Journal of Mathematical Economics 107 (2023) 102871

Contents lists available at ScienceDirect

Journal of Mathematical Economics

journal homepage: www.elsevier.com/locate/jmateco

Dynamic bid–ask pricing under Dempster-Shafer uncertainty

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ARTICLE INFO

Article history:
Received 17 October 2022
Received in revised form 21 June 2023
Accepted 26 June 2023
Available online 7 July 2023
Manuscript handled by Editor Ruodu Wang

Keywords:
Time-homogeneous Markov multiplicative binomial process
Belief function
Dynamic pricing rule
Bid–ask spread

ABSTRACT

We deal with the problem of pricing in a multi-period binomial market model, allowing for frictions in the form of bid–ask spreads. We introduce and characterize time-homogeneous Markov multiplicative binomial processes under Dempster-Shafer uncertainty together with the induced conditional Choquet expectation operator. Given a market formed by a frictionless risk-free bond and a non-dividend paying stock with frictions, we prove the existence of an equivalent one-step Choquet martingale belief function. We then propose a dynamic Choquet pricing rule with bid–ask spreads showing that the discounted lower price process of a European derivative contract on the stock is a Choquet super-martingale. We finally provide a normative justification in terms of a dynamic generalized no-arbitrage condition relying on the notion of partially resolving uncertainty due to Jaffray.

1. Introduction

One of the main hypotheses of classical no-arbitrage pricing theory is the absence of frictions in the market, which essentially materializes in the linearity and time-consistency of the dynamic pricing rule. In turn, this translates in a discounted conditional expectation representation of prices that relies on martingale theory (see, e.g., Harrison and Kreps, 1979; Harrison and Pliska, 1981). Despite this simplifying assumption, it is well-known that real markets show frictions, most evidently in the form of bid–ask spreads (see, e.g., Amihud and Mendelson, 1986, 1991). Therefore, several researches faced the problem of modeling frictions in a pricing problem (see, e.g., Bion-Nadal, 2009; Jouini, 2000; Jouini and Kallal, 1995; Roux, 2011).

The majority of the quoted approaches takes probability theory as the natural environment, eventually switching to sets of probability measures for expressing lower/upper prices (Beissner and Riedel, 2019). On the other hand, a different way for dealing with the problem is to abandon the probabilistic setting and refer to the purely non-additive framework of Choquet theory (Choquet, 1954) leading to non-linear pricing rules. Indeed, starting from Chateauneuf et al. (1996), a stream of research explored this path in pricing (Cerreia-Vioglio et al., 2015; Chateauneuf and Cornet, 2022a,b; Cinfrignini et al., 2023; Kast et al., 2014), thanks to its connection with decision theory (see, e.g., Aouani et al., 2021; Chateauneuf, 1991). Most of such proposals focus on the single period case.

By following the ideas developed in Chateauneuf et al. (1996) and Cerreia-Vioglio et al. (2015), we consider Choquet pricing rules relying on the Dempster-Shafer theory of evidence (Dempster, 1967; Shafer, 1976) as basic framework for modeling uncertainty. In this context, as well as in decision theory and artificial intelligence, several proposals are available for introducing conditioning (see Coletti et al., 2016; Denneberg, 2002; Eichberger et al., 2007; Gilboa and Schmeidler, 1993; Horie, 2013) and a suitable notion of expectation (see Wang and Klir, 2009). Here, we refer to the *product (or geometric) conditioning rule* proposed in Suppes and Zanotti (1977) and to the Choquet integral (Denneberg, 1994; Grabisch, 2016). The choice of the conditioning rule has a direct impact both on computational aspects and bid–ask pricing intervals (see Remark 1). The product conditioning rule takes the non-additive setting as reference, by relegating the probabilities in the core of a belief function to a marginal role. A different approach would be to refer to the so-called *Bayesian conditioning rule* for belief functions (Fagin and Halpern, 1991), which, however, gives rise to computational difficulties and to the so-called dilation effect on bid–ask price intervals.

We define a time-homogeneous Markov multiplicative binomial process (namely, a *DS-multiplicative binomial process*) by fixing the structure of its set of t-step transition belief functions, for which we show the existence of a consistent global belief function. In turn, this allows us to introduce a corresponding conditional Choquet expectation operator. We stress that our notion of DS-multiplicative binomial process differs from other proposals that aim to introduce "imprecision" in a Markov process (see, e.g., Kast et al., 2014; Krak et al., 2019; Nendel, 2021;

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https://doi.org/10.1016/j.jmateco.2023.102871

Škulj, 2016; T'Joens et al., 2021). To the best of our knowledge, the notion of DS-multiplicative binomial process is new, even if it is based on concepts already known in the literature, such as the product conditioning rule for belief function. Indeed, other proposals usually refer to more general non-additive uncertainty measures and to different notions of conditioning. Moreover, a distinguishing feature of our approach is that we look for a global belief function that generates all the local transition belief functions via the product conditioning rule. On the other hand, many other proposals only pay attention to local transition models. In turn, this makes our Theorem 1 the main result of this paper since it does not follow from previous results in the literature and guarantees the soundness of all the proposed theory.

We show that the introduced DS-multiplicative binomial process is completely determined by the choice of only two parameters that characterize the one-step transition belief function. In turn, this allows us to provide a closed form expression for the conditional Choquet expectation of any function of a variable of the process. We also show that the conditional Choquet expectation operator generally fails the linearity and tower properties, which are recovered in the particular case the two parameters sum up to 1: in this case we get back to the classical multiplicative binomial process appearing in Cox et al. (1979).

Then, we consider a market formed by a frictionless risk-free bond (whose price is modeled by a deterministic process) and a non-dividend paying stock with frictions (whose lower price is modeled by a DS-multiplicative binomial process). In this market we prove an analog of the classical theorem of change of measure relying on the notion of *equivalent one-step Choquet martingale belief function*. With such a global belief function, the discounted lower price process of the stock turns out to be a one-step Choquet martingale, though it is only a Choquet super-martingale when more than one steps are considered. Also this series of results is, to the best of our knowledge, new.

Next, assuming that the payoff of a European derivative only depends on the lower price of the stock, we propose a dynamic pricing rule that accounts for bid–ask spreads. The derivative lower price process is defined as a one-step discounted conditional Choquet expectation, while the upper price process is defined one-step-wise through duality. For this pricing rule, the lower price process is shown to be always dominated by the upper price process and its discounted version turns out to be a Choquet super-martingale.

Finally, we provide a normative justification of the proposed dynamic lower pricing rule by referring to a dynamic generalized no-arbitrage condition. Such condition, introduced in Cinfrignini et al. (2023) in the single period case, is based on the *partially resolving uncertainty* principle due to Jaffray (1989). Given the history up to time n, by partially resolving uncertainty we mean that at time $n+1$ the market agent may not be able to determine which one between the two mutually exclusive events "up" and "down" for the stock has occurred. Thus, he/she needs to consider the set of all the possible pieces of information he/she may acquire once uncertainty is resolved at time $n+1$ that reduce to "up", "down", and "up or down". In other terms, random quantities are extended to the three pieces of information that an agent may acquire. We assume that the agent adopts a *systematically pessimistic behavior* under partially resolving uncertainty as he/she systematically considers the minimum value of a random quantity X defined on "up" and "down", when the piece of information "up or down" is acquired. We further show that the resulting lower pricing rule satisfies time-consistency in the sense of Cheridito and Stadje (2009).

The paper is structured as follows. Section 2 recalls the necessary preliminaries on Dempster-Shafer theory, Choquet integration and the classical binomial pricing model. Section 3 introduces time-homogeneous Markov multiplicative processes under Dempster-Shafer uncertainty (namely, DS-multiplicative binomial processes) and defines the induced conditional Choquet expectation operator. Section 4 considers a market composed by a frictionless risk-free bond and a non-dividend paying stock with frictions, and proves an analog of the classical theorem of change of measure relying on the notion of equivalent one-step Choquet martingale belief function. Section 5 proposes a dynamic pricing rule with bid–ask spreads according to which the discounted lower price process of a European derivative contract on the stock is a Choquet super-martingale. Section 6 shows that the dynamic lower pricing rule of Section 5, though not consistent with the classical one-step no-arbitrage condition, is consistent with a generalized one-step no-arbitrage condition that relies on the notion of partially resolving uncertainty due to Jaffray. Moreover, the same section deals with the time-consistency of the proposed dynamic lower pricing rule, and its connection with dynamic risk measures. Finally, Appendix gathers the proofs of results presented in the previous sections.

2. Preliminaries

2.1. Non-additive measures and integrals

Let $\Omega = \{\omega_1, \ldots, \omega_d\}$ be a finite non-empty set of states of the world and take $\mathcal{F} = \mathcal{P}(\Omega)$, where the latter denotes the power set of Ω. Let $\mathbb{R}^\Omega$ denote the set of all random variables on Ω and $\mathbf{1}_E$ the *indicator* of event E, for every $E \in \mathcal{F}$. In what follows, for every $a \in \mathbb{R}$, we identify $a\mathbf{1}_\Omega$ with a.

The reference framework is the Dempster-Shafer theory of evidence (see Dempster, 1967; Shafer, 1976) where a *belief function* is a mapping $\nu : \mathcal{F} \to [0, 1]$ satisfying:

(i) $\nu(\emptyset) = 0$ and $\nu(\Omega) = 1$;

(ii) $\nu\left(\bigcup_{i=1}^k E_i\right) \geq \sum\limits_{\emptyset \neq I \subseteq \{1,\ldots,k\}} (-1)^{|I|+1} \nu\left(\bigcap\limits_{i \in I} E_i\right)$, for every $k \geq 2$ and every $E_1, \ldots, E_k \in \mathcal{F}$.

Condition *(ii)* is called *complete monotonicity* and together with *(i)* it implies *monotonicity*: $\nu(A) \leq \nu(B)$ when $A \subseteq B$, with $A, B \in \mathcal{F}$. In other terms, by belief function we mean a *completely monotone normalized capacity* (Choquet, 1954; Grabisch, 2016). The function ν is associated with a *dual* set function $\overline{\nu}$ on $\mathcal{F}$ called *plausibility function* and defined, for every $A \in \mathcal{F}$, as $\overline{\nu}(A) = 1 - \nu(A^c)$.

Notice that, if condition *(ii)* is asked to hold for $k = 2$, then ν is termed 2-*monotone* (also called *supermodular* or *convex*) normalized capacity (Grabisch, 2016). Thus, belief functions are particular 2-monotone normalized capacities.

Both ν and $\overline{\nu}$ are completely characterized by the *Möbius inverse* of ν (see, e.g., Grabisch, 2016; Shafer, 1976) that goes also under the name of *basic probability assignment*. Such function $\mu : \mathcal{F} \to [0, 1]$ satisfies $\mu(\emptyset) = 0$ and $\sum\limits_{B \in \mathcal{F}} \mu(B) = 1$, and, for all $A \in \mathcal{F}$, it holds that

$$\nu(A) = \sum_{B \subseteq A} \mu(B) \quad \text{and} \quad \overline{\nu}(A) = \sum_{B \cap A \neq \emptyset} \mu(B). \tag{1}$$

Every belief function induces a non-empty, closed and convex set of probability measures on $\mathcal{F}$ called *core* (see, e.g., Grabisch, 2016):

$$\mathbf{core}(\nu) = \{P : P \text{ is a probability measure on } \mathcal{F}, P \geq \nu\}. \tag{2}$$

Note that $\nu = \min \mathbf{core}(\nu)$ and $\overline{\nu} = \max \mathbf{core}(\nu)$, where minima and maxima are pointwise on $\mathcal{F}$. Therefore, belief and plausibility functions are particular coherent lower and upper probabilities (Walley, 1991; Williams, 2007). We also recall that probability measures are particular belief functions.

A belief function ν, being a particular coherent lower probability, can be the lower envelope of possibly infinitely many closed and convex sets of probability measures on $\mathcal{F}$ (see, e.g., Walley, 1991), where we refer to the space $[0,1]^{\mathcal{F}}$ endowed with the product topology. The set $\mathbf{core}(\nu)$ turns out to be the largest of such sets and, since ν is (at least) 2-monotone, the extreme points of $\mathbf{core}(\nu)$ can be given a combinatorial characterization (see, e.g., Grabisch, 2016). Moreover, since property *(ii)* is preserved under pointwise limits on $\mathcal{F}$, the set of belief functions on $\mathcal{F}$ is a closed and convex subset of $[0,1]^{\mathcal{F}}$.

In decision theory (see Etner et al., 2012; Gilboa and Marinacci, 2016) the term *ambiguity* refers to situations related to partially determined probability measures, like the celebrated Ellsberg's paradox (Ellsberg, 1961). Our motivation for sticking to Dempster-Shafer theory is that belief functions are sufficiently expressive to address these problems and are close enough to probability theory.

The problem of conditioning for belief functions has been deeply investigated in the literature and several proposals have been considered (see, e.g., Dempster, 1967; Suppes and Zanotti, 1977 and Coletti et al., 2016; Coletti and Vantaggi, 2008 for a deeper discussion). In this work we refer to the *product (or geometric) conditioning rule*: for every $E,H \in \mathcal{F}$ with $\nu(H) > 0$

$$\nu(E|H) = \frac{\nu(E\cap H)}{\nu(H)}. \tag{3}$$

Let us stress that, for every $H \in \mathcal{F}$ with $\nu(H) > 0$, $\nu(\cdot|H)$ is a belief function on $\mathcal{F}$, thus it induces a core as in (2), denoted by $\mathbf{core}(\nu(\cdot|H))$.

The product conditioning rule imposes to focus just on the "evidence" implying H, while that compatible with H^c is not taken into account. As a consequence, a conditional belief function cannot be seen as the lower envelope of a family of conditional probabilities under the product conditioning rule.

Given $\nu(\cdot|H)$ on $\mathcal{F}$, then it uniquely extends to a conditional completely monotone functional defined on $\mathbb{R}^{\Omega}$ through the *Choquet integral* (see, e.g., Denneberg, 1994; Grabisch, 2016) by setting, for all $X \in \mathbb{R}^{\Omega}$,

$$\oint X(\omega)\mathrm{d}\nu(\omega|H) = \sum_{i=1}^{d}\left(X(\omega_{\sigma(i)}) - X(\omega_{\sigma(i+1)})\right)\nu(E_i^{\sigma}|H), \tag{4}$$

where σ is a permutation of Ω such that $X(\omega_{\sigma(1)}) \geq \cdots \geq X(\omega_{\sigma(d)})$, $E_i^{\sigma} = \{\omega_{\sigma(1)},\ldots,\omega_{\sigma(i)}\}$ for $i=1,\ldots,d$, and $X(\omega_{\sigma(d+1)}) = 0$. Notice that, by identifying $\nu(\cdot|\Omega)$ with $\nu(\cdot)$, Eq. (4) covers also the definition of the unconditional Choquet integral $\oint X\mathrm{d}\nu$.

We actually have that (see Proposition 3 in Schmeidler, 1986) the above Choquet integral can be given a lower expectation interpretation locally on H, by referring to $\mathbf{core}(\nu(\cdot|H))$, as it holds that

$$\oint X(\omega)\mathrm{d}\nu(\omega|H) = \min_{P\in\mathbf{core}(\nu(\cdot|H))}\int X(\omega)\mathrm{d}P(\omega), \tag{5}$$

where the integrals in the minimum are of Stieltjes type. In the particular case ν reduces to a probability measure, then the conditional Choquet integral reduces to a conditional Stieltjes integral.

Remark 1. Besides the product conditioning rule for belief functions, two other popular choices are the Dempster's rule (Dempster, 1967) and the Bayesian rule (Fagin and Halpern, 1991): for every $E,H \in \mathcal{F}$ with $\nu(H) > 0$ define

$$\nu_D(E|H) = 1 - \frac{\overline{\nu}(E^c\cap H)}{\overline{\nu}(H)} = \frac{\nu((E\cap H)\cup H^c) - \nu(H^c)}{1-\nu(H^c)},$$

$$\nu_B(E|H) = \min\left\{\frac{P(E\cap H)}{P(H)} \,:\, P\in\mathbf{core}(\nu)\right\} = \frac{\nu(E\cap H)}{\nu(E\cap H)+\overline{\nu}(E^c\cap H)}.$$

As shown in Coletti et al. (2016), we have that, for all $E \in \mathcal{F}$, it holds that

$$\nu_B(E|H) \leq \min\{\nu(E|H), \nu_D(E|H)\},$$

which implies that, for all $X \in \mathbb{R}^{\Omega}$, it holds that

$$\oint X(\omega)\mathrm{d}\nu_B(\omega|H) \leq \min\left\{\oint X(\omega)\mathrm{d}\nu(\omega|H), \oint X(\omega)\mathrm{d}\nu_D(\omega|H)\right\},$$

while no dominance relation generally holds between $\nu(\cdot|H)$ and $\nu_D(\cdot|H)$. An axiomatic decision-theoretic characterization of both the product and the Dempster's conditioning rules has been recently provided for conditional preferences in a generalized Anscombe–Aumann setting (Petturiti and Vantaggi, 2022). We notice that, though $\nu_B(\cdot|H)$ can be interpreted as the lower envelope of conditional probabilities computed with respect to $\mathbf{core}(\nu)$, it generally produces a dilation with respect to both $\nu(\cdot|H)$ and $\nu_D(\cdot|H)$, when computing Choquet integrals. The last aspect is particularly relevant in the bid–ask pricing problem.

The choice of the product conditioning rule is motivated by the bid–ask pricing model we develop in the next sections, as this conditioning rule assures that the core of the updated belief function never reduces to a singleton, provided ν is not additive. That is, ambiguity "never vanishes", which is natural given the interpretation in terms of bid–ask spreads and financial frictions. Note that this feature could be questionable in decision models, where (from a normative point of view) it seems more natural to assume an updating rule for which ambiguity vanishes as information accumulates (Marinacci, 2002; Marinacci and Massari, 2019) (see Section 3 for a deeper discussion). As shown in Petturiti and Vantaggi (2022), the choice of the product conditioning rule may give rise to dynamically inconsistent preferences and this has an impact on the ensuing notion of time-consistency in pricing. We discuss the time-consistency issue in Section 6.2 in more detail.

2.2. The classical binomial pricing model

The classical binomial pricing model (see, e.g., Černý, 2009; Pliska, 1997) builds upon the assumption of a perfect (frictionless and competitive) market under the classical no-arbitrage principle. The market is composed by two assets: a risk-free *bond* and a risky *stock* that does not pay dividends, both considered over a discrete set of times $\{0,\ldots,T\}$, for a finite horizon $T \in \mathbb{N}$.

In particular, the hypothesis of absence of frictions in the market implies that the bid and ask prices of securities in the market always coincide. The price evolution of the bond is expressed by the deterministic process $\{B_0,\ldots,B_T\}$ with $B_0 = 1$, and for $n = 1,\ldots,T$,

$$B_n = (1+r)B_{n-1}, \tag{6}$$

where r is the risk-free interest rate over each period. On the other hand, the price evolution of the stock is expressed by the stochastic process $\{S_0,\ldots,S_T\}$ with $S_0 = s_0 > 0$ and, for $n = 1,\ldots,T$,

$$S_n = \begin{cases} uS_{n-1} & \text{if "up"}, \\ dS_{n-1} & \text{if "down"}, \end{cases} \tag{7}$$

where $u > d > 0$ are the "up" and "down" coefficients. Such a process will be called in what follows a *multiplicative binomial process*. The process above gives rise to a filtered measurable space $(\Omega,\mathcal{F},\{\mathcal{F}_n\}_{n=0}^{T})$, where $\Omega = \{1,\ldots,2^T\}$ and $\mathcal{F}_n$ is the algebra generated by random variables $\{S_0,\ldots,S_n\}$, for $n = 0,\ldots,T$, with $\mathcal{F}_0 = \{\emptyset,\Omega\}$ and $\mathcal{F}_T = \mathcal{F} = \mathcal{P}(\Omega)$.

The trajectories of $\{S_0,\ldots,S_T\}$ can be represented graphically on a binomial tree. In particular, every state $\omega \in \Omega$ is identified with the path corresponding to the T-digit binary expansion of number $\omega - 1$.

As usual, for every $0 \leq n \leq T$, a random variable $X : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}_n$*-measurable* if it is constant on the atoms of the algebra $\mathcal{F}_n$. Notice that, all random variables in $\mathbb{R}^\Omega$ are $\mathcal{F}_T$-measurable.

In the classical binomial pricing model, the returns over each period $\frac{S_1}{S_0}, \ldots, \frac{S_T}{S_{T-1}}$ are assumed to be i.i.d. random variables, with range $\{u, d\}$ and probability distribution $b_u, 1 - b_u$, with $b_u \in (0, 1)$. In turn, the i.i.d. hypothesis on the returns gives rise to a unique *real-world probability measure* P on $\mathcal{F}$, strictly positive on $\mathcal{F} \setminus \{\emptyset\}$, which is completely determined by the parameter b_u. Therefore, both the bond and stock price processes are assumed to be defined on the real-world filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, P)$. Under such P, the process $\{S_0, \ldots, S_T\}$ satisfies both the classical Markov and time-homogeneity properties (see, e.g., Černý, 2009; Pliska, 1997).

A *trading strategy* is a bivariate stochastic process $\{\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_{T-1}\}$ with $\boldsymbol{\lambda}_n = (\lambda_n^0, \lambda_n^1)$, where λ_n^0 and λ_n^1 are $\mathcal{F}_n$-measurable random variables expressing, respectively, the number of units of bond and stock to buy (if positive) or short-sell (if negative) at time n up to time $n + 1$. The trading strategy is *self-financing* if, for $n = 1, \ldots, T - 1$, it satisfies

$$\lambda_{n-1}^0 B_n + \lambda_{n-1}^1 S_n = \lambda_n^0 B_n + \lambda_n^1 S_n. \tag{8}$$

Given a self-financing trading strategy $\{\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_{T-1}\}$, we can introduce the corresponding *price process* $\{\Pi_0^{\boldsymbol{\lambda}}, \ldots, \Pi_T^{\boldsymbol{\lambda}}\}$ by setting

$$\Pi_n^{\boldsymbol{\lambda}} = \lambda_n^0 B_n + \lambda_n^1 S_n, \quad \text{for } n = 0, \ldots, T - 1, \tag{9}$$

$$\Pi_T^{\boldsymbol{\lambda}} = \lambda_{T-1}^0 B_T + \lambda_{T-1}^1 S_T. \tag{10}$$

A self-financing strategy $\{\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_{T-1}\}$ is an *arbitrage opportunity* in the classical sense (see, e.g., Černý, 2009; Munk, 2013) if the corresponding price process $\{\Pi_0^{\boldsymbol{\lambda}}, \ldots, \Pi_T^{\boldsymbol{\lambda}}\}$ satisfies one of the following two conditions, where comparisons are intended over Ω:

(a) $\Pi_0^{\boldsymbol{\lambda}} < 0$ and $\Pi_T^{\boldsymbol{\lambda}} = 0$;
(b) $\Pi_0^{\boldsymbol{\lambda}} \leq 0$ and $\Pi_T^{\boldsymbol{\lambda}} \geq 0$ with $\Pi_T^{\boldsymbol{\lambda}} \neq 0$.

The additive formulation of the market above is shown to be *dynamically complete* (see, e.g., Černý, 2009; Pliska, 1997), in the sense that every derivative whose payoff depends only on the stock price history can be replicated by a self-financing strategy. In particular, every simple European-type derivative with payoff $Y_T = \varphi(S_T)$ can be replicated by a self-financing strategy $\{\boldsymbol{\lambda}_0, \ldots, \boldsymbol{\lambda}_{T-1}\}$ whose value process $\{\Pi_0^{\boldsymbol{\lambda}}, \ldots, \Pi_T^{\boldsymbol{\lambda}}\}$ is such that $Y_T = \Pi_T^{\boldsymbol{\lambda}}$. Thus, setting

$$Y_n = \Pi_n^{\boldsymbol{\lambda}}, \quad \text{for } n = 0, \ldots, T - 1, \tag{11}$$

the value of the derivative can be determined via a replication argument. The resulting process $\{Y_0, \ldots, Y_T\}$ is interpreted as the price evolution of the derivative.

In this classical setting, the condition $u > 1 + r > d > 0$ is necessary and sufficient to the absence of arbitrage opportunities and, together with dynamic completeness, implies the existence of a unique *risk-neutral probability measure* $\widehat{P}$, completely specified by the parameter $\widehat{b}_u = \frac{(1+r)-d}{u-d}$, still assuring i.i.d. returns. The measure $\widehat{P}$ is an "artificial" probability measure that comes from the model and shares with the real-world probability measure P only the strict positivity on $\mathcal{F} \setminus \{\emptyset\}$. We have that, for $n = 0, \ldots, T - 1$

$$Y_n = \frac{1}{1 + r} \widehat{\mathbb{E}}[Y_{n+1} | \mathcal{F}_n], \tag{12}$$

where $\widehat{\mathbb{E}}$ denotes the expectation with respect to $\widehat{P}$, and $\widehat{P}$ is also called *equivalent martingale measure* for this last property.

3. DS-multiplicative binomial processes

The aim of this section is to define a multiplicative binomial process in the Dempster-Shafer framework, i.e., by replacing the probability measure P with a belief function ν. This will be the basis for addressing bid–ask pricing in the next sections.

Consider a discrete-time finite-horizon stochastic process $\{S_0, \ldots, S_T\}$ with $T \in \mathbb{N}$, that is assumed to be a multiplicative binomial process defined on the filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T)$, as in Section 2.2.

From now on, we consider a filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \nu)$ with a fixed belief function ν defined on $\mathcal{F}$.

For $n = 1, \ldots, T$, denote

$$\mathcal{A}_n = \{a_k = u^k d^{n-k} \,:\, k = 0, \ldots, n\}, \tag{13}$$

for which we have $a_0 < a_1 < \cdots < a_n$ and, for $i \leq j$, let

$$[a_i, a_j] = \{a_k \in \mathcal{A}_n \,:\, a_i \leq a_k \leq a_j\}. \tag{14}$$

For every $s > 0$ and $A \in \mathcal{P}(\mathcal{A}_n)$, denote

$$As = \{a_k s \,:\, a_k \in A\}, \tag{15}$$

where $As = \emptyset$ if $A = \emptyset$. In particular, each random variable S_n takes values s_n in $\mathcal{S}_n = \mathcal{A}_n s_0$.

Definition 1. Given a filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \nu)$, the process $\{S_0, \ldots, S_T\}$ is said to satisfy the:

Markov property: if for every $0 \leq n \leq T - 1$ and $1 \leq t \leq T - n$, $A \in \mathcal{P}(\mathcal{A}_t)$, and $s_0 \in \mathcal{S}_0, \ldots, s_n \in \mathcal{S}_n$ on a trajectory with positive belief it holds that

$$\nu(S_{n+t} \in As_n | S_0 = s_0, \ldots, S_n = s_n) = \nu(S_{n+t} \in As_n | S_n = s_n);$$

Time-homogeneity property: if for every $0 \leq n \leq T - 1$ and $1 \leq t \leq T - n$, $A \in \mathcal{P}(\mathcal{A}_t)$, and $s_0 \in \mathcal{S}_0, \ldots, s_n \in \mathcal{S}_n$ on a trajectory with positive belief it holds that

$$\nu(S_{n+t} \in As_n | S_0 = s_0, \ldots, S_n = s_n) = \beta_t(A),$$

where $\beta_t : \mathcal{P}(\mathcal{A}_t) \to [0, 1]$ is a fixed belief function.

If the process satisfies both the properties above is called a **DS-multiplicative binomial process** (where DS reads "Dempster-Shafer").

In the particular case where the process $\{S_0, \ldots, S_T\}$ satisfies the Markov property, then the time-homogeneity property reduces to

$$\nu(S_{n+t} \in As_n | S_n = s_n) = \beta_t(A). \tag{16}$$

The properties above are called *one-step* if they hold only for $t = 1$.

Since the purpose of process $\{S_0, \ldots, S_T\}$ is to model a stock price evolution, the ratio behind the Markov property is to assure *weak market efficiency*. On the other hand, the time-homogeneity property rests upon the family of transition belief functions $\{\beta_t : t = 1, \ldots, T\}$, that are considered as partially specified randomizing devices used to evaluate the t-step evolution of the stock. Every β_t conveys ambiguity in analogy to an Ellsberg's partially specified urn (Ellsberg, 1961). Under this interpretation, it is important to notice that the resulting evolution is not subject to learning as in Marinacci (2002), Marinacci and Massari (2019). Indeed, at each node of the tree, the future stock evolution will be determined with the same set of partially specified randomizing devices, i.e., ambiguity does not fade away as time passes by.

The first issue to face is the existence of a belief function ν on $\mathcal{F}$ that makes the process $\{S_0, \ldots, S_T\}$ Markov and time-homogeneous (i.e., a DS-multiplicative binomial process). Notice that a DS-multiplicative binomial process singles out a family of belief functions $\{\beta_t : t = 1, \ldots, T\}$ defined on the family of power sets $\{\mathcal{P}(\mathcal{A}_t) : t = 1, \ldots, T\}$ that, in turn, are determined by the particular ν that is chosen. Let us point out that, if ν is not additive, then we need the entire family of β_t's since the usual *Chapman–Kolmogorov equations* (see, e.g., Çinlar, 1975) do not apply due to the lack of additivity. Such β_t's are actually *t-step transition* belief functions.

In general, we can have infinitely many belief functions on $\mathcal{F}$ that make the process $\{S_0, \ldots, S_T\}$ a DS-multiplicative binomial process (see, e.g., Cinfrignini et al., 2022 for a related discussion). Actually, some choices of ν on $\mathcal{F}$ could lead to a lack of interpretation for the family $\{\beta_t : t = 1, \ldots, T\}$ induced by ν, and to a large amount of parameters that could make difficult a calibration procedure. This is why, in what follows we restrict to a particular family of t-step transition belief functions that guarantee a clear interpretation and a nice parameterization.

Let b_u, b_d be two strictly positive numbers with $b_u + b_d \leq 1$ corresponding, for every $0 \leq n \leq T-1$, and $s_0 \in \mathcal{S}_0, \ldots, s_n \in \mathcal{S}_n$ on a trajectory with positive belief, to

$$\nu(S_{n+1} = us_n | S_0 = s_0, \ldots, S_n = s_n) = b_u, \tag{17}$$

$$\nu(S_{n+1} = ds_n | S_0 = s_0, \ldots, S_n = s_n) = b_d, \tag{18}$$

that can be interpreted as one-step "up" and "down" conditional beliefs. In case $b_u + b_d = 1$, conditions (17) and (18) determine a unique additive belief function that satisfies time-homogeneity and Markov properties. On the other hand, if $b_u + b_d < 1$ we need to characterize ν by means of the t-step transition belief functions β_t's. For that, for $t = 1, \ldots, T$, we consider the belief function $\beta_t : \mathcal{P}(\mathcal{A}_t) \to [0,1]$ defined, for all $A \in \mathcal{P}(\mathcal{A}_t)$, as

$$\beta_t(A) = \sum_{a_k \in A} \binom{t}{k} b_u^k b_d^{t-k} + \sum_{\substack{[a_k, a_{k+j}] \subseteq A \\ j \geq 1}} \binom{t-j}{k} b_u^k b_d^{t-j-k}(1 - (b_u + b_d)). \tag{19}$$

Notice that (19) is consistent with (17) and (18) as it holds that $\beta_1(\emptyset) = 0$, $\beta_1(\{u\}) = b_u$, $\beta_1(\{d\}) = b_d$, and $\beta_1(\mathcal{A}_1) = 1$. This leads to a clear interpretation where ambiguity that amounts to the excessive weight to unity $(1 - (b_u + b_d))$ is attached to the entire frame of evidence $\mathcal{A}_1 = \{d, u\}$.

Proposition 1. *The function $\beta_t : \mathcal{P}(\mathcal{A}_t) \to [0,1]$ defined as in Eq. (19) is a belief function on $\mathcal{P}(\mathcal{A}_t)$.*

The belief function β_t in (19) generalizes the binomial distribution with parameters t and b_u, to which it reduces in case $b_u + b_d = 1$, since the second summation vanishes. On the other hand, if $b_u + b_d < 1$, then the second summation takes into account a contribution of intervals contained in A which receive a binomial-like weighting deflated by the excessive weight to unity $(1-(b_u + b_d))$. More in detail, we have that intervals of length j contribute by weights mimicking the binomial distribution with parameters $t - j$ and b_u, multiplied by the deflator $(1 - (b_u + b_d))$. Looking at the binomial tree representation of process $\{S_0, \ldots, S_T\}$ we get that, starting from a node s_n at time n and looking ahead of t steps, the interval $[a_k, a_{k+j}]$ of length j represents the set of all trajectories starting at node s_n and continuing for t steps that have a fixed state s_{n+t-j} at time $n + t - j$. Indeed, all the continuations of partial trajectory $s_n, \ldots, s_{n+t-j}$ for the remaining j times will end in a state belonging to $[a_k, a_{k+j}]s_n$. Therefore, interpreting such weights as *evidence* in the spirit of Dempster-Shafer theory (Shafer, 1976), $\beta_t(A)$ is obtained by summing the binomial-like weights of all partial trajectories with decreasing length starting from node s_n, that support the evidence of having a final state of the process after t steps belonging to As_n.

The following theorem states that there exists a strictly positive belief function $\nu : \mathcal{F} \to [0,1]$ meeting all the desiderata.

Theorem 1. *There exists a belief function $\nu : \mathcal{F} \to [0,1]$ such that a multiplicative binomial process on the filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \nu)$ meets the following properties:*

(i) $\nu(B) > 0$, for every $B \in \mathcal{F} \setminus \{\emptyset\}$;
(ii) $\{S_0, \ldots, S_T\}$ is a DS-multiplicative binomial process whose transition belief functions $\{\beta_t : t = 1, \ldots, T\}$ satisfy (19).

The extent of Theorem 1 is deeply connected to the product conditioning rule for belief functions we adopt. To the best of our knowledge, this is the first time that imprecise stochastic processes are introduced in Dempster-Shafer theory by relying on the product conditioning rule. Moreover, still to the best of our knowledge, the notion of DS-multiplicative binomial process endowed with the transition belief functions satisfying (19) is new. Therefore, the proof of Theorem 1, which is reported in Appendix, does not follow from previous results already known in the literature.

Assumption 1. From now on, we assume the belief function ν meeting conditions *(i)–(ii)* of Theorem 1 to be fixed. Therefore, we always refer to transition belief functions $\{\beta_t : t = 1, \ldots, T\}$ satisfying (19).

Every DS-multiplicative binomial process can be associated with an additive binomial process through a logarithmic transformation. In detail, we consider the process $\{R_0, \ldots, R_T\}$ where

$$R_n = \ln \frac{S_n}{S_0}, \quad \text{for } n = 0, \ldots, T. \tag{20}$$

Setting $l_u = \ln u$ and $l_d = \ln d$, we have that $R_0 = 0$ and R_n ranges in the set

$$\mathcal{R}_n = \{r_k = kl_u + (n-k)l_d : k = 0, \ldots, n\}. \tag{21}$$

The process $\{R_0, \ldots, R_T\}$ is still a time-homogeneous Markov process under ν, since it satisfies, for every $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$, and $B \in \mathcal{P}(\mathcal{R}_t)$,

$$\begin{aligned}\nu(R_{n+t} \in B | R_0 = 0, \ldots, R_n = r_n) &= \nu(R_{n+t} \in B | R_n = r_n) \\ &= \beta_t(\exp(B)).\end{aligned} \tag{22}$$

From a financial point of view, if $\{S_0, \ldots, S_T\}$ is used to model the price evolution of a stock, then $\{R_0, \ldots, R_T\}$ is the corresponding log-return process. We also notice that $\{R_0, \ldots, R_T\}$ is an example of *DS-random walk* as introduced in Cinfrignini et al. (2022).

Definition 2. Let $\{S_0, \ldots, S_T\}$ be a DS-multiplicative binomial process on the filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \nu)$. Then, for every random variable $X \in \mathbb{R}^\Omega$, define

$$\mathbb{C}[X|S_n = s_n] = \oint X(\omega)\mathrm{d}\nu(\omega|S_n = s_n),$$
$$\mathbb{C}[X|S_0 = s_0, \ldots, S_n = s_n] = \oint X(\omega)\mathrm{d}\nu(\omega|S_0 = s_0, \ldots, S_n = s_n).$$

In turn, we define the random variables $\mathbb{C}[X|S_n]$ and $\mathbb{C}[X|S_0, \ldots, S_n]$ setting, for all $\omega \in \{S_n = s_n\}$,

$$\mathbb{C}[X|S_n](\omega) := \mathbb{C}[X|S_n = s_n], \tag{23}$$

and, for all $\omega \in \{S_0 = s_0, \ldots, S_n = s_n\}$,

$$\mathbb{C}[X|S_0, \ldots, S_n](\omega) := \mathbb{C}[X|S_0 = s_0, \ldots, S_n = s_n]. \tag{24}$$

We also simply write

$$\mathbb{C}[X|\mathcal{F}_n] := \mathbb{C}[X|S_0, \ldots, S_n], \tag{25}$$

which is easily seen to be $\mathcal{F}_n$-measurable. The operator $\mathbb{C}[\cdot|\mathcal{F}_n]$ will be referred to as *conditional Choquet expectation*, in the following. The properties of the Choquet integral with respect to a belief function (see, e.g., Denneberg, 1994; Grabisch, 2016) imply that $\mathbb{C}[\cdot|\mathcal{F}_n]$ is a positively homogeneous, completely monotone, comonotone additive and superadditive conditional functional on $\mathbb{R}^{\Omega}$, which further satisfies the following property.

Proposition 2. *The conditional Choquet expectation $\mathbb{C}[\cdot|\mathcal{F}_n]$ associated with the filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \nu)$ satisfies:*

(conditional constant) *for all $\mathcal{F}_n$-measurable $X \in \mathbb{R}^{\Omega}$ and all $Y \in \mathbb{R}^{\Omega}$,*

$$\mathbb{C}[X|\mathcal{F}_n] = X,$$

and, if $X \geq 0$,

$$\mathbb{C}[XY|\mathcal{F}_n] = X\mathbb{C}[Y|\mathcal{F}_n].$$

In the particular case $b_u+b_d = 1$, the belief function ν reduces to a probability measure P and $\mathbb{C}[\cdot|\mathcal{F}_n]$ becomes a classical conditional expectation $\mathbb{E}[\cdot|\mathcal{F}_n]$. In this case, the conditional constant property in Proposition 2 holds for all $\mathcal{F}_n$-measurable $X \in \mathbb{R}^{\Omega}$, without any sign restriction. Moreover, as is well-known, $\mathbb{E}[\cdot|\mathcal{F}_n]$ satisfies the *tower property*, that is, for $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$, we have that

$$\mathbb{E}[\mathbb{E}[X|\mathcal{F}_{n+t}]|\mathcal{F}_n] = \mathbb{E}[X|\mathcal{F}_n],$$

for all $X \in \mathbb{R}^{\Omega}$.

On the other hand, if $b_u + b_d < 1$, it is easy to verify that the conditional Choquet expectation $\mathbb{C}[\cdot|\mathcal{F}_n]$ may fail to satisfy the tower property

$$\mathbb{C}[\mathbb{C}[X|\mathcal{F}_{n+t}]|\mathcal{F}_n] = \mathbb{C}[X|\mathcal{F}_n], \tag{26}$$

as the following example shows.

Example 1. Let $T = 2$, $u > d > 0$, $s_0 > 0$, $b_u, b_d > 0$, $b_u+b_d \leq 1$, and consider the DS-multiplicative binomial process $\{S_0, S_1, S_2\}$ and the random variable X on $\Omega = \{1, 2, 3, 4\}$ reported below

Ω	Binary $\omega - 1$	S_0	S_1	S_2	X
4	11	s_0	us_0	u^2s_0	1
3	10	s_0	us_0	uds_0	1
2	01	s_0	ds_0	uds_0	1
1	00	s_0	ds_0	d^2s_0	0

Taking $n = 0$ and $t = 1$, we have that

$$\mathbb{C}[X|\mathcal{F}_0](\omega) = b_ub_d + b_u, \quad \text{for all } \omega \in \Omega,$$
$$\mathbb{C}[X|\mathcal{F}_1](\omega) = \begin{cases} 1, & \text{for all } \omega \in \{3, 4\}, \\ b_u, & \text{for all } \omega \in \{1, 2\}, \end{cases}$$
$$\mathbb{C}[\mathbb{C}[X|\mathcal{F}_1]|\mathcal{F}_0](\omega) = 2b_u - b_u^2, \quad \text{for all } \omega \in \Omega,$$

hence, $\mathbb{C}[X|\mathcal{F}_0] = \mathbb{C}[\mathbb{C}[X|\mathcal{F}_1]|\mathcal{F}_0]$ holds if and only if $b_u + b_d = 1$.

We stress that the failure of the tower property (26) implies some important consequences, like the failure of the usual dynamic programming approach.

If $\varphi(x)$ is a real-valued function of one real variable defined on the range of S_{n+t}, then the following proposition characterizes the conditional Choquet expectation when $X = \varphi(S_{n+t})$.

Proposition 3. *Let $\{S_0, \dots, S_T\}$ be a DS-multiplicative binomial process on the filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \nu)$. Then, for every $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$, and every real-valued function of one real variable $\varphi(x)$ defined on the range of S_{n+t}, we have that*

$$\mathbb{C}[\varphi(S_{n+t})|S_n = s_n] = \sum_{h=0}^{t} \varphi(a_h s_n)\binom{t}{h} b_u^h b_d^{t-h}$$
$$+ \sum_{j=1}^{t}\sum_{h=0}^{t-j}\left[\min_{a_i\in[a_h,a_{h+j}]} \varphi(a_i s_n)\right]\binom{t-j}{h} b_u^h b_d^{t-j-h}(1-(b_u+b_d))$$

and $\mathbb{C}[\varphi(S_{n+t})|S_0 = s_0, \dots, S_n = s_n] = \mathbb{C}[\varphi(S_{n+t})|S_n = s_n]$. In particular, if $\varphi(x)$ is non-decreasing

$$\mathbb{C}[\varphi(S_{n+t})|S_n = s_n] = \sum_{h=0}^{t} \varphi(u^h d^{t-h} s_n)\binom{t}{h} b_u^h b_d^{t-h}$$
$$+ \sum_{h=0}^{t-1} \varphi(u^h d^{t-h} s_n)\sum_{j=1}^{t-h}\binom{t-j}{h} b_u^h b_d^{t-j-h}(1-(b_u+b_d)),$$

while, if $\varphi(x)$ is non-increasing

$$\mathbb{C}[\varphi(S_{n+t})|S_n = s_n] = \sum_{h=0}^{t} \varphi(u^h d^{t-h} s_n)\binom{t}{h} b_u^h b_d^{t-h}$$
$$+ \sum_{h=0}^{t-1} \varphi(u^{t-h} d^{h} s_n)\sum_{j=1}^{t-h}\binom{t-j}{h} b_u^{t-j-h} b_d^{h}(1-(b_u+b_d)).$$

In Denk et al. (2018), non-linear expectations are used to introduce a suitable version of the Chapman–Kolmogorov equations, thus, it is interesting to verify if $\mathbb{C}[\cdot|\mathcal{F}_n]$ can be used to derive an analogous result. In our setting, the Chapman–Kolmogorov equations are vacuous for $T = 1$. For $T \geq 2$, we say that the DS-multiplicative binomial process $\{S_0, \dots, S_T\}$ satisfies the *Chapman–Kolmogorov equations* if, for every $0 \leq n \leq T-2$ and $1 \leq t < w \leq T-n$, it holds that

$$\mathbb{C}[\mathbf{1}_{\{S_{n+w}\in AS_n\}}|\mathcal{F}_n] = \mathbb{C}[\mathbb{C}[\mathbf{1}_{\{S_{n+w}\in AS_n\}}|\mathcal{F}_{n+t}]|\mathcal{F}_n], \tag{27}$$

for all $A \subseteq \mathcal{A}_w$, where $\{S_{n+w} \in AS_n\} := \bigcup_{s_n\in\mathcal{S}_n}\{S_{n+w} \in As_n\}$. Let us notice that, due to the Markov and time-homogeneity properties, Eq. (27) can be rewritten explicitly as

$$\beta_w(A) = \mathbb{C}[\mathbb{C}[\mathbf{1}_{\{S_{n+w}\in AS_n\}}|\mathcal{F}_{n+t}]|\mathcal{F}_n].$$

Except for the case $b_u+b_d = 1$, the following example shows that the Chapman–Kolmogorov equations may fail when $b_u + b_d < 1$, and this happens yet for $w = t + 1$. In turn, the failure of (27) is due to the failure of the tower property (26).

Example 2. Consider the process $\{S_0, S_1, S_2\}$ and the random variable X of Example 1. Taking $n = 0$, $t = 1$, $w = 2$ and $A = \{u^2, ud\}$, noticing that $X = \mathbf{1}_{\{S_2\in AS_0\}}$, we have that $\mathbb{C}[\mathbf{1}_{\{S_2\in AS_0\}}|\mathcal{F}_0] = \mathbb{C}[\mathbb{C}[\mathbf{1}_{\{S_2\in AS_0\}}|\mathcal{F}_1]|\mathcal{F}_0]$ holds if and only if $b_u + b_d = 1$.

4. Equivalent one-step Choquet martingale belief functions

As usual, given the filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \nu)$, a process $\{X_0, \dots, X_T\}$ defined on such space is said to be *adapted* if X_n is $\mathcal{F}_n$-measurable, for $n = 0, \dots, T$.

Definition 3. An adapted process $\{X_0, \dots, X_T\}$ on the filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \nu)$ is said to be a:

one-step Choquet martingale if, for $n = 0, \dots, T-1$, it holds that

$$\mathbb{C}[X_{n+1}|\mathcal{F}_n] = X_n.$$

one-step Choquet super[sub]-martingale if, for $n = 0, \dots, T-1$, it holds that

$$\mathbb{C}[X_{n+1}|\mathcal{F}_n] \leq [\geq] X_n.$$

Choquet martingale if, for every $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$, it holds that

$$\mathbb{C}[X_{n+t}|\mathcal{F}_n] = X_n.$$

Choquet super[sub]-martingale if, for every $0 \leq n \leq T-1$ and $1 \leq t \leq T-n$, it holds that

$$\mathbb{C}[X_{n+t}|\mathcal{F}_n] \leq [\geq] X_n.$$

Using the process $\{B_0, \ldots, B_T\}$ as *numeraire*, we can define the *discounted process* $\{S_0^*, \ldots, S_T^*\}$ setting, for $n = 0, \ldots, T$

$$S_n^* = \frac{S_n}{B_n} = \frac{S_n}{(1+r)^n}, \tag{28}$$

which is trivially seen to be adapted.

The following theorem is the analog in the Dempster-Shafer theory of the classical theorem of change of measure for the probabilistic binomial pricing model (Černý, 2009; Pliska, 1997). In what follows, analogously to probability theory, a belief function $\widehat{\nu} : \mathcal{F} \to [0,1]$ is said to be *equivalent* to the belief function ν if, for all $A \in \mathcal{F}$, $\nu(A) = 0 \iff \widehat{\nu}(A) = 0$. In particular, since the reference belief function ν is strictly positive on $\mathcal{F} \setminus \{\emptyset\}$, an equivalent $\widehat{\nu}$ will satisfy the same property. In what follows $\widehat{\mathbb{C}}[\cdot|\mathcal{F}_n]$ denotes the conditional Choquet expectation with respect to $\widehat{\nu}$.

Theorem 2. *The condition $u > 1 + r > d > 0$ is necessary and sufficient to the existence of a belief function $\widehat{\nu} : \mathcal{F} \to [0,1]$ equivalent to ν such that the discounted process $\{S_0^*, \ldots, S_T^*\}$ on the filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \widehat{\nu})$ satisfies the following properties:*

(a) it is a DS-multiplicative binomial process with transition belief functions $\{\beta_t : t = 1, \ldots, T\}$ satisfying (19) with parameters

$$u^* = \frac{u}{1+r},\quad d^* = \frac{d}{1+r},\quad \widehat{b_u} = \frac{(1+r)-d}{u-d} \text{ and } \widehat{b_d} \in (0, 1-\widehat{b_u}],$$

(b) it is a one-step Choquet martingale, i.e., for $n = 0, \ldots, T-1$ it holds that

$$\widehat{\mathbb{C}}[S_{n+1}^*|\mathcal{F}_n] = S_n^*,$$

(c) it is a Choquet super-martingale, i.e., for every $0 \le n \le T-1$ and $1 \le t \le T-n$, it holds that

$$\widehat{\mathbb{C}}[S_{n+t}^*|\mathcal{F}_n] \le S_n^*.$$

Let us stress that, due to the time-homogeneity and Markov properties of the process $\{S_0^*, \ldots, S_T^*\}$ and the fact that $\{S_n^* = s_n^*\} = \{S_n = s_n\}$, properties *(b)* and *(c)* of Theorem 2 reduce to

(b') for $n = 0, \ldots, T-1$ it holds that

$$\widehat{\mathbb{C}}[S_{n+1}^*|S_n^*] = S_n^*,$$

(c') for every $0 \le n \le T-1$ and $1 \le t \le T-n$, it holds that

$$\widehat{\mathbb{C}}[S_{n+t}^*|S_n^*] \le S_n^*.$$

We also have that the original process $\{S_0, \ldots, S_T\}$ continues to be a DS-multiplicative binomial process, seen in the new filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \widehat{\nu})$.

Following the usual terminology of mathematical finance (see, e.g., Černý, 2009; Pliska, 1997), the belief function $\widehat{\nu}$ singled out by the choice of $\widehat{b_u}$ and $\widehat{b_d}$ as in Theorem 2, will be called an *equivalent one-step Choquet martingale belief function* or, simply, *risk-neutral belief function*. By contrast, the original belief function ν will be called *real-world belief function*. We stress that there are actually infinitely many risk-neutral belief functions, depending on the choice of $\widehat{b_d} \in (0, 1-\widehat{b_u}]$. The adjective risk-neutral for such a belief function $\widehat{\nu}$ is justified by the fact that the Choquet expectation at time n of the return of the stock over the period $[n, n+1]$ coincides with the risk-free return $1 + r$, that is

$$\widehat{\mathbb{C}}\left[\frac{S_{n+1}}{S_n}|\mathcal{F}_n\right] = 1 + r. \tag{29}$$

The following corollary is an immediate consequence of the proof of Theorem 2.

Corollary 1. *If $T > 1$ and $u > 1 + r > d > 0$, then the discounted process $\{S_0^*, \ldots, S_T^*\}$ satisfying the properties (a)–(c) of Theorem 2 further satisfies the property:*

(d) it is a Choquet martingale, i.e., for every $0 \le n \le T-1$ and $1 \le t \le T-n$, it holds that

$$\widehat{\mathbb{C}}[S_{n+t}^*|\mathcal{F}_n] = S_n^*,$$

if and only if $\widehat{b_d} = 1 - \widehat{b_u}$, that is $\widehat{\nu}$ is a probability measure.

It is easily seen that our model subsumes the classical binomial pricing model reported in Section 2.2, which can be recovered when $\widehat{b_d} = 1 - \widehat{b_u}$. Giving up on additivity, i.e., for $\widehat{b_d} \in (0, 1-\widehat{b_u})$, we can model bid–ask spreads, but the price we pay is the loss of the tower property (26) that further implies that composing t times a one-step model we do not get the same results of a t-step model.

We point out that we do not require the one-step Choquet martingale property under the real-world belief function ν, but we can always obtain a representation of $\{S_0, \ldots, S_T\}$ as a one-step Choquet martingale, if we switch to a risk-neutral belief function $\widehat{\nu}$. This last fact will be justified through a suitable dynamic no-arbitrage condition in Section 6.1.

5. A dynamic pricing rule with bid–ask spreads

Consider the market introduced in Section 4, described by the processes

$$\{B_0, \ldots, B_T\} \quad \text{and} \quad \{S_0, \ldots, S_T\},$$

defined on the real-world filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \nu)$. We face the problem of finding the lower price of a simple European-type *derivative contract* with maturity T, whose underlying asset is the stock. Such a contract has payoff at the maturity T given by

$$Y_T = \varphi(S_T), \tag{30}$$

where φ is a suitable *contract function* defined on the range of S_T. Let $\widehat{b_u}$ and $\widehat{b_d}$ as in Theorem 2, determining the risk-neutral belief function $\widehat{\nu}$ and the corresponding risk-neutral filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \widehat{\nu})$.

We define a lower price process for the derivative contract by setting, for $n = 0, \ldots, T-1$,

$$Y_n = \frac{1}{1+r}\widehat{\mathbb{C}}[Y_{n+1}|\mathcal{F}_n], \tag{31}$$

where $\widehat{\mathbb{C}}[\cdot|\mathcal{F}_n]$ denotes the conditional Choquet expectation with respect to $\widehat{\nu}$. We actually have that, since $Y_T = \varphi(S_T)$, then $Y_n = \varphi_n(S_n)$ where φ_n is a function on the range of S_n, for $n = 0, \ldots, T-1$, and $\varphi_T = \varphi$, that is all random variables Y_n's turn out to be functions of the corresponding random variables S_n's. In particular, by the time-homogeneity and Markov properties of the process $\{S_0, \ldots, S_T\}$ under the risk-neutral belief function $\widehat{\nu}$, we get that

$$Y_n = \frac{1}{1+r}\widehat{\mathbb{C}}[Y_{n+1}|S_n]. \tag{32}$$

The above construction defines a process $\{Y_0, \ldots, Y_T\}$, still adapted to the risk-neutral filtered belief space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \widehat{\nu})$.

Using again the process $\{B_0, \ldots, B_T\}$ as numeraire, we can define the discounted process $\{Y_0^*, \ldots, Y_T^*\}$ setting, for $n = 0, \ldots, T$

$$Y_n^* = \frac{Y_n}{B_n} = \frac{Y_n}{(1+r)^n}, \tag{33}$$

which is trivially seen to be adapted.

Theorem 3. *The discounted process* $\{Y_0^*, \dots, Y_T^*\}$ *on the risk-neutral filtered belief space* $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \widehat{\nu})$ *satisfies the properties:*

(a) *it is a one-step Choquet martingale, i.e., for* $n = 0, \dots, T-1$*, it holds that*

$$\widehat{\mathbb{C}}[Y_{n+1}^*|\mathcal{F}_n] = Y_n^*,$$

(b) *it is a Choquet super-martingale, i.e., for every* $0 \le n \le T-1$ *and* $1 \le t \le T-n$*, it holds that*

$$\widehat{\mathbb{C}}[Y_{n+t}^*|\mathcal{F}_n] \le Y_n^*,$$

(c) *it is a Choquet martingale, i.e., for every* $0 \le n \le T-1$ *and* $1 \le t \le T-n$*, it holds that*

$$\widehat{\mathbb{C}}[Y_{n+t}^*|\mathcal{F}_n] = Y_n^*,$$

when $\widehat{b}_d = 1 - \widehat{b}_u$.

We stress that the condition $\widehat{b}_d = 1 - \widehat{b}_u$ is sufficient for the discounted process $\{Y_0^*, \dots, Y_T^*\}$ to be a Choquet martingale but it is not necessary. To see this, it is enough to take a constant contract function φ defined on the range of S_T, for which $\{Y_0^*, \dots, Y_T^*\}$ is a Choquet martingale, independently of $\widehat{b}_d \in (0, 1-\widehat{b}_u]$.

From a financial point of view, the undiscounted process $\{Y_0, \dots, Y_T\}$ can be interpreted as the lower price evolution of the derivative with payoff $Y_T = \varphi(S_T)$. In turn, such process can be associated with an upper price process $\{\overline{Y}_0, \dots, \overline{Y}_T\}$ under the assumption that $\overline{Y}_T = Y_T = \varphi(S_T)$, by setting for $n = 0, \dots, T-1$,

$$\overline{Y}_n = -\frac{1}{1+r}\widehat{\mathbb{C}}[-\overline{Y}_{n+1}|\mathcal{F}_n]. \tag{34}$$

The pair of processes $\{Y_0, \dots, Y_T\}$ and $\{\overline{Y}_0, \dots, \overline{Y}_T\}$ can thus be used to model the time evolution of bid–ask spreads in a market with frictions. We point out that, in the single-period case, working with lower prices and upper prices is completely equivalent. Nevertheless, in our setting, due to the product conditioning rule for belief functions, the main focus is on lower prices, since we pay attention to ν only, and the upper prices are generated through duality. A different approach would be to consider the dual plausibility function $\overline{\nu}$ and the Dempster's rule of conditioning, but this would lead to a different model (see Remark 1).

Since our model subsumes the classical binomial pricing model reported in Section 2.2, the choice $\widehat{b}_d = 1 - \widehat{b}_u$, which expresses absence of frictions in the market, turns out to be consistent. In this particular case lower and upper price processes coincide, and we can simply speak of a price process.

Proposition 4. *The following statements hold:*

(i) $Y_n \le \overline{Y}_n$, *for* $n = 0, \dots, T$;
(ii) *if* φ *is non-decreasing then the lower price process* $\{Y_0, \dots, Y_T\}$ *does not depend on the choice of* $\widehat{b}_d \in (0, 1-\widehat{b}_u]$;
(iii) *if* φ *is non-increasing then the upper price process* $\{\overline{Y}_0, \dots, \overline{Y}_T\}$ *does not depend on the choice of* $\widehat{b}_d \in (0, 1-\widehat{b}_u]$.

The following toy example shows the explicit computation of bid–ask price processes for a European put option.

Example 3. Let $T = 3$, $r = 0.04$, S_0 = €100, $u = 1.2$, $d = 0.8$, and consider a European put option on the stock with maturity T and strike price K = €100, whose final payoff is

$$P_T = \max\{K - S_T, 0\}.$$

In this case we have $\widehat{b}_u = 0.6$ and $\widehat{b}_d \in (0, 0.4]$. Fig. 1 shows the binomial tree representation of lower and upper price processes $\{P_0, P_1, P_2, P_3\}$ and $\{\overline{P}_0, \overline{P}_1, \overline{P}_2, \overline{P}_3\}$ for $\widehat{b}_d = 0.4 \cdot 0.999$.

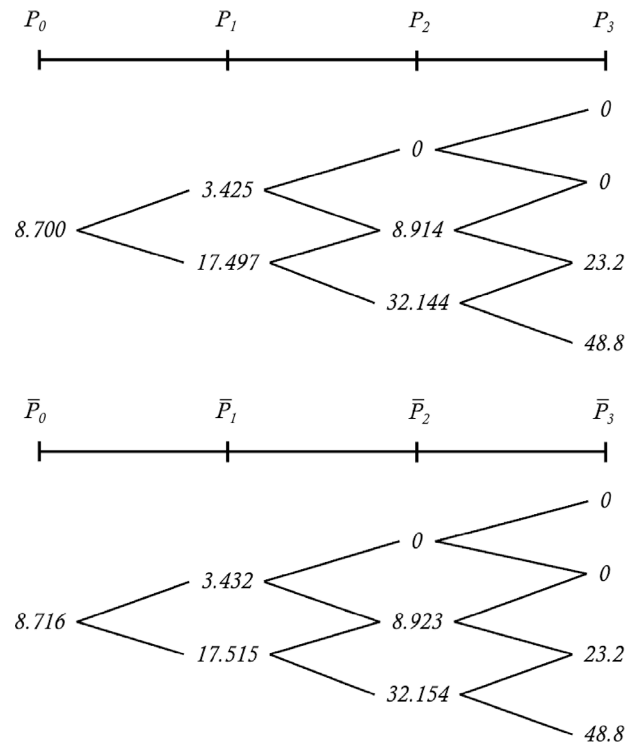

Fig. 1. Binomial tree representation of lower and upper price processes $\{P_0, P_1, P_2, P_3\}$ and $\{\overline{P}_0, \overline{P}_1, \overline{P}_2, \overline{P}_3\}$ for $\widehat{b}_d = 0.4 \cdot 0.999$.

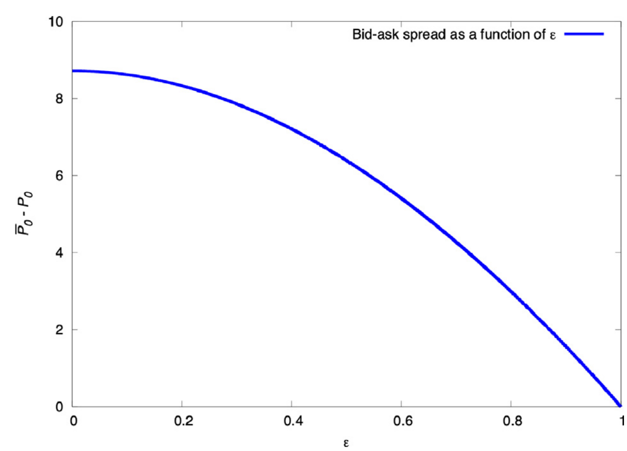

Fig. 2. Bid–ask spread $\overline{P}_0 - P_0$ as a function of $\epsilon \in (0, 1]$.

Setting $\widehat{b}_d = 0.4\epsilon$, we have that

$$P_0 = \frac{11.136\epsilon^2 - 1.3312\epsilon^3}{(1.04)^3} \quad \text{and}$$

$$\overline{P}_0 = \frac{3 \cdot 0.6 \cdot 0.4^2 \cdot 23.2 + 0.4^3 \cdot 48.8}{(1.04)^3},$$

where $\overline{P}_0$ does not depend on ϵ by Proposition 4. Fig. 2 shows the graph of the bid–ask spread $\overline{P}_0 - P_0$ as a function of $\epsilon \in (0, 1]$.

We point out that another possibility for defining a lower price process is to set $\underline{Y}_T = \varphi(S_T)$ and, for $n = 0, \dots, T-1$, define

$$\underline{Y}_n = \frac{1}{(1+r)^{T-n}}\widehat{\mathbb{C}}[\underline{Y}_T|\mathcal{F}_n]. \tag{35}$$

The resulting lower price process $\{\underline{Y}_0, \dots, \underline{Y}_T\}$ coincides with $\{Y_0, \dots, Y_T\}$ if $\widehat{b}_d = 1 - \widehat{b}_u$, while in general we have $\underline{Y}_n \le Y_n$, by virtue of Theorem 2. The fact that $\{\underline{Y}_0,\dots,\underline{Y}_T\}$ gives rise to a greater dilation in lower prices makes us favor the one-step approach given by (31). On the other hand, the lower price process defined through (35) assures that a dynamic version of the *put–call parity relation* introduced in Cerreia-Vioglio et al. (2015) (see also Bastianello et al., 2022) is satisfied. Indeed, denoting by $C_T = \max\{S_T - K, 0\}$ and $P_T = \max\{K - S_T, 0\}$ the payoffs of European call and put options on S_T with strike price K, the decomposition

$$C_T - P_T = S_T - K \tag{36}$$

and the comonotonic additivity of $\widehat{\mathbb{C}}[\cdot|\mathcal{F}_n]$ imply that

$$\widehat{\mathbb{C}}[C_T|\mathcal{F}_n] + \widehat{\mathbb{C}}[-P_T|\mathcal{F}_n] = \widehat{\mathbb{C}}[S_T|\mathcal{F}_n] - K \tag{37}$$

which, after discounting, reduces to

$$\underline{C}_n + \frac{\widehat{\mathbb{C}}[-P_T|\mathcal{F}_n]}{(1+r)^{T-n}} = \underline{S}_n - \frac{K}{(1+r)^{T-n}}, \tag{38}$$

where $\underline{C}_n, \underline{S}_n$ refer to (35). Let us stress that, since the lower stock price process under $\widehat{\nu}$ is only a Choquet super-martingale, we actually have that $\underline{S}_n \leq S_n$. However, under ambiguity, different forms of put–call parity relations arise: for instance, the form introduced in Chateauneuf et al. (1996) is generally not satisfied in our framework (see also Bastianello et al., 2022), as it holds only when $\widehat{\nu}$ is additive, i.e., in absence of frictions.

6. Discussion

6.1. A dynamic no-arbitrage condition under partially resolving uncertainty

The construction carried out in the previous section subsumes the classical linear formulation, obtained when we restrict to work with additive belief functions. In this case, we get back to probability theory where the conditional Choquet expectation operator defined in (24) reduces to the classical conditional expectation operator, which is linear and satisfies the tower property.

The classical construction recalled in Section 2.2 is intrinsically based on the additivity of ν and $\widehat{\nu}$. Indeed, in case of additive belief functions, the one-step Markov and time-homogeneity properties imply the general Markov and time-homogeneity properties. The same also holds for the one-step martingale and general martingale properties. From a normative point of view, additivity can be justified in order to ensure price linearity which is a materialization of absence of frictions in the market.

On the other hand, real markets are quite far from being perfect as they can show frictions, mainly in the form of bid–ask spreads (Amihud and Mendelson, 1986, 1991). If we allow frictions in the market, i.e., we give up on the additivity of ν and $\widehat{\nu}$, the above construction necessarily breaks down since $\widehat{\mathbb{C}}[\cdot|\mathcal{F}_n]$ is not linear and does not satisfy the tower property. In financial terms, the lack of linearity of $\widehat{\mathbb{C}}[\cdot|\mathcal{F}_n]$ translates in the lack of duality between the direct definition of the lower price process $\{Y_0,\dots,Y_n\}$ as a discounted Choquet expectation and the replicating portfolio representation. Furthermore, the failure of the tower property implies that working on single periods $[n, n+1]$ is not equivalent to working on larger periods.

Here, we provide a detailed analysis of implications due to the lack of additivity. If we assume ν and $\widehat{\nu}$ are non-additive belief functions, i.e., $b_d \in (0, 1-b_u)$ and $\widehat{b}_d \in (0, 1-\widehat{b}_u)$, then we can still define the lower price process $\{Y_0,\dots,Y_T\}$ of a simple derivative with payoff $Y_T = \varphi(S_T)$ through (31), for which we have that $Y_n = \varphi_n(S_n)$ with $\varphi_n : \mathcal{S}_n \to \mathbb{R}$. In order to have a replicating strategy, in every period $[n, n+1]$, working conditionally on the history of the stock lower price process up to time n, the random vector $\boldsymbol{\lambda}_n = (\lambda_n^0, \lambda_n^1)$ must be chosen by solving the linear system

$$\begin{cases} \lambda_n^0(1+r)B_n + \lambda_n^1 uS_n = \varphi_{n+1}(uS_n), \\ \lambda_n^0(1+r)B_n + \lambda_n^1 dS_n = \varphi_{n+1}(dS_n), \end{cases} \tag{39}$$

which has a unique solution. In turn, the replication constraint can be compactly rewritten as

$$\lambda_n^0 B_{n+1} + \lambda_n^1 S_{n+1} = Y_{n+1}. \tag{40}$$

The lack of linearity of $\widehat{\mathbb{C}}[\cdot|\mathcal{F}_n]$ implies that the resulting trading strategy $\{\boldsymbol{\lambda}_0,\dots,\boldsymbol{\lambda}_T\}$ is generally not self-financing as we may have

$$\begin{aligned} Y_n &= \frac{1}{1+r}\widehat{\mathbb{C}}[Y_{n+1}|\mathcal{F}_n] \\ &= \frac{1}{1+r}\widehat{\mathbb{C}}[\lambda_n^0 B_{n+1} + \lambda_n^1 S_{n+1}|\mathcal{F}_n] \\ &= \frac{1}{1+r}\left(\lambda_n^0(1+r)B_n + \widehat{\mathbb{C}}[\lambda_n^1 S_{n+1}|\mathcal{F}_n]\right) \\ &= \lambda_n^0 B_n + \frac{1}{1+r}\widehat{\mathbb{C}}[\lambda_n^1 S_{n+1}|\mathcal{F}_n] \\ &\neq \lambda_n^0 B_n + \lambda_n^1 S_n = \Pi_n^{\lambda}, \end{aligned}$$

where $\widehat{\mathbb{C}}[\lambda_n^1 S_{n+1}|\mathcal{F}_n] \neq \lambda_n^1\widehat{\mathbb{C}}[S_{n+1}|\mathcal{F}_n]$ unless $\lambda_n^1 \geq 0$. This shows that we generally lose the replicating strategy representation of the lower price process.

On the other hand, by virtue of Theorem 3, the failure of the tower property of $\widehat{\mathbb{C}}[\cdot|\mathcal{F}_n]$ implies that the discounted process $\{Y_0^*,\dots,Y_T^*\}$ is only a one-step Choquet martingale and a Choquet super-martingale, but it is generally not a Choquet martingale. In particular, we only have that

$$Y_0 \geq \frac{1}{(1+r)^T}\widehat{\mathbb{C}}[Y_T|\mathcal{F}_0]. \tag{41}$$

We now investigate further how the choice of $\widehat{b}_d \in (0, 1-\widehat{b}_u)$ can be justified from a normative point of view. Indeed, as already highlighted, the classical no-arbitrage principle is inconsistent with this choice, as the only admissible choice is to set $\widehat{b}_d = 1-\widehat{b}_u$. To see this, we reformulate the no-arbitrage condition restricting to every single period $[n, n+1]$. At this aim, working conditionally on the history $\{S_0 = s_0,\dots,S_n = s_n\}$, we can define the events $U(s_n) = \{S_{n+1} = us_n\}$ and $D(s_n) = \{S_{n+1} = ds_n\}$, which are functions of the value S_n can take, thus we can write $U(S_n)$ and $D(S_n)$ to stress this fact. In turn, the one-period market formed by the bond and the stock over $[n, n+1]$ can be augmented by adding the artificial securities whose payoff at time $n+1$ is

$$A_{n+1}^u = \mathbf{1}_{U(S_n)} \quad \text{and} \quad A_{n+1}^d = \mathbf{1}_{D(S_n)}, \tag{42}$$

that turn out to be *Arrow–Debreu securities* (Černý, 2009). Pricing through (31), the prices at time n of Arrow–Debreu securities are set equal to

$$A_n^u = \frac{1}{1+r}\widehat{\mathbb{C}}[A_{n+1}^u|\mathcal{F}_n] = \frac{\widehat{b}_u}{1+r}, \tag{43}$$

$$A_n^d = \frac{1}{1+r}\widehat{\mathbb{C}}[A_{n+1}^d|\mathcal{F}_n] = \frac{\widehat{b}_d}{1+r}. \tag{44}$$

In the augmented one-period market over $[n, n+1]$, a portfolio is a vector $\boldsymbol{\delta}_n = (\delta_n^0, \delta_n^1, \delta_n^2, \delta_n^3)$, where the δ_n^i's are $\mathcal{F}_n$-measurable random variables expressing, respectively, the number of units of bond, stock and Arrow–Debreu's securities to buy (if positive) or short-sell (if negative) at time n up to time $n+1$. Furthermore, we can define a local price process $\{\pi_n^{\delta}, \pi_{n+1}^{\delta}\}$ associated with $\boldsymbol{\delta}_n$ over $[n, n+1]$ by defining the random variables

$$\pi_n^{\delta} = \delta_n^0 B_n + \delta_n^1 S_n + \delta_n^2 A_n^u + \delta_n^3 A_n^d, \tag{45}$$

$$\pi_{n+1}^{\delta} = \delta_n^0 B_{n+1} + \delta_n^1 S_{n+1} + \delta_n^2 A_{n+1}^u + \delta_n^3 A_{n+1}^d. \tag{46}$$

Given the history $\{S_0 = s_0, \ldots, S_n = s_n\}$, $U(s_n)$ and $D(s_n)$ form a partition of each event $\{S_n = s_n\}$, moreover, the random variables π_n^{δ} and π_{n+1}^{δ} can be simply regarded as functions with domain $\mathcal{W}(s_n) = \{U(s_n), D(s_n)\}$, where π_n^{δ} is actually constant over $\mathcal{W}(s_n)$.

If we are at time n, the tacit assumption of the classical no-arbitrage condition concerning time $n+1$ is to work under *completely resolving uncertainty*. This means that, given the history $\{S_0 = s_0, \ldots, S_n = s_n\}$, at time $n+1$ the market agent will be always able to determine which one between the mutually exclusive events $U(s_n)$ and $D(s_n)$ has occurred. In this setting, we define a *one-step arbitrage opportunity* as a portfolio $\boldsymbol{\delta}_n$ that satisfies one of the following two conditions, where comparisons are intended over $\mathcal{W}(s_n)$ given the history $\{S_0 = s_0, \ldots, S_n = s_n\}$:

(a') $\pi_n^{\delta} < 0$ and $\pi_{n+1}^{\delta} = 0$;
(b') $\pi_n^{\delta} \le 0$ and $\pi_{n+1}^{\delta} \ge 0$ with $\pi_{n+1}^{\delta} \ne 0$.

It is easily shown that the absence of one-step arbitrage opportunities is equivalent to $u > 1 + r > d > 0$, $\widehat{b_u} = \frac{(1+r)-d}{u-d}$ and $\widehat{b_d} = 1 - \widehat{b_u}$. In turn, this is equivalent to the classical no-arbitrage condition and, therefore, to the existence of a unique strictly positive additive risk-neutral belief function $\widehat{\nu}$ that reduces to the already quoted probability measure $\widehat{P}$.

Hence, choosing $\widehat{b_d} \in (0, 1 - \widehat{b_u})$ we can always build a one-step arbitrage opportunity. Therefore, to justify the choice of $\widehat{b_d} \in (0, 1 - \widehat{b_u})$ from a normative point of view, we need to generalized the one-step no-arbitrage condition by working under *partially resolving uncertainty*, as done in Cinfrignini et al. (2023). The concept of partially resolving uncertainty goes back to Jaffray (1989) and in the present context means that, given the history $\{S_0 = s_0, \ldots, S_n = s_n\}$, at time $n+1$ the market agent may not be able to determine which one between the two mutually exclusive events $U(s_n)$ and $D(s_n)$ has occurred. Thus, he/she needs to consider the set of all the possible pieces of information he/she may acquire once uncertainty is resolved at time $n+1$ which form the set $\mathcal{U}(s_n) = \{U(s_n), D(s_n), U(s_n) \cup D(s_n)\}$.

To address partially resolving uncertainty, the local price process needs to be changed to $\{\tilde{\pi}_n^{\delta}, \tilde{\pi}_{n+1}^{\delta}\}$ by defining its components as functions defined over $\mathcal{U}(s_n)$ instead of over $\mathcal{W}(s_n)$, given the history up to time n. To do so, we notice that, given the history up to time n, B_n, S_n, A_n^u, A_n^d as well as $B_{n+1}, S_{n+1}, A_{n+1}^u, A_{n+1}^d$ can be seen as functions with domain $\mathcal{W}(s_n)$. Given a function X defined on $\mathcal{W}(s_n)$, the market agent adopts a *systematically pessimistic behavior* under partially resolving uncertainty if he/she considers in place of X the quantity $[X]^{\mathbf{L}}$ defined on $\mathcal{U}(s_n)$ by setting, for every $E \in \mathcal{U}(s_n)$

$$[X]^{\mathbf{L}}(E) = \min\{X(F) : F \subseteq E, F \in \mathcal{W}(s_n)\}. \tag{47}$$

We finally define

$$\tilde{\pi}_n^{\delta} = \delta_n^0 [B_n]^{\mathbf{L}} + \delta_n^1 [S_n]^{\mathbf{L}} + \delta_n^2 [A_n^u]^{\mathbf{L}} + \delta_n^3 [A_n^d]^{\mathbf{L}}, \tag{48}$$

$$\tilde{\pi}_{n+1}^{\delta} = \delta_n^0 [B_{n+1}]^{\mathbf{L}} + \delta_n^1 [S_{n+1}]^{\mathbf{L}} + \delta_n^2 [A_{n+1}^u]^{\mathbf{L}} + \delta_n^3 [A_{n+1}^d]^{\mathbf{L}}. \tag{49}$$

Also in this case, $\tilde{\pi}_n^{\delta}$ is actually constant over $\mathcal{U}(s_n)$.

In agreement with Cinfrignini et al. (2023), we define a *generalized one-step arbitrage opportunity* a portfolio $\boldsymbol{\delta}_n$ that satisfies one of the following two conditions, where comparisons are intended over $\mathcal{U}(s_n)$ given the history $\{S_0 = s_0, \ldots, S_n = s_n\}$:

(a") $\tilde{\pi}_n^{\delta} < 0$ and $\tilde{\pi}_{n+1}^{\delta} \ge 0$ with $\tilde{\pi}_{n+1}^{\delta} = 0$ over $\mathcal{W}(s_n)$;
(b") $\tilde{\pi}_n^{\delta} \le 0$ and $\tilde{\pi}_{n+1}^{\delta} \ge 0$ with $\tilde{\pi}_{n+1}^{\delta} \ne 0$ over $\mathcal{W}(s_n)$.

As an immediate consequence of Theorem 5 in Cinfrignini et al. (2023), avoiding generalized one-step arbitrage opportunities is equivalent to the existence of a conditional belief function $\widehat{\nu}(\cdot|S_0 = s_0, \ldots, S_n = s_n)$ defined on the ring generated by $\mathcal{W}(s_n)$ such that

$$\widehat{\nu}(S_{n+1} = us_n | S_0 = s_0, \ldots, S_n = s_n) = \widehat{b_u}, \tag{50}$$

$$\widehat{\nu}(S_{n+1} = ds_n | S_0 = s_0, \ldots, S_n = s_n) = \widehat{b_d}, \tag{51}$$

$$\frac{1}{1+r}\widehat{\mathbb{C}}[S_{n+1} | S_0 = s_0, \ldots, S_n = s_n] = s_n. \tag{52}$$

In other terms, the choice of $\widehat{b_d} \in (0, 1 - \widehat{b_u})$ is consistent with the generalized one-step no-arbitrage condition, i.e., it does not produce generalized one-step arbitrage opportunities. It is important to notice that abandoning additivity we lose the self-financing property and, therefore, dynamic completeness. We also stress that in the additive case the one-step no-arbitrage principle alone assures the uniqueness of the global $\widehat{P}$ defined on the whole $\mathcal{F}$. On the other hand, this is not the case for the generalized one-step no-arbitrage principle since we generally have infinitely many non-additive risk-neutral belief functions $\widehat{\nu}$ compatible with the fixed one-step transition belief functions.

6.2. Induced dynamic risk measures and time-consistency

Consider $\{B_0, \ldots, B_T\}$ and $\{S_0, \ldots, S_T\}$, where the latter is a DS-multiplicative binomial process on $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n=0}^T, \nu)$. The real-world belief function ν and the associated conditional Choquet expectation operator $\mathbb{C}[\cdot|\mathcal{F}_n]$ allow us to define a *dynamic risk measure* by setting, for $n = 0, \ldots, T$ and all $X_T \in \mathbb{R}^{\Omega}$,

$$\rho_n(X_T) = -\frac{1}{(1+r)^{T-n}} \mathbb{C}[X_T | \mathcal{F}_n], \tag{53}$$

where X_T is taken as a risky position at time T. In turn, this implies that ρ_n is an $\mathcal{F}_n$-measurable, positively homogeneous, translation invariant, monotone and sub-additive conditional operator, that is, for $n = 0, \ldots, T$:

(i) $\rho_n(X_T)$ is an $\mathcal{F}_n$-measurable random variable, for all $X_T \in \mathbb{R}^{\Omega}$;
(ii) $\rho_n(\lambda X_T) = \lambda \rho_n(X_T)$, for all $X_T, \lambda \in \mathbb{R}^{\Omega}$ where λ is $\mathcal{F}_n$-measurable and $\lambda \ge 0$;
(iii) $\rho_n(X_T + \alpha) = \rho_n(X_T) - \frac{\alpha}{(1+r)^{T-n}}$, for all $X_T, \alpha \in \mathbb{R}^{\Omega}$ where α is $\mathcal{F}_n$-measurable;
(iv) $\rho_n(X_T) \ge \rho_n(Y_T)$, for all $X_T, Y_T \in \mathbb{R}^{\Omega}$ with $X_T \le Y_T$;
(v) $\rho_n(X_T + Y_T) \le \rho_n(X_T) + \rho_n(Y_T)$, for all $X_T, Y_T \in \mathbb{R}^{\Omega}$.

The above properties imply that the family $\{\rho_n\}_{n=0}^T$ is a *coherent* dynamic risk measure according to Riedel (2004), where we restrict to risky positions that can be non-null only at time T. Noticing that $\rho_T(X_T) = -X_T$, for $T = 1$, then ρ_0 reduces to a static coherent risk measure in the sense of Artzner et al. (1999).

Following Epstein and Schneider (2003) in the axiomatization of inter-temporal multiple-priors utility (see also Amarante and Siniscalchi, 2019), the notion of *time (or dynamic) consistency* can be formulated for $\{\rho_n\}_{n=0}^T$. In particular, as in Cheridito and Stadje (2009), we say that $\{\rho_n\}_{n=0}^T$ is *time-consistent* if for $n = 0, \ldots, T-1$ and $X_T, Y_T \in \mathbb{R}^{\Omega}$,

$$\rho_{n+1}(X_T) \ge \rho_{n+1}(Y_T) \text{ implies } \rho_n(X_T) \ge \rho_n(Y_T). \tag{54}$$

The next example shows that the coherent dynamic risk measure $\{\rho_n\}_{n=0}^T$ built through a DS-multiplicative binomial process is generally not time-consistent.

Example 4. Let $T = 2$, $u > d > 0$, $s_0 > 0$, $b_u = b_d = \frac{1}{4}$, $1+r = 1$ and consider the DS-multiplicative binomial process $\{S_0, S_1, S_2\}$ and the random variables X_2, Y_2, Z_2 on $\Omega = \{1, 2, 3, 4\}$ reported below

Ω	Binary $\omega - 1$	S_0	S_1	S_2	X_2	Y_2	Z_2
4	11	s_0	us_0	u^2s_0	1	2	4
3	10	s_0	us_0	uds_0	1	$\frac{1}{6}$	$\frac{1}{2}$
2	01	s_0	ds_0	uds_0	1	2	4
1	00	s_0	ds_0	d^2s_0	1	$\frac{1}{6}$	$\frac{1}{2}$

Taking $n = 0$, we have that

$$\mathbb{C}[X_2|\mathcal{F}_0](\omega) = \mathbb{C}[X_2|\mathcal{F}_1](\omega) = \mathbb{C}[Y_2|\mathcal{F}_1](\omega) = 1, \quad \text{for all } \omega \in \Omega,$$
$$\mathbb{C}[Y_2|\mathcal{F}_0](\omega) = \frac{19}{48}, \quad \text{for all } \omega \in \Omega,$$

therefore, we get $\rho_1(X_2) = \rho_1(Y_2)$ but $\rho_0(X_2) < \rho_0(Y_2)$, so equalities are not preserved. Moreover, since

$$\mathbb{C}[Z_2|\mathcal{F}_1](\omega) = \frac{11}{8}, \quad \text{for all } \omega \in \Omega,$$
$$\mathbb{C}[Z_2|\mathcal{F}_0](\omega) = \frac{15}{16}, \quad \text{for all } \omega \in \Omega,$$

we have that $\rho_1(X_2) > \rho_1(Z_2)$ but $\rho_0(X_2) < \rho_0(Z_2)$, so inequalities can even be inverted.

Actually, the failure of time-consistency is not surprising and is precisely due to the product conditioning rule for belief functions (3). Indeed, this has been already highlighted in Petturiti and Vantaggi (2022) in terms of conditional preferences in a generalized Anscombe–Aumann setting under Dempster-Shafer uncertainty.

The failure of time-consistency is a well-known problem in risk measurement (see, e.g., Cheridito and Stadje, 2009): such property is known to hold when $\{\rho_n\}_{n=0}^T$ can be expressed in terms of a closed and convex set $\mathcal{P}$ of probability measures on $\mathcal{F}$, satisfying a suitable version of the tower property, called *rectangularity* in Epstein and Schneider (2003) or *consistency* in Riedel (2004). In this setting, conditioning is intended element-wise on $\mathcal{P}$ by relying on the classical Bayesian conditioning rule for probabilities.

In our setting we can refer to $\mathcal{P} = \mathbf{core}(\nu)$ but we consider the product conditioning rule (3) that only pays attention to the lower envelope ν and not to the elements of $\mathcal{P}$ (see Remark 1). On one hand, our approach gives rise to an easy parameterization of the family of transition belief functions $\{\beta_t : t = 1, \dots, T\}$ and, therefore, of the conditional Choquet expectation operator. On the other hand, the resulting conditional Choquet expectation operator is generally not the lower envelope of the set of conditional expectation operators obtained from $\mathcal{P}$ as in Epstein and Schneider (2003), Riedel (2004) and, further, it fails the tower property (26). Hence, in our setting, under the definition (54), time-consistency holds when $b_d = 1 - b_u$.

Nevertheless, following Cheridito and Stadje (2009), time-consistency can be recovered in our setting by composing one-period risk measures over time. This gives rise to the dynamic risk measure $\{\rho_n^C\}_{n=0}^T$ defined, for $n = 0, \dots, T-1$ and all $X_T \in \mathbb{R}^\Omega$, as

$$\rho_n^C(X_T) = -\frac{1}{1+r}\mathbb{C}[-\rho_{n+1}^C(X_T)|\ \mathcal{F}_n], \tag{55}$$

where $\rho_T^C(X_T) := -X_T$. In this case, $\{\rho_n^C\}_{n=0}^T$ is easily verified to satisfy the properties *(i)*–*(v)* above and to be time-consistent by construction. Therefore, $\{\rho_n^C\}_{n=0}^T$ is a time-consistent coherent dynamic risk measure.

If we refer to a risk-neutral belief function $\widehat{\nu}$ according to Theorem 3, then the dynamic risk measures $\{\widehat{\rho}_n\}_{n=0}^T$ and $\{\widehat{\rho}_n^C\}_{n=0}^T$, computed with respect to $\widehat{\nu}$ and $\widehat{\mathbb{C}}[\cdot|\mathcal{F}_n]$ as in (53) and (55), give rise to two dynamic lower pricing rules $\{\widehat{\Psi}_n^C\}_{n=0}^T$ and $\{\widehat{\Psi}_n\}_{n=0}^T$ obtained, for all $Y_T \in \mathbb{R}^\Omega$, as

$$\widehat{\Psi}_n^C(Y_T) = -\widehat{\rho}_n^C(Y_T) \quad \text{and} \quad \widehat{\Psi}_n(Y_T) = -\widehat{\rho}_n(Y_T), \tag{56}$$

that actually correspond to Eqs. (31) and (35), respectively. The previous discussion, shows that $\{\widehat{\Psi}_n^C\}_{n=0}^T$ always satisfies the analog of property (54), while $\{\widehat{\Psi}_n\}_{n=0}^T$ does not. In turn, time-consistency is another motivation in favor of the lower pricing rule $\{\widehat{\Psi}_n^C\}_{n=0}^T$ generated through (31).

7. Conclusion

We introduce the novel notion of DS-multiplicative binomial process and use it to build a bid–ask pricing model. All the construction relies on the product rule of conditioning for belief functions (3), that leads to a nice parameterization of the resulting conditional Choquet expectation operator. The proposed pricing model subsumes the classical binomial pricing model, which is extended in a way to allow for frictions in the market. Many properties of the additive case are preserved, while the general failure of the tower property (26) implies the failure of the Chapman–Kolmogorov equations and the usual dynamic programming approach. Nevertheless, the one-step construction allows us to preserve time-consistency.

This pricing model, though simple, can be easily calibrated on market data, due to its significant parameterization. Nevertheless, the research carried out in this paper naturally looks towards more complex models, whose development is reserved to the future. Below we report some of the possible future expansions of the present model:

- Define a more complex market model where more stocks evolve as DS-multiplicative binomial processes. This would require to express dependencies between the processes by referring, for instance, to a suitable notion of correlation for belief functions (Jiang, 2018).
- Define a DS-multiplicative n-nomial process where the stock is allowed to have $n \geq 2$ future developments after one step. This would require to define and characterize a suitable family of "canonical" transition belief functions, in analogy with (19).
- Consider the convergence of a DS-multiplicative binomial process in continuous time. This would require to recur to results on Choquet weak convergence in a way to get a sort of DS-geometric Brownian motion in the limit (Feng and Nguyen, 2007).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgments

The second author was supported by Università degli Studi di Perugia, Italy, Fondo Ricerca di Base 2019, project "Modelli per le decisioni economiche e finanziarie in condizioni di ambiguità ed imprecisione". The third author was supported by "La Sapienza" Università di Roma, Bandi 2020, project "Modelli decisionali in condizioni di incertezza e ambiguità" (RP120172B81E33DD) and by MUR PNRR project "Future Artificial Intelligence Research" (PE0000013-FAIR).

Appendix. Proofs

Proof of Proposition 1. Define $\mu_t : \mathcal{P}(\mathcal{A}_t) \to [0,1]$ setting, for $k = 0, \dots, t$,

$$\mu_t(\{a_k\}) = \binom{t}{k} b_u^k b_d^{t-k}, \tag{A.1}$$

for $j = 1, \dots, t$ and $k = 0, \dots, t-j$,

$$\mu_t([a_k, a_{k+j}]) = \binom{t-j}{k} b_u^k b_d^{t-j-k}(1-(b_u+b_d)), \tag{A.2}$$

and μ_t is 0 otherwise. The function μ_t is easily seen to range in $[0,1]$, moreover we have that it sums up to 1 since

$$\sum_{k=0}^{t} \mu_t(\{a_k\}) + \sum_{j=1}^{t}\sum_{k=0}^{t-j} \mu_t([a_k, a_{k+j}]) = (b_u+b_d)^t + \sum_{j=1}^{t}(b_u+b_d)^{t-j}(1-(b_u+b_d)) = 1.$$

Finally, the claim follows since, for all $A \in \mathcal{P}(\mathcal{A}_t)$, we have that

$$\beta_t(A) = \sum_{a_k \in A} \mu_t(\{a_k\}) + \sum_{\substack{[a_k, a_{k+j}] \subseteq A \\ j \geq 1}} \mu_t([a_k, a_{k+j}]),$$

that is μ_t is the Möbius inverse of the belief function β_t. □

Proof of Theorem 1. Let $\mu : \mathcal{F} \to [0,1]$ be such that:

(a) $\mu(\{S_0 = s_0, \dots, S_T = s_T\}) = b_u^k b_d^{T-k}$, for $s_T = u^k d^{T-k} s_0 \in \mathcal{S}_T$;
(b) $\mu(\{S_0 = s_0, \dots, S_n = s_n\}) = b_u^k b_d^{n-k}(1-(b_u+b_d))$, for $0 < n < T$ and $s_n = u^k d^{n-k} s_0 \in \mathcal{S}_n$;
(c) $\mu(\{S_0 = s_0\}) = \mu(\Omega) = 1-(b_u+b_d)$;
(d) μ is zero otherwise.

We prove statement *(i)*. We have that $\mu(B) \geq 0$, for all $B \in \mathcal{F}$, moreover

- $\sum_{s_T \in \mathcal{S}_T} \mu(\{S_0 = s_0, \dots, S_T = s_T\}) = (b_u+b_d)^T$,
- for all $0 < n < T$, $\sum_{s_n \in \mathcal{S}_n} \mu(\{S_0 = s_0, \dots, S_n = s_n\}) = (b_u+b_d)^n(1-(b_u+b_d))$,
- $\mu(\{S_0 = s_0\}) = \mu(\Omega) = 1-(b_u+b_d)$,

while μ is zero otherwise. Hence, we get that

$$\sum_{B \in \mathcal{F}} \mu(B) = (b_u+b_d)^T + \sum_{0<n<T}(b_u+b_d)^n(1-(b_u+b_d)) + (1-(b_u+b_d)) = 1,$$

that is μ is the Möbius inverse of a belief function. Moreover, since elements of Ω can be identified with the trajectories on the binomial tree, i.e., with events $\{S_0 = s_0, \dots, S_T = s_T\}$, μ is such that $\mu(\{\omega\}) > 0$, for all $\omega \in \Omega$. In turn, this implies that ν is such that $\nu(B) > 0$, for every $B \in \mathcal{F} \setminus \{\emptyset\}$.

We prove statement *(ii)*. For every $0 \leq n \leq T$, we let $s_n = u^k d^{n-k} s_0 \in \mathcal{S}_n$ and prove that

$$\nu(S_0 = s_0, S_1 = s_1, \dots, S_n = u^k d^{n-k} s_0) = b_u^k b_d^{n-k}. \tag{A.3}$$

In order to get the events with strictly positive μ contained in the event $\{S_0 = s_0, S_1 = s_1, \dots, S_n = u^k d^{n-k} s_0\}$, the corresponding partial trajectory on the binomial tree must be completed for the remaining $T-n$ times indexed by l with $l = T-n, T-n-1, \dots, 0$, working backward.

For $l = T-n$ we have to add $i_{T-n} = 0, \dots, T-n$ movements to the state of the random variable S_n. For a fixed i_{T-n}, by summing over all the possible completions of the trajectory, we have that

$$\sum_{s_{n+1},\dots,s_{T-1}} \mu(S_0 = s_0, \dots, S_n = u^k d^{n-k} s_0, \dots, S_T = u^{k+i_{T-n}} d^{n-k+(T-n)-i_{T-n}} s_0) = \binom{T-n}{i_{T-n}} b_u^{k+i_{T-n}} b_d^{n-k+(T-n)-i_{T-n}}.$$

Then, summing over i_{T-n} we have that

$$\sum_{i_{T-n}=0}^{T-n} \binom{T-n}{i_{T-n}} b_u^{k+i_{T-n}} b_d^{n-k+(T-n)-i_{T-n}}. \tag{A.4}$$

For a generic $0 \leq l \leq T-n-1$ we need to add $i_l = 0, \dots, l$ movements to the state of the random variable S_n. For a fixed i_l, by summing over all the possible completions of the trajectory, we have that

$$\sum_{s_{n+1},\dots,s_{n+l-1}} \mu(S_0 = s_0, \dots, S_n = u^k d^{n-k} s_0, \dots, S_{n+l} = u^{k+i_l} d^{n-k+l-i_l} s_0) = \binom{l}{i_l} b_u^{k+i_l} b_d^{n-k+l-i_l}(1-(b_u+b_d)).$$

Then, summing over i_l we have that

$$\sum_{i_l=0}^{l} \binom{l}{i_l} b_u^{k+i_l} b_d^{n-k+l-i_l}(1-(b_u+b_d)). \tag{A.5}$$

Therefore we obtain that

$$\begin{aligned}
\nu(S_0 = s_0, S_1 = s_1, \dots, S_n = u^k d^{n-k} s_0) &= \sum_{i_{T-n}=0}^{T-n} \binom{T-n}{i_{T-n}} b_u^{k+i_{T-n}} b_d^{n-k+(T-n)-i_{T-n}} \\
&\quad + \sum_{l=0}^{T-n-1}\sum_{i_l=0}^{l} \binom{l}{i_l} b_u^{k+i_l} b_d^{n-k+l-i_l}(1-(b_u+b_d)) \\
&= b_u^k b_d^{n-k}\left[(b_u+b_d)^{T-n} + (1-(b_u+b_d))\sum_{l=0}^{T-n-1}(b_u+b_d)^l\right] \\
&= b_u^k b_d^{n-k}.
\end{aligned}$$

Now we prove that

$$\nu(S_n = u^k d^{n-k} s_0) = \binom{n}{k} b_u^k b_d^{n-k}. \tag{A.6}$$

Eq. (A.4) considers the trajectory from time n to time T, having fixed the part before n. Summing over all the possible completions of the trajectory before time n, we get

$$\binom{n}{k}\sum_{i_{T-n}=0}^{T-n} \binom{T-n}{i_{T-n}} b_u^{k+i_{T-n}} b_d^{n-k+(T-n)-i_{T-n}}.$$

Analogously, for a generic $0 \leq l \leq T-n-1$, Eq. (A.5) considers the trajectory from time n to time $n+l$, having fixed the part before n. For a fixed l, summing over all the possible completions of the trajectory before time n, we get

$$\binom{n}{k}\sum_{i_l=0}^{l} \binom{l}{i_l} b_u^{k+i_l} b_d^{n-k+l-i_l}(1-(b_u+b_d)).$$

Hence, we obtain

$$\nu(S_n = u^k d^{n-k} s_0) = \binom{n}{k}\sum_{i_{T-n}=0}^{T-n} \binom{T-n}{i_{T-n}} b_u^{k+i_{T-n}} b_d^{n-k+(T-n)-i_{T-n}}$$

$$+\sum_{l=0}^{T-n-1}\binom{n}{k}\sum_{i_l=0}^{l}\binom{l}{i_l}b_u^{k+i_l}b_d^{n-k+l-i_l}(1-(b_u+b_d))$$
$$=\binom{n}{k}b_u^k b_d^{n-k}.$$

Now let $1 \le t \le T-n$ and $A \subseteq \mathcal{A}_t = \{a_h = u^h d^{t-h} : h = 0,\dots,t\}$. Let μ_t be the Möbius inverse of β_t defined in Proposition 1 through (A.1) and (A.2). We prove that

$$\nu(\{S_{n+t} \in Au^k d^{n-k}s_0\} \cap \{S_0 = s_0,\dots,S_n = u^k d^{n-k}s_0\}) = b_u^k b_d^{n-k}\beta_t(A). \tag{A.7}$$

If $a_h \in A$, summing over the partial trajectories from time $n+1$ to time $n+t-1$, we get that

$$\sum_{s_{n+1},\dots,s_{n+t-1}} \mu(S_0 = s_0,\dots,S_n = u^k d^{n-k}s_0,\dots,S_{n+t} = u^{h+k}d^{n+t-(h+k)}s_0)$$
$$=\binom{t}{h}b_u^{h+k}b_d^{n+t-(h+k)}$$
$$=b_u^k b_d^{n-k}\mu_t(\{a_h\}).$$

If $j \ge 1$ and $[a_h, a_{h+j}] \subseteq A$, summing over the partial trajectories from time $n+1$ to time $n+t-j-1$, we get that

$$\sum_{s_{n+1},\dots,s_{n+t-j-1}} \mu(S_0 = s_0,\dots,S_n = u^k d^{n-k}s_0,\dots,S_{n+t-j} = u^{h+k}d^{n+t-j-(h+k)}s_0)$$
$$=\binom{t-j}{h}b_u^{h+k}b_d^{n+t-j-(h+k)}(1-(b_u+b_d))$$
$$=b_u^k b_d^{n-k}\mu_t([a_h,a_{h+j}]).$$

Hence

$$\nu(\{S_{n+t} \in Au^k d^{n-k}s_0\} \cap \{S_0 = s_0,\dots,S_n = u^k d^{n-k}s_0\})$$
$$=\sum_{a_h\in A}\binom{t}{h}b_u^{h+k}b_d^{n+t-h-k}$$
$$+\sum_{\substack{[a_h,a_{h+j}]\subseteq A\\ j\ge 1}}\binom{t-j}{h}b_u^{h+k}b_d^{n+t-j-h-k}(1-(b_u+b_d))$$
$$=b_u^k b_d^{n-k}\left[\sum_{a_h\in A}\mu_t(\{a_h\}) + \sum_{\substack{[a_h,a_{h+j}]\subseteq A\\ j\ge 1}}\mu_t([a_h,a_{h+j}])\right]$$
$$=b_u^k b_d^{n-k}\beta_t(A).$$

Proceeding in analogy with the derivation of Eq. (A.6) we get that

$$\nu(\{S_{n+t} \in Au^k d^{n-k}s_0\} \cap \{S_n = u^k d^{n-k}s_0\}) = \binom{n}{k}b_u^k b_d^{n-k}\beta_t(A). \tag{A.8}$$

Finally, Markovianity and time-homogeneity follow from Eqs. (A.1), (A.2), (A.6), (A.7) and (A.8) since we obtain

$$\nu(S_{n+t} \in Au^k d^{n-k}s_0 | S_0 = s_0,\dots,S_n = u^k d^{n-k}s_0)$$
$$=\nu(S_{n+t} \in Au^k d^{n-k}s_0 | S_n = u^k d^{n-k}s_0)$$
$$=\beta_t(A). \quad \square$$

Proof of Proposition 3. Conditionally on $\{S_n = s_n\}$, the random variable S_{n+t} takes values in $\mathcal{A}_t s_n$ and has belief distribution given by β_t on $\mathcal{P}(\mathcal{A}_t)$. Let μ_t be the Möbius inverse of β_t defined in Proposition 1 through (A.1) and (A.2). The general expression of $\mathbb{C}[\varphi(S_{n+t})|S_n = s_n]$ easily follows by the properties of the Choquet integral (see, e.g., Denneberg, 1994; Gilboa and Schmeidler, 1994; Grabisch, 2016). We have that

$$\mathbb{C}[\varphi(S_{n+t})|S_n = s_n] = \oint \varphi(S_{n+t}(\omega))\mathrm{d}\nu(\omega|S_n = s_n)$$
$$= \oint_{\mathcal{A}_t}\varphi(as_n)\mathrm{d}\beta_t(a)$$
$$=\sum_{h=0}^{t}\varphi(a_h s_n)\mu_t(\{a_h\})$$
$$+\sum_{j=1}^{t}\sum_{h=0}^{t-j}\left[\min_{a_i\in[a_h,a_{h+j}]}\varphi(a_i s_n)\right]\mu_t([a_h,a_{h+j}]),$$

and the claim follows by (A.1) and (A.2). The special cases of a non-decreasing or non-increasing $\varphi(x)$ are obtained by computing minima and gathering terms. Finally, the equality $\mathbb{C}[\varphi(S_{n+t})|S_0 = s_0,\dots,S_n = s_n] = \mathbb{C}[\varphi(S_{n+t})|S_n = s_n]$ follows by the time-homogeneity and Markov properties of the process. $\square$

Proof of Theorem 2. We prove only sufficiency as necessity is readily verified. Hence, suppose $u > 1+r > d > 0$. Property *(a)* follows immediately, by taking the discounted "up" and "down" coefficients $u^* = \frac{u}{1+r}$ and $d^* = \frac{d}{1+r}$ and taking

$$\widehat{b_u} = \frac{(1+r)-d}{u-d} \quad \text{and} \quad \widehat{b_d} \in (0, 1-\widehat{b_u}].$$

Property *(b)* follows by Proposition 3, noticing that $\{S_0^* = s_0^*,\dots,S_n^* = s_n^*\} = \{S_0 = s_0,\dots,S_n = s_n\}$, since

$$\widehat{\mathbb{C}}[S_{n+1}^*|S_0^* = s_0^*,\dots,S_n^* = s_n^*]$$
$$= d^*s_n^*[\widehat{b_d} + 1 - (\widehat{b_u}+\widehat{b_d})] + u^*s_n^*\widehat{b_u}$$
$$= s_n^*\left[\frac{d(u-(1+r))}{(1+r)(u-d)} + \frac{u((1+r)-d)}{(1+r)(u-d)}\right]$$
$$= s_n^*.$$

We prove property *(c)* by conditioning on $\{S_0^* = s_0^*,\dots,S_n^* = s_n^*\}$. By Proposition 3, we have that

$$\widehat{\mathbb{C}}[S_{n+t}^*|S_0^* = s_0^*,\dots,S_n^* = s_n^*] = \sum_{h=0}^{t}\delta_h {u^*}^h {d^*}^{t-h}s_n^*,$$

where $\delta_0,\dots,\delta_t \ge 0$ and $\sum_{h=0}^{t}\delta_h = 1$, and the δ_h's are defined, for $h = 0,\dots,t$, as

$$\delta_h = \binom{t}{h}\widehat{b_u}^h\widehat{b_d}^{t-h} + \sum_{j=1}^{t-h}\binom{t-j}{h}\widehat{b_u}^h\widehat{b_d}^{t-j-h}(1-(\widehat{b_u}+\widehat{b_d})),$$

in which the second summation is 0 for $h = t$. Moreover, by well-known results on the classical binomial model (see, e.g., Černý, 2009; Pliska, 1997) we have that

$$s_n^* = \sum_{h=0}^{t}\alpha_h {u^*}^h {d^*}^{t-h}s_n^*,$$

where $\alpha_0,\dots,\alpha_t \ge 0$ and $\sum_{h=0}^{t}\alpha_h = 1$, and the α_h's are defined, for $h = 0,\dots,t$, as

$$\alpha_h = \binom{t}{h}\widehat{b_u}^h(1-\widehat{b_u})^{t-h}.$$

If $\widehat{b_d} = 1-\widehat{b_u}$, then $\delta_h = \alpha_h$, for $h = 0,\dots,t$, and so

$$\widehat{\mathbb{C}}[S_{n+t}^*|S_0^* = s_0^*,\dots,S_n^* = s_n^*] = s_n^*.$$

Thus, suppose $\widehat{b_d} \in (0, 1-\widehat{b_u})$. If $t = 1$, then by property *(b)* we still have that $\widehat{\mathbb{C}}[S_{n+1}^*|S_0^* = s_0^*,\dots,S_n^* = s_n^*] = s_n^*$. Therefore, suppose $t > 1$. In this case, after a straightforward algebraic manipulation we have that, for $h = 0, \ldots, t-1$,

$$\delta_h = \widehat{b_u}^h \left\{ \sum_{j=1}^{t-h} \left[\binom{t-j+1}{h} - \binom{t-j}{h} \right] \widehat{b_d}^{t-j-h+1} + 1 - \widehat{b_u} \sum_{j=1}^{t-h} \binom{t-j}{h} \widehat{b_d}^{t-j-h} \right\}$$

and $\delta_t = \widehat{b_u}^t$. From this, since $\widehat{b_d} < 1 - \widehat{b_u}$, we get that

$$\delta_0 = 1 - \widehat{b_u} \sum_{j=1}^{t} \widehat{b_d}^{t-j} > 1 - \widehat{b_u} \sum_{j=1}^{t} (1-\widehat{b_u})^{t-j} = \alpha_0,$$

moreover,

$$\begin{aligned}
\delta_0 + \delta_1 &= 1 - \widehat{b_u} \sum_{j=1}^{t} \widehat{b_d}^{t-j} \\
&\quad + \widehat{b_u} \left\{ \sum_{j=1}^{t-1} \widehat{b_d}^{t-j} + 1 - \widehat{b_u} \sum_{j=1}^{t-1} (t-j) \widehat{b_d}^{t-j-1} \right\} \\
&= 1 - \widehat{b_u}^2 \sum_{j=1}^{t-1} (t-j) \widehat{b_d}^{t-j-1} \\
&> 1 - \widehat{b_u}^2 \sum_{j=1}^{t-1} (t-j)(1-\widehat{b_u})^{t-j-1} = \alpha_0 + \alpha_1.
\end{aligned}$$

More generally, for $k = 0, \ldots, t-2$, we have that

$$\begin{aligned}
\sum_{h=0}^{k} \delta_h &= 1 - \widehat{b_u}^{k+1} \sum_{j=1}^{t-k} \binom{t-j}{k} \widehat{b_d}^{t-j-k} \\
&> 1 - \widehat{b_u}^{k+1} \sum_{j=1}^{t-k} \binom{t-j}{k} (1-\widehat{b_u})^{t-j-k} = \sum_{h=0}^{k} \alpha_h,
\end{aligned}$$

while we get that

$$\sum_{h=0}^{t-1} \delta_h = 1 - \widehat{b_u}^t = \sum_{h=0}^{t-1} \alpha_h,$$

$$\sum_{h=0}^{t} \delta_h = \quad 1 \quad = \sum_{h=0}^{t} \alpha_h.$$

Hence, we have shown that $\delta_0, \ldots, \delta_t$ and $\alpha_0, \ldots, \alpha_t$ are probability distributions on $\mathcal{A}_t s_n$ such that $\alpha_0, \ldots, \alpha_t$ first-order stochastically dominates $\delta_0, \ldots, \delta_t$. In turn, this implies that

$$\widehat{\mathbb{C}}[S^*_{n+t} | S^*_0 = s^*_0, \ldots, S^*_n = s^*_n] < s^*_n,$$

and this concludes the proof. $\square$

Proof of Theorem 3. Property *(a)* is an immediate consequence of (33) and the positive homogeneity property of the conditional Choquet expectation, indeed

$$\begin{aligned}
Y^*_n = \frac{Y_n}{(1+r)^n} &= \frac{1}{(1+r)^n} \frac{1}{1+r} \widehat{\mathbb{C}}[Y_{n+1} | \mathcal{F}_n] \\
&= \widehat{\mathbb{C}}\left[\frac{Y_{n+1}}{(1+r)^{n+1}} | \mathcal{F}_n \right] = \widehat{\mathbb{C}}[Y^*_{n+1} | \mathcal{F}_n].
\end{aligned}$$

We prove property *(b)*. Due to its definition, the discounted process $\{Y^*_0, \ldots, Y^*_T\}$ can be expressed as $Y^*_n = \psi_n(S_n)$ for a suitable $\psi_n : \mathcal{S}_n \to \mathbb{R}$, for $n = 0, \ldots, T$, where $\psi_T(S_T) = \frac{\varphi(S_T)}{(1+r)^T}$. Fix $0 \leq n \leq T-1$, $1 \leq t \leq T-n$, and $s_n \in \mathcal{S}_n$. By Proposition 3, it holds that

$$\begin{aligned}
\widehat{\mathbb{C}}[Y^*_{n+t} | S_n = s_n] &= \sum_{h=0}^{t} \mu_t(\{a_h\}) \psi_{n+t}(a_h s_n) \\
&\quad + \sum_{j=1}^{t} \sum_{h=0}^{t-j} \mu_t([a_h, a_{h+j}]) \min_{a_i \in [a_h, a_{h+j}]} \psi_{n+t}(a_i s_n)
\end{aligned}$$

We also have that, for $j = 0, \ldots, t-1$ and $s_{n+j} \in \mathcal{S}_{n+j}$

$$\begin{aligned}
\psi_{n+j}(s_{n+j}) &= \widehat{b_d} \psi_{n+j+1}(d s_{n+j}) + \widehat{b_u} \psi_{n+j+1}(u s_{n+j}) \\
&\quad + \min\{\psi_{n+j+1}(d s_{n+j}), \psi_{n+j+1}(u s_{n+j})\}(1 - (\widehat{b_u} + \widehat{b_d})).
\end{aligned}$$

Since

$$\min\{\psi_{n+j+1}(d s_{n+j}), \psi_{n+j+1}(u s_{n+j})\} \geq \min_{a_i \in \mathcal{A}_{t-j}} \psi_{n+t}(a_i s_{n+j}),$$

starting from $\psi_n(s_n)$, an iterative substitution and minorization shows that

$$\begin{aligned}
\psi_n(s_n) &\geq \sum_{h=0}^{t} \mu_t(\{a_h\}) \psi_{n+t}(a_h s_n) \\
&\quad + \sum_{j=1}^{t} \sum_{h=0}^{t-j} \mu_t([a_h, a_{h+j}]) \min_{a_i \in [a_h, a_{h+j}]} \psi_{n+t}(a_i s_n) \\
&= \widehat{\mathbb{C}}[Y^*_{n+t} | S_n = s_n],
\end{aligned}$$

thus the claim follows.

Property *(c)* is an immediate consequence of well-known results on the classical binomial model (see, e.g., Černý, 2009; Pliska, 1997). $\square$

Proof of Proposition 4. Statement *(i)* is an immediate consequence of (31) and (34). If φ is non-decreasing, statement *(ii)* follows by (31) and Proposition 3 since, for $n = 0, \ldots, T$, it is easy to show that $Y_n = \varphi_n(S_n)$, where $\varphi_n : \mathcal{S}_n \to \mathbb{R}$ is non-decreasing. If φ is non-increasing, statement *(iii)* follows by (34) and Proposition 3 since, for $n = 0, \ldots, T$, it is easy to show that $\overline{Y}_n = \varphi_n(S_n)$, where $\varphi_n : \mathcal{S}_n \to \mathbb{R}$ is non-increasing. $\square$

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