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Efficient reinsurance strategies considering counterparty default risk

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Abstract

Insurance companies pursue the objective of increasing their technical profit, but in doing so, they expose themselves to more risks, increasing the variability of their result. In order to balance the potential profitability deriving from the underwriting activity with the related risks, insurers typically resort to reinsurance treaties. In this context arises the problem of finding the optimal treaty which jointly satisfies multiple objectives, typically represented by risk and return metrics. The classical approaches consider only the characteristics of the treaty, neglecting the ones of the reinsurance provider. However, this approach could lead to sub-optimal choices, since it does not consider counterparty default risk.

The purpose of this thesis is threefold. Firstly, we extend classical formulas of technical profit of an insurance company to a partial internal model of Solvency II, including the potential default of the reinsurance counterparty. Secondly, we develop a stochastic simulation approach that includes counterparty default risk and potentially other features, for estimating the efficient frontier of reinsurance strategies for a non-life insurance company. Finally, we propose the application of a neural network model for finding the efficient frontier in a multi-objective optimization problem, requiring limited observations and preserving the possibility of deriving the strategies which generate the Pareto front.

Numerical applications are performed assuming a multi-line non-life insurer with parameters from the Italian market. The results show the importance of the rating of reinsurers, i.e. counterparty default risk, for the assessment of the optimal reinsurance strategies. Moreover, we show how this risk could become an opportunity in case the reinsurer with high risk offers a discounted price that more than compensate the potential default effect. Finally, the neural network model offers another perspective for determining optimal reinsurance strategies, which can be especially useful in case of high number of potential combinations defining each strategy.

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Chapter 1

Introduction

The choice of optimal strategies in an uncertain setting is a topic object of extensive academic and professional studies in many different areas. The general objective consists in finding the strategies which maximize or minimize a certain metric. The complexity of this analysis increases when the interest is in the joint maximization/minimization of more than one metric, defining a multi-objective optimization (MOO) problem. The reason is that typically there is a trade-off between the multiple objectives, which move in different directions. This kind of problem has been addressed also in the actuarial sector, where one relevant field of application is in the definition of optimal reinsurance strategies. The standard approach for dealing with this problem considers reinsurance treaties as a separate "object" with respect to their reinsurer provider, neglecting the characteristics of the firm. In this context, this thesis has the objective of analyzing optimal reinsurance in a coherent framework which considers both the characteristics of the treaties and also of the reinsurers. The thesis addresses the problem under multiple aspects and contribute to the literature in a threefold way. In particular, presenting advantages and drawbacks for each method, we start from extending closed formulas of non-life insurance stochastic capital calculation in a context which considers reinsurance treaties and counterparty default risk. Then, we develop a stochastic model which permits to calculate the efficient frontier of reinsurance strategies considering default risk in a multi-objective optimization framework. Finally, we present a neural network approach allowing to obtain the efficient frontier in the same multi-objective optimization framework, but that is general enough to be employed also in other contexts.

The thesis is organized as follows. Chapter 2 provides a general overview of the area and topics object of this thesis. The chapter is divided in three sections, each one focusing on a specific topic, object of the analysis. We start from a global presentation of reinsurance treaties, their characteristics and the most common pricing methodologies. Then we move to the description of counterparty default risk, starting from the general framework and then moving to two approaches for modeling this risk, one related to the financial/banking sector and the other to the insurance one. Finally, we present the current legislative framework which regulates the insurance sector in EU and its approach to the calculation of risks, with a focus on underwriting, reinsurance and counterparty, since it will be a benchmark for our model. Hence, in this chapter we give all the instruments for understanding the models developed in the following chapters and the related innovations.

The following chapters present the original research results developed in this thesis. Chapter 3 describes the general framework of the risk reserve equation and the main results under classical assumptions. Hence, we present three extensions of the risk reserve equation accounting for reinsurance treaties, and considering the counterparty default risk of the reinsurance provider. These extensions are developed in detail for the case of excess of loss reinsurance, considering specifically an insurance working in a single segment with a single reinsurer, working in multiple segments with a single reinsurer and working in a single segment with multiple reinsurers. After developing and describing the main two moments of these closed formulas for the extended model, we present a numerical analysis showing some practical empirical applications of the models. In this way we show how a non-life insurance company could use these closed formulas for analyzing different reinsurance strategies in a risk/return framework, enabling to obtain a result in a complex setting without the need of simulations.

Chapter 4 presents a simulating approach for the estimation of efficient frontier of reinsurance treaties in a risk/return framework. We present, step by step, a general framework for simulating the different stochastic components of the extended version of risk reserve equation and the approach for computing the efficient frontier of reinsurance strategies under different scenarios. Departing from classical approaches we consider both the characteristics of reinsurance treaties and their reinsurance providers. Hence, we present a numerical analysis showing how the general framework developed in this chapter can be employed in a practical context for efficient strategies selection.

In Chapter 5 we propose a strategy which takes advantage of Artificial Neural Network (ANN) in order to estimate the efficient frontier of reinsurance strategies in the same setting defined in the previous chapter, but requiring a much lower amount of data. The neural network developed in this chapter provides a feasible approach for finding efficient combinations in a multi-objective optimization problem, relying on a much lower amount of data, compared to the stochastic approach of previous chapter, keeping a high precision. The simplicity and advantages of this model are then presented in a numerical analysis.

Finally, the Conclusion chapter summarizes the results obtained and the future potential extensions of this work.

Chapter 2

Reinsurance, counterparty default risk and Solvency II

2.1 Reinsurance

2.1.1 General characteristics and main reinsurance contracts

A typical definition of reinsurance is “insurance for insurance companies”. Indeed, reinsurance is a contract where one party, the insurer, transfers a portion of the risk related to its activity to another party, the reinsurer. There are many reasons that justify the use of a reinsurance contract by an insurer. The most relevant are:

- **Increase underwriting capacity:** an insurer can decrease its technical risks by means of reinsurance treaties. In this way the insurance company can increase its underwriting capacity at the same or at a lower risk compared to the “gross” situation.

- **Substitute the equity capital:** connected to the previous point, the insurer can exploit reinsurance for decreasing its equity capital constraint. Indeed, insurance companies are required to hold a certain amount of capital connected with their specific riskiness. By means of reinsurance, the insurer can decrease its risks and consequently free up part of its capital (e.g. the effect of reduction of underwriting risk under Solvency II is described in Section 2.3.6).

- **Stabilize the variability of technical result:** the technical result of the insurance entity is affected by a certain degree of uncertainty. Thanks to reinsurance it is possible to reduce this uncertainty, at the cost of a lower expected result.

- **Reduce the unearned premium reserve:** the technical liability of the insurance entity is reduced by the quota of risk ceded to the reinsurance company.

- **Provide protection against a catastrophic loss:** in case the insurer works in some specific lines of business[1], where it is exposed to the risk of impact of catastrophic losses on its result, reinsurance is able to provide a protection against these extreme events.

Traditionally, it is usually made a distinction of reinsurance contracts depending on some of their characteristics. A first way for dividing reinsurance contracts is according to their "type". Under this categorization we distinguish facultative and treaty reinsurance.

In facultative reinsurance the parties negotiate and agree on the specific terms for each of the risks offered by the insurer. In treaty reinsurance the parties negotiate and agree on a given set of policies which are in the scope of the contract. In this case it is possible that only one of the two party has the "obligatory" condition as in the "facultative-obligatory" reinsurance, which permit the insurer to choose which kind of risk to cede, with the reinsurer obligated to accept.

Another relevant distinction in reinsurance context is according to methods used for sharing the losses between the two parties. Under this categorization, as reported in the following list, we distinguish two main categories:

- **Proportional reinsurance:** contract under which the reinsurer undertakes to reimburse the insurer a percentage of the cost of claim, equal to the percentage of risk transferred.
- **Non-proportional reinsurance:** contract under which the reinsurer undertakes to reimburse the insurer for losses over a certain amount and up to a certain limit, according to the conditions defined in the contract.

Under proportional reinsurance the two most relevant treaties employed in the market are *Quota Share* (QS) e *Surplus* (SP).

Definition 1. *A quota share is a proportional reinsurance contract, through which the primary insurance cedes premiums and losses with a cession rate* $(1 - \alpha) \in [0, 1]$ *to the reinsurer.*

In a QS treaty the insurer cedes a constant percentage of premium and losses for each of its risks. Hence, given α the quota of risk retained by the insurer and $(1 - \alpha)$ the cession rate, the amount of gross premium ceded from the insurer to the reinsurer is:

$$B^{re} = \sum_k B_k^{re} = \sum_k (B_k(1 - \alpha)) = B(1 - \alpha)$$

where B denotes the premium earned by the insurer and the sum over k considers all the contracts in the scope.

Similarly, the amount of ceded claims is:

$$\tilde{X}^{re} = \sum_k \tilde{X}_k^{re} = \sum_k \left(\tilde{X}_k(1 - \alpha)\right) = \tilde{X}(1 - \alpha) \,. \tag{2.1}$$

where X denotes the claims borne by the insurer and the sum over k considers all the claims in the scope.

[1] In this thesis we use interchangeably the term line of business and segment.

The second form of proportional reinsurance is represented by the surplus.

Definition 2. *A surplus contract is a proportional reinsurance contract, where the cession rate* $(1-\alpha_k) \in [0,1]$ *varies by policy k. The share* $(1-\alpha_k)$ *is a function of the insured sum* SI_k*, the Retention* $R > 0$ *and the number of lines of cover NL (usually integer,* > 0*) :*

$$(1-\alpha_k) = \frac{1}{SI_k} \min\left(\max(SI_k - R, 0), NL\ R\right)$$

In surplus reinsurance the insurer cedes a specific percentage of premium and losses for each of its risks. Hence, given α_k the quota of risk retained by the insurer for risk k and $(1-\alpha_k)$ the corresponding cession rate, the amount of gross premium ceded from the insurer to the reinsurer is:

$$B^{re} = \sum_k B_k^{re} = \sum_k B_k \left(1-\alpha_k\right)$$

and the amount of ceded claims is:

$$\tilde{X}^{re} = \sum_k \tilde{X}_k^{re} = \sum_k \tilde{X}_k \left(1-\alpha_k\right) .$$

A fundamental element in pricing of proportional reinsurance is the ceded commission, which represents a quota of premium that is returned from the reinsurer to the insurer. From a theoretical point of view the ceded commission C^{re} reimburses the insurer for the expenses that the firm incurred, in particular those arising from the acquisition of the contracts. Indeed, the proportional reinsurer is granted a percentage of the whole gross premium, which includes both a safety and an expense loading. However, since the reinsurance company actually did not incur in the acquisition expenses it should not be entitled of that portion of gross premiums. In practice, the ceded commission works as the main negotiation element in proportional reinsurance, determining the expected profitability of the contract for the reinsurer.

The simplest method of setting the ceded commission is by means of a Fixed Commission rate c^{re}. Under this approach, as reported in Formula (2.2), the ceded commission is deterministic and it is obtained by applying the fixed commission rate to the ceded claims, irrespectively of the (future) actual amount of claims

$$C^{re} = c^{re} B^{re} . \tag{2.2}$$

In this way, both insurer and reinsurer know already at the inception of the contract the final amount of ceded commission.

An alternative method, much more used in practice, is the Sliding Scale Commission. This approach is a way to link the ceded commission with the performance of the treaty. Indeed, the commission is a function of the actual loss ratio (LR) experience of the portfolio object of reinsurance. The ceded commission is then defined as a random variable $\tilde{c}^{re}$, whose value will be known only at the settlement of all the claims[2]. In Formula (2.3) the equation of stochastic commission rate is reported:

$$\tilde{c}^{re} = c^{re}\left[1 + \left(1 - \frac{\widetilde{LR}}{\mathbb{E}[\widetilde{LR}]}\right)\right] \tag{2.3}$$

where $\widetilde{LR}$ represents the stochastic loss ratio and c^{re} the "base" commission rate, obtained in case the observed loss ratio is equal to the expected one.

In a practical perspective, the following conditions represent a more common structure of stochastic commission:

$$\tilde{c}^{re} = \begin{cases} c_1^{re} & \text{if } \widetilde{LR} \le LR_1 \\ c_1^{re} + \left(\frac{c_2^{re} - c_1^{re}}{LR_2 - LR_1}\right)\left(\widetilde{LR} - LR_1\right) & \text{if } LR_1 < \widetilde{LR} \le LR_2 \\ c_2^{re} & \text{if } \widetilde{LR} > LR_2 \end{cases} \tag{2.4}$$

where c_1^{re} and c_2^{re} represent the minimum and maximum commission rate, respectively. Similarly, LR_1 and LR_2 represent two thresholds for the loss ratio. In practice, under this structure, commission rate is obtained by means of a linear interpolation of the minimum and maximum commission rate over the corresponding loss ratio interval. Instead, in case the loss ratio is below (above) the minimum (maximum) threshold the corresponding minimum (maximum) commission rate is selected.

As reported in Formula (2.5), regardless of the chosen structure, under this approach the commission is not deterministic, but function of the technical performance of the insurer.

$$\tilde{C}^{re} = \tilde{c}^{re} B^{re} \, . \tag{2.5}$$

Another possible alternative methods for defining the ceded commission, often used in the insurance practice also together with fixed or sliding commission, is the Profit Commission, which represents a fixed quota of the reinsurer technical result that is given back to the insurer. In the base case where the commission is established by only the profit commission we have a deterministic commission rate, applied to a stochastic quantity, the technical performance, which depends on the stochastic claims amount.

$$\tilde{C}^{re} = c^{re} \max\left(\left(B^{re} - \tilde{X}^{re}\right), 0\right) \, . \tag{2.6}$$

Under non-proportional reinsurance the two most relevant treaties employed in the market are *Excess of loss* (XL) and *Stop loss* (SL).

Before describing these two non-proportional treaties it is useful to define a fundamental element for describing non-proportional reinsurance contracts.

[2]In practice at periodic dates (usually quarterly or yearly) insurer and reinsurer adjust the commission according to the observed performance at the date.

Definition 3. Given a random loss $\tilde{X}$, the layer function $\mathcal{L}(\cdot)$ with deductible[3] d and limit[4] l is defined as:

$$\mathcal{L}_{d,l}\left(\tilde{X}\right) = \min\left(\max\left(\tilde{X} - d, 0\right), l\right) = \min\left(\tilde{X}, d + l\right) - \min\left(\tilde{X}, d\right) . \tag{2.7}$$

It is possible to notice that the layer function summarizes the case of modification of random variables, where a lower and upper limit is imposed (see for instance [52] for details on effect of coverage modification on claims amount random variable). Indeed we can write (2.7) as:

$$\tilde{X}_{d,l} = \left(\tilde{X} \wedge (d + l)\right) - \left(\tilde{X} \wedge d\right) = \begin{cases} 0 & \tilde{X} \leq d \\ \tilde{X} - d & d < \tilde{X} \leq d + l \\ l & \tilde{X} > d + l \,. \end{cases} \tag{2.8}$$

In non-proportional reinsurance the loss paid by reinsurer is commonly defined by means of the layer function, with specific differences according to the contract.

Definition 4. Given a random loss $\tilde{X}$, the "per risk" excess of loss contract, with deductible d and limit l, is defined as:

$$\tilde{X}^{re} = \sum_{r\ in\ risk} \mathcal{L}_{d,l}\left(\tilde{X}_r\right) \tag{2.9}$$

where the sum over r *in risk* indicates that the reinsurance layer is applied to each of the risks in the treaty. For this characteristic it is particularly useful for protecting against large losses generated from a higher number of exposures.

Definition 5. Given a random loss $\tilde{X}$, the "per event" excess of loss contract, with deductible d and limit l, is defined as:

$$\tilde{X}^{re} = \sum_{e\ in\ event} \mathcal{L}_{d,l}\left(\sum_{loss\ by\ event\ e} \tilde{X}_{e,i}\right) \tag{2.10}$$

where the first sum over e *in event* indicates that all the events included in the treaty are considered, while the second sum over the *loss by event* e indicates that the reinsurance layer is applied to all the losses caused by the same "event" e.

This type of contract is often defined as CAT XL, since the insurer seeks a coverage against the effects (in term of claims) caused by catastrophic events. The treaty applies to claims aggregated by event and is useful for protecting against accumulations originating from catastrophic events, whatever their origin.

[3]For describing the same element it is often used the term "priority" or "retention". In this thesis we will use these names indifferently.

[4]For describing the same element it is often used the term "cover". In this thesis we will use these names indifferently.

Definition 6. Given a random loss $\tilde{X}$, the stop loss contract, with deductible d and limit l, is defined as:

$$\tilde{X}^{re} = \mathcal{L}_{d,l}\left(\sum_k \tilde{X}_k\right) \tag{2.11}$$

where the sum over k considers all the claims of the portfolio in the scope of the treaty.

This treaty applies to the whole portfolio and is then useful for protecting against accumulations of losses, whatever originated from large losses or accumulation of small losses, at portfolio level. Under this contract, since we are considering the whole portfolio and the corresponding aggregated losses, the values of deductible and limit are typically defined as a function of loss ratio. The deductible is usually set at the level of loss ratio (multiplied by gross premium) that the insurer expect to be able to afford, while the limit is set to a value such that the probability of incurring in a higher loss is considered “small enough” by the insurer[5].

An important characteristic of stop loss treaty is that they could be considered, in a general sense, optimal. Indeed, in [64] it is shown theoretically that any optimal reinsurance is such it applies directly to the aggregate claim amount, as a SL treaty does.

For all the treaties that we have defined above, but in particular for stop loss reinsurance (due to the size of the risk underwritten), a common practice in the market is that the risk is underwritten by a pool of reinsurers rather than a single firm. In practice, a common approach for the insurer that wants to transfer (part of) its underwriting risks consists in employing a broker for finding the potential reinsurers interested and the share of risk in which they are interested. Hence, after the usual negotiation, the insurer, depending on the number of reinsurers and their conditions, can finalize the treaty. Under this contract each reinsurance company is liable for the specific layer of risk it has underwritten.

The pricing of non-proportional reinsurance is much more complex compared to proportional reinsurance. Indeed, for instance in per risk excess of loss, the reinsurer has to estimate at least the expected value of a part of the distribution of the single claim amount, but clearly it will be interested also in the higher-order moments. For these reasons in practice there are some specific methods that can be used for pricing non-proportional contracts that will be described in detail in Section 2.1.4.

An additional complexity in the pricing of XL reinsurance is related to the typical presence of specific clauses which modifies the base structure described in Definition (4)-(5). Annual Aggregate Deductible (AAD) and Annual Aggregate Limit (AAL) are two of the most common clauses employed in this context. These features modify the base structure of excess of loss coverage by adding another layer function, as described in Formula (2.12):

$$\tilde{X}^{re} = \mathcal{L}_{AAD,AAL}\left(\sum_{i=1}^{N} \mathcal{L}_{d,l}\left(\tilde{X}_i\right)\right) \tag{2.12}$$

[5]It shall be noted that a higher limit comes also with a strong increase in the price of the stop loss treaty. Hence, the insurer has to also consider this constraint when choosing the value.

where $\mathcal{L}_{AAD,AAL}$ is the "external" layer function of AAD and AAL at aggregate level, $\mathcal{L}_{d,l}$ is the layer function of deductible and limit at single claim level and N the number of claims.

In practice, the name of these clauses already explains their functionality in a XL treaty. They limit the coverage offered by the insurance in the year by a certain level of cumulative deductible and cumulative limit. As expected, since these clauses limit the exposure of the reinsurance company they result in a reduction of the price required for the coverage. Simulation approaches are the typical methods for an insurance company to evaluate the potential advantages and drawbacks of AAD and AAL, eventually including only one of the two.

Reinstatements are another typical clause present in most of the XL reinsurance treaties. They represent the number of times that the cover (i.e. limit) offered by the treaty can be used. In practice a XL treaty with deductible d and limit l offering no reinstatement implies that the aggregate cost of claims that are ceded to the reinsurer could be at maximum l. After then, the reinsurance company does not cover any other claim exceeding the deductible. Actually, a common approach consists in having the "first" premium (i.e. the premium paid at the inception of the contract) working as first time for the aggregate cost to exceed the limit. Then, it is possible to "reinstate" the coverage by paying additional premium, called reinstatement premium. More precisely, it possible to reinstate just a portion of the whole cover by paying the proportional quota of reinstatement premium. Regarding the specificity of this clause it is common practice to have a limited number of reinstatements that can be bought by the insurance company but, especially for "low" layers they could also be unlimited. In some countries (e.g. UK), instead, it is common to have unlimited free reinstatements for some contracts, like motor third-party liability (MTPL) [62].

2.1.2 General description of pricing of non-life policies

A non-life insurance contract requires the policyholder to pay a premium for the services offered by the insurer. It could be a single premium, paid upon the signing of the contract, or consists in multiple premium, paid at fixed dates over the coverage period. The payment in installments usually takes into account the time value of money and the probability of insolvency during the following payments, thus being higher than the corresponding single premium.

In the context of non-life business we typically use three different definitions of premium depending on the specific element considered: fair premium, pure premium and tariff premium.

Fair premium is the expected value of the claims that the insurer shall reimburse to his policyholders, pure premium is the risk premium from the point of view of the insurer and tariff premium is the amount asked by the insurer for issuing the policy.

More detailed, fair premium represents the amount that the insurer expects to pay, on average, for a given contract. In practice, no insurer could ask a premium equal to only its risk component. Indeed, for the "gambler ruin theory" an insurance company which sells policies at fair premium would go bankrupt with probability equal to 1 (see for instance [29] for the mathematical details). Hence, the insurance company requires an additional amount over fair premium, defined safety loading, in order to avoid certain ruin. Safety loading can be seen, *nomen omen*, as this additional margin needed for covering the cost of claims in excess of the expectation, in case of an unfavorable development. At the same time, it can also be interpreted as a "premium for the risk", i.e. the amount of profit required by the insurer in order to undertake the risk from the policyholder. Pure premium is then obtained as the sum of fair premium and safety loading.

Actually, an insurance company bears also expenses connected to its activity, where the most relevant are acquisition, collection and management costs. Acquisition and collection expenses are connected with the activity of selling the policies, in particular the first is linked with the remuneration of the agents selling insurance contracts and the latter with the costs for the emission of the contract. Finally, management expenses are connected with all the costs of the insurance activity from the issuing to the expiration of a policy. For this reasons, an insurance company requires an additional amount, on top of pure premium, defined expense loading, in order to have enough resources for covering the expenses it bears. Tariff premium is defined as the sum of risk premium, safety loading (i.e. pure premium) and expense loading.

Finally, there is a last premium, which represents the amount actually paid by the policyholder, and consists in the sum of taxes and other contributions (e.g. road victim fund) to tariff premium.

Computation of fair premium

There are two main methods for calculating fair premium: empirical approach and theoretical approach. The first one is based on the empirical observations of a given portfolio of policies, premiums and claims, in order to calibrate the premium. The second one is based on the selection and application of a theoretical model for describing the distribution of the aggregate amount of claims of a given portfolio of risks.

Hence, under the empirical approach, taking advantage of the historical claim information, the insurer can compute the fair premium according to:

$$P^E = \frac{\sum_{i=1}^{m} Z_i}{E} = \underbrace{\frac{m}{E}}_{\substack{expected\\frequency}} \underbrace{\frac{1}{m}\sum_{i=1}^{m} Z_i}_{severity} \tag{2.13}$$

where m represents the observed number of claims, Z_i the cost of the i-th claim and E the exposure of the whole portfolio of policies. The two terms of the product represent (observed) frequency and (observed) average cost of claim.

Hence, Formula (2.13) shows that it is possible to decompose the fair premium P^E according to its components of frequency and severity, where the first term represents the expected number of claims for unit of exposure and the second one the expected cost of the single claim.

The theoretical approach assumes that the (random variable) aggregate claim amount can be described according to a collective risk model (see for instance [29] for details).

This approach makes the following assumptions:

(i) The claim sizes are independent of each other: $\tilde{Z}_i \perp \tilde{Z}_j \forall i, j$.

(ii) The claim sizes are identically distributed: $F(\tilde{Z}_i) \stackrel{d}{=} F(\tilde{Z}) \forall i$

(iii) The number of claims and the claim sizes are independent: $\tilde{K} \perp \tilde{Z}_i$.

in order to describe the aggregate claim amount according to:

$$\tilde{X} = \sum_{i=1}^{\tilde{K}} \tilde{Z}_i \,.$$

Under these hypotheses it is possible to obtain the moments of the aggregate claim amount. In particular, remembering the definition of fair premium, we have that the first moment (i.e. the mean) of this random variable corresponds to the fair premium under the theoretical approach, and is obtained as reported in Formula (2.14).

$$P^T = \mathbb{E}\left[\tilde{X}\right] = \mathbb{E}\left[\tilde{K}\right] \mathbb{E}\left[\tilde{Z}\right] \,. \tag{2.14}$$

Under this modelling assumptions it is also possible to derive the other moments. In Formula 2.15 and 2.16 two of the most relevant moments are reported: variance and skewness

$$Var\left[\tilde{X}\right] = \mathbb{E}\left[\tilde{K}\right] Var\left[\tilde{Z}\right] + Var\left[\tilde{K}\right] \mathbb{E}\left[\tilde{Z}\right]^2 \,, \tag{2.15}$$

$$\gamma(\tilde{X}) = \frac{\mathbb{E}\left[\tilde{K}\right] Var\left[\tilde{Z}\right]^{3/2} \gamma\left[\tilde{Z}\right] + 3Var\left[\tilde{K}\right] \mathbb{E}\left[\tilde{Z}\right] Var\left[\tilde{Z}\right] + Var\left[\tilde{K}\right]^{3/2} \gamma\left[\tilde{K}\right] \mathbb{E}\left[\tilde{Z}\right]^3}{Var\left[\tilde{X}\right]^{3/2}} \,. \tag{2.16}$$

Risk measure and premium principles

In order to describe the premium principles that can be used to price a risk, we need to give a proper definition of risk measure. Following the work of Artzer et al. [7] we can give a proper definition.

Definition 7. A risk measure of the random loss $\tilde{X}$, denoted by $\varrho(\tilde{X})$ is a real-valued function $\varrho : \tilde{X} \to \Re$ where $\Re$ is the set of real numbers.

Moreover, [7] suggest four axioms that a risk measure is required to satisfy in order to be a *coherent* measure of risk. These are:

- **Translational invariance:** given a loss variable $\tilde{X}$ and a (non-negative) constant a, $\varrho\left(\tilde{X}+a\right)=\varrho\left(\tilde{X}\right)+a$.
- **Subadditivity:** given loss variables $\tilde{X}$ and $\tilde{Y}$, $\varrho\left(\tilde{X}+\tilde{Y}\right)\leq\varrho\left(\tilde{X}\right)+\varrho\left(\tilde{Y}\right)$.
- **Positive homogeneity:** given loss variable $\tilde{X}$ and a (non-negative) constant a, $\varrho(aX)=a\varrho\left(\tilde{X}\right)$.
- **Monotonicity:** given loss variable $\tilde{X}$ and $\tilde{Y}$ such that $\tilde{X}\leq\tilde{Y}$ under all states of nature, $\varrho\left(\tilde{X}\right)\leq\varrho\left(\tilde{Y}\right)$.

We can now present the most common premium principles, namely expected value, standard deviation and variance[6].

Under the expected value premium principle, the pure premium is computed as the sum between the expected value of the random variable and a safety loading, where also the latter is a function of the mean. In Formula (2.17) it is reported the expression

$$\pi_{ev}\left(\tilde{X}\right)=\left(1+\lambda_{ev}\right)\mathbb{E}\left[\tilde{X}\right] \tag{2.17}$$

where the parameter λ_{ev} represents the safety loading coefficient.

In general, this approach can be used in traditional insurance, but it is not practicable for a reinsurer. Indeed, using only the first moment of the random variable we do not consider the variability and the other characteristics of the distribution. This is critical for reinsurance companies, especially in non-proportional treaties, where the variability of random loss is a fundamental element.

It shall be noted that, in case we set $\lambda_{ev}=0$, we obtain a risk measure that return the mean of the random variable. This special case, where the premium correspond to the expected loss, represents the fair premium.

Under the standard deviation premium principle, the pure premium is function of two moments of the random variable: expected value and the squared root of the variance. Hence, in this case, the safety loading depends on the standard deviation of X, $\sigma\left[\tilde{X}\right]$. In Formula (2.18) it is reported the expression

$$\pi_{sd}\left(\tilde{X}\right)=\mathbb{E}\left[\tilde{X}\right]+\lambda_{sd}\sigma\left[\tilde{X}\right] \tag{2.18}$$

where the parameter λ_{sd} represents the safety loading coefficient.

Finally, the variance premium principle is function of mean and variance of the random variable. In Formula (2.19) it is reported the expression

$$\pi_{var}\left(\tilde{X}\right)=\mathbb{E}\left[\tilde{X}\right]+\lambda_{var}\sigma^2\left[\tilde{X}\right] \tag{2.19}$$

[6] Another relevant premium principle, especially in the reinsurance context, is the one presented in [80] and based on the Wang transform.

where the parameter λ_{var} represents the safety loading coefficient.

In practice, we can observe that the premium principles presented above are all function of the same random variable. Hence, it is possible to derive the equivalent safety loading coefficient according to different risk measures. This can be useful in case we want to charge the same loading according to different premium principles.

These premium principles are all based on the moments of the same distribution, then if we choose one premium principle we can find the loading parameters of the other principles that return the same premium. In Formula (2.20) we show the relationship between each safety loading coefficient of the three premium principles

$$\begin{aligned}
&\pi_{ev}\left(\tilde{X}\right) = \pi_{sd}\left(\tilde{X}\right) = \pi_{var}\left(\tilde{X}\right)\\
&\mathbb{E}\left[\tilde{X}\right](1+\lambda_{ev}) = \mathbb{E}\left[\tilde{X}\right] + \lambda_{sd}\sigma\left[X\right] = \mathbb{E}\left[\tilde{X}\right] + \lambda_{var}\sigma^2\left[X\right]\\
&\lambda_{ev}\mathbb{E}\left[\tilde{X}\right] = \lambda_{sd}\sigma\left[\tilde{X}\right] = \lambda_{var}\sigma^2\left[\tilde{X}\right]\\
&\lambda_{ev} = \lambda_{sd}CoV(X) = \lambda_{var}\frac{\sigma^2\left[\tilde{X}\right]}{\mathbb{E}\left[\tilde{X}\right]}\\
&\lambda_{sd} = \frac{\lambda_{ev}}{CoV(\tilde{X})} = \lambda_{var}\sigma\left[\tilde{X}\right]\\
&\lambda_{var} = \frac{\lambda_{sd}}{\sigma\left[\tilde{X}\right]} = \lambda_{ev}\frac{\mathbb{E}\left[\tilde{X}\right]}{\sigma^2\left[\tilde{X}\right]}\,.
\end{aligned} \tag{2.20}$$

In practice it is possible to obtain the same pure premium under the three different premium principles presented by modifying the safety loading coefficients accordingly. These equivalences will then be used in the numerical part of Chapter 4 in order to calibrate the reinsurer safety loading coefficient using a different premium principle compared to the insurer one.

An important analysis that shall be made on these premium principles regards how they are affected by changes in the aggregate claim amount $\tilde{X}$. If we compute the derivative of each premium principle with respect to $\mathbb{E}\left[X\right]$ we obtain the effect that a change in the expected value of the random aggregate loss has on the charged premium. If we instead compute the derivative of the premium principle with respect to $Var\left[X\right]$ we obtain the effect of a change in the variance on the charged premium.

It follows that the effect of a change in expected value and variance on the expected value premium principle are:

$$\begin{aligned}
\frac{\mathrm{d}\pi_{ev}(\tilde{X})}{\mathrm{d}\mathbb{E}\left[\tilde{X}\right]} &= \frac{\mathrm{d}\,(1+\lambda_{ev})\,\mathbb{E}\left[\tilde{X}\right]}{\mathrm{d}\mathbb{E}\left[\tilde{X}\right]} = (1+\lambda_{ev})\ ,\\
\frac{\mathrm{d}\pi_{ev}(\tilde{X})}{\mathrm{d}\sigma^2\left[\tilde{X}\right]} &= \frac{\mathrm{d}\,(1+\lambda_{ev})\,\mathbb{E}\left[\tilde{X}\right]}{\mathrm{d}\sigma^2\left[\tilde{X}\right]} = 0
\end{aligned}$$

which means that the price increases linearly for an increase in the expected loss, but it is unaffected by changes in variance.

Finally, the effect of a change in expected value and variance on the standard deviation premium principle are:

$$\frac{\mathrm{d}\pi_{var}(\tilde{X})}{\mathrm{d}\mathbb{E}\left[\tilde{X}\right]} = \frac{\mathrm{d}\mathbb{E}\left[\tilde{X}\right] + \lambda_{sd}\sigma\left[\tilde{X}\right]}{\mathrm{d}\mathbb{E}\left[\tilde{X}\right]} = 1\,,$$

$$\frac{\mathrm{d}\pi_{var}(\tilde{X})}{\mathrm{d}\sigma^2\left[\tilde{X}\right]} = \frac{\mathrm{d}\mathbb{E}\left[\tilde{X}\right] + \lambda_{sd}\sigma\left[\tilde{X}\right]}{\mathrm{d}\sigma^2\left[\tilde{X}\right]} = \frac{1}{2}\lambda_{sd}\,.$$

Conversely, the effect of a change in expected value and variance on the variance premium principle are:

$$\frac{\mathrm{d}\pi_{var}(\tilde{X})}{\mathrm{d}\mathbb{E}\left[\tilde{X}\right]} = \frac{\mathrm{d}\mathbb{E}\left[\tilde{X}\right] + \lambda_{var}\sigma^2\left[\tilde{X}\right]}{\mathrm{d}\mathbb{E}\left[\tilde{X}\right]} = 1\,,$$

$$\frac{\mathrm{d}\pi_{var}(\tilde{X})}{\mathrm{d}\sigma^2\left[\tilde{X}\right]} = \frac{\mathrm{d}\mathbb{E}\left[\tilde{X}\right] + \lambda_{var}\sigma^2\left[\tilde{X}\right]}{\mathrm{d}\sigma^2\left[\tilde{X}\right]} = \lambda_{var}\,.$$

We can observe that the expected value premium principle increases linearly with the mean (with a slope of $1 + \lambda_{ev}$). Similarly, also the other two premium principles increase linearly with the mean, but at a lower slope (1). However, more importantly, we observe that the expected value premium principle is not affected by changes in the standard deviation/variance of the random variable. In practice, a change in the value of the standard deviation do not provide any change in the expected value premium principle. The standard deviation and variance premium principle, instead, increase linearly for changes in the variance by $1/2\lambda_{sd}$ and λ_{var} respectively.

These results show why for an excess of loss reinsurer, whose risk is much more related to the variability of the random loss rather than the mean, it is much more coherent to apply a standard deviation/variance premium principle.

2.1.3 Pricing of a quota share treaty

The main element used in the pricing of a quota share treaty is the ceded commission. Indeed, since QS works through a proportional cession of premium and claims, ceded commission is the only element for creating a discrimination in the pricing.

In general we can assume that the reinsurer has a certain target of technical profit from the treaty. In order to compare different possible treaties, the "monetary" technical profit can be expressed as a margin on premium. The reinsurer can then calibrate the ceded commission as a function of this margin. Since we are dealing with random variables this will be more precisely an expected margin, and will be obtained as reported in Formula (2.21).

$$
\begin{aligned}
\mathbb{E}[M] &= \frac{\mathbb{E}\left[B^{re} - \tilde{X}^{re} - \tilde{C}^{re}\right]}{B^{re}} = \frac{\mathbb{E}\left[B^{re} - \tilde{X}^{re} - B^{re}\tilde{c}^{re}\right]}{B^{re}} \\
&= \frac{B^{re} - \mathbb{E}\left[\tilde{X}^{re}\right] - B^{re}\mathbb{E}\left[\tilde{c}^{re}\right]}{B^{re}} \\
&= 1 - \frac{\mathbb{E}\left[\tilde{X}^{re}\right]}{B^{re}} - c^{re,*}
\end{aligned}
\tag{2.21}
$$

where M is the margin.

In previous formula we have assumed the most general case of stochastic ceded commission, but clearly the same logic holds also for the fixed commission case. At this point, as reported in Formula (2.22), it is possible to derive the ceded commission as a function of the required margin:

$$
c^{re} = \min\left(0, 1 - \frac{\mathbb{E}\left[\tilde{X}^{re}\right]}{B^{re}} - \mathbb{E}[M]\right) \tag{2.22}
$$

which also includes a constraint, imposing a non-negative value.

It shall be noted that this formula does not consider any “external” expense to pay, for instance, for the brokerage activity. Anyhow, these eventual costs do not change the logic of the formula, but simply add other elements reducing the technical profit.

Differently, in practice a QS treaty could include some additional loss sensitive features (e.g. profit commission, loss corridor, etc.) which modify the base structure. In those cases, it is not always possible to express the commission rate directly in a closed form, but it could be necessary to perform a numerical simulation. In [26] some examples of pricing of a quota share treaty with stochastic ceded commission and other features are reported.

2.1.4 Pricing of an excess of loss treaty

Differently from quota share, the pricing of an excess of loss treaty requires a much more technical analysis. Indeed, as already explained in Section 2.1.1 a XL treaty covers the risks related to the tail of the single claim distribution. As a consequence, an incorrect pricing could have huge impacts on the reinsurer technical result and then on its capital.

Typically, the pricing of a XL reinsurance contract is based on the following two approaches[7]:

- **Experience pricing:** it based on the use of past observations to calibrate the premium.
- **Exposure pricing:** it is based on the use of the so-called “exposure curves” to calibrate the premium.

[7] In the reinsurance terminology these approaches are more often reported as experience rating and exposure rating. In this thesis we use the terms interchangeably.

Experience rating, coherently with its name, is an approach for pricing reinsurance treaties based on the assumption that future expectation can be predicted by means of past experience (properly adjusted). For instance, assuming that the reinsurer calibrates the premium according to a traditional "premium principle" approach, it shall calculate the moments of the random variable "cost of single claim" (also called claim size) $\tilde{Z}$ in the layer where reinsurance operates. More specifically, remembering the definition of layer and Formula (2.8), for a layer of range $(d, d+l)$, with d deductible and l limit respectively, the random variable describing the cost of single claim borne by the reinsurer is defined as:

$$\tilde{Z}_{d,l} = \min\left(\max\left(\tilde{Z} - d, 0\right), l\right) = \begin{cases} 0 & \tilde{Z} \leq d \\ \tilde{Z} - d & d < \tilde{Z} \leq d + l \\ l & \tilde{Z} > d + l \end{cases}$$

Hence, expected value and variance of the aggregate claim amount borne by the reinsurer can be derived using the decomposition between number of claims and severity, as reported below:

$$\mathbb{E}\left[\tilde{X}_{d,l}\right] = \mathbb{E}\left[\sum_{i=1}^{\tilde{K}} \tilde{Z}_{i,d,l}\right] = \mathbb{E}\left[\tilde{K}\right] \mathbb{E}\left[\tilde{Z}_{d,l}\right] ,$$

$$Var\left[\tilde{X}_{d,l}\right] = Var\left[\sum_{i=1}^{\tilde{K}} \tilde{Z}_{i,d,l}\right] = \mathbb{E}\left[\tilde{K}\right] Var\left[\tilde{Z}_{d,l}\right] + Var\left[\tilde{K}\right] \mathbb{E}\left[\tilde{Z}_{d,l}\right]^2 .$$

At this point, the reinsurance company can choose the premium principle for calibrating pure premium to charge to the insurer for underwriting its risks. In practice, the reinsurer requires past observations in order to fit the appropriate frequency and severity model necessary for estimating the mean and variance of the aggregate claim amount and consequently the required premium

Exposure pricing does not rely on previous experience history, but on the so-called "exposure curves", which are based on the severity curves of a large number of insurer. In practice the reinsurer takes advantage of the past information obtained from all its "clients" and use them to calibrate these curves. Hence, it applies this approach in particular when the data of the specific insurer is not sufficient or enough reliable to be used for experience pricing.

Formally, exposure curves indicate the quota of retained risk by the insurance company for each level of the deductible, expressed in percentage of the insured sum. The construction of exposure curves and the strength of their application relies on the fact that there is a relation between these and severity curves which permits to calculate one given the other. In particular, defined with x the percentage of loss respect to the insured sum (maximum value), in Formula 2.23 it is described the relationship between exposure and severity curves:

$$G(u) = \frac{\int_0^u (1 - F(x))\,dx}{\mathbb{E}\left[\tilde{X}\right]} \tag{2.23}$$

where $G(u)$ and $F(x)$ are the cumulative distribution function (c.d.f.) of exposure and severity respectively (for a more detailed description of the relation between exposure curve and severity curve and how these curves are constructed refer to [62]). Hence, it is possible to exploit this relation for pricing non-proportional treaties.

It shall be noted that, similarly to QS, even more in non-proportional treaty as XL could be present some specific clauses that modify the result and for which it is not always possible to develop a closed model. Hence, it is usually necessary to take advantage of Monte Carlo simulation approaches in order to estimate the effect of the many possible clauses present in the contract (e.g. reinstatement, annual aggregate limit, etc.) and compute the premium accordingly. See for instance [56] for an explicative description of excess of loss pricing with loss sensitive features in the context of exposure rating.

Effect of reinsurance layer segmentation

We now analyze the effect of segmenting a given reinsurance layer into multiple sub-layers. In practice, we move from a case where there is only one reinsurer covering a loss layer to the case where there are multiple reinsurers covering "together" the same layer. According to the premium principle chosen by the reinsurer there will be a different effect on the cost of coverage charged to the insurer.

In order to describe these differences on the pure premium, first of all we show that fair premium is not affected by the number of reinsurers. Indeed, since we are simply dividing the same layer into multiple sub-layers, we have no impact on the total premium. Formula (2.24) shows that the sum of the expected aggregate claim amount of N sub-layer is equivalent to the expected aggregate claim amount of the layer as a whole.

$$\sum_{h=0}^{N-1} \mathbb{E}\left[\tilde{X}^{re}_{d+h\frac{l}{N},\frac{l}{N}}\right] = \mathbb{E}\left[\sum_{h=0}^{N-1} \tilde{X}^{re}_{d+h\frac{l}{N},\frac{l}{N}}\right] = \mathbb{E}\left[\tilde{X}^{re}_{d,l}\right] \tag{2.24}$$

where, for simplicity, we have assumed that each layer has the same limit.

However, as anticipated, the cost of XL premium in presence of multiple reinsurers compared to the single reinsurer case depends on the used premium principle. For the case of the expected value premium principle and a single reinsurer for the whole layer the premium is obtained according to Formula (2.25).

$$B^{re} = \mathbb{E}\left[\tilde{X}^{re}\right] + \beta\mathbb{E}\left[\tilde{X}^{re}\right] \tag{2.25}$$

where β represents the safety loading coefficient.

Instead, if there are R reinsurers, each one covering a portion of the whole layer, and all applying the expected value premium principle, the total premium paid by the insurer is obtained according to Formula (2.26)

$$\begin{aligned} B^{re} &= \sum_{r=1}^{R} \left(\mathbb{E}\left[\tilde{X}^{re(r)}\right] + \beta \mathbb{E}\left[\tilde{X}^{re(r)}\right] \right) \\ &= \sum_{r=1}^{R} \mathbb{E}\left[\tilde{X}^{re(r)}\right] + \beta \sum_{r=1}^{R} \mathbb{E}\left[\tilde{X}^{re(r)}\right] \\ &= \mathbb{E}\left[\tilde{X}^{re}\right] + \beta \mathbb{E}\left[\tilde{X}^{re}\right] . \end{aligned} \tag{2.26}$$

This result shows that, for the properties of the expectation, the segmentation of the layer has no effect in case the reinsurance companies apply the expected value premium principle.

For the case of the standard deviation premium principle and a single reinsurer for the whole layer, the premium is obtained according to Formula (2.27)

$$B^{re} = \mathbb{E}\left[\tilde{X}^{re}\right] + \beta \sigma \left[\tilde{X}^{re}\right] . \tag{2.27}$$

Instead, if there are R reinsurers, each one for a portion of the whole layer, and all applying the standard deviation premium principle, the total premium paid by the insurer is obtained according to Formula (2.28)

$$\begin{aligned} B^{re} &= \sum_{r=1}^{R} \left(\mathbb{E}\left[\tilde{X}^{re(r)}\right] + \beta \sigma \left[\tilde{X}^{re(r)}\right] \right) \\ &= \sum_{r=1}^{R} \mathbb{E}\left[\tilde{X}^{re(r)}\right] + \beta \sum_{r=1}^{R} \sigma \left[\tilde{X}^{re(r)}\right] \\ &= \mathbb{E}\left[\tilde{X}^{re}\right] + \beta \sum_{r=1}^{R} \sigma \left[\tilde{X}^{re(r)}\right] . \end{aligned} \tag{2.28}$$

Hence, the two approaches lead to a different “total premium”. It is possible to show that, in this case, segmenting the layer creates an increase in the premium charged to the insurer equal to:

$$\begin{aligned}
&\beta \sum_{r=1}^{R} \sigma \left[\tilde{X}^{re(r)}\right] - \beta\sigma \left[\tilde{X}^{re}\right] \geq 0 \\
&\sum_{r=1}^{R} \sigma \left[\tilde{X}^{re(r)}\right] - \sigma \left[\tilde{X}^{re}\right] \geq 0 \\
&\sum_{r=1}^{R} \sigma \left[\tilde{X}^{re(r)}\right] - \sigma \left[\sum_{r=1}^{R} \tilde{X}^{re(r)}\right] \geq 0 \\
&\sum_{r=1}^{R} \sigma \left[\tilde{X}^{re(r)}\right] - \sqrt{\sum_{r=1}^{R} \sigma^2 \left[\tilde{X}^{re(r)}\right] + \sum_{r\neq s} cov\left[X^{re(r)}, X^{re(s)}\right]} \leq 0
\end{aligned} \tag{2.29}$$

where the equivalence is reached only in case the correlation coefficient between the aggregate claim amount of reinsurer r and s is $\rho_{r,s} = 1, \forall r, s$.

For the case of the variance premium principle and a single reinsurer for the whole layer, the premium is obtained according to Formula (2.30).

$$B^{re} = \mathbb{E}\left[\tilde{X}^{re}\right] + \beta\sigma^2\left[\tilde{X}^{re}\right] . \tag{2.30}$$

Instead, if there are R reinsurers, each one for a portion of the whole layer, and all applying the variance premium principle, the total premium paid by the insurer is obtained according to Formula (2.31)

$$\begin{aligned}
B^{re} &= \sum_{r=1}^{R} \left(\mathbb{E}\left[\tilde{X}^{re(r)}\right] + \beta\sigma^2\left[\tilde{X}^{re(r)}\right]\right) \\
&= \sum_{r=1}^{R} \mathbb{E}\left[\tilde{X}^{re(r)}\right] + \beta \sum_{r=1}^{R} \sigma^2\left[\tilde{X}^{re(r)}\right] \\
&= \mathbb{E}\left[\tilde{X}^{re}\right] + \beta \sum_{r=1}^{R} \sigma^2\left[\tilde{X}^{re(r)}\right] .
\end{aligned} \tag{2.31}$$

Hence, the two approaches lead to a different "total premium". It is possible to show that the segmentation of the layer creates a decrease of cost for the insurer equal to:

$$\begin{aligned}
&\beta \sum_{r=1}^{R} \sigma^2 \left[\tilde{X}^{re(r)}\right] - \beta\sigma^2 \left[\tilde{X}^{re}\right] \leq 0 \\
&\sum_{r=1}^{R} \sigma^2 \left[\tilde{X}^{re(r)}\right] - \sigma^2 \left[\tilde{X}^{re}\right] \leq 0 \\
&\sum_{r=1}^{R} \sigma^2 \left[\tilde{X}^{re(r)}\right] - \sigma^2 \left[\sum_{r=1}^{R} \tilde{X}^{re(r)}\right] \leq 0 \\
&\sum_{r=1}^{R} \sigma^2 \left[\tilde{X}^{re(r)}\right] - \sum_{r=1}^{R} \sigma^2 \left[\tilde{X}^{re(r)}\right] - \sum_{r\neq s} cov\left[X^{re(r)}, X^{re(s)}\right] \leq 0
\end{aligned}$$

where the equivalence is reached only in case $\rho_{r,s} = 0, \forall r, s$.

2.2 Counterparty default risk

2.2.1 General description of counterparty default risk

All the time an agent enters in a contractual agreement it is exposed to the risk that the other party will not fulfill their contractual obligations. In a more economic context, we usually refer to this situation when the agent is exposed to a "monetary" risk, like a receivable, a bond, a load, etc. and consequently the breach of the deal or the default of the counterparty generates a loss to the agent. This kind of risk is evidently present in all the business, but it has a major relevance for a financial institution, and indeed, most of the models for describing and assessing it comes from financial literature. Generally, the aim of these models is to measure the loss in case of the occurrence of the default event. There exist also some specific approaches with other additional objectives, but the main elements for modeling counterparty risk are typically the same. In the following we will briefly present these elements and then explore in more detail a financial and an insurance model for counterparty default risk.

A quite general formula for describing the loss related to counterparty default is expressed Formula (2.32) and presents the three main elements of this risk:

$$Loss = EAD\ (1 - RR)\ p = LGD\ p \tag{2.32}$$

where EAD represents the *Exposure At Default*, p the *Probability of Default* and RR the *Recovery Rate* in case of default. Sometimes, instead of using the first equality, it is used directly the second formulation, which considers together the exposure at default and the percentage not recovered in the *Loss Given Default* (LGD) term.

The starting point of counterparty risk consists in a monetary exposition against another party, which could default on its obligations. In practice, we are not interested in the exposition in general but only on the exposure at default, which represents the exposition that the agent has against the other party at the moment of default.

In the financial context, the exposure is often the residual value of a financial instrument. In the insurance context, the exposition is represented by the credit that the insurer holds against the reinsurer, which, in turn for the premium received, shall indemnify the insurer for claims in the scope of the reinsurance treaty[8].

The probability of default, indicated with p can be generally defined as the probability that the borrower or debtor defaults on its payment obligations. This element has a fundamental importance for estimating the expected loss that could derive from a credit position against a single or a group of counterparties, and consequently

[8]In this context we are limiting our analysis to the main counterparty risk for a non-life insurance company. Clearly, an insurer holds a credit and then a counterparty risk also against other subject, like agents or other providers, but these are typically less relevant.

also impacts the pricing of the loan/credit. However, it is not easy to estimate the value of this quantity for a generic firm. Some of the most relevant reasons are the limited number of events for building a model, the change in the "definition of default" over time (making past data less comparable) and unavailability of specific firm information.

There are three main methodologies which have been developed in literature for the estimation of probability of default: "accounting analytic approach" (rating agencies), statistical methods and option-theoretic approach. The first approach, as suggested by the name, consists in the analysis of the accounting information of a firm in order to assess its credit quality. In particular, the approach uses financial ratios, coming from accounting/financial data (e.g. leverage ratio, stability of earning and cashflows, etc.) and complement the evaluation with the judgement of industry experts, in order to derive a rating which takes into account also human expertise. Rating agencies, whose main activity consists in the evaluation of the credit quality of firms, typically follow an approach based on the accounting analytic one (with much more elaborated procedures) for assigning the rating based on their estimated probability of default. The outcome of this approach is a label (usually connected with an alphabetic ordering) which identify the credit quality of a firm, where a "low" letter (e.g. AAA) corresponds to a low probability of default. However, a rating purely based on a qualitative (categorical) element could not be applicable for instance for the simulation of default events. Hence, in practice the main rating agencies (e.g. Moody's Investor Service, Standard & Poor's and Fitch Ratings) publish historical data on the default probability for each of their rating, creating then a matching between the (qualitative) rating and the corresponding (quantitative) probability of default. Moreover, they also provide data on the transition probability to different ratings classes, at different time horizons. In a multi-year perspective this transition matrix is particularly helpful, since we are able to analyze the possible migration from the initial credit rating and the consequent change in the probability of default over time. For instance, in the context of corporate bond it could be a really relevant element, since a change of rating would also impact the price of the financial instrument.

The rationale behind the creation of a matrix of transition probabilities is that a firm, as a normal consequence of the randomness of its business, could change its financial performance (in better and worse), consequently impacting its rating. Indeed, in general the rating of a firm is not fixed forever, but it changes according to the outlook of the firm. This element can be seen exactly as a transition matrix of a Markov chain representing the probability that a firm with a certain rating at time t would move to another rating at time $t + 1$. An example of this transition matrix, as developed by Standard & Poor's [76], is reported in Table 2.1.

Coming back to the methodologies for estimating the probability of default, the statistical methods, differently from rating, are quantitative models which aim at estimating the default "likelihood" of a firm. In general, these models use past information in order to "understand" which elements are the main drivers that determine a default. In the literature there are many approaches developed for estimating the probability of default of a firm, based on historical data. Linear

Table 2.1. Average One-Year Corporate Transition Rates (1981-2021) - Europe (in %), as from [76].

	AAA	**AA**	**A**	**BBB**	**BB**	**B**	**CCC/C**	**D**	**NR**
AAA	87.13	9.11	0.46	0.11	0.00	0.00	0.11	0.00	3.08
AA	0.26	86.27	9.49	0.52	0.00	0.00	0.00	0.00	3.46
A	0.01	1.75	88.02	5.24	0.14	0.03	0.00	0.03	4.78
BBB	0.00	0.08	3.91	85.70	3.34	0.26	0.08	0.06	6.56
BB	0.00	0.00	0.09	5.05	74.88	6.86	0.37	0.34	12.40
B	0.00	0.00	0.02	0.20	4.77	73.52	5.06	1.96	14.47
CCC/C	0.00	0.00	0.00	0.20	0.00	12.55	44.33	25.91	17.00

discriminant analysis was one of the first developed and consists in a classification model with the objective of identifying the variables that are better able to classify firms according to the occurrence of default event. The most famous example is represented by the Z-score developed by Altman in [1], which proposes a score, based on a linear function of the relevant firm variables, which can be used for obtaining default likelihoods. Other approaches consist in principal components analysis, logistic regression, hierarchical classification models, etc., all of these aiming at grouping firms according to their degree of default likelihood. More recently, models based on neural network and other machine learning approaches started to be developed also in this context. However, one of the most relevant limit in their application is connected with the fact that they perform the best when there is an high amount of data to be feed. Consequently, while these models are generally developed successfully in "customer" credit risk (where more data is available), the limited amount of data at firm level reduces their potential applicability in this context.

Finally, the last most relevant element in the context of counterparty default risk is recovery rate, which represents the quota of exposure that is recovered in case of occurrence of the default event. The estimation of this element for modeling purposes is considered even more difficult than probability of default. In practice, one of the most relevant problem related to their estimation is that there are usually no "objective valuations" of recoveries. Moreover, the recovered value could change in the time between the "announcement of default" to the full settlement (which could take a relevant amount of time), increasing the complexity in creating an estimate based on homogeneous data. In general, the empirical analysis on the recovery rates of corporate bonds (see for instance [22] or [2]) show that there is a correlation between the expected recovery and the seniority of the bond. In particular, a higher bond seniority also implies a higher expected recovery. The volatility, measured by the standard deviation, instead does not seem to show a clear dependence. Regarding the distribution of recovery rate, Beta distribution is typically the most appropriate, since it assures a support between 0 and 1 (coherently with the constraint of loss rates) and its parameters can be defined according to mean and standard deviation.

2.2.2 CreditMetrics: a financial model for counterparty default risk

Most of the models for credit risk derives from financial literature, since it is the area where this kind of risk has the most relevance. Indeed, financial institutions, like banks, lend money to different counterparties with a certain (unknown) probability of default, which in turn could not return them back. For an insurance company instead this risk, in principle, is less relevant for its main business, since, thanks to inversion of the economic cycle, it finds themselves in the situation of borrower, rather than lander, against policyholders. In practice, however, counterparty default risk has a certain relevance also for insurance companies because, in their management of business, they also find themselves in the situation of lenders, for instance when they cede premiums to a third-party (e.g. a reinsurance company) in order to reduce part of their risks. CrediMetrics is one of the many methodology developed for assessing counterparty default risk.

The two main risks of a financial institution are market risk and credit risk, where the first one is defined as "the risk of losses arising from movements in market prices" and the second one as "the potential that a bank borrower or counterparty will fail to meet its obligations in accordance with agreed terms" [15].

Indeed, historically CrediMetrics follows the logic of another model also developed by JP Morgan, RiskMetrics, which had the objective of estimating the exposure to market risk. However, while the logic underlying the two models is similar, the approach followed for the development of CreditMetrics is different, because of the characteristics of counterparty risk compared to other risks, like the market one. Indeed, for assessing market risk there is "an abundance of daily liquid pricing data on which to construct a model of conditional volatility", while for counterparty risk the data is "relatively sparse and infrequently priced data on which to construct a model of unconditional volatility" [41]. Hence, the necessity of building a model which depends on characteristics not directly observable.

In the Technical Document of CreditMetrics the procedure employed by the model for assessing the counterparty risk is summarized as reported below: "*looks to a horizon and constructs a distribution of historically estimated credit outcomes (rating migrations including potentially default). Each credit quality migration is weighted by its likelihood (transition matrix analysis). Each outcome has an estimate of change in value (given by either credit spreads or studies of recovery rates in default). We then aggregate volatilities across the portfolio, applying estimates of correlation*" [41].

Regarding the rating of firms and their related probability of default, CreditMetrics does not make any assessment, but it applies the credit quality provided by the specific rating providers.

At this point, having the possibility to associate a rating to each firm, CreditMetrics construct a "transition matrix" for rating, based for instance of the available from Standards&Poor's [76]. This matrix reports the transition probability of a firm from a given rate to another, with default being just of the possible states. It should be noted that this transition matrix is estimated in such a way to guarantee coherence between the time interval of the matrix and the horizon over which to estimate the event (change in rating, default). CreditMetrics, by means of transition matrix rather than just probability of default, assures to consider not only the risk related to the default event, but also the one from downgrades.

Using the empirical evidences of the relation between bond seniority and recovery rate reported in [22] and [2], the model determines the parameters of the Beta distribution of recovery rate as a function of the seniority of the bond.

A last analysis necessary in order to perform a valuation of counterparty risk at portfolio level consists in the assessment of the correlation between counterparty at credit quality level. CreditMetrics, following Merton option theoretic mode, proposes an approach which bases the changes in credit rating with asset value. Indeed, in a broad sense, the value of a firm assets is connected with the ability to pay debit (actually it depends on many factor, first of all the type of asset, their liquidity, etc.). Hence, it is assumed that the distribution of the assets value of a firm can be divided in many ranges, each one representing a credit rating, where the values of the thresholds are linked with the probabilities of the transition matrix. In this way, given an appropriate model describing the change in asset value over the time horizon of interest it is possible to determine the credit rating of a firm. In the specific case of CreditMetrics the assumption for modeling the asset value is that the asset return is normally distributed.

2.2.3 Common shock approach: a counterparty default risk model in insurance

In the insurance context, a great part of the literature on counterparty risk derives from the works developed in the different Quantitative Impact Studies (QIS) for the definition of the structure of Solvency II. The approaches developed can be divided in three macro-areas: derived from bank credit risk modeling, based on dynamic financial analysis (DFA) and based on common shock.

To the first category belongs the work developed in [79], which was also one of the first approaches proposed for the default risk module of Solvency II during QIS3 and QIS4. This work, following the credit risk modeling developed for banks under Basel II (see [14]) was based on the Vasicek portfolio model, which, however has the strong drawback of being a limiting form. Hence, while it proves to be valid in case of a large number of firms, it shall not be applied in the insurance case, where the number of reinsurer is small and heterogeneous.

One example of the second approach for modeling the reinsurance default risk, the dynamic financial analysis, is represented by the work developed in [17], which propose a unifying stochastic approach to modeling the reinsurance credit risk in a DFA environment.

Here, coherently with the model chosen already from QIS5 and, with some modification, in the final version of the Standard Formula (SF) of the counterparty default risk module, we will present the common shock approach. The seminal work behind its application in Solvency II consists in the work of ter Berg in [78].

The logic of the common shock approach in a certain sense comes from studies of default likelihood at macro-economic level. Indeed, some studies have shown that there exists a correlation between the aggregate default likelihood and some macro-economic measure, like business cycle, GDP, etc.

Following the empirical finding which relates the probability of default with macroeconomic variable, the common shock approach assumes that there exists a common shock that affects the probability of default of the reinsurers and that, being common to all the market, also their correlation.

More formally we describe the common shock, which affects all the reinsurers in the scope, as a random variable distributed according to a distribution with domain from 0 to 1. Hence, the approach suggested in [78] is to model the common shock variable as a special case of Beta distribution with monotone decreasing probabilities, with the implicit assumption that shocks of increasing size are less and less likely. Mathematically, this is expressed by the probability density function (pdf) reported in Formula (2.33)

$$f(s) = \alpha s^{\alpha-1} \quad 0 < s < 1 \qquad 0 < \alpha < 1\,. \tag{2.33}$$

The practical meaning of this formulation, as anticipated, is that small shocks have (a certain) high probability, which declines for shocks of greater intensity. The parameter α governs the speed of decay of the probability.

Having defined the common shock as an element affecting all reinsurers, the effect is that the probability of default of each reinsurer is driven by the common shock, creating in this way a dependence between the reinsurers.

In order to formally define this connection, in [78] it is proposed to assume that each insurer has a "baseline" probability of default (connected with its characteristics) and that the common shock influences the "shock-modified default probability" according to the modeling defined in Formula (2.34)

$$p(s) = b + (1-b)s^{\tau/b} \quad 0 < b < 1 \qquad \tau > 0 \tag{2.34}$$

where b represents the baseline probability of default and τ is a shape parameter governing the intensity of the shock impact.

It is possible to notice that, under this modeling approach, the shock effect depends on b. In particular, the lower b, the lower the shock. This characteristic is well-explained by ter Berg, saying that "*reinsurance companies with low baseline default probabilities will be rather immune to shocks as long as these remain non-extreme, whereas large baseline levels increase the sensitivity for shocks even if these are of modest size.*" [78]. Moreover, the ratio α/τ determines the difference between the observed and baseline probability of default, increasing the difference for higher ratios.

At this point it is possible to calculate the expected default probability as the expected value of the "shock-modified default probability" over the shock sizes.

$$p = \mathbb{E}\left[p(\tilde{S})\right] = \int_0^1 p(s)f(s)ds = \frac{(\tau+\alpha)b}{\tau+\alpha b} = \frac{(\tau/\alpha+1)b}{1+\tau/\alpha b}$$

where $\tilde{S}$ is the random variable defining the shock size, with pdf defined in Formula (2.33).

The idea at this point is that the value of p can be obtained from external rating agencies and consequently the baseline default probability derived according to Formula (2.35)

$$b = \frac{\tau p}{\alpha(1-p)+\tau} = \frac{\tau/\alpha p}{(1-p)+\tau/\alpha}\,. \tag{2.35}$$

Formula (2.35) shows that, under these modelling assumptions, the baseline default probability of a firm depends on three elements: the observable (from rating agencies) probability and two parameters governing the intensity of the impact of shocks, α and τ. Consequently, these will be the elements that will also affect the variability of the loss in a multiple reinsurance case.

Indeed, an insurer is typically exposed to counterparty default risk not just from one, but from many reinsurers. Hence, it is possible to define the total loss, arising from the whole credit portfolio, according to the following equation:

$$\tilde{L} = \sum_{r=1}^{R} LGD^{(r)}\tilde{I}^{(r)}$$

where R represents the number of reinsurers to which the insurer is exposed to counterparty default, $LGD^{(r)}$ the loss given default of the r-th reinsurer and $\tilde{I}^{(r)}$ the random variable governing the default event of the r-th reinsurer.

The first two moments of the credit portfolio loss are equal to Formula (2.36) and Formula (2.37):

$$\mathbb{E}\left[\tilde{L}\right] = \mathbb{E}\left[\sum_{r=1}^{R} LGD^{(r)}\tilde{I}^{(r)}\right] = \sum_{r=1}^{R} LGD^{(r)}\mathbb{E}\left[\tilde{I}^{(r)}\right] = \sum_{r=1}^{R} LGD^{(r)}p^{(r)}\,, \tag{2.36}$$

$$Var\left[\tilde{L}\right] = Var\left[\sum_{r=1}^{R} LGD^{(r)}\tilde{I}^{(r)}\right] = \sum_{r=1}^{R}\sum_{s=1}^{R} LGD^{(r)}LGD^{(s)}\sigma_{r,s} \tag{2.37}$$

with

$$\sigma_{r,r} = p^{(r)}\left(1-p^{(r)}\right) \tag{2.38}$$

and

$$\sigma_{r,s} = \frac{\alpha\left(1-b^{(r)}\right)\left(1-b^{(s)}\right)}{\alpha+\tau\left(b^{(r)}\right)^{-1}+\left(\tau b^{(s)}\right)^{-1}} - \left(p^{(r)}-b^{(r)}\right)\left(p^{(s)}-b^{(s)}\right)\,. \tag{2.39}$$

Here, having determined mean and variance of the loss distribution, it is possible to use some approximations in order to determine the capital requirement for counterparty default risk. In his work, ter Berg suggests that the simplest assumption would be to use the moments obtained from the model and apply the 99.5% quantile of the normal distribution for determining the capital requirement for Solvency II scopes

$$RC_{def} = \Phi^{-1}(99.5\%) \sqrt{Var\left[\tilde{L}\right]}.$$

In practice, this is a "first-order approximation", which assumes a normal distribution of this variable. This assumption was then refined during QIS5, where, in Appendix A.10 of [28], was specified that:

> *"The capital requirement charge for counterparty default risk is the product of a quantile factor q and the square root of the variance, whereby the results is subject to a ceiling by the sum of the loss given default."*

$$SCR_{def} = \min\left(\sum_{i=1}^{k} y_i, q\sqrt{V}\right)$$

where y_i represents the loss given default of the i-th counterparty and V the variance of the credit portfolio loss[9].

In the Standard Formula of Solvency II, it has undergone a further change, leading to the approach reported below:

$$SCR_{def} = \begin{cases} 3\sigma & \sigma \leq 7\% \\ 5\sigma & 7\% < \sigma \leq 20\% \\ \text{LGD} & \sigma > 20\% \end{cases}$$

In addition to the approach defined in the Standard Formula of Solvency II for estimating the capital requirement for default risk, we can propose 2 alternatives, one based on a distributional assumption and the other on a simulative approach[10]:

(i) LogNormal distributional assumption of total losses.

(ii) Simulation of default (and eventually recovery) events.

Under approach (i) we follow the same procedure described in the common shock approach for the estimation of the mean and variance parameters of the loss random variable. Then, instead of using the approach described in Solvency II Standard Formula, we assume a LogNormal distribution and derive the capital requirement accordingly, as reported in the formula below.

[9] Note that here we have also reported the formula with the notation used in [28], that is why it is not coherent with the one used in the rest of the thesis.

[10] Starting from the models presented in QIS and Standard Formula of Solvency II, in [44] the authors propose other approaches for the estimation of the loss due to counterparty default risk, based on both closed formulas and simulation approaches.

$$SCR_{def} = q_{99.5\%}\left(\tilde{L}\right) - \mathbb{E}\left[\tilde{L}\right]$$

where q represents the quantile function of LogNormal distribution.

Also in this case it is appropriate to constraint the capital required such that it is at most equal to the sum of the loss given default.

Approach (ii) is based on Monte Carlo simulation, so it does not make any distributional assumption on total loss, but it uses the data of each counterparty to simulate a high number of scenarios and consequently estimate the capital requirement. In particular, we use the probability of default of each counterparty to simulate their default events. Then, in case we assumed that also the recovery rate is a stochastic variable, we simulate the random recovery rate in the event of default to derive the corresponding loss given default. Summing the total LGD in each simulation provides us the empirical distribution of the total loss due to default of the counterparty. Hence, the capital required is derived by means of the formula below, where the estimators of quantile and mean of the distribution are obtained by means of the Monte Carlo simulation approach just described

$$SCR_{def} = \hat{q}_{99.5\%}\left(\tilde{L}\right) - \hat{\mathbb{E}}\left[\tilde{L}\right] .$$

2.3 Solvency II

2.3.1 General framework of Solvency II

Solvency II is the latest regulation on the European insurance sector. The legislative reference is the Legislative Decree 138/2009 (in force from 1/1/2016) [34], which was subsequently modified in some of its aspects until reaching the current version of 30/06/2021. Its main objective is the protection of policyholders and beneficiaries, with financial stability and fair and stable markets as secondary objectives, which shall be reached without prejudice of the main one. This new regulation drastically modifies the previous system in force, with the objective of solving its drawbacks and trying to give harmonized rules, at European level, for the regulation and supervision of insurance and reinsurance companies.

The general framework of Solvency II is based on a three-pillar approach consisting of: *Quantitative Requirements*, *Governance requirements and Supervisory review process* and *Reporting and disclosure requirements.*

The first pillar deals with the quantitative requirements which an insurance company has to comply to. One of the main elements of this pillar consist in the definition of Solvency Capital Requirement (SCR) and Minimum Capital Requirement (MCR). These two elements represent the required and the minimum[11] capital that the company shall hold in order to be able to pursue the activity, calibrated ensuring that all quantifiable risks to which the firm is exposed are taken into account.

[11] They are to be intended as the minimum capital necessary under normal circumstance and the minimum capital necessary before the activation of the supervisory authority.

Another important element is represented by the definition of a specific balance sheet, the so-called Solvency II Economic Balance Sheet, necessary for the calculation of the capital, according to Solvency II regulation. In this way it is possible to assess the solvency position of the insurance company through the Solvency Ratio (SR).

The second pillar regards to the qualitative aspects of the insurance business. In fact, it is not enough to look only at the quantitative aspects, but it is also necessary to look at the qualitative elements. The aim is to improve the internal aspects of an insurance company, such as management, etc. The main aspect of this pillar regards the definition of four key functions and their respective activities, which are necessary for the pursuing of the insurance activity and the requirements for the management of the company. Moreover, it also sets out a new process for the internal assessment of the own risks and solvency, called Own Risk and Solvency Assessment (ORSA). This process must be an integral part of the business strategy, shall be performed periodically and after any significant change in the risk profile.

The third pillar regards the requirements of reporting and disclosure both to the market and to the Supervisory Authority, in the form of periodic or extraordinary information. In particular, one of the main public report is the Solvency Financial Condition Report (SFCR), published yearly with a relevant number of detailed information on the insurance business, performance, governance and solvency position. The rationale of Pillar III is to improve the transparency and push companies to a better management of the risks in order to not decrease their market image.

2.3.2 The Economic Balance Sheet and valuation of technical provisions

The Economic Balance Sheet prescribed by Solvency II aims at a market consistent valuation of assets and liabilities, which means evaluating assets and liabilities by means of their market value. More specifically, Articles 75, 1.(a) and 1.(b) of the Solvency II Directive [36] establish that:

> *"Assets shall be valued at the amount for which they could be exchanged between knowledgeable willing parties in an arm's length transaction"*

> *"Liabilities shall be valued at the amount for which they could be transferred, or settled, between knowledgeable willing parties in an arm's length transaction"*

The most relevant elements in the liability side of the balance sheet are the technical provisions. However, for these elements it is not possible to directly apply the definitions of Articles 76, 1.(b) for their valuation. Hence, Article 76, 2 of the directive states that:

> *"The value of technical provisions shall correspond to the current amount insurance and reinsurance undertakings would have to pay if they were to transfer their insurance and reinsurance obligations immediately to another insurance or reinsurance undertaking"*

This definition is coherent with the current exit value, which means that the technical provision is the price that an insurer would have to pay to another party in order to transfer the risk related to the liability. Hence, it means that the market consistent valuation of a technical liability can be quantified by mean of its current exit value.

In the context of Solvency II we have different methodologies for the valuation of liability depending if they are hedgeable, and then can be replicated reliably using financial instruments, or not.

In case of hedgeable liabilities, since they can be fully replicated by means of financial instruments, it is possible to use their market value for the valuation. However only a limited portion of liabilities are hedgeable, and mostly in the life business (e.g. unit and index-linked, in particular when they do not provide any guarantees).

Non-hedgeable liabilities are the greatest portion of liabilities. For them, Solvency II directive, in Article 77, 1. [34], prescribes that: *"The value of technical provisions shall be equal to the sum of a best estimate and a risk margin [...]"*

Best estimate

The first element of technical provision, best estimate, is defined in Article 77, 2. [34] as:

> *"The best estimate shall correspond to the probability-weighted average of future cash-flows, taking account of the time value of money (expected present value of future cash-flows), using the relevant risk-free interest rate term structure. The calculation of the best estimate shall be based upon up-to-date and credible information and realistic assumptions and be performed using adequate, applicable and relevant actuarial and statistical methods.[...]"*

In practice, the best estimate shall consider all the cash inflow and outflow related to the insurance obligation over all its lifetime.

A general formula for describing the best estimate can be represented by (2.40):

$$BE = \sum_{t \geq 1} \frac{\mathbb{E}\left[CF_t\right]}{\left(1 + r(0,t)\right)^t} \tag{2.40}$$

where CF_t represents the cashflow at time t and $r(0,t)$ is the spot (risk-free) rate between time 0 and time t.

This formula, as anticipated, represents the expected present value of the future cash flows, related to the insurance obligation. Indeed, the numerator represents the expected value of the cashflows at each future time and the denominator the discounting effect for each time period.

The valuation of best estimate shall be performed gross of reinsurance, with these elements calculated separately, and taking into account the counterparty default probability. The best estimate net of reinsurance then is simply obtained by the difference between the gross best estimate and the expected recoverables from reinsurance:

$$BE^{net} = BE^{gross} - \text{recoverables}$$

with the latter adjusted for taking into account also the expected losses due to default of the counterparty.

In non-life business, the element at numerator of Formula (2.40), the expected cashflows, consists simply in the expectation of the future (stochastic) amounts related to the insurance obligation.

For the denominator of (2.40), consisting of the discounting component of the best estimate, it is necessary to provide a more detailed description, since it is specifically described in the regulation.

In Solvency II framework discounting rates are provided every month by the European Insurance and Occupational Pensions Authority (EIOPA) at market level, which means that all the insurers apply the same curve for discounting their cashflows. More specifically, there is a different curve for each currency and the insurers are required to apply the currency-specific curve depending on the currency of their assets/liabilities.

The three main elements of Solvency II discounting rates are represented by: Basic risk-free interest rates term structure, Volatility adjustment (VA) and Matching adjustment (MA).

The first element represents the basic risk-free curve, while the others represent two possible adjustments that can be applied to the base curve in order to obtain the curve for discounting cashflows. The adjustments consist in a spread added to the basic risk-free rate, which aims at reducing the artificial volatility caused by a stressed situation in the financial market. The theoretical justification is that, when the market is stressed, there could be a situation of low liquidity which typically leads to an increase in the interest rates. Hence, the adjustment tries to replicate this effect also on the liability side by introducing a liquidity spread. An insurer can decide to use either volatility or matching adjustment, but it cannot use both for the same portion of business. Instead it is possible to use volatility adjustment for a portion of the business and matching for the others.

As described in “Technical documentation of the methodology to derive EIOPA’s risk-free interest rate term structures” [33] the calibration of the basic risk-free interest rate term structure is based on swap rates. The decision of using swap rates instead of government bonds was taken in order to avoid possible macroeconomic effects, but it requires additional adjustments in order to remove the effect of credit risk in swap rates and obtain the risk-free component.

The procedure for defining the basic risk-free interest rate term structure is composed of three steps. The first one consists in calibrating the risk-free rate by using the financial swap available in the market until a given maturity, called last liquid point (LLP), which represents the last maturity where there is a significant amount of swap in the market in order to have a calibration that is consistent form a statistical point of view.

Hence, in order to define the term structure also for maturities where there are not swap rates available in the market, two other steps are necessary.

The second step consists in the definition and calculation of the ultimate forward rate (UFR), the convergence point and the convergence tolerance.

Then, it is used an extrapolation procedure, based on Smith-Wilson methodology, for estimating the rates for the periods after the last liquid point until convergence to the ultimate forward rate.

The market consistent methodology for the valuation of technical provision introduced by Solvency II also creates an additional volatility in their estimation, which affects both Solvency Capital Requirement and Own Funds. In this way, we have a full-fair-value balance sheet, but with Solvency Ratio (both numerator and denominator) that could vary over time in a significant way just due to a change in the discounting curve. Clearly, this change will be strictly related to the duration of the liabilities.

The volatility adjustment aims at mitigating the effect of a stressed situation in the financial market by providing the same, or a similar, effect in the discounting curve used to discount liabilities.

The volatility adjustment is provided by EIOPA and is updated over time according to the financial conditions, increasing in stressed situations and decreasing in stable periods. Differently from the basic risk-free rate interest rate term structure, volatility adjustment depends on both currency-specific spread and country-specific spread. The general procedure for obtaining the volatility adjustment is described in [33]. This adjustment is added to the liquid part of the risk-free rate curve, which is the part up to the last liquid point. After that also the curve obtained adding the volatility adjustment converges to the same ultimate forward rate of the risk-free rate one.

The matching adjustment is a parallel shift applied to the entire basic risk-free term structure and serves the same purpose as the volatility adjustment. It is an entity specific adjustment, not based on a reference portfolio but on the specific portfolio of the insurer, and requires approval from the supervisory authority. Similarly to the volatility adjustment, also in this case it depends on a risk adjusted spread, but the two components are the annual effective rate of the portfolio of assigned assets and the annual effective basic risk-free rate. It is then adjusted by applying a risk correction, provided by EIOPA, that takes into account the credit spread. Differently from volatility adjustment, it is applied to all the basic risk-free curve without any distinction between periods before and after the last liquid point.

In order for a company to apply the matching adjustment, it is necessary that the portfolio satisfies several very strict constraints. In particular, Article 77b [35] defines a list of conditions that shall be met in order to apply for matching adjustment. Among these conditions, letter (e) establishes that: *"The only underwriting risks connected to the portfolio of insurance or reinsurance obligations are longevity risk, expense risk, revision risk and mortality risk"* which, in practice, excludes the application of matching adjustment for non-life technical provisions. Also for life policies the application is limited to a very low portion of the business of the insurer, which generally correspond to annuities and in particular to only with-profit annuities.

Risk margin

Articles 77, 3. and 5. of Solvency II Directive [34] provide the definition of risk margin as:

> *"The risk margin shall be such as to ensure that the value of the technical provisions is equivalent to the amount that insurance and reinsurance undertakings would be expected to require in order to take over and meet the insurance and reinsurance obligations. Where insurance and reinsurance undertakings value the best estimate and the risk margin separately, the risk margin shall be calculated by determining the cost of providing an amount of eligible own funds equal to the Solvency Capital Requirement necessary to support the insurance and reinsurance obligations over the lifetime thereof.[...]"*

From this definition, we derive that the general criterion for computing risk margin is based on the cost of capital approach. Article 37 of the Delegated Acts [36] provides an explicit formula for its computation:

$$RM(P) = \sum_{t \geq 0} CoC \frac{SCR_t^{RM}}{(1 + r(0, t+1))^{t+1}} \tag{2.41}$$

where P indicates that the risk margin is calculated at portfolio level.

The elements present in the formula represent CoC the cost of capital rate, SCR_t^{RM} the Solvency capital requirement at time t and $r(0, t+1)$ the risk-free rate between 0 and $t+1$. In practice, we have a sum of the expected discounted value of cost of capital over the future years. In Formula (2.41) it is not specified the last year of computation of the SCR since it will depend on the year in which we have the complete settlement of all the existing liabilities.

The current percentage of cost of capital rate CoC provided by EIOPA, and equal for all the market, is 6% (Article 39 [35]). It shall be noted that in the calculation of risk margin we discount only by means of the risk-free rate, without applying any volatility/matching adjustment. Another important point is that the value of the SCR_t^{RM} is discounted by using a discounting factor at $t+1$, which implicitly assumes that the capital requirement is due at the end of the year, and leads to a lower RM respect to the case of requirement at the beginning of the year (in case of positive risk-free rate).

The Solvency capital requirement SCR_t^{RM} present in the formula is different from the one used for capital requirement purposes since it considers only a specific subset of risks, as prescribed in Article 38 (i) of the Delegated Regulation [36]:

> *"the Solvency Capital Requirement of the reference undertaking captures all of the following risks:*
>
> > *(i) underwriting risk with respect to the transferred business,*
> > *(ii) where it is material, the market risk referred to in point (h), other than interest rate risk,*

> *(iii) credit risk with respect to reinsurance contracts, arrangements with special purpose vehicles, intermediaries, policyholders and any other material exposures which are closely related to the insurance and reinsurance obligations,*
> *(iv) operational risk;"*

The insurer should compute the risk margin separately for each liability. However, in order to reduce the complexity, Guidelines 63 [32] establish the possibility of computing the risk margin considering the whole SCR and then splitting it between lines of business (LoB), obtaining an implicit diversification benefit. The only diversification that is not allowed is between life and non-life business.

Hence, the total risk margin computed as a whole is split according to the different lines of business, in order to obtain the total amount of technical provision for each line of business. As a general approach, the allocation of risk margin can be done in proportion of the SCR, as illustrated in Formula (2.42), or, as an extreme case, in proportion of the best estimate

$$RM(LoB) = \frac{SCR_{RU,LoB}(0)}{\sum\limits_{LoB} SCR_{RU,LoB}(0)} RM(P)\,. \tag{2.42}$$

Article 58 of the Delegated Acts [36] specify that it is possible to use simplified approaches for the calculation of risk margin in order to avoid the complexity coming from the computation of future solvency capital requirements as follow:

> *"Without prejudice to Article 56, insurance and reinsurance undertakings may use simplified methods when they calculate the risk margin, including one or more of the following:*
>
> *(a) methods which use approximations of the amounts denoted by the terms SCR(t) referred to in Article 37(1);*
> *(b) methods which approximate the discounted sum of the amounts denoted by the terms SCR(t) as referred to in Article 37(1) without calculating each of those amounts separately."*

In particular, Guideline 61 and Technical Annex IV of EIOPA guidelines on valuation of technical provisions [32], transposed into Ivass regulation [47], defines four simplified approaches, ordered by degree of simplification, for the calculation of risk margin. In practice they consist in:

- **Method 1:** Approximate the individual risks or sub-risks within some or all modules and sub-modules to be used for the calculation of future Solvency Capital Requirements.
- **Method 2:** Approximate the whole Solvency Capital Requirement for each future year by using the ratio of the best estimate at that future year to the best estimate at the valuation date.
- **Method 3:** To approximate the discounted sum of all future Solvency Capital Requirements in a single step without approximating the Solvency Capital

Requirements for each future year separately by using the modified duration of the insurance liabilities as a proportionality factor.

- **Method 4:** To approximate the risk margin by calculating it as a percentage of the best estimate.

In non-life business Method 2 is the most used, which can be written in formula as:

$$SCR_t = SCR_0 \frac{BE_t^{net}}{BE_0^{net}}$$

where BE^{net} represents the best estimate of liabilities, net of reinsurance.

The underlying assumption of this method is that the future SCR of existing business reduces over time with the same pattern of best estimate.

Non-life technical provisions: premium and claims reserve

The two technical provisions of a non-life insurance company are premium and claims provisions[12].

Premium reserve is the amount that an insurance company needs to hold in order to cover future claims for existing contracts.

As already explained, in Solvency II the computation of best estimate depends on the discounted expected value of the future cash in- and out-flows related to the specific provision.

The three main *cash outflows* that the insurer needs to evaluate in order to compute the premium reserve are: future claims for existing policies, loss adjustment expenses and management costs.

The main *cash inflows* that the insurer needs to evaluate in order to compute the premium reserve are future premiums (net of acquisition costs) for existing annual and multi-annual contracts.

In [32] Technical Annex III is described also a simplified formula for the calculation of premium reserve under Solvency II, similar to the approach used under Italian globally accepted accounting principles (GAAP) balance sheet, defined as:

$$BE = CR \cdot VM + (CR - 1)PVFP + AER \cdot PVFP$$

where CR represents the Combined Ratio (without the run-off effect), VM is the Volume measure for unearned premiums (gross of acquisition expenses), $PVFP$ is the Present Value of Future Premiums (for existing contracts) and AER is the Acquisition Expenses Ratio.

The best estimate of claims reserve is obtained as the discounted expected value of payments and expenses for claims already incurred at the valuation date. The general approach consists in the application of claims reserving methods on a run-off

[12] In this thesis we use the term provision and reserve indifferently when referring to these two elements.

triangle in order to estimate ultimate cost and claims reserve. Hence, the approach used in Solvency II is similar to the Italian GAAP one, with the two main differences concerning discounting and recoverables. In fact, following the definition of best estimate, explicated in Formula (2.40), claims reserve is computed as the discounted expected value of the future cashflows, net of reinsurance recoverables.

According to the discounted cash outflows the insurer has to estimate the lower triangle using a deterministic method. In particular, in order to introduce the discounting effect, the insurer needs to have at his disposal the lower triangle of incremental payments.

According to the discounted expected value of cash inflows it is necessary to compute the recoverables that are the amounts that the insurer can recover from the policyholders. A recoverables is typically a phenomenon observed in motor; for instance, if the amount of the claim is greater than the policy limits, retention, usually the insurer pays all the claims. Then it has a credit to the policyholder whose amount is the recoverables. The same logic applies in case of deductible where the insurer pays and then asks the portion back to the policyholder. Since Solvency II asks for a valuation net of recoverables it is necessary to include this cash-outflow in the estimation of claims reserve.

2.3.3 Solvency Capital Requirement and Standard Formula

From the valuation of the elements in their Economic balance sheet, according to the prescriptions outlined in Section 2.3.2, insurance companies derive the value of their own funds. Own funds can be considered an element corresponding to the capital of a classic balance sheet. Insurance companies are required (Article 100 of [34]) to have enough eligible own funds to cover the Solvency Capital Requirement. This element represents the capital required, considering all the risks to which the company is exposed, for assuring the solvency of the company under a certain confidence level.

More formally, Article 101 3. of [34] provides the definition of Solvency Capital Requirement as follows:

> *"[...] It shall correspond to the Value-at-Risk of the basic own funds of an insurance or reinsurance undertaking subject to a confidence level of 99,5 % over a one-year period"*

Article 101 4. of [34] states that:

> *The Solvency Capital Requirement shall cover at least the following risks:*
>
> *(a) non-life underwriting risk;*
> *(b) life underwriting risk;*
> *(c) health underwriting risk;*
> *(d) market risk;*
> *(e) credit risk;*
> *(f) operational risk.*

Insurance companies have different alternatives for calculating the SCR:

- **Standard Formula:** risk map, aggregation approach, modules and parameters as established in the regulation.
- **Standard Formula with Undertaking Specific Parameters (USP):** as under Standard Formula approach, but with the application of company-specific parameters instead of standard ones, according to the prescriptions reported in Article 218 of [35]. Requires supervisory authority approval.
- **Partial Internal Model:** Internal Model developed for the calculation of the capital requirement for some specific modules, as under one of the previous two approaches for all the other modules. Requires supervisory authority approval.
- **Full Internal Model:** calculation of capital requirement fully based on Internal Model developed by the insurance company. Requires supervisory authority approval.

More specifically, the insurer has to calculate a specific SCR for each sub-module using a scenario-based approach or a factor-based approach, as established in the directive. Then, the sub-modules are aggregated by means of specific correlation matrices, defined in the directive, which then give rise to a diversification benefit. The modules obtained are then aggregated again by means of a correlation matrix, which also this time leads to a diversification benefit, obtaining the Basic Solvency Capital Requirement (BSCR). Finally, the "global" SCR of the insurance company is obtained as the sum of the BSCR, Operational risk OP_{risk} and the adjustment for the loss-absorbing capacity of technical provisions and deferred taxes Adj, as reported in Formula (2.43)

$$SCR = BSCR + OP_{risk} + Adj\,. \tag{2.43}$$

It is possible to observe that the capital requirement for operational risk is not diversified with any other module, but it is added on top to the BSCR.

The first term is the most relevant and consists of the capital requirement for all the modules except operational risk. It is computed according to Formula (2.44).

$$BSCR = \sqrt{\sum_{i,j} Corr_{i,j} SCR_i SCR_j} + SCR_{intangibles} \tag{2.44}$$

where SCR_i and SCR_j are the solvency capital requirements of modules i and j and $Corr_{i,j}$ is the correlation matrix between the two modules i and j as reported in Table 2.2. $SCR_{intangibles}$ represents the capital requirement for intangible asset risk, which is simply added to the other risks, and then does not provide any benefit in term of diversification.

In the analysis for the estimation of the multi-objective optimal reinsurance the two main risk modules that we will consider are non-life underwriting risk (focusing only on the premium and reserve risk sub-module) and counterparty default risk (focusing on the type 1). We will study the capital requirement both according to the Standard Formula approach and according to a Partial Internal Model for

Table 2.2. Correlation matrix between risk modules for BSCR, according to Annex IV of [34].

$\mathbf{Corr}_{i,j}$	Market	Default	Life	Health	Non-life
Market	1	0.25	0.25	0.25	0.25
Default	0.25	1	0.25	0.25	0.5
Life	0.25	0.25	1	0.25	0
Health	0.25	0.25	0.25	1	0
Non-life	0.25	0.5	0	0	1

Table 2.3. Correlation matrix between sub-modules of Non-life underwriting risk, according to Article 114 of [36].

	Non-life premium and reserve risk	Non-life catastrophe risk	Non-life lapse risk
Non-life premium and reserve risk	1	0.25	0
Non-life catastrophe risk	0.25	1	0
Non-life lapse risk	0	0	1

underwriting risk (premium and reserve risk component) and counterparty default risk.

2.3.4 Non-life underwriting risk

The non-life underwriting risk considers the risks arising from the underwriting of non-life insurance and reinsurance contracts. It is composed by three sub-modules: non-life premium and reserve risk, non-life catastrophe risk and non-life lapse risk.

The capital requirement is calculated as:

$$SCR_{non-life} = \sqrt{\sum_{i,j} \left(CorrNL_{(i,j)} SCR_i SCR_j \right)}$$

where SCR_i and SCR_j denotes the capital requirements for the sub-modules i and j and $CorrNL_{(i,j)}$ denotes the correlation parameter between sub-modules i and j as reported in Table 2.3.

Non-life premium and reserve risk

Premium and reserve risk sub-module estimates *"the risk of loss, or of adverse change in the value of insurance liabilities, resulting from fluctuations in the timing, frequency and severity of insured events, and in the timing and amount of claim settlements"* (Article 105 2. (a) [34])

The capital requirement is calculated as reported in Formula (2.45) (Article 115 of [35]):

$$SCR_{nl\ prem,res} = 3\sigma_{nl}V_{nl} \tag{2.45}$$

where σ_{nl} represent the standard deviation parameter and V_{nl} the volume measure.

The volume measure for non-life premium and reserve risk V_{nl} is defined as the sum of the volume measures for premium risk and reserve risk of all the segments set out in Table 2.4, where the volume measure of a generic segment s is defined as:

$$V_s = (V_{(prem;s)} + V_{(res;s)}) * (0.75 + 0.25 * DIV_s) \tag{2.46}$$

with $V_{(prem;s)}$ the volume measure for premium risk, $V_{(res;s)}$ the volume measure for reserve risk and DIV_s the factor for geographical diversification.

As defined in Article 116 3. of [35]:

> *For all segments set out in Annex II, the volume measure for premium risk of a particular segment s shall be equal to the following:*
>
> $$V_{(prem;s)} = \max[P_s; P_{(last;s)}] + FP_{(existing;s)} + FP_{(future;s)}$$
>
> *where:*
>
> *(a) P_s denotes an estimate of the premiums to be earned by the insurance or reinsurance undertaking in the segment s during the following 12 months;*
> *(b) $P_{(last,s)}$ denotes the premiums earned by the insurance or reinsurance undertaking in the segment s during the last 12 months;*
> *(c) $FP_{(existing,s)}$ denotes the expected present value of premiums to be earned by the insurance or reinsurance undertaking in the segment s after the following 12 months for existing contracts;*
> *(d) $FP(future, s)$ denotes the following amount with respect to contracts where the initial recognition date falls in the following 12 months[...]*

The volume measure for reserve risk of a particular segment s is instead defined as the net best estimate of the provision for claim outstanding.

The standard deviation for non-life premium and reserve risk is defined in Article 117 1. of [35] as:

$$\sigma_{nl} = \frac{1}{V_{nl}}\sqrt{\sum_{s,t} CorrS_{(s,t)}\sigma_s V_s \sigma_t V_t}$$

where V_{nl} represents the volume measure for non-life premium and reserve risk, $CorrS_{(s,t)}$ the correlation parameter for non-life premium and reserve risk for segment s and segment t as defined in Table 2.5, σ_s and σ_t the standard deviations for non-life premium and reserve risk of segments s and t depending on the aggregation between

Table 2.4. Segmentation of Non-Life insurance lines of business, according to Annex II of [36].

Number	**Segment**
1	Motor vehicle liability insurance and proportional reinsurance
2	Other motor insurance and proportional reinsurance
3	Marine, aviation and transport insurance and proportional reinsurance
4	Fire and other damage to property insurance and proportional reins.
5	General liability insurance and proportional reinsurance
6	Credit and suretyship insurance and proportional reinsurance
7	Legal expenses insurance and proportional reinsurance
8	Assistance and its proportional reinsurance
9	Miscellaneous financial loss insurance and proportional reinsurance
10	Non-proportional casualty reinsurance
11	Non-proportional marine, aviation and transport reinsurance
12	Non-proportional property reinsurance

Table 2.5. Correlation matrix between Non-Life premium and reserve risk, according to Annex IV of [36].

Corr	1	2	3	4	5	6	7	8	9	10	11	12
1	1.00											
2	0.50	1.00										
3	0.50	0.25	1.00									
4	0.25	0.25	0.25	1.00								
5	0.50	0.25	0.25	0.25	1.00							
6	0.25	0.25	0.25	0.25	0.50	1.00						
7	0.50	0.50	0.25	0.25	0.50	0.50	1.00					
8	0.25	0.50	0.50	0.50	0.25	0.25	0.25	1.00				
9	0.50	0.50	0.50	0.50	0.50	0.50	0.50	0.50	1.00			
10	0.25	0.25	0.25	0.25	0.50	0.50	0.50	0.25	0.25	1.00		
11	0.25	0.25	0.50	0.50	0.25	0.25	0.25	0.25	0.50	0.25	1.00	
12	0.25	0.25	0.25	0.50	0.25	0.25	0.25	0.50	0.25	0.25	0.25	1.00

Table 2.6. Market-wide volatility factors for the Non-Life premium and reserve risk sub-module, according to Annex II of [36].

Segment	Premium risk volatility factors	Reserve risk volatility factors
Motor vehicle liability insurance and proportional reinsurance	10%	9%
Other motor insurance and proportional reinsurance	8%	8%
Marine, aviation and transport insurance and proportional reinsurance	15%	11%
Fire and other damage to property insurance and proportional reinsurance	8%	10%
General liability insurance and proportional reinsurance	14%	11%
Credit and suretyship insurance and proportional reinsurance	12%	19%
Legal expenses insurance and proportional reinsurance	7%	12%
Assistance and its proportional reinsurance	9%	20%
Miscellaneous financial loss insurance and proportional reinsurance	13%	20%
Non-proportional casualty reinsurance	17%	20%
Non-proportional marine, aviation and transport reinsurance	17%	20%
Non-proportional property reinsurance	17%	20%

premium and reserve risk that will be detailed below, and V_s and V_t the volume measures for premium and reserve risk of segments s and t as defined in 2.46.

For a generic segment s the standard deviation for non-life premium and reserve risk is defined in Article 117 2. of [35] as:

$$\sigma_s = \frac{\sqrt{\sigma^2_{(prem,s)}V^2_{(prem,s)} + \sigma_{(prem,s)}V_{(prem,s)}\sigma_{(res,s)}V_{(res,s)} + \sigma^2_{(res,s)}V^2_{(res,s)}}}{V_{(prem,s)} + V_{(res,s)}}$$

where $\sigma_{(prem,s)}$ and $\sigma_{(res,s)}$ represents the standard deviations for non-life premium risk and reserve risk respectively, as defined in 2.6, $V_{(prem,s)}$ and $V_{(res,s)}$ the volume measure for premium risk and reserve risk respectively.

Non-life catastrophe risk

Catastrophe (CAT) risk is *"the risk of loss, or of adverse change in the value of insurance liabilities, resulting from significant uncertainty of pricing and provisioning assumptions related to extreme or exceptional events"* ((Article 105 2. (b) [34])

The capital requirement is calculated as reported in Formula (2.47) (Article 119 of [35]):

$$SCR_{nl\ CAT} = \sqrt{(SCR_{nat\ CAT} + SCR_{np\ property})^2 + SCR_{mm\ CAT} + SCR_{other\ CAT}} \tag{2.47}$$

where $SCR_{nat\ CAT}$ represents the SCR for risks related to natural catastrophe, $SCR_{np\ property}$ to non-proportional property reinsurance catastrophe, $SCR_{mm\ CAT}$ to man-made catastrophe and $SCR_{other\ CAT}$ to other non-life catastrophe.

It is possible to observe that in Formula (2.47) we have an underlying assumption of full correlation between $SCR_{nat\ CAT}$ and $SCR_{np\ property}$ and full independence between the resulting SCR and $SCR_{mm\ CAT}$ and $SCR_{other\ CAT}$.

The approaches for computing each capital requirement are described from Article 120 to Article 135 of [35].

Non-life lapse risk

The lapse risk is the risk of negative impact on the insurance company related to policyholders' options eventually included in non-life policies, as the option to lapse the contract before maturity. In general it is really marginal or totally absent in non-life business, because it mainly regards multi-annual policies. In practice, this risk is negligible in lines like motor, while can be relevant in specific accident/property contracts.

The capital requirement for lapse risk is based on a scenario-based approach, where the insurer is asked to calculate the loss of basic own funds resulting from the combination of two instantaneous events (Article 118 [34]):

> *"(a) the discontinuance of 40% of the insurance policies for which discontinuance would result in an increase of technical provisions without the risk margin;*
> *(b) where reinsurance contracts cover insurance or reinsurance contracts that will be written in the future, the decrease of 40% of the number of those future insurance or reinsurance contracts used in the calculation of technical provisions."*

2.3.5 Counterparty default risk

According to Article 105 6. of [34] the counterparty default risk module *"shall reflect possible losses due to unexpected default, or deterioration in the credit standing, of the counterparties and debtors of insurance and reinsurance undertakings over the following 12 months.[...]"*.

The main elements considered by the counterparty default risk module are reinsurance treaties, securitisations, derivatives and receivables, but also all the credit exposures not covered in the spread risk sub-module. Moreover, it is specified that the undertaking shall consider the overall risk exposure towards his counterparty irrespective of the legal form of the contractual obligation. It means that the insurer has to consider the "substance" of the treaty rather than its "form", when assessing if it exposes him to a counterparty risk.

The capital requirement for counterparty default risk module is calculated as reported in Formula (2.48):

$$SCR_{def} = \sqrt{SCR^2_{(def,1)} + 1.5SCR_{(def,1)}SCR_{(def,2)} + SCR^2_{(def,2)}} \tag{2.48}$$

where: $SCR_{def,1}$ represents the capital requirement for counterparty default risk on type 1 exposures and $SCR_{def,2}$ on type 2 exposures.

Points 3.21 and 3.22 of [28] report the characteristics of the exposure for which they should be considered under type 1 or type 2. Specifically type 1 exposure should consist of *"exposures which may not be diversified and where the counterparty is likely to be rated"*, while type 2 of *"exposures which are usually diversified and where the counterparty is likely to be unrated".* As for the rating, Solvency II admits only the use of an external credit assessment *"only where it has been issued by an External Credit Assessment Institution (ECAI)"* (Article 4 of [32]).

Articles 189 2. and 3. of [35] translates these definitions in the list of exposure which shall be considered type 1 or type 2. In particular, the most relevant exposures of type 1 are represented by risk-mitigation contracts, including reinsurance arrangements, special purpose vehicles, insurance securitisations and derivatives, cash at bank and deposits with ceding undertakings. For type 2 exposures instead we have all the exposure not reported as type 1, where the most relevant are receivables from intermediaries, policyholder debtors and mortgage loans.

As reported in Article 200 of [35] the capital requirement for type 1 exposure is determined according to Formula (2.49).

$$SCR_{def,1} = \begin{cases} 3\sigma & \sigma \leq 7\% \\ 5\sigma & 7\% < \sigma \leq 20\% \\ \text{LGD} & \sigma > 20\% \end{cases} \tag{2.49}$$

where the standard deviation σ is equal to the squared root of the variance, which is in turn equal to the sum of two components, as reported in the following formula

$$\sigma = \sqrt{V} = \sqrt{V_{inter} + V_{intra}}\,.$$

This decomposition of the total variance between two components derives from the assumption of dependence from a common shock of the probability of default of each counterparty (see Section 2.2). V_{inter} and V_{intra} represent respectively the variance among counterparty having different rating and the variance between counterparty with the same rating.

In Formula (2.50) the first variance term is reported.

Table 2.7. Probability of default for each credit quality step by a nominated ECAI, according to Article 199 2. of [36].

Credit quality step	0	1	2	3	4	5	6
Probability of default (%)	0.002	0.01	0.05	0.24	1.20	4.20	4.20

$$V_{inter} = \sum_{(j,k)} \frac{\mathrm{PD}_k(1-\mathrm{PD}_k)\mathrm{PD}_j(1-\mathrm{PD}_j)}{1.25(\mathrm{PD}_k+\mathrm{PD}_j)-\mathrm{PD}_k\mathrm{PD}_j}\mathrm{TLGD}_k\mathrm{TLGD}_j \tag{2.50}$$

where, as anticipated, we have a sum covering all possible combinations of different probabilities of default, with TLGD_k and TLGD_j representing the sum of LGD on type 1 exposures from counterparties with probability of default PD_k and PD_j respectively.

In Formula (2.51) the second variance term is reported.

$$V_{intra} = \sum_{j} \frac{1.5\mathrm{PD}_j(1-\mathrm{PD}_j)}{2.5-\mathrm{PD}_j} \sum_{\mathrm{PD}_j} \mathrm{LGD}_i^2 \tag{2.51}$$

where we sum, for each group of probability of default, the variance of the given group.

For these type 1 exposures the approach prescribed by the regulation for calculating the probability of default consists in using the credit quality step (CQS) of the counterparty and derive the corresponding PD by means of the specific correspondence, as reported in Table 2.7. In case the counterparty is an insurance or reinsurance company, but for which it is not available a credit assessment by a nominated ECAI, then the approach suggested by the regulation consists in assigning a probability of default function of its Solvency Ratio, where the relation is reported in Table 2.8.

As reported in Article 202 of [35] the capital requirement for type 2 exposures shall be equal to the loss of basic own funds that would result from an instantaneous decrease in value of type 2 exposures by the amount defined in Formula (2.52).

$$SCR_{def,2} = 90\%\mathrm{LGD}_{receivables>\,3month} + 15\% \sum_{i} \mathrm{LGD}_i \tag{2.52}$$

where the first term, $\mathrm{LGD}_{receivables>\,3month}$, denotes the total LGD on all receivables from intermediaries which have been due for more than three months. The second term denotes instead the sum of LGD for all type 2 exposures other than receivables from intermediaries which have been due for more than three months.

$$LGD = \max\left[50\%\left(REcoverables + 50\%RM_{re}\right) - F\ Collateral, 0\right]\ . \tag{2.53}$$

Table 2.8. Probability of default for each Solvency Ratio when CQS by a nominated ECAI is not available, according to Article 199 3. of [36].

Solvency Ratio (%)	196	175	150	125	122	100	95	75
Probability of default (%)	0.01	0.05	0.1	0.2	0.24	0.5	1.20	4.20

2.3.6 Reinsurance effect in Solvency II Standard Formula

Solvency II Standard Formula takes into account the effect provided by reinsurance contracts in both technical provisions and capital requirement.

For the technical provisions, as explained in Section 2.3.2, both premium and claims provision are evaluated net of reinsurance. In this way we have a decrease of technical provisions, which produces an increase of the own funds and then a benefit for the solvency of the insurance company.

For the capital requirement, in Article 101 5. of [34] it is explicitly stated that:

> *"When calculating the Solvency Capital Requirement, insurance and reinsurance undertakings shall take account of the effect of risk- mitigation techniques, provided that credit risk and other risks arising from the use of such techniques are properly reflected in the Solvency Capital Requirement"*

which means that it is necessary to calculate both the positive effects of risk transfer and the negative effect of counterparty risk generated by risk-mitigation techniques, like reinsurance.

Considering a non-life company, the two main modules affected by reinsurance contracts are non-life underwriting risk module and counterparty default risk module.

In non-life underwriting risk module we have a relevant impact on the premium and reserve risk sub-module. In fact, we have that both the volume measure and standard deviation measure of $SCR_{nl\ prem,res} = 3\sigma_{nl}V_{nl}$ are reduced by reinsurance treaties.

In particular, volume measure is computed net of reinsurance for both its premium and reserve component, as stated in Article 116 5. and 6. of [35]:

> *"For the purposes of the calculations set out in paragraphs 3 and 4, premiums shall be net, after deduction of premiums for reinsurance contracts[...]"*

> *"For all segments set out in Annex II, the volume measure for reserve risk of a particular segment shall be equal to the best estimate of the provisions for claims outstanding for the segment, after deduction of the amounts recoverable from reinsurance contracts and special purpose vehicles[...]"*

The reduction of these volume measures is independent on the kind of reinsurance treaty, holding for both proportional and non-proportional reinsurance.

For the standard deviation measure, Article 117 3. of [35] states that:

> *"For all segments set out in Annex II, the standard deviation for non-life premium risk of a particular segment shall be equal to the product of the standard deviation for non-life gross premium risk of the segment set out in Annex II and the adjustment factor for non-proportional reinsurance. For segments 1, 4 and 5 set out in Annex II the adjustment factor for non-proportional reinsurance shall be equal to 80 %. For all other segments set out in Annex the adjustment factor for non-proportional reinsurance shall be equal to 100 %."*

It means that the standard deviation measure of the indicated segments for the premium risk is reduced by 20% in case the insurer has a non-proportional reinsurance treaty for those segments.

The rationale behind this adjustment factor is that non-proportional reinsurance removes a part of risk related to extreme events (usually the tail part of the loss distribution), decreasing in this way the volatility of the net-of-reinsurance distribution of the insurer and then its standard deviation parameter. Proportional reinsurance treaties, instead, as explained in Section 2.1.1, do not modify the moments of the loss distribution and then do not provide any benefit in term of volatility.

Another important characteristic of this adjustment is that regards only premium risk, since in practice it is not feasible to estimate the reduction of volatility in claims reserve from the presence of a non-proportional reinsurance treaty.

Reinsurance treaties also affect another module of the Solvency II Standard Formula, counterparty default risk, but in a opposite direction. In fact, the insurer, transferring a portion of its premiums to the reinsurer, becomes exposed to the risk of default of its counterparty. Hence, while it benefits from the reduction of the capital requirement related to underwriting risk, at the same time it also is impacted by the increase of capital requirement related to counterparty default risk.

Chapter 3

Risk reserve in presence of defaultable reinsurers

3.1 Introduction

In this chapter we present closed formulas for the first two moments of the risk reserve equation for a non-life insurer in a one-year time horizon. These closed formulas extend the results available in the literature to the case of an insurance company which cedes part of its risks to defaultable reinsurers, under the usual framework described for instance in [72]. In particular, we present three results which differentiates for the assumptions on the number of segments in which the insurance company underwrites business and the number of reinsurance counterparties to which risks are ceded.

More precisely, we start from a model which consider just the possibility for the insurer to underwrite risks in a single line of business and to cede risks to a single reinsurer. Then we extend this simple initial version to two possibilities: cede the risk of a single LoB to multiple reinsurers or cede the risks of multiple LoBs to a single reinsurer. The first extension permits to analyze the effect of the number of reinsurer, and their rating, on the risk reserve of the insurance company, given the single LoB in which it underwrites business. The second extension permits to analyze the effect of the lines of business, in term of number and “type”, on the risk reserve of the insurance company, given the single reinsurer to which it cedes the risks.

It shall be noted that there should be a final model, which considers the possibility for the insurance company to underwrite business in multiple LoBs and to cede risks to multiple reinsurers. In practice, however, we do not treat this case in the chapter, since it is not easily manageable in a theoretical setting. Indeed, given the multiple combinations of LoBs, reinsurers, ratings, threshold of the XL treaties, etc. it is more manageable in a simulative setting. Hence, we treat it, and actually extend it, in the following chapter, where we develop a simulative approach for the analysis of the risk reserve of an insurance company in presence of defaultable reinsurers.

Finally, we propose some numerical analyses for showing how these closed formulas can be used for evaluating different reinsurance strategies and derive an empirical efficient frontier, without the computational burden coming from simulations.

3.2 General modeling of risk reserve in non-life insurance

3.2.1 Risk reserve equation

In order to present the extended models developed in this thesis it is first necessary to provide a description of the general modeling of risk reserve and the main assumptions used.

Following the approach also employed in Solvency II, we are interested in modeling the stochastic capital[1] of an insurance company in a one-year time horizon. We start from defining in Formula (3.1) the "base" equation of this stochastic process, which considers only the randomness deriving from claims and do not allow for reinsurance treaties

$$\tilde{U}_{t+1} = U_t(1+j) + \left[\left(B_{t+1} - \tilde{X}_{t+1} - E_{t+1}\right)\right](1+j)^{1/2}\,. \tag{3.1}$$

In practice, we can observe that the risk reserve at the end of year $t+1$ depends on two components: the initial risk reserve (risk reserve at the end of year t) and the total "technical result" of the year. The first component is defined as the initial capital U_t invested at the deterministic rate j for one year. The second component is defined as the difference between earned premium B_{t+1} and claims $\tilde{X}_{t+1}$ and expenses E_{t+1} (both paid and reserved), invested at deterministic rate j for half a year.

The only random variable present in this equation is the stochastic aggregate amount of claims. This means that, as shown for instance in [29] or [72], in this initial model (where we are implicitly assuming that the insurance company operates in a single segment) we can obtain the moments of $\tilde{U}_{t+1}$ from the corresponding moments of $\tilde{X}_{t+1}$.

More specifically, the mean, variance and skewness of the risk reserve at time $t+1$ are reported in (3.2)

$$\begin{aligned}
&\mathbb{E}\left[\tilde{U}_{t+1}\right] = U_t(1+j) + \left[\left(B_{t+1} - \mathbb{E}\left[\tilde{X}_{t+1}\right] - E_{t+1}\right)\right](1+j)^{1/2}\,,\\
&Var\left[\tilde{U}_{t+1}\right] = Var\left[\tilde{X}_{t+1}\right](1+j)\,,\\
&Skew\left[\tilde{U}_{t+1}\right] = -Skew\left[\tilde{X}_{t+1}\right]\,.
\end{aligned} \tag{3.2}$$

[1] In this thesis we use the term capital and risk reserve interchangeably.

3.2.2 Collective risk model for aggregate claim amount

The random variable[2] $\tilde{X}$ is defined by means of a collective risk model. This model assumes that the aggregate claim amount can be described according to Formula (3.3) (see for instance [29] for details):

$$\tilde{X} = \sum_{i=1}^{\tilde{K}} \tilde{Z}_i \tag{3.3}$$

based on the following assumptions:

(i) The claim sizes are independent of each other: $\tilde{Z}_i \perp \tilde{Z}_j \forall i, j$.

(ii) The claim sizes are identically distributed: $F(\tilde{Z}_i) \stackrel{d}{=} F(\tilde{Z}) \forall i$.

(iii) The number of claims and the claim sizes are independent: $\tilde{K} \perp \tilde{Z}_i$.

Under these hypotheses it is possible to obtain the moments of the aggregate claim amount. In particular, recalling the definition of fair premium, we know that the first moment (i.e. the mean) of this random variable corresponds to the fair premium under the theoretical approach, and it is obtained as reported in Formula (3.4)

$$\mathbb{E}\left[\tilde{X}\right] = \mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}\right] . \tag{3.4}$$

Variance and skewness are equal to Formula (3.5) and Formula (3.6)

$$Var\left[\tilde{X}\right] = \mathbb{E}\left[\tilde{K}\right] Var\left[\tilde{Z}\right] + Var\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}\right]^2 , \tag{3.5}$$

$$\gamma(\tilde{X}) = \frac{\mathbb{E}\left[\tilde{K}\right] Var\left[\tilde{Z}\right]^{3/2} \gamma\left[\tilde{Z}\right] + 3Var\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}\right] Var\left[\tilde{Z}\right] + Var\left[\tilde{K}\right]^{3/2} \gamma\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}\right]^3}{Var\left[\tilde{X}\right]^{3/2}} . \tag{3.6}$$

Making some specific assumptions on the distributions of number of claims and single claim amount it is possible to find a more compact version of the moments reported above. For instance, as shown in [29], assuming an over-dispersed Poisson distribution for number of claims and a LogNormal distribution for claim amount, in Formula (3.7) we report the first three moments of the aggregate claim amount:

[2]From now on we remove the time reference in the random variables in order to lighten the notation.

$$\mathbb{E}\left[\tilde{X}\right] = \mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}\right] ,$$
$$Var\left[\tilde{X}\right] = \mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}^2\right] + \mathbb{E}\left[\tilde{K}\right]^2\mathbb{E}\left[\tilde{Z}\right]^2 Var\left[\tilde{Q}\right] ,$$
$$\gamma\left[\tilde{X}\right] = \frac{\mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}^3\right] + 3\mathbb{E}\left[\tilde{K}\right]^2\mathbb{E}\left[\tilde{Z}\right]\mathbb{E}\left[\tilde{Z}^2\right]Var\left[\tilde{Q}\right] + \mathbb{E}\left[\tilde{K}\right]^3\mathbb{E}\left[\tilde{Z}\right]^3\gamma\left[\tilde{Q}\right]Var\left[\tilde{Q}\right]^{3/2}}{Var\left[\tilde{X}\right]^{3/2}} \tag{3.7}$$

where $\tilde{Q}$ represents a structure variable for describing short-term fluctuation in the number of claims.

Regarding the other elements of Formula (3.1), the gross premium earned by insurer in the year $t+1$, B_{t+1}, is obtained as the sum of fair premium, safety loading and expense loading, as reported in Formula (3.8)

$$\begin{aligned} B_{t+1} &= P_{t+1}(1+\lambda_{t+1}) + c_{t+1}B_{t+1} \\ &= \mathbb{E}\left[\tilde{X}_{t+1}\right](1+\lambda_{t+1}) + E_{t+1} \end{aligned} \tag{3.8}$$

where, as explained in Section 2.1.2, fair premium P_{t+1} is the expected value of the aggregate claim amount random variable, λ_{t+1} represents the safety loading rate and c_{t+1} the expense loading rate, assumed to be equal to the actual expense rate.

3.2.3 Risk reserve equation in presence of reinsurance

In case we allow for the possibility of purchasing reinsurance, the equation described in (3.1) becomes:

$$\tilde{U}_{t+1} = U_t(1+j) + \left[\left(B_{t+1} - \tilde{X}_{t+1} - E_{t+1}\right) - \left(B^{re}_{t+1} - \tilde{X}^{re}_{t+1} - C^{re}_{t+1}\right)\right](1+j)^{1/2} \tag{3.9}$$

where the new terms B^{re}_{t+1}, $\tilde{X}^{re}_{t+1}$, C^{re}_{t+1} represent ceded premium, ceded claims and ceded commission respectively. The difference between these terms can be interpreted as the “technical result” of the reinsurer, which corresponds to the profit ceded by the insurance company.

In a general case, the gross premium ceded to the reinsurer is composed by the sum of ceded premium for the different possible reinsurance treaties. Considering only QS and XL reinsurance, we obtain the element reported in Formula (3.10)

$$B^{re} = B^{xl} + B^{qs} . \tag{3.10}$$

The gross premium of an excess of loss treaty, as described in Section 2.1, can be obtained by means of experience or exposure rating. Assuming that the reinsurance company employs experience pricing and that calibrates safety loading by means of the standard deviation premium principle, the gross premium is obtained according to Formula (3.11)

$$B^{xl} = \mathbb{E}\left[\tilde{X}^{xl}\right] + \beta^{xl}\sigma\left[\tilde{X}^{xl}\right] . \tag{3.11}$$

As described in Section 2.1, gross premium of a quota share reinsurance can be obtained as reported below:

$$B^{qs} = (1-\alpha)B \tag{3.12}$$

while the (deterministic) ceded commission is defined as:

$$C^{qs} = B^{qs}c^{qs} = (1-\alpha)Bc^{qs} . \tag{3.13}$$

Regarding the aggregate claim amount borne by the reinsurer, $\tilde{X}^{re}_{t+1}$, we recall the formulas presented in Section 2.1 for excess of loss and quota share treaties. In particular, following the usual proportional approach, for a QS treaty the aggregate claim amount ceded to reinsurer is equal to:

$$\tilde{X}^{qs} = (1-\alpha)\,\tilde{X} . \tag{3.14}$$

Mean, variance and skewness of this random variable are reported in Formula (3.15), (3.16) and (3.17) respectively.

$$\mathbb{E}\left[\tilde{X}^{qs}\right] = (1-\alpha)\,\mathbb{E}\left[\tilde{X}\right] , \tag{3.15}$$

$$Var\left[\tilde{X}^{qs}\right] = (1-\alpha)^2\,Var\left[\tilde{X}\right] , \tag{3.16}$$

$$\gamma\left[\tilde{X}^{qs}\right] = \gamma\left[\tilde{X}\right] . \tag{3.17}$$

In Formula (3.18) it is instead reported the aggregate claim amount borne by the reinsurer for a XL treaty.

$$\tilde{X}^{xl} = \sum_{i=1}^{\tilde{K}} \tilde{Z}^{xl} = \sum_{i=1}^{\tilde{K}} \min\left(\max\left(\tilde{Z}_i - d, 0\right), l\right) \tag{3.18}$$

with d deductible, l limit and where $\tilde{Z}^{xl}$ is distributed according to:

$$\tilde{Z}^{xl} = \tilde{Z}_{d,l} = \left(\tilde{Z}\wedge l\right) - \left(\tilde{Z}\wedge d\right) = \begin{cases} 0 & \tilde{Z} \le d \\ \tilde{Z} - d & d < \tilde{Z} \le d + l \\ l & \tilde{Z} > d + l . \end{cases}$$

Under the assumption of LogNormal distribution of $\tilde{Z}$ we can derive the moments of this random variable in a closed form.

For instance, as already reported in Section 2.1 the mean of $\tilde{Z}^{xl}$ is equal to:

$$\begin{aligned}\mathbb{E}\left[\tilde{Z}_{d,l}\right] &= \int_d^{d+l}(z-d)f_Z(z)dz + l\int_{d+l}^{\infty} f_Z(z)dz \\ &= \int_d^{d+l} zf_Z(z)dz - d\int_d^{d+l} f_Z(z)dz + l\int_{d+l}^{\infty} f_Z(z)dz \\ &= e^{\mu+\frac{\sigma^2}{2}}\left[\Phi_{\mu+\sigma^2,\sigma^2}(\ln(d+l)) - \Phi_{\mu+\sigma^2,\sigma^2}(\ln(d))\right] - \\ &\qquad d\left[\Phi_{\mu,\sigma^2}(\ln(d+l)) - \Phi_{\mu,\sigma^2}(\ln(d))\right] + l\left[1-\Phi_{\mu,\sigma^2}(\ln(d+l))\right]\end{aligned} \tag{3.19}$$

where $f_Z(z)$ represents the probability density function (p.d.f.) of the claims amount r.v. and $\Phi_{\mu,\sigma^2}(\cdot)$ the cumulative distribution function (c.d.f.) of a Normal distribution with mean μ and variance σ^2.

As a general result, it is possible to prove that the k-mean of $\tilde{Z}^{xl}$ (i.e. the k-th moment about the origin) can be expressed as reported in Formula (3.20):

$$\begin{aligned}\mathbb{E}\left[\left(\tilde{Z}_{d,l}\right)^k\right] &= \int_d^{d+l}(z-d)^k f_Z(z)dz + l^k\int_{d+l}^{\infty} f_Z(z)dz \\ &= \int_d^{d+l}\sum_{i=0}^{k}\binom{k}{i} z^i d^{k-i} f_Z(z)dz + l^k\int_{d+l}^{\infty} f_Z(z)dz \\ &= \sum_{i=0}^{k}\binom{k}{i}\left(e^{i\mu+\frac{(i\sigma)^2}{2}} - d^{(k-i)}\right)\left[\Phi_{\mu+(i\sigma)^2,\sigma^2}(\ln(d+l)) - \Phi_{\mu+(i\sigma)^2,\sigma^2}(\ln(d))\right] + \\ &\qquad l\left[1-\Phi_{\mu,\sigma^2}(\ln(d+l))\right].\end{aligned} \tag{3.20}$$

Consequently, we can derive the formula of the variance of single claim amount borne by the reinsurer as:

$$Var\left[\tilde{Z}_{d,l}\right] = \mathbb{E}\left[\left(\tilde{Z}_{d,l}\right)^2\right] - \left(\mathbb{E}\left[\tilde{Z}_{d,l}\right]\right)^2 \tag{3.21}$$

and eventually all the other moments.

Mean, variance and skewness of the aggregate claim amount of the XL reinsurer are reported in Formula (3.22), (3.23) and (3.24) respectively.

$$\mathbb{E}\left[\tilde{X}^{xl}\right] = \mathbb{E}\left[\sum_{i=1}^{\tilde{K}}\tilde{Z}_{i,d,l}\right] = \mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}_{d,l}\right] \tag{3.22}$$

where $\mathbb{E}\left[\tilde{Z}_{d,l}\right]$ is equal to (3.19).

$$Var\left[\tilde{X}^{xl}\right] = Var\left[\sum_{i=1}^{\tilde{K}}\tilde{Z}_{i,d,l}\right] = \mathbb{E}\left[\tilde{K}\right]Var\left[\tilde{Z}_{d,l}\right] + Var\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}_{d,l}\right]^2 \tag{3.23}$$

where $\mathbb{E}\left[\tilde{Z}_{d,l}\right]$ is equal to (3.19) and $Var\left[\tilde{Z}_{d,l}\right]$ to (3.21).

$$\gamma\left[\tilde{X}^{xl}\right] = \frac{\mathbb{E}\left[\tilde{K}\right] Var\left[\tilde{Z}_{d,l}\right]^{3/2} \gamma\left[\tilde{Z}_{d,l}\right] + 3Var\left[\tilde{K}\right] \mathbb{E}\left[\tilde{Z}_{d,l}\right] Var\left[\tilde{Z}_{d,l}\right] + Var\left[\tilde{K}\right]^{3/2} \gamma\left[\tilde{K}\right] \mathbb{E}\left[\tilde{Z}_{d,l}\right]^{3}}{Var\left[\tilde{X}^{xl}\right]^{3/2}}. \tag{3.24}$$

In case we assume that QS reinsurance is the only applicable treaty, Formula (3.9) simplifies to:

$$\tilde{U}_{t+1} = U_t(1+j) + \left[\alpha\left(B_{t+1} - \tilde{X}_{t+1}\right) - \left(E_{t+1} + \alpha B_{t+1}\tilde{c}^{re}_{t+1}\right)\right](1+j)^{1/2}.$$

On the other hand, if we instead assume that XL reinsurance is the only applicable treaty, Formula (3.9) becomes:

$$\tilde{U}_{t+1} = U_t(1+j) + \left[\left(B_{t+1} - B^{re}_{t+1}\right) - \left(\tilde{X}_{t+1} - \tilde{X}^{re}_{t+1}\right) - E_{t+1}\right](1+j)^{1/2}.$$

3.3 Risk reserve with one segment and one reinsurer

Following the theoretical approaches developed in previous section, we start our extensions of the risk reserve equation allowing for default of the reinsurance company.

Hence, in Formula (3.25) we define a new random variable “aggregate claim amount returned from reinsurer to insurer”, which considers this possible event.

$$\tilde{X}^{re,d} = \tilde{X}^{re} - (1-q)\tilde{X}^{re}\tilde{I} = \begin{cases} \tilde{X}^{re} & 1-p \\ \tilde{X}^{re}q & p \end{cases} \tag{3.25}$$

where $\tilde{I}$ represents an indicator variable for the realization of the default event, with probability p, and q the recovery rate in case of default, assumed deterministic. In particular, the default event is modeled as a binomial random variable, with probability of default equal to p. Moreover, it is possible to interpret $(1-q)\tilde{X}^{re}$ as a (stochastic) loss given default.

The three main moments of this random variable are reported in Formula (3.26), (3.27) and (3.28).

$$\mathbb{E}\left[\tilde{X}^{re,d}\right] = \mathbb{E}\left[\tilde{X}^{re}\right]\left(1 - \mathbb{E}\left[\tilde{I}\right](1-q)\right) \tag{3.26}$$

$$
\begin{aligned}
&Var\left[\tilde{X}^{re,d}\right] = \mathbb{E}\left[\left(\tilde{X}^{re,d}\right)^2\right] - \mathbb{E}\left[\tilde{X}^{re,d}\right]^2 \\
&= Var\left[\tilde{X}^{re}\right]\left(1 - \mathbb{E}\left[\tilde{I}\right](1-q^2)\right) + \mathbb{E}\left[\tilde{X}^{re}\right]^2 Var\left[\tilde{I}\right](1-q)^2 \\
&= Var\left[\tilde{X}^{re}\right] + (1-q)\left((1-q)\,\mathbb{E}\left[\tilde{X}^{re}\right]^2 Var\left[\tilde{I}\right] - (1+q)\,Var\left[\tilde{X}^{re}\right]\mathbb{E}\left[\tilde{I}\right]\right)
\end{aligned}
\tag{3.27}
$$

$$
\begin{aligned}
&\gamma\left[\tilde{X}^{re,d}\right] \\
&= \frac{\mathbb{E}\left[\left(\tilde{X}^{re,d}\right)^3\right] - 3\mathbb{E}\left[\tilde{X}^{re,d}\right]\mathbb{E}\left[\left(\tilde{X}^{re,d}\right)^2\right] + 3\mathbb{E}\left[\tilde{X}^{re,d}\right]^2\mathbb{E}\left[\left(\tilde{X}^{re,d}\right)\right] - \mathbb{E}\left[\tilde{X}^{re,d}\right]^3}{Var\left[\tilde{X}^{re,d}\right]^{3/2}} \\
&= \frac{\mathbb{E}\left[\left(\tilde{X}^{re,d}\right)^3\right] - 3\mathbb{E}\left[\tilde{X}^{re,d}\right]Var\left[\tilde{X}^{re,d}\right] - \mathbb{E}\left[\tilde{X}^{re,d}\right]^3}{Var\left[\tilde{X}^{re,d}\right]^{3/2}}\,.
\end{aligned}
\tag{3.28}
$$

An important assumption that we made in this context is that we assumed independence between the default event and the aggregate claim amount borne by the reinsurer. The rational of this choice is that the aggregate claim amount borne by the reinsurer deriving by the specific insurance company are just a portion of its whole exposure. Hence, we assume that the effect of the single insurance company is negligible compared to the size of the portfolio of the reinsurer and then there is independence between the two random variables.

It shall be noted that, from the assumptions we made regarding the collective risk model for describing the aggregate claim amount, we are able to compute these moments for both quota share and excess of loss cases. Indeed, we simply have to substitute the generic X^{re} random variables with the corresponding one for the two reinsurance cases, for which we have already reported the main moments above.

At this point we can define the equation of the risk reserve in a scenario where the insurer operates in a single segment and with only a (defaultable) reinsurer. In Formula (3.29) it is reported this equation.

$$
\begin{aligned}
\tilde{U}_1 &= U_0(1+j) + \left[\left(B - \tilde{X} - E\right) - \left(B^{re} - \tilde{X}^{re,d} - C^{re}\right)\right](1+j)^{1/2} \\
&= U_0(1+j) + \left[(B - B^{re}) - \left(\tilde{X} - \tilde{X}^{re,d}\right) - (E - C^{re})\right](1+j)^{1/2}\,.
\end{aligned}
\tag{3.29}
$$

In Formula (3.30) it is reported the mean of this random variable.

$$\begin{aligned}\mathbb{E}\left[\tilde{U}_1\right] &= U_0(1+j) + \left[(B - B^{re}) - \left(\mathbb{E}\left[\tilde{X}\right] - \mathbb{E}\left[\tilde{X}^{re,d}\right]\right) - (E - C^{re})\right](1+j)^{1/2} \\ &= U_0(1+j) + \left[(B - B^{re}) - \left(\mathbb{E}\left[\tilde{X}\right] - \mathbb{E}\left[\tilde{X}^{re}\right]\left(1 - \mathbb{E}\left[\tilde{I}\right](1-q)\right)\right) - E\right](1+j)^{1/2}\end{aligned} \tag{3.30}$$

where we simply apply Formula (3.26).

As expected, the mean of this equation differentiates from Formula (3.9) just for the presence a term accounting for the potential non-payment by the reinsurer in case of default. Indeed, the expected claims borne by the reinsurer is multiplied by one minus the expected probability of default multiplied by the eventual non-recovered quota.

In Formula (3.31) it is reported the variance of this risk reserve:

$$\begin{aligned}Var\left[\tilde{U}_1\right] &= Var\left[U_0(1+j) + \left[(B - B^{re}) - \left(\tilde{X} - \tilde{X}^{re,d}\right) - (E - C^{re})\right](1+j)^{1/2}\right] \\ &= \left\{Var\left[\tilde{X}\right] + Var\left[\tilde{X}^{re,d}\right] - 2Cov\left[\tilde{X}, \tilde{X}^{re,d}\right]\right\}(1+j)\,.\end{aligned} \tag{3.31}$$

Appendix A.1 reports in detail the steps for obtaining the variance and the derivation of the elements of Formula (3.31).

Clearly, from a theoretical point of view we have two extreme possible situations. In case the reinsurer has probability of default equal to 0, then we come back to the results of the model reported in Formula 3.9. The opposite situation occurs if we assume that the reinsurer will surely default (i.e. probability of default equal to 1) and will not return any amount (i.e. recovery rate equal to 0). In that case, the variance of capital at end of time 1 is equal to the gross of reinsurance case. However, the expected capital is lower due to the payment of reinsurance premium. Hence, this strategy will never be pursued due to its inefficiency.

3.3.1 Numerical application

At this point we can present some practical case studies showing the application of the model just described for a non-life insurance company. Table 3.1 shows the parameters of segment, assumed GTPL, where the insurance company carries out its underwriting activity. The other necessary parameters for calculating the risk reserve are instead reported in Table 3.2. From these information we can derive that this line of business has a quite high volatility of single claim amount, indicated by the coefficient of variation (CoV), which represents the ratio between standard deviation and mean of the random variable, equal to $CoV(\tilde{Z}) = 10$. Moreover, the LoB provides on average a positive technical result, with an average loss ratio below 60%. The characteristics of this line of business are well suited for the use of an excess of loss reinsurance. Indeed, the expected loss ratio is quite low, but the insurance company is exposed to a strong variability in the single claim amount, which could be reduced by means of a non-proportional reinsurance treaty. As for

Table 3.1. Parameters of the line of business.

LoB	$\mathbb{E}\left[\tilde{K}\right]$	$\sigma\left[\tilde{Q}\right]$	$\mathbb{E}\left[\tilde{Z}\right]$	$CoV\left[\tilde{Z}\right]$	pl	ER	LR	CR
GTPL	$15,000$	15.39%	$6,000$	10	$10,000,000$	32.7%	59.6%	92.3%

Table 3.2. Parameters of the insurance company, market interest rate and reinsurance deductible.

P	λ	c	B	U	j	d
$90,000,000$	12.9%	32.7%	$150,980,681$	$15,098,068$	1%	$1,000,000$

the remaining parameters of the case study, we assume an initial capital of the insurer equal to 10% of the gross written premium of the year, an annual interest rate of 1% and a deductible of the excess of loss reinsurance equal to 1 million (M). The value of the reinsurance limit is not defined, since it will be the element that we will analyze for choosing the optimal contract.

Finally, Table 3.3 shows the parameters of reinsurer for different values of the credit quality step. In particular, in order to present more distinct results, we have assumed a quite strong discount in the safety loading and a strong impact on the recovery rate for worse values of the credit quality step. In practice, however, the insurance company should use the actual information available for the price offered by the reinsurer of different ratings and its expectation for the recovery rate.

In Figure 3.1 are reported the mean and coefficient of variation of the risk reserve for different values of limit l, fixed all the other parameters. Moreover we compare these metrics for different ratings of the reinsurance company.

We can observe that, for all the values of CQS, there is a decreasing trend in the expected capital at year-end as the limit increases. Indeed the explanation is straightforward: for a fixed deductible, a higher limit implies a higher expected cost for the reinsurer, which would lead to an increase in the price of the XL treaty, reducing the insurance technical result. Comparing the expected risk reserve for different reinsurer CQSs we can observe that the worse the reinsurer rating the higher the insurer result. This is in line with the theoretical expectation, since a

Table 3.3. Parameters of the reinsurance company for different credit quality steps.

CQS	Probability of default	Discount factor	Recovery rate
0	0.002%	87.5%	60.0%
1	0.01%	75.0%	51.4%
2	0.05%	62.5%	42.9%
3	0.24%	50.0%	34.3%
4	1.2%	37.5%	25.7%
5	4.2%	25.0%	17.1%
6	4.2%	12.5%	8.6 %

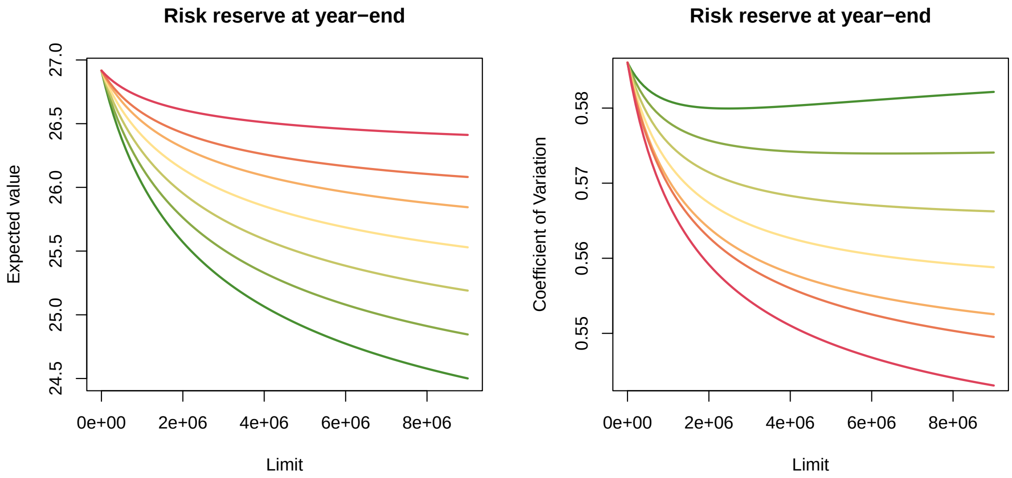

Figure 3.1. Expected value and coefficient of variation of the risk reserve for different values of XL limit. The colors represent the value of CQS of the reinsurance company, ranging from green for a CQS of 0 to red for a CQS of 6.

reinsurer with a worse rating should offer a lower price, at the same conditions, to compensate for its higher probability of default. Moreover the difference in the expected risk reserve for different reinsurer CQSs amplifies as the limit increases. In Figure 3.1 these effects are quite marked, due to the specific assumptions on the reinsurer parameters, reported in Table 3.3.

Analyzing the coefficient of variation of the risk reserve for the different values of the limit of the XL reinsurance we can observe that the general effect is a decrease in the relative variability for an increase in the limit. The reason is that increasing the limit produces a higher reduction of the standard deviation of the risk reserve compared to the expected value. However, we can observe that in case of a reinsurer with CQS equal to 0 the reduction of the CoV of the insurance company is limited to a certain value of limit after which there is an increase. In practice, that specific value corresponds to the limit which minimizes the coefficient of variation of the risk reserve in case the insurance company underwrites a treaty with a reinsurer with CQS equal to 0. Instead, in all the other cases, the minimization of the coefficient of variation is reached at the maximum value of limit, i.e. when the reinsurance operates on all the losses above the deductible. The specific trends of the CoV as function of limit for each CQS value are analyzed in more detail in the following figure.

In Figure (3.2) it is reported the coefficient of variation of the risk reserve for different values of limit l and for each CQS of the reinsurance company. If we are interested in a single objective optimization, consisting for instance in choosing the limit that minimize the coefficient of variation of the risk reserve (fixed the CQS of the reinsurance company) we could analyze these figures. We observe that in case we have a reinsurance with probability of default equal to 0 (CQS $= -1$) the optimal limit corresponds to $1,000,000$. After that value the CoV starts increasing,

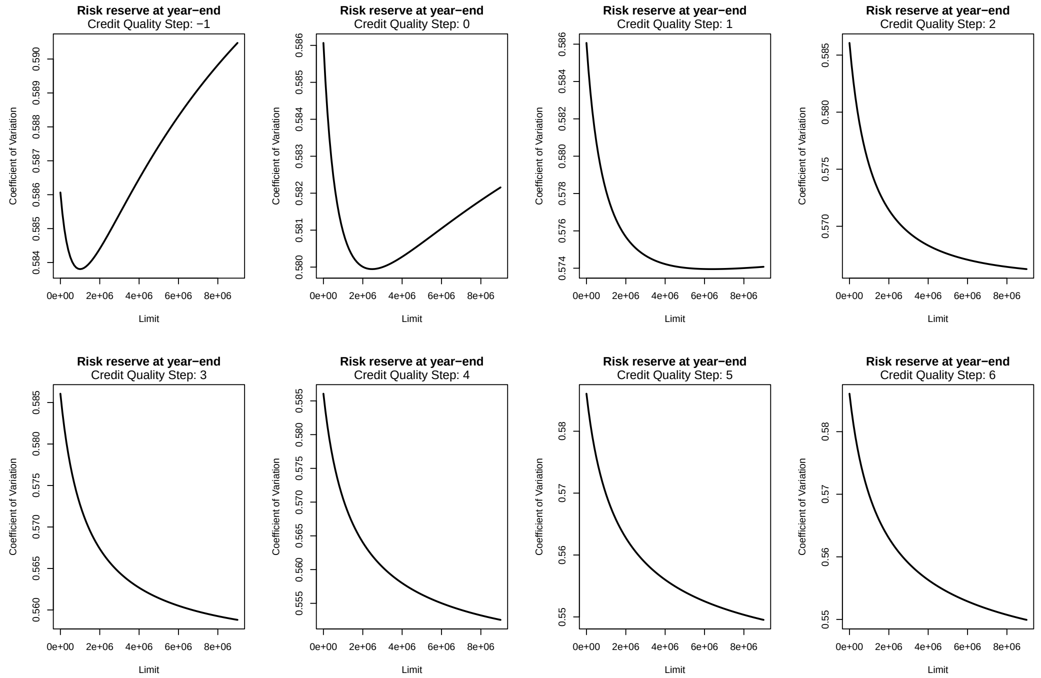

Figure 3.2. Coefficient of variation of the risk reserve for different values of XL limit for each credit quality step. Credit quality step of -1 indicates the theoretical reinsurer with probability of default equal to 0.

even more than the gross of reinsurance case[3]. The reason is that, assuming these parameters, the reduction of premium deriving from the purchase of reinsurance is less than compensated by the corresponding reduction of the standard deviation. A similar situation occurs for the case of reinsurance company with CQS equal to 0, for which the insurer reaches the minimum value of CoV for a limit of $2,500,000$. For all the other CQSs, there is a decreasing trend of the coefficient of variation for increasing limit. This means that in these cases the optimal value of the limit is reached at its maximum, which corresponds to the difference between the contractual limit and the deductible.

A final interesting result that we can derive from these analyses is that, under the specific parameters assumptions that we made, the optimal choice of reinsurance strategy for the insurance company consists in ceding the risk to a reinsurer with CQS equal to 6. Indeed, because of the strong discount that we assumed compared to the relative low probability of default, the reinsurer with the worst credit quality step leads to the highest expected risk reserve and lowest coefficient of variation for the insurance company compared to the other alternatives, for each value of the limit.

Hence, the insurance company shall limit its analysis to the CQS-6 reinsurer only, for finding the value of the limit that optimizes its objectives. One classical multi-

[3]It should be noted that the difference between smallest and the highest values of the coefficient of variation is less than 0.6%.

objective optimization problem consists in jointly maximizing the expected value and minimizing the coefficient of variation. By limiting the analysis to the case of reinsurer with CQS equal to 6 we can already observe in Figure 3.1 that the optimization problem would not lead to a unique solution. Indeed, while the expected value decreases for increasing limit, the coefficient of variation shows the opposite trend. Hence, as in most multi-objective optimization problems we do not have a unique solution, but a set of efficient solutions determining a so-called Pareto frontier. Given the set of optimal solutions the insurance company should then choose the value which better describes its trade-off preference between the two metrics.

3.4 Risk reserve with one segment and multiple reinsurers

At this point we can extend the model to allow for the presence of more than one reinsurance company. Hence, as reported in Formula (3.32), we develop the risk reserve equation in a scenario where the insurer operates in a single segment and with many (defaultable) reinsurers.

$$\begin{aligned}\tilde{U}_1 &= U_0(1+j) + \left[\left(B - \tilde{X} - E\right) - \left(B^{re} - \tilde{X}^{re,d} - C^{re}\right)\right](1+j)^{1/2} \\ &= U_0(1+j) + \left[(B - B^{re}) - \left(\tilde{X} - \tilde{X}^{re,d}\right) - (E - C^{re})\right](1+j)^{1/2} \\ &= U_0(1+j) + \left[(B - B^{re}) - \left(\tilde{X} - \sum_{r=1}^{R} \tilde{X}^{re,d(r)}\right) - \left(E - \sum_{r=1}^{R} C^{re(r)}\right)\right](1+j)^{1/2}\end{aligned} \tag{3.32}$$

where the sums are over the R reinsurers.

In Formula (3.33) and (3.34) the mean and variance of this risk reserve are reported.

$$\begin{aligned}\mathbb{E}\left[\tilde{U}_1\right] &= U_0(1+j) + \left[(B - B^{re}) - \mathbb{E}\left[\tilde{X} - \sum_{r=1}^{R} \tilde{X}^{re,d(r)}\right] - \left[E - \sum_{r=1}^{R} C^{re(r)}\right]\right](1+j)^{1/2} \\ &= U_0(1+j) + \left[(B - B^{re}) - \left(\mathbb{E}\left[\tilde{X}\right] - \sum_{r=1}^{R} \mathbb{E}\left[\tilde{X}^{re,d(r)}\right]\right) - E\right](1+j)^{1/2} \\ &= U_0(1+j) + \Bigg[(B - B^{re}) - \\ &\qquad \left(\mathbb{E}\left[\tilde{X}\right] - \sum_{r=1}^{R} \mathbb{E}\left[\tilde{X}^{re(r)}\right]\left(1 - \mathbb{E}\left[\tilde{I}^{(r)}\right]\left(1 - q^{(r)}\right)\right)\right) - E\Bigg](1+j)^{1/2}\end{aligned} \tag{3.33}$$

where $I^{(r)}$ represents the default event of the r-th reinsurer, modeled as a binomial random variable, with probability of default $p^{(r)}$. The term $q^{(r)}$ is instead the quota of credit (assumed deterministic) that the insurer holds against the reinsurer r which is recovered in case of default. The dependence between reinsurers is assumed to follow the “common shock approach” described in Section 2.2.3.

Table 3.4. Parameters of the line of business.

LoB	$\mathbb{E}\left[\tilde{K}\right]$	$\sigma\left[\tilde{Q}\right]$	$\mathbb{E}\left[\tilde{Z}\right]$	$CoV\left[\tilde{Z}\right]$	pl	ER	LR	CR
GTPL	15,000	15.39%	6,000	10	10,000,000	32.7%	59.6%	92.3%

$$
\begin{aligned}
Var\left[\tilde{U}_1\right] &= Var\left[U_0(1+j) + \left[(B - B^{re}) - \left(\tilde{X} - \sum_{r=1}^{R} \tilde{X}^{re,d(r)}\right) - \right.\right.\\
&\qquad \left.\left.\left(E - \sum_{r=1}^{R} \tilde{C}^{re(r)}\right)\right](1+j)^{1/2}\right]\\
&= Var\left[\tilde{X} - \sum_{r=1}^{R} \tilde{X}^{re,d(r)}\right](1+j)\\
&= Var\left[\tilde{X} - \sum_{r=1}^{R} \left(\tilde{X}^{re(r)} - \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1 - q^{(r)}\right)\right)\right](1+j)\\
&= Var\left[\tilde{X} - \tilde{X}^{re} + \sum_{r=1}^{R} \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1 - q^{(r)}\right)\right](1+j)\\
&= \left\{Var\left[\tilde{X}\right] + Var\left[\tilde{X}^{re}\right] + Var\left[\sum_{r=1}^{R} \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1 - q^{(r)}\right)\right] - \right.\\
&\qquad 2Cov\left(\tilde{X}, \tilde{X}^{re}\right) + 2Cov\left(\tilde{X}, \sum_{r=1}^{R} \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1 - q^{(r)}\right)\right) -\\
&\qquad \left. 2Cov\left(\tilde{X}^{re}, \sum_{r=1}^{R} \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1 - q^{(r)}\right)\right)\right\}(1+j).
\end{aligned}
\tag{3.34}
$$

Appendix A.2 reports in detail the steps for obtaining the variance and the derivation of the elements of Formula (3.34).

3.4.1 Numerical application

Similar to what we did for the model with one line of business and one reinsurer, also in this case we present some practical case studies showing how this second model could be employed for estimating the main moments of the risk reserve and for choosing the optimal reinsurance strategy. Table 3.4 shows the parameters of the GTPL line of business, which we assume being the segment in which the insurance company carries out its underwriting activity. The other necessary parameters for calculating the risk reserve are instead reported in Table 3.5. They are based on the same assumptions of the numerical analysis of the previous chapter, so the comments are valid also in this case. Here, we have also defined the value of the reinsurance limit, because for this model we will analyze the number of reinsurance companies as the variable for optimization.

Also for the reinsurance companies, as reported in Table 3.6, we used the same parameters described in the numerical analysis of the previous section.

Table 3.5. Parameters of the insurance company, market interest rate and reinsurance deductible and limit (amounts in k).

P	λ	c	B	U	j	d	l
$90,000$	12.9%	32.7%	$150,980.7$	$15,098.1$	1%	$1,000$	$2,000$

Table 3.6. Parameters of the reinsurance company for different credit quality steps.

CQS	Probability of default	Discount factor	Recovery rate
0	0.002%	87.5%	60.0%
1	0.01%	75.0%	51.4%
2	0.05%	62.5%	42.9%
3	0.24%	50.0%	34.3%
4	1.2%	37.5%	25.7%
5	4.2%	25.0%	17.1%
6	4.2%	12.5%	8.6 %

Figure 3.3 reports the mean and coefficient of variation of the risk reserve for different values of the number of reinsurers R, fixed all the other parameters. Moreover we compare these metrics for different ratings of the reinsurance company.

We can observe that, for all the values of CQS, there is a decreasing trend in the expected capital at year-end for an increasing number of reinsurance companies. Indeed, for this analysis we have assumed that the reinsurers use the standard deviation premium principle to calibrate the premium to be charged to the insurer. Hence, as proved in Formula (2.29), segmenting the same layer into multiple sub-layers (as in this case with multiple reinsurers), leads to an increase in the premium. In line with the theoretical expectation, comparing the expected risk reserve for different values of the CQS it is possible to observe that a worse rating implies a higher result. In this case, the marginal reduction in the expected risk reserve for an increase in the number of reinsurers is the same for each value of the CQS. Hence, differently from the trend observed in Figure 3.1 for increasing limits, the difference in the expected risk reserve for different reinsurer CQSs remains constant as the number of reinsurers increases.

Analyzing the coefficient of variation of the risk reserve for different number of reinsurers, we can observe an interesting dynamic. There is a decrease in the relative variability when we move from the gross of reinsurance case to the scenario with one reinsurer. After then, increasing the number of reinsurers also produces an increase in the CoV, in some cases even higher than the gross case. This effect is present for all the CQS values[4], but it is particularly relevant for the CQS-6 reinsurer. In practice, when we move from the gross of reinsurance case to the single reinsurer case, we have a reduction in the standard deviation that is greater than the corresponding reduction in the expected value of the risk reserve. However, this effect is limited to the transition from zero to one reinsurer. At that point, under the specific parameter

[4] In Figure 3.1 the red line hides the other ones

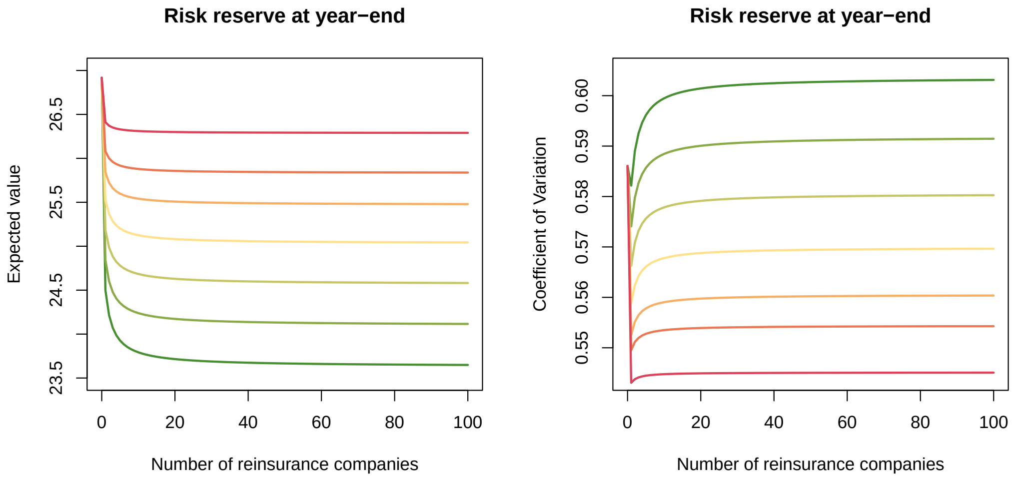

Figure 3.3. Expected value and coefficient of variation of the risk reserve for different number of reinsurance companies. The colors represent the value of CQS of the reinsurance company, ranging from green for a CQS of 0 to red for a CQS of 6.

assumptions that we made, the diversification of risk produced by the increase in the number of counterparties is more than offset by the reduction in expected value.

Also in this case, since we used the same parameters as in the previous section, the optimal choice of reinsurance strategy for the insurance company consists in ceding the risk to a reinsurer with CQS equal to 6. In particular, in Figure 3.4 we show the frontier of reinsurance strategies for the different CQS according to the expected value and the coefficient of variation of the risk reserve. Here, it possible to observe that, as anticipated, the reinsurer with CQS equal to 6 is the optimal choice in all the cases. Moreover, the reinsurance strategy that minimizes the coefficient of variation is reached with a single reinsurer with CQS equal to 6 and the value obtained is 0.54. In Figure 3.4 it is also possible to observe that the strategies determining an expected value lower than 24.5 are always inefficient, except for the cases with CQS equal to 0 and 1. Indeed, combinations to the right of 24.5 produce a higher expected value for the same values of the coefficients of variation. For the cases of reinsurers with CQS equal to 0 and 1, however, this is only partially true.

Finally, in order to present the importance of considering the actual counterparty default risk and how the specific assumptions affect the choice of reinsurance strategies, in Figure 3.5 we report the same analysis of Figure 3.3, but under the assumption that the reinsurance companies offer the same price, regardless of their rating.

Under this setting, the reinsurer with CQS equal to 0 is clearly the preferred one, since in that case the insurance company obtains the highest expected capital and the lowest relative variability. On the other hand, the reinsurer with the highest probability of default (CQS $=6$) provides the worst results, since in this case to the insurance company always reaches the lowest expected value and the highest coefficient of variation. This result is exactly in line with what we expect from the discount removal of lower rated reinsurers. Indeed, if all the reinsurers offer the

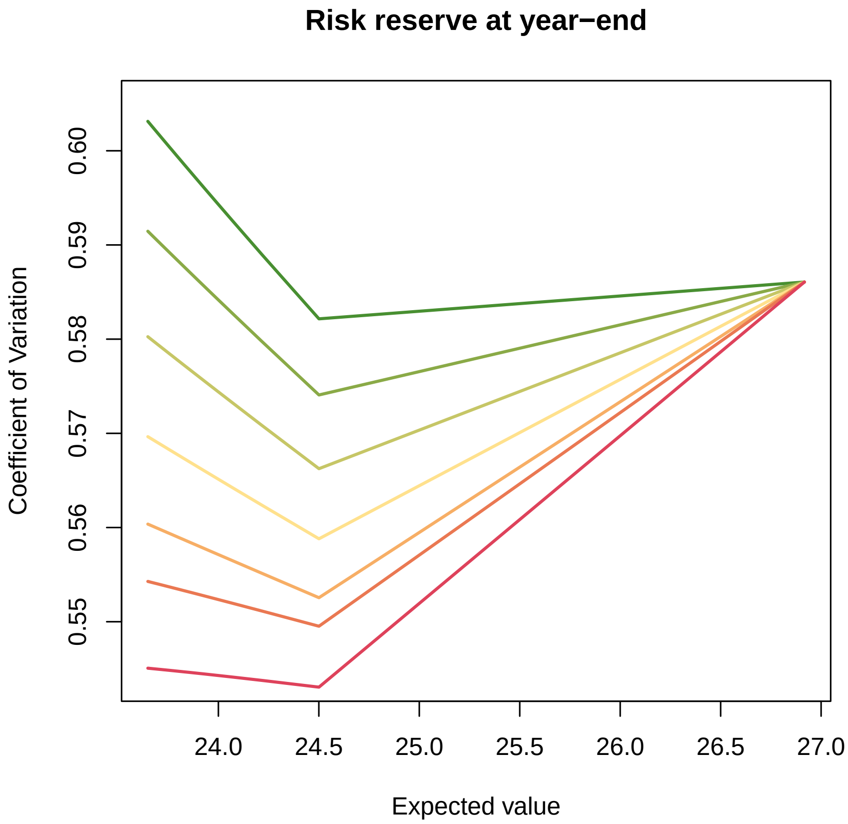

Figure 3.4. Combinations of expected value and coefficient of variation of the risk reserve for difference reinsurance strategies. The colors represent the value of CQS of the reinsurance company, ranging from green for a CQS of 0 to red for a CQS of 6.

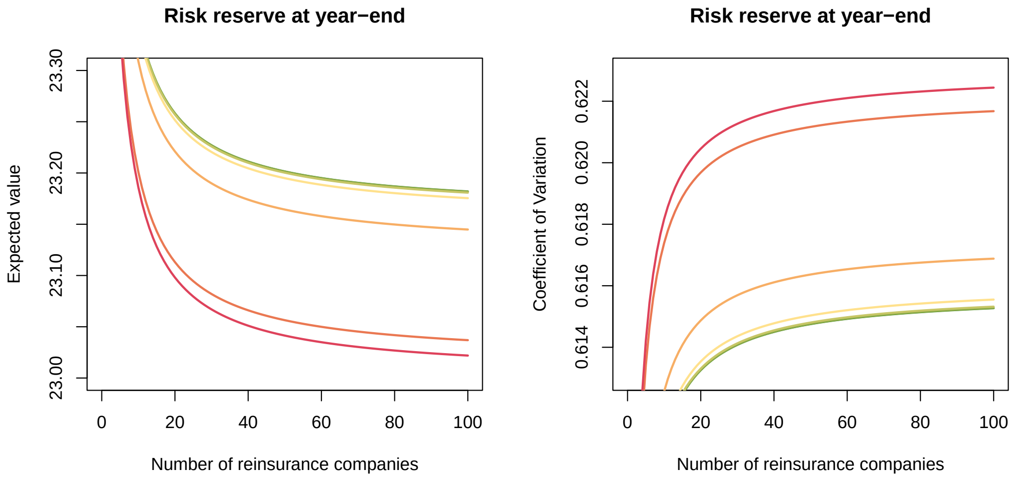

Figure 3.5. Expected value and coefficient of variation of the risk reserve for different values of XL limit. The colors represent the value of CQS of the reinsurance company, ranging from green for a CQS of 0 to red for a CQS of 6.

same price regardless of their rating, the only effect that modifies the risk reserve of the insurance company is the probability of default of the reinsurance companies.

3.5 Risk reserve with multiple segments and one reinsurer

In general an insurance company does not underwrite business of just one insurance segment, but it operates in multiple lines of business. Hence, we extend the base model to allow multiple lines of business. In Formula (3.35) we report the risk reserve equation in a scenario where the insurer operates in multiple segments, with only one (defaultable) reinsurer.

$$
\begin{aligned}
\tilde{U}_1 &= U_0(1+j) + \left[\left(B - \tilde{X} - E\right) - \left(B^{re} - \tilde{X}^{re,d} - C^{re}\right)\right](1+j)^{1/2} \\
&= U_0(1+j) + \sum_{l=1}^{L}\left[\left(B_l - \tilde{X}_l - E_l\right) - \left(B_l^{re} - \tilde{X}_l^{re,d} - C_l^{re}\right)\right](1+j)^{1/2} \\
&= U_0(1+j) + \sum_{l=1}^{L}\left[(B_l - B_l^{re}) - \left(\tilde{X}_l - \tilde{X}_l^{re,d}\right) - (E_l - C_l^{re})\right](1+j)^{1/2}
\end{aligned}
\tag{3.35}
$$

where the sums are over the L lines of business.

In Formula (3.36) and (3.37) are reported the mean and variance of this risk reserve.

Table 3.7. Parameters of the lines of business.

LoBs	$\mathbb{E}\left[\tilde{K}_l\right]$	$\sigma\left[\tilde{Q}_l\right]$	$\mathbb{E}\left[\tilde{Z}_l\right]$	$CoV\left[\tilde{Z}_l\right]$	pl_l	ER_l	LR_l	CR_l
MTPL	50,000	7.47%	4,500	6	10,000,000	21.4%	77.7%	99.1%
MOD	25,000	7.01%	1,500	2	1,000,000	31.6%	61.9%	93.5%
GTPL	15,000	15.39%	6,000	10	10,000,000	32.7%	59.6%	92.3%

$$\begin{aligned}
\mathbb{E}\left[\tilde{U}_1\right] &= U_0(1+j) + \left[(B-B^{re}) - \mathbb{E}\left[\tilde{X} - \sum_{l=1}^{L} \tilde{X}_l^{re,d}\right] - \mathbb{E}\left[E - \sum_{l=1}^{L} C_l^{re}\right]\right](1+j)^{1/2} \\
&= U_0(1+j) + \left[(B-B^{re}) - \left(\mathbb{E}\left[\tilde{X}\right] - \sum_{l=1}^{L} \mathbb{E}\left[\tilde{X}^{re,d}\right]\right) - E\right](1+j)^{1/2} \\
&= U_0(1+j) + \left[(B-B^{re}) - \left(\mathbb{E}\left[\tilde{X}\right] - \left(1-\mathbb{E}\left[\tilde{I}\right](1-q)\right)\sum_{l=1}^{L} \mathbb{E}\left[X_l^{re}\right]\right) - E\right](1+j)^{1/2}
\end{aligned} \tag{3.36}$$

$$\begin{aligned}
Var\left[\tilde{U}_1\right] &= Var\left[U_0(1+j) + \sum_{l=1}^{L}\left[(B_l - B_l^{re}) - \left(\tilde{X}_l - \tilde{X}_l^{re,d}\right) - (E_l - C_l^{re})\right](1+j)^{1/2}\right] \\
&= Var\left[\sum_{l=1}^{L}\left(\tilde{X}_l - \tilde{X}_l^{re,d}\right)\right](1+j) \\
&= \left\{\sum_{l=1}^{L} Var\left[\tilde{X}_l\right] + \sum_{l=1}^{L} Var\left[\tilde{X}_l^{re,d}\right] + \sum_{l=1}^{L}\sum_{m=1,m\neq l}^{L} Cov\left[\tilde{X}_l, \tilde{X}_m\right]\right. \\
&\left. + \sum_{l=1}^{L}\sum_{m=1,m\neq l}^{L} Cov\left[\tilde{X}_l^{re,d}, \tilde{X}_m^{re,d}\right] - 2\sum_{l=1}^{L}\sum_{m=1}^{L} Cov\left[\tilde{X}_l, \tilde{X}_m^{re,d}\right]\right\}(1+j).
\end{aligned} \tag{3.37}$$

Appendix A.3 reports in detail the steps for obtaining the variance and the derivation of the elements of Formula (3.37).

3.5.1 Numerical application

In the following we present some practical case studies showing the application of this model for estimating the main moments of the risk reserve in case the insurance company pursues its activity in more than one segment and an empirical approach for choosing the optimal reinsurance strategy. Table 3.7 reports the parameters of the 3 LoBs in which the insurance company carries out its underwriting activity. The other necessary parameters for calculating the risk reserve are instead reported in Table 3.8, while regarding the correlations between lines of business we assume the same parameters as in Standard Formula of Solvency II (see Table 2.5).

Regarding the characteristics of the reinsurer for different CQS, we assume the same

Table 3.8. Parameters of the insurance company, market interest rate and reinsurance deductible and limit (amounts in k).

P	λ	c	B	U	j	d	l
$90,000$	12.9%	32.7%	$150,980.7$	$15,098.1$	1%	$1,000$	$2,000$

parameters reported in the previous two models (see Table 3.3). Hence, also in this case, we assume that the reinsurance company prices the treaty accounting for its probability of default and consequently discounting the technical price according to its rating. Finally, the value of the limit l that we reported in the table refers only to the MTPL line of business. For MOD we assume that there is no reinsurance, while the limit of GTPL is the object of the optimization problem.

Figure 3.6 shows the mean and coefficient of variation of the risk reserve for different values of the number of the XL limit, for the GTPL line of business, fixed all the other parameters. Moreover we compare these metrics for different ratings of the reinsurance company.

As expected, for all the values of CQS, increasing the limit of the GTPL XL treaty results in a reduction of the expected risk reserve, mainly driven by the higher cost of reinsurance. As already explained for the previous cases, the specific choice of the parameters leads to a situation where the reinsurance with CQS equal to 6 provides the highest expected capital compared to the other CQS for all the values of the limit. Moreover, we can observe that increasing the limit amplifies the difference in the expected risk reserve for different reinsurer CQS.

Analyzing the coefficient of variation of the risk reserve for increasing GTPL XL limit we can observe a specific dynamic for each CQS. The cases of CQS equal to 0, 1 and 2 show a decreasing trend up to a minimum value, which is then followed by an increase. In particular, for the CQS equal to 0 we can observe that high values of the limit correspond to a coefficient of variation even higher than the gross of reinsurance one. Due to the use of the same parameters as in previous examples, also in this case the optimal choice of reinsurance strategy for the insurance company according to the minimization of the coefficient of variation consists in ceding the risk to a reinsurer with CQS equal to 6.

Figure 3.7 shows the detail of coefficient of variation of the risk reserve for different limit of the XL reinsurance for the GTPL segment. This analysis can be used in case we are interested, for instance, in the optimization of this single metric. We can observe different dynamics according to the specific rating of the reinsurance company. For CQSs from -1 to 2 there is a decrease of the CoV to a minimum followed by a subsequent increase. For CQSs from 3 to 6, instead, there is a decrease that reaches its minimum at the extreme of the limit domain. More specifically, for the case of the non-defaultable reinsurer the minimum CoV is reached with a limit of $600,000$. For the defaultable reinsurers the minimum CoV is reached with a value of limit equal to $1,300,000$, $2,500,000$ and $5,300,000$ for CQS of 0, 1 and 2 respectively. As anticipated, in all the other cases the minimum coefficient of variation is reached for the maximum value of the limit, equal to $9,000,000$.

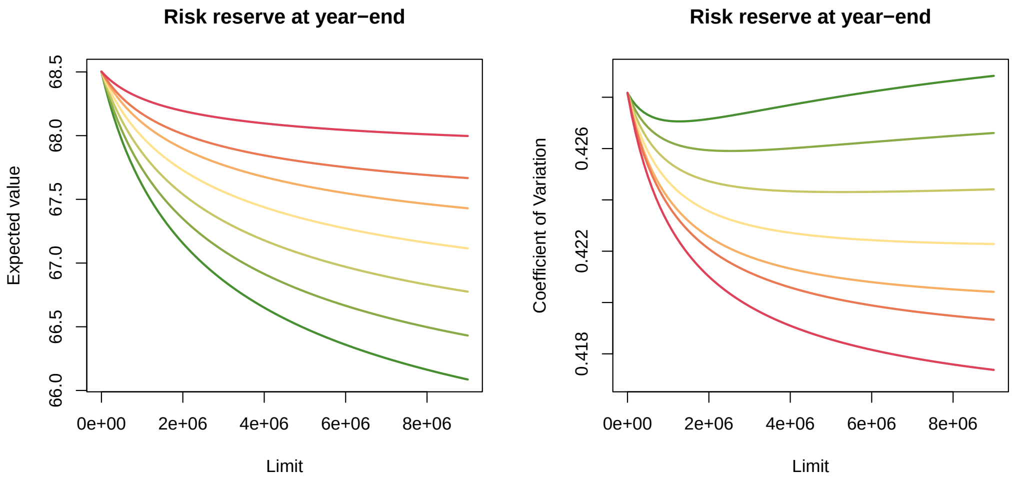

Figure 3.6. Expected value and coefficient of variation of the risk reserve for different values of XL limit for the GTPL line of business. The colors represent the value of CQS of the reinsurance company, ranging from green for a CQS of 0 to red for a CQS of 6.

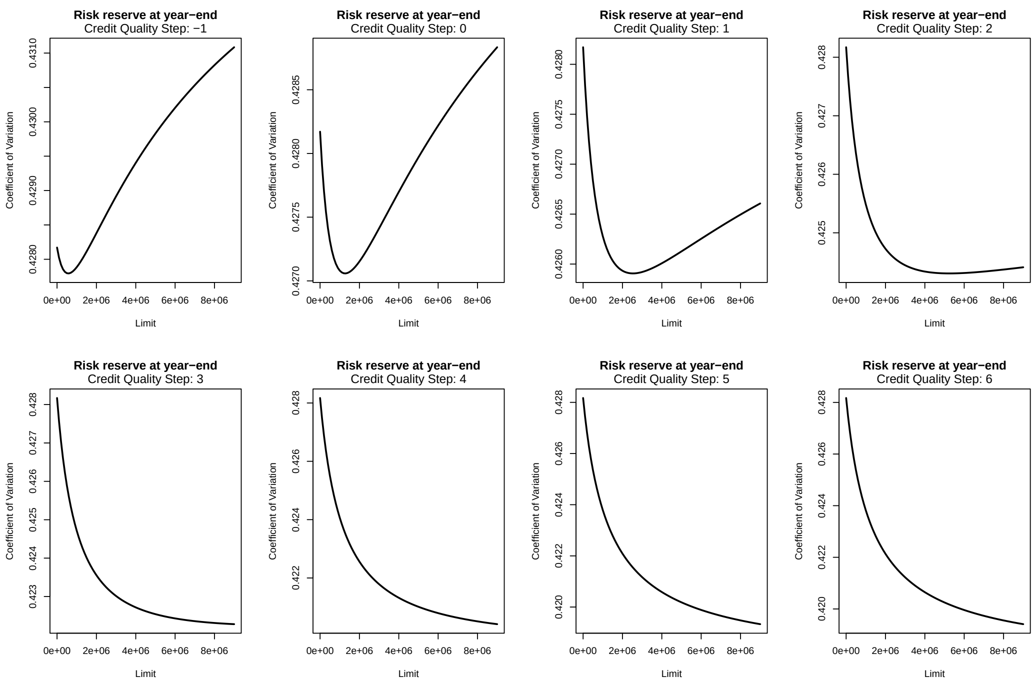

Figure 3.7. Coefficient of variation of the risk reserve for different values of XL limit for the GTPL line of business, for each credit quality step. Credit quality step of −1 indicates the theoretical reinsurer with probability of default equal to 0.

Appendix A

Proofs of risk reserve equations

A.1 Model 1: Risk reserve with one segment and one reinsurer

$$
\begin{aligned}
\tilde{U}_1 &= U_0(1+j) + \left[\left(B - \tilde{X} - E\right) - \left(B^{re} - \tilde{X}^{re,d} - C^{re}\right)\right](1+j)^{1/2} \\
&= U_0(1+j) + \left[(B - B^{re}) - \left(\tilde{X} - \tilde{X}^{re,d}\right) - (E - C^{re})\right](1+j)^{1/2}
\end{aligned}
$$

Variance of risk reserve with one segment and one reinsurer. In the following formula it is reported the variance of the risk reserve:

$$
\begin{aligned}
Var\left[\tilde{U}_1\right] &= Var\left[U_0(1+j) + \left[(B - B^{re}) - \left(\tilde{X} - \tilde{X}^{re,d}\right) - (E - C^{re})\right](1+j)^{1/2}\right] \\
&= \left\{Var\left[\tilde{X}\right] + Var\left[\tilde{X}^{re,d}\right] - 2Cov\left[\tilde{X}, \tilde{X}^{re,d}\right]\right\}(1+j)
\end{aligned}
$$

where we have simply applied the properties of the variance, in order to write the variance of the difference between two dependent random variables as the sum of the variances minus 2 times the covariance.

At this point there are the following variables:

(i) $Var\left[\tilde{X}\right]$: Variance of the aggregate claim amount $\tilde{X}$, see Formula (3.5).

(ii) $Var\left[\tilde{X}^{re,d}\right]$: Variance of the aggregate claim amounted paid back by the (defaultable) reinsurer $\tilde{X}^{re,d}$, see Formula (3.27).

(iii) $Cov\left[\tilde{X}, \tilde{X}^{re,d}\right]$: Covariance between the aggregate claim amount and the aggregate claim amount paid back by the (defaultable) reinsurer.

Since we have already developed the first two elements, we can focus only on the last one. Moreover, for this last element, because of the assumption of independence between the aggregate claim amount (gross of the reinsurer) and the default event, we can write the covariance term as follow:

$$\begin{aligned}
Cov\left[\tilde{X}, \tilde{X}^{re,d}\right] &= Cov\left[\tilde{X}, \tilde{X}^{re}\left(1 - \tilde{I}\,(1-q)\right)\right] \\
&= \left(1 - \mathbb{E}\left[\tilde{I}\right](1-q)\right) Cov\left[\tilde{X}, \tilde{X}^{re}\right] \\
&= \left(1 - \mathbb{E}\left[\tilde{I}\right](1-q)\right)\left[\mathbb{E}\left[\tilde{X}\tilde{X}^{re}\right] - \mathbb{E}\left[\tilde{X}\right]\mathbb{E}\left[\tilde{X}^{re}\right]\right].
\end{aligned}$$

At this point we know all the elements except for the first term of the squared bracket, which consists in the expected value of the product between the “gross” aggregate claim amount and the aggregate claim amount borne by the reinsurer. Focusing on this term we have:

$$\mathbb{E}\left[\tilde{X}\tilde{X}^{re}\right] = \mathbb{E}\left[\sum_{i=1}^{\tilde{K}} \tilde{Z}_i \sum_{j=1}^{\tilde{K}} \tilde{Z}_j^{re}\right] = \mathbb{E}\left[\sum_{i=1}^{\tilde{K}}\sum_{j=1}^{\tilde{K}} \tilde{Z}_i \tilde{Z}_j^{re}\right]$$

where we write both $\tilde{X}$ and $\tilde{X}^{re}$ according to their collective risk theory formula (3.3). Finally, we rearrange the product of the sum as the “double-sum” of the products.

$$\begin{aligned}
\mathbb{E}\left[\sum_{i=1}^{\tilde{K}}\sum_{j=1}^{\tilde{K}} \tilde{Z}_i \tilde{Z}_j^{re}\right] &= \mathbb{E}\left[\sum_{i=1}^{\tilde{K}} \tilde{Z}_i \tilde{Z}_i^{re} + \sum_{i=1}^{\tilde{K}}\sum_{j=1, i\neq j}^{\tilde{K}} \tilde{Z}_i \tilde{Z}_j^{re}\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{\tilde{K}} \tilde{Z}_i \tilde{Z}_i^{re}\right] + \mathbb{E}\left[\sum_{i=1}^{\tilde{K}}\sum_{j=1, i\neq j}^{\tilde{K}} \tilde{Z}_i \tilde{Z}_j^{re}\right] \\
&= \mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}_i \tilde{Z}_i^{re}\right] + \left(\mathbb{E}\left[\tilde{K}^2\right] - \mathbb{E}\left[\tilde{K}\right]\right)\mathbb{E}\left[\tilde{Z}_i \tilde{Z}_j^{re}\right] \\
&= \mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}_i \tilde{Z}_i^{re}\right] + \left(\mathbb{E}\left[\tilde{K}^2\right] - \mathbb{E}\left[\tilde{K}\right]\right)\mathbb{E}\left[\tilde{Z}\right]\mathbb{E}\left[\tilde{Z}^{re}\right].
\end{aligned}$$

First of all, we decompose the double sum in the sum of element with the same index and element with different index. The interpretation of the two elements is straightforward: the first sum represents the products of same random variables, while the double-sum represents the products of different random variables. From this considerations it is possible to further decompose the second term. At first stage we use the law of iterated expectations to separate the number of claims from the claims amount. Then we observe that we are dealing with different random variables for $\tilde{Z}_i$ and $\tilde{Z}_j^{re}$ and consequently, for the independence between $\tilde{Z}_i$ and $\tilde{Z}_j$ (and consequently with $\tilde{Z}_j^{re}$), we can write the expected value of the product as the product of the expected values.

For the first expectation, instead, we can just split between the random variables number of claims and claims amount, but we cannot directly decompose further the expectation of the product between $\tilde{Z}_i$ and $\tilde{Z}_i^{re}$, since there exists a dependence.

Anyhow, under the assumption of LogNormal distribution of claim amount, we can still derive a closed expression of this expectation (dependent on the c.d.f. of the Standard Normal distribution), as reported in the following formula:

$$
\begin{aligned}
\mathbb{E}\left[\tilde{Z}_i \tilde{Z}_i^{re}\right] &= \int_0^\infty z z^{re} f(z) dz \\
&= \int_0^d z0 f(z) dz + \int_d^{d+l} z(z-d) f(z) dz + \int_{d+l}^\infty z l f(z) dz \\
&= \int_d^{d+l} z^2 f(z) dz - d \int_d^{d+l} z f(z) dz + l \int_{d+l}^\infty z f(z) dz \\
&= e^{2\mu+2\sigma^2} \left[\Phi_{\mu+2\sigma^2,\sigma^2}\left(\ln(d+l)\right) - \Phi_{\mu+2\sigma^2,\sigma^2}\left(\ln(d)\right)\right] \\
&\quad - d e^{\mu+\frac{\sigma^2}{2}} \left[\Phi_{\mu+\sigma^2,\sigma^2}\left(\ln(d+l)\right) - \Phi_{\mu+\sigma^2,\sigma^2}\left(\ln(d)\right)\right] \\
&\quad + l e^{\mu+\frac{\sigma^2}{2}} \left[1 - \Phi_{\mu+\sigma^2,\sigma^2}\left(\ln(d+l)\right)\right] .
\end{aligned}
$$

□

A.2 Model 2: Risk reserve with one segment and multiple reinsurers

$$\begin{aligned}\tilde{U}_1 &= U_0(1+j) + \left[\left(B - \tilde{X} - E\right) - \left(B^{re} - \tilde{X}^{re,d} - C^{re}\right)\right](1+j)^{1/2} \\ &= U_0(1+j) + \left[(B - B^{re}) - \left(\tilde{X} - \tilde{X}^{re,d}\right) - (E - C^{re})\right](1+j)^{1/2} \\ &= U_0(1+j) + \left[(B - B^{re}) - \left(\tilde{X} - \sum_{r=1}^{R} \tilde{X}^{re,d(r)}\right) - \left(E - \sum_{r=1}^{R} C^{re(r)}\right)\right](1+j)^{1/2}\end{aligned}$$

Variance of risk reserve with one segment and multiple reinsurers.

$$\begin{aligned}Var\left[\tilde{U}_1\right] &= Var\left[U_0(1+j) + \left[(B - B^{re}) - \left(\tilde{X} - \sum_{r=1}^{R} \tilde{X}^{re,d(r)}\right) - \right.\right. \\ &\qquad \left.\left.\left(E - \sum_{r=1}^{R} C^{re(r)}\right)\right](1+j)^{1/2}\right] \\ &= Var\left[\tilde{X} - \sum_{r=1}^{R} \tilde{X}^{re,d(r)}\right](1+j) \\ &= Var\left[\tilde{X} - \sum_{r=1}^{R}\left(\tilde{X}^{re(r)} - \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1 - q^{(r)}\right)\right)\right](1+j) \\ &= Var\left[\tilde{X} - \tilde{X}^{re} + \sum_{r=1}^{R} \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1 - q^{(r)}\right)\right](1+j) \\ &= \left\{Var\left[\tilde{X}\right] + Var\left[\tilde{X}^{re}\right] + Var\left[\sum_{r=1}^{R} \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1 - q^{(r)}\right)\right] - \right. \\ &\qquad 2Cov\left(\tilde{X}, \tilde{X}^{re}\right) + 2Cov\left(\tilde{X}, \sum_{r=1}^{R} \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1 - q^{(r)}\right)\right) - \\ &\qquad \left. 2Cov\left(\tilde{X}^{re}, \sum_{r=1}^{R} \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1 - q^{(r)}\right)\right)\right\}(1+j).\end{aligned}$$

At this point we have the following elements, which for some cases are already known:

(i) $Var\left[\tilde{X}\right]$: Variance of the aggregate claim amount $\tilde{X}$, see Formula (3.5).

(ii) $Var\left[\tilde{X}^{re}\right]$: Variance of the aggregate claim amounted transferred to the reinsurer $\tilde{X}^{re}$, see Formula (3.23).

(iii) $Var\left[\sum_{r=1}^{R} \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1 - q^{(r)}\right)\right]$: Variance of the loss (related to the default event of the counterparty).

(iv) $Cov\left(\tilde{X}, \tilde{X}^{re}\right)$: Covariance between the aggregate claim amount and the aggregate claim amount transferred to the reinsurer, see result in Appendix A.1.

(v) $Cov\left(\tilde{X}, \sum_{r=1}^{R} \tilde{X}^{re(r)} \tilde{I}^{(r)} \left(1 - q^{(r)}\right)\right)$: Covariance between the "gross" aggregate claim amount and the total loss variable of the defaultable reinsurers.

(vi) $Cov\left(\tilde{X}^{re}, \sum_{r=1}^{R} \tilde{X}^{re(r)} \tilde{I}^{(r)} \left(1 - q^{(r)}\right)\right)$: Covariance between the reinsurer aggregate claim amount and the total loss variable of the defaultable reinsurers.

At this point we analyze the elements not yet known. Here, we start developing the element (iii) of the list. First of all we can observe that this term consists in the variance of the loss of the insurer related to the possible default event of the counterparties. Indeed, this formula

$$Var\left[\sum_{r=1}^{R} \tilde{X}^{re(r)} \tilde{I}^{(r)} \left(1 - q^{(r)}\right)\right] = Var\left[\sum_{r=1}^{R} \widetilde{LGD}^{re(r)} \tilde{I}^{(r)}\right]$$

is exactly as Formula (2.37) with the only difference that in this case we are also dealing with a stochastic loss given default. Hence, we expect to obtain a result connected with the variance of loss from counterparty risk (see Formula (2.37) of Section 2.2), but with an additional element to account for the variability of loss given default.

First of all we decompose the initial formula as follow:

$$\begin{aligned} Var\left[\sum_{r=1}^{R} \tilde{X}^{re(r)} \tilde{I}^{(r)} \left(1 - q^{(r)}\right)\right] &= \sum_{r=1}^{R} Var\left[\tilde{X}^{re(r)} \tilde{I}^{(r)} \left(1 - q^{(r)}\right)\right] \\ &+ \sum_{r=1}^{R} \sum_{s=1, r \neq s}^{R} Cov\left(\tilde{X}^{re(r)} \tilde{I}^{(r)} \left(1 - q^{(r)}\right), \tilde{X}^{re(s)} \tilde{I}^{(s)} \left(1 - q^{(s)}\right)\right) \end{aligned} \tag{A.1}$$

where we just split the variance of the sum in two components: the sum of the variances and the double sum of covariances except for $r = s$ (or equivalently the double sum of covariances).

Studying the variance term of Formula (A.1) we derive:

$$\begin{aligned}
&\sum_{r=1}^{R} Var\left[\tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right] = \\
&= \sum_{r=1}^{R}\left[Var\left(\tilde{X}^{re(r)}\right)Var\left[\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right] + Var\left[\tilde{X}^{re(r)}\right]\left(\mathbb{E}\left[\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right]\right)^2 + \right.\\
&\qquad \left.\left(\mathbb{E}\left[\tilde{X}^{re(r)}\right]\right)^2 Var\left[\tilde{I}^{(r)}(1-q^{(r)})\right]\right]\\
&= \sum_{r=1}^{R} Var\left[\tilde{X}^{re(r)}\right]\left(1-q^{(r)}\right)^2 Var\left[\tilde{I}^{(r)}\right] + \sum_{r=1}^{R} Var\left[\tilde{X}^{re(r)}\right]\left(1-q^{(r)}\right)^2\left(\mathbb{E}\left[\tilde{I}^{(r)}\right]\right)^2 + \\
&\qquad \sum_{r=1}^{R}\left(\mathbb{E}\left[\tilde{X}^{re(r)}\right]\right)^2(1-q^{(r)})^2 Var\left[\tilde{I}^{(r)}\right]\\
&= \sum_{r=1}^{R} Var\left[\tilde{X}^{re(r)}\right]\left(1-q^{(r)}\right)^2\left[Var\left[\tilde{I}^{(r)}\right] + \mathbb{E}\left[\tilde{I}^{(r)}\right]^2\right] + \\
&\qquad \sum_{r=1}^{R}\left(\mathbb{E}\left[\tilde{X}^{re(r)}\right]\right)^2(1-q^{(r)})^2 Var\left[\tilde{I}^{(r)}\right]\\
&= \sum_{r=1}^{R} Var\left[\tilde{X}^{re(r)}\right]\left(1-q^{(r)}\right)^2\mathbb{E}\left[\tilde{I}^{2(r)}\right] + \sum_{r=1}^{R}\left(\mathbb{E}\left[\tilde{X}^{re(r)}\right]\right)^2\left(1-q^{(r)}\right)^2 Var\left[\tilde{I}^{(r)}\right]
\end{aligned}$$

where we use the decomposition of variance of the product of independent random variables[1] at first step and then algebra.

Studying the covariance term of Formula (A.1) we derive:

$$\begin{aligned}
&\sum_{r=1}^{R}\sum_{s=1,r\neq s}^{R} Cov\left(\tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right),\tilde{X}^{re(s)}\tilde{I}^{(s)}\left(1-q^{(s)}\right)\right)\\
&=\sum_{r=1}^{R}\sum_{s=1,r\neq s}^{R}\left(1-q^{(r)}\right)\left(1-q^{(s)}\right)Cov\left(\tilde{X}^{re(r)}\tilde{I}^{(r)},\tilde{X}^{re(s)}\tilde{I}^{(s)}\right)\ .
\end{aligned} \tag{A.2}$$

Focusing only on the covariance term we have:

$$\begin{aligned}
&Cov\left(\tilde{X}^{re(r)}\tilde{I}^{(r)},\tilde{X}^{re(s)}\tilde{I}^{(s)}\right)\\
&=\mathbb{E}\left[\tilde{X}^{re(r)}\tilde{I}^{(r)}\tilde{X}^{re(s)}\tilde{I}^{(s)}\right]-\mathbb{E}\left[\tilde{X}^{re(r)}\tilde{I}^{(r)}\right]\mathbb{E}\left[\tilde{X}^{re(s)}\tilde{I}^{(s)}\right]\\
&=\mathbb{E}\left[\tilde{X}^{re(r)}\tilde{X}^{re(s)}\right]\mathbb{E}\left[\tilde{I}^{(r)}\tilde{I}^{(s)}\right]-\mathbb{E}\left[\tilde{X}^{re(r)}\right]\mathbb{E}\left[\tilde{I}^{(r)}\right]\mathbb{E}\left[\tilde{X}^{re(s)}\right]\mathbb{E}\left[\tilde{I}^{(s)}\right]\\
&=\mathbb{E}\left[\tilde{X}^{re(r)}\tilde{X}^{re(s)}\right]\mathbb{E}\left[\tilde{I}^{(r)}\tilde{I}^{(s)}\right]-\mathbb{E}\left[\tilde{X}^{re(r)}\right]\mathbb{E}\left[\tilde{I}^{(r)}\right]\mathbb{E}\left[\tilde{X}^{re(s)}\right]\mathbb{E}\left[\tilde{I}^{(s)}\right]+\\
&\qquad\mathbb{E}\left[\tilde{X}^{re(r)}\right]\mathbb{E}\left[\tilde{X}^{re(s)}\right]\mathbb{E}\left[\tilde{I}^{(r)}\tilde{I}^{(s)}\right]-\mathbb{E}\left[\tilde{X}^{re(r)}\right]\mathbb{E}\left[\tilde{X}^{re(s)}\right]\mathbb{E}\left[\tilde{I}^{(r)}\tilde{I}^{(s)}\right]\\
&=Cov\left[\tilde{X}^{re(r)},\tilde{X}^{re(s)}\right]\mathbb{E}\left[\tilde{I}^{(r)}\tilde{I}^{(s)}\right]+Cov\left[\tilde{I}^{(r)},\tilde{I}^{(s)}\right]\mathbb{E}\left[\tilde{X}^{re(r)}\right]\mathbb{E}\left[\tilde{X}^{re(s)}\right]\ .
\end{aligned}$$

[1] Given two independent random variables X and Y the variance of the product between X and Y is equal to: $Var\left[XY\right]=Var\left[X\right]Var\left[Y\right]+Var\left[X\right]\mathbb{E}\left[Y\right]^2+Var\left[Y\right]\mathbb{E}\left[X\right]^2$.

Putting this result in Formula (A.2) we have:

$$\sum_{r=1}^{R}\sum_{s=1,r\neq s}^{R}\left(1-q^{(r)}\right)\left(1-q^{(s)}\right)Cov\left[\tilde{X}^{re(r)},\tilde{X}^{re(s)}\right]\mathbb{E}\left[\tilde{I}^{(r)}\tilde{I}^{(s)}\right]+$$
$$\sum_{r=1}^{R}\sum_{s=1,r\neq s}^{R}\left(1-q^{(r)}\right)\left(1-q^{(s)}\right)Cov\left[\tilde{I}^{(r)},\tilde{I}^{(s)}\right]\mathbb{E}\left[\tilde{X}^{re(r)}\right]\mathbb{E}\left[\tilde{X}^{re(s)}\right].$$

Hence, we can observe that we have obtained again the formula of the variance of the loss (see Formula (2.37)) with the additional term accounting for the variability in the loss given default. We can then rewrite the previous results as:

$$\begin{aligned}
&Var\left[\sum_{r=1}^{R}\tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right]\\
&=\sum_{r=1}^{R}\sum_{s=1}^{R}\left(1-q^{(r)}\right)\left(1-q^{(s)}\right)Cov\left[\tilde{X}^{re(r)},\tilde{X}^{re(s)}\right]\mathbb{E}\left[\tilde{I}^{(r)}\tilde{I}^{(s)}\right]+\\
&\quad\sum_{r=1}^{R}\sum_{s=1}^{R}\left(1-q^{(r)}\right)\left(1-q^{(s)}\right)\mathbb{E}\left[\tilde{X}^{re(r)}\right]\mathbb{E}\left[\tilde{X}^{re(s)}\right]\left[Cov\left[\tilde{I}^{(r)},\tilde{I}^{(s)}\right]\right].
\end{aligned}$$

The last two elements that we need to develop in previous formula are $Cov\left[\tilde{X}^{re(r)},\tilde{X}^{re(s)}\right]$ and $\mathbb{E}\left[\tilde{I}^{(r)}\tilde{I}^{(s)}\right]$. For the second one, since we already know from Formula (2.39) the covariance term, we just need to rewrite the formula as: $\mathbb{E}\left[\tilde{I}^{(r)}\tilde{I}^{(s)}\right]=Cov\left[\tilde{I}^{(r)},\tilde{I}^{(s)}\right]+\mathbb{E}\left[\tilde{I}^{(r)}\right]\mathbb{E}\left[\tilde{I}^{(s)}\right]$.

For $Cov\left[\tilde{X}^{re(r)},\tilde{X}^{re(s)}\right]$ we need to study the term for the expectation of the product between the two random variables. The steps are a bit more elaborated and are reported below:

$$\begin{aligned}
\mathbb{E}\left[\tilde{X}^{re(r)}\tilde{X}^{re(s)}\right]&=\mathbb{E}\left[\sum_{i=1}^{\tilde{K}}\tilde{Z}_i^{re(r)}\sum_{j=1}^{\tilde{K}}\tilde{Z}_j^{re(s)}\right]=\mathbb{E}\left[\sum_{i=1}^{\tilde{K}}\sum_{j=1}^{\tilde{K}}\tilde{Z}_i^{re(r)}\tilde{Z}_j^{re(s)}\right]\\
&=\mathbb{E}\left[\sum_{i=1}^{\tilde{K}}\tilde{Z}_i^{re(r)}\tilde{Z}_i^{re(s)}+\sum_{i=1}^{\tilde{K}}\sum_{j=1,i\neq j}^{\tilde{K}}\tilde{Z}_i^{re(r)}\tilde{Z}_j^{re(s)}\right]\\
&=\mathbb{E}\left[\sum_{i=1}^{\tilde{K}}\tilde{Z}_i^{re(r)}\tilde{Z}_i^{re(s)}\right]+\mathbb{E}\left[\sum_{i=1}^{\tilde{K}}\sum_{j=1,i\neq j}^{\tilde{K}}\tilde{Z}_i^{re(r)}\tilde{Z}_j^{re(s)}\right]\\
&=\mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}_i^{re(r)}\tilde{Z}_i^{re(s)}\right]+\left(\mathbb{E}\left[\tilde{K}^2\right]-\mathbb{E}\left[\tilde{K}\right]\right)\mathbb{E}\left[\tilde{Z}_i^{re(r)}\tilde{Z}_j^{re(s)}\right]\\
&=\mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}_i^{re(r)}\tilde{Z}_i^{re(s)}\right]+\left(\mathbb{E}\left[\tilde{K}^2\right]-\mathbb{E}\left[\tilde{K}\right]\right)\mathbb{E}\left[\tilde{Z}^{re(r)}\right]\mathbb{E}\left[\tilde{Z}^{re(s)}\right].
\end{aligned}\tag{A.3}$$

Hence, we analyze the first expectation of Formula (A.3), assuming that the layer of the s-th reinsurer is above the r-th. In particular, we assume that the reinsurer r

covers the claims in layer $(d_1, d_1 + l_1)$ and the reinsurer s in layer $(d_2, d_2 + l_2)$, with $d_2 \geq d_1 + l_1$.

$$\begin{aligned}\mathbb{E}\left[\tilde{Z}_i^{re(r)}\tilde{Z}_i^{re(s)}\right] &= \int_0^{d_1} 00f(z)dz + \int_{d_1}^{d_1+l_1}(z-d_1)0f(z)dz + \\ &\int_{d_1+l_1}^{d_2} l_1 0f(z)dz + \int_{d_2}^{d_2+l_2} l_1(z-d_2)f(z)dz \int_{d_2+l_2}^{\infty} l_1 l_2 f(z)dz \\ &= l_1\left(\int_0^{d_2} 0f(z)dz + \int_{d_2}^{d_2+l_2}(z-d_2)f(z)dz \int_{d_2+l_2}^{\infty} l_2 f(z)dz\right) \\ &= l_1\mathbb{E}\left[\tilde{Z}^{re(s)}\right].\end{aligned}$$

Consequently, we have:

$$\begin{aligned}\mathbb{E}\left[\tilde{X}^{re(r)}\tilde{X}^{re(s)}\right] &= \mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}_i^{re(r)}\tilde{Z}_i^{re(s)}\right] + \left(\mathbb{E}\left[\tilde{K}^2\right]-\mathbb{E}\left[\tilde{K}\right]\right)\mathbb{E}\left[\tilde{Z}^{re(r)}\right]\mathbb{E}\left[\tilde{Z}^{re(s)}\right] \\ &= \mathbb{E}\left[\tilde{K}^2\right]\mathbb{E}\left[\tilde{Z}^{re(r)}\right]\mathbb{E}\left[\tilde{Z}^{re(s)}\right] + \mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}^{re(s)}\right]\left(l_1 - \mathbb{E}\left[\tilde{Z}^{re(r)}\right]\right).\end{aligned}$$

At this point we analyze the (v) element. We study the covariance between the "gross" aggregate claim amount and the loss term.

$$\begin{aligned}&Cov\left(\tilde{X}, \sum_{r=1}^{R}\tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right) = \\ &\mathbb{E}\left[\tilde{X}\sum_{r=1}^{R}\tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right] - \mathbb{E}\left[\tilde{X}\right]\mathbb{E}\left[\sum_{r=1}^{R}\tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right] \\ &\sum_{r=1}^{R}\mathbb{E}\left[\tilde{X}\tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right] - \mathbb{E}\left[\tilde{X}\right]\sum_{r=1}^{R}\mathbb{E}\left[\tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right] \\ &\sum_{r=1}^{R}\mathbb{E}\left[\tilde{I}^{(r)}\right]\left(1-q^{(r)}\right)\mathbb{E}\left[\tilde{X}\tilde{X}^{re(r)}\right] - \mathbb{E}\left[\tilde{X}\right]\sum_{r=1}^{R}\mathbb{E}\left[\tilde{I}^{(r)}\right]\left(1-q^{(r)}\right)\mathbb{E}\left[\tilde{X}^{re(r)}\right] \\ &\sum_{r=1}^{R}\mathbb{E}\left[\tilde{I}^{(r)}\right]\left(1-q^{(r)}\right)\left[\mathbb{E}\left[\tilde{X}\tilde{X}^{re(r)}\right] - \mathbb{E}\left[\tilde{X}\right]\mathbb{E}\left[\tilde{X}^{re(r)}\right]\right].\end{aligned}$$

Hence, we know all the elements, since the development of $\mathbb{E}\left[\tilde{X}\tilde{X}^{re(r)}\right]$ is reported in Appendix (A.1).

At this point we study the element (vi) of the list, which represents the covariance between the reinsurer aggregate claim amount and the total loss variable of the defaultable reinsurers.

$$Cov\left(\tilde{X}^{re}, \sum_{r=1}^{R} \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right) =$$
$$\mathbb{E}\left[\tilde{X}^{re}\sum_{r=1}^{R} \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right] - \mathbb{E}\left[\tilde{X}^{re(r)}\right]\mathbb{E}\left[\sum_{r=1}^{R} \tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right]$$
$$\sum_{r=1}^{R}\mathbb{E}\left[\tilde{X}^{re}\tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right] - \mathbb{E}\left[\tilde{X}^{re}\right]\sum_{r=1}^{R}\mathbb{E}\left[\tilde{X}^{re(r)}\tilde{I}^{(r)}\left(1-q^{(r)}\right)\right]$$
$$\sum_{r=1}^{R}\mathbb{E}\left[\tilde{I}^{(r)}\right]\left(1-q^{(r)}\right)\mathbb{E}\left[\tilde{X}^{re}\tilde{X}^{re(r)}\right] - \mathbb{E}\left[\tilde{X}^{re}\right]\sum_{r=1}^{R}\mathbb{E}\left[\tilde{I}^{(r)}\right]\left(1-q^{(r)}\right)\mathbb{E}\left[\tilde{X}^{re(r)}\right]$$
$$\sum_{r=1}^{R}\mathbb{E}\left[\tilde{I}^{(r)}\right]\left(1-q^{(r)}\right)\left[\mathbb{E}\left[\tilde{X}^{re}\tilde{X}^{re(r)}\right] - \mathbb{E}\left[\tilde{X}^{re}\right]\mathbb{E}\left[\tilde{X}^{re(r)}\right]\right]$$
$$\sum_{r=1}^{R}\mathbb{E}\left[\tilde{I}^{(r)}\right]\left(1-q^{(r)}\right)\left[\mathbb{E}\left[\sum_{i=1}^{\tilde{K}}\sum_{j=1}^{\tilde{K}}\tilde{Z}_i^{re}\tilde{Z}_j^{re(r)}\right] - \mathbb{E}\left[\tilde{X}^{re}\right]\mathbb{E}\left[\tilde{X}^{re(r)}\right]\right]$$
$$\sum_{r=1}^{R}\mathbb{E}\left[\tilde{I}^{(r)}\right]\left(1-q^{(r)}\right)\left[\mathbb{E}\left[\sum_{i=1}^{\tilde{K}}\tilde{Z}_i^{re}\tilde{Z}_i^{re(r)} + \sum_{i=1}^{\tilde{K}}\sum_{j=1,i\neq j}^{\tilde{K}}\tilde{Z}_i^{re}\tilde{Z}_j^{re(r)}\right] - \mathbb{E}\left[\tilde{X}^{re}\right]\mathbb{E}\left[\tilde{X}^{re(r)}\right]\right]$$
$$\sum_{r=1}^{R}\mathbb{E}\left[\tilde{I}^{(r)}\right]\left(1-q^{(r)}\right)\Bigg[\Bigg(\mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}^{re}\tilde{Z}^{re(r)}\right] +$$
$$\left(\mathbb{E}\left[\tilde{K}^2\right] - \mathbb{E}\left[\tilde{K}\right]\right)\mathbb{E}\left[\tilde{Z}^{re}\right]\mathbb{E}\left[\tilde{Z}^{re(r)}\right]\Bigg) - \mathbb{E}\left[\tilde{X}^{re}\right]\mathbb{E}\left[\tilde{X}^{re(r)}\right]\Bigg].$$

The last element that we need to develop is the expected value of the product between $\tilde{Z}^{re}$ and $\tilde{Z}^{re(r)}$ for the same claim. The general formulation is reported below:

$$\mathbb{E}\left[\tilde{Z}_k^{re}\tilde{Z}_k^{re(r)}\right] = \int_0^d 00 f_Z(z)dz + \int_d^{d_i} 0(z-d_i)f_Z(z)dz +$$
$$\int_{d_i}^{d_i+l_i}(z-d)(z-d_i)f_Z(z)dz + \int_{d_i+l_i}^{d+l}(z-d)l_i f_Z(z)dz + \int_{d+l}^{\infty} ll_i f_Z(z)dz$$
$$= \int_{d_i}^{d_i+l_i} z^2 f_Z(z)dz - \int_{d_i}^{d_i+l_i}(d+d_i)z f_Z(z)dz + \int_{d_i}^{d_i+l_i} dd_i f_Z(z)dz +$$
$$\int_{d_i+l_i}^{d+l} zl_i f_Z(z)dz - \int_{d_i+l_i}^{d+l} dl_i f_Z(z)dz + ll_i(1-F_Z(d+l))dz$$
$$= \int_{d_i}^{d_i+l_i} z^2 f_Z(z)dz - \int_{d_i}^{d_i+l_i}(d+d_i)z f_Z(z)dz + dd_i(F_Z(d_i+l_i) - F_Z(d_i)) +$$
$$\int_{d_i+l_i}^{d+l} zl_i f_Z(z)dz - dl_i(F_Z(d+l) - F_Z(d_i+l_i)) + ll_i(1-F_Z(d+l))dz$$

which, under the assumption of LogNormal distribution of claim amount, reduces to:

$$
\begin{aligned}
\mathbb{E}\left[\tilde{Z}_k^{re}\tilde{Z}_k^{re(k)}\right] = {} & e^{2\mu+2\sigma^2}\left[\Phi_{\mu+2\sigma^2,\sigma^2}\left(\ln(d_i+l_i)\right)-\Phi_{\mu+2\sigma^2,\sigma^2}\left(\ln(d_i)\right)\right] \\
& -(d+d_i)e^{\mu+\frac{\sigma^2}{2}}\left[\Phi_{\mu+\sigma^2,\sigma^2}\left(\ln(d_i+l_i)\right)-\Phi_{\mu+\sigma^2,\sigma^2}\left(\ln(d_i)\right)\right] \\
& +l_i e^{\mu+\frac{\sigma^2}{2}}\left[\Phi_{\mu+\sigma^2,\sigma^2}\left(\ln(d+l)\right)-\Phi_{\mu+\sigma^2,\sigma^2}\left(\ln(d_i+l_i)\right)\right] \\
& +dd_i\left[F_Z(d_i+l_i)-F_Z(d_i)\right] \\
& -dl_i\left[F_Z(d+l)-F_Z(d_i+l_i)\right] \\
& +ll_i\left[1-F_Z(d+l)\right]
\end{aligned}
$$

□

A.3 Model 3: Risk reserve with multiple segments and one reinsurer

$$\begin{aligned}\tilde{U}_1 &= U_0(1+j) + \left[\left(B - \tilde{X} - E\right) - \left(B^{re} - \tilde{X}^{re,d} - C^{re}\right)\right](1+j)^{1/2} \\ &= U_0(1+j) + \sum_{l=1}^{L}\left[\left(B_l - \tilde{X}_l - E_l\right) - \left(B_l^{re} - \tilde{X}_l^{re,d} - C_l^{re}\right)\right](1+j)^{1/2} \\ &= U_0(1+j) + \sum_{l=1}^{L}\left[(B_l - B_l^{re}) - \left(\tilde{X}_l - \tilde{X}_l^{re,d}\right) - (E_l - C_l^{re})\right](1+j)^{1/2}.\end{aligned} \tag{A.4}$$

Variance of risk reserve with multiple segments and one reinsurer.

$$\begin{aligned}Var\left[\tilde{U}_1\right] &= Var\left[U_0(1+j) + \sum_{l=1}^{L}\left[(B_l - B_l^{re}) - \left(\tilde{X}_l - \tilde{X}_l^{re,d}\right) - (E_l - C_l^{re})\right](1+j)^{1/2}\right] \\ &= Var\left[\sum_{l=1}^{L}\left(\tilde{X}_l - \tilde{X}_l^{re,d}\right)\right](1+j) \\ &= \left\{\sum_{l=1}^{L} Var\left[\tilde{X}_l\right] + \sum_{l=1}^{L} Var\left[\tilde{X}_l^{re,d}\right] + \sum_{l=1}^{L}\sum_{m=1,m\neq l}^{L} Cov\left[\tilde{X}_l, \tilde{X}_m\right]\right. \\ &\quad \left. + \sum_{l=1}^{L}\sum_{m=1,m\neq l}^{L} Cov\left[\tilde{X}_l^{re,d}, \tilde{X}_m^{re,d}\right] - 2\sum_{l=1}^{L}\sum_{m=1}^{L} Cov\left[\tilde{X}_l, \tilde{X}_m^{re,d}\right]\right\}(1+j).\end{aligned}$$

At this point we have the following elements, which for some cases are already known:

(i) $Var\left[\tilde{X}\right]$: Variance of the aggregate claim amount $\tilde{X}$, see Formula (3.5).

(ii) $Var\left[\tilde{X}^{re}\right]$: Variance of the aggregate claim amount paid back by the (defaultable) reinsurer $\tilde{X}^{re,d}$, see Formula (3.27).

(iii) $Cov\left[\tilde{X}_l, \tilde{X}_m\right]$: Covariance of the gross aggregate claim amount between two different lines of business.

(iv) $Cov\left[\tilde{X}_l^{re,d}, \tilde{X}_m^{re,d}\right]$: Covariance between aggregate claim amount borne by the reinsurer for different segments.

(v) $Cov\left[\tilde{X}_l, \tilde{X}_m^{re,d}\right]$: Covariance between "gross" aggregate claim amount of segment l and aggregate claims borne by reinsurer of segment m.

In order to be able to develop the remaining elements, first of all we shall remark some concept regarding correlation and covariance of aggregate claim amount and add an additional assumption on the correlation between lines of business.

The base hypothesis used for modeling the correlation between different segments is that there exists a dependence at aggregate claim amount level. For instance, in Solvency II this assumption is developed in the use of a correlation matrix for aggregating the lines of business. In practice we have $\rho\left(\tilde{X}^{LoB_i}, \tilde{X}^{LoB_j}\right) = c$.

However, from this approach we do not know the implicit correlation that there exists at number of claims and severity level. Hence, for being able to subsequently decompose these dependences we shall make an additional hypothesis. In particular, we assume that there is independence in the severity of different lines of business, which means $\tilde{X}^{LoB_i} \perp\!\!\!\perp \tilde{X}^{LoB_j}$. In this way we implicitly assume that there is a dependence in the number of claims of different lines of business (in order to preserve the dependence at aggregate claims level), which means $\rho\left(\tilde{K}^{LoB_i}, \tilde{K}^{LoB_j}\right) = k$.

An important consequence of this additional assumption is that now the correlations at number level and at aggregate claim amount level are strictly dependent each other, since all the other terms present are "fixed". In this way it is possible to explicit the correlation at aggregate claim level (for instance using the correlation structure of Solvency II) and obtaining the connected correlation at number level.

In this way we can develop the element (iii) as reported below:

$$
\begin{aligned}
&Cov\left(\tilde{X}_l, \tilde{X}_m\right) = \rho\left(\tilde{X}_l, \tilde{X}_m\right)\sigma\left(\tilde{X}_l\right)\sigma\left(\tilde{X}_m\right) \\
&= \mathbb{E}\left[\tilde{X}_l\tilde{X}_m\right] - \mathbb{E}\left[\tilde{X}_l\right]\mathbb{E}\left[\tilde{X}_m\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{\tilde{K}_l}\tilde{Z}_{l,i}\right)\left(\sum_{i=1}^{\tilde{K}_m}\tilde{Z}_{m,i}\right)\right] - \mathbb{E}\left[\tilde{X}_l\right]\mathbb{E}\left[\tilde{X}_m\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\left(\sum_{i=1}^{\tilde{K}_l}\tilde{Z}_{l,i}\right)\left(\sum_{i=1}^{\tilde{K}_m}\tilde{Z}_{m,i}\right)\right]|\tilde{K}_l = K_l, \tilde{K}_m = K_m\right] - \mathbb{E}\left[\tilde{X}_l\right]\mathbb{E}\left[\tilde{X}_m\right] \\
&= \mathbb{E}\left[\tilde{K}_l\tilde{K}_m\mathbb{E}\left[\tilde{Z}_l\tilde{Z}_m\right]\right] - \mathbb{E}\left[\tilde{X}_l\right]\mathbb{E}\left[\tilde{X}_m\right] \\
&= \mathbb{E}\left[\tilde{K}_l\tilde{K}_m\mathbb{E}\left[\tilde{Z}_l\right]\mathbb{E}\left[\tilde{Z}_m\right]\right] - \mathbb{E}\left[\tilde{X}_l\right]\mathbb{E}\left[\tilde{X}_m\right] \\
&= \mathbb{E}\left[\tilde{K}_l\tilde{K}_m\right]\mathbb{E}\left[\tilde{Z}_l\right]\mathbb{E}\left[\tilde{Z}_m\right] - \mathbb{E}\left[\tilde{X}_l\right]\mathbb{E}\left[\tilde{X}_m\right] \\
&= \mathbb{E}\left[\tilde{K}_l\tilde{K}_m\right]\mathbb{E}\left[\tilde{Z}_l\right]\mathbb{E}\left[\tilde{Z}_m\right] - \mathbb{E}\left[\tilde{K}_l\right]\mathbb{E}\left[\tilde{Z}_l\right]\mathbb{E}\left[\tilde{K}_m\right]\mathbb{E}\left[\tilde{Z}_m\right] \\
&= \mathbb{E}\left[\tilde{Z}_l\right]\mathbb{E}\left[\tilde{Z}_m\right]\left(\mathbb{E}\left[\tilde{K}_l\tilde{K}_m\right] - \mathbb{E}\left[\tilde{K}_l\right]\mathbb{E}\left[\tilde{K}_m\right]\right) \\
&= \mathbb{E}\left[\tilde{Z}_l\right]\mathbb{E}\left[\tilde{Z}_m\right]Cov\left(\tilde{K}_l, \tilde{K}_m\right) .
\end{aligned}
$$

Moreover, we observe that if we know the correlation between lines of business at aggregate claim amount level, we can derive the covariance and correlation between lines of business at number of claims level, and viceversa.

$$\rho\left(\tilde{X}_l, \tilde{X}_m\right) = \frac{Cov\left(\tilde{X}_l, \tilde{X}_m\right)}{\sigma\left[\tilde{X}_l\right]\sigma\left[\tilde{X}_m\right]}$$
$$= \frac{Cov\left(\tilde{K}_l, \tilde{K}_m\right)\mathbb{E}\left[\tilde{Z}_l\right]\mathbb{E}\left[\tilde{Z}_m\right]}{\sigma\left[\tilde{X}_l\right]\sigma\left[\tilde{X}_m\right]}$$
$$= \frac{\rho\left(\tilde{K}_l, \tilde{K}_m\right)\sigma\left[\tilde{K}_l\right]\sigma\left[\tilde{K}_m\right]\mathbb{E}\left[\tilde{Z}_l\right]\mathbb{E}\left[\tilde{Z}_m\right]}{\sigma\left[\tilde{X}_l\right]\sigma\left[\tilde{X}_m\right]}$$
$$= \rho\left(\tilde{K}_l, \tilde{K}_m\right)\frac{\sigma\left[\tilde{K}_l\right]\sigma\left[\tilde{K}_m\right]\mathbb{E}\left[\tilde{Z}_l\right]\mathbb{E}\left[\tilde{Z}_m\right]}{\sigma\left[\tilde{X}_l\right]\sigma\left[\tilde{X}_m\right]}.$$

Consequently we are able to determine the covariance between the reinsurer aggregate claim amount of different segments (element (iv)) as reported below.

$$Cov\left(\tilde{X}_l^{re,d}, \tilde{X}_m^{re,d}\right) = Cov\left(\tilde{X}_l^{re}\left(1-(1-q)\tilde{I}\right), \tilde{X}_m^{re}\left(1-(1-q)\tilde{I}\right)\right)$$
$$= (1-q)^2 Var\left[\tilde{I}\right] Cov\left(\tilde{X}_l^{re}, \tilde{X}_m^{re}\right)$$

$$Cov\left(\tilde{X}_l^{re}, \tilde{X}_m^{re}\right)$$
$$= \mathbb{E}\left[\left(\sum_{i=1}^{\tilde{K}_l}\tilde{Z}_{l,i}^{re}\right)\left(\sum_{i=1}^{\tilde{K}_m}\tilde{Z}_{m,i}^{re}\right)\right] - \mathbb{E}\left[\tilde{X}_l^{re}\right]\mathbb{E}\left[\tilde{X}_m^{re}\right]$$
$$= \mathbb{E}\left[\mathbb{E}\left[\left(\sum_{i=1}^{\tilde{K}_l}\tilde{Z}_{l,i}^{re}\right)\left(\sum_{i=1}^{\tilde{K}_m}\tilde{Z}_{m,i}^{re}\right)\right]|\tilde{K}_l = K_l, \tilde{K}_m = K_m\right] - \mathbb{E}\left[\tilde{X}_l^{re}\right]\mathbb{E}\left[\tilde{X}_m^{re}\right]$$
$$= \mathbb{E}\left[\tilde{K}_l\tilde{K}_m\mathbb{E}\left[\tilde{Z}_l^{re}\tilde{Z}_m^{re}\right]\right] - \mathbb{E}\left[\tilde{X}_l^{re}\right]\mathbb{E}\left[\tilde{X}_m^{re}\right]$$
$$= \mathbb{E}\left[\tilde{K}_l\tilde{K}_m\mathbb{E}\left[\tilde{Z}_l^{re}\right]\mathbb{E}\left[\tilde{Z}_m^{re}\right]\right] - \mathbb{E}\left[\tilde{X}_l^{re}\right]\mathbb{E}\left[\tilde{X}_m^{re}\right]$$
$$= \mathbb{E}\left[\tilde{K}_l\tilde{K}_m\right]\mathbb{E}\left[\tilde{Z}_l^{re}\right]\mathbb{E}\left[\tilde{Z}_m^{re}\right] - \mathbb{E}\left[\tilde{X}_l^{re}\right]\mathbb{E}\left[\tilde{X}_m^{re}\right]$$
$$= \mathbb{E}\left[\tilde{K}_l\tilde{K}_m\right]\mathbb{E}\left[\tilde{Z}_l^{re}\right]\mathbb{E}\left[\tilde{Z}_m^{re}\right] - \mathbb{E}\left[\tilde{K}_l\right]\mathbb{E}\left[\tilde{Z}_l^{re}\right]\mathbb{E}\left[\tilde{K}_m\right]\mathbb{E}\left[\tilde{Z}_m^{re}\right]$$
$$= \mathbb{E}\left[\tilde{Z}_l^{re}\right]\mathbb{E}\left[\tilde{Z}_m^{re}\right]\left(\mathbb{E}\left[\tilde{K}_l\tilde{K}_m\right] - \mathbb{E}\left[\tilde{K}_l\right]\mathbb{E}\left[\tilde{K}_m\right]\right)$$
$$= \mathbb{E}\left[\tilde{Z}_l^{re}\right]\mathbb{E}\left[\tilde{Z}_m^{re}\right]Cov\left(\tilde{K}_l, \tilde{K}_m\right).$$

Finally, we determine the covariance between the reinsurer aggregate claim amount and the insurer aggregate claim amount of another segment (element (v) of the list) as reported below:

$$Cov\left(\tilde{X}_l^{re}, \tilde{X}_m\right) = Cov\left(\tilde{K}_l, \tilde{K}_m\right)\mathbb{E}\left[\tilde{Z}_l^{re}\right]\mathbb{E}\left[\tilde{Z}_m\right].$$

□

Chapter 4

A simulative approach for estimating optimal reinsurance strategies

4.1 Introduction

In the actuarial literature, the first works on the topic of optimal reinsurance can be considered the seminal papers of Borch [16] and Arrow [6]. Since then, this field has been object of an intense activity of research. The many contributes, coming from both academics and practitioners, range from closed formulas, under general and specific settings, to simulating approaches. Regarding closed results, optimal reinsurance under mean and variance premium principles is analyzed in [50] which derives closed formulas for a quite general structure and in [25], which use as optimization criterion the minimization of a general two-dimensional function of the mean and variance of the insurer's total risk exposure. Optimal reinsurance under general risk measures is instead studied in [13], where are provided necessary and sufficient conditions for a wide family of risk measures. Quantile-based risk measures are instead employed in the analysis of optimal risk transfer to multiple reinsurer in [9]. The developed methodology allows to analyze high-dimensional problems in which the insurance company diversifies its risk with multiple reinsurance counterparties, where the insurer risk position and the premium charged by the reinsurers are functions of the underlying risk quantile. Closed form solutions are elaborated for some particular settings.

Optimal retention for a stop-loss reinsurance, with the objective of minimizing the value-at-risk (VaR) and the conditional tail expectation (CTE) of the total risks of an insurer is analyzed in [20]. Necessary and sufficient conditions for the existence of the optimal retentions are analyzed in both individual and collective risk model, showing that the optimal retention depends only on the assumed loss distribution and the reinsurer's safety loading factor. These results are then extended in [21] for a class of increasing convex ceded loss functions, under the expected value premium principle. The problem of minimization of VaR and CTE of the insurer retained risk by means of quota share and stop loss treaties is investigated also in [84]. Explicit forms are obtained under a constrained reinsurance premium in the case of quota-share after stop-loss reinsurance and viceversa. Considering optimal reinsurance for both parties is also a topic analyzed in the literature, as in [12], where the decision variable is the sensitivity of the retained (ceded) with respect to the total claims.

The relatively recent development of the European framework of Solvency II moved the focus of analysis of optimal reinsurance also in this context. In [11] optimal reinsurance strategies aiming at the minimization of risk exposure under Solvency II are analyzed. Two optimal reinsurance problems are formulated, depending on approach used for calculating the risk margin. In [27] the authors develop a Partial Internal Model based on the Solvency II framework, extending the classical collective risk model to also consider expense volatility. It is analyzed the effect of quota share and excess of less treaties on the exact moment of the distribution of technical result and investigated the effect of QS commission rates on the variability of distribution.

Following the interest in analyzing optimal reinsurance in the context of Solvency II, we also propose a stochastic model coherent with this framework. However, rather than focusing on a single objective, we base our model on a multi-objective optimization problem. This kind of problems aims at finding the solutions that optimize two or more objectives, typically conflicting. In the context of optimal reinsurance, most of the approaches used for solving a multi-objective problem are based on evolutionary strategies. Under the objective of maximizing the expected value of profit and minimizing the risk of retained losses, [69] follows this paradigm by building an evolutionary strategy (ES) that approximates the efficient frontier using a combination of four classical reinsurance structures: surplus, quota share, excess of loss and stop loss. [83], instead, approaches the topic in a different way, by introducing a flexible and efficient multi-objective simulation-based optimization framework for a multiline non-life insurance company under the Solvency II Directive, with the objective of jointly maximizing the solvency ratio and the profitability under specific constraints. A Pareto frontier is then employed to delete inefficient reinsurance treaties.

The use of Pareto frontier in the context of optimal reinsurance has been employed also in other research. In [49] Pareto-optimal reinsurance is studied in a framework that considers both insurer's and reinsurer's risks and return. The approach assumes to employ distortion risk measures with different distortion operators for determining the risks of insurer and reinsurer, as well as the reinsurance premium. Explicit expressions for the optimal reinsurance policies are derived with some constraint on reinsurance policy. Simple numerical methods are found for constructing insurance contracts that are Pareto and robust optimal in [10]. In this work, departing from classical formulation of reinsurance problem assuming that the underlying risk distribution is known, the authors aim at identifying a robust optimal contract that is not sensitive to the chosen risk distribution, obtaining closed-form solutions for the VaR and Worst-case scenario cases.

In our model we employ a simulative approach for the multi-objective optimization framework as in [83]. However, we differ from the existing literature by extending the analysis to consider the effect of default of the reinsurer in the simulative environment.

Moreover, we allow for the presence of multiple lines of business for the insurance company, multiple reinsurance treaties by multiple reinsurance companies and with potentially different ratings. In this way the result of the proposed optimization problem will not be limited to the characteristics of the reinsurance treaty, but it will take into account also the number of counterparties for each treaty and their respective credit rating. Moreover, the extended approach developed is enough general to be potentially extended to also consider additional characteristics of the different parties involved.

In [8] the optimal reinsurance arrangement is identified considering the presence or not of counterparty default risk. Closed form solutions are elaborated when the insurer's objective function is set via some well-known risk measures. In this work is found that optimal reinsurance contract does not usually change if the counterparty default risk is taken into account, unless buyer and seller have very different views on the reinsurer's recovery rate. [19] investigates the problem of optimal reinsurance from an insurer point of view, under a setting with presence of regulatory initial capital and counterparty default risk. Optimal reinsurance strategies that maximize the expected utility of an insurer's terminal wealth or minimize the VaR of an insurer's total retained risk are derived.

The following sections of this Chapter are organized as follows. In Section 4.2 we describe the general framework of the risk reserve equation and the algorithms for simulating the different elements of the equation. Moreover, we describe the multi-objective optimization framework for the selection of efficient reinsurance strategies. In Section 4.3 we present the numerical application of the simulative framework described before. We start from the description of the parameters used and the methodological choice for the various steps of the simulating procedure. We then propose a numerical analysis, developed for a multi-line non-life insurance company in order to describe the approach proposed. In this part we analyze the multi-objective efficient frontier under both a unconstrained and a constrained case, providing numerical and graphical evidence of the results. We compare the efficient frontier according to the Solvency Ratio obtained by means of the partial internal model we developed and according to the Standard Formula of Solvency II. Finally we refer to the Appendix for additional graphical analysis and sensitivities.

4.2 Stochastic approach for modeling risk reserve

In the previous chapter we showed the use of closed formulas in order to estimate the first two moments of the risk reserve distribution. These formulas have the advantage of not requiring any approximation derived by simulating approach. However, on the other hand, they limit the potential structure of reinsurance treaties, because of the complexity inherit in their modeling. For instance, we did not explore the possibility of having the presence of quota share and excess of loss treaties together or multiple lines of business and multiple reinsurers. Finally, while we could use distributional approximations (e.g. Normal approximation) for describing the distribution of risk reserve, in practice this assumption would be generally not respected. Moreover, being an approximation not precise for the tail of the distribution, it should not be employed for estimating the Solvency Capital Requirement of the firm. Simulating approaches, on the other hand, while requiring appropriate modelling assumptions and a much higher amount of computational resources, are able to derive the empirical distribution of risk reserve also for complex combinations of treaties with multiple features.

In the following we present an extension of the risk reserve equation, which considers the presence of multiple treaties for multiple lines of business and taking into account the default risk of reinsurance companies.

Hence, we describe a simulative approach for estimating the distribution of the risk reserve in a one-year time horizon and consequently the estimation of the metrics of risk and return related to this random variable. Finally, we show how this approach can be used for computing the efficient frontier of optimal reinsurance strategies.

In Formula (4.1) it is described this complete risk reserve equation:

$$\tilde{U}_1 = U_0(1+j) + \sum_{l=1}^{L} \left[(B_l - B_l^{re}) - \left(\tilde{X}_l - \sum_{r=1}^{R} \tilde{X}_l^{re,d(r)} \right) - \left(E_l - \sum_{r=1}^{R} \tilde{C}_l^{re(r)} \right) \right] (1+j)^{1/2} \,. \tag{4.1}$$

4.2.1 Pricing of insurance contracts

In our modeling we assume that the insurer determines the premium according to the theoretical pricing approach described in Section 2.1.2. Assuming that the insurer uses the expected value premium principle (for the safety loading) and a loading for the expenses equal to their expectation we derive the usual formula for gross premium reported below

$$B = \mathbb{E}\left[\tilde{X}\right](1+\lambda) + cB = \mathbb{E}\left[\tilde{K}\right]\mathbb{E}\left[\tilde{Z}\right]\frac{(1+\lambda)}{(1-c)}$$

where the elements are explained in Section 2.1.2.

Under this approach, the insurer only needs the first central moment of the aggregate claim amount, which means (remembering collective risk theory approach) the first central moment of number of claims and claims amount random variables.

In our model we follow the approach described in Algorithm 1 for defining the gross premium of the insurer, necessary for calculating the risk reserve of Formula (4.1).

4.2.2 Pricing of reinsurance contracts

For the reinsurance company, the premium is obtained according to the approaches described in Section 2.1.3 and 2.1.4. In particular, the pricing of QS is based on ceded commission, while XL on experience pricing.

For the QS treaty, we assume that the reinsurer wants to achieve a certain margin on the ceded premium and sets the ceded commission accordingly. In the general

Algorithm 1 Pricing of insurance contracts.

L: number of segments
for l = 1 to L **do**
  Calculate the mean of random variables: number of claims $\tilde{K}_l$ and single claim amount $\tilde{Z}_l$
  Calculate the gross premium of segment l as: $B_l = \mathbb{E}\left[\tilde{K}_l\right]\mathbb{E}\left[\tilde{Z}_l\right]\frac{(1+\lambda_l)}{(1-c_l)}$
end for

case, where the reinsurer uses a sliding scale commission, the pricing is still based on the same approach. However, in this case the reinsurer shall define an upper and lower threshold such that the expected ceded commission is equal to the ceded commission which assures the achievement of the desired margin.

The use of experience pricing in XL reinsurance is described in Section 2.1.4. As already explained, using the expected value principle could lead to a underestimation of the risk, since it neglects the variability. For this reason, we assume that the XL reinsurer bases its pricing on the standard deviation principle. In particular, since we are in a simulative framework and do not have at disposal the real pricing structure of the reinsurers, we assume that all the firms calibrate their safety loading coefficient by means of the "equivalent loading" approach described in Formula (2.20) under the standard deviation premium principle.

It shall be noted that we are assuming a pricing separate for each reinsurer. An important consequence is that, in case there is a "layer segmentation", since we assumed the standard deviation premium principle, there is an increase in the total price of reinsurance with an increase of segmentation of the total layer (see Formula (2.28) in Section 2.1).

A last important point concerns the effect of default risk in pricing of QS and XL treaties. Indeed, the possibility that the reinsurance company will default represents a risk for the insurer, which reduces the potential benefit deriving from the reinsurance contract. For this reason a coherent reinsurance pricing should account for its probability of default in the price charged to the insurer.

For defining the reinsurance pricing structure in presence of default we start from these considerations:

- An "ideal" reinsurer which has a probability of default equal to 0 should set its pricing in line with the theoretical approach described.
- A reinsurer with a probability of default lower than 1 should set its pricing in order to have a positive (> 0) technical result[1].

Related to the second point, we assume that it is not possible to have a treaty with a reinsurer with a probability of default equal to 1. Indeed, if it were possible, it

[1]In practice, both insurer and reinsurer could price a contract with an expected loss, for instance in case of expansion in a new line of business, commercial reasons, etc. However this approach should be followed for a limited time/scope since, as described in [30], it would lead to a default with probability 1.

would have no cost since it is equivalent to a no-reinsurance case from the point of view of the insurer.

Hence, we build a pricing for reinsurer in order to satisfy the constraints described above. In particular, we create a connection between the rating of the reinsurer (and its probability of default) with the discount applied in pricing.

The pricing of an excess of loss treaty for a defaultable reinsurer is described in Formula (4.2).

$$B^{re} = \begin{cases} \mathbb{E}\left[\tilde{X}^{re}\right] + \beta\sigma\left[\tilde{X}^{re}\right] & p = 0 \\ \mathbb{E}\left[\tilde{X}^{re}\right] + \delta\beta\sigma\left[\tilde{X}^{re}\right] & p \in (0,1) \end{cases} \tag{4.2}$$

where p represents the probability of default of the reinsurance company and $\delta \in (0,1)$ the "discount factor" for compensating for the probability of default. In practice, we assume that in case the XL reinsurer has a probability of default greater than 0 then the loading component is decreased accordingly.

The main element in the pricing of an quota share treaty is the ceded commission, or equivalently the ceded commission rate. In our model, we assume that the QS reinsurer wants to achieve a certain margin from the contracts, then as described in Formula (2.22), the corresponding ceded commission (for a non-defaultable reinsurer) is reported below.

$$c^{re*} = \min\left(0, 1 - \frac{\mathbb{E}\left[\tilde{X}^{re}\right]}{B^{re}} - \mathbb{E}\left[M\right]\right)$$

Instead, for the general case of a defaultable reinsurer, in Formula (4.3) it is reported the corresponding pricing of ceded commission.

$$c^{re} = \begin{cases} c^{re*} & p = 0 \\ \min\left(0, 1 - \frac{\mathbb{E}\left[\tilde{X}^{re}\right]}{B^{re}} - \delta\mathbb{E}\left[M\right]\right) & p \in (0,1) \end{cases} \tag{4.3}$$

where c^{re*} represents the "base" ceded commission of the non-defaultable reinsurer case and $\delta \in (0,1)$ the "discount factor" for compensating for the probability of default. In practice, we assume that in case the QS reinsurer has a probability of default greater than 0 then the reinsurer technical margin component is decreased accordingly.

It shall be remarked that we allow the insurer to underwrite both XL and QS treaties in the same line of business. Following the usual market practice, we build our model assuming that the order of application of the treaties consists in quota share following excess of loss. Hence, the pricing of quota share treaties modifies as reported Formula (4.4)-(4.5)-(4.6).

$$B_l^{qs} = (1-\alpha_l)\left(B_l - B_l^{xl}\right) \tag{4.4}$$

$$c_l^{qs*} = \min\left(0, 1 - \frac{\mathbb{E}\left[\tilde{X}_l^{qs}\right]}{B_l^{qs}} - \mathbb{E}\left[M_l\right]\right) \tag{4.5}$$

$$c^{re} = \begin{cases} c^{qs*} & p = 0 \\ \min\left(0, 1 - \frac{\left(\mathbb{E}[\tilde{X}] - \mathbb{E}[\tilde{X}_l^{xl}]\right)}{\left(B_l - B_l^{xl}\right)} - \delta\mathbb{E}\left[M_l\right]\right) & p \in (0,1)\ . \end{cases} \tag{4.6}$$

In our model we follow the approach described in Algorithm 2 for defining the gross premium of the reinsurers, necessary for calculating the risk reserve of Formula (4.1).

There are many possible ways for defining the discount factor δ as a function of the probability of default/rating of the counterparty. In our model we choose to link the δ factor with the CQS of the reinsurance company, in order to be consistent with the metric used in the Standard Formula of Solvency II. The function we choose for linking these elements is described in Formula (4.7).

$$\delta_{pd} = f(CQS) = (1 - (CQS+1)\,Q)^D \tag{4.7}$$

where $Q < 1$ is the discount quota and $D > 0$ is a power function for modeling the strength of the increase in discount for an increase in CQS. For $D = 1$ we have a linear decrease of the delta factor. For $D > 1$ we have a decrease of the delta factor more than proportional to the increase in CQS, while the opposite holds for $D < 1$. Hence, we can set the parameters Q and D in order to create the effect we want.

Simulation of the aggregate claim amount

Since we are developing a stochastic model for the risk reserve, our main interest is focused on modeling the random variables of Formula (4.1) and subsequently computing numerical estimations of their moments.

The aggregate cost of claims, as described in detail in Section 3.2.2, is modeled through a collective risk model depending on two other random variables: number of claims and cost of single claim.

Once we have chosen an appropriate distribution for the two random variables, we can follow the approach described in Algorithm 3 for retrieving the vector of simulations of the aggregate claim amount.

It shall be noted that if we follow this approach we lose the information on the cost of single claim, which is necessary for the subsequent application of reinsurance treaties. At the same time, however, it would be inefficient to store the information on each claim, since a QS treaty (using the same proportion on each claim) can be applied directly to the aggregate claim amount and XL works only for claims over the deductible. For these reasons, in addition to aggregate claim amount, we

Algorithm 2 Pricing of reinsurance contracts.

L: number of segments
R: number of reinsurers
for l = 1 to L **do**
  for r = 1 to R **do**
    if Reinsurer r provides a XL treaty for the segment l **then**
      Apply Formula (4.2) using the specific credit quality step (or the probability of default) of the reinsurer $CQS^{(r)}$ for calibrating $\delta_{pd}\left(CQS^{(r)}\right)$
    end if
  end for
  for r = 1 to R **do**
    if Reinsurer r provides a QS treaty for the segment l **then**
      Calculate the ceded commission to return to insurer in order to satisfy the required margin, according to Formula (4.5)
      Apply Formula (4.4) and (4.6) using the specific credit quality step (or the probability of default) of the reinsurer $CQS^{(r)}$ for calibrating $\delta_{pd}\left(CQS^{(r)}\right)$
    end if
  end for
end for

Algorithm 3 Simulation of the aggregate claim amount.

L: number of segments
n_{sim}: number of simulations
for l = 1 to L **do**
  for s = 1 to n_{sim} **do**
    Simulate the number of claims of the l-th line of business $\tilde{K}_l^s$ from its distribution, obtaining a certain number k_l^s
    for i = 1 to k_l^s **do**
      Simulate the cost of the i-th claim $\tilde{Z}_{l,i}^s$ from its distribution, obtaining a certain amount $Z_{l,i}^s$
    end for
    Calculate the aggregate claim amount of the s-th simulation as: $X_l^s = \sum_{i=1}^{k_l^s} Z_{l,k}^s$
  end for
end for

can store just the information on the claims whose cost is higher than a certain threshold, set as the lowest deductible that could be applied for an XL treaty.

Hence, in Algorithm 4 we present a modified version of Algorithm 3 for simulating the random variables and storing information of large claims and aggregate claim amount.

Algorithm 4 Simulation of the aggregate claim amount and storing of results.

L: number of segments
n_{sim}: number of simulations
$threshold$: threshold from which consider a claim as large
$output$: list for storing output from the simulations
$large$: list for storing large claims
for l = 1 to L **do**
  for s = 1 to n_{sim} **do**
    Simulate the number of claims of the l-th line of business $\tilde{K}_l^s$ from its distribution, obtaining a certain number k_l^s
    for k = 1 to k_l^s **do**
      Simulate the cost of the i-th claim $\tilde{Z}_{l,i}^s$ from its distribution, obtaining a certain amount $Z_{l,i}^s$
      if $Z_{l,k}^s > threshold$ **then**
        $large_{l,s} = c(large_{l,s}, Z_{l,k}^s)$
      end if
    end for
    Calculate the aggregate claim amount of the s-th simulation as: $X_l^s = \sum_{i=1}^{k_l^s} Z_{l,k}^s$
    $output_{l,s} = X_l^s$
  end for
end for

At this point we have obtained the simulations of claims for the different lines of business where the insurer operates. However, the modeling described up to this point makes an implicit assumption which was empirically observed not correct. The simulation of claims is performed separately for each lines of business, which means that the aggregate claims cost of each line of business is independent of the others. In practice, both conceptual and empirical evidences show that there is a positive correlation between non-life segments, with a specific, different, intensity. In Solvency II, for instance, is defined a correlation matrix (calibrated on impact studies based on market data) for describing the dependence between the claims of each line of business (see Annex IV of [36] and the corresponding table for Premium risk reported in 2.5).

Copula function is one of the approaches most used in practice for creating a dependence structure between random variables. It is an instrument for describing multivariate distributions, with strong flexibility on the modeling of the dependence between marginals. The wide diffusion of copulas is mostly related to the results of Sklar theorem [75], which is described below.

Theorem 4.1. *Every multivariate cumulative distribution function $F(x_1, ..., x_d)$ can be expressed in terms of its marginals $F(x_i)$ $(i = 1, ...d)$ and a copula function C. The cumulative distribution function of a generic multivariate distribution can be expressed as:*

$$F(x_i, ..., x_d) = C(F(x_i), ..., F(x_d)) \ . \tag{4.8}$$

The converse is also true: for any copula C and marginals $F_i(x)$ then the function $C(F_1(x_1), ..., F_d(x_d))$ defines a d-dimensional cumulative distribution function with margins $F_i(x)$.

The importance of Sklar theorem depends on the fact that it allows a simple approach for simulating a multivariate distribution, needing just its marginal distributions and a copula function.

In literature there are many types of copulas, which can be divided in two categories: elliptical (or Gaussian) and Archimedean.

The first group is represented by copulas "related with multivariate normal distribution", which includes Gaussian and Student's t copula. The characteristic of this family is that the dependence between marginals is described by means of a single "correlation matrix" which holds for the whole distribution.

Archimedean copulas are more "flexible", in the sense that they allow the modeling of dependence in specific areas of the distributions, for instance in the tails. However, differently from elliptical copulas, the dependence can be defined only between two marginals.

In the actuarial context copulas are often applied for the aggregation of dependent risks. In our specific case we are interested in the application of copula for modeling the dependence between claims of different (but dependent) lines of business. Having already the marginal distributions of each segment from the simulations described in Algorithm 4 it is now necessary to choose an appropriate dependence structure.

As already explained, a Gaussian copula needs just a correlation matrix between the segments, but it assumes the same dependence for all the distribution. On the other hand, an Archimedean copula requires the definition of the "area of dependence" (e.g. whole distribution, right tail, left tail, etc.) and create a hierarchical flow of dependence for couple of segments.

In our model we choose to use a Gaussian copula for defining the dependence between claims of different lines of business. In this way we need just a single correlation matrix and we avoid the needs for choosing the order of aggregation between LoBs, as would be necessary in case of Archimedean copulas.

In Algorithm 5 is described the approach that shall be followed for creating a dependence structure between the aggregate claim amount of the lines of business by means of a Gaussian copula with a given correlation matrix.

At this point we have obtained all the elements necessary for assessing the risk reserve of the insurance company in the gross of reinsurance case. Hence, we can

Algorithm 5 Creation of dependence by means of Copula.

L: number of segments
n_{sim}: number of simulations
$corrMatrix$: correlation matrix for describing the dependence between lines of business
Simulate a Gaussian Copula C by means of the correlation matrix $corrMatrix$
for l = 1 to L **do**
  Extract the vector of rank of the l-th column of C
  Sort the $output_l$ object (containing the figures of simulation of l-th line of business) in ascending order according to the value of the aggregate claim amount X_l^s
  Reorder the sorted elements according to the vector $C[, l]$
end for

now move to the description of the algorithms for simulating the elements related to the reinsurance companies.

Simulation of reinsurer claims, default and recovery event

Having at disposal the simulations of the aggregate claim amount, in Algorithm 6 we describe the approach for computing the aggregate claim amount borne by each reinsurance company. As expected, we have two different approaches for XL and QS treaties, where the first requires the single claims for each simulation (saved in the *large* object as described in Algorithm 4) while the latter only needs the total amount and the claims paid by XL reinsurers just derived. Indeed, as for the premium, also the claims are regulated with QS net of XL, i.e. it is first necessary to apply all the excess of loss treaties before the quota share ones.

In the event of default, the reinsurance company is no more able to meet its contractual obligations, meaning that the insurer could not receive back all the credits it holds. Indeed, usually just a portion of the whole credit is recovered in case of default of a counterparty. This percentage, called recovery rate, is typically very volatile since it depends on many circumstances and it is not easily modeled. Standard Formula of Solvency II, for instance, assumes the recovery rate as a deterministic quantity and implicitly equal to 50%, regardless of the rating of the counterparty (see Formula (2.53) of Section 2.3.5).

In the literature, also due to the low number of observations, there is not a standard assumption on the link between recovery rate and other elements of the (defaulted) firm, especially in the insurance sector. In the banking sector it is common practice to link the recovery rate with the “quality” of a financial instrument, represented by its seniority. In CreditMetrics, for instance, it is assumed a Beta distribution for the recovery rate, whose parameters are linked with the seniority of the financial instrument. This assumption is based on the results of the analysis reported in [22] and in [2] which report the recovery statistics as a function of the seniority class of corporate bonds. These results indeed show the (empirical) existence of a relation between the quality of a bond (measured by seniority) and the expected recovery

Algorithm 6 Simulation of the aggregate claim amount borne by the reinsurer.

L: number of segments
R: number of reinsurers
n_{sim}: number of simulations
$output$: list storing output from the simulations
$large$: list storing large claims
for l = 1 to L **do**
  for s = 1 to n_{sim} **do**
    for r = 1 to R **do**
      if Reinsurer r provides a XL treaty for the segment l **then**
        Apply Formula (2.9) for calculating the cost of claims for the excess of loss reinsurance: $X^{xl,(r)s} = \sum \min\left(\max\left(0, large_l^s - d\right), l\right)$
      end if
    end for
    for r = 1 to R **do**
      if Reinsurer r provides a XL treaty for the segment l **then**
        Apply Formula (2.1): for calculating the cost of claims for QS reinsurer after the XL cession: $X_l^{qs,(r)s} = \alpha_l^{(r)} \left(X_l^s - \sum_{r=1}^{R} X^{xl,(r)s} \right)$
      end if
      Obtain the total amount of claims due to reinsurer r according to: $X_l^{(r)s} = X_l^{xl,(r)s} + X_l^{qs,(r)s}$
    end for
  end for
end for

rate in case of default. In our context, however, we do not have an element similar to the seniority of a bond to define a functional dependence with the recovery rate, so we seek a different relation.

Another possibility that we could explore is the link between probability of default and recovery rate. This dependence, although not present in most of the credit models, included CreditMetrics, is empirically suggested by an increasing number of research. Among others, [42] shows evidences of the presence of a negative correlation between probability of default and recovery rate, while [3] provides an extensive review of the findings regarding this relationship and the impact of the different modeling choices on procyclicality of regulatory capital. Finally, the more recent work [18] proposes an econometric model in which default rates and recovery rate distributions are modeled by means of the same underlying process, interpreted as the "credit cycle". Hence, we also follow these findings and model the recovery rate as a stochastic variable with a negative dependence with the probability of default.

In the context of our model we extend Formula (3.25) of the risk reserve equation for including this additional stochasticity, as reported in Formula (4.9):

$$\tilde{X}^{re,d} = \tilde{X}^{re} - (1-\tilde{q})\tilde{X}^{re}\tilde{I} = \begin{cases} \tilde{X}^{re} & 1-p \\ \tilde{X}^{re}\tilde{q} & p \end{cases} \tag{4.9}$$

where $\tilde{q}$ is a random variable describing the recovery rate by means of an appropriate distribution conforming to the assumed support between 0 and 1

Similarly to the "discount factor", also in this case we create a functional (negative) dependence between the mean of recovery rate and the probability of default/CQS of the reinsurance company. We report the relation in Formula (4.10)

$$\delta_q = f(CQS) = base_q \left(1 - (CQS+1)\,Q\right)^D \tag{4.10}$$

where $base_q$ represents the base value of recovery rate that we want to set for a firm with CQS $= 0$, $Q < 1$ is the discount quota and $D > 0$ is a power function for modeling the strength of the decrease of recovery rate for an increase in CQS. It is possible to observe that the modeling we chose to employ for describing the relation between recovery rate and probability of default has the same structure of the one used for modeling the discount connected with the CQS.

An important remark about the assumption of negative dependence between probability of default and recovery rate is that this hypothesis could have a strong impact on the selection of the reinsurer by the insurance company. Indeed, this modeling "amplifies the extremes", since it creates a double penalization to the reinsurance company with a high probability of default, for which it also assumes a low recovery rate, and the opposite effect for firms with a low probability of default.

In Algorithm 7 we describe the approach for simulating the default event and the recovery rate event in case of default. One relevant point is that we should underline is that, for a given simulation, the probability of default of all the reinsurance company shall be based on the same common shock. Indeed, in the algorithm we have an initial simulation of the common shock variable and only after then the definition of the "shocked" probability of default of each reinsurer.

Algorithm 7 Simulation of default event and recovery rate.

R: number of reinsurers
n_{sim}: number of simulations
b: baseline probability of default
for s = 1 to n_{sim} **do**
  Simulate the common shock variable, using the distribution function defined in Formula (2.33): $f(c) = \alpha c^{\alpha-1}$
  for r = 1 to R **do**
    Calculate the probability of default of the r-the reinsurer under common shock variable c, by means of Formula (2.34): $p(c)^{(r)} = b^{(r)} + (1 - b^{(r)})c^{\tau/b^{(r)}}$
    Simulate the default event of r-the reinsurer in the s-th simulation: $\tilde{I}^{(r)s} \sim Bernoulli\left(p(c)^{(r)}\right)$
    if $I^{(r)s} = 1$ **then**
      Apply Formula (4.10) to the credit quality step (or the probability of default) of the reinsurer to calibrate the expected value parameter of its recovery rate distribution
      Derive the parameters α and β of the Beta distribution from the calibrated expected value and standard deviation of recovery rate
      Simulate the recovery rate of the corresponding default event: $\tilde{q}^{(r)s} \sim Beta\left(\alpha^{(r)}, \beta^{(r)}\right)$
    end if
  end for
end for

4.2.3 Risk/return metrics, multi-objective optimization and Pareto frontier

Following the risk reserve equation of Formula (4.1) we are now able to compute its value for the different reinsurance strategies considered. Hence, we can propose the metrics of risk and return for the evaluation of the result of the insurance company under the different choice of reinsurance strategies (both from treaty perspective and counterparty quality).

Regarding the metrics of return, the most common one that we are interested in analyzing is the expected Return on Equity (RoE), as reported in Formula (4.11)

$$\mathbb{E}\left[\widetilde{RoE}_{t+1}\right] = \frac{\mathbb{E}\left[\tilde{U}_{t+1}\right]}{U_t} - 1 \,. \tag{4.11}$$

Regarding risk there are two main metrics that we can propose, which are connected each other. The first one is the ruin probability (RP), which represents the probability that the insurance company goes bankrupt in the specific (in our case one-year) time horizon. As reported in Formula (4.12) it is computed as the probability that the insurer capital goes below 0.

$$RP = 1 - F_{U_{t+1}}(0) \tag{4.12}$$

where $F_{U_{t+1}}(u)$ represents the c.d.f. of the $\tilde{U}_{t+1}$.

The other metric is the Solvency Capital Requirement, which represents the capital that the company has to hold for being solvent at 99.5% confidence level in a one-year time horizon. This metric can be computed both at an "internal model" level, using the estimate of the risk reserve equation that we defined or at "standard formula" level, using the approach prescribed by Solvency II Standard Formula and outlined in Section 2.3. In practice, we propose a transformation of the SCR which is more interpretable and permits the comparability between different insurance entities which is the solvency ratio. It represents the ratio between the initial capital and the SCR and is computed as reported in Formula (4.13) or Formula (4.14), in case of internal model or standard formula respectively.

$$SR_{IM} = \frac{U_t}{SCR_{IM}} = \frac{U_t}{-q_{0.5\%}\left(\tilde{U}_{t+1} - U_t\right)} \tag{4.13}$$

$$SR_{SF} = \frac{U_t}{SCR_{SF}} \,. \tag{4.14}$$

The objective of our multi-objective optimization framework is to find the reinsurance strategies that maximize/minimize all the chosen metrics simultaneously. In order to do so, we define a preference structure for choosing between the different strategies. We say that strategy S_1 dominates another strategy S_2 and indicate it with $S_1 \succeq S_2$ if it is at least better in one objective function and equal in all the others:

$$\begin{aligned} O_1(S_1) &\geq O_1(S_2) \\ O_2(S_1) &\geq O_2(S_2) \\ &\cdot \quad \cdot \quad \cdot \\ O_n(S_1) &\geq O_n(S_2) \end{aligned}$$

where $O_i(S_k)$ represents the value of the i-th objective function in case of application of strategy S_k.

More formally, defining M the set of possible reinsurance strategies and m a generic strategy, we have the optimization problem reported in (4.15).

$$\max_{m} \left(O_1, O_2, ..., O_n\right) \tag{4.15}$$

subject to:

$$\begin{cases} m \in M \\ f_1(m) \in [l_1, u_1] \\ f_2(m) \in [l_2, u_2] \\ \cdots \\ f_k(m) \in [l_k, u_k] \end{cases}$$

where $f_i(\cdot)$ represents the i-th constraint function, while l_i and u_i the lower and upper bound of the i-th constraint.

The strategies that are not dominated by any other are called efficient and define the Pareto Frontier.

In Algorithm 8 we report the procedure for obtaining the efficient frontier.

Algorithm 8 Efficient frontier.

M: matrix containing all the combinations of input (reinsurance treaty) and corresponding output (metrics of risk and return)
$Risk$: vector containing all the risk values for each combination of inputs
$Return$: vector containing all the return values for each combination of inputs
row_M: number of combinations (rows) in the matrix M
for $i = 1$ to row_M **do**
  Extract the subset of treaties M_i where $Risk = Risk_i$
  Among the subset of treaties select the one which has the highest return: $\max(Return|M_i)$
end for
Order the vector of treaties obtained by $Risk$ in increasing order
M^*: matrix containing all the combinations of input (reinsurance treaty) and corresponding output (metrics of risk and return) after previous step
row_M^*: number of combinations (rows) in the matrix M^*
for $j = 1$ to row_M^* **do**
  if $Risk_j \leq Risk_{j-1}$ **then**
    Remove the corresponding treaty M_j^* from the list of the efficient treaties
  end if
end for
The remaining elements in the list correspond to the optimal treaties

At this point the insurance company can choose one of the strategies on the efficient frontier. In practice, another relevant aspect concerns the criteria for choosing the preferred strategy. Typically, insurance companies have a certain risk appetite which could help in choosing the best trade-off between risk and return. Here, we propose a simple approach that could be employed in case of "agnostic" preference. In Formula (4.16) we report the equation of the marginal increase (MI) in Solvency Ratio for a marginal reduction in the expected Return on Equity with respect to the gross of reinsurance scenario.

$$\begin{aligned} MI &= \frac{\left(\frac{SR(m)-SR(gross)}{SR(gross)}\right)}{\left(\frac{\mathbb{E}\left[\widetilde{RoE}(gross)\right]-\mathbb{E}\left[\widetilde{RoE}(m)\right]}{\mathbb{E}\left[\widetilde{RoE}(gross)\right]}\right)} \\ &= \frac{SR(m)-SR(gross)}{\mathbb{E}\left[\widetilde{RoE}(gross)\right]-\mathbb{E}\left[\widetilde{RoE}(m)\right]}\frac{\mathbb{E}\left[\widetilde{RoE}(gross)\right]}{SR(gross)}\,. \end{aligned} \tag{4.16}$$

Hence, the criterion which we propose is to choose the reinsurance strategy with the maximum marginal increase in SR for a marginal reduction in the expected RoE with respect to the gross of reinsurance case.

4.3 Numerical application

4.3.1 Description and calibration of insurer parameters

We calibrate the parameters of our "average" insurer by means of the market data available from a report [4] of the "Associazione Nazionale fra le Imprese Assicuratrici" (ANIA).

First of all, for each of the L segments where the insurer operates, we calculate the "market" combined ratio (CR) and expense ratio (ER) from historical data, according to Formula (4.17)-(4.18).

More specifically, we estimate these figures as weighted average of the observations of the last 5 years, with weights equal to the gross premium of the year[2].

$$\widehat{CR} = \frac{\sum_{y=y_1}^{y_n} B_y CR_y}{\sum_{y=y_1}^{y_n} B_y} \tag{4.17}$$

$$\widehat{ER} = \frac{\sum_{y=y_1}^{y_n} B_y ER_y}{\sum_{y=y_1}^{y_n} B_y}\,. \tag{4.18}$$

where $y_1, ..., y_n$ indicate the years used for estimating the two metrics.

The estimate of loss ratio is then obtained as the difference between the previously estimated combined ratio and expense ratio: $\widehat{LR} = \widehat{CR} - \widehat{ER}$.

Hence, remembering the relation $P + \lambda P = B(1 - c)$ the safety loading coefficient λ is obtained according to Formula (4.19), using the estimates of combined ratio and loss ratio obtained at previous stage.

$$\widehat{\lambda} = \frac{1 - \widehat{CR}}{\widehat{LR}}\,. \tag{4.19}$$

The last elements that we need for simulating the claims of the segments we are interested in is the standard deviation of the over-dispersion parameter. In order to estimate this element by means of market data, remembering the relation described in Formula (4.20), first we need to estimate the standard deviation of the loss ratio.

$$\lim_{k\to\infty} \sigma\left[\widetilde{LR}\right] = \lim_{k\to\infty} \sigma\left[\frac{\tilde{X}}{B}\right] = \lim_{k\to\infty} \sigma\left[\frac{\tilde{X}}{P}\right]\frac{P}{B} = \lim_{k\to\infty} \sigma\left[\tilde{Q}\right]\frac{P}{B} \tag{4.20}$$

[2] In order to have a coherent value of these figures we excluded accounting year 2020 from the analysis since it was strongly affected by the Covid effects.

Table 4.1. Parameters of the lines of business.

LoBs	$\mathbb{E}\left[\tilde{K}_l\right]$	$\sigma\left[\tilde{Q}_l\right]$	$\mathbb{E}\left[\tilde{Z}_l\right]$	$CoV\left[\tilde{Z}_l\right]$	pl_l	ER_l	LR_l	CR_l
MTPL	50,000	7.47%	4,500	6	10,000,000	21.4%	77.7%	99.1%
MOD	25,000	7.01%	1,500	2	1,000,000	31.6%	61.9%	93.5%
GTPL	15,000	15.39%	6,000	10	10,000,000	32.7%	59.6%	92.3%

Table 4.2. Correlation matrix between lines of business.

Corr	MTPL	MOD	GTPL
MTPL	1.00		
MOD	0.50	1.00	
GTPL	0.50	0.25	1.00

$$\sigma\left[\tilde{Q}\right] = \sigma\left[\widetilde{LR}\right]\frac{B}{P} = \sigma\left[\widetilde{LR}\right]\frac{(1+\lambda)}{(1-c)}.$$

Hence, we used the observed loss ratio in the last 15 years in order to estimate its standard deviation. In this way, using the elements previously estimated, we derive the standard deviation of the over-dispersion parameter as reported in Formula (4.21).

$$\hat{\sigma}\left[\tilde{Q}\right] = \hat{\sigma}\left[\widetilde{LR}\right]\frac{(1+\hat{\lambda})}{(1-\hat{c})}. \tag{4.21}$$

Following the approach just described we can derive the parameters of all the lines of business of the Italian market. However, in order to keep the focus of our numerical analyses on the specific problem covered in this chapter, we assume an insurer operating in only three segments. In particular, we assume a non-life insurer operating in the motor third-party liability (MTPL), motor own damage (MOD) and general third-party liability (GTPL) lines of business. The parameters of these three LoBs we are interested in, obtained according to the methodology described above, are then reported in Table 4.1. In addition to the elements we commented before there are two additional terms: the coefficient of variation of the severity random variable $CoV\left[\tilde{Z}_l\right]$ and the policy limit pl_l. These elements cannot be estimated by means of the data available in ANIA report, so were derived from more general market reports and following the assumptions used by other papers on the same subject (see for instance [27] and [83]).

The dependence structure between lines of business is obtained by means of a Gaussian copula, where the parameters are chosen to match the correlation matrix of Solvency II Standard Formula. In our specific context, the dependence matrix between the lines of business is reported in Table 4.2

At this point, we shall define the size of the insurer of our analysis. We assume it is an "average insurer" in the Italian market, setting its volume measures (number of

Table 4.3. Parameters of the insurance company.

LoBs	P_l	λ_l	c_l	B_l
MTPL	225,000,000	1.10%	21.4%	289,408,397
MOD	37,500,000	10.5%	31.6%	60,581,140
GTPL	90,000,000	12.9%	32.7%	150,980,681

claims, premium, etc.) in the range between the 10-th and 15-th insurer of each line of business. Table 4.3 reports the parameters of the modeled insurer (premium and safety/expense loading) used for the simulations.

Finally, regarding the initial capital available by the insurance company we set it equal to 15% of the gross earned premium of next year: $U_t = 15\% B_{t+1}$.

4.3.2 Simulation of the aggregate claim amount

Given the equation of aggregate claim amount of a generic line of business l, reported in Formula 4.22, we have to make specific assumptions on the distribution of the random variable $\tilde{K}$ and $\tilde{Z}$ in order to apply Algorithm 4 for the simulation.

$$\tilde{X}_l = \sum_{k=1}^{\tilde{K}_l} \tilde{Z}_{l,k} \,. \tag{4.22}$$

Regarding number of claims a usual assumption is that they are distributed according to a Poisson distribution. However, simply using a Poisson distribution has the disadvantage of the implicit assumption that the variance is equal to the mean. In practice, it has been empirically shown that the variance is greater than the mean, and that this variability is strictly connected with the line of business. Hence, we accordingly move from a standard Poisson process to a compound Poisson process, assuming that the parameter of Poisson distribution is itself a random variable.

More precisely, we assume that there is a perturbation (over-dispersion) parameter, governing the volatility of risk propensity, which is applied to the number of claims parameter of the Poisson distribution.

The assumption is that the random variable number of claims is distributed as an over-dispersed Poisson with parameter $n\tilde{Q}$ as reported in (4.23).

$$\tilde{K} \sim Poisson\left(n\tilde{Q}\right) \tag{4.23}$$

where n represents the expected number of claims and $\tilde{Q}$ the perturbation parameter.

This last random variable is assumed to be distributed as a Gamma. In particular, as reported in (4.24), we assume a Gamma distribution of equal parameters, which does not change the expected number of claims, since $\mathbb{E}\left[\tilde{Q}\right] = 1$, but it creates a "non-diversifiable variability" $\sigma\left[\tilde{Q}\right]$.

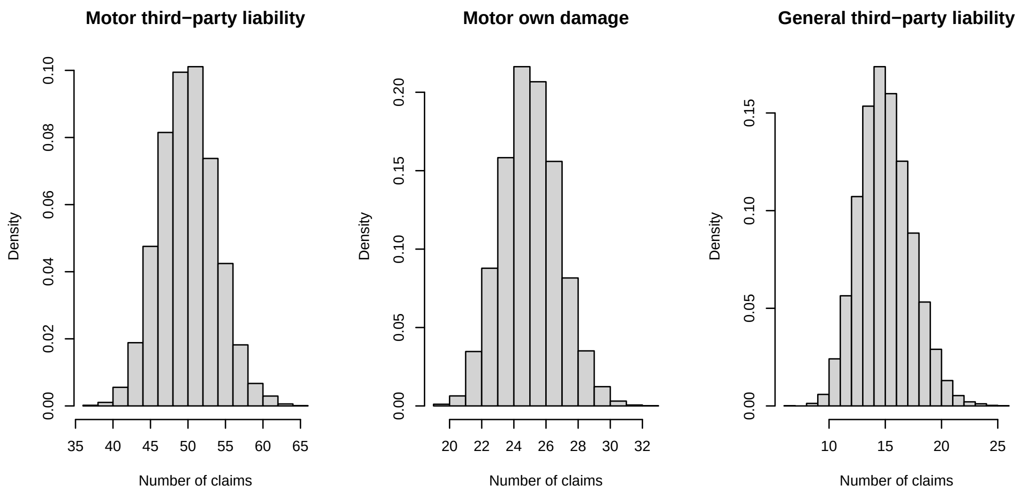

Figure 4.1. Distribution of simulated number of claims distribution (values in k).

$$\tilde{Q} \sim Gamma\left(h, h\right) . \tag{4.24}$$

Figure 4.1 shows the distribution of number of claims obtained by simulation assuming the over-dispersed Poisson distribution with the parameters reported in Table 4.3. We can notice the effect of the over-dispersion parameter on the three distributions, with the GTPL one having a relevant spread despite the lower expected value for this reason.

Regarding the random variable "cost of single claim", a usual assumption, that we follow in the numerical application, is to describe it by means of LogNormal distribution, as in Formula (4.25).

$$\tilde{Z} \sim LogNormal\left(\mu_Z, \sigma_Z\right) . \tag{4.25}$$

At this point we just need to choose the threshold value from which to consider a claims as large. Then, we are able to apply Algorithm 4 for simulating the aggregate claim amount according to the collective risk model approach, by means of the distributional assumptions that we made and using the parameters reported in Table 4.1. We set $threshold_{large} = 500.000$ since it could be considered an amount "big enough" for the presence of an XL treaty, despite usually in the market it is more common to have an XL treaty with deductible of $1M$.

We follow the steps outlined in previous section and, after obtaining the lists of simulations of aggregate claim amount and large claims, we follow Algorithm 5 and create a dependence in the segments simulated via a Copula function. In particular we use a Gaussian Copula and set the dependence matrix between the lines of business equal to the correlation matrix of Solvency II Standard Formula reported in Table 4.2 for the segments of interest.

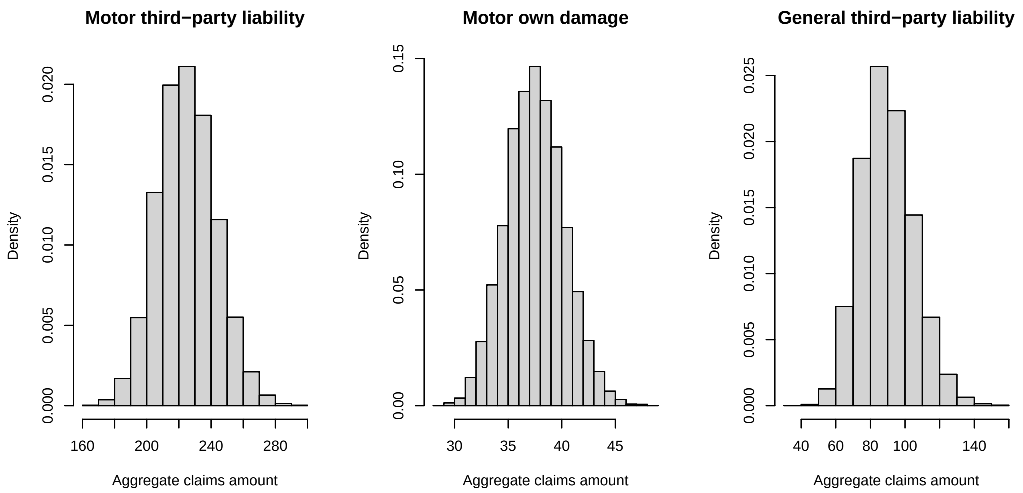

Figure 4.2. Distribution of simulated aggregate claim amount distribution (values in M).

In Figure 4.2 we report the distributions of the aggregate claim amount of the three LoBs, obtained using the parameters reported in Table 4.3, before the application of the Copula function to modify their correlation. As already expected looking at their parameters, we can observe that the MTPL line of business has the highest expected loss. Hence, we expect that the risk of this LoB will be quite relevant in defining the optimal reinsurance strategies. However, another LoB that the insurer adequately consider is the GTPL one. Indeed, this line of business shows a distribution ranging from 40M to more than 160M, due to the strong volatility of both the number of claims and the single claim amount distributions. The MOD line of business shows instead a quite small expected loss and low variability, compared to the other distributions. Moreover, since we already know from its parameters that the LR is also quite low, we expect that in most efficient strategies the insurer will keep most of the risk coming from this LoB.

Figure 4.3 reports the distributions and correlation coefficients between the three lines of business of the insurance company after the application of the Copula function, using the parameters reported in Table 4.3 and 4.2. In practice, we can observe that there is no change in the distributions of the aggregate claim amount of the three lines of business. Indeed, the Copula function "modifies the order of the simulations" underlying the distributions, affecting only the overall distribution, composed by the sum of the three.

4.3.3 Description of reinsurer parameters and simulation of default and recovery event

Similarly to what we did for the insurance company, we also have to define the characteristics of the potential reinsurance counterparties. In particular, in the context of the framework that we presented, the parameters of the reinsurers are a function of their credit quality step. In Table 4.4 we report these parameters, where discount factor and recovery rate are obtained by means of Formula (4.7) and (4.10)

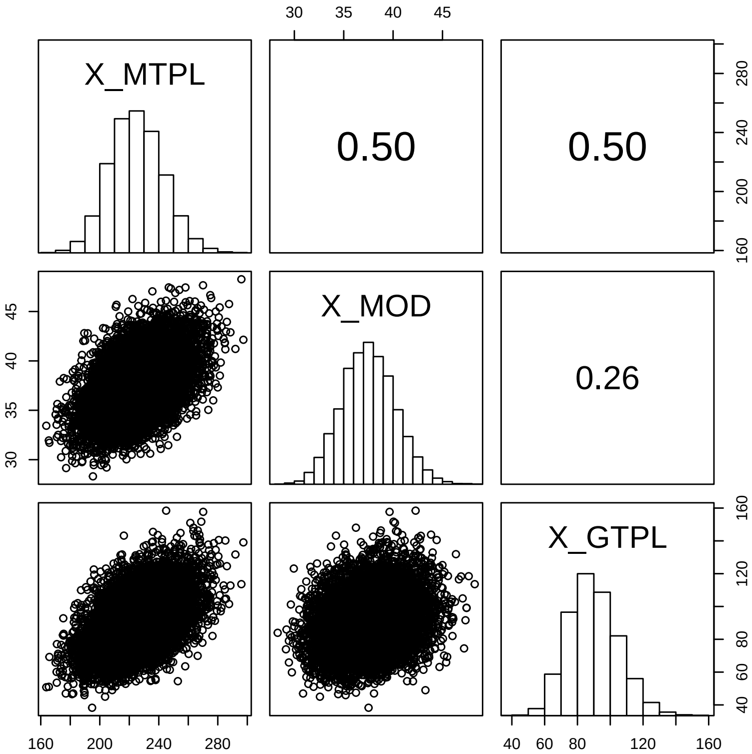

Figure 4.3. Aggregate claim amount distribution and correlation between lines of business after the application of Copula function (claims amount in M).

Table 4.4. Parameters of the reinsurance company for different credit quality steps.

CQS	Probability of default	Discount factor	Recovery rate
0	0.002%	91.7%	60.0%
1	0.01%	83.3%	54.5%
2	0.05%	75.0%	49.1%
3	0.24%	66.7%	43.6%
4	1.2%	58.3%	38.2%
5	4.2%	50.0%	32.7%
6	4.2%	41.7%	27.3%

with $base_q = 60\%$, $Q = 1/12$ and $D = 1$.

At this point, in order to determine the price charged by the reinsurance companies we apply the approach described in Algorithm 2 using the parameters just reported in previous table.

Having determined the pricing elements, the last remaining element related to reinsurance companies that we shall calculate is the aggregate claim amount borne by the reinsurer that is returned to the insurer. Regarding the cost of claims borne by reinsurer, it can be easily determined by applying the specific conditions of the treaty to the aggregate claim amount, as described in Algorithm 6. This element represents just the amount of claims that the insurer cedes to the reinsurer. However it does not consider the potential default of the counterparty and consequently the negative effect on the risk reserve of the insurance company. In order to determine the distribution of the risk reserve, described in Formula (4.1), we have to determine $\tilde{X}^{re,d}$, hence taking into account the potential default in the calculation of the claims ceded to the reinsurers.

Following the common shock approach described in Section 2.2.3 we model the default event by means of a Bernoulli variable with probability of default dependent on the common shock. Numerically, we use the approach of Standard Formula of Solvency II and model the "observed probability of default" according to the CQS of the firm. Then, we set the parameters of common shock variable α and τ equal to 0.8 and 0.2, respectively, as suggested in QIS5 [28]. In this way it is possible to derive the corresponding "baseline probability", from which to apply Algorithm 7 for the simulation.

Regarding the recovery rate we assume a stochastic model. For the definition of its distribution, a classical assumption that we follow is to use of a Beta distribution, since it is coherent with the constraint of having a support between 0 and 1 (as the recovery rate should). The parameters of the Beta distribution are calibrated in order to have a mean equal to the expected recovery rate, which is assumed to be function of the probability of default/CQS of the reinsurance company. The standard deviation is instead assumed fixed, regardless of the rating of the counterparty, coherently with the results reported in [22] and [2] and the more recent paper [18].

Numerically, we assume that the expected recovery rate of a counterparty with CQS equal to 0 is 60%, then we apply Formula (4.10) with $Q = 1/12$ and $D = 1$ for computing the recovery rate associated with the other values of CQS. Regarding the standard deviation of the recovery rate, we assume a fixed value irrespectively of the CQS and equal to 25%.

4.3.4 Combination of reinsurance strategies and multi-objective optimization

In order to find the optimal reinsurance strategies the approach that we propose consists in the estimation of all the possible combinations of the reinsurance treaty characteristics, number and rating of the reinsurance companies. In practice we apply the algorithms described in Section 4 and by means of Formula (4.1) we compute the risk and return metrics for each combination.

Table 4.5 shows the variables that will be used for creating the different combinations of reinsurance strategies that we will be evaluated in the risk/return framework. The table describes the minimum, maximum and step for each of the variables. The specific choices are in line with the characteristics of potential reinsurance strategies that could be employed by an insurance company. The only constraint that we impose in the case study is to have the same credit quality step for all the reinsurers of the same segment. This choice is done to simplify the analysis of the results and because, given already this setting we arrive at a total of more than $2,000,000$ combinations of different reinsurance strategies to evaluate.

In Appendix B.1 are reported the characteristics of the different input variables and the histograms of the risk and return output metrics.

At this point we shall define the risk and return metrics that we want to use for evaluating the different reinsurance strategies. Regarding the risk metric we choose to use the Solvency Ratio according to the partial internal model developed by means of the general formula of the risk reserve reported above. This choice is in line with the interest of the insurance company of having a risk metric which best catches its actual solvency position. The return metric chosen for the analysis is the return on equity, since it is one of the most common metrics for evaluating the performance of the company in the year.

At this point we have all the instruments for applying Algorithm 8 and select the efficient frontier of reinsurance strategies from the list of all the possible combinations of strategies.

4.3.5 Efficient frontier analysis considering expected RoE and SR

In Figure 4.4 it is reported the combination of expected Return on Equity and Solvency Ratio for the different reinsurance strategies considered. All these combination of RoE and SR are obtained by applying the algorithms described in Section 4.2. In particular, the efficient frontier, indicated in red, is obtained by means of Algorithm 8.

Table 4.5. Description of the reinsurance variables used for defining the different strategies.

Variable	Minimum	Maximum	Step	Description
D_{MTPL}	$1,000,000$	$3,000,000$	$1,000,000$	Deductible value of XL treaty for the MTPL segment
L_{MTPL}	$2,000,000$	$10,000,000$	$2,000,000$	Limit value of the XL treaty for the MTPL segment
D_{GTPL}	$1,000,000$	$3,000,000$	$1,000,000$	Deductible value of XL treaty for the GTPL segment
L_{GTPL}	$2,000,000$	$10,000,000$	$2,000,000$	Limit value of the XL treaty for the GTPL segment
α_{MTPL}	20%	100%	10%	Retention quota for the MTPL segment
α_{GTPL}	20%	100%	10%	Retention quota for the GTPL segment
α_{MOD}	20%	100%	10%	Retention quota for the MOD segment
$multi_{RE}$	0	1	1	Indicator for single/ multiple reinsurer
CQS_{MTPL}	0	6	1	CQS of the reinsurer(s) for the MTPL segment
CQS_{GTPL}	0	6	1	CQS of the reinsurer(s) for the GTPL segment
CQS_{MOD}	0	6	1	CQS of the reinsurer(s) for the MOD segment

As already explained, we do not have a single solution but a set of optimal combinations. The efficient frontier represents this set of optimal strategies, also called Pareto efficient, which maximize/minimize the selected metrics. In this case, the efficient frontier is composed by the treaties which provide the highest RoE for a fixed SR (or analogously the highest SR for a fixed RoE). In the figure it is possible to observe the well-known trade-off effect between the optimization of the two metrics: the more we try to maximize the RoE the more we have to reduce the SR and conversely, the more we try to maximize the SR the more we have to reduce the RoE.

Clearly, a portion of these efficient strategies are optimal according to the formal definition, but they could be not interesting in a practical context. For instance, looking at the efficient frontier reported in Figure 4.4, in practice no insurance company would choose a strategy producing an expected RoE of less than -10% with a SR of around 300%, while being formally efficient according to the optimization criterion. Indeed, an insurance company typically has a certain “risk appetite” which broadly represents the amount of risk that it is willing to assume in order to reach its performance objective and that limits the choice of possible combinations. In practice, the insurance company choose the reinsurance strategy, among those efficient, that is most in line with its preference structure.

In this context, however, we do not make any assumption on the risk appetite of the insurance company, but we propose a simpler approach for determining a single optimal solution among different efficient strategies. This approach consists in the maximization of the marginal increase described in Formula 4.16 and the corresponding solution is indicated by the green dot in Figure 4.4. It shall be noted that this approach implicitly assumes that the current situation is represented by the gross of reinsurance case and the objective of the insurance company is to find the reinsurance strategy with the best marginal improvement in the SR for a reduction in the RoE. In practice, if the current situation of the insurance company were different, it should be appropriate to consider that specific situation as the starting point of the marginal increase approach. In that case, however, the insurer should also choose between the marginal increase in the SR, at the cost of RoE, or viceversa, the marginal increase in the RoE, at the cost of SR.

In Table 4.6 it is reported a subset of the efficient strategies, determined by the application of Algorithm 8. The first efficient strategy is represented by the gross of reinsurance case. Indeed, given the parameters of the three LoBs and the assumptions that we made on the pricing of the reinsurance treaties, this strategy is optimal since it is not possible to achieve a higher return on equity by means of reinsurance. This result is also in line with the theoretical description of reinsurance treaties as a way for reducing the risk at the cost of a decrease in the profitability. However, it shall be underlined that with different LoB parameters and pricing assumption it would have been possible to have a reinsurance strategy with the highest RoE and consequently that the gross of reinsurance case would have been inefficient. For instance, in case the asymmetry in the information between insurer and reinsurer lead to different pricing assumption and the expected LR were greater than 100%. Indeed, ceding risks of a unprofitable LoB at a profitable cost would improve both

Table 4.6. Subset of the efficient strategies maximizing expected Return on Equity and Solvency Ratio.

D_{MTPL}	L_{MTPL}	D_{GTPL}	L_{GTPL}	α_{MTPL}	α_{GTPL}	α_{MOD}	$multi_{RE}$	CQS_{MTPL}	CQS_{GTPL}	CQS_{MOD}	SR	RoE
–	–	–	–	100%	100%	100%	–	–	–	–	118.9%	25.7%
100,000	600,000	–	–	20%	20%	30%	0	3	3	3	293.1%	−10.6%
100,000	800,000	–	–	30%	20%	60%	0	3	3	3	291.1%	−8.9%
100,000	800,000	–	–	50%	20%	80%	0	3	3	3	220.2%	−0.0%
100,000	600,000	200,000	200,000	100%	80%	100%	0	4	4	4	133.0%	22.0%
100,000	400,000	100,000	200,000	10%	70%	100%	1	4	4	4	138.5%	20.4%
100,000	200,000	–	–	100%	100%	100%	0	4	4	4	120.9%	25.5%

the solvency position and the technical performance of the insurance company.

The following two efficient solutions reported in the table are evident examples of strategies which a company would not follow in practice. Indeed, as already explained, these strategies lead to an excessively high SR at the cost of a strongly negative RoE. Moreover, while they could seem prudent approaches in the short term, they could actually become riskier in the medium/long term, since each year they produce a negative result which reduces the capital and consequently the following SR.

The fifth and sixth solution reported in the table could be considered appropriate also in a practical perspective. Indeed, they lead to a SR higher than 130% and a positive RoE of more than 20%. Interestingly, it is possible to observe that, while these two solutions are quite close, they are obtained with two quite different reinsurance strategies.

Finally, the last strategy reported in the table consists in the optimal solution according to the marginal increase approach. We can observe that the combination of SR and RoE obtained following this strategy is quite close to the gross of reinsurance one. Indeed, the only reinsurance treaty present in this case consists in the 2M xs 1M (2 million limit in excess of 1 million deductible) for the MTPL line of business.

Another important possibility offered by the methodology that we developed for estimating the efficient frontier consists in the possibility of making inference on the characteristics of the efficient treaties for assessing the relations between variables and other characteristics. In Figure 4.5 it is reported a partial dependence plots between the variables related to the reinsurance treaties. In this way we can understand the relations between variables for the reinsurance strategies on the efficient frontier.

Regarding the deductible for MTPL we observe that it is mostly concentrated on 1M, while the limit is more uniform in its range. Interestingly, the correlation between deductible and limit for MTPL is negative. This effect could actually depend on the fact that there is a policy limit and consequently a coverage on the entire right-hand tail is associated with a lower limit if we increase the deductible. Regarding the deductible for GTPL we observe that it is mostly concentrated on 1M. The limit is instead concentrated in 0M, meaning that there is mostly no XL reinsurance for GTPL.

Regarding QS treaties we can observe that in almost all the efficient strategies the

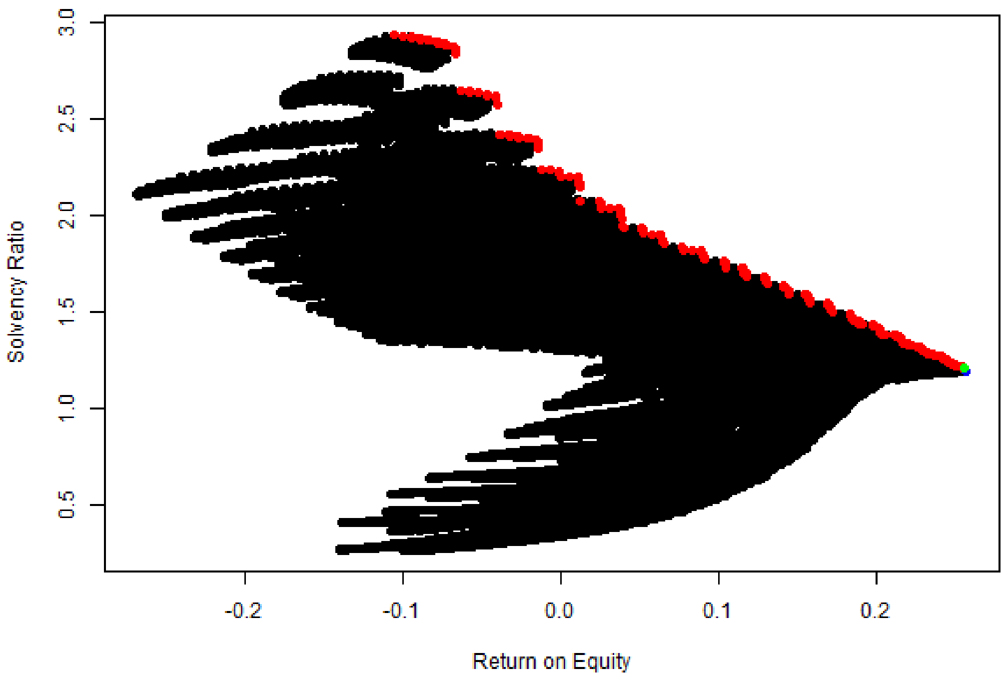

Figure 4.4. Efficient frontier of alternative reinsurance strategies in terms of expected Return on Equity and Solvency Ratio. In black are reported expected RoE and SR for all the combinations of reinsurance strategies, in red the efficient treaties, in blue the no-reinsurance case and in green the optimal solution according to the marginal increase approach.

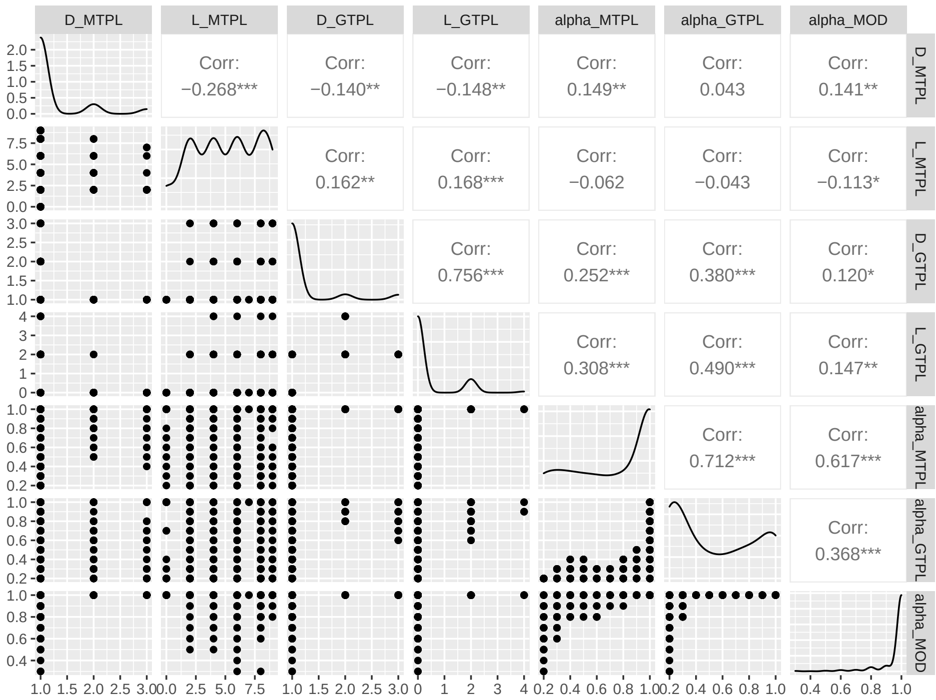

Figure 4.5. Partial dependence plot between the quantitative variables for determining the characteristics of the quota share and excess of loss treaties (deductibles and limits in M).

insurance company retains all the risks related to the MOD line of business. Indeed, it is in line with the expectation since the LR of this LoB is relatively low. Also for the MTPL line of business we can observe a quite similar result, with most of the strategies prescribing a high retention of the risk. On the opposite side is instead the GTPL line of business. Indeed, for this LoB we observe that there is a higher number of strategies suggesting to cede most of the risk to the reinsurance companies. In a certain sense this could be expected, since we had observed almost no XL reinsurance for this LoB, but we know from its parameters that it actually has a quite high loss ratio and volatility. Hence, one possible explanation is that the pricing of the QS treaty for GTPL is more efficient for the insurance company compared to the XL one.

Typically, an insurance company is not interested in determining the "unconstrained" efficient frontier of the reinsurance strategies, since some of these combinations could be actually not viable. Indeed, as anticipated, some extreme solutions would be actually "inefficient" in a practical sense. For instance, it would be sub-optimal to have a SR too high at the cost of a low or negative expected RoE. The reason is that a too high SR could be considered an inefficient allocation of capital. Moreover, the low or negative return amplifies the inefficiency of the strategy, despite its efficiency from a mathematical point of view, since the shareholders are allocating more capital than the company needs but "losing money". On the other hand, it would be problematic, even just considering the regulatory aspect, aiming at a too

Table 4.7. Constraints for the variables of the multi-objective optimization model.

Variable	Minimum	Maximum
SR	150%	$+\infty$
RoE	1%	$+\infty$

high return while having a low solvency ratio. In practice the insurance company has the objective of choosing the efficient strategies under some constraint, which could come from its risk appetite framework, from regulatory requirements or other needs.

Table 4.7 shows the constraints that we assumed for the analysis of a constrained case. Under this approach the insurance company wants to obtain a return on equity at least equal to the market interest rate, so 1%, while keeping its solvency ratio at a level above 150%. The rationale behind these constraints is that the insurer seeks a return greater than the interest rate (typically risk-free in the context of a non-life insurance company) for remunerating the allocated capital. At the same time, for ensuring an adequate solvency against adverse scenarios and potentially also due to supervisory requirements, it aims at a Solvency Ratio greater than 150%.

In Figure 4.6 this is indicated by representing in gray all the reinsurance strategies and in black only the admissible combinations according to the constrained conditions. It is possible to observe that the range of SR narrows from approximately $(30\%, 300\%)$ to $(150\%, 220\%)$. At the same time, also the range of RoE narrows from approximately $(-30\%, 30\%)$ to $(0\%, 15\%)$. The efficient frontier is reported in red and is limited to the constrained strategies.

Also in this case there are many possible approaches for selecting the optimal reinsurance strategies, from which the company has to choose the optimal one according to its risk/return preference structure. Considering also here the approach employed for the unconstrained case we obtain the combination indicated as green dot in Figure 4.6.

4.3.6 Efficient frontier comparison between Standard Formula and Partial Internal Model

Another interesting analysis that it is possible to perform thanks to the simulating framework developed in previous section consists in the comparison of the efficient strategies between the partial internal model and the standard formula. Following this interest, in Figure 4.7 we report the combination of expected Return on Equity and Solvency Ratio, where this latter metric is computed according to both Standard Formula and the partial internal model that we developed by means of Formula (4.1). In particular, the combinations of RoE and SR obtained according to the Standard Formula of Solvency II and our Partial Internal Model are reported in Figure 4.7 in grey and orange, respectively. Finally, in black and red are represented the Pareto Frontiers for the Standard Formula and Partial Internal Model respectively.

We can observe that the structure of the combination of treaties is quite different in

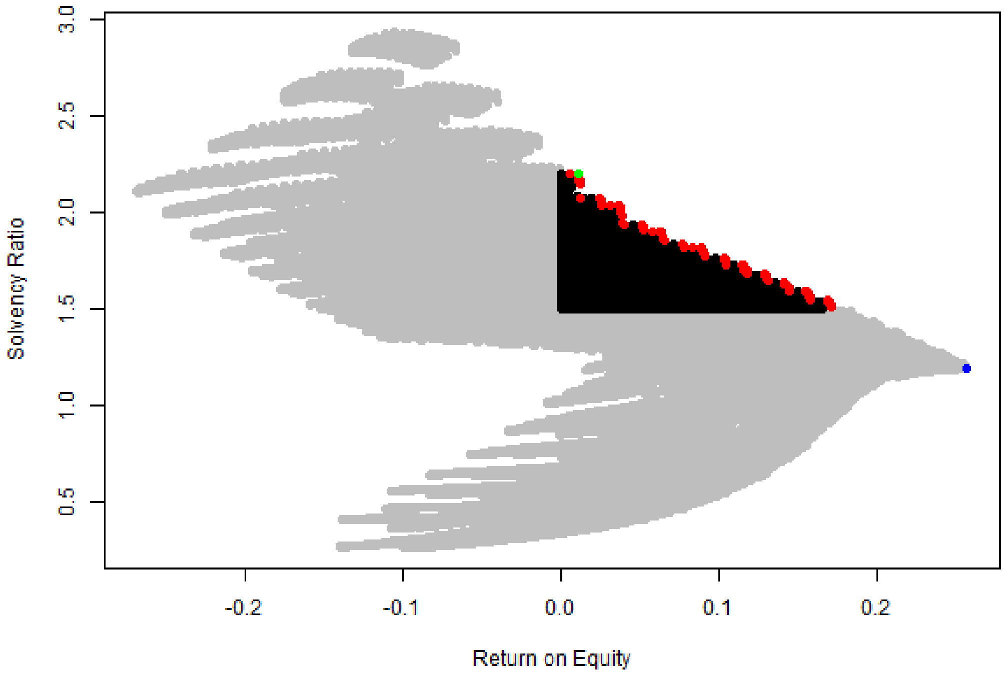

Figure 4.6. Constrained efficient frontier of alternative reinsurance strategies in terms of expected Return on Equity and Solvency Ratio. In gray are reported expected RoE and SR for all the combinations of reinsurance strategies, in black the constrained combinations, in red the efficient treaties, in blue the no-reinsurance case and in green the optimal solution according to the marginal increase approach.

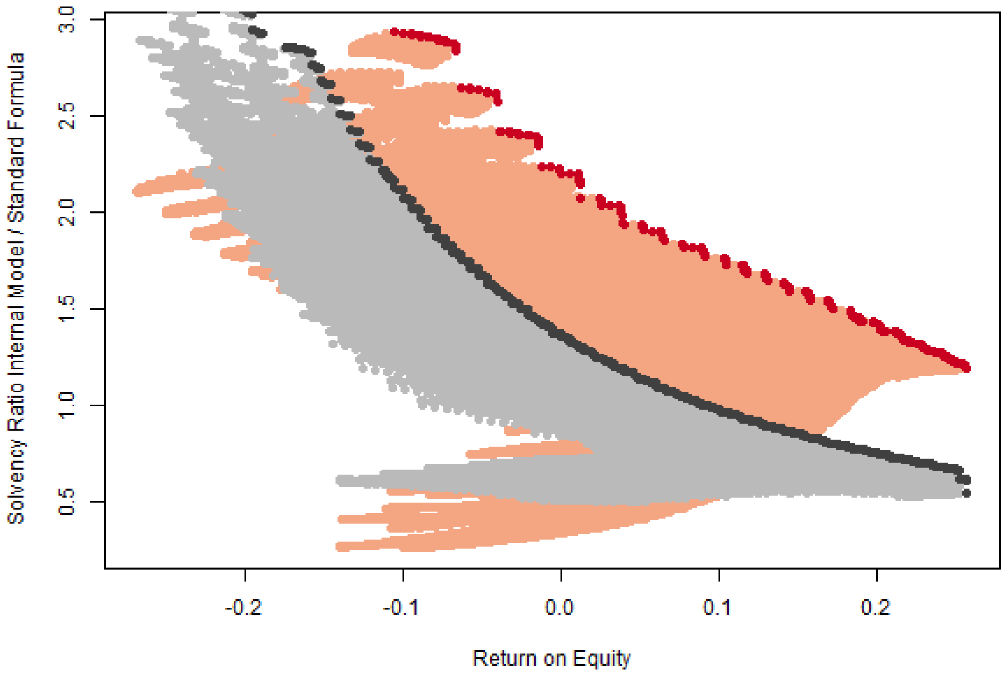

Figure 4.7. Efficient frontier of alternative reinsurance strategies in terms of expected Return on Equity and Solvency Ratio. The SR metric is calculated according to both Standard Formula and Internal Model. In grey are reported the combinations considering SF, while in orange considering IM. The corresponding efficient frontier is reported in black and red, respectively.

case we assume to estimate the SCR, and consequently the SR, according to standard formula or partial internal model. The reason is that the two approaches are based on completely different methodologies for the estimation of the capital requirement. In particular, standard formula approach estimates the capital requirement of a firm by means of factor/scenario-based approaches, calibrated in specific impact studies on market data, with the objective of an applicability to all the companies of the market. For this reason they could be not always appropriate to adequately describe the actual risk position of a firm. Partial internal model approach is instead based on the specific information of the company and it estimates the capital requirement by means of value-at-risk at 99.5% quantile, determined by means of simulated scenarios.

Regarding the efficient frontier for the two approaches there are 2586 Pareto optimal reinsurance strategies in case we use as risk metric the Standard Formula SR and 404 if we use the Partial Internal Model SR. Moreover there are 70 cases where the same strategy is efficient under both the approaches, which represents 3% and 17% of efficient strategies under standard formula and partial internal model, respectively.

As we partially showed with the previous analysis, it shall be remarked that for an insurance company it could be particularly useful to consider both the risk estimated from the application of Standard Formula and internal approach, like the partial internal model of our case. Indeed, an insurance company could be in the situation where it still needs to employ the Standard Formula for the assessment of its solvency position, but it has already implemented some methodologies for calculating its actual riskiness, based on its specific characteristics. In this case, the insurer could be interested in finding the optimal reinsurance strategy according to the partial internal model, since it better represents its actual risk position, but constraining the result on the compliance with some thresholds calculated according to standard formula, as it is the measure used by the supervisory authority.

For instance, in Figure 4.8 we analyze the case of multi-objective optimization of expected Return on Equity and Partial Internal Model SR, constrained on the SR from Standard Formula greater than 130%. It is possible to observe that the constraint condition on SF also reduces the combinations on partial internal model to cases where the SR is greater than 130%. However, there are still some cases where the SR according to the partial internal model is even less than 100%, but none of them are efficient.

Looking at the figure we can observe that only a couple of combinations guarantee a positive technical result, while most reinsurance strategies produce a SR greater than 200%, but with a negative RoE. In this case, if we apply the simple MI approach, described in Formula (4.16) and already applied in previous analyses, we obtain as optimal solution the reinsurance strategy which produces a RoE of -6.71% and a SR of 286.79% (indicated by the green dot in the figure). Finally, the no-reinsurance case, indicated in the figure by the blue dot, in this case is a combination which does not satisfy the constraint on the SR from Standard Formula and consequently cannot be an efficient solution.

4.3.7 Efficient frontier comparison by rating of reinsurers

The framework developed in this model also allows to perform a separate analysis of the reinsurance strategies according to the rating (expressed in term of CQS) of the counterparties.

Indeed, Figure 4.9 reports the combinations of expected return on equity and solvency ratio derived by different reinsurance strategies, assuming that the risks from all the lines of business are ceded to a single reinsurer. We can observe that the specific parametric assumptions lead to a situation where the reinsurer with CQS equal to 0 (dark green dots) shall never be chosen, since none of its combinations lies on the efficient frontier. In particular, the only values of credit quality step for which there is no combination of treaties on the efficient frontier are 0 or 1. Indeed, the efficient frontier is composed only by treaties from reinsurers with a CQS from 2 to 6, with the value of 3 taking up most of its part. The reason is connected with the specific assumptions that we made on the discount connected with the credit quality step. In particular, the combination of discount and implied probability of default for reinsurers with credit quality step equal to 3 results in the best trade-off most of the time for the insurance company.

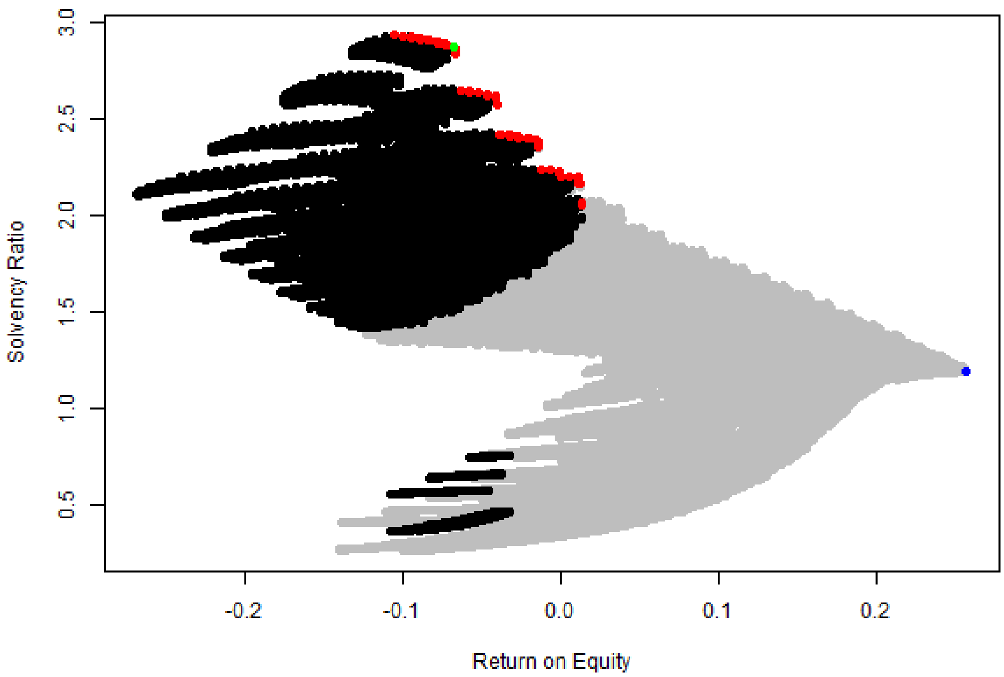

Figure 4.8. Constrained efficient frontier of alternative reinsurance strategies in terms of expected Return on Equity and Solvency Ratio. In gray are reported expected RoE and SR for all the combinations of reinsurance strategies, in black the constrained combinations, in red the efficient treaties, in blue the no-reinsurance case and in green the optimal solution according to the marginal increase approach.

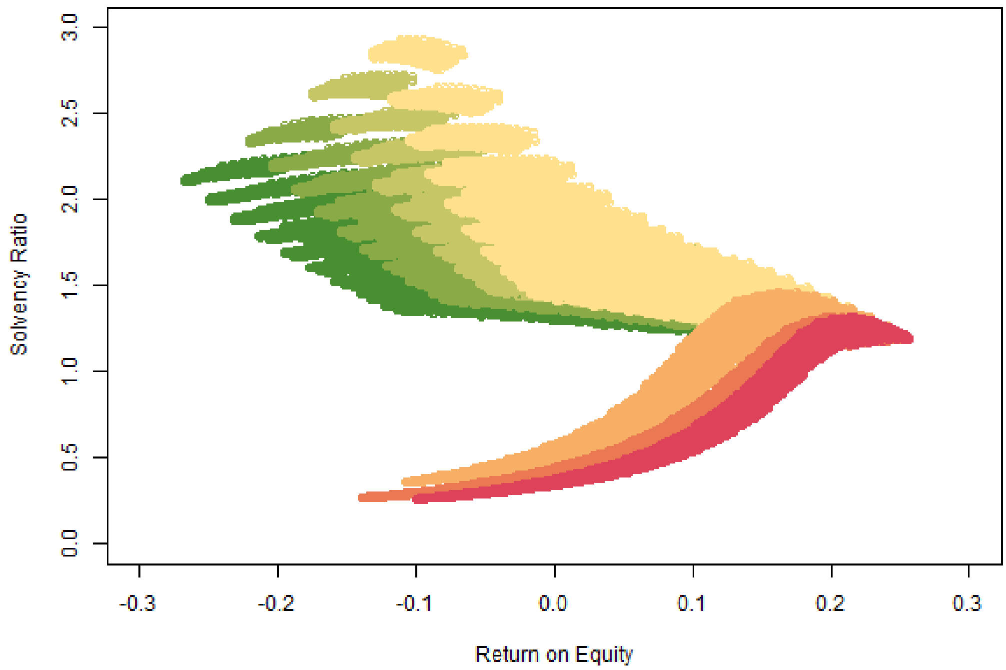

Figure 4.9. Combinations of expected Return on Equity and Solvency Ratio for different reinsurance strategies in case of a single reinsurer for all the segments. The colors represent the CQS of the reinsurance company, ranging from green for a CQS of 0 to red for a CQS of 6.

Another interesting characteristic that we can observe is that many combinations can be obtained indifferently by means of strategies using reinsurers with different CQS. However, it shall be noted that here we are analyzing only these two metrics. In practice, the insurer could be interested in considering also other metrics, like the coefficient of variation of the technical result or the ruin probability.

In Figure 4.10 it is reported the same analysis described above, but considering the case where there is a different reinsurer for each line of business, but all with the same rating. We can observe a different shape of the combinations of strategies especially for reinsurers with CQS greater than or equal to 4. In general it is not possible to identify a single effect in the combination of SR and RoE with respect to the single-reinsurer case. Actually, we know that, given a reinsurance strategy, the RoE obtained by ceding the risks to a single reinsurer or to multiple reinsurer with the same rating gives the same result. However, the effect of splitting the risk to 3 different reinsurers (one for each LoB) on the SR is not so direct, and actually depends on the specific CQS of the counterparty, as we will show in the following analysis.

Regarding the structure of the efficient frontier also in this case most of the area is taken up by the combinations of treaties from reinsurers with CQS equal to 3. It

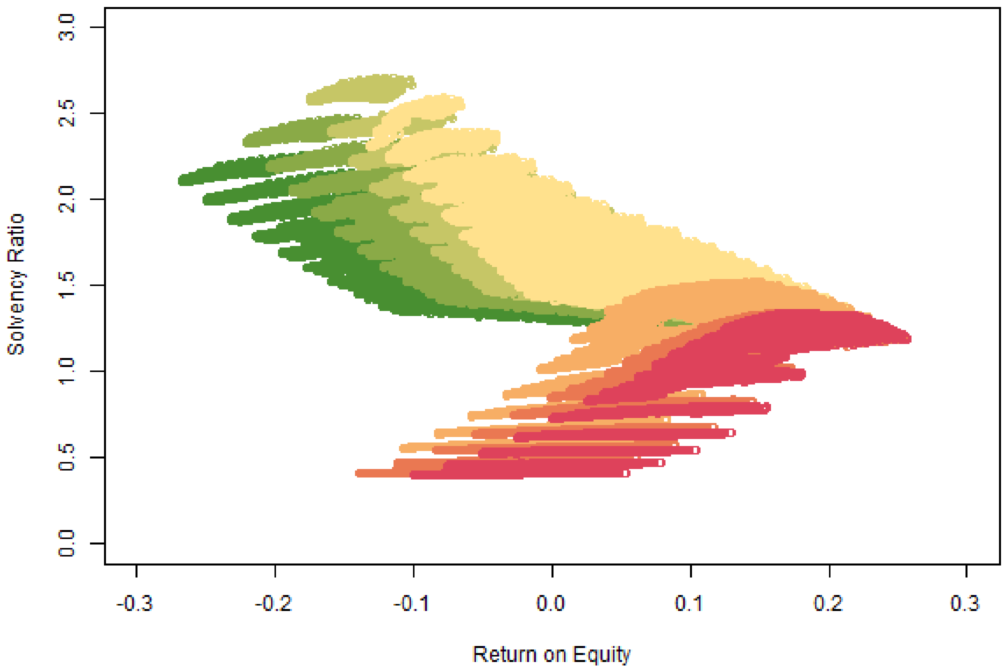

Figure 4.10. Combinations of expected Return on Equity and Solvency Ratio for different reinsurance strategies. The colors represent the CQS of the reinsurance company, ranging from green for a CQS of 0 to red for a CQS of 6.

is possible to observe that, while in the single-reinsurer case the strategy of ceding risk to a reinsurer with CQS equal to 2 did not provide any optimal solution, in this case it is part of the efficient frontier.

The comparison between the case of single and multiple reinsurers with the same CQS is presented in more detail in Figure 4.11, with the combinations of SR and RoE according to the two cases indicated by black and red dots, respectively. Here it is possible to observe more clearly the different effect of the two cases on the SR depending on the CQS. We can observe that for CQS equal to 4, 5 and 6 ceding the risks to more reinsurers give a higher SR compared to the single-reinsurer case. The reason is that, for these values of the probability of default, the diversification produced by the higher number of counterparties provides an improvement in the extreme scenario ($VaR_{99.5\%}$). On the other hand, we can observe that for the lower values of CQS the effect is the opposite, with the case of CQS equal to 3 the most evident. Indeed, in these cases we notice that the single-reinsurer case produces a higher SR for the same level of RoE. The reason is that, for these values of the probability of default, the increase in the likelihood of a default event produced by the higher number of counterparties leads to a worsening in the extreme scenario ($VaR_{99.5\%}$).

In general, we have two main effects when we increase the number of reinsurer

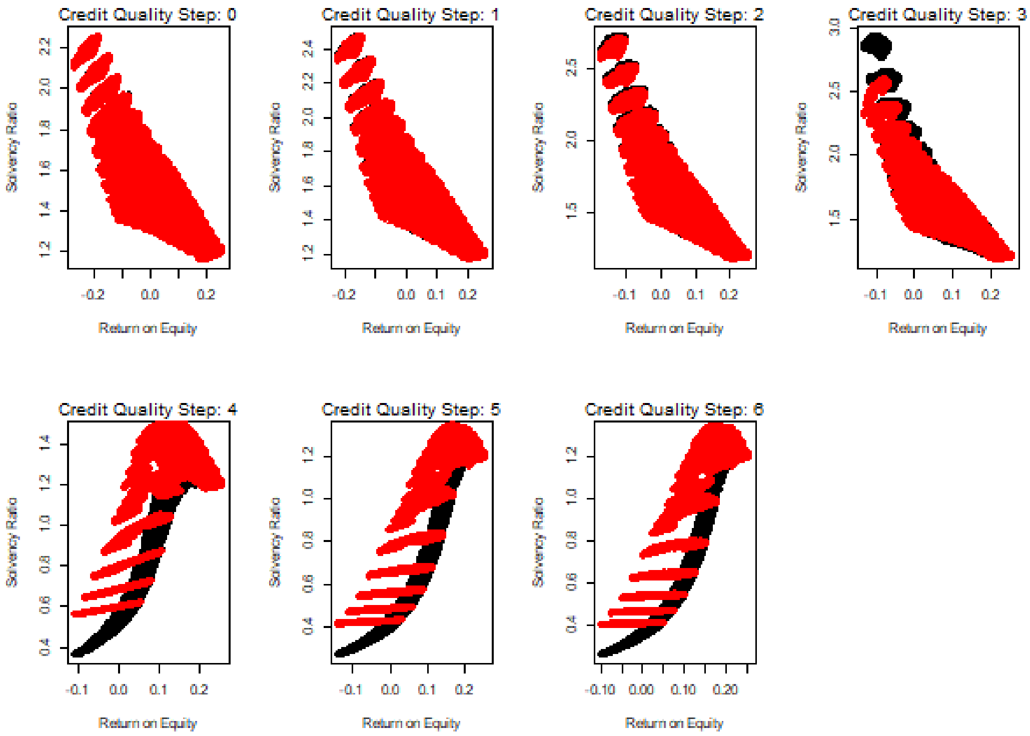

Figure 4.11. Combinations of expected Return on Equity and Solvency Ratio for different reinsurance strategies, for each CQS. The black dots represent the single reinsurer case, while the red ones the multiple reinsurers case.

keeping fixed their probability of default. They are represented by the increase in the number of default events and in the reduction of the total loss (in case of a default event) due to the split of the total amount (under risk) to multiple parties. It is possible to notice that these effects move the total loss in the extreme scenario in opposite directions. Hence, depending on the specific probability of default of the counterparty we have the prevalence of one or the other effect and consequently the benefit or disadvantage from increasing the number of counterparties.

Finally, we can observe that in most of the cases there is a quite small difference between the single and multiple reinsurer(s) case, especially when the probability of default is low. However, the more we move to high CQS the more we observe a difference in the efficient frontier.

Appendix B

Efficient frontier

B.1 Characteristics of inputs and outputs

In Figure B.1 we report the distribution of XL deductible and limit for MTPL and GTPL. It is possible to observe that these distributions are exactly the same. Indeed, as reported in Table 4.5, we have assumed the same minimum, maximum and step size.

Regarding the deductible we can observe that the threshold of 1M is the most represented compared to the other ones. The reason is that, given the chosen step size for the limit, this deductible has an additional combination. For the same reason explained for deductible, also the distribution of limit is not uniform, but it depends on the specific number of possible combination, given the step size assumed for this numerical analysis. Finally, the value of $l = 0$ represents the absence of a XL reinsurance for the specific line of business.

In Figure B.2 we report the distribution of QS retention for MTPL, GTPL and MOD. It is possible to observe that these distributions are exactly the same. Indeed, as reported in Table 4.5, we have assumed the same minimum, maximum and step size for the three lines of business. In this case we have exactly a uniform distribution between 20% and 100%, where the right-hand extreme represents the case of full retention, i.e. the absence of a QS treaty.

In Figure B.3 we report the distribution of expected value, standard deviation and skewness of the risk reserve obtained from the simulated scenarios. Regarding the expected risk reserve we clearly have a distribution with a quite large range, due to the different effect of each reinsurance strategy on the technical result. The peak of the distribution is close to the initial value of the risk reserve, meaning that most strategies produce a technical result close to 0. Finally, looking at the shape of the distribution we can observe that it presents a negative skewness, having more extreme negative results compared to the positive ones.

Regarding the distribution of the standard deviation of the risk reserve we observe a positive skewed distribution with the mode at approximately 2.5M.

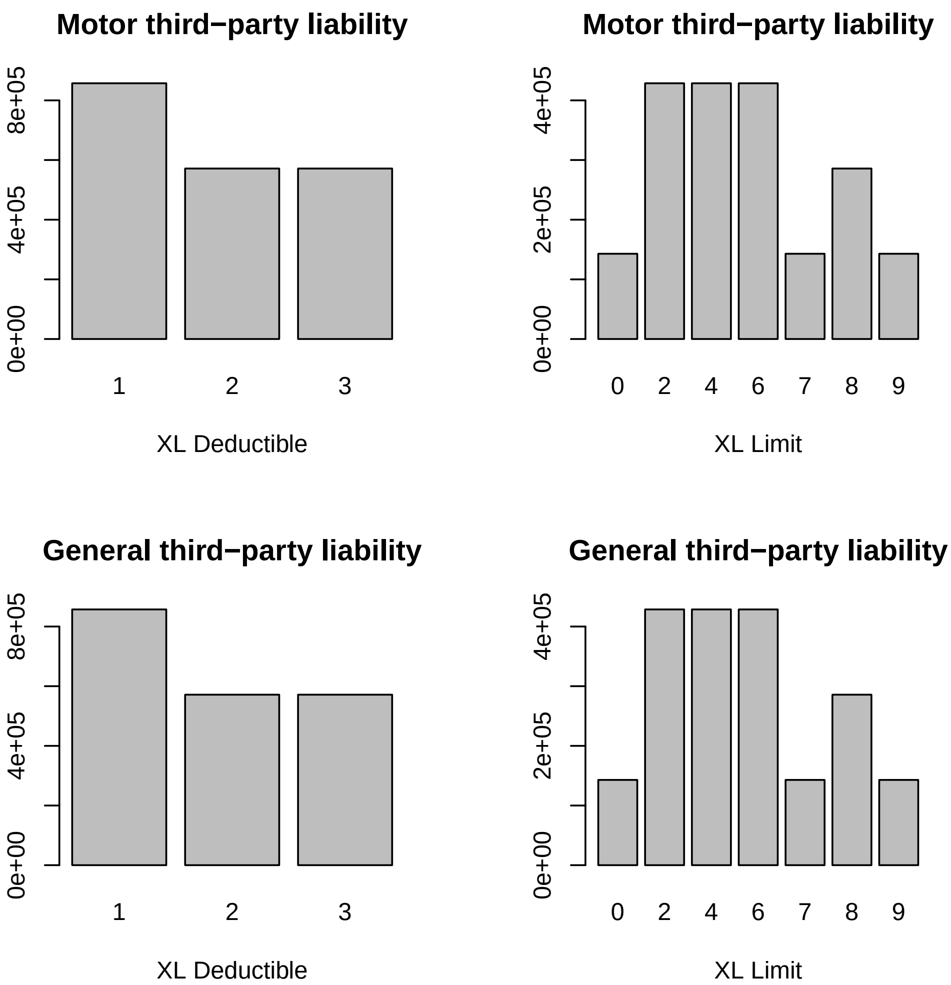

Figure B.1. Distribution of input parameters for the excess of loss reinsurance (values in M).

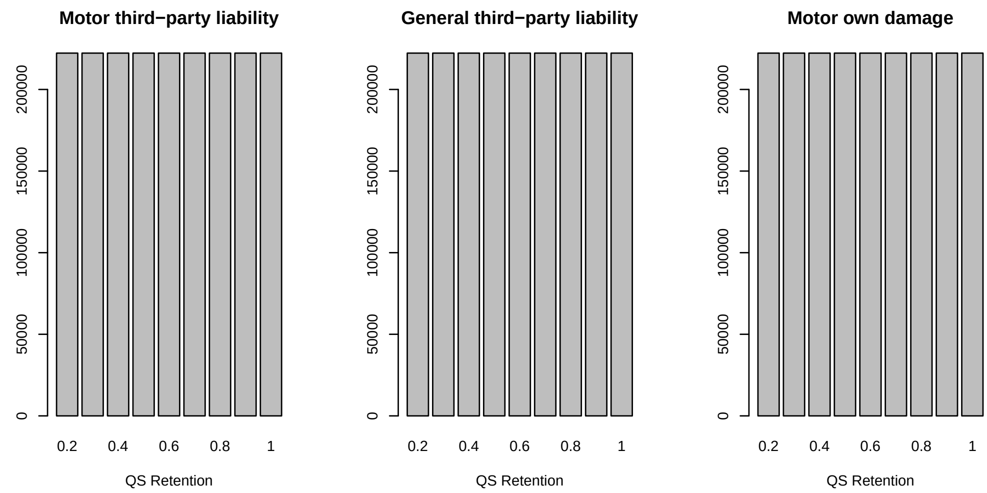

Figure B.2. Distribution of input parameters for the quota share reinsurance.

The distribution of the skewness of the risk reserve is instead mostly concentrated around 0, determining that on average the distribution of the risk reserve has a low asymmetry. However, it is also possible to observe the presence of a fat left tail which reaches quite extreme values, like −15.

In Figure B.4 we report the distribution of the SCR and SR according to the Standard Formula of Solvency 2 and the Partial Internal Model that we proposed in this chapter, obtained from the simulated scenarios. Comparing the two distributions of SCR it is possible to observe that they are quite different. In particular, the partial internal model one shows a positive skewed distribution with peak at approximately 50M and it reaches a capital requirement of more than 250M in some extreme strategies. On the other hand, the standard formula one shows a bimodal distribution with peaks at approximately 80M and 110M, but it reaches a much lower extreme capital requirement.

Regarding the distributions of SR we can observe that under both approaches the range is approximately the same, between 0% and 350%. In this case, the one obtained by applying the partial internal model is bimodal, with peaks at 50% and 150%. The distribution obtained by applying the standard formula instead shows a sort of LogNormal distribution shape. The reason for these quite different shapes is related to the difference in how these approaches measure the same risks. In particular, as already described in the respective sections, the capital requirement obtained by means of the standard formula is based on a modular approach where the risk of each module is obtained by means of a factor/scenario-based approach. The capital requirement obtained by means of the partial internal model developed in this chapter is instead based on a VaR measure at 99.5% quantile, calculated by means of the simulated scenarios which consider the risks to which the company is exposed.

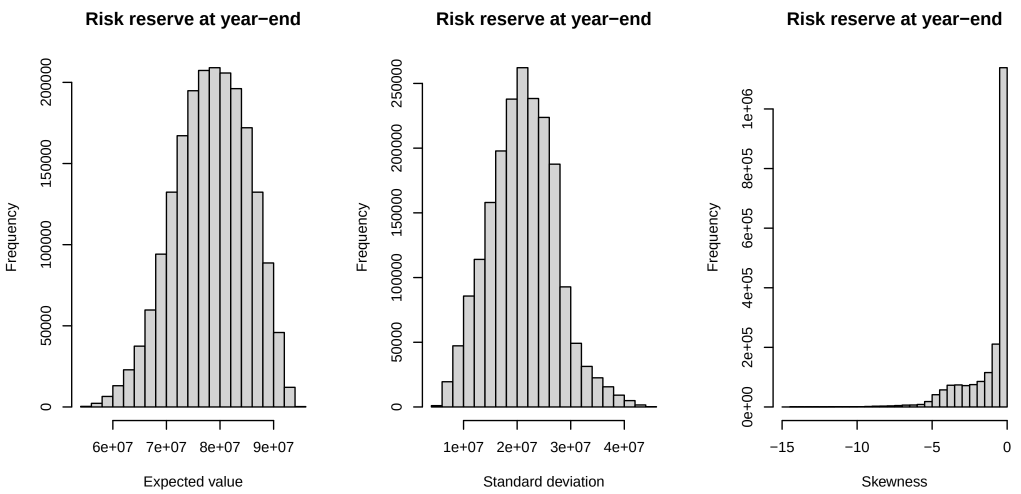

Figure B.3. Distribution of moments of the simulated risk reserve.

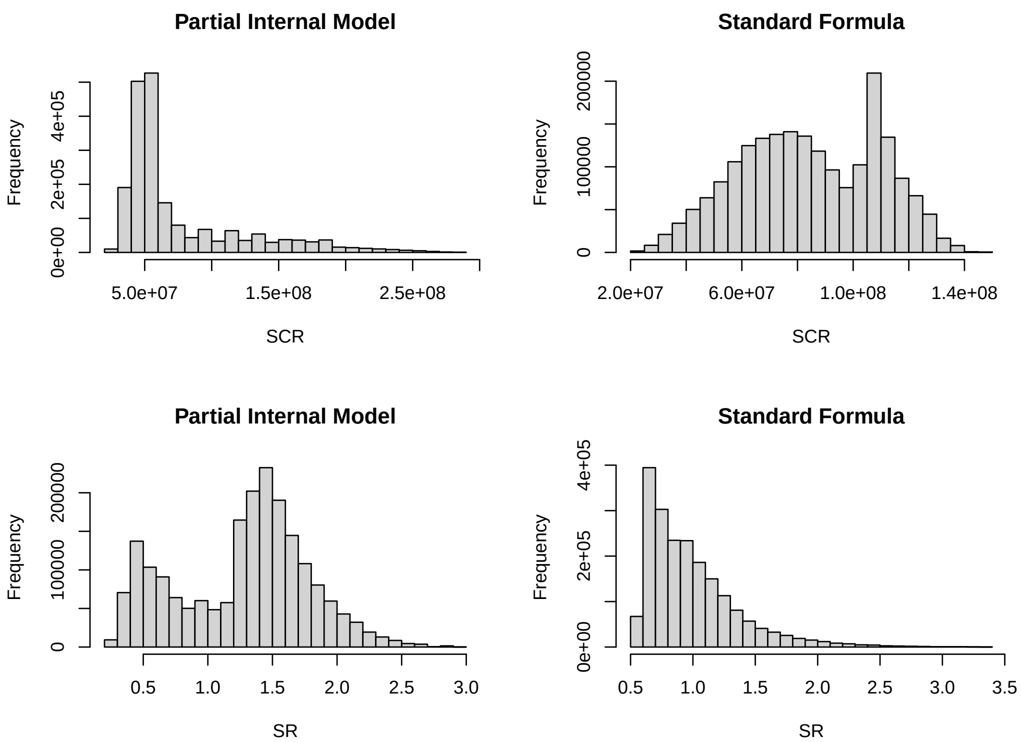

Figure B.4. Distribution of simulated risk measures.

B.2 Efficient frontier analysis considering expected RoE and CoV

The insurance company could also be interested in using other metrics of risk and return as drivers for choosing the optimal reinsurance strategies. In Figure B.5, for instance, it is reported the set of combinations obtained using the expected RoE as return metric and the Coefficient of Variation of the risk reserve as risk metric. It is possible to observe that the shape of the combinations is quite different compared to the case where the optimization metrics were the RoE and SR. Moreover, in this case, the objective of the insurance company is not to maximize both the metrics, but it aims at maximizing the expected RoE, while minimizing the CoV of the risk reserve. For this reason we can also observe that the efficient frontier, reported in red in the figure, is on the “south” part of the figure.

Similarly to the previous analysis, also here we indicate in blue the combination of the two metrics obtained in case the insurance company decides to not purchase any reinsurance. Clearly, this strategy produces the highest RoE, but also the highest CoV among the combinations on the efficient frontier.

Regarding the choice of a single optimal strategy, we propose also in this case the simple approach already used in the previous analysis and based on the choice of the strategy with the highest marginal increase of SR for a reduction in the expected RoE. However, we need to make a slight modification to Formula 4.16, since we are interested in the “marginal decrease” of the CoV. Hence, in Formula B.1 it is reported the modified formula to be maximized in case the insurance company is interested in the optimal strategy according to the marginal decrease of the CoV.

$$\begin{aligned} MI_{CoV} &= \frac{\left(\frac{CoV\left[\tilde{U}_{t+1}(gross)\right] - CoV\left[\tilde{U}_{t+1}(m)\right]}{CoV\left[\tilde{U}_{t+1}(gross)\right]}\right)}{\left(\frac{\mathbb{E}\left[\widetilde{RoE}(gross)\right] - \mathbb{E}\left[\widetilde{RoE}(m)\right]}{\mathbb{E}\left[\widetilde{RoE}(gross)\right]}\right)} \\ &= \frac{CoV\left[\tilde{U}_{t+1}(gross)\right] - CoV\left[\tilde{U}_{t+1}(m)\right]}{\mathbb{E}\left[\widetilde{RoE}(gross)\right] - \mathbb{E}\left[\widetilde{RoE}(m)\right]} \frac{\mathbb{E}\left[\widetilde{RoE}(gross)\right]}{CoV\left[\tilde{U}_{t+1}(gross)\right]} . \end{aligned} \tag{B.1}$$

In Figure B.5 this optimal treaty is reported in green and we can observe that it is quite close to the no-reinsurance case. Interestingly, this is in line with the result obtained considering expected RoE and SR and reported in Figure 4.4. This means that there is consistency in the marginal increase/decrease of the SR and CoV compared to the gross-of-reinsurance case.

In practice, an insurance company has many possible alternative metrics for defining the efficient frontier and choose the optimal strategy. Among the many alternatives some of the most easily obtainable from the framework here developed are ruin/survival probability, TVaR-based measures and risk-adjusted measures (e.g. RAROC). Moreover, rather than focusing on the optimization of two metrics, the insurer could aim at jointly optimizing three or more objectives.

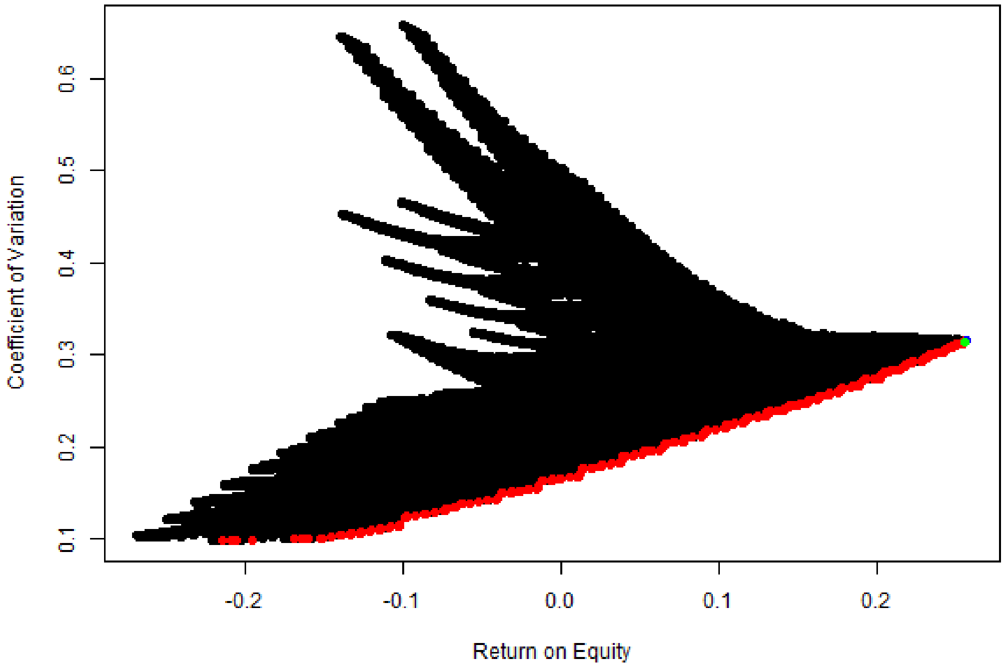

Figure B.5. Efficient frontier of alternative reinsurance strategies in terms of expected Return on Equity and Coefficient of Variation of the risk reserve. In black are reported the combinations of RoE e CoV for all the reinsurance strategies, in red the efficient frontier, in blue the no-reinsurance case and in green the optimal solution according to the marginal increase approach.

Chapter 5

A neural network approach for selecting efficient reinsurance strategies

5.1 Introduction

In the last decades, the application of statistical and machine learning algorithms has seen a strong increase in many different fields, boosted by the improved processing capacity of modern computer. Neural networks, in particular, are one of the most versatile machine learning algorithm, being employable in supervised, unsupervised and reinforcement learning tasks, and have provided many successful results in different areas of research. Relatively recently also the actuarial science community has started to increase the use of this algorithm in many of its areas of application. In [66] and [67] is reported an extensive review of the main machine learning models developed and employed in actuarial science.

In the non-life sector we find an increasing literature on neural network models in both pricing and reserving. In the context of ratemaking, in [77] are compared different machine learning approaches, among which neural network, in a pricing optimization framework, showing their advantages and disadvantages against classical GLM. In [73] it is developed a neural network, based on classical Generalized Linear Model (GLM) approach, in order to provide an improvement in the performance of non-life insurance pricing. [82], instead, after showing the limit of neural networks in providing an unbiased estimate at portfolio level, presents two techniques for overcoming this issue.

Regarding reserving one of the first works that started showing the strength of machine learning technique in this area has been [81], which illustrates how these technique can be employed in individual claims reserving. In the context of neural network, [31] applies a modeling architecture, based on six different neural networks with specific modeling purpose, to individual claims data. Using as benchmark classical Chain-Ladder method shows the potential improvement given by neural network approaches. [37] brings an improvement of classical over-dispersed Poisson model in claims reserving, by embedding it in a neural network architecture. [55] propose DeepTriangle, an approach based on triangular data, in which are jointly modeled paid and outstanding claims by a deep neural network architecture incorporating additional heterogeneous inputs.

Also in the context of life insurance we have seen the development of many approaches based on machine learning and neural network, in particular in the modeling of mortality and lapses. Regarding mortality modeling, in [61] it is proposed a deep learning integrated Lee-Carter model, in which the ARIMA process used to describe the mortality trend over time is modeled by means of a recurrent neural network. [63] develops a generalization of the structure of Lee-Carter model by means of a shallow convolutional neural network mortality rates forecasting. Moreover, it shows that in the specific case the use of a deep network does not provide any significative improved compared to shallow case. In [68] the extension of the Lee-Carter model to multiple population is instead developed using neural network, showing an improvement in forecasting performance, while [74] introduces a neural network approach for fitting the Lee-Carter and the Poisson Lee-Carter models. Finally, [51] describes theoretical models for surrender risk in life insurance, analyzing the performance of different machine learning approaches.

Finally, the use of neural network is employed also for the development of synthetic dataset. Indeed, in [38] the authors use neural networks to developed a stochastic simulation machine for generating individual claims simulation, based on a real non-life insurance dataset.

Here we differentiate from these approaches since our objective is to apply the neural network in the context of reinsurance treaties selection. In this area of actuarial science, there is much lower literature on the application of neural networks. [24] studies deep learning approaches to find optimal reinsurance and dividend strategies, in an infinite-horizon optimal control problem. Our objective is instead to analyze the problem of optimal reinsurance selection in a finite time-horizon, specifically in a one-year time-horizon, and with multiple objectives.

Regarding multi-objective optimization and neural network the literature is quite limited and mainly focused on approaches for building a single network that is able to have a good performance on multiple objective metrics (e.g. trade-off in the accuracy of multiple outputs predicted by the same neural network). Most of the approaches to the problem of multi-objective optimization in deep learning are based on optimizing a new network for every point on the Pareto front, or by using hypernetworks conditioned on modifiable preferences [60]. [71] deals with this problem in a different way, by conditioning the network directly on the preferences by augmenting them to the feature space. Moreover, using a penalization based on cosine similarity penalty is ensured a well-spread Pareto front.

Here, we still employ neural network in the context of multi-objective optimization, but for a different scope. Indeed, we are interested in determining the combinations of inputs which maximize multiple outputs jointly.

The following sections of this Chapter are organized as follows. In Section 5.2 we

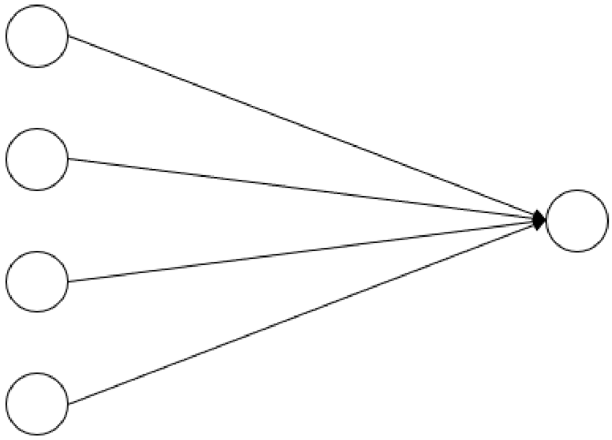

Figure 5.1. Illustrative representation of Perceptron.

provide a general description of neural network, focusing on the aspects most relevant for our problem. In Section 5.3 we present the problem and the theoretical description of the model developed for its solution. We start from describing the limits of the stochastic approach presented in the previous chapter. Hence, we describe our proposal, based on a neural network approach, for managing the problem, focusing on each of its parts. In Section 5.4 we present a numerical analysis, based on the same data as the one developed in the previous chapter. We start from the description of the parameters used and the methodological choice for the various components of the neural networks. We then show the application of the model in the specific context and the result, providing numerical and graphical evidences. Finally we refer to the Appendix for additional sensitivities.

5.2 General description of neural networks

Artificial Neural Networks are one of the most popular machine learning algorithms. They are directed graphs consisting of nodes (also called neurons or units) connected by links, inspired by the biological system of neurons of the human brain (nodes and links are the corresponding analogous of neurons and synapses). Neural networks were described for the first time in the seminal work of McCulloch and Pitts [57]. Then, following the theoretical system presented in that work, Rosenblatt proposed a "learning" model for neural networks in his paper on perceptron [70]. The structure of this model is reported in Figure 5.1.

Here, in order to provide a better understanding of the elements related to perceptron, and in general to neural network models, we briefly describe the terminology and meaning connected to the layers of a neural network.

- **Input layer:** it is the first layer of the neural network. It passes the inputs and a bias term to the next hidden layer. The elements contained in the input layer can be interpreted in the same way as the "explanatory variables" of linear regression models, with the bias representing an analogous of the "intercept term".

- **Hidden layer:** it consists of all layers in the middle between the first and the last one. In each hidden layer are weighted the inputs coming from previous layer and is applied a non-linear transformation by means of an activation function. The weights can be interpreted as the coefficients of linear regression, so are the elements which are "modified" in order to obtain the minimization of the loss function.
- **Output layer:** it is the last layer of the neural network. It receives the elements of the previous hidden layer as input and apply an activation function. The elements of the output layer are the dependent/response variable of the model (using the jargon of linear regression).

From these descriptions we derive that perceptron is composed by just one input and one output layer, not having any other (hidden) layers between the two. This operator is applied on binary a classification problem, performing the operations described below. Computationally, as reported in Formula (5.1), the first step consists in taking the vector of explanatory variables and calculating their weighted linear combination adding a bias.

$$z = \mathbf{w}^T\mathbf{x} + b = \sum_{i=1}^{n} w_i x_i + b \tag{5.1}$$

where $\mathbf{x} = (x_1, ...x_n)$ represents the vector of inputs, $\mathbf{w} = (w_1, ...w_n)$ the vector of weights and b the bias term.

Then, as reported in Formula (5.2), the output $f(x)$ is derived by applying the activation function $\phi(z)$ to the previously calculated value

$$f(x) = \phi(z) = \phi\left(\mathbf{w}^T\mathbf{x} + b\right) = \phi\left(\sum_{i=1}^{n} w_i x_i + b\right) . \tag{5.2}$$

The activation function used in the context of Perceptron is called the Heaviside (or unit step) function and it is reported below.

$$\phi\left(\mathbf{w}^T\mathbf{x} + b\right) = \begin{cases} 1 & \text{if } \mathbf{w}^T\mathbf{x} + b \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{5.3}$$

which, as evident from Formula (5.3), returns a binary output[1].

Activation functions are one of the most relevant components of a neural network and consist in a function which is applied to a node given the set of inputs received. The importance of this element lies in the possibility of using non-linear functions, which

[1] A more detailed description of Perceptron and artificial neural network, with a focus on their relationship with statistical models can be found in [23].

give the possibility to the network of "learning" non-linear dependencies between variables.

Feed-forward neural networks are the natural evolution of perceptron. Their name derives from the fact that information is "fed" to each layer only in a forward direction. The evolution consists in the use of an activation function that returns a continuous value, rather than a binary value. In this way the output is more sensitive to changes in weights respect to the other alternatives. Moreover, in its general formulation, this network could have one or more hidden layers.

One of the most relevant theoretical findings related to neural network, which had a strong impact on the increase of interest in this operator, is the so-called "Universal approximation theorem". Indeed, in [45]-[46] it is proved that a single hidden-layer feed-forward neural network can approximate any measurable function arbitrarily well regardless of the activation function, the dimension of the input space and the input space environment.

The modeling of a neural network can be summarized by two main components:

- **Architecture:** it is the "graph structure" of the network. It encompasses the choice on the number of layers and number of neurons, how they are connected, which is the formulation of activation functions, but also how the information flow withing the network.
- **Learning algorithm:** it is a procedure that update iteratively the set of parameters of the network in order to find the combination that minimize the chosen loss metric.

Regarding the architecture of a neural network, in the literature many other alternatives have been proposed for each of the element defining its structure. We briefly give an outline of the most relevant concepts. Regarding the number of hidden layers, a neural network is called "shallow" when there is only one hidden layer, while it is called as "deep" when there are more than one. In general there is not an "optimal" number of hidden layer for a given problem. Indeed, while input and output layer are determined by the dimension of input and output vectors, there is not a rule for setting the number of hidden layers and the number of nodes. In practice, a common approach for these choices consists in the use of a grid search, where the network is trained on a limited size of data (typically the validation set) for different values of parameters. The network with the best performance is then selected.

Regarding the "flow of information", a feed-forward neural network, as explained above, allows only a forward flow of information. A recurrent neural network, instead, also permits the flow of information backward between nodes. In the financial/actuarial sector, this characteristic of recurrent neural networks makes their application especially useful in the context of time series forecasting and related problems.

Once we have defined the structure of a neural network, it is necessary to determine the learning algorithm, i.e. the approach for calibrating the parameters in order to minimize the loss function. In the context of neural network, the most used learning algorithm is the so-called backpropagation. This approach consists in an efficient computation of the gradient of the loss function with respect to each of the parameters of the neural network. The name derives from the fact that this approach "propagates" the error of the loss function backward, calculating its gradient respect to each of the weights by means of the chain rule. The necessity of this operation is related to the fact that the gradient of a function at a given point defines the direction where its increase is maximized and the relative "speed"; as a consequence, moving in the opposite direction leads to its minimization. Hence, in the context of neural network, where the objective is the minimization of a loss function, backpropagation employs the gradient in order to update the weights in the opposite direction with respect to them. The gradient of loss function can be calculated according to many different approaches, with gradient descent being one of the first optimization algorithms to be employed for finding the (local) minimum of the loss function of a neural network. An important characteristic of the "base" version of gradient descent is that it requires the observation of the whole training set before updating the set of parameters. Consequently it comes with a strong computational burden, especially with large datasets, due to the fact that it is necessary to pass through the whole training set in order to perform a single update of the parameters. In order to solve this important drawback many extensions have been developed with stochastic gradient descent being one of the most known. This optimization algorithm has the opposite behavior of "base" gradient descent, since it updates the set of parameters after iterating a single sample of the whole dataset. In this way, at the cost of an initial higher uncertainty, the learning speed of the algorithm is strongly increased. Moreover, theoretical results assure that also stochastic gradient descent converges to a (local) minimum, usually making a preferred approach. Among the many extensions developed, an approach coming as a sort of compromise between gradient descent and stochastic gradient descent is the so-called batch gradient descent. Under this approach the gradient of the loss function and the consequent update of the parameters is computed against a set of observations, called mini-batch, at each step. In this way, the update of parameters will be faster than gradient descent and more stable than stochastic gradient descent[2].

In the following, in order to show more clearly all the elements of neural networks described until this point, we present an illustrative example. We base this example on a shallow (one hidden layer) feed-forward neural network, as depicted in Figure 5.2.

This network consists of one input layer, composed by $i = 1, ..., n$ inputs, one hidden layer, composed by $h = 1, ...H$ hidden nodes and one output node. The objective is to apply a learning algorithm in order to find the set of weights $\mathcal{W}$ that minimizes the loss function.

Basing our approach on stochastic gradient descent, one iteration of the learning algorithm can be described by the following steps. We start from a certain input vector $\mathbf{x}$ and with the value of the set of weights as obtained from previous iteration

[2]It shall be noted that many other evolution have been developed. A detailed reference of these approaches can be found in [40].

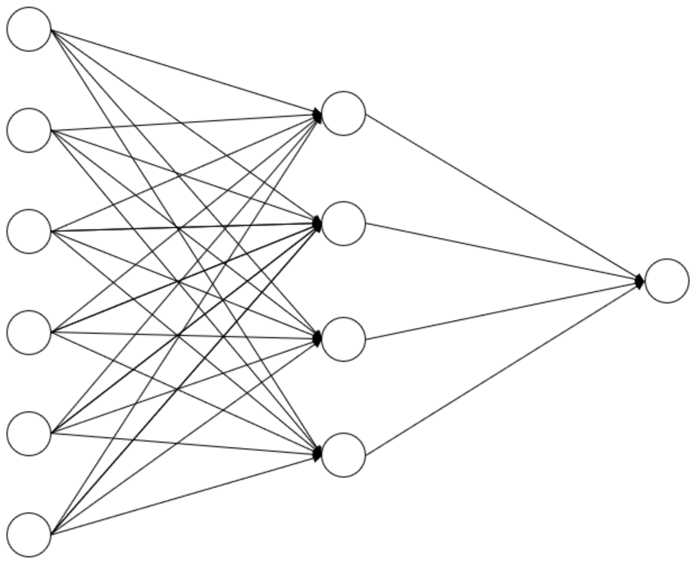

Figure 5.2. Illustrative representation of a shallow feed-forward neural network.

or according to their specific initialization in case of the first iteration[3].

As explained in the context of perceptron, each neuron h of the hidden layer is obtained as a weighted sum of the input neurons of the previous layer (in this case the input layer) to which an activation function is applied. In Formula (5.4) we have:

$$\phi_1\left(z_h\right)=\phi_1\left(\mathbf{w}_h^T\mathbf{x}\right)=\phi_1\left(\sum_{i=0}^{n}w_{ih}x_i\right), \forall h=1,...H \tag{5.4}$$

where $\mathbf{w}_h=(w_{h,0},...w_{h,n})$ is the vector of weights connecting the input nodes to the node h of the hidden layer, and x_0 represent the bias term.

At this point the intermediate outputs of the hidden layer are forward propagated to the next layer (in this case the output layer). As in previous step the forward propagation consists in the application of a weighted sum of the inputs to which a new activation function is applied. In Formula (5.5) we have:

$$f\left(\mathbf{x},\mathcal{W}\right)=\phi_2\left(\mathbf{w}^T\phi_1\left(\mathbf{z}\right)\right)=\phi_2\left(\sum_{h=0}^{H}w_h\phi_1\left(z_h\right)\right) \tag{5.5}$$

where $\mathbf{w}$ represents the set of weights connecting the nodes of the hidden layer to the output node.

Hence, this result consists in the estimate of the output of the neural network for the specific set of input with the weights value at the specific iteration. It is then possible to compare the estimate of the model with the actual value. As usual, we employ a loss function, whose formulation is strictly connected with the problem, for measuring the error/loss between the estimated and actual value. As reported in

[3]For an extensive review on the approaches employed for weights initialization refer to [59].

Formula (5.6), the loss is then computed according to

$$L = \mathcal{L}\left(y, f\left(\mathbf{x}, \mathcal{W}\right)\right) \tag{5.6}$$

where $\mathcal{L}\left(\cdot, \cdot\right)$ denotes the loss function.

At this point, the objective of learning algorithm is to move the weights in such a way to minimize the loss, calculated according to the specific loss function. The basic approach for adjusting the weights of the neural network, which we also report below, is the backpropagtion by means of gradient descent approach. We are then interested in back-propagating the error from the output layer and updating the weights according to the gradient of the loss with respect to them. In practice, the update of the weights according to the gradient descent approach can be described as reported in Formula (5.7):

$$\begin{aligned} w_h^{(i)} &= w_h^{(i-1)} + \Delta w_h^{(i)} \\ &= w_h^{(i-1)} - \gamma \nabla w_h^{(i)} \end{aligned} \tag{5.7}$$

where $\nabla w_h^{(i)}$ represents the gradient of the loss function with respect to the weight w_h, γ the learning rate and (i) the iteration index. It means that the initial weights are adjusted by a certain amount, indicated by $\Delta w_h^{(i)}$, in order to reduce the loss function. According to the gradient descent approach, the generic delta term is substituted by the term $-\gamma \nabla w_h^{(i)}$. Hence, the weights are moved in the opposite direction (negative sign) of a portion γ of the gradient of the loss function $\nabla w_h^{(i)}$.

Regarding the computation of the gradient of the weights, we follow the back-propagation approach. Hence, we start from the error obtained in the output layer and calculate the gradient of the weights from hidden to output layer as reported in Formula (5.8) for a generic node h:

$$\nabla w_h = \frac{\partial L}{\partial w_h} \left(y - f(\mathbf{x}, \mathcal{W})\right) \phi_2^{'} \left(\sum_{h=0}^{H} w_h \phi_1(z_h) \right) \phi_1(z_h)\,. \tag{5.8}$$

The errors are then back-propagated, adjusting the weights of previous layer proportionally to their contribution to the whole error, according to the approach described in Formula (5.9).

$$\begin{aligned} \nabla w_{ih} &= \frac{\partial L}{\partial w_{ih}} \\ &= \frac{\partial L}{\partial \phi\left(z_h\right)} \frac{\partial \phi\left(z_h\right)}{\partial w_{ih}} \\ &= \left(y - f(\mathbf{x}, \mathcal{W})\right) \phi_2^{'} \left(\sum_{h=0}^{H} w_h \phi_1(z_h) \right) w_h \phi_1^{'} \left(\sum_{i=0}^{n} w_{ih} x_i \right) x_i\,. \end{aligned} \tag{5.9}$$

Once the gradient of all the weights is obtained it is possible to apply the update rule described in Formula (5.7) until convergence. Regarding the update rule it shall be remarked the importance of the learning rate parameter γ, governing the speed in which the algorithm learns from observations. In practice the setting of this hyper-parameter[4] plays a crucial role in the converge of the algorithm to the minimization of the loss and its speed.

It shall be noted that the application of gradient descent assures the convergence to a global minimum solution only in case the loss function is a convex function. In the general case this is not true, which means that the solution reached is assured to be just a local minimum. In complex applications of neural network reaching a local minimum solution could be considered sufficient, since it could mean that it is not overfitting the training set. On the other hand, saddle points, which are another possible solution reached by the learning algorithm, are more problematic. Indeed, similar to a local minimum point also saddle points have a gradient close to 0, which imply that the weights will be stuck in that region.

In case the network is composed by more hidden layers the procedure would still be the same, with the only difference of having a backward propagation of the error along all these layers in order to arrive at the input layer.

The last relevant point is related to the potential overfitting of the training set, which would lead to a reduced performance of the test set. For solving this problem many different techniques have been developed. Regularization techniques have the objective of improving the performance of the neural network by reducing the variance at the cost of a slightly increase in the bias. Indeed, given the universal approximation theorem, a neural network could learn any relation on the set of observations, meaning that the error in the training set can be made as small as possible (small bias). However, the resulting model obtained in this way would not generalize well, because it has overfitted the training set. Hence it would lead to bad performance on non-observed data (high variance), explaining the necessity of using regularization techniques. The most common regularization techniques employed in this context are called L_1 and L_2 regularization. They derive from Ridge and Lasso regression, which add a penalty term to a regression problem aiming at reducing the variance at the cost of a small bias. The general formulation of the loss function subject to a regulation technique based on norm penalty term is reported in the following formula:

$$L_{reg} = L - \lambda ||w||_p$$

where $||w||_p = (\sum_{n=1}^{N} ||w_n||_p)^{1/p}$ is the norm of order p and λ is a hyper-parameter governing the intensity of the regularization penalty[5]. Note that L_1 and L_2 are indeed the regularization technique were the penalty term is based on the absolute value (order-1 norm) and squared value (order-2 norm) of the weights. Other common regularization technique employed in the context of neural network are dropout and

[4]It is called hyper-parameter since it is not a parameter updated during the training step as are the weights, but it is fixed from the beginning. Usually it is estimated in a previous phase on the validation set (a set of observations held out from training and test ones) on a grid search of possible values.

[5]As for the other hyper-parameters, also λ is determined in a procedure before the training phase and based on a separate set. As evident, the higher the value of λ, the higher will be the penalty to big changes of weights.

early stopping[6]. The first one consists in a stochastic transformation of the structure of the network which randomly drops some of the connections. The latter instead consists in the employment of a stopping criterion for the selection of the moment in which to terminate the learning algorithm, in order to avoid overfitting.

5.3 Neural network model for efficient frontier

The starting point of the neural network model for obtaining reinsurance strategies on the efficient frontier that we propose in this section is the stochastic simulation approach described in Chapter 4. The approach developed in the previous chapter takes as input all the possible combinations of reinsurance strategies, considering both treaties and reinsurers' characteristics, and return as output the metrics of risk and return that the insurance company typically considers in order to define optimal strategies. By comparing these metrics for all the combinations, it can be easily obtained the efficient frontier and the corresponding strategies. However, one drawback of this stochastic simulation is that it could become impracticable in case we use a too small step size for quota share and excess of loss treaties or if we consider a high number of additional components or characteristics of the counterparties. Indeed, in order to obtain the optimal reinsurance strategies, the model needs to compute all the possible combinations between the considered variables. Hence, while the application of the functions for estimating the metrics of risk and return on the basis of the characteristic analyzed could be so efficient to not require a computational burden, the problem relies on the allocation of a high amount of memory in order to manage a vector of so many combinations. Aside from the technical details, it is evident how the number of combinations scales up really quickly with an increase in the details, etc. As an example considering only quota share treaties and a company underwriting in three lines of business, if we choose a step size of 10%, starting from a minimum retention of 20%, we have 9 different values of quota share retention for each line of business. Hence, the combinations of reinsurance strategies to consider are $9^3 = 729$. However, if we just increase the level of detail and choose a step size of 1% we have 81 different values for each segment, which means $81^3 = 531,441$. Finally, just to show how much this scales up, if we consider a step size of 0.1% we have 801 different values of quota share retention for each segment, meaning that the number of possible combinations between the three segments become $801^3 = 513,922,401$. Clearly, this level of detail is not something we are interested in a practical perspective, but it is just to show how considering all the possible combinations could lead to some problem, since we still need to account for the different excess of loss treaties, credit quality steps of the reinsurance companies and number of reinsurance companies covering these risks.

Indeed, while the stochastic approach developed in the previous chapter assures to reach the optimal reinsurance strategies at the chosen step size, it could become a solution not practicable if we are interested in a level of detail too high, since it requires the computation of all the possible combinations. At the same time, in case we just want to obtain the reinsurance strategies on the efficient frontier and are not

[6]For a detailed taxonomy and description of regularization techniques in deep learning refer to [54].

interested in the details, relationship and other characteristics of the “sub-optimal” treaties, this approach is inefficient. Indeed, considering all the possible combinations of reinsurance strategies, we also analyze a high number of combinations that are already expected to be sub-optimal.

For the reasons described above, we propose a different approach for estimating the efficient frontier and the corresponding reinsurance strategies. This approach comes with many important advantages as will be presented in the following. It employs two neural networks and requires a much lower number of combinations, making it efficient and easily extendable also in case we have a high number of input variables which defines the different strategies.

In this context, compared to other machine learning approaches, the characteristics and properties of neural networks make them the most appropriate model for many reasons. One of their useful characteristics is the ability to approximate also complex functions, thanks to the discovery of potential non-linear relations between variables. Moreover, a single neural network model is able to minimize the loss related to more than one metric, hence returning more than one output. Finally, while they are still considered a black-box model, especially compared to classical statistical approaches when it comes to the interpretation of parameters[7], their structure allows to derive the values in all the nodes, for a given input vector. These characteristics will be of fundamental importance in the model we are going to define in the following.

In practice, in order to obtain the efficient frontier of the reinsurance strategies for the insurance company, the approach that we develop requires the following four steps:

(i) **Simplified simulation approach:** Define an approach for simulating a low number of input variables with low computational burden, while preserving an adequate exploration of the input space.

(ii) **Neural network model 1:** Build a function approximation of the relation between inputs (characteristics of the reinsurance treaties and of the reinsurer(s)) and outputs (corresponding metrics of risk and return).

(iii) **Neural network model 2:** Build a model which, by means of previous neural network, takes as input one objective metric and returns the same metric while maximizing the other one(s).

(iv) **Efficient strategies determination:** Find the parameters of reinsurance strategies on the efficient frontiers defined by previous neural network.

5.3.1 Simplified simulation approach

Differently from the model developed in the previous chapter, in this case we do not need to consider all the possible combinations between input variables for determining the corresponding outputs. Indeed, we will employ a neural network for constructing a functional approximation of the relation between inputs and outputs. For this

[7]Actually, it shall be noted that a relevant research area is related to Explainable AI (XAI), which has the objective of making machine learning models interpretable.

Table 5.1. Input variables defining the reinsurance strategies.

Variable	Type	Support	Description
D_{MTPL}	Numerical	$[100,000; 1,000,000]$	Deductible value of XL treaty for the MTPL segment
L_{MTPL}	Numerical	$[1,000,000; 10,000,000]$	Limit value of the XL treaty for the MTPL segment
D_{GTPL}	Numerical	$[100,000; 1,000,000]$	Deductible value of XL treaty for the GTPL segment
L_{GTPL}	Numerical	$[1,000,000; 10,000,000]$	Limit value of the XL treaty for the GTPL segment
α_{MTPL}	Numerical	$[0,1]$	Retention quota for the MTPL segment
α_{GTPL}	Numerical	$[0,1]$	Retention quota for the GTPL segment
α_{MOD}	Numerical	$[0,1]$	Retention quota for the MOD segment
$multi_{RE}$	Categorical	$\{0,1\}$	Indicator for single/ multiple reinsurer
CQS_{MTPL}	Categorical	$\{0,1,2,3,4,5,6\}$	CQS of the reinsurer(s) for the MTPL segment
CQS_{GTPL}	Categorical	$\{0,1,2,3,4,5,6\}$	CQS of the reinsurer(s) for the GTPL segment
CQS_{MOD}	Categorical	$\{0,1,2,3,4,5,6\}$	CQS of the reinsurer(s) for the MOD segment

reason we need a much lower number of combinations, just sufficient for the neural network to have an adequate predictive capacity.

Hence, we fix the total number of combinations of input variables at $100,000$, just to ensure an adequate coverage of the input space. Regarding the input variables which defines the different reinsurance strategies we have the same elements of previous model and we report them in Table 5.1. In practice, we have the first 7 variables describing the characteristics of the excess of loss and quota share treaties for each line of business. Then, we have the remaining 4 variables which defines the presence of single/multiple reinsurer(s) and the credit quality step of the reinsurer(s) for each line of business.

It is possible to observe that these variables are both numerical and categorical, each one defined on a specific support. In order to cover adequately the input space we employ a straightforward approach. For numerical variables we simply sample $100,000$ values from the respective support by using a uniform distribution. In this way it is guaranteed that the sample of each variable cover the specific support. For categorical variables we simply sample $100,000$ values from a so-called generalized Bernoulli distribution with equal probability for each element. At this point, since the sampling of each variable has been obtained randomly, we can define a "dataframe" built in such a way that the s-th input vector consists in the s-th sample from each input variable. In this way it is possible to obtain the corresponding metrics of risk and return for each of the $100,000$ samples, by means of the Algorithms described in Chapter 4.

Having defined the dataframe of inputs/outputs we can now move to the description of the first neural network.

5.3.2 Neural network model 1

The objective of the first neural network (from now on called NN_1) is to model the dependence between inputs and outputs, where the former represent the characteristics of reinsurance strategies and the latter the corresponding metrics of risk and return. In practice, NN_1 aims at approximating, as accurately as possible, the relation connecting inputs and outputs. In this way it allows us to estimate risk and return also for non-observed combinations of reinsurance strategies in the training set.

The architecture of this network is composed as depicted in Figure 5.3[8]. It is a deep neural network with 30 input nodes, 3 hidden layers, each one consisting of 64 nodes, and 2 output nodes. The setting of the hyper-parameters for number of hidden layers and number of nodes in the different layers were not object of a specific assessment. They were set using some knowledge of the domain, in order to still have a deep neural network, since it usually guarantees better performance, but without having too many parameters, since the task should not be too complex.

At this point we provide a description of the pre-processing and some remarks regarding the management of input variables. As in most of the statistical and machine learning models, also in case of neural network it is possible to have both numerical and categorical variables. In our context, as anticipated, the input variables of the neural network model are reported in Table 5.1.

Regarding numerical variables, a common operation typically performed in the set-up of a neural network, before the training phase, consists in the scaling of input variables such that their magnitude is similar. Actually, this operation is not strictly necessary from a theoretical point of view, but in practice it helps the convergence of loss function toward its minimum in a lower time and without potential problems related to the backpropagation procedure (e.g. vanishing gradient). In our model, following this logic, we scale the numerical variables according to the min-max scaling approach. Given the vector of observations from a given input variable $\mathbf{x}$, in Formula (5.10) we describe the rescaling according to the min-max approach.

$$x'_i = \frac{x_i - \min(\mathbf{x})}{\max(\mathbf{x}) - \min(\mathbf{x})} \tag{5.10}$$

where x_i is a generic observation of the input vector $\mathbf{x}$ and x'_i is the corresponding rescaled element. It is possible to observe that, applying this normalization procedure to all the numerical input variables, the range of their scaled correspondences becomes $[0, 1]$ for all of them.

[8]It shall be underlined that the neural network depicted in the figure is just an illustrative representation, especially considering the number of nodes.

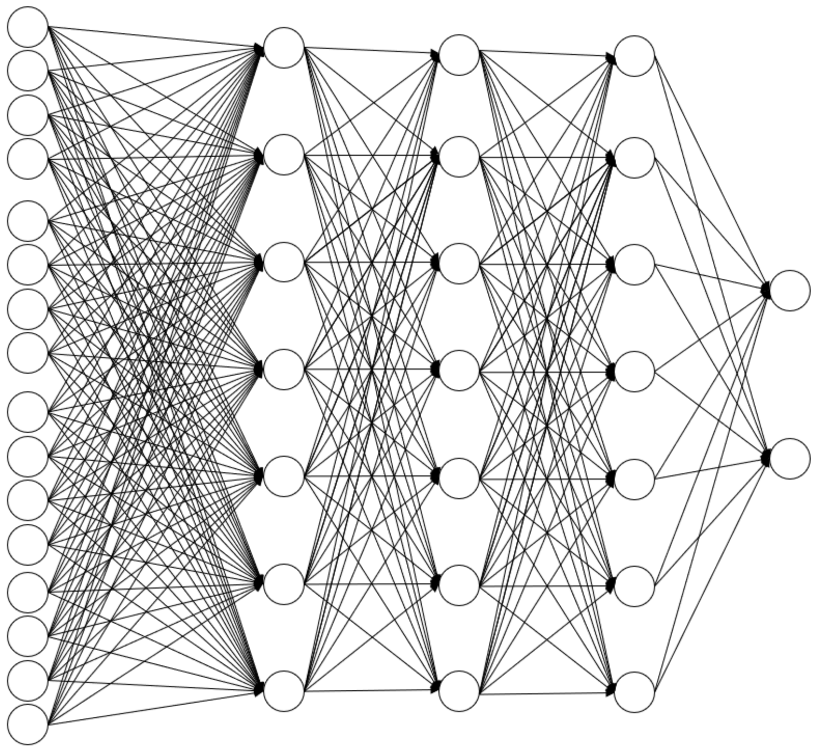

Figure 5.3. Illustrative representation of NN_1.

In practice, there are many possible alternative for scaling approaches, each one with advantages and drawbacks. We choose min-max scaling for its simplicity and also because it returns elements in a bounded interval between 0 and 1, with 0 for the minimum observation and 1 for the maximum one. Indeed, since the sampled observations for each variable include their minimum and maximum we are ensuring a correspondence between the minimum and maximum of each variable and 0 and 1 of their scaled version.

Also regarding categorical variables we have many possible approaches for managing them in the context of neural network (an extensive description of the techniques employed for representing categorical variables in neural network is reported in [43]). One of the first possibility, also employed in statistics, is the “one-hot encoding”. This approach consists in “encoding” the categorical variables in a new set of vectors where 1 indicates the presence and 0 the absence of the categorical variable (as the creation of “dummy variables” in regression analysis). Hence, this solution transforms the input vector of categorical variables in a matrix with number of columns equal to the number of unique variables. Another approach for managing categorical variables, developed specifically in the field of neural network, is the so-called embedding layer. This approach consists in the creation of a layer between the one-hot encoded variables and the hidden layers, with a lower cardinality compared to the number of levels of the categorical variable and whose weights are also “learned” during the training phase. Embedding layers were borne in the context of natural language processing, but they are useful in any context where the number of levels of categorical variables is so high that one-hot encoding creates a high dimensional sparse vector, with the potential problem of the "curse of dimensionality". In our context, since the number of levels for each categorical variable is not so high, we employ directly one-hot encoding. However, it shall be noted that in this way we have that the variable $multi_{RE}$ is now represented as two separate vectors the first one with 1 in case of single reinsurer and 0 in case of multiple reinsurers and viceversa for the second one. Similarly, also the other three categorical variables one-hot encoded are now composed of seven separate vectors. Hence, compared to the list of variables described in Table 5.1, after the pre-processing phase, input layer is actually composed by 30 variables, of which 7 numerical and 23 encoded representations of categorical variables.

At this point, having defined the structure of neural network and the pre-processing of input features, we can move to the description of the other elements of the neural network. In particular, we shall define the activation function for each layer. In this context we choose to employ a non-linear activation function, specifically the rectified linear unit (ReLU). This activation function, together with its variants (e.g. Leaky ReLU, Parametric ReLU) is one of the most commonly employed non-linear functions in deep neural networks. As reported in Formula (5.11) it is simply defined as the maximum between 0 and its argument:

$$\phi(x) = \max(0, x) = \begin{cases} 0 & x \leq 0 \\ x & x > 0 \,. \end{cases} \tag{5.11}$$

Finally, the last and fundamental element of NN_1 is represented by the loss function. As anticipated, we are interested in a neural network that takes as inputs the variables defining the reinsurance strategies, returning the corresponding metrics of risk and return. Hence, we want to adopt a loss function that considers the errors with respect to both the outputs. Indicating with $\mathbf{y}$ and $\hat{\mathbf{y}}$ the vector of true and predicted outputs respectively, in Formula (5.12) we define the loss function as:

$$\begin{aligned} L &= \mathcal{L}\left(\mathbf{y}, \hat{\mathbf{y}}\right) \\ &= \mathcal{L}\left(\mathbf{y}, f\left(\mathbf{x}, \widehat{\mathcal{W}}\right)\right) \\ &= p_1 \mathcal{L}_1\left(y_1, \hat{y}_1\right) + (1 - p_1)\mathcal{L}_2\left(y_2, \hat{y}_2\right) \end{aligned} \tag{5.12}$$

where p_1 is the weight associated to the error respect to the first metrics and $(1 - p_1)$ respect to the second one. As expected, for high values of p_1 we will have a neural network predicting more precisely the first metric, since the relative error would have an higher weight. On the other hand, for small values of p_1 the neural network will produce a more precise outcome for the second metric. In practice, there is typically a trade-off between the precision of multiple outputs and consequently it is necessary to choose the weight according to the required compromise and taking into account also the relative scale of the different outputs.

In this specific case we choose to employ as error measure the mean absolute error (MAE), obtaining the loss function described in Formula (5.13):

$$L = p_1 \frac{1}{n} \sum_{i=1}^{n} \left| y_{1,i} - \hat{y}_{1,i} \right| + (1 - p_1) \frac{1}{n} \sum_{i=1}^{n} \left| y_{2,i} - \hat{y}_{2,i} \right| . \tag{5.13}$$

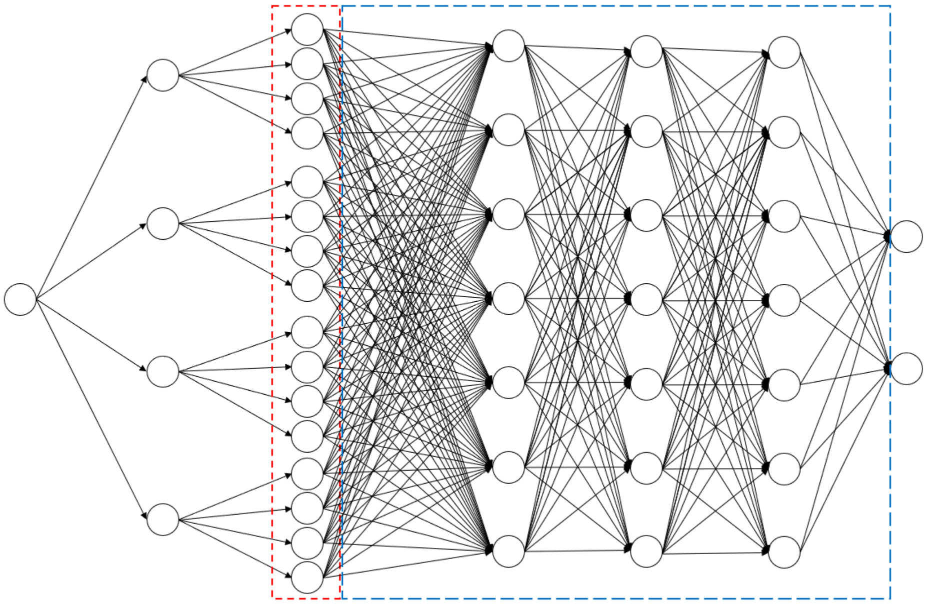

Figure 5.4. Illustrative representation of NN_2. The part surrounded by red dots consists in a concatenation layer of separate inputs, while the part surrounded by blue dots consists in NN_1.

The parameters of NN_1 are calibrated by means of a training set, representing 90% of total observations. During the training phase 10% of the observations are used as validation set to ensure that the model is generalizing also on non-observed data. In order to avoid overfitting of training data early stopping rule is included, while learning rate schedule is employed for trying to avoid that the model learning “stagnates”.

The generalization ability of the model is then assessed by measuring its performance on the test set, built on the remaining 10% of total observations.

5.3.3 Neural network model 2

The objective of the second neural network (from now on called NN_2) is to estimate the efficient frontier of the multi-objective optimization problem. In practice, NN_2 takes as input one of the output metrics and aims at returning the same value of the input and the maximum possible value of the other metric. In this specific case, it takes as input a value of the expected RoE and return two outputs; where the first should be the same value of the expected RoE metric and the other one the maximum possible SR, for the given expected RoE. In order to do this, it takes advantage of the relation between inputs and outputs learned by the previous neural network as we will describe in the following.

For simplifying the explanation of NN_2, first of all we report in Figure 5.4 the illustrative representation of its architecture.

It is a deep neural network with 1 input vector, 5 hidden layers, the first two consisting of 5 and 30 nodes, while the other ones with 64 nodes each, and 2 output nodes. As outlined by the dotted regions, NN_2 is actually composed of more parts. The first layer is the input node, which is then replicated in 5 separate nodes, all with the same value and non-trainable. Then, each of these "second-stage" inputs is connected to another layer, which is then concatenated together, as indicated by the red dots. The remaining part of the network, instead, consists exactly in NN_1, as it is possible to observe from the common structure.

In the following we briefly explain the idea behind this modeling and the details of the structure of NN_2. This neural network takes as input the expected RoE and replicate it 5 times in the subsequent layer, which simply consists in a non-trainable layer with weights equal to 1, no bias and linear activation function[9]. These 5 nodes represent respectively the numerical features (one node) and the categorical features (one node for each feature) reported in Table 5.1 before the pre-processing phase. Hence, each of these 5 nodes is connected to a specific layer which can be interpreted as the input layer of NN_1 for the specific variable. In more detail, the first node propagates in 7 nodes, which represent the 7 numerical variables of NN_1. The other nodes are linked with a number of nodes equal to the number of levels of the corresponding categorical variable in NN_1, representing in this way the one-hot encoded version of these variables. Clearly, once all these layers are concatenated, as indicated by the red dotted region of Figure 5.4, we obtain the exact analogous of the input layer of NN_1. Indeed, our objective is that the concatenated layer represents the parameters of reinsurance strategies used as input for finding the corresponding metrics of risk and return in NN_1. In this way we take advantage of the already learned relation between these inputs and outputs and build the remaining part of the neural network exactly as NN_1. It shall be underlined that the equivalence between the structures is not limited to the architecture, but it also regards the parameters. Moreover these weights are set as non-trainable in order to not modify the relation learned by NN_1.

At this point, the idea behind this model is straightforward. Once NN_2 has been calibrated on training set, reaching a sufficient performance, we just need to choose a value of expected RoE for obtaining the corresponding Solvency Ratio on the efficient frontier (from the output) and the reinsurance strategy generating this combination (from the concatenated layer). Repeating this procedure for a sequence of expected RoE at a certain step size we derive the efficient frontier and the corresponding reinsurance strategies.

Having provided an overall description of the structure of NN_2 it is now necessary to deep-dive on some details and potential issues. As anticipated, the first node of the first hidden layer is connected with 7 other nodes, representing the 7 numerical input features of NN_1. Recalling that we employed a min-max normalization of numerical variables in NN_1, we have to provide inputs on the same range also in this case. This requirement can be easily satisfied by simply using a sigmoid activation function, since it returns a value between 0 and 1. In Formula (5.14) we report one

[9]It is possible to notice that it is equivalent to have a neural network directly with 5 input nodes all equals, but we preferred this architecture.

of the most commonly employed sigmoid functions, represented by the logistic:

$$\phi(x) = \frac{1}{1 + e^{-x}} \,. \tag{5.14}$$

Differently from numerical variables, the procedure for deriving the one-hot encoding of categorical variables, as in NN_1 is much more complex. As a first step, it is evident that the most appropriate activation function in this context is represented by the softmax. Indeed, this function, as reported in Formula (5.15), applies the exponential to each element and returns the normalization of each element with respect to the sum of these exponentials:

$$\phi\left(x_i\right) = \frac{e^{x_i}}{\sum_{n=1}^{N} e^{x_n}} \,. \tag{5.15}$$

It is typically employed as activation function of the output layer in classification problem, since the returned values can be interpreted as probabilities. Also in this context, we are interested in transforming the input in probabilities in order to apply at the next step the one-hot encoding and obtain the same structure as the input layer of NN_1. However, differently from what we did in the pre-processing of NN_1, in this case we cannot apply the one-hot encoding between these two layers. The reason is that, as a function, one-hot encoding can be interpreted as *arg max*, which is a not differentiable function. Hence, it cannot be employed, since we have to apply back-propagation also to the loss in previous layer. The solution that allows us to still obtain a one-hot encoding of the variables from softmax comes from the application of a "soft" version of one-hot encoding: the Gumbel-Softmax estimator. It is a (approximate) differentiable version of one-hot encoding, which then permits the application of backpropagation algorithm[10].

Finally, the last and fundamental element of NN_2 is represented by the loss function. Given the presence of multiple outputs, also in this case we are interested in a loss function that considers the errors with respect to multiple losses. The specific loss function is reported in Formula 5.16

$$\begin{aligned} L &= \mathcal{L}\left(\mathbf{y}, \hat{\mathbf{y}}\right) \\ &= \mathcal{L}\left(\mathbf{y}, f\left(\mathbf{x}, \widehat{\mathcal{W}}\right)\right) \\ &= p_1 \mathcal{L}_1\left(y_1, \hat{y}_1\right) + (1 - p_1)\mathcal{L}_2\left(y_2, \hat{y}_2\right) \\ &= p_1 \frac{1}{n} \sum_{i=1}^{n} \max\left(0, y_{1,i} - \hat{y}_{1,i}\right) - (1 - p_1)\hat{y}_2 \end{aligned} \tag{5.16}$$

where the value of p_1 is set in a way to ensure a balance between the losses during the learning phase.

[10] For a detailed explanation of Gumbel-Softmax distribution and its application refer to the original work [48].

5.3.4 Efficient strategies determination

At this point, we have calibrated NN_2 such that it is able to determine the combinations of expected RoE and SR on the efficient frontier. In order to determine the corresponding reinsurance strategies, reminding the architecture of NN_2 reported in Figure 5.4, we just need to compute the first step of the forward propagation, given the input value (representing the expected RoE). In practice for numerical variables we employ the sigmoid function reported in the following formula:

$$z_i = \frac{1}{1 + e^{-(w_0 + w_i x)}} .$$

It shall be noted that, since we applied the sigmoid function, the returned outputs will be in the range $(0, 1)$. Hence, as last step, in order to obtain the corresponding unscaled variables it is necessary to apply the inverse of the min-max transformation described in Formula (5.10).

Similarly, for categorical variables we employ the GumbelSoftmax function reported in the following formula:

$$z_i = GumbelSoftmax\left(\frac{e^{(w_0 + w_i x_i)}}{\sum_{n=1}^{N} e^{(w_0 + w_n x_n)}}\right)$$

Also in this case, we have an additional final step consisting in retrieving back the categorical variable corresponding to its numerical encoding obtained from previous formula.

5.4 Numerical application

At this point we can assess the performance of the approach proposed in the previous section on a practical case study. For this numerical analysis we employ the same setting described in Section 4.2. The main difference is instead related to the data for the analysis. Indeed, as explained in Section 5.3.1, we have $100,000$ combinations of reinsurance strategies on which to calibrate the model. As anticipated in the same section we use 90% of this data for training NN_1 and the remaining 10% for testing purposes.

The total loss that we want to minimize is composed as a weighted average between the loss respect to the expected RoE and respect to SR, with weights of 20% and 80% respectively. Indeed, our objective is that NN_1 is able to predict both the metrics. The choice of the different weights for the average of the two losses is related to the different scale of the variables and some cross-validation analyses. In practice, a better way for defining these weights is to consider them as hyper-parameters and consequently follow the approach for their calibration.

Regarding the loss metric, we choose the MAE for both the losses. The reason of this choice is that the support of both SR and RoE is $\mathbb{R}$ and consequently we have to select a loss metric which do not compensate positive and negative errors.

For instance, the mean error would have been not appropriate for this scope, while another possible alternative would have been the mean squared error (MSE).

Finally, employing an early stopping rule we reached convergence before 1000 epochs, using a batch size of 4096 samples.

In Figure 5.5 we report the combinations of expected Return on Equity and Solvency Ratio for the reinsurance strategies in the test set according to actual results (in black) and predictions from NN_1 (in red). It is possible to observe that the NN_1 adequately reproduces the actual shape of the combinations of RoE and SR according to the parametric assumptions of the analysis. However, we can notice that there are still some areas where the prediction has a lower precision and there is some room for improvement. In practice, it could be possible to improve the predictive capability of the model by a specific assessment of the "best" value of each hyper-parameter (e.g. number of nodes, number of hidden layers, improvement in the loss for the early stopping rule, etc.).

Numerically, we register a total loss of 1.3%, where the loss from the expected RoE is equal to 0.7% and the one from the SR is 1.4%. It shall be noted that we decided to not normalize the values of outputs, having in this way a difference in their scale. This is the reason for choosing the specific weights between the two losses, and also why the error in the estimate of the solvency ratio is much higher than the one related to the expected return on equity.

In Figure 5.6 we report a more detailed comparison of actual and predicted values for RoE and SR. It is possible to observe that the model has more difficulty in estimating the actual solvency ratio, despite the greater weight we associated to its loss. This is particularly relevant for smaller values of the SR, while the more we reach higher values of the SR the more we observe an improvement in the prediction. Moreover, it is also possible to observe that, while for the RoE we have a quite symmetric prediction compared to the actual values, for the SR the overestimation prevails. A fine-tuning of the model, like increasing the weight associated with the error in estimating SR or including an additional weight for under and over-estimation, could help improve model performance.

For NN_2 we set the input and output in the following way. For the input, we sampled $500,000$ random variables from uniform distribution in the interval $(-0.3, 0.3)$. It represents the range of potential values of RoE, so we choose an interval centered at 0 and larger than the observed one, although in practice we do not expect values too extreme. Regarding the outputs we set the output for RoE equal to the input value, while for the SR we set all the elements equal to 400%. In practice, this value represents an extreme (unreachable) value of SR to which the estimates should aim. In practice, it shall be noted that, according to the loss function we defined in Formula (5.16), this value has no impact on the total loss.

After training the neural network on the training set described above, we are then able to make a first assessment of the efficient frontier produced by the model compared to the observations. We expect that it is at least at the same level as the highest observation for each value of the Return on Equity. Indeed, as shown in Figure 5.7, the efficient frontier produced by NN_2 is above the set of observations.

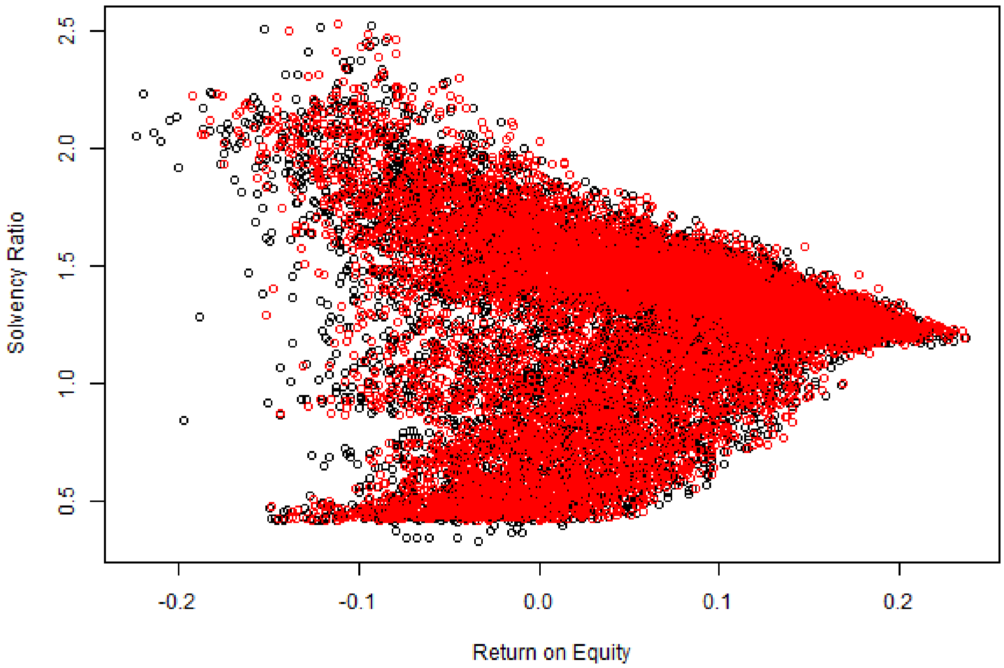

Figure 5.5. Scatterplot of the combinations of expected RoE and SR. In black are reported actual values from the training set, while in red are reported predicted values.

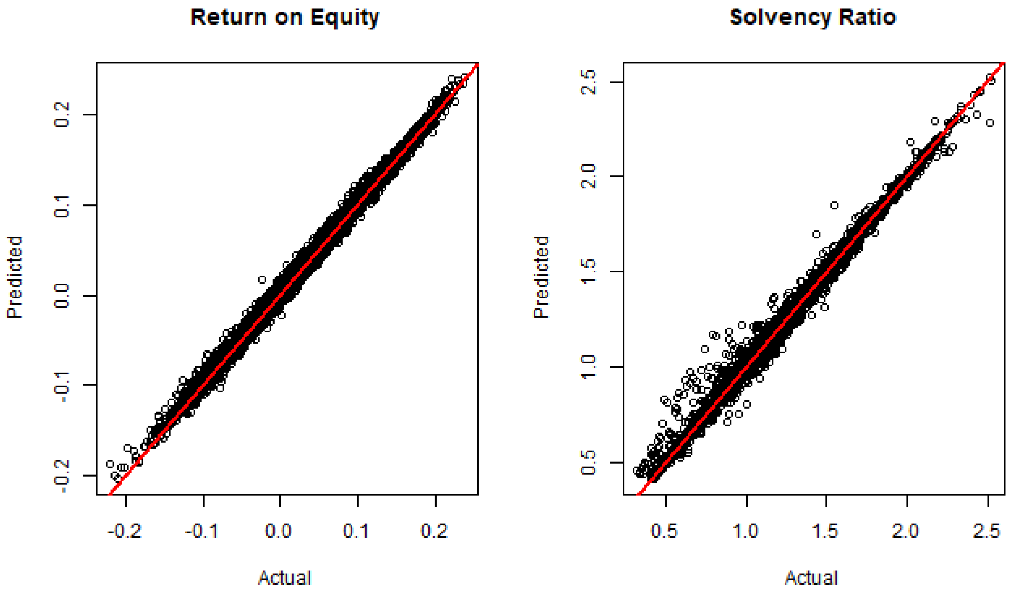

Figure 5.6. Actual vs Predicted plot for expected Return on Equity (left) and Solvency Ratio (right).

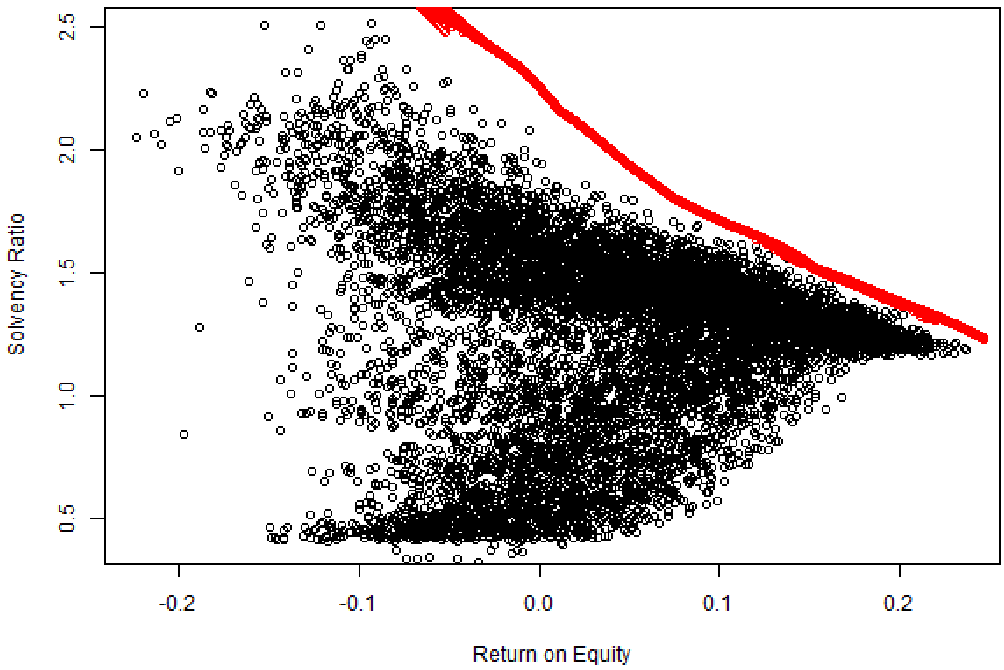

Figure 5.7. Scatterplot of the combinations of expected RoE and SR. In black are reported actual values from the training set, while the red line represents the estimated efficient frontier according to the model.

This means that the model has found a set of strategies that provide a higher Pareto frontier than to the one given by the observed data. This result is expected since we know that the reinsurance strategies of the observed data are based on a limited number of combinations at a given step-size. In practice, the assumption is that if we were able to consider all the possible reinsurance strategies at the smallest step size, we would obtain an efficient frontier very close to the one produced by the neural network model.

In order to derive the efficient strategies corresponding to each point of the efficient frontier we have to follow the last step, described in Section 5.3.4. In practice, it is necessary to apply the sigmoid and GumbelSoftmax functions, depending on whether the variable is numeric or categorical, in order to derive the "transformed" variables. Then, it is applied the inverse transformation of the preprocessed variables in order to obtain each element of the reinsurance strategy "unscaled".

In Table 5.2 we report a subset of efficient reinsurance strategies, obtained using as input a sequence of expected Returns on Equity and deriving the corresponding Solvency Ratio and reinsurance strategy.

It is possible to observe that, in line with the findings of the previous chapter (e.g. see 4.5), the strategies defining the efficient frontier prescribe a complete retention

Table 5.2. Subset of the efficient strategies maximizing expected Return on Equity and Solvency Ratio.

D_{MTPL}	L_{MTPL}	D_{GTPL}	L_{GTPL}	α_{MTPL}	α_{GTPL}	α_{MOD}	$multi_{RE}$	CQS_{MTPL}	CQS_{GTPL}	CQS_{MOD}	SR	RoE
1,000,268	3,651,094	4,999,605	1,000,518	20.1%	20.0%	100%	1	7	3	4	260.5%	−8.5%
1,000,263	1,315,931	4,997,786	1,000,360	23.7%	20.0%	100%	1	7	3	4	255.3%	−7.0%
1,000,262	1,141,071	4,993,484	1,000,321	31.2%	20.0%	100%	1	2	3	4	242.1%	−4.6%
1,000,261	1,062,496	4,979,344	1,000,286	48.4%	20.1%	100%	1	2	3	4	215.0%	0.1%
1,000,260	1,027,831	4,933,352	1,000,257	71.9%	20.7%	100%	1	2	3	4	182.4%	5.9%
1,000,259	1,012,662	4,788,748	1,000,231	89.0%	26.1%	100%	1	2	3	4	164.5%	10.3%
1,000,258	1,006,049	4,378,577	1,000,209	96.4%	53.9%	100%	1	2	5	4	144.6%	15.3%
1,000,257	1,003,169	3,488,852	1,000,189	98.9%	89.3%	100%	1	2	5	4	128.7%	22.5%
1,000,256	1,001,372	1,525,285	1,000,158	99.9%	99.8%	100%	1	2	5	5	124.4%	24.6%

of the risk associated with the MOD line of business[11].

Interestingly, we observe that there are efficient reinsurance strategies for different values of CQS, rather than having almost only CQS equal to 3 as it was for the efficient frontier obtained in previous chapter. This could also be related to the fact that, under this neural network approach, we are allowing for different CQS for each line of business. Indeed, in order to have a manageable number of combinations, in Chapter 4 we had to consider only the case of same CQS for each reinsurer of the three LoBs. The possibility of ceding the risks to reinsurers with different CQS for each LoB allows a better management of each risk by the insurance company and leads to a higher presence of $multi_{RE}$ in the efficient frontier.

[11]It shall be specified that even for the strategies not reported in the table there is a strong prevalence of high retention values for the MOD line of business.

Appendix C

Neural network models

C.1 Sensitivities

In order to show the strength of the described approach, we show how it can also be employed in case a smaller number of observations is available. Indeed, if we observe that NN_1 has a high enough performance with respect to our objectives, then NN_2 will also typically be able to estimate the efficient frontier. Moreover, while it was not needed for the specific case, since the task was simple enough and we wanted to clearly show the formulas for obtaining the parameters of the reinsurance strategy from the single input, it is possible to add other neurons and layers to both the neural networks in order to improve their performance.

In the following, we present the results of the neural network approach developed in Chapter 5 in case we calibrate it with half of the data used in the numerical example, hence with just $50,000$ observations.

In Figure C.1 we report the combinations of expected Return on Equity and Solvency Ratio for the reinsurance strategies in the test set according to actual results (in black) and the predictions from NN_1 (in red). We can observe that NN_1 adequately reproduces the actual shape of the combinations of RoE and SR according to the parametric assumptions of the analysis, despite the strong reduction on the available data on which to make inferences. Indeed, if we ensure that the sample space of each input variable is sufficiently represented in the training set, we can reduce the number of observations without a too strong worsening in the predictive ability of the model. Clearly, in this case there are even more areas, compared to Section 5.4 where the prediction provided by the model has a low accuracy.

Numerically, we register a total loss of 1.8%, with the one from expected RoE equal to 1.0% and the one from SR to 1.9%. We can notice that, as expected, we obtain a higher loss for both the metrics compared to the model based on $100,000$ observations, due to the lower number of data on which to calibrate the model. However, it shall be remarked that the MAE for both the metrics, as well as the total one, is still quite low.

In Figure C.2 we report a more detailed comparison of actual and predicted values for

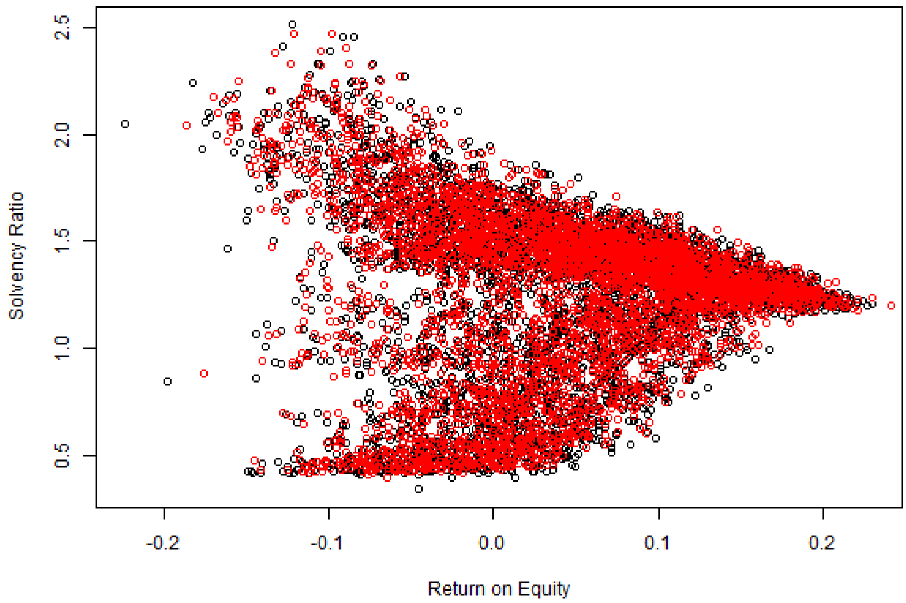

Figure C.1. Scatterplot of the combinations of expected RoE and SR. In black are reported actual values from the training set, while in red are reported predicted values.

RoE and SR. We can observe that, differently from the model described in Section 5.4, this one has a worse (relative) predictive capability in estimating the RoE, rather than the SR. Looking at the red line (representing the 45-degree line), we can notice that the estimates provided by the model are fairly symmetric. Hence, even with the limited data available the model does not seem to produce biased estimates. The only portion of prediction that seems to produce poor fitting concerns the area of actual SR below 100%, while we observe an improvement the more we move towards higher thresholds. Finally, also in this case an appropriate adjustment of the weight associated with the loss of each metric could help improve model performance.

Regarding the steps for producing the efficient frontier, we follow the same approach developed for the "full" model. In Figure C.3 we then report the combinations of expected RoE and SR of the dataset and the efficient frontier estimated by means of the "reduced" model (red dots). We can observe that the efficient frontier lies above all the combinations of treaties, meaning that the model was able to find an improvement in the strategies compared to observed data, but also compared to the whole (not observed dataset).

As expected, if we compare Figure C.3 with Figure 5.7 we can notice that the efficient frontier produced by this model is at a lower position compared to the one defined by the full model. The reason could be that, having to rely on half of the data used by the "full" model, it is less able to generalize the relationship between the variables, as confirmed by the lower predictive performance. Hence, this model produces an efficient frontier which is closer to the actual observed one.

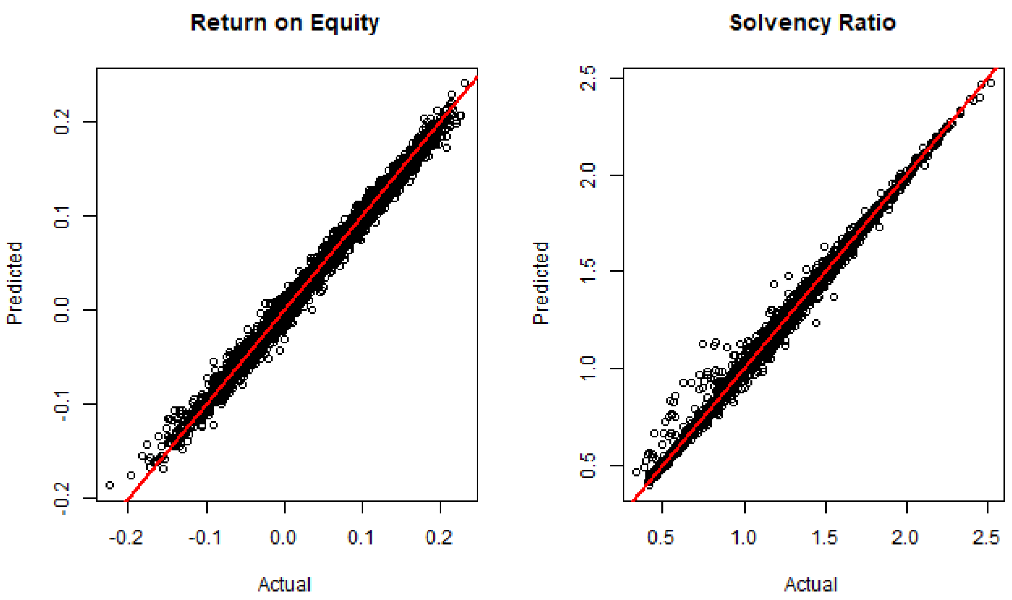

Figure C.2. Actual vs Predicted plot for expected Return on Equity (left) and Solvency Ratio (right).

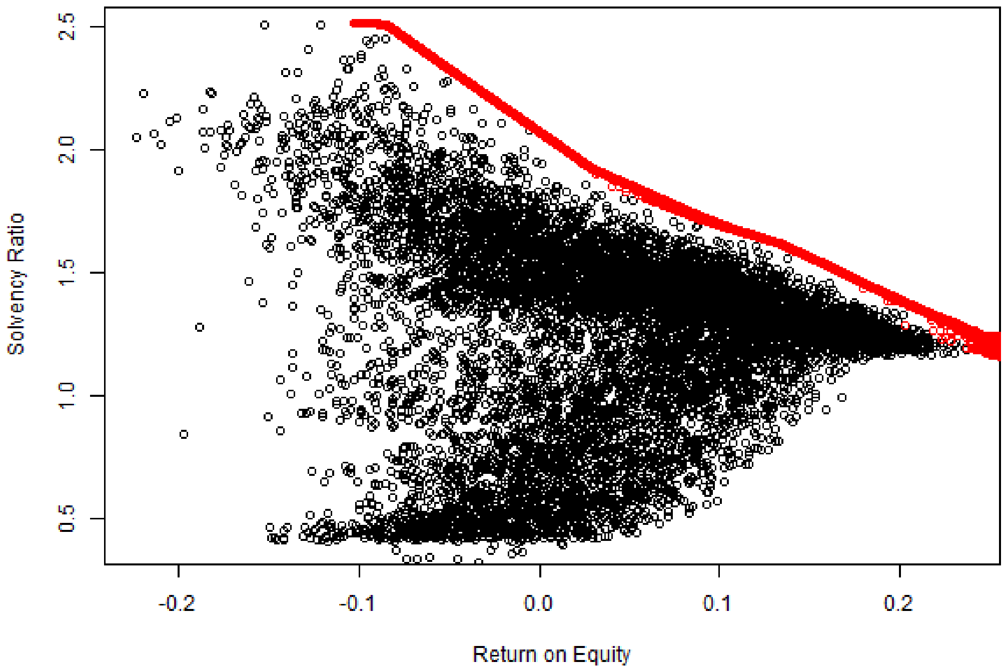

Figure C.3. Scatterplot of the combinations of expected RoE and SR. In black are reported actual values from the training set, while the red line represents the estimated efficient frontier according to the model.

Chapter 6

Conclusions

The main activity of an insurance company consists in covering another party from the financial effects connected to the occurrence of a risk. In pursuing its activity, the insurer shall always balance the potential profitability deriving from its underwriting activity with the related risk. Reinsurance treaties are the main instruments used by insurance companies for reducing their risks and balancing technical performance. These contracts work as a cession of risks from the insurer to the reinsurer, which in turn ask for a remuneration that reduce the insurer expected profit. However, the cession of risks to reinsurance companies is not free, but it generates another potential risk: the risk of default of the counterparty.

This thesis has three main objectives. The first one consists in presenting an extension of classical formulas of technical profit of an insurance company which considers the potential default of the reinsurance counterparty. The second objective of this thesis is to present a stochastic simulation approach that considers counterparty default risk, and enough general to potentially include other features, for estimating the efficient frontier of reinsurance strategies for a non-life insurance company. The last objective consists in showing how neural network models can be easily employed for finding the efficient frontier in a multi-objective optimization problem, requiring a lower amount of data compared to other approaches and still preserving the possibility of deriving the corresponding strategies generating the Pareto front.

In order to develops these points we propose two approaches. The first one is based on closed formulas for determining expected value and standard deviation of the risk reserve, including the effect of reinsurance counterparty default risk. The second one proposes a more general extension of the risk reserve equation considering different possible combinations of reinsurance treaties and number/rating characteristics of reinsurance companies. For both these approaches numerical results, based on Italian market data, are presented in order to show the potential practical application for a non-life insurance company. In particular, the results of Chapter 3 and 4 show that the rating of the reinsurance counterparty could be a relevant driver for the decision of the reinsurance strategy by an insurance company. The relevance of this driver increases the more the reinsurance companies offer a different “discount” based on their rating. From these findings we can conclude that the rating of a counterparty and in general counterparty default risk should not be neglected when considering the selection of optimal reinsurance strategies. Finally, the neural network model, developed in Chapter 5, shows the possibility of employing these approaches in the context of optimal selection also in the actuarial sector.

The models presented in this thesis are clearly not the definitive conclusion of the problems we are interested in answering, but they offer new point of view and potential improvements. In particular, regarding the closed formula of Chapter 3 a future challenge is to derive a general formulation which permits to consider multiple lines of business and multiple reinsurance companies for different reinsurance treaties. Regarding the stochastic simulation approach of Chapter 4 a future challenge consists in extending the model, without suffering for the computational burden related to the increasing number of combinations. Finally, for the neural network approach developed in Chapter 5 future applications are possible in many different areas of actuarial science, but another interesting area of research consists in employing explainable methods for deriving information from the machine learning model.

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