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Rectifiability of stationary varifolds branching set with multiplicity at most 2

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Abstract

This thesis deals with regularity and rectifiability properties on the branching set of stationary varifolds that can be represented as the graph of a two-valued function. In the first chapter I briefly show the Simon and Wickramasekera's work in which they introduce a frequency function monotonicity formula for two-valued $C^{1,\alpha}$ functions with stationary graph that leads to an estimate of the Hausdorff dimension of the branching set. In the second chapter I build upon Simon and Wickramasekera's work and introduce several relaxed frequency functions in order to get an estimate of the Minkowski's content of the branching set. I then use their result to prove the local $(n-2)$-rectifiablility of the branching set.

Contents

Chapter 1

Introduction

In [1] Almgren introduced for the first time the frequency function and used it to prove that the singular set of Q-valued Dir-minimizing functions on $\mathbb{R}^n$ has Hausdorff dimension at most $n-2$. In particular, Dir-minimizing functions are functions that minimize an alternative version of Dirichlet energy adapted for multivalued functions.

His approach has been later readapted and simplified in [8] in which an intrinsic version of Almgren's theory of Q-valued Dir-minimizing functions is developed using many results on metric space-valued Sobolev functions (developed in more detail in [2, 20, 21]).

For a single-valued harmonic function v the frequency centered in x with radius $r > 0$ is defined in the following way:

$$N\left(v,x,r\right) = \frac{r^{2-n}\int_{B_r(x)} \left|\nabla v\right|^2 \ dy}{r^{1-n}\int_{\partial B_r(x)} \left|v\right|^2 \ d\mathcal{H}^{n-1}} \ .$$

When v is Q-valued its frequency is pretty similiar to the formula mentioned above and differs only in the definition of the gradient.

The key property of $N\left(v,x,r\right)$ is the monotonicity with respect to the radius r which guarantees the existence of the limit $N\left(v,x\right) = \lim_{r\to 0^+} N\left(v,x,r\right)$. Moreover, if $N\left(v,x,r\right)$ is constant in a neighborhood of some r, then v is homogeneous in x and $N\left(v,x,r\right)$ is equal to its homogeneity order for every r.

The frequency function played a fundamental role for the estimate on the dimension of the singular set of mass-minimizing currents given by Almgren (indeed the frequency was conceived for this purpose). However, its use in the case of currents is quite involved and indirect, and passes through the approximation of currents by multi-valued functions ([1, 4, 5, 6, 7, 8]).

At present there is no known way to define a frequency intrinsically for a mass-minimizing current, in other words indipendently from an approximation procedure around selected points. However, a frequency function can be sometimes found when additional regularity conditions are verified that allow to represent the mass-minimizing current as the graph of some well-suited function. For example, Wickramasekera in [24] stated that, under certain a priori regularity conditions, near singular points of a codimension 1 stable mass-minimizing current with flat tangent and multiplicity at most 2 the current is itself the graph of some two-valued $C^{1,\alpha}$

function. This last concept is highly nontrivial, there are indeed very few results about the possibility to represent a minimal current as a graph of a multi-valued function.

Under such hypothesis Simon and Wickramasekera have been able to define in [23] a suitable frequency function directly defined for stationary n-dimensional varifolds depictable as the graph of some two-valued $C^{1,\alpha}$ function. In this way they are able to adapt Almgren's results for Q-valued Dir-minimizing functions and prove that Hausdorff dimension of the branching set is at most $n-2$ as for Q-valued Dir-minimizing functions. This result extends Almgren's one because it deals with stationary varifolds instead of mass-minimizing currents.

In particular, they proved that the symmetric part $v = \pm\frac{1}{2}(u_1 - u_2)$ of a stationary two-valued function u is locally $C^{1,1/2}$ near singular points, therefore v is the solution of an elliptic partial differential equation of second order with the coefficients of the principal part Lipschitz regular. This allows to use the results proved by Garofalo and Lin in [13] for elliptic operators in order to get a quasimonotonicity formula for such frequency (their approach will be expanded on in chapter 1 of this thesis).

The frequency function is useful to get even stronger results: Krummel and Wickramasekera in [17] proved that the singular set of two-valued harmonic functions is locally $(n-2)$-rectifiable while in [15] and in [16] they proved that the singular set of Q-valued harmonic and stationary two-valued function respectively is countably $(n-2)$-rectifiable. For Q-valued harmonic functions on a n-dimensional domain a stronger result was obtained by Camillo De Lellis, Andrea Marchese, Emanuele Spadaro and Daniele Valtorta in [9] in which they prove the set of multiplicity Q points has a (locally) finite $(n-2)$-dimensional Minkowski's content, from which follows the local $(n-2)$ rectifiability of the whole singular set. They've applied the tecniques developed by Aaron Naber and Daniele Valtorta in [18, 19]. This approach has been later adapted also to other settings, e.g. the thin obstacle problems in [10, 11, 12].

In this thesis I've followed the same approach to prove local $(n-2)$-rectifiability of the singular set of two-valued $C^{1,\alpha}$ functions with stationary graph, proceding further the analysis started by Simon & Wickramasekera ([23]) and Krummel & Wickramasekera ([16]). In particular, the starting point is the work [23] on an intrinsic frequency for two-valued stationary graphs, with the aim of exploiting it for the analysis of the structure of the singular set à la Naber & Valtorta.

However, their frequency has an important drawback: it heavily depends on the geometry near the point in which it's centered, making more difficult to estimate its spatial oscillation which is fundamental for Naber and Valtorta's approach.

Indeed, in order to get an intrinsic frequency for a stationary two-valued graph, it's necessary to adapt them to the local geometry of the surface, which changes from point to point. Specifically, it is necessary to take into account the change of coordinates in the analysis. This is done by exploiting an intermediate notion of frequency function, namely the fixed coefficients frequency, and estimate its oscillation in two steps:

1. in terms of the fixed coefficients frequency first, thus adapting the works in [9, 11, 12];
2. comparing the fixed coefficients frequency with the intrinsic one, controlling the various errors in the estimates in terms of the frequency itself.

The outcome of this analysis is the following theorem:

Theorem 1.0.1. *Let $u : \Omega \subseteq \mathbb{R}^n \to \mathcal{A}^2\left(\mathbb{R}^k\right)$ be a two-valued $C^{1,\alpha}$ function with stationary graph, then the branching set $\mathcal{B}_u$ is locally $(n-2)$-rectifiable in $\mathbb{R}^n$. In particular, $\mathcal{B}_u$ has locally finite $\mathcal{H}^{n-2}$ measure.*

This result extends and completes the analysis started in [23]: it provides the structure of the singular set of stationary varifolds under the very restrictive hypothesis of being a two-valued graph.

A natural extension of such analysis is to try to relax some of these assumptions, for example by considering Q-valued functions' graphs that induce stationary varifolds. However, several parts of the present work are restricted to the two-valued case (e.g. the local $C^{1,1/2}$ regularity for the symmetric part, heavily depends on the assumption $Q = 2$ and may be no more valid when $Q \geq 3$) and for such problems only partial results are available at moment.

Chapter 2

Hausdorff dimension of the branching set

In the first section of this chapter I first present some simple results about two-valued functions and stationary varifolds. The following sections are instead devoted to explain Simon and Wickramasekera's results published in [23].

2.1 Preliminaries

Two-valued functions

Let $|\cdot|$ be the standard (Euclideian) norm on $\mathbb{R}^N$ with $N \in \mathbb{N}$. and $\mathbb{M}^{n\times k}$ be the space of all matricies with k rows and n columns, thus if $f : \mathbb{R}^n \to \mathbb{R}^k$ is differentiable in some point x_0 then $\nabla f(x_0) \in \mathbb{M}^{n\times k}$. On $\mathbb{M}^{n\times k}$ I define the following norm

$$|M| = \sup\left\{ M\nu \cdot \eta \,\middle|\, \nu \in \mathbb{S}^{n-1}, \eta \in \mathbb{S}^{k-1} \right\}$$

notice that $n^{-1/2}\sqrt{\sum_{ij} M_{ij}^2} \leq |M| \leq \sqrt{\sum_{i,j} M_{ij}^2}$. Also for every $a, b \in \mathbb{R}^n$ let

$$[a,b] = \{\lambda a + (1-\lambda)b \in \mathbb{R}^n \mid \lambda \in [0,1]\}$$
$$(a,b) = \{\lambda a + (1-\lambda)b \in \mathbb{R}^n \mid \lambda \in (0,1)\}$$

and in a similar way the sets $[a, b)$ and $(a, b]$ are defined.

Let $[\![P]\!]$ be the Dirac measure concentrated at the point $P \in \mathbb{R}^N$, then I indicate with $\mathcal{A}^2\left(\mathbb{R}^N\right)$ the space of measures on $\mathbb{R}^N$ that are the sum of two Dirac measures. On such space I define the following metric

$$\begin{aligned} \mathtt{G}_2(\mathbf{P}, \mathbf{Q}) &= \min_{\sigma \in S_2} \sqrt{\sum_{i=1}^{2} \left|P_i - Q_{\sigma(i)}\right|^2} \\ &= \min\left\{ \sqrt{|P_1 - Q_1|^2 + |P_2 - Q_2|^2}, \sqrt{|P_1 - Q_2|^2 + |P_2 - Q_1|^2} \right\} \end{aligned}$$

where $\mathbf{P} = [\![P_1]\!] + [\![P_2]\!]$, $\mathbf{Q} = [\![Q_1]\!] + [\![Q_2]\!]$ and S_2 is the set of all permutations on $\{1, 2\}$. Let also $\delta(\mathbf{P}) = |P_1 - P_2|$ and $\#(\mathbf{P})$ be equal to the cardinality of $\{P_1, P_2\}$.

Proposition 2.1.1. *The space $\mathcal{A}^2\left(\mathbb{R}^N\right)$ is a complete metric space.*

Proof. Let $\mathbf{P}_i = \llbracket P_i^1 \rrbracket + \llbracket P_i^2 \rrbracket$ be a Cauchy sequence in $\mathcal{A}^2\left(\mathbb{R}^N\right)$ and set $l_i = \delta\left(\mathbf{P}_i\right) = \left|P_i^1 - P_i^2\right|$.

If $l_i \to 0^+$ then all the sequences $\{L_i\}_{i\in\mathbb{N}}$ in $\mathbb{R}^N$ such that $L_i \in \left\{P_i^1, P_i^2\right\}$ for every i are Cauchy sequences, therefore they should converge. But their limits should be both equal to some L and so $\mathbf{P}_i \to 2\llbracket L \rrbracket$.

If instead $l_i \geq \varepsilon > 0$ for every i then I can suppose $\mathtt{G}_2\left(\mathbf{P}_i, \mathbf{P}_j\right) < \varepsilon/3$ for every i, j and assume $\left|P_i^1 - P_1^1\right|, \left|P_i^2 - P_1^2\right| \leq \varepsilon/3$. Thus

$$\mathtt{G}_2\left(\mathbf{P}_i, \mathbf{P}_j\right) = \sqrt{\left|P_i^1 - P_j^1\right|^2 + \left|P_i^2 - P_j^2\right|^2}$$

and P_i^1, P_i^2 are Cauchy sequences converging to some $P^1, P^2 \in M$. For this reason I finally get $\mathbf{P}_i \to \llbracket P^1 \rrbracket + \llbracket P^2 \rrbracket$. □

The metric structure on $\mathcal{A}^2\left(\mathbb{R}^N\right)$ allows me to introduce continuous two-valued functions just as continuous function from some topological space Ω to the metric space $\mathcal{A}^2\left(\mathbb{R}^N\right)$.

Let $\mathfrak{P}$ be a generic class of function with domain Ω and codomain $\mathbb{R}^N$, I define a $\mathfrak{P}$-*decomposition* of a two-valued function $f : \Omega \to \mathcal{A}^2\left(\mathbb{R}^N\right)$, as a choice of two single valued functions $g, h \in \mathfrak{P}$, so that $f(y) = \llbracket g(y) \rrbracket + \llbracket h(y) \rrbracket$ for every $y \in \Omega$. Thanks to this I'm able to talk about continuous, C^1, Lipschitz and Hölder decompositions of two-valued functions.

Proposition 2.1.2. *A two-valued measurable function always admit a measurable decomposition.*

Proof. Let $f : M \to \mathcal{A}^2\left(\mathbb{R}^N\right)$ measurable, define

$$A = \{x \in M \,|\, \#[f(x)] = 1\}$$
$$B = \{x \in M \,|\, \#[f(x)] = 2\}.$$

On A I've $f \equiv 2\llbracket g \rrbracket$ with g measurable and so f restricted on A is measurable too.

Let now $b_1, b_2, \ldots, b_N$ be an orthonormal basis of $\mathbb{R}^N$ and for every $1 \leq i \leq N$ define

$$C_i = \left\{\llbracket P \rrbracket + \llbracket Q \rrbracket \in \mathcal{A}^2\left(\mathbb{R}^N\right) \,\middle|\, |(P - Q) \cdot b_i| > 0\right\}$$
$$G_i : \llbracket P \rrbracket + \llbracket Q \rrbracket \in C_i \to P \text{ if } (P - Q) \cdot b_i > 0$$
$$H_i : \llbracket P \rrbracket + \llbracket Q \rrbracket \in C_i \to Q \text{ if } (P - Q) \cdot b_i > 0$$

notice that C_i is open, both G_i, H_i are continuous and $B = \bigcup_{i=1}^N f^{-1}\left(C_i\right)$. Accordingly, I define a measurable selection g, h of f on B so that $g(x) = G_i[f(x)]$, $h(x) = H_i[f(x)]$ if $x \in f^{-1}(C_i) \setminus \bigcup_{j=0}^{i-1} f^{-1}(C_j)$.

Now I've obtained a measurable decomposition of f on A and on B, since both A and B are measurable subsets of M I immediately get a measurable decomposition on $A \cup B = M$. □

Thanks to this result it's possible to define the space $L^p\left[\Omega, \mathcal{A}^2\left(\mathbb{R}^k\right)\right]$ as the space of all two-valued measurable functions on Ω which have an L^p decomposition.

That kind of approach clearly doesn't work on continuous decompositions, because they must require additional assumptions on the continuous two-valued function. Let $f : x \in \Omega \to [\![f_1(x)]\!] + [\![f_2(x)]\!]\, \mathcal{A}^2\left(\mathbb{R}^N\right)$ be a generic two-valued function defined on some set Ω, I define the following set

$$\begin{aligned}\mathcal{Z}_f &= \{x \in \Omega \,|\, \#[f(x)] = 1\}\\ &= \{x \in \Omega \,|\, f_1(x) = f_2(x)\}\end{aligned}$$

If f is continuous then $\mathcal{Z}_f$ is (relatively) closed in Ω. A simple result on continuous decomposition is contained in the following proposition:

Proposition 2.1.3. *For every $f : \Omega \to \mathcal{A}^2\left(\mathbb{R}^k\right)$ continuous two-valued and for every $x \in \Omega \setminus \mathcal{Z}_f$ there exist $U_x \subseteq \Omega$ open neighborhood of x such that f has a continuous decomposition on U_x.*

Proof. By using the same notation in proof of proposition 2.1.2 I can assume $x \in f^{-1}(C_i)$ for some index i. Then setting $U_x = f^{-1}(C_i)$ open functions $g = G_i \circ f$, $h = H_i \circ f$ lead to a continuous decomposition of f on U_x. □

Lemma 2.1.4. *If f is a continuous two-valued function defined on a simply connected open subset Ω of $\mathbb{R}^n$ and $\Omega \cap \mathcal{Z}_f = \emptyset$ then f has an unique continuous decomposition on all Ω.*

The idea behind the proof is rather simple: on each point of Ω f has a local continuous decomposition in a neighborhood of such point. Fix any point x_0 with $f(x_0) = [\![A]\!] + [\![B]\!]$ and take any continuous curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = x_0$, there exists a finite partition $0 = t_0 < t_1 < t_2 < \cdots < t_{l-1} < t_l = 1$ of $[0, 1]$ such that f has a continuous decomposition on $\gamma\left([t_{i-1}, t_i]\right)$.

Let g_1, h_1 be the local continuous decomposition of f on $\gamma\left([t_0, t_1]\right)$, such that $g_1(x_0) = A$, $h_1(x_0) = B$. I set $A_1 = g_1[\gamma(t_1)]$, $B_1 = h_1[\gamma(t_1)]$ and I can define by induction A_i, B_i in the following way: let g_i, h_i be the continuous decomposition of f on $\gamma\left([t_{i-1}, t_i]\right)$ such that $g_i[\gamma(t_{i-1})] = A_{i-1}$, $h_i[\gamma(t_{i-1})] = B_{i-1}$, then I set $A_i = g_i[\gamma(t_i)]$, $B_i = h_i[\gamma(t_i)]$. Finally I set $A_\gamma = A_l$, $B_\gamma = B_l$.

Now remember I've assumed $\#[f(x)] = 2$ on Ω, this implies the two branches of f never intersect each other, thus the points A_γ, B_γ are well defined and uniquely determined. Also they don't depend on the chosen partition but only on γ.

Let $\tau : [0, 1] \times [0, 1] \to \Omega$ be a continuous homotopy such that $\tau(0, l) = \tau(1, l) = x_0$ for every $0 \leq l \leq 1$ and set $\gamma_l(t) = \tau(t, l)$. Let $\tilde{l}$ and $A_{\gamma_{\tilde{l}}}, B_{\gamma_{\tilde{l}}}$ be the two points determined as before with respect to the curve $\gamma_{\tilde{l}}$, I can find a finite open cover of $\gamma_{\tilde{l}}\left([0, 1]\right)$ such that f has a continuous decomposition on each of these open subsets of the cover. This implies the existence of $\delta > 0$ such that if $\left|l - \tilde{l}\right| < \delta$ then $A_{\gamma_l} = A_{\gamma_{\tilde{l}}}, B_{\gamma_l} = B_{\gamma_{\tilde{l}}}$, thus A_{γ_l} and B_{γ_l} remain unchanged for little perturbations of l.

However, because Ω is simply connected, I've $A_\gamma = A, B_\gamma = B$ for each continue curve γ such that $\gamma(0) = \gamma(1) = x_0$. Therefore, I can define a global continuous decomposition g, h of f, so that

$$g(x) = A_{\gamma_x}$$
$$h(x) = B_{\gamma_x}$$

for every $x \in \Omega$ where γ_x is any continuous curve with $\gamma_x(0) = x_0$ and $\gamma_x(1) = x$. This concludes the proof.

Consider now the following function defined for every $\mathbf{P}, \mathbf{Q} \in \mathcal{A}^2\left(\mathbb{R}^N\right)$

$$\mathfrak{F}(\mathbf{P}, \mathbf{Q}) = \max\left\{\sqrt{|P_1 - Q_1|^2 + |P_2 - Q_2|^2}, \sqrt{|P_1 - Q_2|^2 + |P_2 - Q_1|^2}\right\}.$$

This function is very similar to $\mathtt{G}_2(P, Q)$ with the sole difference that it uses maximum instead of minimum. Clearly it's not a metric but it's still continuous as the following proposition says:

Proposition 2.1.5. *For every* $\mathbf{P}, \mathbf{Q}, \mathbf{R}, \mathbf{S} \in \mathcal{A}^2\left(\mathbb{R}^k\right)$

$$|\mathfrak{F}(\mathbf{P}, \mathbf{Q}) - \mathfrak{F}(\mathbf{R}, \mathbf{S})| \leq \mathtt{G}_2(\mathbf{P}, \mathbf{R}) + \mathtt{G}_2(\mathbf{Q}, \mathbf{S})$$

Proof. For every $\sigma, \eta \in S_2$ I get

$$\sqrt{\sum_i \left|P_i - Q_{\sigma(i)}\right|} \leq \sqrt{\sum_i \left|P_i - R_{\eta(i)}\right|} + \sqrt{\sum_i \left|R_{\eta(i)} - Q_{\sigma(i)}\right|}$$
$$\leq \sqrt{\sum_i \left|P_i - R_{\eta(i)}\right|} + \mathfrak{F}(\mathbf{R}, \mathbf{Q}).$$

By taking the infimum on η and the supremum on σ I get $\mathfrak{F}(\mathbf{P}, \mathbf{Q}) \leq \mathtt{G}_2(\mathbf{P}, \mathbf{R}) + \mathfrak{F}(\mathbf{R}, \mathbf{Q}) \leq \mathtt{G}_2(\mathbf{P}, \mathbf{R}) + \mathtt{G}_2(\mathbf{Q}, \mathbf{S}) + \mathfrak{F}(\mathbf{R}, \mathbf{S})$. ☐

Since $2\delta(\mathbf{P})^2 = \mathfrak{F}(\mathbf{P}, \mathbf{P})^2$ the preceding result implies also

$$|\delta(\mathbf{P}) - \delta(\mathbf{Q})| \leq \sqrt{2}\mathtt{G}_2(\mathbf{P}, \mathbf{Q})$$

Consider now any two-valued function $v : \Omega \subseteq \mathbb{R}^n \to \mathcal{A}^2\left(\mathbb{R}^k\right)$, I say that v is *approximately affine* at $x_0 \in \Omega$ if and only if there exist $M_1, M_2 \in \mathbb{M}^{n \times k}$, so that defined

$$T_{x_0}v(x) = \sum_{i=1}^{2} [\![v_i(x_0) + M_i(x - x_0)]\!]$$

I get

$$\lim_{x \to x_0} \frac{\mathtt{G}_2(v(x), T_{x_0}v(x))}{|x - x_0|} = 0$$

and in that case I write

$$\mathtt{J}^1 v(x_0) = [\![v_1(x_0), M_1]\!] + [\![v_2(x_0), M_2]\!] \in \mathcal{A}^2\left(\mathbb{R}^k \times \mathbb{M}^{n \times k}\right)$$
$$\nabla v(x_0) = [\![M_1]\!] + [\![M_2]\!] \in \mathcal{A}^2\left(\mathbb{M}^{n \times k}\right).$$

Clearly $T_{x_0}v$ is uniquely determined by $\mathtt{J}^1 v(x_0)$, and vice versa, whereas it's not always possible to determine $T_{x_0}v$ from $\nabla v(x_0)$. Furthermore, any two-valued function that admits a differentiable decomposition in x is approximately affine in the same point too.

More generally let $F : \mathcal{A}^2\left(\mathbb{R}^h\right) \to \mathcal{A}^2\left(\mathbb{R}^l\right)$ Lipschitz, I say that F is approximately affine at $P \in \mathcal{A}^2\left(\mathbb{R}^h\right)$ if and only if for every $v : \mathbb{R}^m \to \mathcal{A}^2\left(\mathbb{R}^h\right)$ with $v(0) = P$ and approximately affine at 0 their composition $F \circ v$ is approximately affine too.

Notice that for every function F defined on $\mathbb{R}^m \times \mathbb{R}^m$ so that $F(P_1, P_2) = F(P_2, P_1)$ I can always set $F\left(\llbracket P_1 \rrbracket + \llbracket P_2 \rrbracket\right) = F(P_1, P_2)$ because this definition is not ambiguous. Also, if F is differentiable on $\mathbb{R}^m \times \mathbb{R}^m$ then $\partial_1 F(P_1, P_2) = \partial_2 F(P_2, P_1)$.

Proposition 2.1.6. *A function $F : \mathcal{A}^2\left(\mathbb{R}^h\right) \to \mathcal{A}^2\left(\mathbb{R}^l\right)$ is approximately affine at $Q = \llbracket Q_1 \rrbracket + \llbracket Q_2 \rrbracket \in \mathcal{A}^2\left(\mathbb{R}^h\right)$ if and only function*

$$\tilde{F}(P_1, P_2) = F\left(\llbracket P_1 \rrbracket + \llbracket P_2 \rrbracket\right)$$

is approximately affine at (Q_1, Q_2). In that case I get for every $v : \mathbb{R}^m \to \mathcal{A}^2\left(\mathbb{R}^h\right)$ approximately affine at 0 with $v(0) = Q$

$$\mathtt{J}^1 (F \circ v)(0) = \sum_{i=1,2} \left\llbracket F_i(Q), \sum_{j=1,2} \partial_j \tilde{F}_i(Q) \nabla v_j(0) \right\rrbracket$$

Proof. If F is approximately affine then, since the function $(P_i) \in \left(\mathbb{R}^h\right)^2 \to \sum_i \llbracket P_i \rrbracket$ is approximately affine, $\tilde{F}$ is approximately affine too.

Conversely for every $v : \mathbb{R}^m \to \mathcal{A}^2\left(\mathbb{R}^h\right)$ as in our statement and $y \in \mathbb{R}^m$ sufficiently small I get

$$\begin{aligned}
&\frac{\mathtt{G}_2\left(F\left[v(y)\right], \sum_i \left\llbracket \tilde{F}_i\left[v(0)\right] + \sum_j \partial_j \tilde{F}_i\left[v(0)\right] \nabla v_j(0) y \right\rrbracket\right)}{|y|} \\
&\quad \leq C \frac{\mathtt{G}_2\left(v(y), \sum_j \llbracket v_j(0) + \nabla v_j(0) y \rrbracket\right)}{|y|} \\
&\quad + \frac{\mathtt{G}_2\left(\tilde{F}\left[v_j(0) + \nabla v_j(0) y\right], T_Q \tilde{F}\left[v(0) + \nabla v(0) y\right]\right)}{\mathtt{G}_2\left(\nabla v(0) y, 2\llbracket 0 \rrbracket\right)} \mathtt{G}_2\left(\nabla v(0), 2\llbracket 0 \rrbracket\right) \\
&\quad \to 0
\end{aligned}$$

notice that $T_Q \tilde{F}\left[v(0) + \nabla v(0) y\right] = T_Q \tilde{F}\left[v_1(0) + \nabla v_1(0) y, v_2(0) + \nabla v_2(0) y\right]$ is well defined even when $Q_1 = Q_2$. □

Like the single valued case I have an almost-everywhere differentiability result for Lipschitz two-valued functions:

Theorem 2.1.7 (Rademacher)**.** *Let $f : \Omega \subseteq \mathbb{R}^n \to \mathcal{A}^2\left(\mathbb{R}^k\right)$ Lipschitz, then f is approximately affine for almost every $x \in \Omega$.*

Proof. On $\Omega \setminus \mathcal{Z}_f$ the function f is clearly approximately affine a.e. so I need to prove the result only on $\mathcal{Z}_f$. I have $f = 2 [\![g]\!]$ on $\mathcal{Z}_f$ with g Lipschitz and well-defined on all $\mathbb{R}^n$ due to Lipschitz extension theorem. I'll prove that f is approximately affine on each point $x \in \mathcal{Z}_f$ such that

1. $\mathcal{Z}_f$ has density 1;
2. g is differentiable.

Fixed such x and chose any $y \in \Omega$, set $r = |y - x|$ and for r sufficiently small there exists $y^* \in \mathcal{Z}_f \cap \overline{B_{2r}(x)}$ such that

$$|y - y^*| = \operatorname{dist}\left(y, \mathcal{Z}_f \cap \overline{B_{2r}(x)}\right)$$

and because $x \in \mathcal{Z}_f$ I've $|y - y^*| \leq r$. Let $L > 0$ be such that

$$\begin{aligned} \mathtt{G}_2\left(f(a), f(b)\right) &\leq L\,|a - b| \ \ \forall a, b \in \Omega \\ |g(a) - g(b)| &\leq L\,|a - b| \ \ \forall a, b \in \mathbb{R}^n \end{aligned}$$

then

$$\begin{aligned} &\mathtt{G}_2\left(f(y), 2\,[\![g(x) + \nabla g(x)(y - x)]\!]\right) \\ &\quad\leq \mathtt{G}_2\left(f(y), f(y^*)\right) + \sqrt{2\,|g(y^*) - g(x) - \nabla g(x)(y^* - x)|^2} + \sqrt{2}L\,|y - y^*| \\ &\quad\leq (1 + \sqrt{2})L\,|y - y^*| + o\left(|y^* - x|\right) \\ &\quad= (1 + \sqrt{2})L\,|y - y^*| + o\left(r\right) \end{aligned}$$

Set $\rho = |y^* - y|$ and notice that the ball $B_\rho(y)$ doesn't intersect $\mathcal{Z}_f \cap \overline{B_{2r}(x)}$. Moreover, $B_\rho(y) \subseteq B_{2r}(x) \setminus \mathcal{Z}_f$ and

$$|y - y^*| = \sqrt[n]{\frac{|B_\rho(y)|}{\omega_n}} \leq \sqrt[n]{\frac{|B_{2r}(x) \setminus \mathcal{Z}_f|}{\omega_n}} = \sqrt[n]{2}r\,\sqrt[n]{1 - \frac{|B_{2r}(x) \cap \mathcal{Z}_f|}{|B_{2r}(x)|}} = o(r)$$

since $\mathcal{Z}_f$ has density 1 in x. I finally get

$$\mathtt{G}_2\left(f(y), 2\,[\![g(x) + \nabla g(x)(y - x)]\!]\right) = o\left(r\right)$$

so f is approximately affine in x. □

A point $\mathbf{P} \in \mathcal{A}^2\left(\mathbb{R}^N\right)$ is said to be *symmetric* if and only if $\mathbf{P} = [\![P]\!] + [\![-P]\!]$ for some $P \in \mathbb{R}^N$, and a symmetric function is just a function $f : \Omega \to \mathcal{A}^2\left(\mathbb{R}^k\right)$ with $f(x)$ symmetric for every x. If v is symmetric and approximately differentiable then both ∇v and $\mathtt{J}^1 v\,(x)$ are symmetric, and for every $F : \mathbb{R}^k \times \mathbb{M}^{n\times k} \to \mathbb{R}^l$ *even* function ($F(-p, -M) = F(p, M)$) I can uniquely define the new function $F(v, \nabla v)(x) = F\left[v_i(x), \nabla v_i(x)\right]$ which is well defined and doesn't depend on the choose of $v_i(x)$. In particular, the following expressions $|v(x)|$, $v^i(x)\nabla v^j(x)$, ... are well-placed when the two-valued function v is symmetric.

If v is approximately affine on each point of Ω then I define the following set:

$$\begin{aligned} \mathcal{K}_v &= \mathcal{Z}_{\mathtt{J}^1 v} \\ &= \left\{x \in \Omega \,|\, v_1(x) = v_2(x) \text{ and } \nabla v_1(x) = \nabla v_2(x)\right\}, \end{aligned}$$

and I say that $v \in C^1\left(\Omega, \mathcal{A}^2\left(\mathbb{R}^N\right)\right)$ if and only if

1. v is approximately affine on every point of Ω and $\mathtt{J}^1 v\,(x)$ is continuous on Ω;

2. for every $x \notin \mathcal{K}_v$ there exist an open neighborhood $\Omega' \subseteq \Omega$ of x and a C^1 decomposition of f on Ω'.

I say also that $v \in C^1\left(K, \mathcal{A}^2\left(\mathbb{R}^k\right)\right)$ for a generic not empty set $K \subseteq \mathbb{R}^n$ if and only if there exist $\Omega \supseteq K$ open and exists $w \in C^1\left(\Omega, \mathcal{A}^2\left(\mathbb{R}^k\right)\right)$ with $w = v$ on K.

Like in the continuous case not all C^1 two-valued functions admit a global C^1 decomposition: the following function

$$v : z \in \mathbb{C} \approx \mathbb{R}^2 \to \left[\!\left[z^{3/2}\right]\!\right] + \left[\!\left[-z^{3/2}\right]\!\right] \in \mathcal{A}^2\left(\mathbb{C}\right)$$

is a C^1 two-valued symmetric function but doesn't have a C^1 decomposition in any neighborhood of the origin.

The definition I've introduced for C^1 two-valued functions may seem very restrictive because I require the continuity of the jet function. However, this is equivalent to require only the continuity of the gradient function.

Lemma 2.1.8. *A two-valued function $v : \Omega \to \mathcal{A}^2\left(\mathbb{R}^k\right)$ is of class C^1 on Ω if and only if it's approximately affine on each point of Ω and ∇v is continuous.*

Lemma 2.1.8 is just a consequence of the following two results:

Proposition 2.1.9. *Let v be a two-valued function on $\Omega \subseteq \mathbb{R}^n$ which is approximately affine in each point with ∇v continuous, then $\mathtt{J}^1 v$ is continuous.*

Proof. Take any point $x_0 \in \Omega$, I can assume without loss of generality that $v_1(x_0) \neq v_2(x_0)$, $\nabla v_1(x_0) \neq \nabla v_2(x_0)$.

Then there exists $U \subseteq \Omega$ open convex neighborhood of x_0 and f, g a decomposition of v on U with functions differentiable in every point. Notice also that exist $a \in \mathbb{S}^{k-1}$, $\eta > 0$ and a basis $b_1, b_2, \ldots, b_n \in \mathbb{S}^{n-1}$ of $\mathbb{R}^n$ so that

$$|a \cdot \nabla f(x) b_i - a \cdot \nabla g(x) b_i| > \eta \;\forall i = 1, \ldots, n$$

for every $x \in U$. Since ∇v is continuous for every $\eta/3 > \varepsilon > 0$ exist $A_\varepsilon, B_\varepsilon \subseteq U$ so that

$$\begin{aligned} &A_\varepsilon \cup B_\varepsilon \supseteq B_{\delta_\varepsilon}(x_0) \text{ with } \delta_\varepsilon \to 0^+ \\ &x \in A_\varepsilon \Rightarrow |\nabla f(x) - \nabla f(x_0)| < \varepsilon \\ &x \in B_\varepsilon \Rightarrow |\nabla f(x) - \nabla g(x_0)| < \varepsilon \end{aligned}$$

observe that $x_0 \in A_\varepsilon$ and $A_\varepsilon \cap B_\varepsilon = \emptyset$.

I want to prove that for every $t \in \mathbb{R}$ and every $x \in A_\varepsilon \cap B_{\delta_\varepsilon}(x_0)$ so that $x + t b_i \in B_{\delta_\varepsilon}(x_0)$ then $x + b_i t \in A_\varepsilon$, indeed since b_i are a basis of $\mathbb{R}^n$ this will imply $A_\varepsilon \supseteq B_{\hat{\delta}_\varepsilon}(x_0)$ for some $\hat{\delta}_\varepsilon < \delta_\varepsilon$ and so $\nabla f(x) \to \nabla f(x_0)$.

Let

$$F_{x,i}(s) = \begin{cases} \frac{af(x+sb_i) - af(x)}{s} & \text{if } s \neq 0 \\ a \nabla f(x) b_i & \text{if } s = 0 \end{cases}$$

clearly $F_{x,i}$ is a continuous function, also by the mean value theorem $F_{x,i}(s) = a\nabla f\left(\tilde{x}_s\right) b_i \in B_\varepsilon[a\nabla f(x_0)b_i] \cup B_\varepsilon[a\nabla g(x_0)b_i]$ and since these balls are disjoint $F_{x,i}$ is entirely contained only in one of them.

But $F_{x,i}(0) \in B_\varepsilon[a\nabla f(x_0)b_i]$ by assumption and so

$$F_{x+tb_i,i}(-t) = F_{x,i}(t) \in B_\varepsilon[a\nabla f(x_0)b_i]$$

and for the same reason $a\nabla f(x+tb_i)b_i = F_{x+tb_i,i}(0) \in B_\varepsilon[a\nabla f(x_0)b_i]$ which imply $|\nabla f(x+tb_i) - \nabla f(x_0)| < \varepsilon$. □

Proposition 2.1.10. *Let v two-valued so that $\mathtt{J}^1 v$ is continuous on Ω then for every $x \notin \mathcal{K}_v$ there exists an open neighborhood U of x so that $\mathtt{J}^1 v\,(y) = [\![f(y), \nabla f(y)]\!] + [\![g(y), \nabla g(y)]\!]$ with $f, g \in C^1\left(U, \mathbb{R}^k\right)$.*

Proof. First to all on a neighborhood U of x I've $\mathtt{J}^1 v\,(y) = [\![f(y), F(y)]\!] + [\![g(y), G(y)]\!]$ with f, F, g, G continuous.

If $f(x) \neq g(x)$ then, up to reduce U, I immediately get $F(y) = \nabla f(y)$ and $G(y) = \nabla g(y)$ for every $y \in U$ so I need to consider only the case $f(x) = g(x)$, $F(x) \neq G(x)$.

Let $\eta > 0$ and $b_1, b_2, \ldots, b_n \in \mathbb{S}^{n-1}$ be a basis of $\mathbb{R}^n$ such that

$$|F(y)b_i - G(y)b_i| \geq \eta$$

for every i and every $y \in U$. If $f(y) \neq g(y)$ then $F(y) = \nabla f(y)$ and $G(y) = \nabla g(y)$, when $f(y) = g(y)$ notice that exists $t_y > 0$ so that for every $0 < t < t_y$

$$\begin{aligned} |f(y+tb_i) - g(y+tb_i)| &= \delta\left[v(y+tb_i)\right] \\ &\geq \delta\left[T_y v(y+tb_i)\right] - \sqrt{2}\mathtt{G}_2\left(v(y+tb_i), T_y v(y+tb_i)\right) \\ &\geq |F(y)b_i - G(y)b_i|\, t - \frac{\eta}{2} t > 0 \end{aligned}$$

so $f(y+tb_i)$ is differentiable for every $0 < t < t_y$ and by the mean value theorem for every $a \in \mathbb{S}^{k-1}$

$$\frac{a \cdot f(y+tb_i) - a \cdot f(y) - ta \cdot F(y)b_i}{t} = a \cdot F(y+\tilde{t}b_i)b_i - a \cdot F(y)b_i \to 0$$

which implies $\partial_{b_i} f(y) = F(y)b_i$ for every $y \in U$.

So f has continuous partial derivatives in a open set U, and it's well known that this implies f is C^1 on U with $\nabla f = F$. The same holds for g. □

Like the continuous case I've a global decomposition result for C^1 functions too:

Proposition 2.1.11. *If $v : \Omega \to \mathcal{A}^2\left(\mathbb{R}^k\right)$ is a C^1 two-valued function with Ω simply connected and $\mathcal{K}_v = \emptyset$ then v has a unique C^1 decomposition on all Ω.*

Proof. By lemma 2.1.4 $\mathtt{J}^1 v$ has a continuous decomposition on Ω, and by proposition 2.1.10 this decomposition leads to a C^1 decomposition of v on Ω. □

Let $u : \Omega \to \mathcal{A}^2\left(\mathbb{R}^k\right)$ be a two-valued C^1 function. I say that $x \in \Omega$ is a *regular point* for u if and only if exists an open neighborhood U of x and two $f, g : U \to \mathbb{R}^k$ single valued C^1 functions such that $u(y) = [\![f(y)]\!] + [\![g(y)]\!]$ for every $y \in U$ and one of the following is satisfied:

- $f(y) \neq g(y)$ for every $y \in U$;
- $f(y) = g(y)$ for every $y \in U$.

A point that is not regular is called *singular*, if $x \in \Omega$ is a singular point then x is an *intersection point* if and only if $x \notin \mathcal{Z}_{\nabla u}$ and it's a *branching point* if and only if $x \in \mathcal{Z}_{\nabla u}$.

Let $\mathcal{B}_u$ be the set of all branching points of u, it's simple to show that

$$\mathcal{B}_u \subseteq \mathcal{K}_u \subseteq \mathcal{Z}_u.$$

Clearly $\mathcal{B}_u$ is a Borel set so it's $\mathcal{H}^{n-2}$-measurable too. Therefore, the Hausdorff dimension and the rectifiability of $\mathcal{B}_u$ are inherithed from those of $\mathcal{K}_u$. This is the reason why I focus on the regularity of $\mathcal{K}_u$.

I define on $C^1\left(\overline{\Omega}, \mathcal{A}^2\left(\mathbb{R}^k\right)\right)$ the following metric

$$\mathtt{G}_{C^1}(v, w) = \max_{x \in \overline{\Omega}} \mathtt{G}_2\left(\mathtt{J}^1 v(x), \mathtt{J}^1 w(x)\right)$$

and $v_n \to v$ in C^1 if and only if $\lim_{n \to +\infty} \mathtt{G}_{C^1}(v_n, v) = 0$.

The next theorem is an extension of a well-know result for C^1 single valued functions to tw -valued ones, which proof is a bit tricker that the single valued case:

Theorem 2.1.12. *Let v_n be a two-valued function in $C^1\left[\overline{\Omega}, \mathcal{A}^2\left(\mathbb{R}^k\right)\right]$ so that v_n and ∇v_n converge uniformly on $\overline{\Omega}$, then exists $v \in C^1\left(\overline{\Omega}, \mathcal{A}^2\left(\mathbb{R}^k\right)\right)$ such that $v_n \to v$ in C^1.*

Moreover, for every $x \in \Omega \setminus \mathcal{K}_v$ there exist Ω' open neighborhood of x and $a_n, b_n, a, b \in C^1(\Omega', \mathbb{R}^k)$ so that

$$\mathtt{J}^1 v_n(y) = [\![a_n(y), \nabla a_n(y)]\!] + [\![b_n(y), \nabla b_n(y)]\!]$$
$$\mathtt{J}^1 v(y) = [\![a(y), \nabla a(y)]\!] + [\![b(y), \nabla b(y)]\!]$$
$$a_n \overset{C^1}{\to} a,\ b_n \overset{C^1}{\to} b$$

In particular $C^1\left[\overline{\Omega}, \mathcal{A}^2\left(\mathbb{R}^k\right)\right]$ is a complete metric space with respect to the metric $\mathtt{G}_{C^1}$.

The proof of theorem 2.1.12 requires additional work because some of well-known results for single-valued functions aren't true for two-valued functions anymore. For example, a sequence of vectors in $\mathbb{R}^k$ converges if and only if each component converges as a sequence of scalars. This is no longer true when the sequence belongs to $\mathcal{A}^2\left(\mathbb{R}^k\right)$ since it's possible to take sequences in the form $P_n = [\![(-1)^n, 1]\!] + [\![(-1)^{n+1}, -1]\!] \in \mathcal{A}^2\left(\mathbb{R}^2\right)$.

Nevertheless, it's possible to infer a uniform convergence criterion for pointwise convergent sequences which components converges uniformly.

Proposition 2.1.13. *Let $f_n(x) = [\![A_{1,n}, A_{2,n}]\!] + [\![B_{1,n}, B_{2,n}]\!] \in \mathcal{A}^2\left(\mathbb{R}^k \times \mathbb{R}^h\right)$ be a sequence of continuous functions defined on a compact subset K of $\mathbb{R}^n$ such that $f_n \to f$ pointwise.*

Set $g_n(x) = [\![A_{1,n}]\!] + [\![B_{1,n}]\!]$ and $h_n(x) = [\![A_{2,n}]\!] + [\![B_{2,n}]\!]$ if both g_n and h_n converge uniformly to g and h respectively then $f_n \to f$ uniformly too.

Proof. I first prove that $f_n(x_n) \to f(x)$ for every $x_n \to x$ by assuming $\#[g(x)] = \#[h(x)] = 2$, otherwise the statement is trivial. Set $f(x) = [\![A_g, A_h]\!] + [\![B_g, B_h]\!]$, $\varepsilon_g = |A_g - B_g|/3$ and $\mathfrak{B}_g$ the open ball in $\mathcal{A}^2(\mathbb{R}^k)$ with center $g(x) = [\![A_g]\!] + [\![B_g]\!]$ and radius ε_g (in the same way I set ε_h and $\mathfrak{B}_h$).

By uniform convergence of g and h there exists Ω' open connected neighborhood of x and $N \in \mathbb{N}$ so that $g_n(y), g(y) \in \mathfrak{B}_g$ and $h_n(y), h(y) \in \mathfrak{B}_h$ for every $n > N$ and $y \in \Omega'$. In particular there exist $g_{n,A}, g_{n,B}$ single-valued continuous on Ω' such that $g_n(y) = [\![g_{n,A}(y)]\!] + [\![g_{n,B}(y)]\!]$, $g_{n,A}(y) \in B_{\varepsilon_g}(A)$ and $g_{n,A}$ converges uniformly. In the same way I define $h_{n,A}, h_{n,B}$.

For every $n > N$ and $y \in \Omega'$ let

$$\tilde{f}_n(y) = [\![g_{n,A}(y), h_{n,A}(y)]\!] + [\![g_{n,B}(y), h_{n,B}(y)]\!]$$
$$\hat{f}_n(y) = [\![g_{n,A}(y), h_{n,B}(y)]\!] + [\![g_{n,B}(y), h_{n,A}(y)]\!]$$

clearly $f_n(y)$ is equal to $\tilde{f}_n(y)$ or to $\hat{f}_n(x)$. Therefore, the set

$$\Omega'' = \left\{ y \in \Omega' \,\middle|\, f_n(y) = \tilde{f}_n(y) \right\}$$

is both closed and open in Ω'. Now since $f_n(x) \to f(x)$ for n large the set Ω'' is not empty. But Ω' is connected; thus, $f_n = \tilde{f}_n$ and $f_n(x_n) \to f(x)$.

Next I prove that f is continuous. Take any $x_n \to x$ and set N_n so that $|f_m(x_n) - f(x_n)| < \varepsilon$ for every $n \in \mathbb{N}$ and $m \geq N_n \geq n$. But for n large I get

$$|f_{N_n}(x_n) - f(x)| < \varepsilon$$

and so $|f(x_n) - f(x)| < 2\varepsilon$ and f is continuous.

Finally I prove that $f_n \to f$ uniformly. Take any sequence $x_n \to x$, then

$$|f_n(x_n) - f(x_n)| \leq |f_n(x_n) - f(x)| + |f(x) - f(x_n)| \to 0$$

that implies our convergence is uniform. □

Corollary 2.1.14. *Let f_n, g_n be continuous functions from $\overline{\Omega} \subseteq \mathbb{R}^n$ to $\mathbb{R}^k$ that converge pointwise. If $v_n = [\![f_n]\!] + [\![g_n]\!]$ and v_n converges uniformly on $\overline{\Omega}$ then f_n and g_n converge uniformly too.*

Proof. Let $A, B \in \mathbb{R}$ be distinct values and set

$$F_n(x) = [\![f_n(x), A]\!] + [\![g_n(x), B]\!] \in \mathcal{A}^2\left(\mathbb{R}^{k+1}\right)$$

clearly F_n converges pointwise and by lemma 2.1.13 converges uniformly too. Since $A \neq B$ both f_n and g_n converge uniformly. □

Remark 2.1.15. Corollary 2.1.14 is optimal and its conditions can't be further relaxed, in fact you can choose $f_n(x)$ and $g_n(x) = -f_n(x)$ on $\Omega = [0,1]$ where f_n is one of the following:

$$f_n(x) = \begin{cases} (2x-1)^2 & \text{if } n \text{ is even} \\ (2x-1)\,|2x-1| & \text{if } n \text{ is odd} \end{cases} \tag{A}$$

$$f_n(x) = \begin{cases} -1 & \text{if } 0 \leq x < 1 - \frac{1}{n} \\ 1 & \text{if } 1 - \frac{1}{n} \leq x \leq 1 \end{cases} \tag{B}$$

$$f_n(x) = x^n \tag{C}$$

none of them converges uniformly. Sequence (A) doesn't converge pointwise, functions (B) aren't continuous and for (C) notice that $v_n = [\![f_n]\!] + [\![-f_n]\!]$ doesn't converge uniformly.

Proof of theorem 2.1.12. Suppose $v_n \to v$ and $\nabla v_n \to \psi$ and set, with a slight abuse of notation, $\mathcal{K}_v = \{v_1(z) = v_2(x), \psi_1(z) = \psi_2(z)\}$. Take any $x \in \Omega$, if $x \notin \mathcal{K}_v$ then there exists Ω' open neighborhood of x such that $v_n(y) = [\![a_n(y)]\!] + [\![b_n(y)]\!]$ on Ω' with a_n, b_n C^1 functions.

If $\psi_1(x) \neq \psi_2(x)$ I can choose Ω' so that $\nabla a_n \to \tilde{a}$ uniformly and $a_n(x)$ converges. This implies $a_n \to a$ in C^1 to some function a on Ω'. If instead $\psi_1(x) = \psi_2(x)$ and $v_1(x) \neq v_2(x)$ I can choose Ω' so that $a_n \to a$ uniformly. Suppose now that exists $y \in \Omega'$ so that $\nabla a_n(y)$ doesn't converge, then the sequence $\nabla a_n(y)$ has two different accumulation points A and B.

Thanks to continuity of ∇a_n and uniform convergence of ∇v_n there exists an open neighborhood U of y and two subsequences i_n, j_n so that $|\nabla a_{i_n}(z) - A| + |\nabla a_{j_n}(z) - B| < \varepsilon$ for every $z \in U$ and ε sufficiently small. However, this is impossible due to uniform convergence of a_n.

Therefore, ∇a_n and ∇b_n converge pointwise, but by corollary 2.1.14 they converge uniformly too. In both cases $a_n \to a$ and $\nabla a_n \to \nabla a$ and the same holds for b_n, in particular $\mathtt{J}^1 v_n(x) \to \mathtt{J}^1 v(x)$ and v can be decomposed in two single valued C^1 functions near x.

Suppose now that $x \in \mathcal{K}_v$, since $[\![v_1(y), \psi_1(y)]\!] + [\![v_2(y), \psi_2(y)]\!] \to 2\,[\![v_1(x), \psi_1(x)]\!]$ when $y \to x$ I need to prove only that $\mathtt{J}^1 v(x) = 2\,[\![v_1(x), \psi_1(x)]\!]$. Notice that the function $F_u : \Omega \to \mathbb{R}$, defined as $F_u(y) = \mathtt{G}_2\left(u(y), T_x u(y)\right)$, is Lipschitz continuous and by Rademacher theorem a.e. differentiable with

$$
\begin{aligned}
|\nabla F_u(y)| &= \left| \sum_i \frac{\left[u_{\sigma(i)}(y) - u_i(x) - \nabla u_i(x)(y-x)\right] \cdot \left[\nabla u_{\sigma(i)}(y) - \nabla u_i(x)\right]}{\mathtt{G}_2\left(u(y), T_x u(y)\right)} \right| \\
&\leq \max_{\eta} \sqrt{\sum_j \left|\nabla u_{\eta(j)}(y) - \nabla u_j(x)\right|^2} = \mathfrak{F}\left[\nabla u(y), \nabla u(x)\right]
\end{aligned}
$$

where $\sigma \in S_2$ is defined as $\mathtt{G}_2\left(u(y), T_x u(y)\right) = \sqrt{\sum_i \left|u_{\sigma(i)}(y) - u_i(x) - \nabla u_i(x)(y-x)\right|^2}$. Next,

$$
\frac{\mathtt{G}_2\left(v_n(y), T_x v_n(y)\right)}{|y-x|} \leq \left\|\mathfrak{F}\left[\nabla v_n(z), \nabla v_n(x)\right]\right\|_{L^\infty([y,x])}
$$

and by uniform convergence of v_n and ∇v_n I get

$$
\frac{\mathtt{G}_2\left(v(y), T_x v(y)\right)}{|y-x|} \leq \left\|\mathtt{G}_2\left(\psi(z), \psi(x)\right)\right\|_{L^\infty([y,x])}
$$

where $T_x v(y) = 2\,[\![v_1(x) + \psi_1(x)(y-x)]\!]$ since $\mathfrak{F}$ is continuous and coincides with $\mathtt{G}_2$ because $\psi_1(x) = \psi_2(x)$. Letting $y \to x$ I prove that v is approximately affine in x.

These statements prove also that $\mathtt{J}^1 v_n(x) \to \mathtt{J}^1 v(x)$ pointwise on $\overline{\Omega}$, then applying proposition 2.1.13 this convergence is uniform too. □

I say $u \in C^{1,\alpha}\left[\overline{\Omega}, \mathcal{A}^2\left(\mathbb{R}^k\right)\right]$ if and only if $u \in C^1\left[\overline{\Omega}, \mathcal{A}^2\left(\mathbb{R}^k\right)\right]$ and

$$\sup_{x\in\overline{\Omega}} \mathtt{G}_2\left(u(x), 2\,[\![0]\!]\right) + \sup_{x\in\overline{\Omega}} \mathtt{G}_2\left(\nabla u(x), 2\,[\![0]\!]\right) + [\nabla u]_{\alpha,\Omega} < +\infty$$

where

$$[f]_{\alpha,\Omega} = \sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{\mathtt{G}_2\left(f(x), f(y)\right)}{|x-y|^\alpha}$$

Geometric measure theory

A set $M \subseteq \mathbb{R}^{n+k}$ is said to be *countably* n-rectifiable if and only if M is $\mathcal{H}^n$-measurable, where $\mathcal{H}^n$ is the n-dimensional Hausdorff outer measure on $\mathbb{R}^{n+k}$, there exists a sequence of Lipschitz mappings $f_i : \mathbb{R}^n \to \mathbb{R}^{n+k}$ such that

$$\mathcal{H}^n\left(M \setminus \bigcup_{i=1}^{\infty} f_i\left(\mathbb{R}^n\right)\right) = 0.$$

A set M is instead *locally* n-rectifiable if and only if it's countably n-rectifiable and the Borel measure $\mathcal{H}^n \llcorner M$ is finite on compact subsets of $\mathbb{R}^{n+k}$. Moreover, M is just n-rectifiable if and only if it's countably n-rectifiable and $\mathcal{H}^n(M) < +\infty$.

If M is a countably n-rectifiable set and $\theta : M \to \mathbb{Z}$ is an $\mathcal{H}^n$-measurable function then couple $V = (M, \theta)$ is a *(rectifiable) n-varifold* on $\mathbb{R}^{n+k}$ with integral multiplicity. The function θ is sometimes called the *multiplicity function* of V and $\theta(x)$ is the multiplicity of V at x.

Let

$$\mu_V(A) = \int_{M\cap A} \theta \; d\mathcal{H}^n \;\; \forall A\; \mathcal{H}^n\text{-measurable}$$
$$\mathbb{M}\left(V\right) = \mu_V\left(\mathbb{R}^{n+k}\right) = \int_M \theta \; d\mathcal{H}^n$$

Every two-valued C^1 function $u : \Omega \subseteq \mathbb{R}^n \to \mathcal{A}^2\left(\mathbb{R}^k\right)$ induces an n-dimensional varifold: let $\mathcal{G}_u$ be the graph of u defined as follows

$$M = \mathcal{G}_u = \left\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^k \,\middle|\, y = u_1(x) \text{ or } y = u_2(x)\right\}.$$

Further, for every $(x,y) \in \mathcal{G}_u$,

$$\theta_u(x,y) = 3 - \#[u(x)] = \begin{cases} 2 & \text{if } u_1(x) = u_2(x) \\ 1 & \text{if } u_1(x) \neq u_2(x) \end{cases}.$$

The couple $(\mathcal{G}_u, \theta_u)$ is a n-dimensional varifold since

Proposition 2.1.16. *The set $\mathcal{G}_u$ is locally n-rectifiable.*

Proof. On $\Omega \setminus \mathcal{K}_u$ function u admits a local C^1 decomposition near each point so $\mathcal{G}_u \cap \left[(\Omega \setminus \mathcal{K}_u) \times \mathbb{R}^k\right]$ is locally n-rectifiable. On $\mathcal{K}_u$ I've $u = 2\,[\![f]\!]$ with f locally Lipschitz and $\mathcal{G}_u \cap \left(\mathcal{K}_u \times \mathbb{R}^k\right) = \mathcal{K}_u \times f(\mathcal{K}_u)$ which is locally n-rectifiable too. □

In particular, it's possible to use area formula and get for every positive Borel measurable function $g : \mathcal{G}_u \to \mathbb{R}$

$$\int_{\mathcal{G}_u} g(x,y)\theta_u(x,y)\ d\mathcal{H}^n\,(x,y) = \int_\Omega \sum_{i=1,2} g\,[x,u_i(x)]\,\mathcal{J}[\nabla u_i(x)]\ dx$$

where for every $L \in \mathbb{M}^{n\times k}$

$$\mathcal{J}(L) = \sqrt{\det\left(I + \sum_{j=1}^k L^j \otimes L^j\right)}.$$

Let now $U \subseteq \mathbb{R}^{n+k}$ be an open set, I say $\{\Phi_t : U \to U\}_{t\in(-\varepsilon,\varepsilon)}$ is a 1-*parameter family of diffeomorphisms* if and only if

- $\Phi_0(x) = x$ for every x;
- Φ_t is a diffeomorphism for every t and exists $K \subseteq U$ compact such that $\Phi_t(x) = x$ for every t and every $x \in U \setminus K$;
- $(x,t) \to \Phi_t(x)$ is of class C^1.

a varifold V is said to be *stationary* if and only if

$$\frac{d}{dt}\mathbb{M}\left(\Phi_t V\right)\bigg|_{t=0} = 0$$

where $\Phi_t V = \left[\Phi_t(M), \theta\circ\Phi_t^{-1}\right]$, in particular a varifold V is stationary if and only if for every $X \in C_c^1\left(U,\mathbb{R}^{n+k}\right)$ I've

$$\int_M \operatorname{div} X\theta\ d\mathcal{H}^n = 0$$

A two-valued C^1 function is said to be *stationary* if and only if its associated varifold $(\mathcal{G}_u,\theta_u)$ is stationary. In particular, given a 1-parameter family of diffeomorphisms on $\mathbb{R}^n\times\mathbb{R}^k$ which on $\mathcal{G}_u$ is equal to

$$\phi_t(x,y) = [x, y + t\nu\phi(x)]$$

with $\nu \in \mathbb{R}^k$ and ϕ a real-valued C^1 function with compact support in some $B_r(x) \subseteq \Omega\setminus\mathcal{K}_u$, a stationary two-valued function u must satisfy

$$\int_\Omega G[\nabla u_i(y)]\nabla u_i^j(y)\cdot\nabla\phi(y)\ dy = 0$$

for every $i = 1,2$, $j = 1,\ldots,k$ where

$$G(L) = \sqrt{\det P(L)}P(L)^{-1}$$

$$P(L) = I + \sum_{l=1}^k L^l \otimes L^l.$$

The goal of all the following sections of this chapter is to provide regularity properties on coefficients of $G(x) = G[\nabla u_i(x)]$.

2.2 Harmonic functions

Any two-valued function $u : \Omega \subseteq \mathbb{R}^n \to \mathcal{A}^2\left(\mathbb{R}^k\right)$ can be decomposed in an *average part* and in a *symmetric part*:

$$u_a(x) = \frac{u_1(x) + u_2(x)}{2} \qquad\qquad v(x) = \left[\!\left[\pm \frac{u_1(x) - u_2(x)}{2} \right]\!\right]$$

the average part u_a of u is a single-valued function that preserves some of the good properties the original function may have. Moreover, may happen that u_a is more regular than u because the symmetric (two-valued) part v of u takes some of u irregolarities. For this reasons much of the analysis on u regularity descends from regularity of its symmetric part v.

In this section I introduce harmonic two-valued symmetric functions and exhibit some results that Simon and Wickramasekera have proved in their work [23]. Consider now a partial differential equation in the following form

$$\sum_{i,j=1,\ldots,n} \sum_{l=1,\ldots,k} a^l_{ij} \partial_i \partial_j u^l + \sum_{i=1,\ldots,n} \sum_{l=1,\ldots,k} b^l_i \partial_i u^l + \sum_{l=1,\ldots,k} c^l u^l = 0 \tag{2.2.1}$$

with a^l_{ij}, b^l_i, c^l generic functions defined on $\Omega \subseteq \mathbb{R}^n$. A two-valued symmetric function v satisfies (2.2.1) on Ω if and only if v is C^1 in Ω and near each point of $\Omega \setminus \mathcal{K}_v$ has a local C^2 decomposition that satisfies (2.2.1).

In this chapter I'll assume without loss of generality that $\Omega = B_1$ and v belongs to C^1 on $\overline{B_1}$ in order to simplify our proofs.

Definition 2.2.1. A continuous two-valued symmetric function v is *weakly* C^1 on Ω if and only if it's continuous on Ω and has a C^1 decomposition in a neighborhood of each point of $\Omega \setminus \mathcal{Z}_v$.

If v is weakly C^1 then I define its weak gradient Dv on Ω in the following way: if $x \notin \mathcal{Z}_v$ then $Dv(x) = \nabla v(x)$, if $x \in \mathcal{Z}_v$ then $Dv(x) = 2\,[\![0]\!]$.

Analogously v is weakly C^2 if and only if $v \in C^1$ on Ω and ∇v is weakly C^1, in that case let $D^2 v = D\left(\nabla v\right)$.

When $k = 1$ weakly C^1 functions have a global Lipschitz nonnegative decomposition $f, -f$ and $\nabla f, -\nabla f$ is for a.e. point a decomposition of the weak gradient. The next result is a simple approximation result for nonnegative single valued functions that is useful when I need to approximate weak C^1 two-valued functions.

Proposition 2.2.2. *Let f be a nonnegative continuous function on $\overline{\Omega}$ where Ω is an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary, then there exists a sequence of $C^\infty_c\left(\mathbb{R}^n\right)$ functions φ_m such that $\varphi_m \to f$ uniformly on $\overline{\Omega}$ and $\{f = 0\} \cap \operatorname{supp} \varphi_m = \emptyset$ for every $m \in \mathbb{N}$.*

If f is of class C^1 on $\Omega' = \{f > 0\}$ and $\nabla f \in L^2\left(\Omega'\right)$ then $\varphi_m \to f$ in $W^{1,2}(\Omega)$.

Proof. Let

$$\tau : t \geq 0 \to \begin{cases} 0 & \text{if } t \leq \frac{1}{2} \\ 2\left(t - \frac{1}{2}\right) & \text{if } \frac{1}{2} < t \leq 1 \\ t & \text{if } t > 1 \end{cases}$$

$$\tau_m(t) = \frac{\tau(mt)}{m} \ \forall m \in \mathbb{N}$$

$$\phi_m = \tau_m \circ f \ \forall m \in \mathbb{N}$$

notice that $\operatorname{supp}\phi_m \cap \{f = 0\} = \emptyset$ and

$$\|\phi_m - f\|_\infty \leq \|f\|_{L^\infty\{f \leq 1/m\}} \leq \frac{1}{m} \to 0$$

so the convergence is uniform. Now since f is uniformly continuous it's possible to use a continuous extension result on ϕ_m to extend it to $\mathbb{R}^n$ and by applying the standard mollifier I get the desired C_c^∞ sequence.

If f is C^1 on $\{f > 0\}$ then ϕ_m is Lipschitz and converges to f uniformly. Furthermore, $F = \nabla f \mathcal{X}_{\{f>0\}}$ is a L^2 function with

$$\|\nabla\phi_m - F\|_{L^2} \leq \|\nabla f\|_{L^2\{0<f<1/m\}} \to 0$$

since $\nabla f \in L^2$. Thanks to Sobolev's extension theorem and standard mollifiers I get the desired convergence also for the sequence $\varphi_m \in C_c^\infty(\mathbb{R}^m)$. □

In particular, Simon and Wickramasekera define two-valued symmetric harmonic functions as functions that own an harmonic decomposition on a neighborhood of each point of $\Omega \setminus \mathcal{K}_v$, or equivalentely v is weakly C^2 and $\operatorname{tr}\left(D^2 v\right) = 0$ on Ω. They first prove some "$W^{2,2}$" estimates for two-valued harmonic functions:

Proposition 2.2.3. *Let v be a two-valued harmonic function on B_1, then $\left|D^2 v\right| \in L^2[B_r(x)]$ for every ball $\overline{B_r(x)} \subseteq B_1$ and for every $\theta \in (0,1)$ exists $C_\theta > 0$ so that*

$$r^{2-n}\int_{B_{\theta r}(x)} \left|D^2 v\right|^2 \, dy + \sup_{B_{\theta r}(x)} |\nabla v|^2 \leq C_\theta r^{-n} \int_{B_r(x)} |\nabla v|^2 \, dy$$

Proof. Set $j = 1, \ldots, k$, $a = 1, \ldots, n$, $\psi \in C_c^\infty(\mathbb{R}^n)$ with

$$\operatorname{supp}\psi \cap \left\{\partial_a v^j = 2\,[\![0]\!]\right\} = \emptyset$$

and let $\Psi : \left\{\partial_a v^j \neq 2\,[\![0]\!]\right\} \to \mathbb{M}^{n\times k}$ be a single valued C^1 function with $\Psi_a^j > 0$ such that $\nabla v = [\![\Psi]\!] + [\![-\Psi]\!]$, notice that

$$\partial_i \Psi_a^j = \partial_a \Psi_i^j$$

for every i and $\operatorname{div}\Psi^j = 0$ by harmonicity.

For every $\phi \in C_c^\infty[B_r(x)]$ with $\|\phi\|_\infty \leq 1$ they have $\psi\phi^2 \in C_c^1\left(\Omega \setminus \left\{\partial_a v^j = 0\right\}\right)$ and so

$$\int_{\mathbb{R}^n} \partial_a \Psi^j \cdot \nabla\left(\psi\phi^2\right) \, dy = \int_{B_1} \operatorname{div}\Psi^j \partial_a(\psi\phi^2) \, dy = 0$$

then

$$\int_\Omega \nabla\Psi_a^j \cdot \nabla\psi\phi^2 \, dy = \int_\Omega \nabla\Psi_a^j \cdot \left[\nabla\left(\psi\phi^2\right) - 2\psi\phi\nabla\phi\right] \, dy$$
$$= \int_\Omega \partial_a\Psi^j \cdot \nabla\left(\psi\phi^2\right) \, dy - 2\int_\Omega \left(\nabla\Psi_a^j\phi\right)\cdot\psi\nabla\phi \, dy = -2\int_\Omega \left(\nabla\Psi_a^j\phi\right)\cdot\psi\nabla\phi \, dy$$

Fix now any $\varepsilon > 0$, according to proposition 2.2.2 thery choose a sequence ψ_n that converges uniformly and in $W^{1,2}[B_r(x)]$ to $\max\left\{\Psi_a^j - \varepsilon, 0\right\}$. Set $\Omega_\varepsilon = \left\{\Psi_a^j > \varepsilon\right\}$ then

$$\left\|\nabla\Psi_a^j\phi\right\|_{L^2(\Omega_\varepsilon)}^2 = 2\int_{B_r(x)\cap\Omega_\varepsilon} \nabla\Psi_a^j \cdot \nabla\phi\left(\Psi_a^j - \varepsilon\right)\phi \, dy$$
$$\leq \|\nabla\phi\|_\infty \left\|\Psi_a^j\right\|_{L^2[B_r(x)]} \left\|\nabla\Psi_a^j\phi\right\|_{L^2(\Omega_\varepsilon)}$$

At this point it's possible to assume ϕ Lipschitz on $B_r(x)$ and

$$\xi(t) = \begin{cases} 1 & \text{se } 0 < t < \theta \\ \frac{1-t}{1-\theta} & \text{se } \theta < t < 1 \\ 0 & \text{se } t > 1 \end{cases}$$
$$\phi(y) = \xi\left(\frac{|y-x|}{r}\right)$$

then $|\nabla\psi(x)| \leq Cr^{-1}(1-\theta)^{-1}$ and so they finally get

$$\left\|\nabla\Psi_a^j\right\|_{L^2[B_{\theta r}(x)\cap\Omega_\varepsilon]}^2 \leq C_\theta r^{-2}\left\|\Psi_a^j\right\|_{L^2[B_r(x)]}^2$$

Remember that $\nabla\Psi_a^j = D\left(\partial_a v^j\right)$ which is equal to 0 a.e. on $\left\{\partial_a v^j = 2\,[\![0]\!]\right\}$, thus

$$\left\|D^2 v\right\|_{L^2[B_{\theta r}(x)]}^2 \leq C_\theta r^{-2}\|\nabla v\|_{L^2[B_r(x)]}^2$$

For the L^∞ estimate let $\varepsilon > 0$ and $\phi \in C_c^1(\Omega)$ be a nonnegative function. Then

$$\int_{\left\{\Psi_a^j > \varepsilon\right\}} \left(\Psi_a^j - \varepsilon\right)\nabla\Psi_a^j \cdot \nabla\phi \, dy = -\int_{\left\{\Psi_a^j > \varepsilon\right\}} \phi\left|\nabla\Psi_a^j\right|^2 \, dy \leq 0$$

so $\max\left\{\Psi_a^j - \varepsilon, 0\right\}^2$ is (weakly) subharmonic. By mean value property they get $\left|\Psi_a^j(z)\right|^2 \leq C_\theta r^{-n}\int_{B_r(x)}\left|\Psi_a^j\right|^2 \, dy$ for every $z \in B_{\theta r}(x)$. □

Definition 2.2.4. Let v be a two-valued symmetric C^1 harmonic function on B_1, let $B_r(x) \subseteq B_1$ so that v is not identically zero on it. Simon and Wickramasekera define the *frequency function* of v centered in x with radius r the following quantity

$$N(v,x,r) = \frac{r^{2-n}\int_{B_r(x)}|\nabla v(y)|^2 \, dy}{r^{1-n}\int_{\partial B_r(x)}|v(y)|^2 \, d\mathcal{H}^{n-1}(y)}$$

This function is the main ingredient to estimate the Hausdorff dimension of $\mathcal{K}_v$, especially thanks to its monotonicity property:

Lemma 2.2.5. *If v is not identically zero on B_1 then $\mathcal{Z}_v$ has empty interior and frequency function $N(v,x,r)$ is nondecreasing in $r > 0$.*

Let $H(v,x,r) = r^{1-n} \int_{\partial B_r(x)} |v(y)|^2 \, d\mathcal{H}^{n-1}(y)$, it's not hard to prove that

$$N(v,x,r) = \frac{rH(v,x,r)'}{H(v,x,r)}.$$

Consequently, lemma 2.2.5 implies that

$$\left(\frac{s}{r}\right)^{N(v,x,r)} \leq \sqrt{\frac{H(v,x,s)}{H(v,x,r)}} \leq \left(\frac{s}{r}\right)^{N(v,x,s)} \leq \left(\frac{s}{r}\right)^{N(v,x)} \tag{2.2.2}$$

where $N(v,x) = \lim_{r\to 0^+} N(v,x,r) = \inf_{r>0} N(v,x,r)$. From (2.2.2) derives immediately the following doubling property for v:

$$\left(\frac{r}{R}\right)^{N(v,x,R)} R^{-n/2} \|v\|_{L^2[B_R(x)]} \leq r^{-n/2} \|v\|_{L^2[B_r(x)]} \leq \left(\frac{r}{R}\right)^{N(v,x)} R^{-n/2} \|v\|_{L^2[B_R(x)]} \tag{2.2.3}$$

for $0 < r < R$.

Proof of lemma 2.2.5. Let ψ be a $C^\infty(B_r(x))$ function with $\operatorname{supp}\psi \cap \mathcal{Z}_{v^j} = \emptyset$ then

$$\int_{B_r(x)} \nabla v^j \cdot \nabla \psi \, dy = \int_{\partial B_r(x)} \nabla v^j \cdot \frac{y-x}{r} \psi \, d\mathcal{H}^{n-1}(y)$$

by proposition 2.2.2

$$\begin{aligned}\int_{B_r(x)} \left|\nabla v^j\right|^2 \, dy = \int_{\partial B_r(x)} \nabla v^j \cdot \frac{y-x}{r} v^j(y) \, d\mathcal{H}^{n-1}(y) \\ = \frac{1}{2} \int_{\partial B_r(x)} \nabla \left(\left|v^j\right|^2\right) \cdot \frac{y-x}{r} \, d\mathcal{H}^{n-1}\end{aligned}$$

which implies

$$r^{2-n} \int_{B_r(x)} \left|\nabla v^j\right|^2 \, dy = \frac{1}{2} \frac{d}{dr} H(v,x,r)$$

also if v is identically zero on $\partial B_r(x)$ then $\nabla v = 2 [\![0]\!]$ on $B_r(x)$ and so v is identically zero so $N(v,x,r)$ is well defined.

In the same way, with the help of proposition 2.2.3, they prove that

$$\begin{aligned}\int_{B_r(x)} \left(|\nabla v|^2 \delta_{ij} - 2\partial_i v \cdot \partial_j v\right) \partial_i \xi^j \, dy \\ = \int_{\partial B_r(x)} \left(|\nabla v|^2 \frac{(y-x)\cdot\xi}{r} - 2\partial_r v \cdot \partial_j v \xi^j\right) \, d\mathcal{H}^{n-1}\end{aligned}$$

for every Lipschitz function $\xi : B_1 \to \mathbb{R}^n$ where $\partial_r v = \nabla v \cdot (y-x)/r$.

In particular, let $\xi(y) = (y - x)$, then

$$\frac{d}{dr}\left(r^{2-n}\int_{B_r(x)}|\nabla v|^2\ dy\right) = r^{2-n}\int_{\partial B_r(x)}|\partial_r v|^2\ d\mathcal{H}^{n-1}.$$

Putting everything together they have

$$\frac{d}{dr}N(v,x,r) = 2rH(v,x,r)^{-2}$$
$$\left[\int_{\partial B_r(x)}|v|^2\ d\mathcal{H}^{n-1}\int_{\partial B_r(x)}|\partial_r v|^2\ d\mathcal{H}^{n-1} - \left(\int_{\partial B_r(x)}v\partial_r v\ d\mathcal{H}^{n-1}\right)^2\right] \geq 0$$

by Cauchy-Schwartz's inequality for every r if v is not identically zero on $\partial B_r(x)$.

Next they'll prove that $\mathcal{Z}_v$ has empty interior. By contraddiction exist $x \in B_1$, $\tilde{r} > 0$ so that v is identically zero on $B_{\tilde{r}}(x)$. However,

$$\sup_{y\in B_{\tilde{r}+\delta}(x)}|v(y)| > 0$$

for every $\delta > 0$. Chosen δ, δ_0 with $0 < \delta < \delta_0$ they have

$$H(v,x,\tilde{r}+\delta) \geq \left(\frac{\tilde{r}+\delta}{\tilde{r}+\delta_0}\right)^{2N(v,x,\tilde{r}+\delta)} H(v,x,\tilde{r}+\delta_0) > 0$$

which leads to a contraddiction when $\delta \to 0^+$ since $\lim_{\delta\to 0^+} N(v,x,\tilde{r}+\delta)$ exists finite. □

Remark 2.2.6. If $N(v,x,r)$ is constant for $r_1 < r < r_2$ then by Cauchy-Schwartz's inequality $\tilde{v}(y) = v(y+x)$ is β-homogeneous for some $\beta > 0$ and $N(v,x,r) = \beta$ for every $r > 0$.

2.3 Schauder estimates

Let $v : B_1 \subseteq \mathbb{R}^n \to \mathcal{A}^2\left(\mathbb{R}^k\right)$ a two-valued $C^{1,\alpha}$ harmonic function with $0 < \alpha < 1/2$, in order to get an estimate on the Harsdorff's dimension of $\mathcal{K}_v$ they need first to get some Schauder estimates on v. More generally, let v be a two-valued symmetric function of class $C^{1,\alpha}$ that satisfies the following linear homogeneous equation

$$\Delta v^l + \sum_{i,j=1}^{n}\sum_{h=1}^{k}\partial_i\left(a_{lh}^{ij}\partial_j v^h\right) + \sum_{i=1}^{n}\sum_{h=1}^{k}b_{lh}^i\partial_i v^h + \sum_{h=1}^{k}c_{lh}v^h = 0 \tag{2.3.1}$$

in the weak sense, that is for every $B_r(x) \subseteq \Omega \setminus \mathcal{K}_v$ v has a C^1 decomposition that satisfies (2.3.1) in the weak sense.

Theorem 2.3.1. *Exists $\delta > 0$ depending only on n such that for every $0 < \alpha < \delta$ exists $\epsilon_\alpha \in (0, 1/2)$ that satisfies the following statement: for every $\beta > 0$ if*

$$\left\|a_{lh}^{ij}\right\|_{L^\infty[B_r(x)]} \leq \epsilon_\alpha \quad r^\alpha\left[a_{lh}^{ij}\right]_{\alpha,B_r(x)} + r\left\|b_{lh}^i\right\|_{L^\infty[B_r(x)]} + r^2\left\|c_{lh}\right\|_{L^\infty[B_r(x)]} \leq \beta$$

then exists C depending only on n, k, β such that

$$\|v\|_{L^\infty[B_{r/2}(x)]} + r\,\|\nabla v\|_{L^\infty[B_{r/2}(x)]} + r^{1+\alpha}\,[\nabla v]_{\alpha,B_{r/2}(x)} \le C r^{-n/2}\,\|v\|_{L^2[B_r(x)]}$$

In particular if

$$R^x_{r,K}v(y) = \frac{v(x+ry)}{r^{-\frac{n}{2}}\,\|v\|_{L^2[B_{Kr}(x)]}}$$

then $\left[\nabla R^x_{r,1}v\right]_{\alpha,B_{1/2}} \le 1$.

I show an outline of the proof of theorem 2.3.1 given by Simon and Wickramasekera. First to all they prove that inequality $[\nabla v]_{\alpha,B_{1/2}} \le \|v\|_{L^2(B_1)}$ is a consequence of the following statement: for every $\eta > 0$ exists $C_\eta > 0$ such that

$$[\nabla v]_{\alpha,B_{1/2}} \le \eta\,[\nabla v]_{\alpha,B_1} + C_\eta\left(\|v\|_{L^\infty[B_1]} + \|\nabla v\|_{L^\infty[B_1]}\right).$$

Then they proceed to prove this statement.

By contraddiction exists $\eta, \beta > 0$ and a sequence of solutions v_n of (2.3.1) with $\left|a^{ij}_{lh,n}\right| \le 1/n$ and exists $x_n \ne y_n$ in $B_{1/2}$ such that

$$\begin{aligned} 2\frac{\mathtt{G}_2\left(\nabla v_n(x_n), \nabla v_n(y_n)\right)}{|x_n - y_n|^\alpha} &> [\nabla v_n]_{\alpha,B_{1/2}} \\ &> \eta\,[\nabla v_n]_{\alpha,B_1} + n\left(\|v_n\|_{L^\infty[B_1]} + \|\nabla v_n\|_{L^\infty[B_1]}\right) \end{aligned}$$

if $r_n = |x_n - y_n| > \varepsilon$ then $[\nabla v_n]_{\alpha,B_{1/2}} < C_\varepsilon\,\|\nabla v_n\|_{L^\infty(B_1)}$ which leads to a contraddiction for n large, so they can assume $r_n \to 0$. Let

$$w_n(y) = \frac{v_n(y_n + r_n y)}{r_n^{1+\alpha}\,[\nabla v_n]_{\alpha,B_{1/2}}}$$

for every y with $|y| < r_n^{-1}/2$. If $\xi_n = (x_n - y_n)/r_n$ then $[\nabla w_n]_{\alpha,B_{1/(2r_n)}} \le 1/\eta$ and $\mathtt{G}_2\left(\nabla w_n(\xi_n), \nabla w_n(0)\right) \ge 1/2$.

Consider the following cases:

1. exists $C > 0$ such that $\operatorname{dist}\left([x_n, y_n], \mathcal{K}_{v_n}\right) r_n^{-1} \le C$;
2. $\operatorname{dist}\left([x_n, y_n], \mathcal{K}_{v_n}\right) r_n^{-1} \to +\infty$

If case 1 holds then exists $z_n \in \mathcal{K}_{w_n}$ with $|z_n|$ bounded so that $|w_n(x)| + |x - z_n|\,|\nabla w_n(x)| \le C\eta^{-1}\,|x - z_n|^{1+\alpha}$. Next they use Ascoli-Arzelà to prove that $w_n \to \varphi$ in C^1_{loc} with $\varphi \in C^{1,\alpha}$ two-valued symmetric harmonic.

However, when α is sufficiently small thet prove that φ is an affine function and contraddicts the inequality $\mathtt{G}_2\left(\nabla\varphi(\xi), \nabla\varphi(0)\right) \ge 1/2$ for $\xi_n \to \xi$.

If case 2 holds then w_n has a $C^{1,\alpha}$ decomposition $\tilde{w}_n, -\tilde{w}_n$ on B_{R_n} with $R_n \to +\infty$, in particular $\hat{w}_n(x) = \tilde{w}_n(x) - \tilde{w}_n(0) - \nabla\tilde{w}_n(0)x$ converges to a single valued $C^{1,\alpha}$ harmonic function, again for α sufficiently small this leads to a contraddiction.

Thanks to theorem 2.3.1 sequences in the form

$$R^{x_i}_{r_i,K} v_i(y) = \frac{v_i(x_i + r_i y)}{r^{-n/2} \|v_i\|_{L^2[B_{Kr_i}(x_i)]}}$$

with $x_i \to x$, $r_i \to 0^+$ have a subsequence that converge in $C^1_{loc}(B_K)$ to some harmonic function when v satisfies (2.3.1) if $a^{ij}_{lh}(x) = 0$.

In particular if v is $C^{1,\alpha}$ two-valued symmetric harmonic then for every $x \in \mathcal{K}_v$, $0 < r < R$

$$\left(\frac{r}{R}\right)^{N(v,x,R)} R^{-n/2} \|v\|_{L^2[B_R(x)]} \leq r^{-n/2} \|v\|_{L^2[B_r(x)]} \leq Cr^{1+\alpha}$$

which implies $N(v, x, R) \geq 1 + \alpha$ for every $R > 0$ and so $N(v, x) \geq 1 + \alpha$.

Let now v be a two-valued symmetric harmonic homogeneous function on $\mathbb{R}^n$ with $v(0) = 0$, I define its *spine* as its maximal subspace of invariance:

$$S(v) = \{x \in \mathbb{R}^n \mid v(x + y) = v(y)\ \forall y \in \mathbb{R}^n\}$$

clearly $S(v)$ is a vector subspace of $\mathbb{R}^n$ because for every $y \in S(v)$, $x \in \mathbb{R}^n$, $\gamma > 0$

$$\begin{gathered} v(x - y) = v[(x - y) + y] = v(x) \\ v(x + \gamma y) = \gamma^l v(x/\gamma + y) = \gamma^l v(x/\gamma) = v(x) \end{gathered}$$

Also $\dim S(v) \geq n - h$ if and only if, up to a linear transformation in $\mathbb{R}^n$, v depends only on h variables:

$$v(x_1, x_2, \ldots, x_n) = w(x_1, x_2, \ldots, x_h)$$

with w again harmonic and homogeneous on $\mathbb{R}^h$, in this way estimating the dimension of spines is equivalent to study harmonic homogeneous functions in lower dimension.

Theorem 2.3.2. *If φ is a $C^{1,\alpha}$ two-valued harmonic symmetric function on B_1 then the Hausdorff dimension of $\mathcal{K}_\varphi$ is at most $n - 2$.*

Proof. In this proof Simon and Wickramasekera use the dimension reduction argument from the monotonicity of $N(v, x, r)$. By contraddiction exists $\varepsilon > 0$ such that $\mathcal{H}^{n-2+\varepsilon}(\mathcal{K}_\varphi) > 0$, let

$$\mathcal{H}^{n-2+\varepsilon}_\infty(K) = \inf\left\{\sum_{i=1}^{\infty} \omega_{n-2+\varepsilon} r_i^{n-2+\varepsilon} \,\middle|\, K \subseteq \bigcup_{i=1}^{\infty} B_{r_i}(x_i)\right\}$$

notice that $\mathcal{H}^{n-2+\varepsilon}_\infty(\mathcal{K}_\varphi) > 0$ and exist $x \in \mathcal{K}_\varphi$, $\sigma_l \to 0^+$ such that

$$\lim_{l \to +\infty} \frac{\mathcal{H}^{n-2+\varepsilon}_\infty(\mathcal{K}_\varphi \cap B_{\sigma_l}(x))}{\sigma_l^{n-2+\varepsilon}} > 0$$

Thanks to Schauder estimates and the doubling property exists $r_j \to 0^+$ such that $R^x_{r_j,1}\varphi \to \phi$ locally in $C^1_{loc}(\mathbb{R}^n)$ and

$$\|\phi\|_{L^2(B_{1/2})} = \lim_{j\to+\infty} \left\| R^x_{r_j,1}\varphi \right\|_{L^2(B_{1/2})}$$
$$= \lim_{j\to+\infty} \frac{(r/2)^{-n/2}\|\varphi\|_{L^2[B_{r/2}(x)]}}{r^{-n/2}\|\varphi\|_{L^2[B_r(x)]}} \geq \left(\frac{1}{2}\right)^{N(\varphi,x)} > 0$$

so ϕ is not trivial and

$$N(\phi,0,\rho) = \lim_{r_n\to+\infty} N\left(R^x_{r_n,1}\varphi,0,\rho\right) = \lim_{r_n\to+\infty} N(\varphi,x,\rho r_n) = N(\varphi,x) \geq 1+\alpha$$

for every $\rho > 0$ so ϕ is homogeneous. Also for every $\delta > 0$ exist a finite number of $y_j \in \overline{B_1}$, $\rho_j > 0$ such that $\mathcal{K}_\phi \cap \overline{B_1} \subseteq \bigcup_j B_{\rho_j}(y_j)$ and $\sum_j \omega_{n-2+\varepsilon}\rho_j^{n-2+\varepsilon} \leq \mathcal{H}^{n-2+\varepsilon}_\infty\left(\mathcal{K}_\phi \cap \overline{B_1}\right)+\delta$, the C^1 convergence implies also that $\mathcal{K}_{R^x_{r_j,1}\varphi} \cap \overline{B_1} \subseteq \bigcup_l B_{\rho_l}(y_l)$ for j sufficiently large so

$$0 < \limsup_{j\to+\infty} \frac{\mathcal{H}^{n-2+\varepsilon}_\infty\left(\mathcal{K}_\varphi \cap B_{r_j}(x)\right)}{\omega_{n-2+\varepsilon} r_j^{n-2+\varepsilon}}$$
$$\leq \limsup_{j\to+\infty} \mathcal{H}^{n-2+\varepsilon}_\infty\left(\mathcal{K}_{R^x_{r_j,1}\varphi} \cap \overline{B_1}\right) \leq \mathcal{H}^{n-2+\varepsilon}_\infty\left(\mathcal{K}_\phi \cap \overline{B_1}\right)$$

Since $\mathcal{H}^{n-2+\varepsilon}_\infty(\mathcal{K}_\phi \cap \overline{B_1}) > 0$ they can chose a new point $x_1 \in \mathcal{K}_\phi \setminus \{0\}$. Exists another $r_n \to 0^+$ such that $R^{x_1}_{r_n,1}\phi \to \phi_1$ uniformly on compact subsets of B_1 and ϕ_1 is again two-valued homogeneous harmonic nontrivial and $\mathcal{H}^{n-2+\varepsilon}_\infty(\mathcal{K}_{\phi_1} \cap \overline{B_1}) > 0$. For every $y \in \mathbb{R}^n$ and every $s > 0$ by uniform convergence

$$\phi_1(y+sx_1) = \lim_{j\to+\infty} C_j\phi\left[x_1 + r_j(y+sx_1)\right]$$
$$= \lim_{j\to+\infty} C_j\phi\left[x_1(1+r_js)+r_jy\right] = \lim_{j\to+\infty} C_j(1+r_js)^\beta \phi\left(x_1 + r_j\frac{y}{1+r_js}\right)$$
$$= \lim_{j\to+\infty}(1+r_js)^\beta \phi_1\left(\frac{y}{1+r_js}\right) = \phi_1(y)$$

for some $\beta > 0$, so $S(\phi_1)$ has dimension at least 1.

Repeating this argument $n-2$ more times they finally get an harmonic homogeneous function $\tilde{\phi}$ and $\dim S(\tilde{\phi}) \geq n-1$. But then $\tilde{\phi}$ is an affine function and since $0 \in \mathcal{K}_{\tilde{\phi}}$ this implies $\tilde{\phi}$ is identically zero, which is impossible. So $\mathcal{K}_\varphi$ has Hausdorff dimension at most $n-2$. □

2.4 $C^{1,1/2}$ regularity

It's possible to show that $C^{1,\alpha}$ two-valued symmetric harmonic functions are also of class $C^{1,1/2}$. Since $\mathcal{K}_\varphi$ has dimension at most $n-2$ when φ is a nontrivial $C^{1,\alpha}$ two-valued symmetric harmonic map they can use the dimension resuction argument to φ for each point $x \in \mathcal{K}_\varphi$ to get a $C^{1,\alpha}$ two-valued homogeneous harmonic function that depends only on 2 variables.

The next result characterize two-valued homogeneous harmonic maps in $\mathbb{R}^n$

Proposition 2.4.1. *Let w be a two-valued C^1 symmetric harmonic λ-homogeneous function on $\mathbb{R}^2$. Then $\mathcal{K}_w$ is equal to $\emptyset$, $\{0\}$ or $\mathbb{R}^2$. If $\mathcal{K}_w = \{0\}$ then $\lambda = m/2$ for some $m \in \mathbb{N}$, $m \geq 3$ and, in polar coordinates,*

$$w^i(r,\theta) = r^{m/2} \left[\!\left[\pm C_i \sin\left(\frac{m}{2}\theta + \eta_i\right) \right]\!\right]$$

for every $i = 1, \dots, k$.

Proof. Suppose $\mathcal{K}_w$ is not equal to $\emptyset$, $\{0\}$ or $\mathbb{R}^2$. Since $\mathcal{K}_w$ is a closed cone they could assume $(r,0) \notin \mathcal{K}_w$ for every $r > 0$ and set

$$\theta_1 = \max\{\theta \in (-2\pi, 0) \mid (r,\theta) \in \mathcal{K}_w \ \forall r > 0\}$$
$$\theta_2 = \min\{\theta \in (0, 2\pi) \mid (r,\theta) \in \mathcal{K}_w \ \forall r > 0\}$$

and notice that $\theta_1 < 0 < \theta_2$ (may happen that $\theta_2 - \theta_1 = 2\pi$ but this is not a problem).

Then $(r,\theta) \notin \mathcal{K}_w$ for every $r > 0$, $\theta_1 < \theta < \theta_2$ and proposition 2.1.11 implies

$$w(r,\theta) = \left[\!\left[\pm r^\lambda g(\theta) \right]\!\right] \ \ \forall r > 0, \theta_1 < \theta < \theta_2$$

for some $g \in C^2\left((\theta_1,\theta_2), \mathbb{R}^k\right)$.

Now remember that $\Delta = \frac{\partial^2}{\partial r^2} + r^{-1}\frac{\partial}{\partial r} + r^{-2}\frac{\partial^2}{\partial \theta^2}$ then for every $\theta_1 < \theta < \theta_2$ they have

$$g'' + \lambda^2 g = 0$$

which implies

$$g^i(\theta) = C_i \sin(\lambda\theta + \eta_i) \ \ \forall \theta_1 < \theta < \theta_2$$

with $C_i, \eta_i \in \mathbb{R}$ and $C_i \neq 0$ for at least one i.

Because $w \in C^1$ and $(r,\theta_1) \in \mathcal{K}_w$ these would imply $\lim_{\theta \to \theta_1^+} g(\theta), g'(\theta) = 0$ which is impossible unless $C_i = 0$ for every i, but that possibility has already been ruled out.

If $\mathcal{K}_w = \{0\}$ then $w^i(r,\theta) = \left[\!\left[\pm r^\lambda g^i(\theta) \right]\!\right]$ for $0 < \theta < 2\pi$ where $g^i(\theta) = C_i \sin(\lambda\theta + \eta_i)$. Notice that

$$\left| w^i(r,\theta) \right|^2 = r^{2\lambda} \frac{C_i^2}{2} \left[1 - \cos(2\lambda\theta + 2\eta_i)\right]$$

is a single valued C^1 function on $\mathbb{R}^n$ and $\nabla w(0) = 2\left[\!\left[0\right]\!\right]$, and this is possible if and only if $2\lambda = m$ for some $m \in \mathbb{N}$ with $m \geq 3$. $\square$

In particular $N(\varphi, x) \geq 3/2$ for every $x \in \mathcal{K}_\varphi$ even if φ is not homogeneous and also

$$|\varphi(x)| \leq C \|\varphi\|_{L^2} d(x)^{3/2} \qquad\qquad |\nabla\varphi(x)| \leq C \|\varphi\|_{L^2} d(x)^{1/2}$$

where $d(x) = \operatorname{dist}(x, \mathcal{K}_\varphi)$. For every $x, \neq y$ in B_1 $r = |x - y|$ they have from Schauder estimates and from (2.2.3)

$$r\mathtt{G}_2(\nabla\varphi(x), \nabla\varphi(y)) \leq C(2r)^{-n/2} \|\varphi\|_{L^2[B_{2r}(z)} \leq Cr^{3/2} \|\varphi\|_{L^2(B_5)}$$

so $\varphi \in C^{1,1/2}$ near each point in $\mathcal{K}_\varphi$. The $C^{1,1/2}$ estimates are optimal for two-valued harmonic functions, take for example $\varphi(z) = [\![\pm z^{3/2}]\!]$ for every $z \in \mathbb{C} \equiv \mathbb{R}^2$.

The next step is to provide a $C^{1,1/2}$ estimate for two-valued $C^{1,\alpha}$ functions $u : \overline{\Omega} \to \mathcal{A}^2\left(\mathbb{R}^k\right)$ with stationary graph where Ω is a open bounded convex set and $[\nabla u]_{\alpha,\Omega} \leq \epsilon_0$ for some dimensional constant ϵ_0 depending only on n, for that reason they should write u as the solution of a partial differential equation. Remember that u can be decomposed in an average u_a and in a two-valued symmetric part v and the two-valued function u satisfies the following equation

$$\text{div}\left[G[\nabla u(y)]\nabla u^j(y)\right] = 0 \ \forall j$$

on each point $y \in \Omega \setminus \mathcal{K}_u$.

So they define for every $M, N \in \mathbb{M}^{n\times k}$

$$A(M,N) = G(M+N) + G(M-N)$$
$$E(M,N)[B] = \int_{-1}^{1} dG(M+tN)[B]\ dt$$

where $dG(M)[B] = \lim_{r\to 0} \frac{G(M+rB)-G(M)}{r}$ and $E(M,N)$ is a linear application from $\mathbb{M}^{n\times k}$ to $\mathbb{M}^{n\times n}_{\text{sym}}$ for every $M, N \in \mathbb{M}^{n\times k}$.

Clearly $A(M,N) = A(M,-N)$, $E(M,N) = E(M,-N)$ and

$$E(M,N)[N] = G(M+N) - G(M-N)$$

so the functions defined as $A(x) = A\left(\nabla u_a(x), \nabla v(x)\right)$, $E(x) = E\left(\nabla u_a(x), \nabla v(x)\right)$ are well defined on all Ω. Functions u_a and v satisfy the following equations:

$$\text{div}\left[A(x)\nabla v^j(x) + E(x)[\nabla v(x)]\nabla u_a^j(x)\right] = 0 \tag{2.4.1}$$
$$\text{div}\left[A(x)\nabla u_a^j(x) + E(x)[\nabla v(x)]\nabla v^j(x)\right] = 0 \tag{2.4.2}$$

for every $x \in \Omega \setminus \mathcal{K}_v$. The second equation holds weakly on all Ω too since $E(x)[\nabla v(x)]\nabla v^j(x)$ is well defined everywhere.

Next since the mean curvature of u graph is zero they have

$$\text{div}\left[G^j[\nabla u(x)]\right] = 0 \ \forall j = 1, \ldots, n$$

on $\Omega \setminus \mathcal{K}_u$, so (2.4.1) and (2.4.2) become

$$\sum_{i,j=1}^{n} \left[A^{ij}(x)\partial_i\partial_j v^h(x) + E^{ij}(x)[\nabla v(x)]\partial_i\partial_j u_a^h(x)\right] = 0$$
$$\sum_{i,j=1}^{n} \left[A^{ij}(x)\partial_i\partial_j u_a^h(x) + E^{ij}(x)[\nabla v(x)]\partial_i\partial_j v^h(x)\right] = 0$$

for every $h = 1, \ldots, k$.

The proof that u_a is $C^{1,1}$ and v is $C^{1,1/2}$ when u is stationary is more difficult than the harmonic case and proceeds by degree. By harmonic approximations it's possible to prove some weaker estimates: for every $\overline{B_{2r}(x)} \subseteq \Omega$ and every $\varepsilon > 0$ exists $C_\varepsilon > 0$ such that for every $y \in B_r(x)$

$$|\nabla v(y)| \leq C_\varepsilon d(y)^{1/2-\varepsilon}$$
$$[\nabla u_a]_{1-\varepsilon, B_r(x)} \leq C_\varepsilon$$
$$\left|D^2 u_a(y)\right| \leq C d(y)^{-\varepsilon} \text{ if } y \notin \mathcal{K}_v$$

By standard regularity theory for partial differential equations it's possible to prove also "$W^{2,2}$" estimates: functions $D^2 u_a^j$, $D^2 v^j$ belongs to L^2_{loc} for every $j = 1, \ldots, k$ and also

$$\left\|D^2 u_a\right\|_{L^2[B_r(x)]} \leq C r^{-2} \left\|u_a\right\|_{L^2[B_{2r}(x)]}$$
$$\left\|D^2 v\right\|_{L^2[B_r(x)]} \leq C r^{-2} \left\|v\right\|_{L^2[B_{2r}(x)]}$$

In order to get $C^{1,1/2}$ estimates from the $C^{1,1/2-\varepsilon}$ ones for v they need to reuse the frequency function $N(v, x, r)$ but this time assuming the doubling condition

$$\|v\|_{B_{2r}(x)} \leq C \|v\|_{B_r(x)} \tag{2.4.3}$$

is satisfied in x by v for r sufficiently small in order to prove that $r \to e^{Cr} N(v, x, r)$ is nondecreasing and $\lim_{r \to 0^+} N(v, x, r) \geq \frac{3}{2}$. If v was harmonic then they wouldn't need (2.4.3) to prove frequency monotonicity, and in the final part of this chapter they'll provide a new frequency for v such that doesn't requires the doubling precondition. However, that frequency strictly requires $C^{1,1/2}$ regularity for v.

Nevertheless, if v doesn't satisfy (2.4.3) for some r, more precisely if $\|v\|_{L^2[B_{r/2}(x)]} \geq 2^{\gamma+n/2} \|v\|_{L^2[B_{r/4}(x)]}$ for some well-suited $\gamma > 3/2$, then $\|v\|_{L^2[B_r(x)]} \geq 2^{\gamma+n/2} \|v\|_{L^2[B_{r/2}(x)]}$ which in turn implies

$$r^{-n/2} \|v\|_{L^2[B_r(x)]} \leq C r^\gamma \|v\|_{L^2(\Omega)} \leq C r 3/2 \|v\|_{L^2(\Omega)}$$

that applied to Schauder estimates proves that $v \in C^{1,1/2}$. In the same way it's possible to prove that $u_a \in C^{1,1}$.

2.5 A frequency function for stationary functions

From the $C^{1,1}$ estimates for u_a and the $C^{1,1/2}$ estimates for v Simon and Wickramasekera use them to estimate better the terms A and E defined in (2.4.1). First to all notice that

$$E^{ij}(x)[\nabla v] \partial_i \partial_j u_a^h = \sum_{l=i}^{k} \sum_{s=1}^{n} \tilde{E}_l^{ijs}(x) \partial_s v^l \partial_i \partial_j u_a^h$$

and notice that $\tilde{E}_l^{ijs}(x) \partial_i \partial_j u_a^h \in L^\infty$, then for the second order coefficients A they have the following result:

Proposition 2.5.1. *Application* $A : \Omega \to \mathbb{M}^{n \times n}_{\text{sym}}$ *is Lipschitz.*

Proof. Take any $x, y \in \Omega$ if both belong to $\mathcal{K}_v$ then $A(x) = G(\nabla u_a(x))$ so A is Lipschitz on $\mathcal{K}_v$. If instead $[x,y] \cap \mathcal{K}_v = \emptyset$ then for every $z = tx + (1-t)y$ they get

$$\begin{aligned}\nabla A(z) \cdot (x-y) = dA(z)[(x-y)] = d_1 A\left(\nabla u_a(z), \nabla v(z)\right)\left[\nabla^2 u_a(z) \cdot (x-y)\right] \\ + d_2 A\left(\nabla u_a(z), \nabla v(z)\right)\left[\nabla^2 v(z) \cdot (x-y)\right]\end{aligned}$$

by $C^{1,1}$ estimates for u_a thet have

$$\left| d_1 A\left(\nabla u_a(z), \nabla v(z)\right)\left[\nabla^2 u_a(z) \cdot (x-y)\right]\right| \leq C\left|x-y\right|$$

so they need only to estimate the second term.

Now $A(M,N) = f\left[(M+N)^T(M+N)\right] + f\left[(M-N)^T(M-N)\right]$ with $f \in C^\infty$ and set

$$\begin{aligned}\mathcal{A} &= (M+N)^T(M+N) \\ \mathcal{B} &= (M-N)^T(M-N)\end{aligned}$$

then $\mathcal{A} - \mathcal{B} = 2\left(M^T N + N^T M\right)$. Suppose exists $R > 0$ so that $|M|, |N| < R$ then

$$\begin{aligned}&\left|d_2 A(M,N)[B]\right| \\ &= \left|\nabla f(\mathcal{A})\left[B^T(M+N) + (M+N)^T B\right] - \nabla f(\mathcal{B})\left[B^T(M-N) + (M-N)^T B\right]\right| \\ &\qquad \leq C\left|\nabla f(\mathcal{A}) - \nabla f(\mathcal{B})\right|\left|M\right|\left|B\right| + C\left|\nabla f(\mathcal{A}) + \nabla f(\mathcal{B})\right|\left|N\right|\left|B\right| \\ &\qquad\qquad \leq C_R\left|N\right|\left|B\right|\end{aligned}$$

for some constant $C_R > 0$ depending on n, k and R. Finally

$$\begin{aligned}\left|d_2 A\left(\nabla u_a(z), \nabla v(z)\right)\left[\nabla^2 v(z) \cdot (x-y)\right]\right| &\leq C\left|\nabla v(z)\right|\left|\nabla^2 v(z)(x-y)\right| \\ &\leq C d(z)^{1/2} d(z)^{-1/2}\left|x-y\right| = C\left|x-y\right|\end{aligned}$$

where $d(x) = \inf\left\{|x-y| \,|\, v(y) = 0\right\}$, so $|\nabla A(z)| \leq C$ for every $z \in [x,y]$.

Now suppose that exists $z \in [x,y] \cap \mathcal{K}_v$ and $[x,z) \cap \mathcal{K}_=\emptyset$, let $z_n \in [x,z)$ be any sequence converging to z. Since the Lipschitz constant C doesn't depend on x or z they get

$$\left|A(x) - A(z_n)\right| \leq C\left|x - z_n\right| \leq C\left|x - z\right|$$

so A is Lipschitz on Ω. □

So the symmetric part v of u satisfies the following equation

$$\operatorname{div}\left(A \nabla v^j\right) + \sum_{l=1}^{k} E^{lj} \cdot \nabla v^l = 0$$

for every $j = 1, \ldots, k$ with $A : \Omega \to \mathbb{M}^{n\times n}_{\text{sym}}$ Lipschitz symmetric definite positive and $E^{lj} : \Omega \to \mathbb{R}^n$ in L^∞. The next step is to introduce a new system of local coordinates Γ in an open neighborhood U of $x_0 \in \mathcal{K}_v$ such that $\hat{v} = v \circ \gamma^{-1}$ satisfies

$$\operatorname{div}\left(\hat{A} \nabla \hat{v}^j\right) + \sum_{l=1}^{k} \hat{E}^{lj} \cdot \nabla \hat{v}^l = 0$$

with $\tilde{A}$ Lipschitz symmetric definite positive with $\tilde{A}(x_0) = I$ and $\tilde{A}(y)(y - x_0) = \mu(y)(y - x_0)$ where $\mu : U \to \mathbb{R}^+$ satisfies

$$\frac{1}{C} \leq \mu(y) \leq C \qquad\qquad |\partial_r \mu| \leq C$$

where C depends only on n and α.

Thanks to L^2 estimates on $D^2\hat{v}$ they can define

$$C^* (\hat{v}, x, r) \, r^{2-n} \int_{\partial B_r(x)} \mu \partial_r \hat{v} \hat{v} \; d\mathcal{H}^{n-1} = r^{2-n} \int_{B_r(x)} \left[\hat{A} \nabla \hat{v} \cdot \nabla \hat{v} + \sum_{j=1}^{k} E_j \cdot \nabla \hat{v}^j \hat{v} \right] \, dy$$

$$N(v, x, r) = \frac{r^{2-n} \int_{\partial B_r(x)} \mu \partial_r \hat{v} \hat{v} \; d\mathcal{H}^{n-1}}{r^{1-n} \int_{\partial B_r(x)} \mu \left| \hat{v} \right|^2 \; d\mathcal{H}^{n-1}}$$

and it's possible to prove that $N(v, x, r)$ is nonnegative for r small and

$$r \to e^{Cr} N(v, x, r)$$

is an increasing function for r small and for some $C > 0$, in particular the limit $N(v, x) = \lim_{r \to 0^+} N(v, x, r)$ exists finite. This monotonicity result will be proved in the first part of next chapter for a "relaxed" version of $N(v, x, r)$ but the idea behind it is the same.

Thanks to $C^{1,1/2}$ and "$W^{2,2}$" estimates if they let

$$\hat{v}_r(y) = \frac{v(x + ry)}{r^{-n/2} \left\| v \right\|_{L^2[B_r(x)]}}$$

then they have a subsequence converging to some $C^{1,1/2}$ harmonic two-valued function φ. Also $N(\varphi, 0, r) = N(v, x)$ for every r so φ is also homogeneous. We're able so to use the dimension reduction argument to prove that

Theorem 2.5.2. *If u is a nontrivial two-valued $C^{1,\alpha}$ stationary function on B_1 with $[\nabla u]_{\alpha, B_1} \leq \epsilon_0$ for some dimensional constant ϵ_0 then $u_a \in C^{1,1}$, $v \in C^{1,1/2}$ and $\mathcal{K}_u = \mathcal{K}_v$ has Hausdorff dimension at most $n - 2$.*

Chapter 3

Structure of the branching set

This chapter is devoted to show a proof of theorem 1.0.1. In order to prove it I'll first prove a stronger result in the last section

Theorem 3.0.1. *Let u be a two-valued $C^{1,\alpha}$ function with stationary graph on $\mathbb{R}^n$, then for every compact set $K \subseteq \mathbb{R}^n$ exists a constant $C_K > 0$ depending on v and K such that for every $x \in \mathcal{K}_u$ and every $0 < r < 1$*

$$|(\mathcal{K}_v \cap K) + B_r| \leq C_K r^2$$

3.1 Natural frequency function

Let v be a two-valued symmetric function in $C^{1,1/2}\left(\overline{\Omega}, \mathcal{A}^2\left(\mathbb{R}^k\right)\right)$ where the set Ω is open bounded and convex with $\operatorname{diam}(\Omega) \leq 1$. I say that v belongs to $\mathcal{WQ}\left(n,k,\Lambda\right)$ for some $\Lambda > 0$ if and only if exist $A : \Omega \to \mathbb{M}^{n\times n}_{\text{sym}}$, $L^{ij} : \Omega \to \mathbb{R}^m$ for every $1 \leq i, j \leq k$ that satisfy the following statements:

- for every $x \notin \mathcal{K}_v$ exist an open neighborhood $U \subseteq \Omega \setminus \mathcal{K}_v$ and a C^2 function $u : U \to \mathbb{R}^k$ with $v(y) = [\![u(y)]\!] + [\![-u(y)]\!]$ and

$$\int_U A(y)\nabla u^i(y) \cdot \nabla\phi(y)\, dy + \sum_{j=1}^{k} \int_U L^{ij}(y) \cdot \nabla u^j(y)\phi(y)\, dy = 0 \tag{3.1.1}$$

  for every $\phi \in C^1_c(U)$ and for every $1 \leq i \leq k$;

- for every $x \in \Omega$ and every $\nu \in \mathbb{R}^n$

$$(\Lambda+1)^{-1}|\nu|^2 \leq A(x)\nu\cdot\nu \leq (1+\Lambda)|\nu|^2 \tag{3.1.2}$$
$$\left|L^{ij}(x)\right| \leq \Lambda$$

- A is Lipschitz on Ω with

$$|A(x) - A(y)| \leq \Lambda\,|x-y| \qquad \forall x,y \in \Omega \tag{3.1.3}$$

Let $\mathcal{Q}\left(n,k,\Lambda\right)$ be its subset in which each function v satisfies, in addition to preceding conditions, $0 \in \mathcal{K}_v$ and $A(0) = I$

Lemma 3.1.1. *Let* $v \in \mathcal{WQ}(n,k,\Lambda)$ *and* $\phi \in C_c^1(\Omega)$ *then*

$$\int_\Omega A(y)\nabla v^i(y)\cdot\nabla\left[v^l(y)\phi(y)\right]\,dy+\sum_{j=1}^k\int_\Omega L^{ij}(y)\cdot\nabla v^j(y)v^l(y)\phi(y)\,dy=0 \quad (3.1.4)$$

Proof. For every $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp}\varphi\cap\left\{v^j=2\,[\![0]\!]\right\}=\emptyset$ I have

$$\varphi\phi\in C_c^1(\Omega\setminus\mathcal{K}_v)$$

then

$$\int_\Omega A(y)\nabla v^i(y)\cdot\nabla\left[\varphi(y)\phi(y)\right]\,dy+\sum_{j=1}^k\int_\Omega L^{ij}(y)\cdot\nabla v^j(y)\varphi(y)\phi(y)\,dy=0.$$

The proof then follows immediately from proposition 2.2.2. □

From this point, in order to make clearer our notation, any constant I'll introduce will depend only by n,k and Λ unless additional parameters are specified.

Theorem 3.1.2. *Let* v_n *be a function in* $\mathcal{WQ}(n,k,\Lambda)$ *with* $\left(A_n,L_n^{ij}\right)$ *as in the very definition of* $\mathcal{WQ}(n,k,\Lambda)$. *Let also* $x_n\in\Omega$, $b_n\in\mathbb{R}$, $r_n>0$ *with* $r_n\to0$, $x_n\to x\in\overline{\Omega}$ *and suppose that*

$$w_n(y)=b_nv_n(x_n+r_ny)\ \ \forall y\in B_1$$

converges in $C^1_{loc}(B_1)$ *to some two-valued function* w.

Then exists $\tilde{A}\in\mathbb{M}^{n\times n}_{\text{sym}}$ *so that, up to a subsequence,* $A_n(x_n)\to\tilde{A}$ *and either* $w\equiv2\,[\![0]\!]$ *or*

$$\operatorname{div}\left(\tilde{A}\nabla\tilde{w}^j\right)=0$$

for every $j=1,\dots,k$.

Also if w^j *is equal to* $2\,[\![0]\!]$ *on an open not empty subset of* B_1 *then it's identically zero on all* B_1.

Proof. Suppose w is not identically equal to $2\,[\![0]\!]$, thanks to theorem 2.1.12 I can assume, with a slight abuse of notation, that $v_n=[\![\pm v_n]\!]$, $w=[\![\pm w]\!]$ and the preceding convergence is a convergence of C^1 single-valued functions on an open subset $U\subseteq B_1\setminus\mathcal{K}_w$.

For every $\phi\in C_c^1(U)$ I define the function

$$\phi_n(x)=\phi\left(\frac{x-x_n}{r_n}\right)$$

and

$$\int_{\tilde{U}}A_n(x_n+r_ny)\nabla w_n^i(y)\nabla\phi(y)\,dy+\sum_{j=1}^n r_n\int_{\tilde{U}}L_n^{ij}(x_n+r_ny)\cdot\nabla w_n^j(y)\phi(y)\,dy=0.$$

Notice that $\tilde{A}_n(y) = A_n(x_n + r_n y)$ are equibounded and equilipschitz with Lip $\tilde{A}_n \leq r_n \Lambda$. Therefore, exists $\tilde{A} \in \mathbb{M}^{n\times n}_{\text{sym}}$ so that $\tilde{A}_n \to \tilde{A}$ uniformly on compact subsets of B_1 (up to a subsequence). Since $\left\| L^{ij}_n \right\|_{L^\infty} \leq \Lambda$ when $n \to +\infty$ I finally get

$$\int_{\tilde{U}} \tilde{A}\,(y)\, \nabla w^i(y) \nabla \phi(y)\ dy = 0.$$

The last statement is a direct consequence of lemma 2.2.5. □

Notice now that if $w_n \in \mathcal{Q}\,(n,k,\Lambda)$ and $x_n = 0$ for every n then $w \in \mathcal{Q}\,(n,k,\Lambda)$ is also harmonic. Thanks to Shauder estimates proved in chapter 1 I have the following convergence result:

Theorem 3.1.3. *For every $\theta \in (0,1)$ exist $\tilde{r}_\theta > 0$ and $C_\theta > 0$ such that for every $v \in \mathcal{Q}\,(n,k,\Lambda)$, $x \in \Omega$ with $|x| < \tilde{r}_\theta$ and $0 < r < \tilde{r}_\theta$ so that $B_r(x) \subseteq \Omega$ I get*

$$\left[\nabla R^x_{r,1} v\right]_{1/2, B_{\theta r}} \leq C_\theta$$

In particular I get

$$\|v\|_{L^\infty[B_{\theta r}(x)]} + r\,\|\nabla v\|_{L^\infty[B_{\theta r}(x)]} + r^{3/2}[\nabla v]_{1/2, B_{\theta r}(x)} \leq C_\theta r^{-\frac{n}{2}}\,\|v\|_{L^2[B_r(x)]}$$

In order to introduce the first frequency I need to define some auxiliary functions. First to all I define $\phi : [0,+\infty) \to [0,1]$ in this way:

$$\phi(t) = \begin{cases} 1 & \text{if } 0 \leq t < \frac{1}{2} \\ 2(1-t) & \text{if } \frac{1}{2} \leq t < 1 \\ 0 & \text{if } t \geq 1 \end{cases}.$$

Let also

$$\mathbf{\Phi}_{x,r}(M) = \int_M \phi\left(\frac{|y-x|}{r}\right)\ dy$$
$$\rho_{x,r}(M) = -\int_M \phi'\left(\frac{|y-x|}{r}\right) \frac{1}{|y-x|}\ dy$$

it's very simple to prove that for every $\varphi \in C^1_c(\Omega)$, $\Phi \in C^1_c\,(\Omega, \mathbb{R}^n)$

$$\int_\Omega \text{div}\Phi(y)\ d\mathbf{\Phi}_{x,r}\,(y) = \frac{1}{r}\int_\Omega \Phi(y)\cdot(y-x)\ d\rho_{x,r}\,(y)$$
$$\frac{d}{dr}\int_\Omega \varphi(y)\ d\mathbf{\Phi}_{x,r}\,(y) = \frac{1}{r^2}\int_\Omega \varphi(y)\,|y-x|^2\ d\rho_{x,r}\,(y)$$
$$\nabla_x \int_\Omega \varphi(y)\ d\mathbf{\Phi}_{x,r}\,(y) = \frac{1}{r}\int_\Omega \varphi(y)(y-x)\ d\rho_{x,r}\,(y)$$

Also for every $A : \Omega \to \mathbb{M}^{n\times n}_{\text{sym}}$, $x \in \Omega$ I define the function $\mu_{A,x}$ as in [10]:

$$\mu_{A,x} : y \in \Omega \to \begin{cases} A(y)\frac{y-x}{|y-x|}\cdot\frac{y-x}{|y-x|} & \text{if } y \neq x \\ 1 & \text{if } y = x \end{cases}$$

Proposition 3.1.4. *If A satisfies* (3.1.2), (3.1.3) *and $A(x) = I$ then also $\mu_{A,x}$ is well defined, Lipschitz with Lipschitz constant depending only on n and Λ and*

$$\frac{1}{C} \leq \mu_{A,x}(y) \leq C$$

for every y with $C > 0$ depending only on n and Λ.

Proof. Take any $a, b \in \Omega$ with $a \neq b$ and both different from x, suppose $|a - x| \leq |b - x|$ and set

$$b' = x + \frac{|a - x|}{|b - x|}(b - x)$$

clearly I get

$$\left|\mu_{A,x}(b) - \mu_{A,x}(b')\right| \leq \left|A(b) - A(b')\right| \leq \Lambda \left||a - x| - |b - x|\right| \leq \Lambda \left|a - b\right|.$$

Next since $A(x) = I$, $|a - x| = |b' - x|$ and A is symmetric

$$\begin{aligned}\left|\mu_{A,x}(a) - \mu_{A,x}(b')\right| &= \frac{|A(a)(a - x) \cdot (a - x) - A(b')(b' - x) \cdot (b' - x)|}{|a - x|^2} \\ &\leq \left|A(a) - A(b')\right| + \frac{|(A(a) \,:\, (a - x) \otimes (a - x) - (b' - x) \otimes (b' - x))|}{|a - x|^2} \\ &= \left|A(a) - A(b')\right| + \frac{|(A(a) - A(x) \,:\, (a - b') \otimes (a + b' - 2x))|}{|a - x|^2} \\ &\leq \Lambda \left|a - b'\right| + n\Lambda \frac{|a - b'| \, |a + b' - 2x|}{|a - x|} \\ &\leq \Lambda \left|a - b'\right| + 2n\Lambda \left|a - b'\right| \leq (1 + 2n) \Lambda \left|a - b'\right|\end{aligned}$$

where $(M \,:\, N) = \sum_{ij} M_{ij} N_{ij}$. Since

$$\begin{aligned}\left|a - b'\right|^2 &= 2 \left|a - x\right|^2 - 2(a - x) \cdot (b - x) \frac{|a - x|}{|b - x|} \\ &= \left[2 \left|a - x\right| \left|b - x\right| - 2(a - x) \cdot (b - x)\right] \frac{|a - x|}{|b - x|} \\ &\leq \left[\left|a - x\right|^2 + \left|b - x\right|^2 - 2(a - x) \cdot (b - x)\right] \frac{|a - x|}{|b - x|} \leq |a - b|^2\end{aligned}$$

I finally have $|\mu_{A,x}(a) - \mu_{A,x}(b)| \leq 2(1 + n)\Lambda \left|a - b\right|$.

If $a = x \neq b$ then I easily get

$$|\mu_{A,x}(b) - 1| = \frac{|[A(b) - A(x)] \, (b - x) \cdot (b - x)|}{|b - x|^2} \leq \Lambda \left|b - x\right|$$

so $\mu_{A,x}$ is Lipschitz. □

For every v two-valued symmetric, $x \in \Omega$, $r > 0$ so that $B_r(x) \subseteq \Omega$ I define the following quantity:

$$\begin{aligned}\mathbb{H}\left(v, x, r\right) &= r^{1-n} \int_{\Omega} |v(y)|^2 \, \mu_{A,x}(y) \; d\rho_{x,r}\left(y\right) \\ &= -r^{1-n} \int_{B_r(x) \setminus B_{r/2}(x)} \phi' \left(\frac{|y - x|}{r}\right) \frac{|v(y)|^2}{|y - x|} \mu_{A,x}(y) \; dy\end{aligned}$$

I want to use the preceding compactness result in order to prove that such quantity scales as well as $r^{-n}\|v\|^2_{L^2[B_r(x)]}$ for r small. However, I first need this simple inequality:

Proposition 3.1.5. *There exist* $\tilde{r}>0, C\in(0,1)$ *so that for every* $v\in\mathcal{WQ}(n,k,\Lambda)$, $x\in\Omega$, $0<r<\tilde{r}$ *with* $B_r(x)\subseteq\Omega$ *I get*

$$\int_{B_{r/2}(x)}|v(y)|^2\ dy\leq C\int_{B_r(x)}|v(y)|^2\ dy$$

Proof. By contradiction for every n exist $r_n>0$ converging to 0, $x_n\in\Omega_n\subseteq\overline{B_2}$, $v_n\in\mathcal{WQ}(n,k,\Lambda)$ so that

$$\int_{B_{r_n/2}(x_n)}|v_n(y)|^2\ dy\geq\left(1-\frac{1}{n}\right)\int_{B_{r_n}(x_n)}|v_n(y)|^2\ dy$$

Let $w_n=R^{x_n}_{r_n,1}v_n$, since $w_n\to w$ in C^1 on $\overline{B_{2/3}}$ I have

$$\int_{B_{1/2}}|w(y)|^2\ dy=1$$
$$\int_{B_{2/3}\setminus B_{1/2}}|w(y)|^2\ dy=0$$

that's impossible since div $\left(\tilde{A}\nabla w\right)=0$ on $B_{2/3}$. □

This result implies that exist $\tilde{r},C>0$ depending only by n,k,Λ (since the diameter of Ω is less than or equal to 1) so that for every $0<r<\tilde{r}$

$$\frac{1}{C}r^{-n}\|v\|^2_{L^2[B_r(x)]}\leq\mathbb{H}(v,x,r)\leq Cr^{-n}\|v\|^2_{L^2[B_r(x)]}$$

also I have the following estimate: there exist $\tilde{r},C>0$ so that for every $v\in\mathcal{WQ}(n,k,\Lambda)$, every $|x|<\tilde{r}$, every $\theta\in(0,1)$ and every $0<r<\tilde{r}$

$$\mathbb{H}(v,x,\theta r)\leq C\mathbb{H}(v,x,r)$$

For every $v\in\mathcal{WQ}(n,k,\lambda)$ I define the following quantities:

$$\mathbb{D}(v,A,x,r)=\sum_j r^{2-n}\int_{\mathbb{R}^n}A(y)\nabla v^j(y)\cdot\nabla v^j(y)\ d\mathbf{\Phi}_{x,r}(y)$$
$$\partial^F_r f(y)=\nabla f(y)\cdot\frac{A(y)(y-x)}{\mu_{A,x}(y)\,|y-x|}$$
$$\mathbb{C}^*(v,x,r)=\sum_j r^{1-n}\int_{\mathbb{R}^n}|y-x|\,v^j(y)\partial^F_r v^j(y)\mu_{A,x}(y)\ d\rho_{x,r}(y)$$
$$\mathbb{E}(v,A,x,r)=\sum_j r^{1-n}\int_{\mathbb{R}^n}|y-x|^2\left|\partial^F_r v^j(y)\right|^2\mu_{A,x}(y)\ d\rho_{x,r}(y)$$

notice that by the Cauchy-Schwartz inequality

$$\mathbb{C}^*(v,x,r)^2 \leq \mathbb{H}(v,x,r)\,\mathbb{E}(v,x,r).$$

Let $f \in C^1(B_r,\mathbb{R})$, for every $0<\eta<r$ I set

$$f_\eta = \frac{1}{\omega_n \eta^n}\int_{B_\eta} f(y)\;dy$$

it's well known (see for example [14, Lemma 7.16]) that for every $0<\theta<t$, $y\in B_{rt}$

$$|f(y)-f_{r\theta}| \leq \frac{2^n t^n}{n\omega_n \theta^n}\int_{B_{rt}} |\nabla f(z)|\,|z-y|^{1-n}\;dz$$

and so

$$\int_{B_{rt}} |f(y)-f_{r\theta}|\;dy \leq C_\theta t^{n+1} r\int_{B_{rt}} |\nabla f(y)|\;dy$$

Now I introduce a generalized version of ϕ:

$$\phi_{a,b}(t) = \begin{cases} 1 & \text{if } 0<t\leq a \\ \frac{b-t}{b-a} & \text{if } a<t\leq b \\ 0 & \text{if } t>b \end{cases}$$

where $1/4<a<b<2$, in particular $\phi=\phi_{1/2,1}$. Notice that for every F measurable nonnegative function

$$\int_{\mathbb{R}^n}\phi_{a,b}\left(\frac{|y|}{r}\right)F(y)\;dy = \frac{1}{b-a}\int_a^b\int_{B_{rt}} F(y)\;dy\;dt$$

which implies:

$$\int_{\mathbb{R}^n}\phi_{a,b}\left(\frac{|y|}{r}\right)|f(y)-f_{r\theta}|\;dy \leq C_\theta r\int_{\mathbb{R}^n}\phi_{a,b}\left(\frac{|y|}{r}\right)|\nabla f(y)|\;dy \tag{3.1.5}$$

Now let v be a two-valued as before, I want to apply (3.1.5) to function $f(y)=|v(x+y)|^2$. By theorem 3.1.3 there exists $\tilde{r}>0$ so that for every $0<\theta<1/4$, $0<2r<\tilde{r}$ and every $|x|<\tilde{r}$ exists $\hat{y}\in B_{r\theta}$ such that

$$\begin{aligned} f_{r\theta} = |v(x+\hat{y})| &\leq 2r\theta\sup_{y\in B_{r\theta}}|v(x+y)|\,|\nabla v(x+y)| + |v(x)|^2 \\ &\leq r^2\theta\sup_{y\in B_{r\theta}}|\nabla v(x+y)|^2 + \theta\sup_{y\in B_{r\theta}}|v(x+y)|^2 + |v(x)|^2 \\ &\leq \theta C_{1/3} r^{-n}\int_{B_{ar}(x)}|v(z)|^2\;dz + |v(x)|^2 \\ &\leq \theta C_{1/3} r^{-n}\int_{\mathbb{R}^n}\phi_{a,b}\left(\frac{|z-x|}{r}\right)|v(z)|^2\;dz + |v(x)|^2 \end{aligned}$$

exists also $C_\theta>0$ such that

$$\begin{aligned} \int_{\mathbb{R}^n}\phi_{a,b}\left(\frac{|y-x|}{r}\right)|v(y)|^2\;dy &\leq \omega_n\theta K\int_{\mathbb{R}^n}\phi_{a,b}\left(\frac{|z-x|}{r}\right)|v(z)|^2\;dz \\ &+ \omega_n r^n|v(x)|^2 + 2C_\theta r\int_{\mathbb{R}^n}\phi_{a,b}\left(\frac{|y-x|}{r}\right)|v(y)|\,|\nabla v(y)|\;dy \end{aligned}$$

where K is a constant not depending on θ.

Let $\varepsilon > 0$ and set

$$\theta_\varepsilon = \min\left\{\frac{1}{4}, \frac{\varepsilon}{(1+\varepsilon)\omega_n K}\right\}$$

and $C_\varepsilon = C_{\theta_\varepsilon}$, I obtain

$$\begin{aligned}\int_{\mathbb{R}^n} \phi_{a,b}\left(\frac{|y-x|}{r}\right)|v(y)|^2\ dy &\le 2(1+\varepsilon)C_\varepsilon r\int_{\mathbb{R}^n}\phi_{a,b}\left(\frac{|y-x|}{r}\right)|v(y)|\,|\nabla v(y)|\ dy \\ &\quad + (1+\varepsilon)\omega_n r^n |v(x)|^2 \\ &\le C_\varepsilon^2(1+\varepsilon)^2 r^2\frac{\varepsilon+1}{\varepsilon}\int_{\mathbb{R}^n}\phi_{a,b}\left(\frac{|y-x|}{r}\right)|\nabla v(y)|^2\ dy \\ &\quad + \frac{\varepsilon}{\varepsilon+1}\int_{\mathbb{R}^n}\phi_{a,b}\left(\frac{|y-x|}{r}\right)|v(y)|^2\ dy + (1+\varepsilon)\omega_n r^n|v(x)|^2\end{aligned}$$

Rescaling in ε I finally get the following two-valued version of the Poincaré's inequality

Lemma 3.1.6 (Poincaré's inequality)**.** *There exists $\tilde{r} > 0$ so that for every $\varepsilon > 0$ esists $C_\varepsilon > 0$ such that for every $v \in \mathcal{WQ}(n,k,\Lambda)$, $|x| < \tilde{r}$, $1/3 < a < b < 2$ and every $0 < r < \tilde{r}$ I get*

$$\begin{aligned}\int_{\mathbb{R}^n}\phi_{a,b}\left(\frac{|y-x|}{r}\right)|v(y)|^2\ dy \le &C_\varepsilon r^2\int_{\mathbb{R}^n}\phi_{a,b}\left(\frac{|y-x|}{r}\right)|\nabla v(y)|^2\ dy \\ &+ (1+\varepsilon)\omega_n r^n|v(x)|^2\end{aligned} \tag{3.1.6}$$

From (3.1.6) I'm able to prove nonnegativity of $\mathbb{C}^*(v,x,r)$:

Theorem 3.1.7. *There exist $\tilde{r} > 0$, $C > 0$ such that for every $v \in \mathcal{WQ}(n,k,\Lambda)$, $x \in \mathcal{K}_v \cap B_{\tilde{r}}$ and every $0 < r < \tilde{r}$*

$$|\mathbb{D}(v,x,r) - \mathbb{C}^*(v,x,r)| \le Cr\mathbb{D}(v,x,r)$$

In particular I can choose $\tilde{r}$ so that $C\mathbb{D}(v,x,r) \ge \mathbb{C}^*(v,x,r) \ge C^{-1}\mathbb{D}(v,x,r)$.

Proof. Let $\tilde{r}$ be as in preceding statement and $\varepsilon = 1$, then

$$\begin{aligned}|\mathbb{D}(v,x,r) - \mathbb{C}^*(v,x,r)| &= \left|\sum_j r^{2-n}\int_{\mathbb{R}^n}\operatorname{div}\left[A(y)\nabla v^j(y)\right]v^j(y)\ d\mathbf{\Phi}_{x,r}(y)\right| \\ &\le Cr^{2-n}\int_{\mathbb{R}^n}|\nabla v(y)|\,|v(y)|\ d\mathbf{\Phi}_{x,r}(y) \\ &\le C\sqrt{\mathbb{D}(v,x,r)}\sqrt{r^{2-n}\int_{\mathbb{R}^n}|v(y)|^2\ d\mathbf{\Phi}_{x,r}(y)} \\ &\le Cr\mathbb{D}(v,x,r)\end{aligned}$$

□

Theorem 3.1.8. *Exist $\tilde{r}, C > 0$ such that for every $v \in \mathcal{WQ}(n,k,\Lambda)$, $x \in B_{\tilde{r}} \cap \mathcal{K}_v$ and $0 < r < \tilde{r}$ I get*

$$\frac{1}{|b-a|} r^{1-n} \int_{B_{br}(x)\setminus B_{ar}(x)} \frac{|v(y)|^2}{|y-x|}\, dy \leq C r^{2-n} \int_{\mathbb{R}^n} \phi_{a,b}\left(\frac{|y-x|}{r}\right) |\nabla v(y)|^2\, dy \tag{3.1.7}$$

for every $1/3 < a < b < 2$, in particular I've

$$\mathbb{D}(v,x,r) \geq C\mathbb{H}(v,x,r) \tag{3.1.8}$$

Proof. By divergence theorem I get

$$\begin{aligned}
&\frac{1}{r|b-a|} \int_{B_{br}(x)\setminus B_{ar}(x)} |v(y)|^2 |y-x|\, dy \\
&\qquad = \int_{\mathbb{R}^n} \phi_{a,b}\left(\frac{|y-x|}{r}\right) \operatorname{div}\left[|v(y)|^2 (y-x)\right]\, dy \\
&= 2\int_{\mathbb{R}^n} \phi_{a,b}\left(\frac{|y-x|}{r}\right) v(y)\partial_r v(y) |y-x|\, dy + n\int_{\mathbb{R}^n} \phi_{a,b}\left(\frac{|y-x|}{r}\right) |v(y)|^2\, dy \\
&\leq C\int_{\mathbb{R}^n} \phi_{a,b}\left(\frac{|y-x|}{r}\right) |v(y)|^2\, dy + Cr^2 \int_{\mathbb{R}^n} \phi_{a,b}\left(\frac{|y-x|}{r}\right) |\nabla v(y)|^2\, dy \\
&\qquad \leq Cr^2 \int_{\mathbb{R}^n} \phi_{a,b}\left(\frac{|y-x|}{r}\right) |\nabla v(y)|^2\, dy
\end{aligned}$$

our thesis follows by multiplying each term by r^{-n}. □

Remark 3.1.9. If $b = a + \varepsilon$ then letting $\varepsilon \to 0^+$ I get

$$r^{1-n} \int_{\partial B_{ar}(x)} |v(y)|^2\, d\mathcal{H}^{n-1}(y) \leq Cr^{2-n} \int_{B_{ar}(x)} |\nabla v(y)|^2\, dy$$

Definition 3.1.10. Let $v \in \mathcal{WQ}(n,k,\Lambda)$, $x \in \mathcal{K}_v$, $r > 0$ so that $B_r(x) \subseteq \Omega$, I define the *natural frequency function* of v centered at x and with radius r the following quantity:

$$\mathbb{I}(v,x,r) = \frac{\mathbb{C}^*(v,x,r)}{\mathbb{H}(v,x,r)}$$

The main goal this section is to prove a monotonicity result for the natural frequency function when $v \in \mathcal{Q}(n,k,\Lambda)$ and $x = 0$. In particular,

- $0 \in \mathcal{K}_v$ and $A(0) = I$;
- A is Lipschitz,

so $\mu_{A,0}$ isa Lipschitz function and can be differentiated almost everywhere.

The first step is to estimate the derivative of $H(v,r)$.

Lemma 3.1.11. *Exist $\tilde{r}, C > 0$ so that for every $v \in \mathcal{Q}(n,k,\Lambda)$ and $0 < r < \tilde{r}$ I get*

$$\left|\mathbb{H}(v,r)' - \frac{2\mathbb{C}^*(v,r)}{r}\right| \leq C\mathbb{H}(v,r)$$

Proof. Deriving $\mathbb{H}(v,r)$ in r I get

$$\begin{aligned}\mathbb{H}(v,r)' = &\frac{1-n}{r}\mathbb{H}(v,r)\\ &+ 2r^{1-n}\int_{\partial B_r}\frac{|v(y)|^2}{|y|}\mu_{A,0}(y)\ d\mathcal{H}^{n-1}(y)\\ &- r^{1-n}\int_{\partial B_{r/2}}\frac{|v(y)|^2}{|y|}\mu_{A,0}(y)\ d\mathcal{H}^{n-1}(y)\\ = &\frac{1-n}{r}\mathbb{H}(v,r)\\ &+ 2r^{-n}\int_{\partial B_r}|v(y)|^2 A(y)\frac{y}{|y|}\cdot\nu\ d\mathcal{H}^{n-1}(y)\\ &+ 2r^{-n}\int_{\partial B_{r/2}}|v(y)|^2 A(y)\frac{y}{|y|}\cdot\nu\ d\mathcal{H}^{n-1}(y)\\ = &\frac{1-n}{r}\mathbb{H}(v,r) + 2r^{-n}\int_{B_r\backslash B_{r/2}}|v(y)|^2\,\mathrm{div}\left[A(y)\frac{y}{|y|}\right]\ dy\\ &+ 2r^{-n}\int_{B_r\backslash B_{r/2}}2v(y)\nabla v(y)\cdot A(y)\frac{y}{|y|}\ dy\\ = &\frac{1-n}{r}\mathbb{H}(v,r) + 2r^{-n}\int_{B_r\backslash B_{r/2}}|v(y)|^2\,\mathrm{div}\left[A(y)\frac{y}{|y|}\right]\ dy\\ &+ \frac{2}{r}\mathbb{C}^*(v,r)\end{aligned}$$

since $v \in W^{1,2}(B_1)$.

Next notice that

$$\begin{aligned}\mathrm{div}\left[A(y)\frac{y}{|y|}\right] &= \mathrm{div}A(y)\cdot\frac{y}{|y|} + \sum_{ij}a_{ij}\frac{\delta_{ij}|y| - (y_j)(y_i)|y|^{-1}}{|y|^2}\\ &= \mathrm{div}A(y)\cdot\frac{y}{|y|} + \frac{\mathrm{tr}\,A(y) - \mu_{A,0}(y)}{|y|}\end{aligned}$$

where vector $\mathrm{div}A(y)$ is the column-wise divergence of $A(y)$, then

$$\begin{aligned}&\left|\frac{1-n}{r}\mathbb{H}(v,r) - 2r^{-n}\int_{B_r\backslash B_{r/2}}|v(y)|^2\,\mathrm{div}\left[A(y)\frac{y}{|y|}\right]\ dy\right|\\ &\leq 2r^{-n}\int_{B_r\backslash B_{r/2}}|v(y)|^2\,|\mathrm{div}A(y)|\ dy\\ &\quad+ 2r^{-n}\int_{B_r\backslash B_{r/2}}\frac{|v(y)|^2}{|y|}\,|\mathrm{tr}\,A(y) - n\mu_{A,0}(y)|\ dy\\ &\leq C\mathbb{H}(v,r) + 2r^{-n}\int_{B_r\backslash B_{r/2}}\frac{|v(y)|^2}{|y|}\,|\mathrm{tr}\,A(y) - n\mu_{A,0}(y)|\ dy.\end{aligned}$$

Let y be any point in $B_r\backslash B_{r/2}$, there exists $\eta \in \mathbb{S}^{n-1}$ so that $\mathrm{tr}\,A(y) = nA(y)\eta\cdot\eta$. Let

$$z = \frac{2}{3}r\eta + x$$

then $z \in B_r \setminus B_{r/2}$ and so $|z-y| \leq 2r$. Also

$$|\operatorname{tr} A(y) - n\mu_{A,0}(y)| \leq n\,\|A(z)-A(y)\| + \|A(y)\|\,|\mu_{A,0}(z)-\mu_{A,0}(y)| \leq Cr$$

and our result is so proved. □

To estimate the derivative of $\mathbb{C}^*(v,r)$ is far more difficult than of $\mathbb{H}(v,r)$. Therefore, I proceed in multiple steps.

Lemma 3.1.12. *Exist $\tilde{r}, C>0$ such that for every $v \in \mathcal{Q}(n,k,\Lambda)$ and $0<r<\tilde{r}$ I have*

$$\left|\mathbb{D}(v,x,r)' - \frac{2}{r}\mathbb{E}(v,x,r)\right| \leq C\mathbb{D}(v,x,r)$$

Proof. Let $\psi \in C^1(\Omega,\mathbb{R}^n)$ I get

$$\begin{aligned}
\int_{\mathbb{R}^n} A(y)\nabla v^j \cdot \nabla v^j \operatorname{div}\psi \; dy &= \sum_{khl}\int_{\mathbb{R}^n} a_{kh}\partial_k v^j \partial_h v^j \partial_l \psi^l \; dy \\
&= -\sum_{khl}\int_{\mathbb{R}^n} \partial_l a_{kh}\partial_k v^j \partial_h v^j \psi^l \; dy \\
&\quad - 2\sum_{khl}\int_{\mathbb{R}^n} a_{kh}\partial_k\partial_l v^j \partial_h v^j \psi^l \; dy \\
&= -\sum_{khl}\int_{\mathbb{R}^n} \partial_l a_{kh}\partial_k v^j \partial_h v^j \psi^l \; dy \\
&\quad + 2\sum_{khl}\int_{\mathbb{R}^n} \partial_k\left(a_{kh}\partial_h v^j\right)\partial_l v^j \psi^l \; dy \\
&\quad + 2\sum_{khl}\int_{\mathbb{R}^n} a_{kh}\partial_l v^j \partial_h v^j \partial_k \psi^l \; dy
\end{aligned}$$

so

$$\begin{aligned}
\int_{\mathbb{R}^n} A(y)\nabla v^j \cdot \nabla v^j \operatorname{div}\psi \; dy &= -\int_{\mathbb{R}^n}\left[\nabla A \,:\, \nabla v^j \otimes \nabla v^j \otimes \psi^j\right]\; dy \\
&\quad - 2\sum_i \int_{\mathbb{R}^n} L^{ji}\cdot \nabla v^i \psi \cdot \nabla v^j \; dy + 2\int_{\mathbb{R}^n}\left(A\nabla v^j\right)\cdot\left(\nabla\psi\nabla v^j\right)\; dy \qquad (3.1.9)
\end{aligned}$$

Set then

$$\begin{aligned}
F(y) &= \frac{A(y)y}{\mu_{A,0}(y)} \\
\psi(y) &= \phi\left(\frac{|y|}{r}\right)F(y)
\end{aligned}$$

clearly I get

$$\begin{aligned}
\partial_i\psi^j(y) &= \phi'\left(\frac{|y|}{r}\right)\frac{y_i}{r\,|y|}F^j(y) + \phi\left(\frac{|y|}{r}\right)\partial_i F^j(y) \\
\operatorname{div}\psi(y) &= \phi'\left(\frac{|y|}{r}\right)\frac{y\cdot F(y)}{r\,|y|} + \phi\left(\frac{|y|}{r}\right)\operatorname{div}F(y)
\end{aligned}$$

applied to (3.1.9) I get

$$
\begin{aligned}
&- r^{-1}\int_{\mathbb{R}^n} A(y)\nabla v^j\cdot\nabla v^j F(y)\cdot y\ d\rho_{0,r}(y) + \int_{\mathbb{R}^n} A(y)\nabla v^j\cdot\nabla v^j \mathrm{div}F\ d\mathbf{\Phi}_{0,r}(y) \\
&\qquad = -\int_{\mathbb{R}^n}\left[\nabla A : \nabla v^j\otimes\nabla v^j\otimes F\right]\ d\mathbf{\Phi}_{0,r}(y) \\
&- 2\sum_i\int_{\mathbb{R}^n} L^{ji}\cdot\nabla v^i F(y)\cdot\nabla v^j\ d\mathbf{\Phi}_{0,r}(y) + 2\sum_i\int_{\mathbb{R}^n}\left(A\nabla v^j\right)\cdot\left(\nabla F^i\partial_i v^j\right)\ d\mathbf{\Phi}_{0,r}(y) \\
&\qquad - 2r^{-1}\sum_i\int_{\mathbb{R}^n} F^i(y)\partial_i v^j\left(A\nabla v^j\right)\cdot y\ d\rho_{0,r}(y) \qquad (3.1.10)
\end{aligned}
$$

Notice now that $F(y)\cdot y = |y|^2$ and since $\mu_{A,0}$ is Lipschitz with $|F(y)|\le C|y|$, $|\nabla F(y) - I|\le C|y|$ then I have

$$
\begin{aligned}
&-\frac{1}{r}\int_{\mathbb{R}^n} A(y)\nabla v^j\cdot\nabla v^j|y|^2\ d\rho_{0,r}(y) + \int_{\mathbb{R}^n} A(y)\nabla v^j\cdot\nabla v^j\mathrm{div}F\ d\mathbf{\Phi}_{0,r}(y) \\
&\qquad = -\int_{\mathbb{R}^n}\left[\nabla A : \nabla v^j\otimes\nabla v^j\otimes F\right]\ d\mathbf{\Phi}_{0,r}(y) \\
&- 2\sum_i\int_{\mathbb{R}^n} L^{ji}\cdot\nabla v^i\partial_r^F v^j|y|\ d\mathbf{\Phi}_{0,r}(y) + 2\sum_i\int_{\mathbb{R}^n}\left(A\nabla v^j\right)\cdot\left(\nabla F^i\partial_i v^j\right)\ d\mathbf{\Phi}_{0,r}(y) \\
&\qquad -\frac{2}{r}\sum_i\int_{\mathbb{R}^n}\left|\partial_r^F v^j\right|^2|y|^2\mu_{A,0}(y)\ d\rho_{0,r}(y).
\end{aligned}
$$

Since

$$
\mathbb{D}(v,r)' = \frac{2-n}{r}\mathbb{D}(v,r) + r^{-n}\sum_j\int_{\mathbb{R}^n} A(y)\nabla v^j\cdot\nabla v^j|y|^2\ d\rho_{0,r}(y)
$$

I finally get

$$
\begin{aligned}
\left|\mathbb{D}(v,r)' - \frac{2}{r}\mathbb{E}(v,r)\right| &\le Cr^{1-n}\int_{\mathbb{R}^n}\left|\left[\nabla A : \nabla v\otimes\nabla v\otimes F\right]\right|\ d\mathbf{\Phi}_{0,r}(y) \\
&\quad + r^{1-n}\int_{\mathbb{R}^n} A(y)\nabla v\cdot\nabla v\,|\mathrm{div}F - n|\ d\mathbf{\Phi}_{0,r}(y) \\
&\quad + 2r^{1-n}\int_{\mathbb{R}^n}|A(y)\nabla v|\,|\nabla v|\,|I-\nabla F|\ d\mathbf{\Phi}_{0,r}(y) \\
&\quad + Cr^{2-n}\int_{\mathbb{R}^n}|\nabla v|^2\ d\mathbf{\Phi}_{0,r}(y) \\
&\le C\mathbb{D}(v,r)
\end{aligned}
$$

□

Corollary 3.1.13. *With the same assumptions of the preceding lemma*

$$
\left|r^{1-n}\sum_j\int_{\mathbb{R}^n} A(y)\nabla v^j\cdot\nabla v^j|y|^2\ d\rho_{0,r}(y) - 2\mathbb{E}(v,r)\right| \le C\mathbb{D}(v,r)
$$

for every $r<\tilde{r}$.

Next I want to prove that the limit $\lim_{r\to 0^+}\mathbb{I}(v,r)$ exists finite. By (3.1.8) I get $\mathbb{C}^*(v,r)\geq C\mathbb{H}(v,r)$ for some $C>0$ and so

$$\mathbb{C}^*(v,r)\leq \frac{C}{2}\mathbb{H}(v,r)+\frac{1}{2C}\mathbb{E}(v,r)\leq \frac{1}{2}\mathbb{C}^*(v,r)+\frac{1}{2C}\mathbb{E}(v,r)$$

and so

$$\mathbb{C}^*(v,r)\leq \frac{1}{C}\mathbb{E}(v,r)$$

thai implies

$$\begin{aligned}\mathbb{D}(v,r)&\leq C\mathbb{E}(v,r)\\ \mathbb{H}(v,r)&\leq C\mathbb{E}(v,r)\end{aligned}$$

for every $r<\tilde{r}$.

Set $J(v,r)=\mathbb{C}^*(v,r)-\mathbb{D}(v,r)=r^{2-n}\sum_i\int_{\mathbb{R}^n}\operatorname{div}\left[A(y)\nabla v^i(y)\right]v^i(y)\ d\mathbf{\Phi}_{0,r}(y)$ I immediately get

$$\begin{aligned}\left|J(v,r)'\right|&=\left|\frac{2-n}{r}J(v,r)+\sum_{ij}2r^{-n}\int_{\mathbb{R}^n}L^{ij}(y)\cdot\nabla v^j(y)v^i(y)\left|y\right|^2\ d\rho_{0,r}(y)\right|\\ &\leq C\mathbb{D}(v,r)+Cr^{1-n}\int_{\mathbb{R}^n}\left|\nabla v(y)\right|\left|v(y)\right|\left|y\right|\ d\rho_{0,r}(y)\\ &\leq C\mathbb{D}(v,r)+C\sqrt{\mathbb{H}(v,r)}\sqrt{r^{1-n}\sum_j\int_{\mathbb{R}^n}A(y)\nabla v^j\cdot\nabla v^j\left|y\right|^2\ d\rho_{0,r}(y)}\\ &\leq C\mathbb{C}^*(v,r)+C\sqrt{\mathbb{H}(v,r)}\sqrt{\mathbb{E}(v,r)}\end{aligned}$$

where I've applied the preceding inequalities.

Finally I get

$$\mathbb{C}^*(v,r)'\geq \frac{2}{r}\mathbb{E}(v,r)-C\mathbb{C}^*(v,r)-C\sqrt{\mathbb{H}(v,r)\,\mathbb{E}(v,r)}$$

Now I'm able to estimate $\mathbb{I}(v,r)'$. First to all I get

$$\begin{aligned}\mathbb{I}(v,r)'&=\frac{\mathbb{C}^*(v,r)'\,\mathbb{H}(v,r)-\mathbb{C}^*(v,r)\,\mathbb{H}(v,r)'}{\mathbb{H}(v,r)^2}\\ &\geq \mathbb{H}(v,r)^{-2}\left[2\frac{\mathbb{H}(v,r)\,\mathbb{E}(v,r)-\mathbb{C}^*(v,r)^2}{r}-C\mathbb{H}(v,r)^{\frac{3}{2}}\,\mathbb{E}(v,r)^{\frac{1}{2}}\right]\\ &\qquad -C\mathbb{I}(v,r)\end{aligned}$$

Fix $r>0$ then one of the following inequalities is true:

1. $\mathbb{H}(v,r)\,\mathbb{E}(v,r)\leq 2\mathbb{C}^*(v,r)^2$;
2. $\mathbb{H}(v,r)\,\mathbb{E}(v,r)>2\mathbb{C}^*(v,r)^2$.

In the first case I get

$$\mathbb{H}(v,r)^{\frac{3}{2}}\,\mathbb{E}(v,r)^{\frac{1}{2}} \le C\mathbb{H}(v,r)\,\mathbb{C}^*(v,r)$$

and so $\mathbb{I}(v,r)' \ge -C\mathbb{I}(v,r)$. In the second case I get instead

$$\mathbb{H}(v,r)^{\frac{3}{2}}\,\mathbb{E}(v,r)^{\frac{1}{2}} \le C\mathbb{H}(v,r)\,\mathbb{E}(v,r)$$

and so

$$2\frac{\mathbb{H}(v,r)\,\mathbb{E}(v,r) - \mathbb{C}^*(v,r)^2}{r} - C\mathbb{H}(v,r)\,\mathbb{E}(v,r) \ge \frac{1}{r}\left[(2-Cr)\mathbb{H}(v,r)\,\mathbb{E}(v,r) - 2\mathbb{C}^*(v,r)^2\right] > 0$$

because C doesn't depend on r and so I can set $\tilde{r} < 1/C$.

I've just proved the following theorem:

Theorem 3.1.14. *Exists $\tilde{r} > 0$ so that for every $v \in \mathcal{Q}(n,k,\Lambda)$ and every $0 < r < \tilde{r}$ I get*

$$\mathbb{I}(v,r)' \ge -C\mathbb{I}(v,r)$$

in particular for every $0 < s < r < \tilde{r}$

$$\mathbb{I}(v,s) \le e^{C(r-s)}\mathbb{I}(v,r)$$

and $\limsup_{r\to 0^+} \mathbb{I}(v,r) < +\infty$.

From this theorem I can easily prove existence of $\lim_{r\to 0^+}\mathbb{I}(v,r)$. If by contradiction that limit doesn't exists then I can take two sequences $r_k, s_k > 0$ converging to 0 with $s_k < t_k < s_{k-1}$ and $\mathbb{I}(v,s_k) - \mathbb{I}(v,t_k) > \varepsilon$ for some $\varepsilon > 0$.

But since $\mathbb{I}(v,r)$ is bounded above I get for some $M > 0$ and some $l_k \in (s_k, t_k)$

$$-M < \mathbb{I}(v,l_k)' = \frac{\mathbb{I}(v,t_k) - \mathbb{I}(v,s_k)}{t_k - s_k} \le -\frac{\varepsilon}{t_k - s_k} \to -\infty$$

that's impossible.

Next I need to obtain some additional estimates estimates on function $\mathbb{H}(v,r)$ I'll need later. Thanks to lemma 3.1.11 I immediately get

$$\left|\frac{\mathbb{H}(v,r)'}{\mathbb{H}(v,r)} - \frac{2}{r}\mathbb{I}(v,r)\right| < C$$

for every $0 < r < \tilde{r}$. So if $0 < r \le s < \tilde{r}$ then

$$e^{-C(s-r)}e^{\int_r^s \frac{2}{t}\mathbb{I}(v,t)\,dt}\mathbb{H}(v,r) \le \mathbb{H}(v,s) \le e^{C(s-r)}e^{\int_r^s \frac{2}{t}\mathbb{I}(v,t)\,dt}\mathbb{H}(v,r)$$

and I obtain the following monotonicity result:

Proposition 3.1.15. *Let $M, N \ge 0$ be such that $N \le \mathbb{I}(v,t) \le M$ for every $0 < r < t < s < \tilde{r}$ I get*

$$e^{-C(s-r)}\left(\frac{s}{r}\right)^{2N}\mathbb{H}(v,r) \le \mathbb{H}(v,s) \le e^{C(s-r)}\left(\frac{s}{r}\right)^{2M}\mathbb{H}(v,r) \tag{3.1.11}$$

Also

$$\mathbb{D}(v,r) \le C\frac{M}{N}e^{C(s-r)}\left(\frac{r}{s}\right)^{2N}\mathbb{D}(v,s)$$

Proof. The estimates on $\mathbb{H}(v,r)$ follows immediately from lemma 3.1.11. The estimate on $\mathbb{D}(v,r)$ instead follows by this inequality:

$$\mathbb{C}^*(v,r) = \mathbb{I}(v,r)\,\mathbb{H}(v,r) \leq \frac{M}{N}\mathbb{I}(v,s)\,\mathbb{H}(v,r)$$

□

3.2 Fixed coefficients frequency function: monotonicity

The next step is to introduce a coefficient-free frequency function that doesn't involve matrix A. Let

$$\begin{aligned} H(v,x,r) &= r^{1-n}\int_{\mathbb{R}^n} |v(y)|^2 \; d\rho_{x,r}(y) \\ D(v,x,r) &= r^{2-n}\int_{\mathbb{R}^n} |\nabla v(y)|^2 \; d\mathbf{\Phi}_{x,r}(y) \\ E(v,x,r) &= r^{1-n}\int_{\mathbb{R}^n} |y-x|^2\,|\partial_r v(y)|^2 \; d\rho_{x,r}(y) \\ I(v,x,r) &= \frac{D(v,x,r)}{H(v,x,r)} \end{aligned}$$

for every $v \in \mathcal{Q}(n,k,\Lambda)$, in particular $0 \in \mathcal{K}_v$ and $A(0) = I$, and every $x \in \Omega$, $r > 0$ with $B_r(x) \subseteq \Omega$.

The estimates (3.1.2) on A allows me to control $H(v,r)$, $D(v,x)\,, I(v,r)$ with $\mathbb{H}(v,r)$, $\mathbb{D}(v,r)$, $\mathbb{I}(v,r)$ respectively and use the results proved in the preceding section. In particular, exist $\tilde{r} > 0$, $C > 0$ such that

1. $I(v,r) \leq C I(v,R)$ for every $0 < r < R < \tilde{r}$;

2. for every $0 < s < r < \tilde{r}$ I get

$$C^{-1}\left(\frac{r}{s}\right)^{CN} H(v,s) \leq H(v,r) \leq C\left(\frac{r}{s}\right)^{CM} H(v,s) \tag{3.2.1}$$

   where $N \leq I(v,t) \leq M$ for every $s < t < r$.

In order to simplify the notation I set also

$$\mathcal{R}^x_{r,K} v(y) = \frac{v(x+ry)}{\sqrt{H(v,x,Kr)}}.$$

Proposition 3.2.1. *For every $M > 0$ exist $r_M, C_M > 0$ such that for every $v \in \mathcal{Q}(n,k,\Lambda)$, every $0 < r < r_M$ and every $|x| < r$ if $I(v,3r) \leq M$ then*

$$\frac{1}{C_M} \leq \frac{H(v,x,r)}{H(v,r)} \leq C_M$$

and the same holds for $D(v,x,r)$.

Proof. By contradiction exists $M > 0$ such that for every $m \in \mathbb{N}$ exist $v_m \in \mathcal{Q}(n,k,\Lambda)$, $r_m < 1/m$, $|x_m| < r_m$ so that $I(v_m, 3r_m) \leq M$ but

$$\lim_{m \to +\infty} \frac{H(v_m, x_m, r_m)}{H(v_m, r_m)} \in \{0, +\infty\}$$

Let

$$y_m = \frac{x_m}{r_m} \qquad\qquad u_m(y) = \mathcal{R}^0_{r_m,1} v_m(y) = \frac{v_m(r_m y)}{\sqrt{H(v_m, r_m)}}$$

clearly $H(u_m, 1) = 1$ and

$$H(u_m, y_m, 1) = \frac{H(v_m, x_m, r_m)}{H(v_m, r_m)}$$

By (3.2.1) I get $H(u_m, 3) \leq C_M$ so u_m converges uniformly on $\overline{B_2}$ to some function u and $y_m \to y \in \overline{B_1}$. Since $B_1(y_m) \subseteq B_2$ I get

$$H(u,1) = 1$$
$$H(u,y,1) = 0$$

that's impossible.

Now suppose instead that

$$\lim_{m \to +\infty} \frac{D(v_m, x_m, r_m)}{D(v_m, r_m)} \in \{0, +\infty\} \ .$$

Since u_m converges in C^1 to u (up to a subsequence) with $u(0) = 2[\![0]\!]$, $\nabla u(0) = 2[\![0]\!]$ and $u \not\equiv 0$ I get

$$\frac{D(u,y,1)}{D(u,1)} = 0$$

which is impossible too. □

Observe that the only limitation I've imposed on x is that it must be sufficiently near to the origin, in which I have $v(0) = 2[\![0]\!]$, $\nabla v(0) = 2[\![0]\!]$ and $A(0) = I$. But x may not belong to $\mathcal{K}_v$.

Remember now that ∇v is weakly C^1 on Ω and its weak hessian $D^2 v$ is equal to its true hessian $\nabla^2 v$ on $\Omega \setminus \mathcal{Z}_{\nabla v}$ and it's equal to $2[\![0]\!]$ on $\mathcal{Z}_{\nabla v}$. I now provide some L^2 estimates for $D^2 v$ in the same way I've done in chapter 1 for harmonic functions.

Proposition 3.2.2. *If $v \in \mathcal{WQ}(n,k,\Lambda)$ then $D^2 v \in L^2$, also for every $\theta \in (0,1)$ exists $C_\theta > 0$ such that if $\overline{(B_r(x))} \subseteq \Omega$ then*

$$\left\| D^2 v \right\|_{L^2[B_{\theta r}(x)]} \leq C_\theta r^{-1} \left\| \nabla v \right\|_{L^2[B_r(x)]}$$

Proof. Let $j = 1, \ldots, k$, $a = 1, \ldots, n$ and $\psi \in C^\infty_c(\mathbb{R}^n)$ with

$$\operatorname{supp} \psi \cap \left\{ \partial_a v^j = 2[\![0]\!] \right\} = \emptyset$$

and let $\Psi : \Omega' = \left\{\partial_a v^j \neq 2 [\![0]\!]\right\} \to \mathbb{M}^{n\times k}$ be a single-valued C^1 function with $\Psi_a^j > 0$ and $\nabla v(y) = [\![\Psi(y)]\!] + [\![-\Psi(y)]\!]$. Notice that

$$\partial_i \Psi_a^j = \partial_a \Psi_i^j$$

for every $i = 1, \ldots, n$. Let $\overline{B_r(x)} \subseteq \Omega$ and $\phi \in C_c^\infty [B_r(x)]$ with $\|\phi\|_\infty \leq 1$, clearly $\psi\phi^2 \in C_c^\infty \left(B_r(x) \setminus \left\{\partial_a v^j = 0\right\}\right)$ and so

$$\begin{aligned}\int_{\mathbb{R}^n} \partial_a \left(A\Psi^j\right) \cdot \nabla \left(\psi\phi^2\right)\, dy &= \int_{B_1} \operatorname{div}\left(A\Psi^j\right) \partial_a \left(\psi\phi^2\right)\, dy \\ &= \sum_i \int_{B_1} L^{ij} \cdot \Psi^i \partial_a \left(\psi\phi^2\right)\, dy \qquad (3.2.2)\end{aligned}$$

Let $g = \phi\nabla\psi$, then

$$\begin{aligned}\int_\Omega A\nabla\Psi_a^j \cdot g\phi\, dy &= \int_\Omega A\nabla\Psi_a^j \cdot \left[\nabla\left(\psi\phi^2\right) - 2\psi\phi\nabla\phi\right]\, dy \\ &= \int_\Omega A\partial_a \Psi^j \cdot \nabla\left(\psi\phi^2\right)\, dy - 2\int_\Omega A\left(\nabla\Psi_a^j\phi\right) \cdot \psi\nabla\phi\, dy \\ &= \int_\Omega \left[\sum_i L^{ij} \cdot \Psi^i e_a - \partial_a A\Psi^j\right] \cdot \nabla\left(\psi\phi^2\right)\, dy - 2\int_\Omega A\left(\nabla\Psi_a^j\phi\right) \cdot \psi\nabla\phi\, dy \\ &= \int_\Omega \left[\sum_i L^{ij} \cdot \Psi^i e_a - \partial_a A\Psi^j\right] \cdot g\phi\, dy + 2\int_\Omega \left[\sum_i L^{ij} \cdot \Psi^i e_a - \partial_a A\Psi^j\right] \cdot \psi\phi\nabla\phi\, dy \\ &\quad - 2\int_\Omega A\left(\nabla\Psi_a^j\phi\right) \cdot \psi\nabla\phi\, dy\end{aligned}$$

with $e_a \in \mathbb{R}^n$ the a-th element of the standard basis of $\mathbb{R}^n$ (so that $\nabla f \cdot e_a = \partial_a f$).

Fix now any $\varepsilon > 0$, according to proposition 2.2.2 I can choose a sequence ψ_n that converges uniformly and in $W^{1,2}[B_r(x)]$ to $\max\left\{\Psi_a^j - \varepsilon, 0\right\}$ and put it in place of ψ in preceding equations, let $\Omega_\varepsilon = \left\{\Psi_a^j > \varepsilon\right\}$ and notice that

$$g = \begin{cases}\phi\nabla\Psi_a^j & \text{on } B_r(x) \cap \Omega_\varepsilon \\ 0 & \text{otherwise}\end{cases} \quad \text{a.e.}$$

so $A\nabla\Psi_a^j \cdot g\phi = Ag \cdot g$ almost everywhere on Ω_ε.

Then

$$\begin{aligned}C^{-1}\|g\|_{L^2(\Omega_\varepsilon)}^2 \leq \left|\int_{\Omega_\varepsilon} Ag \cdot g\, dy\right| &\leq \left\|\sum_i L^{ij} \cdot \Psi^i e_a - \partial_a A\Psi^j\right\|_{L^2[B_r(x)]} \|g\|_{L^2(\Omega_\varepsilon)} \\ &+ 2\left\|\sum_i L^{ij} \cdot \Psi^i e_a - \partial_a A\Psi^j\right\|_{L^2[B_r(x)]} \left\|\left(\Psi_a^j - \varepsilon\right)\nabla\phi\right\|_{L^2(\Omega_\varepsilon \cap B_r(x))} \\ &+ C\|g\|_{L^2(\Omega_\varepsilon)} \left\|\Psi_a^j - \varepsilon\right\|_{L^2} \|\nabla\phi\|_\infty \\ &\leq C\left(1 + \|\nabla\phi\|_\infty + \|\nabla\phi\|_\infty^2\right)\|\Psi\|_{L^2}^2 + \frac{1}{2C}\|g\|_{L^2(\Omega_\varepsilon)}^2\end{aligned}$$

At this point I can assume ϕ Lipschitz on $B_r(x)$ so I can set

$$\xi(t)=\begin{cases}1 & \text{se } 0<t<\theta\\ \frac{1-t}{1-\theta} & \text{se } \theta<t<1\\ 0 & \text{se } t>1\end{cases}$$

$$\phi(y)=\xi\left(\frac{|y-x|}{r}\right)$$

then $|\nabla\phi(x)|\leq Cr^{-1}(1-\theta)^{-1}$ and so I finally get

$$\left\|\nabla\Psi_a^j\right\|^2_{L^2[B_{\theta r}(x)\cap\Omega_\varepsilon]}\leq C_\theta r^{-2}\left\|\Psi\right\|^2_{L^2[B_r(x)]}$$

letting $\varepsilon\to 0^+$ and adding on a and j I get

$$\left\|D^2v\right\|^2_{L^2[B_{\theta r}(x)]}\leq C_\theta r^{-2}\left\|\nabla v\right\|^2_{L^2[B_r(x)]}$$

□

Now with the same assumptions for every $x\in B_r$ and a.e. $y\in B_r(x)\setminus\mathcal{K}_v$ I have

$$\left|\Delta v^j(y)\right|\leq|\mathrm{div}A(y)|\left|\nabla v^j(y)\right|+|I-A(y)|\left|D^2v^j(y)\right|+C\left|\nabla v(y)\right|$$

and so for every $M>0$ exists $C_M>0$ such that

$$\left\|\Delta v^j\right\|^2_{L^2[B_r(x)]}\leq C_M\left\|\nabla v\right\|^2_{L^2[B_r(x)]}$$

if $I\left(v,3r\right)\leq M$.

Proposition 3.2.3. *For every $M>0$ exist $\tilde{r}_M, C_M>0$ so that for every $v\in\mathcal{Q}\left(n,k,\Lambda\right)$, $0<r<\tilde{r}_M$, $|x|<r$ if $I\left(v,6r\right)\leq M$ then*

$$\left|D\left(v,x,r\right)-C^*\left(v,x,r\right)\right|\leq C_M rH\left(v,x,r\right) \tag{3.2.3}$$

where

$$C^*\left(v,x,r\right)=r^{1-n}\int_{\mathbb{R}^n}|y-x|\,v(y)\partial_r v(y)\ d\rho_{x,r}\left(y\right)$$

In particular I get

$$\left|H\left(v,x,r\right)'-\frac{2}{r}D\left(v,x,r\right)\right|\leq C_M H\left(v,x,r\right)$$

Proof. By divergence theorem I get

$$\begin{aligned}\left|r^{2-n}\int_{\mathbb{R}^n}v(y)\Delta v(y)\ d\mathbf{\Phi}_{x,r}\left(y\right)\right|&\leq r^{2-n}\left\|\Delta v\right\|_{L^2}\left\|v\right\|_{L^2}\\ &\leq C_M\sqrt{D\left(v,x,2r\right)}\sqrt{r^2H\left(v,x,r\right)}\leq C_M rH\left(v,x,r\right)\end{aligned}$$

The last statement is trivial since $H\left(v,x,r\right)'=2r^{-1}C^*\left(v,x,r\right)$. □

In the same way it's possible to prove that

$$\left| D\left(v,x,r\right)' - \frac{2}{r} E\left(v,x,r\right) \right| \leq C_M D\left(v,x,r\right)$$

and get

$$\begin{aligned}
I\left(v,x,r\right)' &= H\left(v,x,r\right)^{-2}\left[D\left(v,x,r\right)' H\left(v,x,r\right) - D\left(v,x,r\right) H\left(v,x,r\right)'\right] \\
&= H\left(v,x,r\right)^{-2}\left[D\left(v,x,r\right)' H\left(v,x,r\right) - \frac{2}{r} D\left(v,x,r\right) C^*\left(v,x,r\right)\right] \\
&\geq H\left(v,x,r\right)^{-2} \frac{2}{r}\left[E\left(v,x,r\right) H\left(v,x,r\right) - D\left(v,x,r\right) C^*\left(v,x,r\right)\right] - C_M I\left(v,x,r\right) \\
&\geq H\left(v,x,r\right)^{-2} \frac{2}{r}\left[E\left(v,x,r\right) H\left(v,x,r\right) - C^*\left(v,x,r\right)^2\right] - C_M I\left(v,x,r\right) \\
&\qquad - C_M H\left(v,x,r\right)^{-1} C^*\left(v,x,r\right) \\
&\geq H\left(v,x,r\right)^{-2} \frac{2}{r}\left[E\left(v,x,r\right) H\left(v,x,r\right) - C^*\left(v,x,r\right)^2\right] \\
&\qquad - C_M I\left(v,x,r\right) - C_M r
\end{aligned}$$

and finally get

$$\begin{aligned}
&I\left(v,x,r\right)' + C_M I\left(v,x,r\right) + C_M r \\
&\qquad \geq \frac{2}{r} H\left(v,x,r\right)^{-2}\left[H\left(v,x,r\right) E\left(v,x,r\right) - C^*\left(v,x,r\right)^2\right] \geq 0
\end{aligned}$$

for every $0 < r < \tilde{r}_M$, $|x| < \tilde{r}_M$, $v \in \mathcal{Q}\left(n,k,\Lambda\right)$ with $I\left(v,6r\right) \leq M$.

Let

$$\begin{aligned}
J_M(v,x,r) &= e^{C_M r} I\left(v,x,r\right) + r e^{C_M r} + C_M^{-1}\left[1 - e^{C_M r}\right] \\
&= e^{C_M r} I\left(v,x,r\right) + V_M(r)
\end{aligned} \tag{3.2.4}$$

then J_M is an increasing function in r with $\lim_{r\to 0^+} J_M(v,r) = \lim_{r\to 0^+} I\left(v,r\right) < +\infty$.

Summarizing all the preceding results I get a monotonicity result for the fixed coefficients frequency function:

Theorem 3.2.4. *For every $M > 0$ exist $\tilde{r}_M$, $C_M > 0$ so that for every $v \in \mathcal{Q}\left(n,k,\Lambda\right)$, $0 < r < \tilde{r}_M$, $|x| < r$ if $I\left(v,6r\right) \leq M$ then let $J_M(v,x,r)$ be as in* (3.2.4) *I have*

$$J_M(v,x,r)' \geq \frac{2}{r} e^{C_M r} H\left(v,x,r\right)^{-2}\left[H\left(v,x,r\right) E\left(v,x,r\right) - C^*\left(v,x,r\right)^2\right] \tag{3.2.5}$$

Notice now that V_M is an increasing convex function with $V_M(0) = 0$ also I have

$$\begin{aligned}
J_M(v,x,r) - J_M(v,x,s) &= e^{C_M s}\left[I\left(v,x,r\right) - I\left(v,x,s\right)\right] \\
&\qquad + \left[e^{C_M r} - e^{C_M s}\right] I\left(v,x,r\right) + \left[V_M(r) - V_M(s)\right]
\end{aligned}$$

I set $\tilde{r}_M$ sufficiently small so that

$$\sup_{r<\tilde{r}_M,|x|<r} C_M e^{C_M r} I(v,x,r) + C_M \tilde{r}_M e^{C_M \tilde{r}_M} \leq 1 \tag{3.2.6}$$

then for $0 < S < s < r < R < \tilde{r}_M$, $|x| < S$ and for $I(v, 6R) \leq M$

$$J_M(v,x,r) - J_M(v,x,s) \leq e^{C_M S} \left[I(v,x,R) - I(v,x,S) \right] + (R - S)$$

and

$$\begin{aligned} e^{C_M s} \left[I(v,x,r) - I(v,x,s) \right] &\leq \left[J_M(v,x,R) - J_M(v,x,S) \right] \\ e^{C_M s} \left| I(v,x,r) - I(v,x,s) \right| &\leq \left[J_M(v,x,R) - J_M(v,x,S) \right] + (R - S) \end{aligned}$$

I can then define

$$\Delta_s^r(v,x) = J_M(v,x,r) - J_M(v,x,s) + (r - s) > 0$$

so that for every $(s,r) \subset\subset (s',r') \subset\subset (s'',r'')$ with $0 < s''$, $r'' < \tilde{r}_M$, $|x| < s''$, $I(v, 6r'') \leq M$

$$\begin{aligned} e^{C_M s} \left| I(v,x,r) - I(v,x,s) \right| &\leq \Delta_{s'}^{r'}(v,x) \\ &\leq e^{C_M s''} \left[I\left(v,x,r''\right) - I\left(v,x,s''\right) \right] + \left(r'' - s''\right) \end{aligned} \tag{3.2.7}$$

3.3 Fixed coefficients frequency function: oscillations

Now I differentiate D and H with respect to the spatial coordinate x to get

$$\begin{aligned} \partial_i H(v,x,r) &= 2r^{1-n} \int_{\mathbb{R}^n} v(y) \partial_i v(y) \, d\rho_{x,r}(y) \\ \partial_i D(v,x,r) &= r^{1-n} \int_{\mathbb{R}^n} |\nabla v|^2 (y_i - x_i) \, d\rho_{x,r}(y) \end{aligned}$$

for every $i = 1, \ldots, n$, I want to approximate $\partial_i D(v,x,r)$ with an integral term that allows us to collect $\partial_i v$ like for $\partial_i H(v,x,r)$. Take so any $\nu \in \mathbb{S}^{n-1}$ I can use proposition 2.2.2 on $|\nabla v|$ to get

$$\begin{aligned} r^{1-n} \int_{\mathbb{R}^n} |\nabla v|^2 \nu \cdot (y - x) \, d\rho_{x,r}(y) &= r^{2-n} \int_{\mathbb{R}^n} \operatorname{div} \left[|\nabla v|^2 \nu \right] \, d\mathbf{\Phi}_{x,r}(y) \\ = -2r^{2-n} \int_{\mathbb{R}^n} \Delta v \nabla v \cdot \nu \, d\mathbf{\Phi}_{x,r}(y) &+ 2r^{1-n} \int_{\mathbb{R}^n} \nabla v \cdot (y - x) \nabla v \cdot \nu \, d\rho_{x,r}(y) \end{aligned}$$

so if I set

$$G(v,x,r) = 2r^{1-n} \int_{\mathbb{R}^n} \nabla v \cdot (y - x) \nabla v \, d\rho_{x,r}(y) \in \mathbb{R}^n$$

then

$$\partial_i D(v,x,r) = -2r^{2-n} \int_{\mathbb{R}^n} \Delta v \partial_i v \, d\mathbf{\Phi}_{x,r} + G(v,x,r) \cdot e_i$$

and when $0 < r < \tilde{r}_M$, $|x| < r$ and $I(v, 6\tilde{r}_M) \leq M$

$$|\nabla D(v,x,r) - G(v,x,r)| \leq C_M D(v,x,r) \tag{3.3.1}$$

Lemma 3.3.1. *For every* $M > 0$ *exist* $C_M, \tilde{r}_M > 0$ *so that for every* $0 < s < r < \tilde{r}_M$, $|x| < s$, $v \in \mathcal{Q}(n,k,\Lambda)$ *with* $I(v, 12r) \leq M$ *I have*

$$r^{1-n} \int_{B_r(x) \setminus B_s(x)} \frac{[\nabla v(y) \cdot (y-x) - I(v,x,s)\, v(y)]^2}{|y-x|}\, dy \\ \leq C_M H(v,x,2r)\, \Delta_s^{2r}(v,x)$$

Before to prove lemma 3.3.1 I give a proof of an elementary inequality:

Proposition 3.3.2. *for every* $0 < s < r$ *and every* $g \geq 0$ *measurable function I have*

$$\int_s^r g(t)\ dt \leq \frac{1}{s} \int_s^{2r} dt \int_{t/2}^t g(q)\ dq \tag{3.3.2}$$

Proof. The integration by parts formula leads to the following relations

$$\begin{aligned}\int_s^{2r} dt \int_{t/2}^t g(q)\ dq &= 2r \int_r^{2r} g(t)\ dt - s \int_{s/2}^s g(t)\ dt - \int_s^{2r} t \left[g(t) - \frac{1}{2} g\left(\frac{t}{2}\right)\right]\ dt \\ &= \int_r^{2r} (2r-t) g(t)\ dt + \int_{s/2}^s (2t-s) g(t)\ dt + \int_s^r t g(t)\ dt \geq s \int_s^r g(t)\ dt\end{aligned}$$

since $g(t) \geq 0$. □

Proof of lemma 3.3.1. Let

$$K(v,x,r) = r^{1-n} \int_{\mathbb{R}^n} [\nabla v(y) \cdot (y-x) - I(v,x,r)\, v(y)]^2\ d\rho_{x,r}(y),$$

I immediately get

$$\begin{aligned}K(v,x,r) &= E(v,x,r) - 2I(v,x,r)\, C^*(v,x,r) + I(v,x,r)^2 H(v,x,r) \\ &= H(v,x,r)^{-1} \left[E(v,x,r)\, H(v,x,r) - 2D(v,x,r)\, C^*(v,x,r) + D(v,x,r)^2\right] \\ &= H(v,x,r)^{-1} \left[E(v,x,r)\, H(v,x,r) - C^*(v,x,r)^2\right] \\ &\quad + H(v,x,r)^{-1} [C^*(v,x,r) - D(v,x,r)]^2 .\end{aligned}$$

I can then use (3.2.5) to get

$$K(v,x,r) \leq \frac{r}{2} H(v,x,r)\, e^{-C_M r} J_M(v,x,r)' + C_M r^2 H(v,x,r)$$

and (3.3.2) to get

$$\int_{B_r(x) \setminus B_s(x)} f(y)\ dy \leq \frac{1}{2s} \int_s^{2r} dt \int_{\mathbb{R}^n} |y-x|\, f(y)\ d\rho_{x,t}(y)$$

if $f \geq 0$ on $B_{2r}(x)$ and $0 < s < r$. Since $1-n<0$ I get $r^{1-n} \leq Ct^{1-n}$ for every $t \leq 2r$ and so if $|x| < s$

$$\begin{aligned}
r^{1-n} & \int_{B_r(x)\setminus B_s(x)} \frac{\left[\nabla v(y)\cdot(y-x) - I\left(v,x,s\right)v(y)\right]^2}{|y-x|}\,dy \\
&\leq s^{-1}\int_s^{2r} K\left(v,x,t\right)\,dt \\
&+ r^{1-n}s^{-1}\int_s^{2r} dt \int_{\mathbb{R}^n} \left[I\left(v,x,t\right) - I\left(v,x,s\right)\right]^2 |v(y)|^2 \, d\rho_{x,t}\left(y\right) \\
&\leq C_M s^{-1}\int_s^{2r} \frac{t}{2} H\left(v,x,t\right) J_M(v,x,t)'\,dt + C_M s^{-1}\int_s^{2r} t^2 H\left(v,x,t\right)\,dt \\
&+ s^{-1}\Delta_s^{2r}\left(v,x\right)^2 \int_s^{2r} H\left(v,x,t\right)\,dt
\end{aligned}$$

Now set $r/8 \leq s \leq r$ and $|x| < r/8$ I immediately get $s^{-1} \leq Cr^{-1}$ the preceding term is controller by $C_M H\left(v,x,2r\right)\Delta_s^{2r}(v,x)$ and the statement is so proved. □

I now use lemma 3.3.1 to prove the following theorem:

Theorem 3.3.3 (Oscillation of frequency)**.** *For every $M > 0$ and for every $K > 6$ exist $C_{M,K}, \tilde{r}_{M,K} > 0$ so that for every $v \in \mathcal{Q}\left(n,k,\Lambda\right)$, $0 < r < \tilde{r}_{M,K}$, $x, y \in \mathcal{K}_v \cap B_r$ if $I\left(v, 12Kr\right) \leq M$ then*

$$\begin{aligned}
&\left|I\left(v,y,Kr\right) - I\left(v,x,Kr\right)\right| \\
&\qquad\qquad \leq C_M\left|y-x\right| + C_{M,K}\left[\sqrt{\Delta^{2(2K+2)r}_{(K/2-2)r}(x)} + \sqrt{\Delta^{2(2K+2)r}_{(K/2-2)r}(y)}\right]
\end{aligned}$$

Consider now $x, y \in \mathcal{K}_v \cap B_r$ with $x \neq y$, let $\eta = y - x$ and a any point in the closed segment $[x, y]$. For every $z \in \Omega$ and every $t > 0$ let

$$\mathcal{E}_{a,t}\left(z\right) = \nabla v(z)\cdot(z-a) - I\left(v,a,t\right)v(z) \qquad \Delta I_t = I\left(v,y,t\right) - I\left(v,x,t\right)$$

$$\Delta\mathcal{E}_t\left(z\right) = \mathcal{E}_{y,t}\left(z\right) - \mathcal{E}_{x,t}\left(z\right) \qquad \partial_{a,r}f(z) = \nabla f(z)\cdot\frac{z-a}{|z-a|}$$

lemma 3.3.1 implies in particular that

$$r^{1-n}\int_{B_r(a)\setminus B_s(a)} \frac{\left|\mathcal{E}_{a,s}\left(z\right)\right|^2}{|z-a|}\,dz \leq C_M H\left(v,a,2r\right)\Delta_s^{2r}\left(v,a\right)$$

when $|x|, |y| < s$.

It's simple to show that

$$\begin{aligned}
\nabla v(z)\cdot\eta =& -\nabla v(z)\cdot(z-y)+\nabla v(z)\cdot(z-x)\\
=& -\nabla v(z)\cdot(z-y)+I\left(v,y,t\right)v(z)\\
&+\nabla v(z)\cdot(z-x)-I\left(v,x,t\right)v(z)\\
&+\left[I\left(v,x,t\right)-I\left(v,y,t\right)\right]v(z)\\
=& -\Delta\mathcal{E}_t\left(z\right)-\Delta I_t v(z)\\
\partial_\eta H\left(v,a,t\right) =& \, 2t^{1-n}\int_{\mathbb{R}^n} v(z)\nabla v(z)\cdot\eta\ d\rho_{a,t}\left(z\right)\\
=& -2\Delta I_t H\left(a,t\right)-2t^{1-n}\int_{\mathbb{R}^n} v(z)\Delta\mathcal{E}_t\left(z\right)\ d\mathbf{\Phi}_{a,t}\left(z\right)\\
G\left(v,a,t\right)\cdot\eta =& \, 2t^{1-n}\int_{\mathbb{R}^n}\nabla v(z)\cdot(z-a)\nabla v(z)\cdot\eta\ d\rho_{a,t}\left(z\right)\\
=& -2\Delta I_t C^*\left(a,t\right)-2t^{1-n}\int_{\mathbb{R}^n}\nabla v(z)\cdot(z-a)\Delta\mathcal{E}_t\left(z\right)\ d\rho_{a,t}\left(z\right)
\end{aligned}$$

and so

$$\begin{aligned}
\partial_\eta I\left(v,a,t\right) &= H\left(v,a,t\right)^{-2}\left[\partial_\eta D\left(v,a,t\right)H\left(v,a,t\right)-\partial_\eta H\left(v,a,t\right)D\left(v,a,t\right)\right]\\
&= H\left(v,a,t\right)^{-1}\left[\partial_\eta D\left(v,a,t\right)-\partial_\eta H\left(v,a,t\right)I\left(v,a,t\right)\right]\\
&= H\left(v,a,t\right)^{-1}\left[\nabla D\left(v,a,t\right)-G\left(v,a,t\right)\right]\cdot\eta\\
&\quad-2H\left(v,a,t\right)^{-1}\Delta I_t\left[C^*\left(v,a,t\right)-H\left(v,a,t\right)I\left(v,a,t\right)\right]\\
&\quad-2H\left(v,a,t\right)^{-1}t^{1-n}\int_{\mathbb{R}^n}\left[\nabla v(z)\cdot(z-a)-I\left(v,a,t\right)v(z)\right]\Delta\mathcal{E}_t\left(z\right)\ d\rho_{a,t}\left(z\right)
\end{aligned}$$

If $t<\tilde{r}_M$, $|a|<t$ then by (3.2.3), (3.3.1) I get

$$\left|H\left(v,a,t\right)^{-1}\left[\nabla D\left(v,a,t\right)-G\left(v,a,t\right)\right]\cdot\eta\right|\leq C_M I\left(v,a,t\right)|x-y|$$

$$\left|H\left(v,a,t\right)^{-1}\Delta I_t\left[C^*\left(v,a,t\right)-D\left(v,a,t\right)\right]\right|\leq C_M t\left|\Delta I_t\right|$$

I set now $K>6$, $t=Kr$, $\tilde{r}_{M,K}\leq\tilde{r}_M/K$ and $|x|,|y|<r<\tilde{r}_{M,K}$. For every $z\in\Omega$ such that $|z-a|<Kr$ I have $|z|<(K+1)r$, so for $\tilde{r}_M$ sufficiently small

$$\begin{aligned}
&\left|\nabla v(z)\cdot(z-a)-I\left(v,a,Kr\right)v(z)\right|\\
&\qquad\leq Kr\left|\nabla v(z)\right|+C_M\left|v(z)\right|\leq C_M\sqrt{H\left(v,2(K+1)r\right)}.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\left|2H\left(v,a,t\right)^{-1}t^{1-n}\int_{\mathbb{R}^n}\left[\nabla v(z)\cdot(z-a)-I\left(v,a,t\right)v(z)\right]\Delta\mathcal{E}_t\left(z\right)\ d\rho_{a,t}\left(z\right)\right|\\
&\quad\leq 2\frac{\sqrt{H\left(v,2(K+1)r\right)}}{H\left(v,a,Kr\right)}(Kr)^{1-n}\int_{B_{Kr}(a)\setminus B_{Kr/2}(a)}\frac{\left|\mathcal{E}_{y,Kr}\left(z\right)\right|+\left|\mathcal{E}_{x,Kr}\left(z\right)\right|}{|z-a|}\ dz\\
&\quad\leq 2\frac{\sqrt{H\left(v,2(K+1)r\right)}}{H\left(v,a,Kr\right)}2(Kr)^{-n}\int_{B_{Kr}(a)\setminus B_{Kr/2}(a)}\left[\left|\mathcal{E}_{y,Kr}\left(z\right)\right|+\left|\mathcal{E}_{x,Kr}\left(z\right)\right|\right]\ dz
\end{aligned}$$

Notice now that since $|u-a|-2r \leq |u-x| \leq |u-a|+2r$ I immediately get

$$B_{Kr}(a) \setminus B_{Kr/2}(a) \subseteq B_{(K+2)r}(x) \setminus B_{(K/2-2)r}(x)$$

and the same holds for y. Then

$$\begin{aligned}\int_{B_{Kr}(a)\setminus B_{Kr/2}(a)} |\mathcal{E}_{x,Kr}(z)|\ dz &\leq (K+2)r \int_{B_{(K+2)r}(x)\setminus B_{(K/2-2)r}(x)} \frac{|\mathcal{E}_{x,Kr}(z)|}{|z-x|}\ dz \\ &\leq (K+2)r\,\mathfrak{R}\sqrt{\int_{B_{(K+2)r}(x)\setminus B_{(K/2-2)r}(x)} \frac{|\mathcal{E}_{x,Kr}(z)|^2}{|z-x|}\ dz} \\ &\leq C_{M,K} r^{(n+1)/2} r^{(n-1)/2} \sqrt{H(x, 2(2K+2)r)} \sqrt{\Delta_{(K/2-2)r}^{2(2K+2)r}(x)}\end{aligned}$$

where

$$\mathfrak{R} = \sqrt{\int_{B_{(K+2)r}(x)\setminus B_{(K/2-2)r}(x)} \frac{1}{|z-x|}\ dz} \leq C_K r^{(n-1)/2}$$

notice also that by proposition 3.2.1

$$\frac{\sqrt{H(v, 2(K+1)r)\, H(v, x, 2(2K+2)r)}}{H(v, a, Kr)} \leq C_{M,K}$$

I finally get our oscillation estimate

$$\begin{aligned}&|\nabla I(v, a, Kr) \cdot \eta| \leq C_M I(v, a, Kr)\, |y-x| \\ &+ C_M r\, |I(v, y, Kr) - I(v, x, Kr)| + C_{M,K}\left[\sqrt{\Delta_{(K/2-2)r}^{2(2K+2)r}(x)} + \sqrt{\Delta_{(K/2-2)r}^{2(2K+2)r}(y)}\right]\end{aligned}$$

by setting $C_M \tilde{r}_{M,K} \leq 1/2$.

3.4 Intrinsic frequency function

So far I've assumed that $v \in \mathcal{Q}(n,k,\Lambda)$ but in general for any two-valued symmetric $C^{1,1/2}$ function v with stationary graph and any $x \in \mathcal{K}_v$ I don't have $A(x) = I$ and so I can't immediately apply the results of preceding sections. In this section I introduce a third frequency function that is similar to the second one but works for any $v \in \mathcal{WQ}(n,k,\Lambda)$.

Let v be any function in $\mathcal{WQ}(n,k,\Lambda)$ and A, L^{ij} define as in the very definition of $\mathcal{WQ}(n,k,\Lambda)$. Take any point $x_0 \in \mathcal{K}_v$ I want to transform v into another function $v_{x_0} \in \mathcal{Q}(n,k,C\Lambda)$ where C is a positive constant depending only by $n,k,\tilde{\Lambda}$ with $\Lambda < \tilde{\Lambda}$.

Let

$$\begin{aligned}\Psi_{x_0}(y) &= A^{-1/2}(x_0)(y-x_0) \\ \Omega_{x_0} &= \Psi_{x_0}(\Omega) \\ v_{x_0} &: y \in \Omega_{x_0} \to v\left[\Psi_{x_0}^{-1}(y)\right] \\ A_{x_0}(y) &= A^{-1/2}(x_0) A\left[\Psi_{x_0}^{-1}(y)\right] A^{-1/2}(x_0) \\ L_{x_0}^{ij}(y) &= A^{-1/2}(x_0) L^{ij}\left[\Psi_{x_0}^{-1}(y)\right]\end{aligned} \tag{3.4.1}$$

It's simple to prove that div $\left(A_{x_0}\nabla v^i_{x_0}\right)+\sum_{j=1}^n L^{ij}_{x_0}\cdot\nabla v^j_{x_0}=0$ weakly on $\Omega_{x_0}\setminus\mathcal{K}_{v_{x_0}}$, $v_{x_0}\in C^2$ on $\Omega_{x_0}\setminus\mathcal{K}_{v_{x_0}}$, $0\in\mathcal{K}_{v_{x_0}}$ and $A_{x_0}(0)=I$. Notice also that

$$(1+\Lambda)^{-1}|\nu|^2\le|A_{x_0}(y)\nu\cdot\nu|\le(1+\Lambda)|\nu|^2$$

and consequently

$$\begin{aligned}\left|L^{ij}_{x_0}(y)\right|&\le(1+\Lambda)^{1/2}\Lambda\\ |A_{x_0}(y)-A_{x_0}(x)|&=\left|A^{-1/2}(x_0)\left[A\left[\Psi^{-1}_{x_0}(y)\right]-A\left[\Psi^{-1}_{x_0}(x)\right]\right]A^{-1/2}(x_0)\right|\\ &\le(1+\Lambda)\Lambda\left|\Psi^{-1}_{x_0}(y)-\Psi^{-1}_{x_0}(x)\right|\le(1+\Lambda)^{3/2}\Lambda|y-x|\end{aligned}$$

and so $v_{x_0}\in\mathcal{Q}(n,k,C\Lambda)$ for some constant C depending only on n,k and $\tilde{\Lambda}$ where $\tilde{\Lambda}>\Lambda>0$.

I can then define the following functions for every $v\in\mathcal{WQ}(n,k,\Lambda)$, $x\in\mathcal{K}_v$

$$\begin{aligned}\mathcal{H}(v,x,r)&=H(v_x,0,r)\\ \mathcal{D}(v,x,r)&=D(v_x,0,r)\\ \mathcal{I}(v,x,r)&=I(v_x,0,r)\end{aligned}$$

in particular $\mathcal{I}(v,x,r)$ is the intrinsic frequency function of v centered at x. I now want to extend our oscillation result to this new frequency function.

For every $x,y\in\mathcal{K}_v$ notice that $v(y)=v_y(0)=v_x[\Psi_x(y)]$ and so

$$\mathcal{I}(v,x,r)-\mathcal{I}(v,y,r)=[I(v_x,r)-I(v_x,\Psi_x(y),r)]+[I(v_x,\Psi_x(y),r)-I(v_y,r)]$$

since $|\Psi_x(y)|\le(1+\Lambda)^{-1/2}|y-x|$ when $|y-x|<r$ I can use the preceding oscillation bound to estimate $I(v_x,r)-I(v_x,\Psi_x(y),r)$, the aim of this section is to estimate $I(v_x,\Psi_x(y),r)-I(v_y,r)$.

Let

$$\begin{aligned}|\nu|^2_x&=A^{-1}(x)\nu\cdot\nu\\ El^x_r(y)&=\Psi^{-1}_x\left[B_r(\Psi_x(y))\right]\end{aligned}$$

and notice that $z\in El^x_r(y)$ if and only if $|z-y|_x<r$. Also for every $x,y\in\Omega$, $\nu\in\mathbb{R}^n\setminus\{0\}$ I have $|\nu|^2_y\ge(1+\Lambda)^{-1}|\nu|^2$ and so

$$\begin{aligned}\left||\nu|_x-|\nu|_y\right|&\le\frac{1}{2(1+\Lambda)^{-1/2}|\nu|}\left|\left[A(x)^{-1}-A(y)^{-1}\right]\nu\cdot\nu\right|\\ &\le\frac{\left|A^{-1}(x)\right|\left|A^{-1}(y)\right||A(x)-A(y)|}{2(1+\Lambda)^{-1/2}}|\nu|\\ &\le\frac{\Lambda}{2}(1+\Lambda)^{3/2}|x-y||\nu|\\ &=J|x-y||\nu|\end{aligned}$$

in particular for every $t>|x-y|$ there exists constants

$$\begin{aligned}a_t&=(1+\Lambda)Jt\in(0,+\infty)\\ b_t&=\frac{(1+\Lambda)Jt}{1+(1+\Lambda)Jt}\in(0,1)\end{aligned}$$

such that

$$El^{y}_{(1-b_t)r}(z) \subseteq El^{x}_{r}(z) \subseteq El^{y}_{(1+a_t)r}(z)$$

Remember I've assumed $\operatorname{diam}(\Omega) \leq 1$, then I can set $t = 1$ and $a_t \leq a_1, b_t \leq b_1$.

Next I obtain

$$\begin{aligned} D(v_x, \Psi_x(y), r) &= r^{2-n} \int_{\mathbb{R}^n} \left| \nabla \Psi_x(z)^{-1} \nabla v \left[\Psi_x^{-1}(z) \right] \right|^2 \, d\mathbf{\Phi}_{\Psi_x(y),r}(z) \\ &= r^{2-n} \int_{\mathbb{R}^n} \phi\left(\frac{|z - \Psi_x(y)|}{r} \right) \left| \nabla \Psi_x(z)^{-1} \nabla v \left[\Psi_x^{-1}(z) \right] \right|^2 \, dz \\ &= r^{2-n} \int_{\mathbb{R}^n} \phi\left(\frac{|u-y|_x}{r} \right) \left| A^{1/2}(x) \nabla v(u) \right|^2 |\det \nabla \Psi_x(u)| \, du \\ &= r^{2-n} \int_{\mathbb{R}^n} A(x) \nabla v(u) \cdot \nabla v(u) \, d\mathbf{\Phi}^x_{y,r}(u) \end{aligned}$$

where

$$\mathbf{\Phi}^x_{y,r}(U) = \left(\Psi_x^{-1} \right)^{\#} \mathbf{\Phi}_{\Psi_x(y),r}(U) = \int_U \phi\left(\frac{|u-y|_x}{r} \right) \frac{1}{\sqrt{\det A(x)}} \, du$$

has support in $El^x_r(y)$. In the same way I get also

$$H(v_x, \Psi_x(y), r) = r^{1-n} \int_{\mathbb{R}^n} \frac{|v(u)|^2}{|u-y|_x} \, d\rho^x_{y,r}(u)$$

where $\rho^x_{y,r}(U) = \left(\Psi_x^{-1} \right)^{\#} \rho_{\Psi_x(y),r}(U)$ has instead support in $El^x_r(y) \setminus El^x_{r/2}(y)$.

Since ϕ is a Lipschitz function I get for every $F \geq 0$ measurable and every $1 > t > 0$ with $|y - x| < t$

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} \phi\left(\frac{|z-y|_x}{r} \right) f(z) \, dz - \int_{\mathbb{R}^n} \phi\left(\frac{|z-y|_y}{r} \right) f(z) \, dz \right| \\ &\qquad \leq \frac{J}{2}(1+\Lambda) |x-y| \int_{El^y_{(1+a_t)r}(y)} \frac{|z-y|_y}{r} f(z) \, dz \\ &\qquad \leq C |x-y| (1+a_t) \int_{\mathbb{R}^n} \phi\left(\frac{|z-y|_y}{2(1+a_t)r} \right) f(z) \, dz \\ &\qquad = C |x-y| (1+a_t) \int_{\mathbb{R}^n} f(z) \sqrt{\det A(y)} \, d\mathbf{\Phi}^y_{y,2(1+a_t)r}(z) \end{aligned}$$

For t sufficiently small I then immediately get

$$|D(v_x, \Psi_x(y), r) - D(v_y, 0, r)| \leq C |y-x| D(v_y, 4r)$$

if $\mathcal{I}(v, y, 4r) \leq M$ then I get $D(v_y, 4r) \leq C_M D(v_y, r)$ when $|y - x| < 1/C$ and $r < \tilde{r}$.

In order to estimate $H(v_x, \Psi_x(y), r) - H(v_y, 0, r)$ I need some extra step. First to all for every $0 < S < R$

$$\begin{aligned} &\left[El^x_{Rr}(y) \setminus El^x_{Sr}(y) \right] \; \Delta \; \left[El^y_{Rr}(y) \setminus El^y_{Sr}(y) \right] \\ &\qquad \subseteq \left(El^y_{(1+a_t)Rr}(y) \setminus El^y_{(1-b_t)Rr}(y) \right) \cup \left(El^y_{(1+a_t)Sr}(y) \setminus El^y_{(1-b_t)Sr}(y) \right) \end{aligned}$$

with again $t = |y - x|$. Notice now that $a_t + b_t \leq Ct$ so the measure of symmetric difference of these annuli goes to 0 as tr^n for fixed R, S.

Using the coarea formula I get

$$\int_{El^x_{Rr}(y)\setminus El^x_{Sr}(y)} f(z)\, dz = \int_{Sr}^{Rr} dt \int_{|z-y|_x = t} f(z) \frac{t}{\left|A^{-1/2}(x)(z-y)\right|_x}\, d\mathcal{H}^{n-1}(z)$$

and so

$$\begin{aligned} 2r^{1-n}\Bigg| \int_{El^x_r(y)\setminus El^x_{r/2}(y)} \frac{|v(z)|^2}{|z-y|_x} \frac{1}{\sqrt{\det A(x)}}\, dz \\ - \int_{El^y_r(y)\setminus El^y_{r/2}(y)} \frac{|v(z)|^2}{|z-y|_x} \frac{1}{\sqrt{\det A(x)}}\, dz \Bigg| \\ \leq Cr^{1-n} \left| \int_{El^y_{(1+a_t)r}(y)\setminus El^y_{(1-b_t)r}(y)} \frac{|v(z)|^2}{|z-y|_y} \frac{1}{\sqrt{\det A(y)}}\, dz \right| \\ + Cr^{1-n} \left| \int_{El^y_{(1+a_t)r/2}(y)\setminus El^y_{(1-b_t)r/2}(y)} \frac{|v(z)|^2}{|z-y|_y} \frac{1}{\sqrt{\det A(y)}}\, dz \right| \end{aligned}$$

Assuming $y \in \mathcal{K}_v$, $t < 1$ and $\mathcal{I}(v, y, 4r) \leq M$ I get

$$\begin{aligned} r^{1-n} \int_{El^x_{(1+a_t)r}(y)\setminus El^x_{(1-b_t)r}(y)} \frac{|v(z)|^2}{|z-y|_y} \frac{1}{\sqrt{\det A(y)}}\, dz \\ \overset{(3.1.7)}{\leq} Ctr^{2-n} \int_{\mathbb{R}^n} \phi_{1-b_t, 1+a_t}\left(\frac{|z-y|_y}{r}\right) \frac{A(y)\nabla v(z)\cdot \nabla v(z)}{\sqrt{\det A(y)}}\, dz \\ \leq Ctr^{2-n} \int_{\mathbb{R}^n} A(y)\nabla v(z)\cdot\nabla v(z)\, d\mathbf{\Phi}^y_{y,2(1+a)r}(z) \\ \leq Ct\mathcal{D}(v, y, 2(1+a)r) \leq C_M t\mathcal{H}(v, y, r) \end{aligned}$$

which implies $|H(v_x, \Psi_x(y), r) - H(v_y, 0, r)| \leq C_M |y - x| H(v_y, 0, r)$. In particular, if $|y - x| \leq 1/(2C_M)$, then I get

$$\frac{1}{2} H(v_y, 0, r) \leq H(v_x, \Psi_x(y), r) \leq \frac{3}{2} H(v_y, 0, r) \tag{3.4.2}$$

Finally I can apply (3.4.2) to get

$$\begin{aligned} |I(v_x, \Psi_x(y), r) - I(v_y, 0, r)| \\ = \left| \left(\frac{D(v_x, \Psi_x(y), r)}{D(v_y, 0, r)} - \frac{H(v_x, \Psi_x(y), r)}{H(v_y, 0, r)} \right) \frac{D(v_y, 0, r)}{H(v_x, \Psi_x(y), r)} \right| \\ \leq C_M |y - x| I(v, y, r) \end{aligned}$$

which proves the following result:

Lemma 3.4.1. *For every $M > 0$ exist $\tilde{r}_M, C_M > 0$ so that for every $v \in \mathcal{WQ}(n, k, \Lambda)$, every $0 < r < \tilde{r}_M$, every $x, y \in \mathcal{K}_v$ with $|x - y| < 1/(2C_M)$ and $\mathcal{I}(v, y, 4r) \leq M$ I get*

$$|I(v_x, \Phi_x(y), r) - I(v_y, 0, r)| \leq C_M |x - y| \mathcal{I}(v, y, r) \tag{3.4.3}$$

Let $\mathcal{J}_M(v,x,r) = J_M(v_x,0,r)$, by using (3.4.3) I immediately get

$$\begin{aligned}
&J_M(v_x,\Psi_x(y),r) - J_M(v_x,\Psi_x(y),s) = \mathcal{J}_M(v,y,r) - \mathcal{J}_M(v,y,s)\\
&\quad + e^{C_M r}\left[I(v_x,\Psi_x(y),r) - I(v_y,0,r)\right] - e^{C_M s}\left[I(v_x,\Psi_x(y),s) - I(v_y,0,s)\right]\\
&\qquad\qquad \leq \mathcal{J}_M(v,y,r) - \mathcal{J}_M(v,y,s) + C_M|x-y|\,\mathcal{I}(v,y,r)
\end{aligned}$$

and get

$$\Delta_s^r[v_x,\Psi_x(y)] \leq \Delta_s^r(v_y,0) + C_M|x-y| \tag{3.4.4}$$

so I can set $\Delta_s^r(v,y) = \Delta_s^r(v_y,0)$ for every $v \in \mathcal{WQ}(n,k,\Lambda)$, $y \in \mathcal{K}_v$. I can then rewrite (3.2.7) in this intrinsic form:

$$\begin{aligned}
&e^{C_M s}\left|\mathcal{I}(v,x,r) - \mathcal{I}(v,x,s)\right| \leq \Delta_s^r(v,x)\\
&\qquad\qquad \leq e^{C_M s}\left[\mathcal{I}(v,x,r) - \mathcal{I}(v,x,s)\right] + (r-s)
\end{aligned}$$

Theorem 3.4.2 (Oscillation of frequency-intrinsic version)**.** *For every $M>0$ and for every $K>6$ exist $C_{M,K},\tilde{r}_{M,K}>0$ such that for every $v \in \mathcal{WQ}(n,k,\Lambda)$, $0 < r < \tilde{r}_{M,K}$, $x,y \in \mathcal{K}_v$ with $|x-y|_x, |x-y|_x < r$ if $\mathcal{I}(v,x,12Kr), \mathcal{I}(v,y,12Kr) \leq M$ then*

$$\begin{aligned}
&\left|\mathcal{I}(v,y,Kr) - \mathcal{I}(v,x,Kr)\right|\\
&\qquad \leq C_{M,K}\left[|y-x|_x + |y-x|^{1/2} + \sqrt{\Delta_{(K/2-2)r}^{2(2K+2)r}(v,x)} + \sqrt{\Delta_{(K/2-2)r}^{2(2K+2)r}(v,y)}\right]
\end{aligned}$$

Proof. Since $|\Psi_x(y)| = |y-x|_x$ I can apply theorem 3.3.3 to v_x to get

$$\begin{aligned}
&\left|I(v_x,0,Kr) - I(v_x,\Psi_x(y),Kr)\right|\\
&\qquad \leq C_{M,K}\left[|y-x|_x + \sqrt{\Delta_{(K/2-2)r}^{2(2K+2)r}(v_x,0)} + \sqrt{\Delta_{(K/2-2)r}^{2(2K+2)r}(v_x,\Psi_x(y))}\right]
\end{aligned}$$

then the preceding inequalities complete the proof. □

3.5 Proofs of main theorems

In this section I explain the proof of theorem 1.0.1. Before to prove it I need to define first the *mean flatnees* of a Radon measure in order to estimate "how much its support differs from an affine subspace".

Definition 3.5.1. Let μ be any Radon measure on $\mathbb{R}^n$, I define the mean flatness of μ in the the following way

$$\beta_{x,r}(\mu) = \inf_W \sqrt{r^{-n}\int_{B_r(x)} \operatorname{dist}(y,W)^2\, d\mu(y)}$$

where the infimum is evaluated among every affine subspace W of $\mathbb{R}^n$.

Notice that if $0 < r < t$ then

$$\beta_{x,r}(\mu) \le \left(\frac{t}{r}\right)^{n/2} \beta_{x,t}(\mu)$$

and

$$\begin{aligned}\int_{B_r} \operatorname{dist}(y,W)^2 \; d\Psi_x^{\#}\mu(y) &= \int_{El_r^x(x)} \operatorname{dist}[\Psi_x(y),W]^2 \; d\mu(y) \\ &\ge \frac{1}{1+\Lambda}\int_{B_{(1+\Lambda)^{-1/2}r}(x)} \operatorname{dist}\left[y, x + A^{1/2}(x)W\right]^2 \; d\mu(y)\end{aligned}$$

which implies

$$\beta_{x,r}(\mu) \le (1+\Lambda)^{(n+2)/4}\beta_{0,\sqrt{1+\Lambda}r}\left(\Psi_x^{\#}\mu\right)$$

The mean flatness $\beta_{x,r}(\mu)$ can be expressed also in a form that takes into account eigenvalues of some symmetric bilinear operator. Let

$$\begin{aligned} m &= \frac{1}{\tilde{\mu}(B_r)}\int_{B_r} y \; d\tilde{\mu}(y) \\ B(\nu,\eta) &= \int_{B_r} [(y-m)\cdot\nu]\,[(y-m)\cdot\eta] \; d\tilde{\mu}(y)\end{aligned}$$

and let λ_i be the eigenvalues of $B_{\mu,x,r}$ in nondecreasing order, it's simple to show that

$$\beta_{0,r}(\tilde{\mu}) = r^{-\frac{n}{2}}\sqrt{\lambda_1+\lambda_2}\;.$$

This new formulation of the mean flatness allows me to prove the fundamental estimate contained in the following theorem

Theorem 3.5.2. *For every $M > 0, R > L$, where L is a positive constant depending only on n, k and Λ, exist $C_{M,R}, \tilde{r}_{M,R} > 0$, $6 < B_R < A_R$ so that for every $v \in \mathcal{WQ}(n,k,\Lambda)$, $x \in \mathcal{K}_v$, $0 < r < \tilde{r}_{M,R}$ that satisfy $\mathcal{I}(v,x,6A_R r) \le M$ and for every μ Radon measure with* $\operatorname{supp}\mu \subseteq \mathcal{K}_v$ *I have*

$$\beta_{0,r}\left(\Psi_x^{\#}\mu\right)^2 \le \frac{C_{M,R}}{r^{n-2}}\int_{El_r^x(x)}\left[\Delta_{B_R r}^{A_R r}(v,z) + r\right] \; d\mu(z)$$

where $\Delta_a^b(v,z) = \Delta_a^b(v_z,0)$.

To prove this result let ν_i be the eigenvector related to eigenvalue λ_i and let $\tilde{\mu} = \Psi_x^{\#}\mu$. I immediately get

$$\begin{aligned} -\lambda_i\nu_i\cdot\nabla v_x(z) &= \int_{B_r}[(y-m)\cdot\nu_i]\,[(z-y)\cdot\nabla v_x(z) - p v_x(z)] \; d\tilde{\mu}(y) \\ &= \int_{El_r^x(x)}[(\Psi_x(y)-m)\cdot\nu_i]\,[(z-\Psi_x(y))\cdot\nabla v_x(z) - p v_x(z)] \; d\mu(y)\end{aligned}$$

for every z in the domain of v_x and $p \in \mathbb{R}$. Hence,

$$\begin{aligned}
&\lambda_i^2 \left|\nu_i \cdot \nabla v_x(z)\right|^2 \\
&\quad \leq \left[\int_{El_r^x(x)} \left[(\Psi_x(y) - m)\cdot \nu_i\right]\left[(z - \Psi_x(y))\cdot \nabla v_x(z) - p v_x(z)\right]\, d\mu\,(y)\right]^2 \\
&\quad \leq \lambda_i \int_{El_r^x(x)} \left[(z - \Psi_x(y))\cdot \nabla v_x(z) - p v_x(z)\right]^2\, d\mu\,(y)
\end{aligned} \tag{3.5.1}$$

I define the tangential differential of v_x in the following way

$$\nabla^T v_x(z) = \sum_{l=1,2} \left[\!\left[\nabla\,(v_x)_l - \partial_{\nu_1}(v_x)_l \otimes \nu_1\right]\!\right] = \sum_{j=2}^{n}\sum_{l=1,2} \left[\!\left[\partial_{\nu_j}(v_x)_l \otimes \nu_j\right]\!\right]. \tag{3.5.2}$$

Next I apply it to (3.5.1) in order to get

$$\begin{aligned}
&r^n \beta_{0,r}\,(\tilde{\mu})^2 \int_{B_{(R+1)r}\setminus B_{Rr}} \left|\nabla^T v_x(z)\right|^2\, dz \\
&\qquad = \lambda_1 \int_{B_{(R+1)r}\setminus B_{Rr}} \left|\nabla^T v_x(z)\right|^2\, dz + \lambda_2 \int_{B_{(R+1)r}\setminus B_{Rr}} \left|\nabla^T v_x(z)\right|^2\, dz \\
&\qquad\qquad \leq 2\sum_{i=2}^{n} \lambda_i \int_{B_{(R+1)r}\setminus B_{Rr}} \left|\nabla v_x(z)\cdot v_i\right|^2\, dz \\
&\qquad = 2n \int_{B_{(R+1)r}\setminus B_{Rr}} \int_{El_r^x(x)} \left[(z - \Psi_x(y))\cdot \nabla v_x(z) - p v_x(z)\right]^2\, d\mu\,(y)\, dz
\end{aligned}$$

Observe now that for every $y \in B_r(x)$ I've $|y-x|_x, |y-x|_y < Lr$ for some constant L depending only on n, k, Λ, thus

$$\begin{aligned}
El^x_{(R+1)r}(x) &\subseteq El^x_{(R+1+L)r}(y) \subseteq El^y_{(1+a_t)(R+1+L)r}(y) \\
El^x_{Rr}(x) &\supseteq El^x_{(R-L)r}(y) \supseteq El^y_{(1-b_t)(R-L)r}(y)
\end{aligned}$$

if $|y - x| \leq t$.

For the sake of clarity I set $\hat{A}_{t,R} = (1+a_t)(R+1+L)$, $\hat{B}_{t,R} = (1-b_t)(R-L)$ when $R > L$. Consequently, for every $y \in El_r^x(x) \cap \mathcal{K}_v$

$$\begin{aligned}
&\int_{B_{(R+1)r}\setminus B_{Rr}} \left[(z - \Psi_x(y))\cdot \nabla v_x(z) - p v_x(z)\right]^2\, dz \\
&\quad \leq \int_{El^y_{\hat{A}_{t,R}r}(y)\setminus El^y_{\hat{B}_{t,R}r}(y)} \left[[w - x - (y-x)]\cdot \nabla v(w) - p v(w)\right]^2 \det A(x)^{-1/2}\, dw \\
&\qquad = \int_{B_{\hat{A}_{t,R}r}\setminus B_{\hat{B}_{t,R}r}} \left[z\cdot \nabla v_y(z) - p v_y(z)\right]^2 \det\left[A(x)^{-1/2}A(y)^{1/2}\right]\, dz \\
&\leq 2\int_{B_{\hat{A}_{t,R}r}\setminus B_{\hat{B}_{t,R}r}} \left[z\cdot \nabla v_y(z) - I\left(v_y, \hat{B}_{t,R}r\right) v_y(z)\right]^2 \det\left[A(x)^{-1/2}A(y)^{1/2}\right]\, dz \\
&\qquad + 2\int_{B_{\hat{A}_{t,R}r}\setminus B_{\hat{B}_{t,R}r}} \left[I\left(v_y, \hat{B}_{t,R}r\right) - p\right]^2 |v_y(z)|^2 \det\left[A(x)^{-1/2}A(y)^{1/2}\right]\, dz.
\end{aligned}$$

Suppose now $\hat{B}_{t,R} > 6$ and consider the first addendum:

$$\int_{B_{\hat{A}_{t,R}r}\setminus B_{\hat{B}_{t,R}r}} \left[z\cdot\nabla v_y(z) - I\left(v_y,\hat{B}_{t,R}r\right)v_y(z)\right]^2\, dz$$
$$\leq \hat{A}_{t,R}r\int_{B_{\hat{A}_{t,R}r}\setminus B_{\hat{B}_{t,R}r}} \frac{\left[z\cdot\nabla v_y(z) - I\left(v_y,\hat{B}_{t,R}r\right)v_y(z)\right]^2}{|z|}\, dz$$
$$\overset{\text{Lemma 3.3.1}}{\leq} r^n H\left(v_y, 2\hat{A}_{t,R}r\right)\Delta^{2\hat{A}_{t,R}}_{\hat{B}_{t,R}}(v,y)$$

by assuming $\mathcal{I}\left(v,y,12\hat{A}_{t,r}\right)\leq M$.

Set now

$$p = \frac{1}{\mu\left[El^x_r(x)\right]}\int_{El^x_r(x)} I\left(v_z,\hat{B}_{t,R}r\right)\, d\mu(z)$$

by Jensen's inequality

$$\int_{El^x_r(x)}\left[I\left(v_y,\hat{B}_{t,R}r\right)-p\right]^2\, d\mu(y)$$
$$\leq \frac{1}{\mu\left[El^x_r(x)\right]}\int_{El^x_r(x)}\int_{El^x_r(x)}\left[I\left(v_y,\hat{B}_{t,R}r\right)-I\left(v_z,\hat{B}_{t,R}r\right)\right]^2\, d\mu(z)\, d\mu(y)$$
$$\leq \frac{C_{M,t,R}}{\mu\left[El^x_r(x)\right]}\int_{El^x_r(x)}\int_{El^x_r(x)}\Bigg[|y-z| + \Delta^{2\left(2\hat{B}_{t,R}+2\right)r}_{\left(\hat{B}_{t,R}/2-2\right)r}(v_y)$$
$$+\Delta^{2\left(2\hat{B}_{t,R}+2\right)r}_{\left(\hat{B}_{t,R}/2-2\right)r}(v_z)\Bigg]d\mu(z)d\mu(y)$$
$$\leq C_{M,R}\int_{El^x_r(x)}\left[\Delta^{2\left(2\hat{B}_{t,R}+2\right)r}_{\left(\hat{B}_{t,R}/2-2\right)r}(v_z)+r\right]\, d\mu(z)$$

and

$$\int_{B_{\hat{A}_{t,R}r}\setminus B_{\hat{B}_{t,R}r}}|v_y(z)|^2\, dz \leq \int_{B_{2\hat{A}_{t,R}r}}|v_y(z)|^2\, dz \leq Cr^nH\left(v_y,2\hat{A}_{t,R}\right)$$

Finally I use (3.4.2) and proposition 3.2.1 to get

$$\mathcal{H}\left(v,y,2\hat{A}_{t,R}r\right)\leq C_M\mathcal{H}\left(v,x,2\hat{A}_{t,R}r\right)$$

when $|y-x|\leq r\leq \tilde{r}_M$. Consequently,

$$\beta_{0,r}(\tilde{\mu})^2\int_{B_{(R+1)r}\setminus B_{Rr}}\left|\nabla^T v_x(z)\right|^2\, dz$$
$$\leq C_{M,t,R}H\left(v_x,2\hat{A}_{t,R}\right)\int_{El^x_r(x)}\Delta^{2\hat{A}_{t,R}r}_{\left(\hat{B}_{t,R}/2-2\right)r}(v_z)\, d\mu(z) + C_M r\mu\left[El^x_r(x)\right] \quad (3.5.3)$$

provided $0<r<t<\tilde{r}_{M,R}$, $|y-x|<r$ and $6<\hat{B}_{t,R}<\hat{A}_{t,R}$. Since I've assumed $\operatorname{diam}(\Omega)\leq 1$ I can assume without loss of generality that $t=1$, hence, I'm able to define the following constants depending only on n, k, Λ and R

$$A_R = \max\left\{2\hat{A}_{1,R}, 2\left(2\hat{B}_{1,R}+2\right), R+2\right\} \qquad B_R = \left(\frac{\hat{B}_{1,R}}{2} - 2\right).$$

To conclude the proof of theorerm 3.5.2 I need only to prove the following proposition

Proposition 3.5.3. *For every $R > 2, M > 0$ exist $C_{M,R}, \tilde{r}_{M,R} > 0$ so that if $0 < r < \tilde{r}_{M,R}$ and $v \in \mathcal{Q}(n,k,\Lambda)$ with $I(v, A_R r) \leq M$ then for every orthonormal basis $\nu_1, \nu_2, ..., \nu_n$ of $\mathbb{R}^n$*

$$D(v, A_R r) \leq C_{M,R} r^{2-n} \int_{B_{(R+1)r} \setminus B_{Rr}} \left|\nabla^T v(z)\right|^2 \, dz$$

where $\nabla^T v$ is defines as in (3.5.2).

Consequently, since I already know that $\mathcal{I}(v,x,r) \geq C$ for every $x \in \mathcal{K}_v$, the inequality (3.5.3) directly proves the initial statement.

Proof. Assume the statement is not true for some R, M, so for every $k \in \mathbb{N}$ exist $v_k \in \mathcal{Q}(n,k,\Lambda)$, $0 < r_k < k^{-1}$ with

$$r_k^{2-n} \int_{B_{(R+1)r_k} \setminus B_{Rr_k}} \left|\nabla^T v_k(z)\right|^2 \, dz \leq \frac{D(v_k, A_R r_k)}{k}$$

where $\nabla^T v_k$ is evaluated with respect to orthonormal basis $\nu_1^k, \nu_2^k, ..., \nu_n^k$ depending on k too. Up taking a subsequence I may assume $\nu_i^k \to \nu_i$ where ν_i is again an orthonormal basis of $\mathbb{R}^n$.

Let

$$u_k(y) = R^0_{r_k, A_R} v_k(y) = \frac{v_k(r_k y)}{\sqrt{H(v_k, A_R r_k)}}$$

clearly $H(u_k, A_R) = 1$ and $D(u_k, A_R) = I(u_k, A_R) \leq M$ so

$$\int_{B_{R+1} \setminus B_R} \left|\nabla^T u_k(z)\right|^2 \, dz \leq \frac{M}{k} \to 0$$

Now, up to a subsequence, u_k converges in C^1 on $\overline{B_{R+1}}$ to some harmonic function u since $A_R \geq R+2$. Notice that $\nabla^T u \equiv 0$ on $B_{R+1} \setminus B_R$, thus,

$$u(y) = [\![ay \cdot \nu_1 + b]\!] + [\![-ay \cdot \nu_1 - b]\!]$$

on all B_{R+1} for some $a, b \in \mathbb{R}$. However, the C^1 convergence implies $u(0) = 0$ and $\nabla u(0) = 0$ so $u \equiv 0$, that is impossible because $H(u_k, R+1) \geq C_{M,R}$ for some positive constant $C_{M,R}$ not depending on k. □

Remark 3.5.4. I have $\lim_{R \to +\infty} B_R = +\infty$ so I can assume B_R arbitarily large.

The next tool to prove theorem 1.0.1 is related on almost homogeneous solutions defined in the following way:

Definition 3.5.5. Let $v \in \mathcal{WQ}(n,k,\Lambda)$, $x \in \mathcal{K}_v$ and $\eta, r > 0$, I say that v is (η, r)-almost homogeneous in x if and only if

$$\mathcal{I}(v,x,r) - \mathcal{I}(v,x,r/2) \leq \eta$$

Notice that when v is harmonic and homogeneous in x then it's automatically (η, r)-almost homogeneous for every η, r.

Remember now the definition of spine of a two-valued harmonic function I've given in chapter 1. The next result allows us to characterize the spine of a two-valued harmonic homogeneous function v in $\mathbb{R}^n$.

Proposition 3.5.6. *Let v be a two-valued harmonic homogeneous symmetric function. For every $x \in \mathbb{R}^n$ the following are equivalent:*

1. *$x \in S(v)$;*
2. *$I(v,x,r)$ is constant in r;*
3. *$I(v,x) = I(v,0)$.*

where $I(v,z) = \lim_{r \to 0^+} I(v,z,r)$ and $S(v)$ is the spine of v.

Proof. First to all notice that when v is harmonic the relaxed frequency $I(v,x,r)$ satisfies the same monotonicity properties of the original frequency $N(v,x,r)$, also v is homogeneous if and only if $I(v,0,r)$ is constant in r.

If $x \in S(v)$ then $I(v,x,r) = I(v,0,r)$ so 1 implies 2 and 3. If 2 holds then exists $r_n \to +\infty$ such that $\mathcal{R}^0_{r_n,1} v \to v_0$ in C^1_{loc} so

$$\begin{aligned} |I(v,x) - I(v,0)| &= \lim_{n \to +\infty} |I(v,x,r_n) - I(v,0,r_n)| \\ &= \lim_{r \to +\infty} \left| I\left(\mathcal{R}^0_{r_n,1} v, \frac{x}{r_n}, 1\right) - I\left(\mathcal{R}^0_{r_n,1} v, 0, 1\right)\right| = |I(v_0,0,1) - I(v_0,0,1)| = 0 \end{aligned}$$

and 3 is satisfied. If 3 holds then

$$\lim_{n \to +\infty} |I(v,x,r_n) - I(v,x)| \leq \lim_{n \to +\infty} |I(v,x,r_n) - I(v,0,r_n)| = 0,$$

so I need to prove only that the statements 2 and 3 imply statement 1. In particular, by 2 and 3 both $y \to v(y)$ and $y \to v(x+y)$ are λ-homogeneous with $\lambda = I(v,x) = I(v,0)$ and for every $y \in \mathbb{R}^n$

$$v(x+y) = 2^\lambda v\left(\frac{x+y}{2}\right) = 2^\lambda v\left(x + \frac{y-x}{2}\right) = v(x+y-x) = v(y),$$

therefore, $x \in S(v)$. □

Lemma 3.5.7. *For every $\delta, M > 0, R > 1$ exist $\eta_{\delta,M,R}, \tilde{r}_{\delta,M,R}$ so that for every $r < \tilde{r}_{\delta,M,R}$, $v \in \mathcal{WQ}(n,k,\Lambda)$ not identically zero, $x \in \mathcal{K}_v$ $(\eta_{\delta,M,R}, r)$-almost homogeneous in x and $\mathcal{I}(v,x,rR) \leq M$ exists w two-valued harmonic homogeneous function not identically zero such that*

$$\mathtt{G}_{C^1}\left(\mathcal{R}^0_{r,R} v_x, w\right) \leq \delta$$

Proof. Fix δ, R, M suppose for every $i \in \mathbb{N}$ exist $0 < r_i < 1/i$, $v_i \in \mathcal{Q}(n,k,\Lambda)$ so that $I(v_i, r_i) - I(v_i, r_i/2) \leq 1/i$ and $\mathtt{G}_{C^1}\left(\mathcal{R}^0_{r_i,R} v_i, w\right) \geq \delta$ for every w two-valued harmonic homogeneous nontrivial function.

But $\mathcal{R}^0_{r_i,R} v_i \to w$ in $C^1_{loc}(B_R)$ with w two-valued harmonic and $I(w,0,1) = N(w,0,1) \leq N(w,0,1/2)$ so w is also homogeneous. Since $I\left(R^0_{r_i,R} v_i, R\right) \leq M$ for every i function w can't be identically zero on B_1, contraddiction. □

This lemma is what I need to prove another fundamental result

Theorem 3.5.8. *For every $M, \varepsilon > 0$ there exist $\eta_{M,\varepsilon}, \tilde{r}_{M,\varepsilon} > 0$ so that for every $0 < r < \tilde{r}_{M,\varepsilon}$, $v \in \mathcal{Q}(n,k,\Lambda)$ $(\eta_{M,\varepsilon}, r)$-almost homogeneous with $I(v, 4r) \leq M$ then at least one of the following sentences is true:*

1. *for every $x \in \mathcal{K}_v \cap B_{2r}$ I get*
$$|I(v,x,r) - I(v,r)| \leq \varepsilon;$$
2. *$n \geq 3$ and exists $V \leq \mathbb{R}^n$ $(n-3)$-dimensional vector subspace so that for every $y \in \mathcal{K}_v \cap B_{2r}$ if $I(v,y,r) - I(v,y,r/2) \leq \eta_{M,\varepsilon}$ then $\operatorname{dist}(y, V) \leq \varepsilon r$.*

Proof. By contradiction exist $r_i < 1/i$, $v_i \in \mathcal{Q}(n,k,\Lambda)$ $(1/i, r_i)$-almost homogeneous with $I(v_i, 4r_i) \leq M$, $x_i \in \mathcal{K}_{v_i}$ with $|x_i| < 2r_i$ and for every V linear $(n-3)$-dimensional subspace exist $y_{i,V} \in \mathcal{K}_{v_i}$ with $|y_{i,V}| < 2r_i$ so that

$$|I(v_i, x_i, r_i) - I(v_i, r_i)| > \varepsilon$$
$$I(v_i, y_{i,V}, r_i) - I(v_i, y_{i,V}, r_i/2) < \frac{1}{i}$$
$$|y_{i,V} - \nu| > \varepsilon r_i \quad \forall \nu \in V$$

Set

$$a_i = \frac{x_i}{r_i} \qquad b_{i,V} = \frac{y_{i,V}}{r_i} \qquad w_i = \mathcal{R}^0_{r_i,4} v_i$$

clearly $w_i \to w$ uniformly in C^1 on compact subsets of B_4 (up to a subsequence) and analogously $a_i \to a$, notice also that w is not identically zero. Lemma 3.5.7 ensures existence of harmonic homogeneous not identically zero functions h_i converging to w too.

If $\dim S(h_i) \geq n-2$ for every i then $\dim S(w) \geq n-2$ and since $x_i \in \mathcal{K}_{v_i}$ I have $a \in \mathcal{K}_w$, because $w \not\equiv 2[\![0]\!]$ proposition 2.4.1 implies $a \in S(w)$. Consequently $I(w,a,1) = I(w,0,1)$, that's impossible because $|I(w,a,1) - I(w,1)| \geq \varepsilon$. In particular $n \geq 3$.

I can then suppose that $\mathtt{G}_{C^1}(w_i, h) \geq \delta$ for every h harmonic homogeneous with $\dim S(h) \geq n-2$, then $\dim S(w) \leq n-3$. Let $V \geq S(w)$ with dimension exactly $n-3$ and take a subsequence so that $b_{i,V} \to b$, then $|b - \nu| \geq \varepsilon$ for every $\nu \in V$. However, this is impossible because $I(w,b,1) = I(v,b,1/2)$ and so $b \in S(w) \leq V$. □

Corollary 3.5.9. *For every $M, \varepsilon > 0$ exist $\eta_{M,\varepsilon}, \tilde{r}_{M,\varepsilon} > 0$ so that for every $v \in \mathcal{WQ}(n,k,\Lambda)$, $x_0 \in \mathcal{K}_v$, $0 < r < \tilde{r}_{M,\varepsilon}$ so that v is $(\eta_{M,\varepsilon}, r)$-almost homogeneous in x_0 and $\mathcal{I}(v, x_0, 4r) \leq M$ then at least one of the following is true:*

1. *for every* $x \in \mathcal{K}_v$ *with* $|x - x_0|_{x_0} < 2r$ *I've*

$$\left|\mathcal{I}(v, x, r) - \mathcal{I}(v, x_0, r)\right| \leq \varepsilon,$$

2. $n \geq 3$ *and exists* $V \leq \mathbb{R}^n$ *an* $(n-3)$*-dimensional vector subspace so that for every* $y \in \mathcal{K}_v$ *with* $|y - x_0|_{x_0} < 2r$ *if* v *is* $(\eta_{M,\varepsilon}, r)$*-almost homogeneous in* y *then* $\operatorname{dist}(y - x_0, V) \leq \varepsilon r$.

Proof of theorem 3.0.1

Now I've all the needed tools to prove theorem 3.0.1. First to all for every $x \in \mathbb{R}^n$, $r > 0$ set

$$\Xi(v, x, r) = \sup_{z \in \mathcal{K}_v \cap B_r(x)} \mathcal{I}(v, z, r).$$

For every $0 < \delta < 1$ let $\mathcal{N}(n, \delta)$ be the maximum number of n-dimensional disjoint open balls of radius δ that can be contained by a ball of radius 1 in $\mathbb{R}^n$. It's trivial to show that

$$\left\lfloor n^{-1/2}\sigma^{-1}\right\rfloor^n \leq \mathcal{N}(n, \sigma) \leq \left\lfloor n^{1/2}\sigma^{-1}\right\rfloor^n \tag{3.5.4}$$

Proposition 3.5.10. *For every* $M > 0$ *exist constants* $C_M, \tilde{r}_M > 0$ *so that for every* $v \in \mathcal{WQ}(n, k, \Lambda)$ *not identically zero and for every* $x \in \Omega$, $0 < \rho < \tilde{r}_M$, $0 < r < \rho/2$ *with* $B_{2\rho}(x) \subseteq \Omega$ *if* $\Xi(x, \rho) \leq M$ *then*

$$\left|[\mathcal{K}_v \cap B_\rho(x)] + B_r\right| \leq C_M r^2 \rho^{n-2} \tag{3.5.5}$$

I first prove that proposition 3.5.10 implies theorem 3.0.1. Let

$$\delta_K = \operatorname{dist}(K, \partial\Omega)/2$$

and set $\tilde{M}_v = \max_{x \in K \cap \mathcal{K}_v} \Xi(v, x, \delta)$. Then $\Xi(x, \rho) \leq C_{\tilde{M}}$ for every $x \in K, \rho \leq \delta_K$.

Set then $M = C_{\tilde{M}}$ and $\rho = \min\{\delta_K, \tilde{r}_M\}$ by Vitali covering theorem you can choose $B_{\rho/3}(x_i)$ disjoint contained in Ω so that $B_\rho(x_i)$ cover K. Then

$$\left|[\mathcal{K}_v \cap K] + B_r\right| \leq \sum_i^{\mathcal{N}(n, \rho/3)} \left|[\mathcal{K}_v \cap B_\rho(x_i)] + B_r\right| \leq C_M \mathcal{N}(n, \rho/3) r^2 \rho^{n-2} \leq C_K r^2$$

where C_K depends only on K, n, k, Λ.

By theorem 3.1.8 that proposition is automatically satisfied for $M < C$ and $\tilde{r}_M = \tilde{r}$ because for every nontrivial two-valued function v with $\mathcal{I}(v, x, \rho) < M$ I must have $x \notin \mathcal{K}_v$.

To prove proposition 3.5.10 I use this simple observation: if $\tilde{M}$ is the supremum of every M that satisfies (3.5.5), and exists ξ positive depending only on $\tilde{M}$ and not on M such that $M + \xi$ satisfies (3.5.5) when also M satisfies it, then $\tilde{M}$ should be equal to $+\infty$ and the proposition is satisfied.

In the next steps I prove that for every $\tilde{M} > 0$ extst $\tilde{r}_{\tilde{M}} > 0$ and $\xi_{\tilde{M}} > 0$ so that if M satisfies proposition 3.5.10 then $M + \xi_{\tilde{M}}$ satisfies it too for some $\tilde{r}_{M+\xi_{\tilde{M}}}$.

First I assume $\xi = \xi_{\tilde{M}} \leq \tilde{M}$ thus $M + \xi \leq 2\tilde{M}$. Let $v \in \mathcal{WQ}(n, k, \Lambda)$ with $\Xi(v, 0, \rho) \leq M + \xi$ where $0 < r < \rho/2$, $\rho < \tilde{r}_{M+\xi} \leq \tilde{r}_M$.

For every $x \in \mathcal{K}_v \cap B_{\rho/2}$ set

$$r_x = \begin{cases} \inf\{0 < t \leq \rho \,|\, \Xi(v,x,t') > M \ \forall t \leq t' \leq \rho\} & \text{if } \Xi(v,x,\rho) > M \\ \rho & \text{if } \Xi(v,x,\rho) \leq M \end{cases}$$
$$s_x = \max\{r_x, 2r\}$$

and for every x with $\Xi(v,x,\rho) > M$ (in particular, $r_x < \rho$) set also $y_x \in \mathcal{K}_v$ so that $|y_x - x| \leq s_x$ and

$$\mathcal{I}(v, y_x, s_x) \geq M$$

Notice also that $s_x \neq 0$ and if $s_x > 2r$ then $s_x = r_x$ and so $\Xi(v,x,s_x) \leq M$. By Vitali's covering lemma there exists $\{x_i\}_{i\in I}$ finite collection of points in $\mathcal{K}_v \cap B_{\rho/2}$ such that

- $B_{s_{x_i}/10}(x_i)$ are disjoint;
- $B_{s_{x_i}/2}(x_i)$ cover $\mathcal{K}_v$.

Then since $s_x \leq \tilde{r}_M$

$$\begin{aligned} \left|\left[\mathcal{K}_v \cap B_{\rho/2}\right] + B_r\right| &\leq \sum_{i\in I} \left|\left[\mathcal{K}_v \cap B_{s_{x_i}/2}(x_i)\right] + B_r\right| \\ &\leq \sum_{s_{x_i}>2r} C_M r^2 s_{x_i}^{n-2} + \sum_{s_{x_i}=2r} \omega_n 2^n r^n \\ &\leq \left[C_M + 4\omega_n\right] r^2 \sum_i s{x_i}^{n-2} \end{aligned}$$

where I've used the assumption that proposition 3.5.10 is satisfied for M. If I prove the existence of some $C_{\tilde{M}}$ so that

$$\sum_i s_{x_i}^{n-2} \leq C_{\tilde{M}} \rho^{n-2} \tag{3.5.6}$$

then proposition 3.5.10 is proved.

Let $\sigma = \sigma_{\tilde{M}} \in (0,1)$ be a constant I'll determine later, for the moment I only assume that σ depends only on n, k, Λ and $\tilde{M}$.

Definition 3.5.11. For every $1 > \sigma > 0$ let $A_\sigma^0, A_\sigma^1, A_\sigma^2, \ldots$ be a partition of $\{x_i\}_{i\in I}$ so that

$$A_\sigma^0 = \left\{x_i \,\middle|\, i \in I, s_{x_i} \geq \sigma^2 \rho\right\}$$
$$x, y \in A_\sigma^{j+1}, |x-y| < \frac{s_y}{\sigma}, x \neq y \text{ implies } s_x \leq \sigma^2 s_y$$

Proposition 3.5.12. *When $\sigma < 10^{-1}$ such partition exists and is made by at most $\mathcal{M}(n,\sigma)$ sets.*

Proof. Assume $A_\sigma^0 =$ and $i < j \Rightarrow s_{x_i} \geq s_{x_j}$, set $x_1 \in A_\sigma^1$ and assume $x_1, x_2, \ldots, x_{i-1}$ have been assigned, I set $x_i \in A_\sigma^l$ if and only if l is the minimum so that for every $j < i$ such that

$$|x_i - x_j| < \frac{s_{x_j}}{\sigma},\ \sigma^2 s_{x_j} \leq s_{x_i} \tag{3.5.7}$$

I have $x_j \notin A^l_\sigma$.

In this way I'm able to build each A^l_σ, also for every $x_i \in A^l_\sigma$ and every $j < i$ satisfying (3.5.7) I've $|x_i - x_j| \leq s_{x_i}/\sigma^3$. Since $B_{s_{x_j}/10}(x_j)$ are disjoint with centers contained in $B_{s_{x_i}/\sigma^3}(x_i)$ there are up to $N_\sigma = \mathcal{N}\left[n, 10^{-1}\left(\sigma^{-3}+10^{-1}\right)^{-1}\right]$ of such j, notice that this number doesn't depend on i. Suppose now that $x_i \in A^l_\sigma$, this means for every $l' < l$ exists $j < i$ satisfying (3.5.7) with $x_j \in A^{l'}_\sigma$, and since only a finite number of such x_j could exists I must have $l - 1 \leq N_\sigma$.

Take now $x_i, x_j \in A^l_\sigma$ with $j < i$ then I have $|x_i - x_j| \geq s_{x_j}/\sigma$ or $s_{x_i} < \sigma^2 s_{x_j}$. Then

- if $|x_i - x_j| < s_{x_j}/\sigma$ then $s_{x_i} \leq \sigma^2 s_{x_j}$ automatically;
- if $|x_j - x_i| < s_{x_i}/\sigma$ then $|x_j - x_i| < s_{x_j}/\sigma$ because $s_{x_i} \leq s_{x_j}$ and so $s_{x_i} < \sigma^2 s_{x_j}$ again. But this is impossible because balls of radius $s_{x_j}/10$ are disjoint:

$$\frac{s_{x_j}}{10} < |x_i - x_j| < \frac{s_{x_i}}{\sigma} < \sigma s_{x_j}$$

Then $|x_i - x_j| \geq s_{x_i}/\sigma$ and A^l_σ is a good partition.

The statement is so proved. □

For every such $0 \leq l \leq N_\sigma + 1$ I define the following measures

$$\mu^l = \sum_{x \in A^l_\sigma} s_x^{n-2} [\![x]\!]$$

(remember that $[\![x]\!]$ is the Dirac measure at $x \in \mathbb{R}^n$). Notice that proving (3.5.6) is equivalent to prove:

Theorem 3.5.13. *There exists $C_{\tilde{M}} > 0, \sigma = \sigma_{\tilde{M}}$ so that for every such l I've*

$$\mu^l\left(B_{\rho/2}\right) \leq C_{\tilde{M}} \rho^{n-2}$$

I first prove it for $l = 0$. This is trivial because $\mathcal{H}^0\left(A^0_\sigma\right) \leq \mathcal{N}(n, 10^{-1}\sigma^2\rho/\left(2^{-1}\rho\right)) = C_\sigma = C_{\tilde{M}}$. Remember that $s_x \leq 2\rho$ so

$$\sum_{x \in A^0_\sigma} s_x^{n-2} \leq C_{\tilde{M}} \rho^{n-2}$$

When $l \geq 1$ theorem 3.5.13 is a direct consequence of

Theorem 3.5.14. *There exists $\sigma = \sigma_{\tilde{M}}$ so that for every $l \geq 1$, $x \in \operatorname{supp} \mu^l$, $s_x < t \leq \sigma^2 \rho$ I have*

$$\mu^l\left[B_t(x)\right] \leq C_{\tilde{M}} t^{n-2}$$

Indeed take any $A' \subseteq A^l_\sigma$ maximal subset so that $|x - y| \geq \sigma^2 \rho$ when $x \neq y$ in A'. Notice that for every $x \in A'$ $B_{\sigma^2\rho/2}(x) \subseteq B_\rho$ are disjoint ($x \in B_{\rho/2}$) so

$$\mathcal{H}^0\left(A'\right) \leq \mathcal{N}\left(n, \sigma^2/2\right)$$

But A' is maximal too, so for every $x \in A^l_\sigma$ exists $x' \in A'$ so that $x \in B_{\sigma^2\rho}(x')$. This means

$$\mu^l\left(B_{\rho/2}\right) \leq \sum_{x \in A'} \mu^l\left[B_{\sigma^2\rho}(x)\right] \leq \mathcal{H}^0\left(A'\right) C_{\tilde{M}} \sigma^{2(n-2)} \rho^{n-2} = C_{\tilde{M}} \rho^{n-2}$$

Proof of theorem 3.5.14

At this point I have $l \geq 1$, σ a constant less than $1/10$ that I should determine in function of $\tilde{M}$, $A^l_\sigma \subseteq \mathcal{K}_v$ and radiuses $s_x > 0$.

Define

$$s^l_{min} = \min_{x \in \operatorname{supp} \mu^l} s_x$$
$$T^l_{max} = \left\lfloor \frac{\ln s^l_{min} - \ln \rho}{\ln \sigma} \right\rfloor$$

where $\lfloor x \rfloor$ is the greatest integer $\leq x$. Since $\sigma < 1$ and $s_x \leq \sigma^2 \rho$ I've $T^l_{max} \geq 2$ and $\sigma^{T^l_{max}+1}\rho < s^l_{min} \leq \sigma^{T^l_{max}}\rho$.

Let

$$t \in \left(\max\left\{\sigma^{k+1}\rho, s^l_{min}\right\},\ \sigma^k\rho\right]$$

I want to prove theorem 3.5.14 for every such t by using induction on $k \in \left\{2, 3, \ldots, T^l_{max}\right\}$.

I start with $k = T^l_{max}$ and $s^l_{min} < t \leq \sigma^k \rho$, choose any $x \in \operatorname{supp} \mu^l$ with $s_x \leq \sigma^k \rho$. Consider any $y \in B_{\sigma^k\rho}(x) \cap A^l_\sigma$ if $y \neq x$ then $|y - x| < \sigma^k \rho = \sigma^{k+1}\rho/\sigma \leq s_x/\sigma$, from the very definition of A^l_σ $s_y \leq \sigma^2 s_x$ that is impossible because

$$\sigma^{k+1}\rho < s^l_{min} \leq s_y \leq \sigma^2 s_x \leq \sigma^{k+2}$$

so $B_{\sigma^k\rho}(x) \cap A^l_\sigma = \{x\}$ which implies

$$\mu^l\left[B_t(x)\right] \leq \mu^l\left[B_{\sigma^k\rho}(x)\right] = s_x^{n-2} \leq \sigma^{(n-2)T^l_{max}}\rho^{n-2} \leq \sigma^{2(n-2)}\rho^{n-2}$$

Suppose now I've proved it for some $k < T^l_{max}$ and every $t \in \left(s^l_{min}, \sigma^{k+1}\rho\right]$, I need to prove it for $t \in \left(\sigma^{k+1}\rho, \sigma^k\rho\right]$.

Take any $x_0 \in A^l_\sigma$ so that $t > s_{x_0}$ (always exists such x_0 because $t > s^l_{min}$), let also $\tau = \tau_{\tilde{M}} > \xi$ be a constant that I'll determine later. I set

$$W = \operatorname{supp} \mu^l \cap B_t(x_0)$$
$$W^1 = \{x \in W \,|\, \mathcal{I}(v, x, s_x) < M - \tau\}$$
$$W^2 = W \setminus W^1$$

and I reorder the elements of W^1 in such a way that $W^1 = \{p_i\}_{i \in \mathbb{N}}$ with $s_{p_i} \geq s_{p_j}$ when $i < j$. Remember now that for every $x \in A^l_\sigma$ I've $r_x \leq s_x \leq \sigma^2\rho < \rho$ and so exists y_x so that $\mathcal{I}(v, y_x, s_x) \geq M$. Define $z_1 = y_{p_1}$ and for every $i \geq 2$ set by induction

$$m_i = \min\left\{ j \in \mathbb{N} \,\middle|\, y_{p_j} \notin \bigcup_{h=1}^{i-1} B_{s_h}(z_h) \right\}$$
$$z_i = y_{p_{m_i}}$$
$$s_i = s_{p_{m_i}}$$

and notice that m_i is strictly increasing: $m_{i+1} > m_i$.

In order to simplify our notation set also

$$Z = \{z_i\}_i$$
$$E = \bigcup_{z_i \in Z} B_{s_i}(z_i)$$

remember now that $\mu^l \llcorner B_t(x_0) = \sum_{x \in W^1 \cup W^2} s_x^{n-2} [\![x]\!]$ I introduce a modified version of μ^l:

$$\mu_1^l = \sum_{z_i \in Z} s_i^{n-2} [\![z_i]\!] + \sum_{x \in W^2 \setminus E} s_x^{n-2} [\![x]\!]$$

I then prove the following properties of μ_1^l:

Proposition 3.5.15. *For every $w \in \operatorname{supp} \mu_1^l$ I've $\mathcal{I}(v, w, \rho) - \mathcal{I}(v, w, s_w) \le \tau + \xi \le 2\tau$.*

Proof. If $w \in Z$ then $w = y_{p_i}$ for some i and so

$$\mathcal{I}(v, y_{p_i}, \rho) - \mathcal{I}(v, y_{p_i}, s_{p_i}) \le M + \xi - M \le \xi$$

Instead if $w \in W^2$ then $\mathcal{I}(v, w, s_w) \ge M - \tau$ and so

$$\mathcal{I}(v, w, \rho) - \mathcal{I}(v, w, s_w) \le M + \xi - M + \tau \le \xi + \tau$$

and the statement is so proved □

Proposition 3.5.16. *If $w, p \in \operatorname{supp} \mu_1^l$ with $w \ne p$ then*

$$\max\{s_p, s_w\} \le 20\,|p - w|$$

Proof. If $w, p \in Z$ assume $w = z_i$, $p = z_j$ with $m_i < m_j$, then $i < j$ and by definition of Z $z_j \notin B_{s_i}(z_i)$ which implies $|p - w| \ge s_i \ge s_j$.

If $p, w \in W^2$, since balls $B_{s_x/10}(x)$ are disjoint I immediately get $\max\{s_p, s_w\} \le 10\,|p - w|$.

So I need to prove only when $p \in Z$, $w \in W^2 \setminus E$. Then $w \notin B_{s_j}(z_j)$ where $z_j = p = y_{p_{m_j}}$ by definition of E so $|p - w| \ge s_j$. Again because $p_{m_j} \ne w$ I get $\left|p_{m_j} - w\right| \ge s_w/10$ and so

$$s_w \le 10\left|p_{m_j} - w\right| \le 10\left(\left|p_{m_j} - y_{p_{m_j}}\right| + |w - z_j|\right) \le 10\,(s_j + |w - z_j|) \le 20\,|p - w|$$

□

Proposition 3.5.17. *I've* $\operatorname{supp} \mu_1^l \subseteq B_{11t}(x_0)$

Proof. First to all for every $x \in W$ with $x \ne x_0$ I've $s_x \le 10\,|x - x_0| \le 10t$ because balls $B_{s_x/10}(x)$ are disjoint, notice that this is true even if $x = x_0$ because I've assumed $s_{x_0} < t$. Then

$$|y_x - x_0| \le |y_x - x| + |x - x_0| \le s_x + |x - x_0| \le 11t$$

□

Lemma 3.5.18. *I've* $\mu^l[B_t(x_0)] \leq 2\mu_1^l[B_{11t}(x_0)]$.

Proof. Set for every $z_i \in Z$

$$W^{z_i} = \left[W^2 \cap \left(B_{s_i}(z_i) \setminus \bigcup_h^{i-1} B_{s_h}(z_h) \right) \right] \cup \left\{ w \in W^1 \,\middle|\, y_w \in B_{s_i}(z_i) \setminus \bigcup_h^{i-1} B_{s_h}(z_h) \right\}$$

since $E = \bigcup_i \left[B_{s_i}(z_i) \setminus \bigcup_h^{i-1} B_{s_h}(z_h) \right]$ and $y_w \in E = \bigcup_i \left[B_{s_i}(z_i) \setminus \bigcup_h^{i-1} B_{s_h}(z_h) \right]$ for every $w \in W^1$ I've

$$W = W^1 \cup W^2 = \bigcup_i W^{z_i} \cup \left(W^2 \setminus E \right)$$

$$\mu^l[B_t(x_0)] = \sum_i \mu^l \left(W^{z_i} \right) + \mu^l \left(W^2 \setminus E \right) = \sum_i \mu^l \left(W^{z_i} \right) + \mu_1^l \left(W^2 \setminus E \right)$$

so I need to prove our inequality only on each W^{z_i}.

Let $\tilde{\imath} = m_i$ be so that $z_i = y_{p_{\tilde{\imath}}}$ with $p_{\tilde{\imath}} \in W^1$ so

$$\mathcal{I}\left(v, z_i, s_i\right) - \mathcal{I}\left(v, p_{\tilde{\imath}}, s_i\right) \geq M - M + \tau = \tau$$

I've $\mathcal{I}\left(v, p_{\tilde{\imath}}, 16 s_i\right) \leq M + \xi \leq 2\tilde{M}$ by assuming $\xi \leq \tilde{M}$ (which doesn't depend on M). By corollary 3.5.9 with $\varepsilon = 2\tau$ I can assume $2\xi \leq \eta_{2\tilde{M}, 2\tau}$

Corollary 3.5.9 then implies for every $x \in \mathcal{K}_v$ with $|x - z_i| < 4s_i$ and I have $\mathcal{I}\left(v, x, 2s_i\right) - \mathcal{I}\left(v, x, s_i\right) \leq \xi$ then $\operatorname{dist}\left(x, V + z_i\right) \leq 2\tau s_i$ for some $(n-3)$-dimensional vector space V.

I want to prove that

$$W^{z_i} \setminus \{p_{\tilde{\imath}}\} \subseteq V + B_{4\sigma^2 s_i}(z_i) \tag{3.5.8}$$

so take any $w \in W^{z_i} \setminus \{p_{\tilde{\imath}}\}$, if $w \in W^2 \cap B_{s_i}(z_i)$ then

$$|w - p_{\tilde{\imath}}| \leq |w - z_i| + |y_{p_{\tilde{\imath}}} - p_{\tilde{\imath}}| \leq 2s_i = 2s_{p_{\tilde{\imath}}}$$

Remember now that $W \subseteq A_\sigma^l$, $w \neq p_{\tilde{\imath}}$ and since $\sigma < 1$ this implies $s_w \leq \sigma^2 s_i$ then

$$|y_w - z_i| \leq |y_w - w| + |w - z_i| \leq \left(\sigma^2 + 1 \right) s_i \leq 2s_i$$

for σ small and so by (3.2.7)

$$\begin{aligned} \left| \mathcal{I}\left(v, y_w, 2s_w\right) - \mathcal{I}\left(v, y_w, s_w\right) \right| &\leq \mathcal{I}\left(v, y_w, \rho\right) - \mathcal{I}\left(v, y_w, s_w\right) + C_{\tilde{M}}(\rho - s_w) \\ &\leq M + \xi - M + C_{\tilde{M}} \tilde{r}_{\tilde{M}} \leq 2\xi \end{aligned}$$

for $\tilde{r}_{\tilde{M}}$ sufficiently small, so by corollary 3.5.9 $y_w \in V + B_{2\tau s_i}(z_i)$. Assuming $\tau \leq \sigma^2$ I get

$$w \in V + B_{2\tau s_i + s_w}(z_i) \subseteq V + B_{4\sigma^2 s_i}(z_i)$$

Assume now that $w = p_j \in W^1$ because $y_{p_j} \notin B_{s_h}(z_h)$ for every $h < i$ I get $\tilde{\imath} = m_i \leq j$ and so $s_i = s_{p_{\tilde{\imath}}} \geq s_{p_j} = s_w$, in particular

$$|w - p_{\tilde{\imath}}| \leq |w - y_w| + |y_w - z_i| + |y_{p_{\tilde{\imath}}} - p_{\tilde{\imath}}| \leq s_w + 2s_i \leq 3s_i \tag{3.5.9}$$

and again $s_w \le \sigma^2 s_i$ and in the same way I get $w \in V + B_{4\sigma^2 s_i}(z_i)$ so (3.5.8) is proved.

Take now $Q' \subseteq W^{z_i} \setminus \{p_{\tilde{i}}\}$ any maximal subset so that

$$\left\{B_{\sigma s_i/20}(p)\right\}_{p\in Q'}$$

are disjoint and set

$$\pi : z \in \mathbb{R}^n \to \pi_V(z - z_i) + z_i$$

where π_V is the orthogonal projection on V, then (3.5.8) implies $|p - \pi(p)| \le 4\sigma^2 s_i$ for every $p \in Q'$. Assuming $\sigma \le 160^{-1}$ I get for every $p \in Q'$

$$B_{\sigma s_i/40}[\pi(p)] \subseteq B_{\sigma s_i/20}(p) \overset{(3.5.9)}{\subseteq} B_{3s_i}(z_i)$$

so $B_{\sigma s_i/40}[\pi(p)] \cap V + z_i$ are disjoint too and so

$$\mathcal{H}^0(Q') \le \mathcal{N}(n-3, \sigma/120)$$

Since Q' is maximal I get

$$\mu^l\left(W^{z_i}\right) \le s_i^{n-2} + \sum_{q\in Q'} \mu^l\left[B_{\sigma s_i/10}(p)\right]$$

I want to use induction hypothesis for $t = \sigma s_i/10$. First to all since $p \in B_{3s_i}(z_i)$ I have $s_p \le \sigma^2 s_i \le \sigma s_i/10 = t$. In order to prove that $t \le \sigma^{k+1}$ remember that balls $B_{s_x/10}(x)$ are disjoint so

$$s_i = s_{p_{\tilde{i}}} \le 10\left|p_{\tilde{i}} - x_0\right| \le 10t \le 10\sigma^k$$

so $t \le \sigma^{k+1}$ and by induction hypothesis

$$\mu^l\left(W^{z_i}\right) \le s_i^{n-2} + \mathcal{N}(n-3, \sigma/120) C_{\tilde{M}} 10^{2-n} \sigma^{n-2} s_i^{n-2} \le 2 s_i^{n-2}$$

where in the last inequality I've used (3.5.4) by assuming

$$\sigma \le (n-3)^{(3-n)/2} 10^{n-2} C_{\tilde{M}}^{-1}$$

From that inequality follows immediately our statement because $z_i \in \operatorname{supp} \mu_i^l \subseteq B_{11t}(x_0)$. ☐

Lemma 3.5.19. *I've*

$$\mu_1^l[B_s(p)] \le C_{\tilde{M}} n^{n/2} 10^{10n-10} \sigma^{2-2n} 41^{n-2} s^{n-2}$$

for every $p \in \operatorname{supp} \mu_1^l$, $s_p/20 \le s \le 10^2 \sigma^k$.

Proof. Take any other $q \in \operatorname{supp} \mu_1^l$ with $|p - q| < s$, let $x_q \in \operatorname{supp} \mu^l$ be so that

$$\begin{cases} q = y_{x_q} & \text{if } q \notin \operatorname{supp} \mu^l \\ q = x_q & \text{if } q \in \operatorname{supp} \mu^l \end{cases}$$

Then if $p \neq q$ by proposition 3.5.16

$$|x_p - x_q| \leq |x_p - p| + |p - q| + |x_q - q| \leq s_p + s + s_q \leq s + 40\,|p - q| < 41s$$

I can then build an injective application $q \in \operatorname{supp} \mu_1^l \cap B_s(p) \to x_q \in \operatorname{supp} \mu^l \cap B_{41s}(x_p)$, since $s_q = s_{y_{x_q}} = s_{x_q}$ I get

$$\mu_1^l[B_s(p)] \leq \mu^l[B_{41s}(x_p)]$$

Now:

1. if $41s \leq \sigma^{k+1}$ then assuming $\sigma < 1/100$ I get $s_p \leq 20s \leq 41s$ and by inductive hypothesis
$$\mu^l[B_s(p)] \leq \mu^l[B_{41s}(p)] \leq C_{\tilde{M}} 41^{n-2} s^{n-2}$$

2. if instead $41s > \sigma^{k+1}$ then let $\left\{B_{\sigma^{k+1}/10}(w)\right\}_{w \in \operatorname{supp} \mu^l}$ be an open cover of $\operatorname{supp} \mu^l \cap B_{41s}(p)$ with $B_{\sigma^{k+1}/80}(w)$ disjoint. If $w \neq x_p$ then $|w - x_p| \geq (s_w + s_p)/10$ so $s_w \leq 20 \cdot 41s \leq 10^5 \sigma^k$, also by assumption $s_p \leq 20s$ so that holds for $w = x_p$ too.

   If $s_w < \sigma^{k+1}$ then by inductive hyphothesis for $t = \sigma^{k+1}$ and get
$$\mu^l \left[B_{\sigma^{k+1}/10}(w)\right] \leq \mu^l[B_{\sigma^{k+1}}(w)] \leq C_{\tilde{M}} \sigma^{(k+1)(n-2)}$$
   otherwise $s_w \geq \sigma^{k+1}$ but since balls $B_{s_w/10}(w)$ are disjoint this would imply $\operatorname{supp} \mu^l \cap B_{\sigma^{k+1}/10}(w) = \{w\}$ and so
$$\mu^l \left[B_{\sigma^{k+1}/10}(w)\right] = s_w^{n-2} \leq 10^{5(n-2)} \sigma^{k(n-2)}$$
   Finally I get
$$\begin{aligned}\mu_1^l\left[B_s(p)\right] \leq \mu^l\left[B_{41s}(x_p)\right] &\leq \mathcal{N}\left(n, \sigma^{k+1}/(41 \cdot 20s)\right) 10^{5(n-2)} C_{\tilde{M}} \sigma^{k(n-2)} \\ \leq \mu^l\left[B_{41s}(x_p)\right] &\leq \mathcal{N}\left(n, \sigma^{k+1}/(41 \cdot 20 \cdot 10^2 \sigma^k)\right) 10^{5(n-2)} C_{\tilde{M}} \sigma^{k(n-2)} \\ &\leq n^{n/2} 10^{5n} \sigma^{-n} 10^{5(n-2)} C_{\tilde{M}} \sigma^{k(n-2)} = C_{\tilde{M}} n^{n/2} 10^{10n-10} \sigma^{2-2n} 41^{n-2} s^{n-2}\end{aligned}$$
   because $\sigma^{k+1} \leq 41s$.

The statement is so proved. □

Now I'm able to to prove theorem 3.5.14. Choose any $R > 0$ so that constant B_R defined in theorem 3.5.2 satisfies $B = B_R \geq 20C$ with $C > 0$ is a constant depending only on n and Λ such that $El_s^w(w) \subseteq B_{C^{1/2}s}\mu$ and $\beta_{x,r}(\mu) \leq \tilde{C} \beta_{0,C^{1/2}r}\left(\Psi_x^{\#}\mu\right)$. Recall that $t \in \left(\sigma^{k+1}, \sigma^k\right]$ and take any $s \in (0, t\rho/B]$, $w \in \operatorname{supp} \mu_1^l$, then by theorem 3.5.2 I've

$$\beta_{0,s}\left(\Psi_w^{\#}\mu_1^l\right)^2 \leq \frac{C_{\tilde{M}}}{s^{n-2}} \int_{El_s^w(w)} \left[\Delta_{Bs}^{As}(v,z) + s\right] \, d\mu_1^l(z)$$

In particular I have

$$\beta_{w,s}\left(\mu_1^l\right)^2 \le \frac{C_{\tilde{M}}}{s^{n-2}} \int_{B_{Cs}(w)} \left[\Delta_{Bs}^{As}(v,z)+s\right] \mathcal{X}_{[0,20Cs]}(s_z)\ d\mu_1^l\,(z)$$

because by proposition 3.5.16

$$s_z \le 20\,|z-w| \le 20Cs \le Bs$$

for every z

Let $\bar{w} \in \operatorname{supp}\mu_1^l$, $T < \sigma^k/B$ and $s < T\rho$, then

$$\begin{aligned}
&\int_{B_{T\rho}(\bar{w})} \beta_{w,s}\left(\mu_1^l\right)^2\ d\mu_1^l\,(w) \\
&\quad\le \frac{C_{\tilde{M}}}{s^{n-2}} \int_{B_{T\rho}(\bar{w})} \int_{B_{Cs}(w)} \left[\Delta_{Bs}^{As}(v,z)+s\right] \mathcal{X}_{[0,20Cs]}(s_z)\ d\mu_1^l\,(z)\ d\mu_1^l\,(w) \\
&\quad\le \frac{C_{\tilde{M}}}{s^{n-2}} \int_{B_{T\rho+Cs}(\bar{w})} \int_{B_{Cs}(z)} \left[\Delta_{Bs}^{As}(v,z)+s\right] \mathcal{X}_{[0,20Cs]}(s_z)\ d\mu_1^l\,(w)\ d\mu_1^l\,(z) \\
&\quad= \frac{C_{\tilde{M}}}{s^{n-2}} \int_{B_{T\rho+Cs}(\bar{w})} \mu_1^l\left[B_{Cs}(z)\right] \left[\Delta_{Bs}^{As}(v,z)+s\right] \mathcal{X}_{[0,20Cs]}(s_z)\ d\mu_1^l\,(z) \\
&\quad\le C_{\tilde{M}} C_\sigma \int_{B_{(1+C)T\rho}(\bar{w})} \left[\Delta_{Bs}^{As}(v,z)+s\right] \mathcal{X}_{[0,20Cs]}(s_z)\ d\mu_1^l\,(z)
\end{aligned}$$

where I've used 3.5.19 and the simple relation

$$w \in B_{T\rho}(\bar{w}), z \in B_{Cs}(w) \Rightarrow z \in B_{T\rho+Cs}(\bar{w}), w \in B_{Cs}(z)$$

Next I consider the oscillation of μ defined in the following way:

$$\operatorname{Osc}_\mu^\sigma(\bar{w},t) = \int_{B_t(\bar{w})} \sum_{j=0}^{+\infty} \beta_{y,\sigma^j t}(\mu)^2\ d\mu\,(y)$$

and set

$$\sigma \le \frac{B}{A}$$

in this way I get

$$\Delta_{B\sigma^j T\rho}^{A\sigma^j T\rho} \le \Delta_{B\sigma^j T\rho}^{B\sigma^{j-1} T\rho}$$

and notice that

$$\begin{aligned}
\sum_{j=0}^{N} \Delta_{B\sigma^j T\rho}^{B\sigma^{j-1} T\rho} &= \sum_{j=0}^{N} \left[\mathcal{J}_M(v,B\sigma^{j-1}T\rho) - \mathcal{J}_M(v,B\sigma^j T\rho) + B\sigma^{j-1}T\rho\,(1-\sigma)\right] \\
&= \mathcal{J}_M(v,B\sigma^{-1}T\rho) - \mathcal{J}_M(v,B\sigma^N T\rho) + B\sigma^{-1}T\rho\left(1-\sigma^{N+1}\right) \\
&\le \Delta_{BT\sigma^N\rho}^{\rho} + \rho
\end{aligned}$$

because $B\sigma^{-1}T \leq \sigma^{k-1} \leq \sigma \leq 1$, in particular

$$\int_{B_{(1+C)T\rho}(\bar{w})} \sum_{j=0}^{N_z} \left[\Delta_{B\sigma^j T\rho}^{B\sigma^{j-1}T\rho}(v,z) + \sigma^j T\rho\right] \mathcal{X}_{[0,20C\sigma^j T\rho]}(s_z)\ d\mu_1^l(z)$$
$$\leq \int_{B_{(1+C)T\rho}(\bar{w})} \left[\mathcal{I}(v,z,\rho) - \mathcal{I}(v,z,s_z) + 2\rho\right]\ d\mu_1^l(z)$$
$$\leq C_{\tilde{M}}\tau\mu_1^l\left[B_{(1+C)T\rho}(\bar{w})\right]$$

for $\tilde{r}_{\tilde{M}}$ sufficiently small, and by proposition 3.5.15 by assuming $s_{\bar{w}}/20 \leq (1+C)T\rho$ I finally have

$$\mathrm{Osc}_{\mu}^{\sigma}(\bar{w}, T\rho) \leq C_{\tilde{M},\sigma}\tau\mu_1^l\left[B_{(1+C)T\rho}(\bar{w})\right] \leq C_{\tilde{M},\sigma}\tau T^{n-2}\rho^{n-2} \tag{3.5.10}$$

Also if $T\rho \leq (1+C)T\rho \leq s_{\bar{w}}/20$ then $\operatorname{supp}\mu_1^l \cap B_{T\rho}(\bar{w}) = \{\bar{w}\}$ so $\mathrm{Osc}_{\mu}^{\sigma}(\bar{w}, T\rho) = 0$ and (3.5.10) holds for every $T < \sigma^k/B$. Notice also that if $T < \sigma^k/(2B)$ then (3.5.10) holds for every $\bar{w}$ because if $\operatorname{supp}\mu_1^l \cap B_{T\rho}(\bar{w}) = \emptyset$ then $\mathrm{Osc}_{\mu}^{\sigma}(\bar{w}, T\rho) = 0$, otherwise let $w' \in \operatorname{supp}\mu_1^l \cap B_{T\rho}(\bar{w})$, then $B_{T\rho}(\bar{w}) \subseteq B_{2T\rho}(w')$ and

$$\mathrm{Osc}_{\mu_1^l}^{\sigma}(\bar{w}, T\rho) \leq C\,\mathrm{Osc}_{\mu_1^l}^{\sigma}\left(w', 2T\rho\right) \leq CC_{\tilde{M},\sigma}\tau T^{n-2}\rho^{n-2}$$

with C depending only on n and Λ.

Assuming τ sufficiently small, in other words $\tau \leq \delta_\sigma C_{\sigma,\tilde{M}}^{-1}$ where δ_σ is the constant in the following theorem:

Theorem 3.5.20 (Naber and Valtorta, [18, Remark 3.9])**.** *There exists a constant $C > 0$ depending only on n such that, for every $\sigma > 0$, exists a constant $\delta_\sigma > 0$ that satisfies the following property: for every finite collection of pairwise disjoint balls $\{B_{r_i}(x_i)\}_i$ on $B_2 \subseteq \mathbb{R}^n$ and $\mu = \sum_i r_i^{n-2}[\![x_i]\!]$ if the following inequality holds*

$$\mathrm{Osc}_{\mu}^{\sigma}(x,\eta) < \delta_\sigma \eta^{n-2}$$

for every $B_\eta(x) \subseteq B_2$ then $\mu(B_2) \leq C$.

I can apply that result to a rescaled version of μ_1^l to get $\mu_1^l[B_t(w)] \leq C_{\tilde{M}}t^{n-2}$

$$\mu^l[B_t(w_0)] \leq 2\mu_1^l[B_{11t}(w_0)] \leq C_{\tilde{M}}t^{n-2}$$

and theorem 3.5.14 is proved.

Proof of theorem 1.0.1

From theorem 3.0.1 I'm able to prove the rectifiability of $\mathcal{K}_v$ by using the following result:

Theorem 3.5.21 (Azzam and Tolsa, [3])**.** *Let $K \subseteq \mathbb{R}^n$ be a Borel set with $\mathcal{H}^l(K) < +\infty$ for some $0 < l \leq n$.*

Then K is l-rectifiable if and only if

$$\int_0^1 \frac{\beta_{x,r}\left(\mathcal{H}^l \llcorner K\right)^2}{r}\,dr < +\infty$$

Consider any $0 < \sigma < 1$, I get for every Borel finite measure μ

$$\int_0^1 \frac{\beta_{x,r}(\mu)^2}{r}\, dr = \sum_j \int_{\sigma^{j+1}}^{\sigma^j} \frac{\beta_{x,r}(\mu)^2}{r}\, dr \le \sum_j \beta_{x,\sigma^j}(\mu)^2 \int_{\sigma^{j+1}}^{\sigma^j} \left(\frac{\sigma^j}{r}\right)^n \frac{1}{r}$$
$$= \frac{1-\sigma^n}{n\sigma^n} \sum_j \beta_{x,\sigma^j}(\mu)^2$$

so if the right hand is finite for $\mu = \mathcal{H}^{n-2} \llcorner \mathcal{K}_v$ then $\mathcal{K}_v \cap B_1$ is $(n-2)$-rectifiable. Remember also that for $\mathcal{H}^l$-a.e. $x \in K$

$$2^{-l} \le \limsup_{r\to 0^+} \frac{\mathcal{H}^l[K \cap B_r(x)]}{\omega_l r^l} \le 1$$

I consider now the following sets

$$E_l = \left\{ x \in \mathcal{K}_v \cap B_1 \,\middle|\, \mathcal{H}^{n-2}\left[\mathcal{K}_v \cap B_r(x)\right] \le 2\omega_{n-2} r^{n-2}\ \forall 0 < r < \frac{1}{l} \right\}$$

clearly E_l is an increasing sequence of sets with $\bigcup_l E_l = (\mathcal{K}_v \cap B_1) \cup N_0$ with $\mathcal{H}^{n-2}(N_0) = 0$. I want to use theorem 3.5.21 to $K = E_l$ in order to prove rectifiability of $\mathcal{K}_v \cap B_1$.

Let $\mu_l = \mathcal{H}^{n-2} \llcorner E_l$, $\sigma \in (0,1)$ and $q \in \mathbb{N}$ so that $\sigma^{q-1} \le 1$. Up to rescale v I've

$$\sum_{j=q}^{+\infty} \int_{B_1} \beta_{x,\sigma^j}(\mu_l)^2\, d\mu_l(x)$$
$$\le \sum_j \frac{C_{\tilde{M}}}{\sigma^{(n-2)j}} \int_{B_1} \int_{B_{C\sigma^j}(y)} \left[\Delta_{A\sigma^j}^{B\sigma^j}(v,z) + \sigma^j\right] d\mu_l(z)\, d\mu_l(y)$$
$$\le \sum_j \int_{B_{3/2}} \mu_l\left[B_{C\sigma^j}(z)\right] \left[\Delta_{A\sigma^j}^{B\sigma^j}(v,z) + \sigma^j\right] d\mu_l(z)$$
$$\le C_{\tilde{M}} \mu_l\left[B_{3/2}\right] \overset{\text{(theorem 3.0.1)}}{<} +\infty$$

for σ sufficiently small.

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