w$, the following relation is obtained.

$$p(x) = \frac{\sqrt{3}}{8\,\pi}\int_{\mu_3-1}^{\mu_3}\int_{\mu_2-1}^{\mu_2}\int_{\mu_1-1}^{\mu_1} \log(|u\,e_1 + v\,e_2 + w\,e_3|^2)\,du\,dv\,dw. \tag{33}$$

Since $|u\,e_1 + v\,e_2 + w\,e_3|^2 = \frac{1}{2}((u-v)^2 + (u-w)^2 + (v-w)^2)$, a function $F(u,v,w)$ for which

$$\frac{\partial}{\partial w}\frac{\partial}{\partial v}\frac{\partial}{\partial u}F(u,v,w) = \log((u-v)^2 + (u-w)^2 + (v-w)^2),$$

will deliver our $p(x) = p(\mu_1\,e_1 + \mu_2\,e_2 + \mu_3\,e_3)$ as follows

$$p(x) = \frac{\sqrt{3}}{8\,\pi}\,(-\log 2 + (I - E_{\mu_3})\,(I - E_{\mu_2})\,(I - E_{\mu_1})F(\mu_1, \mu_2, \mu_3)). \tag{34}$$

Here, I is the identity operator and E_a denotes the backward shift operator with respect to the variabele a, for instance $E_{\mu_1}F(\mu_1, \mu_2, \mu_3) = F(\mu_1 - 1, \mu_2, \mu_3)$.

In this way, our function $p(x)$ is expressed as a third order finite difference of a function $F(u,v,w)$ on a cube in $\mathbb{R}^3$. A function $F(u,v,w)$, satisfying the previous partial differential equation can be found by repeated integration. An explicit formula is given by the expression

$$\begin{aligned} F(u,v,w) &= h_1(u,v,w) + \\ h_2(u,v,w)\,(\log((u-v)^2 &+ (u-w)^2 + (v-w)^2) - \tfrac{11}{3}), \end{aligned} \tag{35}$$

where,

$$\begin{aligned} h_1(u,v,w) \quad = \quad & \frac{\sqrt{3}}{6}\,(u-v)^3\ \arctan(\frac{u+v-2\,w}{\sqrt{3}\ (u-v)}) + \\ & \frac{\sqrt{3}}{6}\,(u-w)^3\ \arctan(\frac{u+w-2\,v}{\sqrt{3}\ (u-w)}) + \\ & \frac{\sqrt{3}}{6}\,(v-w)^3\ \arctan(\frac{v+w-2\,u}{\sqrt{3}\ (v-w)}), \end{aligned} \tag{36}$$

and

$$h_2(u,v,w) = \tfrac{1}{12}\,(2\,v-u-w)\,(2\,u-v-w)\,(2\,w-u-v). \tag{37}$$

Our conclusion is that $p(x)$ can be found by doing elementary computations. In BEM also normal derivatives at some boundary points of Γ of are needed. Formulas for $p_{x_1}(x) = \frac{\partial p}{\partial x_1}(x)$ and $p_{x_2}(x) = \frac{\partial p}{\partial x_2}(x)$ are given below. In these formulas, we use again the barycentric coordinates μ_1, μ_2 and μ_3 of x with respect to e_1, e_2 and e_3 and functions $F_k(\mu_1,\mu_2,\mu_3)$, $(k=1,2)$, which play a similar role as the function $F(\mu_1,\mu_2,\mu_3)$ for $p(x)$. These functions are given by the formulas:

$$\begin{aligned} F_1(u,v,w) &= \big(2\,\sqrt{3}\,(u-v)^2\ \arctan(\frac{u+v-2\,w}{\sqrt{3}\,(u-v)}) + \\ & 2\,\sqrt{3}\,(u-w)^2\ \arctan(\frac{u+w-2\,v}{\sqrt{3}\,(u-w)}) + \\ ((u-v)^2 + (u-w)^2 &- 2\,(v-w)^2)\ \log(((u-v)^2 + (u-w)^2 + (v-w)^2))\big)/4. \end{aligned}$$

$$\begin{aligned} F_2(u,v,w) &= -\frac{1}{2}\,(u-v)^2\ \arctan(\frac{u+v-2\,w}{\sqrt{3}\,(u-v)}) + \\ \frac{1}{2}\,(u-w)^2\ \arctan(\frac{u+w-2\,u}{\sqrt{3}\,(u-w)}) &+ (v-w)^2\ \arctan(\frac{v+w-2\,u}{\sqrt{3}\,(v-w)}) + \\ \frac{\sqrt{3}}{4}\,(v-w)\,(2\,u-v-w)\,(3 &- \log((u-v)^2 + (u-w)^2 + (v-w)^2)). \end{aligned}$$

Finally, the partial derivatives of $p(x)$ are given by

$$p_{x_1}(x) = \frac{\sqrt{3}}{8\pi}(I - E_{\mu_1})(I - E_{\mu_2})(I - E_{\mu_3})F_1(\mu_1, \mu_2, \mu_3),$$
$$p_{x_2}(x) = \frac{\sqrt{3}}{8\pi}(I - E_{\mu_1})(I - E_{\mu_2})(I - E_{\mu_3})F_2(\mu_1, \mu_2, \mu_3).$$

Here μ_1, μ_2 and μ_3 are the barycentric coordinates of x with respect to e_1, e_2 and e_3.

Example
As a numerical test we solve the Dirichlet problem for

$$f(x_1, x_2) = \frac{1}{4}(x_1^2 + x_2^2 - 1)\, e^{-x_1^2 - x_2^2}$$
$$g(x_1, x_2) = e^{-1},$$

on the unit circle $x_1^2 + x_2^2 \leq 1$ with boundary $\Gamma: \ x_1^2 + x_2^2 = 1$. The exact solution equals $u(x_1, x_2) = e^{-x_1^2 - x_2^2}$. We solve this problem by using a BEM method having 10 quadratic boundary elements and used the hexagonal wavelets for deriving a particular solution by the method described in this section with meshsize $h = 0.05$, $n = 50$ and treshold value equals 0.001. The compression factor is equal to 0.25 .The difference between the exact solution and the BEM solution at certain points in the circle are tabulated below

x_1	x_2	wavelet	exact
0.0	0.0	0.165947	0.165979
0.25	0.0	0.198489	0.198554
0.75	0.0	0.22557	0.225667
0.90	0.0	0.00000	0.000000

We conclude that the approximation is of order 10^{-4}, which is smaller then h^2. However, in general we may not expect to have an approximation order h^p, with p less then 2.

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