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Abstract

The problem of simultaneous (partial) feedback linearization and input-output linearization for SISO nonlinear control systems is considered. It is shown that the problem of existence of a linear subsystem of a certain dimension may be reduced to a well-known problem from real algebraic geometry.

1 Introduction and problem statement

In this paper we consider a smooth SISO nonlinear control system Σ of the form

$$\Sigma \left\{ \begin{array}{rcll} \dot{x} & = & f(x) + g(x)u & , x \in I\!R^n, \ u \in I\!R \\ y & = & h(x) & , y \in I\!R \end{array} \right. \tag{1}$$

around a point $x_0 \in I\!R^n$. Further, consider a linear SISO system $\bar{\Sigma}$ of the form

$$\bar{\Sigma} \left\{ \begin{array}{rcll} \dot{\xi} & = & \bar{A}\xi + \bar{B}\bar{u} & , \xi \in I\!R^{\bar{n}}, \ \bar{u} \in I\!R \\ \eta & = & \bar{C}\xi & , \eta \in I\!R \end{array} \right. \tag{2}$$

where $\bar{n} \leq n$. We will call $\bar{\Sigma}$ a (linear) *subsystem* of Σ around x_0 if for Σ around x_0 there exist a regular static state feedback $Q_s : \ u = \alpha(x) + \beta(x)v$ and new coordinates $\bar{x}(x) = (\bar{x}_1(x), \bar{x}_2(x))$ such that in the new coordinates $\bar{x}(x)$ the system $\Sigma \circ Q_s$ around x_0 takes the form

$$\Sigma \circ Q_s \left\{ \begin{array}{rcl} \dot{\bar{x}}_1 & = & \bar{A}\bar{x}_1 + \bar{B}v \\ \dot{\bar{x}}_2 & = & \bar{a}(\bar{x}) + \bar{b}(\bar{x})v \\ y & = & \bar{C}\bar{x}_1 \end{array} \right. \tag{3}$$

In this paper we answer the question whether, given $\bar{n} \in \{1, \cdots, n\}$, the system Σ has a controllable linear subsystem of dimension $\bar{n}$ around x_0. Note that if Σ has a controllable linear subsystem around x_0, then around x_0 one may partially feedback linearize Σ by means of regular static state feedback and coordinate transformation, while at the same time achieving a linear input-output behavior. In this respect the problem considered in this paper may be seen as a combined (partial) feedback linearization problem and input-output linearization problem. The problem of feedback linearization was first solved independently in [10],[8]. In [14], the maximal feedback linearizable subsystem of a nonlinear control system was characterized. The problem of input-output linearization was first tackled in [13] (see also [11]). For a further overview of the literature on (partial) feedback linearization and input-output linearization we refer to [12],[15] and the references therein.

The problem considered in this paper has received some attention in the literature. In [3] the question whether a MIMO system has a linear subsystem of dimension n has been addressed. In [9], SISO systems were studied, and sufficient conditions were given for the existence of a linear subsystem of dimension larger than the relative degree. In [19], the authors characterized for MIMO systems the maximal linear subsystem after an input-output linearizing static state feedback has been applied.

The organization of the paper is as follows. In the next section we will introduce some notation, concepts and results that will be used in the rest of the paper. In Section 3 necessary and sufficient conditions for the existence of a controllable linear subsystem of a given dimension will be derived. Starting from these conditions, it will be shown in Section 4 that the problem under consideration may be reduced to a well known problem from real algebraic geometry. In Section 5, we give an example, and in Section 6 some conclusions are drawn.

2 Preliminaries

2.1 Relative degree of one-forms

In this subsection we give a differential-geometric treatment of the relative degree of one-forms. The concept of relative degree of a one-form was introduced in [2] in an algebraic framework. Define the manifold $M_0 := I\!R^n$ with local coordinates x, and the manifolds $M_k := M_{k-1} \times I\!R$ with local coordinates $(x, u, \cdots, u^{(k-1)})$ $(k = 1, \cdots, 2n+1)$. Clearly, M_k is an embedded submanifold of M_ℓ $(k = 0, \cdots, 2n;\ \ell = k+1, \cdots, 2n+1)$, with the natural embedding $i_{k\ell} : M_k \to M_\ell$ defined by

$$i_{k\ell}(x, u, \cdots, u^{(k-1)}) = (x, u, \cdots, u^{(k-1)}, 0, \cdots, 0) \tag{4}$$

Let Ξ_k denote the codistribution span$\{dx\}$ on M_k $(k = 0, \cdots, 2n+1)$. On M_{2n+1}, we define the extended vector field

$$f^e := (f + gu)\frac{\partial}{\partial x} + \sum_{i=0}^{2n} u^{(i+1)} \frac{\partial}{\partial u^{(i)}} \tag{5}$$

For a one-form ω on M_k $(k = 0, \cdots, n+1)$, we define $\omega^{(\ell)}$ on M_{2n+1} by

$$\omega^{(\ell)} := \mathcal{L}^{\ell}_{f^e}((i_{k2n+1})_* \omega) \quad (\omega \in M_k; k = 0, \cdots, n+1; \ell = 0, \cdots, 2n+1-k) \tag{6}$$

Then $\omega^{(\ell)}$ may be interpreted as a one-form on on $M_{k+\ell}$, in the sense that

$$(i_{k+\ell 2n+1})_*(i_{k+\ell 2n+1})^* \omega^{(\ell)} = \omega^{(\ell)} \quad (\omega \in M_k; k = 0, \cdots, n+1; \ell = 0, \cdots, 2n+1-k) \tag{7}$$

Let $\omega \in \Xi_k$ $(k = 0, \cdots, n)$, and assume that there exists an $\ell \in \{1, \cdots, n\}$ such that $\omega^{(\ell)} \notin \Xi_{2n+1}$. Then the smallest such ℓ is called the *relative degree* of ω, to be denoted by r_ω. If for all $\ell \in \{1, \cdots, n\}$ we have that $\omega^{(\ell)} \in \Xi_{2n+1}$, we define $r_\omega := +\infty$. For a function ϕ

satisfying $d\phi \in \Xi_k$, we define its relative degree by $r_\phi := r_{d\phi}$. Define the codistributions $\mathcal{H}_k^\ell$ $(k = 1, \cdots, n;\ \ell = k-1, \cdots, 2n+1-k)$ by

$$\mathcal{H}_k^\ell := \{\omega \in \Xi_\ell \mid r_\omega \geq k\} \tag{8}$$

Using (7), it may then be shown that $\mathcal{H}_k^\ell$ may be identified with $\mathcal{H}_k^{k-1}$, in the sense that

$$(i_{k-1\ell})_*(i_{k-1\ell})^*\mathcal{H}_k^\ell = (i_{k-1\ell})_*\mathcal{H}_k^{k-1} \quad (k = 1, \cdots, n; \ell = k-1, \cdots, 2n+1-k) \tag{9}$$

We further define the codistribution $\mathcal{H}_\infty^n$ on M_n by

$$\mathcal{H}_\infty^n := \{\omega \in \Xi_n \mid r_\omega = +\infty\} \tag{10}$$

Next, define

$$\mathcal{H}_k := (i_{k-12n+1})_*\mathcal{H}_k^{k-1} \quad (k = 1, \cdots, n) \tag{11}$$

$$\mathcal{H}_\infty := (i_{n2n+1})_*\mathcal{H}_\infty^n \tag{12}$$

We then have the following properties (for a proof, see (mutatis mutandis) [2]).

Lemma 2.1 *Let $x_0 \in \mathbb{R}^n$ be given, and assume that the codistributions $\mathcal{H}_k$ $(k \in \{1, \cdots, n, \infty\})$ have constant dimension around $(x_0, 0, \cdots, 0)$. Then around x_0 these codistributions have the following properties.*

(i) $\mathcal{H}_1 \supset \mathcal{H}_2 \supset \cdots \supset \mathcal{H}_n \supset \mathcal{H}_\infty$.

(ii) $\mathcal{H}_\infty$ *is integrable.*

(iii) Σ *is strongly accessible if and only if* $\mathcal{H}_\infty = \{0\}$.

(iv) $\mathcal{H}_k = \{\omega \in \mathcal{H}_{k-1} \mid ((i_{k-22n+1})^*\omega)^{(1)} \in \mathcal{H}_k\}$ $(k = 1, \cdots, n)$.

(v) $\mathcal{H}_\infty = \{\omega \in \mathcal{H}_n \mid ((i_{n-12n+1})^*\omega)^{(1)} \in \mathcal{H}_n\}$.

(vi) Define

$$\sigma := n + 1 - \dim(\mathcal{H}_\infty) \tag{13}$$

Then

$$\dim(\mathcal{H}_k) = n + 1 - k \quad (k = 1, \cdots, \sigma) \tag{14}$$

and

$$\mathcal{H}_k = \mathcal{H}_\infty \quad (k = \sigma, \cdots, n) \tag{15}$$

(vii) Let $\lambda \in \mathcal{H}_{\sigma-1} \backslash \mathcal{H}_\infty$. Then we have for $k \in \{1, \cdots, \sigma - 1\}$:

$$\mathcal{H}_k = \mathcal{H}_\infty \oplus \operatorname{span}\{((i_{n-22n+1})^*\lambda)^{(\ell)} \mid \ell = 0, \cdots, \sigma - 1 - k\} \tag{16}$$

■

2.2 Parametrized post compensated system

In the sequel, the notion of a parametrized post compensated system will be of key importance. In this subsection we introduce this notion, and give some properties. Consider a smooth SISO system Σ of the form (1), and let $d \in \mathbb{N}$ be given. Let $s_1, \cdots, s_d$ be parameters that take their values in $\mathbb{R}$. We then define a *parametrized post compensated system* $\Sigma^p\langle s_1, \cdots, s_d\rangle$ by

$$\Sigma^p\langle s_1,\cdots,s_d\rangle \begin{cases} \dot{x} &= f(x)+g(x)u \\ \dot{z}_1 &= z_2 \\ &\vdots \\ \dot{z}_{d-1} &= z_d \\ \dot{z}_d &= h(x) - \sum\limits_{k=1}^{d} s_k z_k \end{cases} \tag{17}$$

Similarly to what has been done in the previous subsection, one may define a sequence of parametrized codistributions $\mathcal{H}_k\langle s_1, \cdots, s_d\rangle$ for $\Sigma^p\langle s_1, \cdots, s_d\rangle$. Define $M := M_{2n+1}$, where M_{2n+1} has been defined in the previous subsection, and define $M^p := \mathbb{R}^n \times \mathbb{R}^d \times \mathbb{R}^{2(n+d)+1}$ with local coordinates $(x, z, u, \cdots, u^{(2(n+d))})$. Define the embedding $i : M \to M^p$ by

$$i(x,u,\cdots,u^{(2n)}) := (x,0,u,\cdots,u^{(2n)},0,\cdots,0)$$

Further, let Ξ, Ξ^p denote the codistribution span$\{dx\}$ on M and M^p respectively. For $\Sigma^p\langle s_1, \cdots, s_d\rangle$, we define the codistributions

$$\mathcal{H}_k^e := i_*\mathcal{H}_k \quad (k=1,\cdots,n) \tag{18}$$

$$\mathcal{H}_\infty^e := i_*\mathcal{H}_\infty \tag{19}$$

It then follows from the form of $\Sigma^p\langle s_1, \cdots, s_d\rangle$ that

$$\forall_{s_1,\cdots,s_d\in\mathbb{R}}\ \forall_{k\in\{1,\cdots,n\}}\ \mathcal{H}_k^e \subset \mathcal{H}_k^p\langle s_1,\cdots,s_d\rangle \tag{20}$$

$$\forall_{s_1,\cdots,s_d\in\mathbb{R}}\ \forall_{k\in\{n+1,\cdots,n+d,\infty\}}\ \mathcal{H}_\infty^e \subset \mathcal{H}_k^p\langle s_1,\cdots,s_d\rangle \tag{21}$$

$$\forall_{s_1,\cdots,s_d\in\mathbb{R}}\ \forall_{k\in\{1,\cdots,n\}}\ \mathcal{H}_k^p\langle s_1,\cdots,s_d\rangle \cap \Xi^p = \mathcal{H}_k^e \tag{22}$$

$$\forall_{s_1,\cdots,s_d\in\mathbb{R}}\ \forall_{k\in\{n+1,\cdots,n+d,\infty\}}\ \mathcal{H}_k^p\langle s_1,\cdots,s_d\rangle \cap \Xi^p = \mathcal{H}_\infty^e \tag{23}$$

We now show that the codistributions $\mathcal{H}_k^p\langle s_1, \cdots, s_d\rangle$ $(k = 1, \cdots, \sigma)$ may be parametrized in a polynomial way. Let $\mathcal{S}$ denote the ring of smooth functions of $(x, u, \cdots, u^{(2n)})$, and define the polynomial ring $\mathcal{R} := \mathcal{S}[s_1, \cdots, s_d]$.

Lemma 2.2 *Consider the parametrized post compensated system $\Sigma^p\langle s_1, \cdots, s_d\rangle$ and the sequence of parametrized codistributions $\mathcal{H}_k^p\langle s_1, \cdots s_d\rangle$ $(k = 1, \cdots, \sigma)$, where σ is defined in (13). Let $x_0 \in \mathbb{R}^n$ be given, and assume that the codistributions $\mathcal{H}_k$ $(k = 1, \cdots, n)$ have constant dimension around $(x_0, 0, \cdots, 0)$. Let $\lambda \in \mathcal{H}_n \backslash \mathcal{H}_\infty$ satisfy*

$$(i_{n-12n+1})_*(i_{n-12n+1})^*\lambda = \lambda \tag{24}$$

Define $r := r_h$. Then around $(x_0, 0, \cdots, 0)$ we have that

$$\dim(\mathcal{H}_k^p\langle s_1,\cdots,s_d\rangle) = \dim(\mathcal{H}_k^e) + d \quad (k=1,\cdots,\sigma) \tag{25}$$

and there exist $\phi_{k\ell} \in \mathcal{R}$ $(k = 1,\cdots,d;\ \ell = 0,\cdots,\sigma - r - d - 2 + k)$ *such that*

$$\mathcal{H}_k^p\langle s_1,\cdots,s_d\rangle = \mathcal{H}_\infty^e \oplus \operatorname{span}\{i_*\omega_k\langle s_1,\cdots,s_d\rangle - dz_k \mid k = 1,\cdots,d\} \qquad (k = 1,\cdots,\sigma) \tag{26}$$

where

$$\omega_k := \sum_{\ell=0}^{\sigma-r-d-2+k} \phi_{k\ell}\lambda^{(\ell)} \tag{27}$$

Proof Equality (25) follows straightforwardly from Lemma 2.1 and (20),$\cdots$,(23). It then follows from (21),(23),(25) that there exist parametrized one-forms $\tilde{\omega}_k\langle s_1,\cdots,s_d\rangle \in \Xi^p$ $(k = 1,\cdots,d)$ such that

$$\mathcal{H}_\sigma^p\langle s_1,\cdots,s_d\rangle = \mathcal{H}_\infty^e \oplus \operatorname{span}\{\tilde{\omega}_k\langle s_1,\cdots,s_d\rangle - dz_k \mid k = 1,\cdots,d\} \tag{28}$$

From Lemma 2.1.*(i)* and (20),(22),(28) it then follows that

$$\mathcal{H}_\ell^p\langle s_1,\cdots,s_d\rangle = \mathcal{H}_\ell^e \oplus \operatorname{span}\{\tilde{\omega}_k\langle s_1,\cdots,s_d\rangle - dz_k \mid k = 1,\cdots,d\} \qquad (\ell = 1,\cdots,\sigma) \tag{29}$$

What remains to be shown is that $\tilde{\omega}_k = i_*\omega_k$ $(k = 1,\cdots,d)$, where the ω_k are of the form (27). We give the proof for $d = 2$. The proof for $d > 2$ is analogous. Since $r_h = r$, there exist $\alpha_0,\cdots,\alpha_{\sigma-1-r} \in \mathcal{S}$ such that $\alpha_{\sigma-1-r} \neq 0$, and

$$dh = \sum_{\ell=0}^{\sigma-1-r} \alpha_\ell\lambda^{(\ell)} \tag{30}$$

From Lemma 2.1.*(iv)* and (29) it follows that

$$\dot{\tilde{\omega}}_1 - d\dot{z}_1 = \dot{\tilde{\omega}}_1 - \tilde{\omega}_2 + (\tilde{\omega}_2 - dz_2) \in \mathcal{H}_{\sigma-1}^p\langle s_1,s_2\rangle \tag{31}$$

and

$$\dot{\tilde{\omega}}_2 - d\dot{z}_2 = \dot{\tilde{\omega}}_2 + s_1\tilde{\omega}_1 + s_2\tilde{\omega}_2 - dh - s_1(\tilde{\omega}_1 - dz_1) - s_2(\tilde{\omega}_2 - dz_2) \in \mathcal{H}_{\sigma-1}^p\langle s_1,s_2\rangle \tag{32}$$

Let $\mathcal{S}^p$ denote the ring of smooth functions of $(x,z,u,\cdots,u^{(2(n+d))})$. With Lemma 2.1.*(vii)* it follows from (31),(32) that there exist parametrized functions $\beta_1\langle s_1,s_2\rangle,\beta_2\langle s_1,s_2\rangle$ satisfying $\beta_1\langle s_1,s_2\rangle,\beta_2\langle s_1,s_2\rangle \in \mathcal{S}^p$, $(\forall s_1,s_2 \in I\!\!R)$ and parametrized one-forms $\pi_1\langle s_1,s_2\rangle,\pi_2\langle s_1,s_2\rangle$ satisfying $\pi_1\langle s_1,s_2\rangle,\pi_2\langle s_1,s_2\rangle \in \mathcal{H}_\infty^e$, $(\forall s_1,s_2 \in I\!\!R)$ such that

$$\dot{\tilde{\omega}}_1 = \tilde{\omega}_2 + \beta_1(i_*\lambda) + \pi_1 \tag{33}$$

$$\dot{\tilde{\omega}}_2 = dh - s_1\tilde{\omega}_1 - s_2\tilde{\omega}_2 + \beta_2(i_*\lambda) + \pi_2 \tag{34}$$

From (33),(34) it follows in particular that $r_{\tilde{\omega}_1} = r + 2$, $r_{\tilde{\omega}_2} = r + 1$, and hence there exist parametrized functions $\tilde{\phi}_{k\ell}\langle s_1,s_2\rangle$ $(k = 1,2;\ \ell = 0,\cdots,\sigma - 4 - r + k)$ and parametrized one-forms $\eta_1\langle s_1,s_2\rangle,\eta_2\langle s_1,s_2\rangle$ such that

$$\forall_{s_1,s_2\in I\!\!R}\ \eta_1\langle s_1,s_2\rangle,\eta_2\langle s_1,s_2\rangle \in \mathcal{H}_\infty^e \tag{35}$$

$$\forall_{s_1,s_2\in \mathbb{R}}\ \forall_{k\in\{1,2\}}\ \forall_{\ell\in\{0,\cdots,\sigma-4-r+k\}}\ \tilde{\phi}_{k\ell}\in\mathcal{S}^p \tag{36}$$

$$\tilde{\omega}_k = \sum_{\ell=0}^{\sigma-4-r+k} \tilde{\phi}_{k\ell}(i_*\lambda)^{(\ell)} + \eta_k \ \ (k=1,2) \tag{37}$$

Comparing (30),(33),(34),(37) we then obtain:

$$\dot{\tilde{\phi}}_{10} - \tilde{\phi}_{20} = \beta_1 \tag{38}$$

$$\dot{\tilde{\phi}}_{1\ell} + \tilde{\phi}_{1\ell-1} - \tilde{\phi}_{2\ell} = 0 \ \ (\ell = 1,\cdots,\sigma-3-r) \tag{39}$$

$$\tilde{\phi}_{1\sigma-3-r} - \tilde{\phi}_{2\sigma-2-r} = 0 \tag{40}$$

$$\dot{\tilde{\phi}}_{20} - s_1\tilde{\phi}_{10} - s_2\tilde{\phi}_{20} = \alpha_0 + \beta_2 \tag{41}$$

$$\dot{\tilde{\phi}}_{2\ell} + \tilde{\phi}_{2\ell-1} - s_1\tilde{\phi}_{1\ell} - s_2\tilde{\phi}_{2\ell} = \alpha_\ell \ \ (\ell = 1,\cdots,\sigma-3-r) \tag{42}$$

$$\dot{\tilde{\phi}}_{2\sigma-2-r} + \tilde{\phi}_{2\sigma-3-r} - s_2\tilde{\phi}_{2\sigma-2-r} = \alpha_{\sigma-2-r} \tag{43}$$

$$\tilde{\phi}_{2\sigma-2-r} = \alpha_{\sigma-1-r} \tag{44}$$

From (40),(44) it follows that

$$\tilde{\phi}_{1\sigma-3-r} = \tilde{\phi}_{2\sigma-2-r} = \alpha_{\sigma-1-r} \in \mathcal{S} \subset \mathcal{R} \tag{45}$$

Equalities (43),(45) then give

$$\tilde{\phi}_{2\sigma-3-r} = \alpha_{\sigma-2-r} - \dot{\tilde{\phi}}_{2\sigma-2-r} + s_2\tilde{\phi}_{2\sigma-2-r} \in \mathcal{R} \tag{46}$$

Using an induction argument, it then follows from (39),(42),(45),(46) that

$$\tilde{\phi}_{k\ell} \in \mathcal{R} \ \ (k=1,2; \ell = 1,\cdots,\sigma-4-r+k) \tag{47}$$

It further follows from (38),(41) that $\tilde{\phi}_{10}$, $\tilde{\phi}_{20}$ are arbitrary. Together with (47), this establishes our claim. ■

3 Necessary and sufficient conditions

In this section we derive necessary and sufficient conditions for the existence of a linear subsystem of dimension $\bar{n} \in \{1,\cdots,n\}$ for a *strongly accessible* SISO system Σ. We consider a smooth SISO system Σ of the form (1) around a point $x_0 \in \mathbb{R}^n$. We assume throughout that the relative degree $r := r_h$ of h is well-defined around x_0, and that the codistributions $\mathcal{H}_k$ ($k \in \{1,\cdots,n,\infty\}$) have constant dimension around x_0. We start with some (rather trivial) observations.

Lemma 3.1 *Consider a SISO system Σ of the form (1) around x_0. Let $\bar{n} \in \{1,\cdots,n\}$ be given. Then Σ has a linear subsystem of dimension $\bar{n}$ around x_0* only if *$\bar{n} \geq r$.*

Proof Follows immediately from (3) and the fact that the relative degree of h is invariant under regular static state feedback and coordinate transformations. ■

Lemma 3.2 *Consider a SISO system Σ of the form (1) around x_0. Then Σ has a linear controllable subsystem of dimension r around x_0.*

Proof As is well known (see e.g. [12],[15]), the differentials $dy^{(k)}$ $(k = 0, \cdots, r-1)$ are linearly independent around x_0, and $y^{(r)} = a(x) + b(x)u$, where $b(x) \neq 0$ around x_0. The result then follows by defining $\bar{x}_{1k} = y^{(k-1)}$ $(k = 1, \cdots, r)$ and $v := a(x) + b(x)u$. ■

We next state and prove our main results.

Proposition 3.3 *Consider a SISO system Σ of the form (1) around x_0. Let $\bar{n} \in \{r+1, \cdots, n\}$ be given, and define $d := \bar{n} - r$. Then Σ has a controllable linear subsystem of dimension $\bar{n}$ around x_0 if and only if around x_0 there exist a function $\phi : \mathbb{R}^n \to \mathbb{R}$ and $a_1, \cdots, a_d \in \mathbb{R}$ such that*

$$r_\phi = \bar{n} \tag{48}$$

and

$$h = \sum_{k=1}^{d} a_k \mathcal{L}_f^{k-1}\phi + \mathcal{L}_f^d \phi \tag{49}$$

Proof *(necessity)* Assume that Σ has a controllable linear subsystem $\bar{\Sigma}$ of dimension $\bar{n}$. Since $\bar{\Sigma}$ is controllable, one may assume without loss of generality that the matrices $\bar{A}, \bar{B}$ in (2) are in Brunovsky canonical form. Let $\bar{c}_i$ $(i = 1, \cdots, n)$ denote the entries of $\bar{C}$ in (2). Since the relative degree of h is invariant under coordinate transformations and regular static state feedback, we have that $\bar{c}_{d+1} \neq 0$, and $\bar{c}_{d+2} = \cdots = \bar{c}_n = 0$. Define

$$a_k := \frac{\bar{c}_k}{\bar{c}_{d+1}} \quad (k = 1, \cdots, d) \tag{50}$$

and

$$\phi := \bar{c}_{d+1}\bar{x}_{11} \tag{51}$$

We then have

$$\begin{aligned} h = & \sum_{k=1}^{d+1} \bar{c}_k \bar{x}_{1k} = \sum_{k=1}^{d+1} \bar{c}_k \mathcal{L}_f^{k-1}\bar{x}_{11} = \sum_{k=1}^{d} \tfrac{\bar{c}_k}{\bar{c}_{d+1}} \mathcal{L}_f^{k-1}\phi + \mathcal{L}_f^d\phi = \\ & \sum_{k=1}^{d} a_k \mathcal{L}_f^{k-1}\phi + \mathcal{L}_f^d\phi \end{aligned} \tag{52}$$

which establishes (49). Further, it follows from the fact that $\bar{A}, \bar{B}$ in (2) are in Brunovsky canonical form, that

$$r_{\bar{x}_{1k}} = \bar{n} - k + 1 \quad (k = 1, \cdots, \bar{n}) \tag{53}$$

which establishes (48).
(sufficiency) Assume that there exist a function $\phi : I\!\!R^n \to I\!\!R$ and $a_1, \cdots, a_d \in I\!\!R$ satisfying (48),(49). Since the relative degree of ϕ is finite, we have that the differentials $d\phi, \cdots, d\mathcal{L}_f^{\bar{n}-1}\phi$ are independent. Further, we have

$$\mathcal{L}_f^{\bar{n}}\phi = a(x) + b(x)u \tag{54}$$

where $b(x) \neq 0$. Defining $\bar{x}_{1k} := \mathcal{L}_f^{k-1}\phi$ $(k = 1, \cdots, \bar{n})$ and $v := a(x) + b(x)u$, we then obtain that Σ has a linear controllable subsystem of dimension $\bar{n}$. ■

Remark 3.4 The constants $a_1, \cdots, a_d \in I\!\!R$ appearing in the formulation of Proposition 3.3 may be given an interpretation in terms of the zeros of the linear subsystem in the following way. Note that the transfer function of the linear subsystem constructed in the sufficiency-part of the proof of Proposition 3.3 is given by $p(s)/s^{\bar{n}}$, where

$$p(s) = s^d + \sum_{k=1}^{d} a_k s^{k-1} \tag{55}$$

Conversely, using the same kind of arguments as in the proof of the necessity-part of Proposition 3.3, it may be shown that the existence of a controllable linear subsystem of dimension $\bar{n}$, where the numerator of the transfer function is given by (55), implies the existence of a function $\phi : I\!\!R^n \to I\!\!R$ satisfying (48),(49).

From Proposition 3.3 we obtain the following upper bound for the maximal dimension of a controllable linear subsystem of Σ.

Corollary 3.5 *Consider for Σ around x_0 the sequence of codistributions $\mathcal{H}_k$, and let $\mathcal{H}_k^*$ denote the maximal integrable codistribution contained in $\mathcal{H}_k$ $(k = 1, \cdots, n)$. Assume that $\mathcal{H}_k^*$ has constant dimension around x_0 $(k = 1, \cdots, n)$, and define*

$$\bar{k} := \max\{k \in \{1, \cdots, n\} \mid \mathcal{H}_k^* \neq \{0\}\} \tag{56}$$

Assume that Σ has a controllable linear subsystem of dimension $\bar{n}$ around x_0. Then

$$\bar{n} \leq \bar{k} \tag{57}$$

Proof Assume that Σ has a controllable linear subsystem of dimension $\bar{n}$. It then follows from Proposition 3.3 that there exists a non-zero exact one-form $\omega \in \mathcal{H}_{\bar{n}}$. This implies that $\mathcal{H}_{\bar{n}}^* \neq \{0\}$, which establishes (57) ■

Remark 3.6 In fact, it may be shown that $\bar{k}$ defined in (56) is the dimension of the maximal linearizable subsystem of Σ around x_0. In this respect, Corollary 3.5 is a rephrasement of the main result of [14].

Now consider the following set of parametrized PDE's:

$$\mathcal{L}_f^d \phi = h - \sum_{k=1}^{d} s_k \mathcal{L}_f^{k-1} \phi \tag{58}$$

$$\mathcal{L}_g \mathcal{L}_f^k \phi = 0 \quad (k = 1, \cdots, d) \tag{59}$$

Proposition 3.7 *Consider a SISO system Σ of the form (1) around x_0. Let $\bar{n} \in \{r+1, \cdots, n\}$ be given, and define $d := \bar{n} - r$. Then Σ has a controllable linear subsystem of dimension $\bar{n}$ around x_0 if and only if there exist $a_1, \cdots, a_d \in I\!R$ such that the set of PDE's (58),(59), with $s_k = a_k$ $(k = 1, \cdots, d)$, has a solution around x_0.*

Proof *(necessity)* Assume that Σ has a controllable linear subsystem of dimension $\bar{n}$. Then clearly there exists a ϕ such that (58), with $s_k = a_k$, holds. Further, it follows from (48) that

$$\mathcal{L}_g \mathcal{L}_f^k \phi = 0 \quad (k = 0, \cdots, \bar{n} - 1) \tag{60}$$

which establishes (59).
(sufficiency) Assume that there exist $a_1, \cdots, a_d \in I\!R$ such that the set of PDE's (58),(59), with $s_k = a_k$ has a solution. Then clearly (49) holds. To establish (48), we first show by induction that

$$\mathcal{L}_g \mathcal{L}_f^{d+\ell} \phi = \mathcal{L}_g \mathcal{L}_f^\ell h = 0 \quad (\ell = 0, \cdots, r-2) \tag{61}$$

For $\ell = 0$ we have:

$$\mathcal{L}_g \mathcal{L}_f^d \phi \overset{(58)}{=} \mathcal{L}_g (h - \sum_{k=1}^{d} a_k \mathcal{L}_f^{k-1} \phi) \overset{(59)}{=} \mathcal{L}_g h = 0 \tag{62}$$

and hence (61) holds for $\ell = 0$. Next, assume that (61) holds for $\ell = 0, \cdots, \nu - 1$, where $\nu \in \{1, \cdots, r-2\}$. then

$$\mathcal{L}_g \mathcal{L}_f^{d+\nu} \phi \overset{(58)}{=} \mathcal{L}_g \mathcal{L}_f^\nu (h - \sum_{k=1}^{d} a_k \mathcal{L}_f^{k-1} \phi) \overset{(\mathrm{IH})}{=} \mathcal{L}_g \mathcal{L}_f^\nu h = 0 \tag{63}$$

which establishes (61). Further, we have by definition of r,

$$0 \neq \mathcal{L}_g \mathcal{L}_f^{r-1} h = \mathcal{L}_g \mathcal{L}_f^{r-1} (\mathcal{L}_f^d \phi + \sum_{k=1}^{d} a_k \mathcal{L}_f^{k-1} \phi) \overset{(61)}{=} \mathcal{L}_g \mathcal{L}_f^{\bar{n}-1} \phi \tag{64}$$

From (61),(64) it then follows that (48) holds. ■

From Propositions 3.3 and 3.7 it follows that the question whether Σ has a controllable linear subsystem of dimension $\bar{n} \in \{r+1, \cdots, \bar{n}\}$ is equivalent to the question whether there exist parameter values such that the parametrized set of PDE's (58),(59) has a solution. The following theorem gives the integrability conditions for these PDE's in terms of the parametrized post compensated system $\Sigma^p\langle s_1, \cdots, s_d\rangle$ for a *strongly accessible* system Σ.

Theorem 3.8 *Consider a strongly accessible SISO system of the form (1) around x_0. Let $\bar{n} \in \{r+1, \cdots, n\}$ be given, and define $d := \bar{n} - r$. Consider the parametrized post compensated system $\Sigma^p\langle s_1, \cdots, s_d\rangle$ and the sequence of parametrized codistributions $\mathcal{H}_k^p\langle s_1, \cdots, s_d\rangle$. Then Σ has a controllable linear subsystem of dimension $\bar{n}$ around x_0 if and only if there exist $a_1, \cdots, a_d \in I\!R$ such that around x_0 we have*

$$\mathcal{H}_\infty^p\langle a_1, \cdots, a_d\rangle = \mathcal{H}_{n+1}^p\langle a_1, \cdots, a_d\rangle \tag{65}$$

Proof *(necessity)* Assume that Σ has a controllable linear subsystem of dimension $\bar{n}$. By Proposition 3.3, there exist $a_1, \cdots, a_d \in I\!R$ and a function $\phi : I\!R^n \to I\!R$ such that (48),(49) hold. For the post-compensated system $\Sigma^p\langle a_1, \cdots, a_d\rangle$, we consider new coordinates (x, ξ), where

$$\xi_k := z_k - \mathcal{L}_f^{k-1}\phi \quad (k = 1, \cdots, d) \tag{66}$$

We then have

$$\dot{\xi}_k = z_{k+1} - \mathcal{L}_f^k\phi = \xi_{k+1} \quad (k = 1, \cdots, d-1) \tag{67}$$

and

$$\dot{\xi}_d = (h - \sum_{k=1}^d a_k z_k) - (h - \sum_{k=1}^d a_k \mathcal{L}_f^{k-1}\phi) = -\sum_{k=1}^d a_k \xi_k \tag{68}$$

From (67),(68) it follows that

$$r_{\xi_k} = +\infty \quad (k = 1, \cdots, d) \tag{69}$$

From Lemma 2.1.*(i)* and (21),(23) it then follows that

$$\begin{aligned}\mathcal{H}_\infty^p\langle a_1, \cdots, a_d\rangle &= \mathcal{H}_\infty^e \oplus \text{span}\{d\xi_1, \cdots, d\xi_d\} = \\ &\mathcal{H}_{n+1}^e \oplus \text{span}\{d\xi_1, \cdots, d\xi_d\} = \mathcal{H}_{n+1}^p\langle a_1, \cdots, a_d\rangle\end{aligned} \tag{70}$$

which establishes (65).
(necessity) Assume that there exist $a_1, \cdots, a_d \in I\!R$ such that (65) holds. It then follows from Lemma 2.2 that there exist one-forms $\omega_1, \cdots, \omega_d \in \text{span}\{dx\}$ such that

$$\mathcal{H}_\infty^p\langle a_1, \cdots, a_d\rangle = \text{span}\{\omega_1 - dz_1, \cdots, \omega_d - dz_d\} \tag{71}$$

and

$$\text{d}\omega_i \in \text{span}\{\pi \wedge \rho \mid \pi, \rho \in \text{span}\{dx, du, \cdots, du^{(2n)}\}\} \quad (i = 1, \cdots, d) \tag{72}$$

From (71), Lemma 2.1.*(v)* and the form of $\Sigma^p\langle a_1, \cdots, a_d\rangle$ it follows that

$$\dot{\omega}_i = \omega_{i+1} \quad (i = 1, \cdots, d-1) \tag{73}$$

and

$$dh = \dot{\omega}_d + \sum_{k=1}^d a_k \omega_k \tag{74}$$

Combining (73) and (74), we obtain

$$dh = \omega_1^{(d)} + \sum_{k=1}^{d} a_k \omega_1^{(k-1)} \tag{75}$$

Analogously to what has been done in the proof of Proposition 3.7, it may be shown that

$$r_{\omega_1} = \bar{n} \tag{76}$$

We next show that ω_1 is exact. From Lemma 2.1.*(ii)* we know that $\mathcal{H}_\infty^p\langle a_1, \cdots, a_d\rangle$ is integrable. By the Frobenius Theorem, this implies in particular that

$$\begin{aligned} 0 = \;& \mathrm{d}(\omega_1 - dz_1) \wedge (\omega_1 - dz_1) \wedge \cdots \wedge (\omega_d - dz_d) = \\ & \mathrm{d}\omega_1 \wedge (\omega_1 - dz_1) \wedge \cdots \wedge (\omega_d - dz_d) \end{aligned} \tag{77}$$

From (72),(77) it then follows that $\mathrm{d}\omega_1 = 0$, and hence, by Poincaré's Lemma, there locally exists a function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that $\omega_1 = d\phi$. It then follows from Proposition 3.3 and (74),(76) that Σ has a controllable linear subsystem of dimension $\bar{n}$. ∎

Remark 3.9 Note that in the necessity-part of the proof of Theorem 3.8, we did not use the assumption that Σ is strongly accessible. Thus, the existence of $a_1, \cdots, a_d \in \mathbb{R}$ such that (65) is satisfied is also a necessary condition for the existence of a linear subsystem of dimension $\bar{n}$ when Σ is not strongly accessible. However, it is not a sufficient condition. In fact, it may be shown that (65) is equivalent to the existence of functions $\phi, \psi : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$d\psi \in \mathcal{H}_\infty \tag{78}$$

$$r_\phi = \bar{n} \tag{79}$$

$$h = \psi + \sum_{k=1}^{d} a_k \mathcal{L}_f^{k-1}\phi + \mathcal{L}_f^d \phi \tag{80}$$

This raises the question what extra integrability conditions are needed in the case of not necessarily strongly accessible systems. This remains a topic for future research.

4 Reduction to an algebro-geometric problem

In this section we show that the question whether there exists a linear subsystem of dimension $\bar{n} > r$ is equivalent to a well-known problem from real algebraic geometry. For reasons of clarity of exposition, we first restrict to the case $\bar{n} = r + 1$. At the end of the section we make some remarks about the case $\bar{n} > r + 1$. Let $x_0 \in \mathbb{R}^n$ be given, and assume that Σ is strongly accessible around x_0. Further, assume that the codistributions $\mathcal{H}_k$ $(k = 1, \cdots, n)$ have constant dimension around $(x_0, 0, \cdots, 0)$, and that the relative degree $r := r_h$ of h is well-defined around x_0. Let $\lambda \in \mathcal{H}_n - \{0\}$ be such that (16),(24) hold. Then there exist $\alpha_0, \cdots, \alpha_{n-r} \in \mathcal{S}$ such that $\alpha_{n-r} \neq 0$ and

$$dh = \sum_{\ell=0}^{n-r} \alpha_\ell \lambda^{(\ell)} \tag{81}$$

Consider the parametrized post compensated system $\Sigma^p\langle s\rangle$. It then follows from Lemma 2.2 that there exist $\phi_\ell \in \mathcal{R}$ ($\ell = 0, \cdots, n-r-1$) such that

$$\mathcal{H}^p_{n+1}\langle s\rangle = \text{span}\{\sum_{\ell=0}^{n-r-1} \phi_\ell\langle s\rangle\lambda^{(\ell)} - dz\} \tag{82}$$

Define $\psi_0, \cdots, \psi_{n-r} \in \mathcal{R}$ by

$$\psi_0 := \dot{\phi}_0 + s\phi_0 - \alpha_0 \tag{83}$$

$$\psi_\ell := \dot{\phi}_\ell + \phi_{\ell-1} + s\phi_\ell - \alpha_\ell \quad (\ell = 1, \cdots, n-r-1) \tag{84}$$

$$\psi_{n-r} := \phi_{n-r-1} - \alpha_{n-r} \tag{85}$$

Let $0_{\mathcal{S}}$ denote the zero-function. We now have the following result.

Theorem 4.1 *Consider a strongly accessible SISO system Σ of the form (1) around x_0. Let $\psi_0, \cdots, \psi_{n-r}$ be defined by (83),(84),(85). Then Σ has a linear subsystem of dimension $r+1$ around x_0 if and only if $\psi_0, \cdots, \psi_{n-r}$ have a common real zero, i.e.,*

$$\exists_{a\in\mathbb{R}} \forall_{\ell\in\{0,\cdots,n-r\}} \ \psi_\ell\langle a\rangle = 0_{\mathcal{S}} \tag{86}$$

Proof From Theorem 3.8 it follows that Σ has a linear subsystem of dimension $r+1$ if and only if there exists an $a \in \mathbb{R}$ such that $\mathcal{H}_{n+1}\langle a\rangle = \mathcal{H}_\infty\langle a\rangle$. It is straightforwardly shown that this is equivalent to the existence of an $a \in \mathbb{R}$ such that

$$\frac{d}{dt}(\sum_{\ell=0}^{n-r-1} \phi_\ell\langle a\rangle\lambda^{(\ell)}) + a(\sum_{\ell=0}^{n-r-1} \phi_\ell\langle a\rangle\lambda^{(\ell)}) = dh \tag{87}$$

It then easily follows that this is equivalent to (86). ∎

We next show how (86) may be checked by reducing it to the question whether a set of polynomials in $\mathbb{R}[s]$ has a common real zero. Define $\xi := \text{col}(x, u, \cdots, u^{(2n)}) \in \mathbb{R}^{3n+1}$, and let ν denote the maximal degree in s of the polynomials $\psi_0, \cdots, \psi_{n-r}$. Then there exist functions $\psi_\ell^k \in \mathcal{S}$ such that

$$\psi_\ell\langle s\rangle(\xi) = \sum_{k=0}^{\nu} \psi_\ell^k(\xi)s^k \quad (\ell = 0, \cdots, n-r) \tag{88}$$

Define the $(n-r+1, \nu+1)$-matrix $P(\xi)$ with entries $P_{ij}(\xi) := \psi_i^j(\xi)$ ($i = 0, \cdots, n-r$; $j = 0, \cdots, \nu$). Further, define for $s \in \mathbb{R}$ the vector $v_s := \text{col}(1, s, \cdots, s^\nu)$. Then the question to be considered is whether there exists a *real* solution to the equation $P(\xi)v_s \equiv 0$. Obviously, there exists a real solution to this equation *only if* there exists a $v \in \mathbb{R}^{\nu+1} - \{0\}$ satisfying the equation $P(\xi)v \equiv 0$. Note that this equation may be extended by the equations $(\partial/\partial\xi_i(P(\xi)))v \equiv 0$ ($i = 1, \cdots, 2n$) and equations obtained by taking higher-order partial derivatives. Consider the following algorithm that performs this extension in a controlled way. The algorithm was suggested by [18], and is reminiscent of the Structure Algorithm ([12],[15]).

Algorithm 4.2

Step 0

Define $p^1 := n - r + 1$, $q^1 := \nu + 1$, $P^1(\xi) := P(\xi)$.

Step k

Define $\rho_k := \text{rank} P^k(\xi)$. There exist an invertible (p^k, p^k)-matrix $Q^k(\xi)$ and a (q^k, q^k)-permutation matrix R^k such that

$$Q^k(\xi)P^k(\xi)R^k = \begin{pmatrix} I_{\rho^k} & \bar{P}^k(\xi) \\ 0 & 0 \end{pmatrix} \tag{89}$$

where $\bar{P}^k$ is a $(\rho_k, q^k - \rho_k)$-matrix. If either $\rho_k = q^k$, or $\bar{P}^k(\xi)$ is a constant matrix, we **STOP**. Otherwise, define $p^{k+1} := (3n+1)\rho_k$, $q^{k+1} := q^k - \rho_k$, and

$$P^{k+1} := \begin{pmatrix} \frac{\partial \bar{P}^k}{\partial \xi_1} \\ \vdots \\ \frac{\partial \bar{P}^k}{\partial \xi_{3n+1}} \end{pmatrix} \tag{90}$$

and go to Step $k+1$. ■

It may be shown that Algorithm 4.2 terminates in a finite number, say k^* of steps. We have the following results.

Lemma 4.3 *Assume that* $q^{k^*} - \rho_{k^*} > 0$. *Let for* $k = 1, \cdots, k^*$ *the* (q^k, ρ_k)*-matrix* $\hat{R}^k$ *and the* $(q^k, q^k - \rho_k)$*-matrix* $\tilde{R}^k$ *be such that*

$$R^k = \begin{pmatrix} \hat{R}^k & \tilde{R}^k \end{pmatrix} \quad (k = 1, \cdots, k^*) \tag{91}$$

and define the matrices

$$S^k(\xi) := \tilde{R}^k - \hat{R}^k \bar{P}^k(\xi) \quad (k = 1, \cdots, k^*) \tag{92}$$

Then the matrix $S(\xi)$ *defined by*

$$S(\xi) := S^1(\xi)S^2(\xi)\cdots S^{k^*}(\xi) \tag{93}$$

is constant and left-invertible.

Proof See Appendix. ■

Lemma 4.4 *Assume that there exists a* $v \in \mathbb{R}^{\nu+1} - \{0\}$ *such that* $P(\xi)v \equiv 0$. *Define the matrices*

$$T^k(\xi) := S^1(\xi)\cdots S^k(\xi) \quad (k = 1, \cdots, k^*) \tag{94}$$

Then there exist $\tilde{v}^k \in \mathbb{R}^{q^k - \rho_k} - \{0\}$ $(k = 1, \cdots, k^*)$ *such that*

$$v = T^k(\xi)\tilde{v}^k \quad (k = 1, \cdots, k^*) \tag{95}$$

Proof See Appendix. ∎

Proposition 4.5 *There exists a* $v \in \mathbb{R}^{\nu+1} - \{0\}$ *such that* $P(\xi)v \equiv 0$ *if and only if* $q^{k^*} - \rho_{k^*} > 0$. *Moreover, if* $q^{k^*} - \rho_{k^*} > 0$, *then*

$$\{v \in \mathbb{R}^{\nu+1} \mid P(\xi)v \equiv 0\} = \mathrm{Im}S \tag{96}$$

Proof Assume that $q^{k^*} - \rho_{k^*} = 0$. Then it follows from Lemma 4.4 that $v = 0$, which gives a contradiction. Conversely, if $q^{k^*} - \rho_{k^*} > 0$, it immediately follows from Lemma 4.4 that there exists a $v \in \mathbb{R}^{\nu+1} - \{0\}$ such that $P(\xi)v \equiv 0$. We next prove (96). It follows from Lemma 4.4 that

$$\{v \in \mathbb{R}^{\nu+1} \mid P(\xi)v \equiv 0\} \subset \mathrm{Im}T^{k^*}(\xi) = \mathrm{Im}S \tag{97}$$

Conversely, let $v \in \mathrm{Im}S$, say $v = S\tilde{v}$, where $\tilde{v} \in \mathbb{R}^{q^{k^*}-\rho_{k^*}}$. We have

$$Q^1(\xi)P(\xi)S^1(\xi) = Q^1(\xi)P^1(\xi)(\tilde{R}^1 - \hat{R}^1\bar{P}^1(\xi)) =$$

$$Q^1(\xi)\left[\begin{pmatrix} \bar{P}^1(\xi) \\ 0 \end{pmatrix} - \begin{pmatrix} I \\ 0 \end{pmatrix}\bar{P}^1(\xi)\right] = 0$$

and hence $P(\xi)S^1(\xi) = 0$. This gives

$$P(\xi)v = P(\xi)S\tilde{v} = P(\xi)S^1(\xi)\cdots S^{k^*}(\xi)\tilde{v} \equiv 0$$

which yields

$$\mathrm{Im}S \subset \{v \in \mathbb{R}^{\nu+1} \mid P(\xi)v \equiv 0\}$$

Together with (97) this establishes (96). ∎

We now return to our original problem. Assume that $q^{k^*} - \rho_{k^*} > 0$, and let the matrix S be defined by (93). Let $\bar{P}$ be a right-invertible matrix such that $\mathrm{Im}S = \mathrm{Ker}\bar{P}$, and define the polynomials $\bar{p}_1, \cdots, \bar{p}_{q^{k^*}} \in \mathbb{R}[s]$ by

$$\bar{p}_i(s) := \sum_{j=1}^{\nu+1} \bar{P}_{ij}s^{j-1} \quad (i = 1, \cdots, q^{k^*})$$

It then follows from Proposition 4.5 that $a \in \mathbb{R}$ satisfies (86) if and only if $\bar{P}v_a = 0$, i.e., if and only if a is a common zero of the polynomials $\bar{p}_i$ ($i = 1, \cdots, q^{k^*}$). Let $\langle \bar{p}_1, \cdots, \bar{p}_{q^{k^*}} \rangle$ denote the polynomial ideal in $\mathbb{R}[s]$ spanned by $\bar{p}_1, \cdots, \bar{p}_{q^{k^*}}$. Since $\mathbb{R}[s]$ is a principal ideal domain, there exists a polynomial $\hat{p} \in \mathbb{R}[s]$ with the property that $\langle \bar{p}_1, \cdots, \bar{p}_{q^{k^*}} \rangle = \langle \hat{p} \rangle$ (see e.g. [17]). Thus, we have reduced our problem to the problem whether a monovariable polynomial has a real root. This is a well-known problem from real algebraic geometry, that has received attention since the times of Newton and Descartes. Obviously, there exists a real root when the polynomial $\hat{p}$ is of odd degree. When $\hat{p}$ is of even degree, one can check whether $\hat{p}$ has a real zero (in fact one can even determine the number of real zeros) using the so called Newton sums and Hankel forms associated with the polynomial. We refer to [6] for details on this topic.

In case one is trying to answer the question whether Σ has a real subsystem of dimension $\bar{n} > r+1$, one can proceed roughly in the same way as above. In this case, it may be shown that there exists a linear subsystem of dimension $\bar{n}$ if and only if a set of polynomials $\psi_0, \cdots, \psi_\gamma \in \mathcal{S}[s_1, \cdots, s_d]$, where $d := \bar{n} - r$, has a common real zero. Applying the same kind of algorithm as indicated above, the problem may then reduced to the problem whether a set of polynomials $\bar{p}_1, \cdots, \bar{p}_q \in \mathbb{R}[s_1, \cdots, s_d]$ has a common real zero. This problem has first been solved by Tarski ([16]). Later on, the problem has been considered by Collins ([4], see also [1],[5]) by using the concept of Cylindrical Algebraic Decomposition (CAD) of $\mathbb{R}^n$. By now, MAPLE-implementations of the algorithm for Cylindrical Algebraic Decomposition are available. A drawback, however, is that the complexity of existing algorithms is doubly exponential. Further, with the method of CAD one can also tackle problems in which polynomial equalities as well as polynomial inequalities play a role. By using the polynomial inequalities obtained from the Routh-Hurwitz test, it follows from Remark 3.4 that this also allows to check whether there exist linear subsystems with stable zero dynamics.

5 Example

Consider on $\{x \in \mathbb{R}^3 \mid x_2 \geq 0\}$ the nonlinear SISO system Σ given by

$$\Sigma \begin{cases} \dot{x}_1 &= x_1^2 x_2 + x_1 u \\ \dot{x}_2 &= x_2 - \frac{1}{2} x_1 \\ \dot{x}_3 &= -x_2 + x_3 - x_1 x_2 x_3 - x_3 u \\ y &= x_1 x_2 \end{cases} \tag{98}$$

We have $r := r_h = 1$, and hence Σ has a linear subsystem of dimension 1. We next check whether Σ has a linear subsystem of dimension 2. To this end, we consider the post compensated system $\Sigma^p\langle s\rangle$. Define the one-forms $\omega_1, \omega_2, \omega_3$ by

$$\begin{aligned} \omega_1 &:= dx_2^2 \\ \omega_2 &:= d(x_1 x_3) \\ \omega_3 &:= d(x_1 x_2) \end{aligned} \tag{99}$$

The one-forms ω_1 and ω_2 satisfy

$$\begin{aligned} \dot{\omega}_1 &= 2\omega_1 - \omega_3 \\ \dot{\omega}_2 &= \omega_2 - \omega_3 \end{aligned} \tag{100}$$

For $\Sigma^p\langle s\rangle$ we find

$$\mathcal{H}_4^p\langle s\rangle = \text{span}\{(s+1)\omega_1 - (s+2)\omega_2 - dz\} \tag{101}$$

From (99),(100),(101) it follows that $a \in \mathbb{R}$ satisfies $\mathcal{H}_\infty^p\langle a\rangle = \mathcal{H}_4^p\langle a\rangle$ if and only if it satisfies $a^2 + 3a + 2 = 0$, and hence $a = -1$ or $a = -2$. We have

$$\mathcal{H}_4^p\langle -2\rangle = \text{span}\{\omega_1 - dz\} \tag{102}$$

Defining new coordinates $\bar{x}_1 := x_2^2$, $\bar{x}_2 := \frac{d}{dt}(x_2^2) = 2x_2^2 - x_1 x_2$, $\bar{x}_3 := x_3$, and choosing u in an appropriate way, we then obtain the form (3) for Σ. We further have

$$\mathcal{H}_4^p\langle -1\rangle = \text{span}\{-\omega_2 - dz\} \tag{103}$$

If we now define new coordinates $\bar{x}_1 := x_1x_3$, $\bar{x}_2 := \frac{d}{dt}(x_1x_3) = -x_1x_2 + x_1x_3$, $\bar{x}_3 := x_2$, and choose u in an appropriate way, we also obtain the form (3) for Σ.

We next check whether Σ has a linear subsystem of dimension 3. Considering the post compensated system $\Sigma^p\langle s_1, s_2\rangle$, we obtain

$$\mathcal{H}_4^p\langle s_1, s_2\rangle = \text{span}\{\omega_2 - \omega_1 - dz_1, (s_2 - 2)(\omega_2 - \omega_1) - \omega_1 - dz_1\} \tag{104}$$

It then follows from (99),(100),(104) that $\mathcal{H}_4^p\langle a_1, a_2\rangle = \mathcal{H}_\infty^p\langle a_1, a_2\rangle$ if and only if

$$\begin{aligned} a_2 &= 3 \\ a_2^2 + a_2 + a_1 - 2 &= 0 \\ a_2^2 - a_2 - 2 &= 0 \end{aligned} \tag{105}$$

Clearly, the first and last equation in (105) are contradictory. Hence Σ does not have a linear subsystem of dimension 3. Note, however, that by choosing new coordinates $\bar{x}_1 := x_2^2 - x_1x_2$, $\bar{x}_2 := 2x_2^2 - x_1x_3$, $\bar{x}_3 := 4x_2^2 - x_1x_3 - x_1x_2$, and by choosing u in an appropriate way, we may feedback linearize the state equations of Σ.

6 Conclusions

In this paper we have characterized the linear subsystems of a nonlinear SISO system. Further, it has been shown that the existence of a linear subsystem of a given dimension can be checked by reducing the problem to a well known problem from real algebraic geometry, that can be tackled by means of the so called Cylindrical Algebraic Decomposition (CAD). A drawback of using CAD is that the complexity of existing algorithms is doubly exponential. This brings up the question whether the use of CAD could be circumvented. One way to do this might be to investigate whether or not the polynomial equations obtained have some special (preferably triangular) structure that can be employed. This remains a topic for future research. A more practically oriented way is to come up with an "educated guess" of the possible zeros of a linear subsystem by using the linearization of the system around an equilibrium point. This will be the topic of a forthcoming paper ([7]). In this paper, we have restricted ourselves on the one hand to SISO systems, and on the other hand to regular static state feedback. We expect that an extension of the results in the paper to MIMO systems (using regular static state feedback) is possible. Also an extension to the regular dynamic feedback case (at least for square systems having an invertible decoupling matrix) seems possible. This last extension would be useful in the solution of the model matching problem by means of minimal order dynamic state feedback. These remain topics for future research.

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Appendix

Proof of Lemma 4.3

Note that $S^{k^*}(\xi)$ is constant. We then have for $i = 1, \cdots, 3n+1$:

$$\frac{\partial S}{\partial \xi_i} = \sum_{k=1}^{k^*-1} S^1(\xi)\cdots S^{k-1}(\xi)\frac{\partial S^k}{\partial \xi_i}(\xi)S^{k+1}(\xi)\cdots S^{k^*}(\xi) \tag{106}$$

From (92) we have

$$\begin{aligned} \tfrac{\partial S^k}{\partial \xi_i} S^{k+1} = \quad & -\hat{R}^k \tfrac{\partial \bar{P}^k}{\partial \xi_i} S^{k+1} \\ & -\hat{R}^k P^{k+1}(\tilde{R}^{k+1} - \hat{R}^{k+1}\bar{P}^{k+1} = \\ & -\hat{R}^k Q_{k+1}^{-1}\left[\begin{pmatrix} \bar{P}^{k+1} \\ 0 \end{pmatrix} - \begin{pmatrix} I \\ 0 \end{pmatrix}\bar{P}^{k+1}\right] = 0 \end{aligned} \tag{107}$$

It then follows from (106),(107) and the fact that S^{k^*} is constant that S is constant. Since R^k is invertible, there exists a left-inverse $(\tilde{R}^k)^-$ of $\tilde{R}^k$ satisfying

$$(\tilde{R}^k)^-\hat{R}^k = 0 \tag{108}$$

This gives by (92):

$$(\tilde{R}^k)^- S^k(\xi) = (\tilde{R}^k)^-\tilde{R}^k = I_{q^k-\rho_k} \tag{109}$$

which implies that $S^k(\xi)$ is left-invertible. This immediately implies that also S is left-invertible.

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Proof of Lemma 4.4

By induction. First consider the case $k = 1$. Since $P^1(\xi) \equiv 0$, we also have

$$Q^1(\xi)P^1(\xi)v \equiv 0 \tag{110}$$

Let $\hat{v}^1 \in I\!R^{\rho_1}$, $\tilde{v}^1 \in I\!R^{q^1-\rho_1}$ be such that

$$v = \hat{R}^1\hat{v}^1 + \tilde{R}^1\tilde{v}^1 \tag{111}$$

Then

$$0 \equiv Q^1(\xi)P^1(\xi)v = Q^1(\xi)P^1(\xi)R^1\begin{pmatrix} \hat{v}^1 \\ \tilde{v}^1 \end{pmatrix} = \begin{pmatrix} I & \bar{P}^1(\xi) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \hat{v}^1 \\ \tilde{v}^1 \end{pmatrix} \tag{112}$$

and hence

$$\hat{v}^1 = -\bar{P}^1(\xi)\tilde{v}^1 \tag{113}$$

From (111) and (113) it then follows that

$$v = S^1(\xi)\tilde{v}^1 \tag{114}$$

and hence (95) holds for $k = 1$. Next, assume that (95) holds for $k = 1, \cdots, \ell - 1$, where $\ell \in \{2, \cdots, k^*\}$. We then have in particular that there exists a $\tilde{v}^{\ell-1} \in I\!R^{q^\ell}$ such that

$$v = T^{\ell-1}(\xi)\tilde{v}^{\ell-1} = T^{\ell-2}(\xi)(\tilde{R}^{\ell-1} - \hat{R}^{\ell-1}\bar{P}^{\ell-1}(\xi))\tilde{v}^{\ell-1} \tag{115}$$

Analogously to the proof of Lemma 4.3 it may be shown that

$$\begin{gathered}\frac{\partial}{\partial \xi_i}\left(T^{\ell-2}(\xi)(\tilde{R}^{\ell-1} - \hat{R}^{\ell-1}\bar{P}^{\ell-1}(\xi))\right) = \\ -T^{\ell-2}(\xi)\hat{R}^{\ell-1}\frac{\partial \bar{P}^{\ell-1}}{\partial \xi_i}(\xi) \quad (i = 1, \cdots, 3n+1)\end{gathered} \tag{116}$$

It then follows from (90),(115),(116) that

$$0 \equiv -T^{\ell-1}(\xi)\hat{R}^{\ell-1}P^\ell(\xi)\tilde{v}^{\ell-1} \tag{117}$$

From the fact hat $T^{\ell-2}$ and $\hat{R}^{\ell-1}$ are left-invertible, it then follows that

$$P^\ell(\xi)\tilde{v}^{\ell-1} \equiv 0 \tag{118}$$

Let $\hat{v}^\ell \in I\!R^{\rho_\ell}$, $\tilde{v}^\ell \in I\!R^{q^\ell - \rho_\ell}$ be such that

$$\tilde{v}^{\ell-1} = \hat{R}^\ell\hat{v}^\ell + \tilde{R}^\ell\tilde{v}^\ell \tag{119}$$

It then follows from (118),(119) that

$$0 \equiv Q^\ell(\xi)P^\ell(\xi)R^\ell\begin{pmatrix}\hat{v}^\ell \\ \tilde{v}^\ell\end{pmatrix} = \begin{pmatrix}I & \bar{P}^\ell(\xi) \\ 0 & 0\end{pmatrix}\begin{pmatrix}\hat{v}^\ell \\ \tilde{v}^\ell\end{pmatrix} \tag{120}$$

Together with (119) this implies that

$$\tilde{v}^{\ell-1} = S^\ell(\xi)\tilde{v}^\ell$$

Combining this with (115), we conclude that (95) holds for $k = \ell$. This establishes (95) for all $k \in \{1, \cdots, k^*\}$. ■

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