Technische Universität Berlin
Institut für Mathematik

Data-driven interpolation of dynamical systems with delay

Philipp Schulze   Benjamin Unger

Preprint 17-2015

Preprint-Reihe des Instituts für Mathematik
Technische Universität Berlin
http://www.math.tu-berlin.de/preprints

Preprint 17-2015   August 2015

Data-driven interpolation of dynamical systems with delay*

Philipp Schulze[†‡] Benjamin Unger[†]

January 8, 2016

Abstract

We present a data-driven realization for systems with delay, which generalizes the Loewner framework. The realization is obtained with low computational cost directly from measured data of the transfer function. The internal delay is estimated by solving a least-square optimization over some sample data. Our approach is validated by several examples, which indicate the need for preserving the delay structure in the reduced model.

Keywords: Data-driven model reduction, delay systems, Loewner framework, moment matching, tangential interpolation, delay recovery, structure-preserving model reduction

AMS(MOS) subject classification: 93A30, 37M99, 34K06, 93A15

1 Introduction

Nowadays, it is common to describe a physical or chemical system by a mathematical surrogate model. Such models are often given by (partial) differential equations and may be used for analysis, control, and optimization. The demand for high fidelity models results in large-scale dynamical systems, for which classical numerical methods may be too time or memory consuming. In such cases, often an analytically justified and numerically stable approximation of the input-output map is desirable leading to the field of model order reduction (MOR) (for an overview see [1, 3] and the references therein). Many of these MOR methods require access to the internal dynamics of the full system in terms of a state-space realization. This assumption can be relaxed by employing data-driven MOR techniques that construct low-dimensional models directly from measurements. Let $H(s) \in \mathbb{C}^{p,m}$ denote the transfer function of a system, where m and p are the numbers of inputs and outputs, respectively. Since the input-output behavior of a system is characterized by its transfer function, measurements of H seem appropriate to construct a low-dimensional model. We assume measurements $H(\lambda_i)r_i = w_i$ and $\ell_i H(\mu_i) = v_i$ to be given, i. e., we have

$$\begin{aligned} \text{right interpolatation data} \quad & \{(\lambda_i, r_i, w_i) \mid \lambda_i \in \mathbb{C},\, r_i \in \mathbb{C}^m,\, w_i \in \mathbb{C}^p,\, i = 1, \ldots, \rho\} \quad \text{and} \\ \text{left interpolation data} \quad & \{(\mu_i, \ell_i, v_i) \mid \mu_i \in \mathbb{C},\, \ell_i^T \in \mathbb{C}^p,\, v_i^T \in \mathbb{C}^m,\, i = 1, \ldots, \rho\}. \end{aligned} \tag{1.1}$$

*Research supported by the Collaborative Research Center 910 *Control of self-organizing nonlinear systems: Theoretical methods and concepts of application.*

†Institut für Mathematik, TU Berlin, Germany, {pschulze,unger}@math.tu-berlin.de.

‡The author is supported by the Collaborative Research Center 1029 *Substantial efficiency increase in gas turbines through direct use of coupled unsteady combustion and flow dynamics*

Examples of measurements yielding data in the form (1.1) are scattering parameters for frequency response objects (S-parameters) and admittance parameters for interconnects, which can be obtained by a vector network analyzer [2].

In this paper we assume that the transfer function is based on a system with (possibly unknown) delay and study a *generalized realization problem with internal delay*: Given the data (1.1), construct matrices E_ρ, $A_{1,\rho}$, $A_{2,\rho}$, B_ρ, and C_ρ, such that the transfer function

$$H_\rho(s) = C_\rho \left(sE_\rho - A_{1,\rho} - \mathrm{e}^{-\tau s} A_{2,\rho}\right)^{-1} B_\rho \tag{1.2}$$

with delay $\tau \geq 0$ interpolates the data, i. e.,

$$w_i = H(\lambda_i)r_i = H_\rho(\lambda_i)r_i \qquad \text{and} \qquad v_i = \ell_i H(\mu_i) = \ell_i H_\rho(\mu_i) \qquad \text{for } i = 1, \ldots, \rho.$$

The transfer function (1.2) corresponds to a realization $\Sigma_\rho = (E_\rho, A_{1,\rho}, A_{2,\rho}, B_\rho, C_\rho)$ of the form

$$E_\rho \dot{x}_\rho(t) = A_{1,\rho} x_\rho(t) + A_{2,\rho} x_\rho(t-\tau) + B_\rho u(t), \tag{1.3a}$$

$$y_\rho(t) = C_\rho x_\rho(t), \tag{1.3b}$$

which serves as a low-dimensional model.

A generalized realization problem without delay for the data (1.1) is solved in [16], leading to the Loewner framework. The resulting realization is a generalized state-space representation of the form

$$\begin{aligned} -\mathbb{L}\dot{x}(t) &= -\mathbb{L}_\sigma x(t) + Vu(t), \\ y(t) &= Wx(t), \end{aligned} \tag{1.4}$$

where $\mathbb{L}$ and $\mathbb{L}_\sigma$ are the Loewner matrix and shifted Loewner matrix, respectively, and V and W are matrices consisting of the data (for more details see section 2).

The rate of change of realistic models often depends not only on the current time point, but also on the configuration at previous time instances, which leads to time-delay systems. Popular examples are nonlinear optics, chemical reactor systems, population dynamics, and delayed feedback control (cf. [9] and the references within). Finding a realization of a system with delay by means of the Loewner framework results in the system (1.4), that does not feature the delay term and hence cannot reflect the dynamics of the inherently infinite-dimensional delay system.

In this paper, we propose a generalization of the Loewner realization to systems with delay. The main contributions are described in the following.

- First, we present a generalization to delay systems based on extensions of the Loewner matrix and the shifted Loewner matrix. This approach provides a realization as in (1.3) with the coupling $A_{2,\rho} = -E_\rho$ (see section 3).

- Since the coupling $A_{2,\rho} = -E_\rho$ appears to be rather restrictive, we consider general conditions for interpolating the transfer function (1.2) in section 4. For this purpose, the moment matching framework from [4] (see also [10] for an earlier contribution) is extended to DAE systems with delay.

- Based on the conditions for moment matching, we derive a framework to obtain a realization (1.3) with the coupling $A_{2,\rho} = \alpha E_\rho + \beta A_{1,\rho}$ with scalar parameters α and β (see section 5). Note that this is a more general form than the realization from section 3, which is the special case with $\alpha = -1$ and $\beta = 0$.

- Since the delay is unknown in general, we propose a methodology to determine a delay $\tau_{\kappa,\rho}$, which is optimal in the sense that it minimizes the interpolation error for a set of sampled test data of the transfer function (see section 6). In the same fashion, optimal parameters $\alpha_{\kappa,\rho}$ and $\beta_{\kappa,\rho}$ may be calculated.

In section 7 we apply the proposed framework to several examples. A comparison to alternative techniques from the literature including the original Loewner framework reveals the necessity of preserving the delay structure in the realization.

2 Notation and Preliminary Results

Recall that the realization (1.3) is given by

$$E_\rho \dot{x}_\rho(t) = A_{1,\rho} x_\rho(t) + A_{2,\rho} x_\rho(t-\tau) + B_\rho u(t), \tag{2.1a}$$
$$y_\rho(t) = C_\rho x_\rho(t), \tag{2.1b}$$

where $x_\rho(t) \in \mathbb{R}^r$ for $r \leq \rho$, $u(t) \in \mathbb{R}^m$, and $y_\rho(t) \in \mathbb{R}^p$ denote, respectively, the *state*, *input*, and *output* of the model. As common in the delay literature, the right-hand derivative $\frac{\mathrm{d}}{\mathrm{d}t}$ of a piecewise smooth function f is denoted by $\dot{f}$ [13]. The symbol I_n stands for the identity matrix of dimension $n \times n$. The input u is assumed to be sufficiently smooth and the system (2.1) is equipped with the initial condition (also called *history function*)

$$x(t) = \phi(t) \qquad \text{for } t \in (-\tau, 0], \tag{2.2}$$

which is assumed to be identically zero, i. e., $\phi \equiv 0$. If $\det(sE_\rho - A_{1,\rho} - \mathrm{e}^{-\tau s} A_{2,\rho})$ is not vanishing identically, then the Laplace transformation of (2.1) yields the transfer function

$$H_\rho(s) = C_\rho \left(sE_\rho - A_{1,\rho} - \mathrm{e}^{-\tau s} A_{2,\rho}\right)^{-1} B_\rho. \tag{2.3}$$

For convenience, we call the set $\{s \in \mathbb{C} \mid \det\left(sE_\rho - A_{1,\rho} - \mathrm{e}^{-\tau s} A_{2,\rho}\right) \neq 0\}$ the *resolvent set* of $sE_\rho - A_{1,\rho} - \mathrm{e}^{-\tau s} A_{2,\rho}$. For $\tau > 0$, the system Σ_ρ as in (2.1) can be solved on consecutive time intervals $((K-1)\tau, K\tau]$, such that (2.1a) reduces to the *associated differential-algebraic equation* (DAE) [12]

$$E_\rho \dot{x}_\rho(t) = A_{1,\rho} x_\rho(t) + f_\rho(t) \tag{2.4}$$

with inhomogeneity $f_\rho(t) = A_{2,\rho} x_\rho(t-\tau) + B_\rho u(t)$. Note that we use the terminology DAE here, since we allow E_ρ to be singular. This procedure is known as the (Bellman) method of steps [6, 12]. For $\tau = 0$, (2.1a) reduces to the DAE

$$E_\rho \dot{x}_\rho(t) = A_\rho x(t) + B_\rho u(t) \tag{2.5}$$

with $A_\rho := A_{1,\rho} + A_{2,\rho}$. For convenience, we also refer to (2.5) as *associated DAE*. In both cases, a unique solution of (2.4) and (2.5) is given if and only if the pencils $sE_\rho - A_\rho$ and $sE_\rho - A_{1,\rho}$, respectively are regular [14], i. e., $\det(sE_\rho - A_\rho)$ and $\det(sE_\rho - A_{1,\rho})$ do not vanish identically. For $\tau = 0$, the regularity of the pencil of (2.5) directly implies the existence of the transfer function (2.3). In the case $\tau > 0$ the existence of the transfer function (2.3) is guaranteed by the subsequent lemma.

Lemma 2.1 *If $\tau > 0$ and the pencil $sE_\rho - A_{1,\rho}$ is regular, then* $\det(sE_\rho - A_{1,\rho} - \mathrm{e}^{-\tau s} A_{2,\rho}) \not\equiv 0$.

Proof. Suppose there exists $v \neq 0$ such that $(sE_\rho - A_{1,\rho} - \mathrm{e}^{-\tau s} A_{2,\rho})\, v = 0$ for all $s \in \mathbb{C}$. This implies (since $(E_\rho, A_{1,\rho})$ is regular) that

$$\mathrm{e}^{\tau s} v = (sE_\rho - A_{1,\rho})^{-1} A_{2,\rho} v$$

for all s not in the spectrum of $(E_\rho, A_{1,\rho})$. This contradicts $v \neq 0$, because $(sE_\rho - A_{1,\rho})^{-1}$ is a rational function in s. □

We briefly recall the Loewner realization introduced in [16]. The Loewner matrix $\mathbb{L}$ and shifted Loewner matrix $\mathbb{L}_\sigma$ are defined as solutions of the Sylvester equations

$$\mathbb{L}\Lambda - M\mathbb{L} = LW - VR, \tag{2.6}$$

$$\mathbb{L}_\sigma \Lambda - M\mathbb{L}_\sigma = LW\Lambda - MVR, \tag{2.7}$$

respectively, where

$$\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_\rho), \qquad M = \operatorname{diag}(\mu_1, \ldots, \mu_\rho), \qquad R = \begin{bmatrix} r_1 & \ldots & r_\rho \end{bmatrix},$$

$$W = \begin{bmatrix} w_1 & \ldots & w_\rho \end{bmatrix}, \qquad L = \begin{bmatrix} \ell_1 \\ \vdots \\ \ell_\rho \end{bmatrix}, \qquad \text{and} \qquad V = \begin{bmatrix} v_1 \\ \vdots \\ v_\rho \end{bmatrix}.$$

Direct computation shows that if $\lambda_i \neq \mu_j$ for $i, j = 1, \ldots, \rho$, then the unique solutions of (2.6) and (2.7), respectively, are given by

$$[\mathbb{L}]_{i,j} = \frac{v_i r_j - \ell_i w_j}{\mu_i - \lambda_j} \qquad \text{and} \qquad [\mathbb{L}_\sigma]_{i,j} = \frac{\mu_i v_i r_j - \lambda_j \ell_i w_j}{\mu_i - \lambda_j}. \tag{2.8}$$

Remark 2.2 Both matrices can be assembled efficiently via matrix-matrix operations of size $\rho \times \rho$ using standard tools for scalar, vector, and matrix operations (BLAS). More precisely, the numerator is given by $VR - LW$ for the Loewner matrix and $MVR - LW\Lambda$ for the shifted Loewner matrix. The denominator is implemented as $\mu e^T - e\lambda^T$, where $\mu, \lambda \in \mathbb{C}^\rho$ are vectors containing μ_i and λ_i and $e \in \mathbb{R}^\rho$ is a vector filled with ones.

Theorem 2.3 ([16, Lemma 5.1]) *Let* $\det(\tilde{s}\mathbb{L} - \mathbb{L}_\sigma) \neq 0$ *for all* $\tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}$. *Then the system*

$$\begin{aligned} -\mathbb{L}\dot{x}_\rho(t) &= -\mathbb{L}_\sigma x_\rho(t) + Vu(t), \\ y_\rho(t) &= Wx_\rho(t) \end{aligned} \tag{2.9}$$

is a minimal realization of an interpolant of the data, i. e., its transfer function

$$H_\rho(s) = W(\mathbb{L}_\sigma - s\mathbb{L})^{-1} V$$

interpolates the data (1.1).

Let ε denote the machine precision. If $\det(\tilde{s}\mathbb{L} - \mathbb{L}_\sigma) = \mathcal{O}(\varepsilon)$ for some $\tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}$, then one can use the truncated singular value decomposition (SVD) [11] of $s\mathbb{L} - \mathbb{L}_\sigma$ to truncate the numerically vanishing singular values as in the next theorem, originally stated in [16].

Theorem 2.4 ([16, Theorem 5.1]) *Suppose that*

$$\operatorname{rank}(\tilde{s}\mathbb{L} - \mathbb{L}_\sigma) = \operatorname{rank}\begin{bmatrix}\mathbb{L} & \mathbb{L}_\sigma\end{bmatrix} = \operatorname{rank}\begin{bmatrix}\mathbb{L} \\ \mathbb{L}_\sigma\end{bmatrix} =: r \qquad \textit{for all } \tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}.$$

Then a minimal realization of an interpolant of the data is given by the system

$$\begin{aligned} -Y^*\mathbb{L}X\dot{x}_r(t) &= -Y^*\mathbb{L}_\sigma X x_r(t) + Y^*Vu(t), \\ y_r(t) &= WXx_r(t), \end{aligned} \tag{2.10}$$

where $Y \in \mathbb{C}^{\rho,r}$ and $X \in \mathbb{C}^{\rho,r}$ are the orthogonal factors of the truncated SVD

$$\tilde{s}\mathbb{L} - \mathbb{L}_\sigma = Y\Sigma X^*$$

for any $\tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}$, where $\Sigma \in \mathbb{C}^{r,r}$ is positive definite and diagonal.

Recently, a generalization of the Loewner framework for a special class of delay systems, where $A_{1,\rho} = 0$ in (2.1), was introduced in [18]. There, the interpolant is constructed by means of Theorem 2.3 leading to the subsequent result, written here in slightly different notation.

Theorem 2.5 ([18, Theorem 3]) *Suppose that $\lambda_i \mathrm{e}^{\tau\lambda_i} \neq \mu_j \mathrm{e}^{\tau\mu_j}$ for $i, j = 1, \ldots, \rho$. Let $\mathbb{L}^{(\tau)}$ and $\mathbb{L}_\sigma^{(\tau)}$ denote the Loewner matrix and shifted Loewner matrix, respectively, associated with the transformed data $(\lambda_i \mathrm{e}^{\tau\lambda_i}, r_i, \mathrm{e}^{-\tau\lambda_i} w_i)$ and $(\mu_i \mathrm{e}^{\tau\mu_i}, \ell_i, \mathrm{e}^{-\tau\mu_i} v_i)$ for $i = 1, \ldots, \rho$. If $\det\left(\tilde{s}\mathbb{L}^{(\tau)} - \mathrm{e}^{-\tau\tilde{s}}\mathbb{L}_\sigma^{(\tau)}\right) \neq 0$ for all $\tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}$, then the transfer function*

$$H_\rho(s) = W\mathrm{e}^{-\tau\Lambda}\left(\mathrm{e}^{-\tau s}\mathbb{L}_\sigma^{(\tau)} - s\mathbb{L}^{(\tau)}\right)^{-1}\mathrm{e}^{-\tau M}V$$

of the system

$$\begin{aligned} -\mathbb{L}^{(\tau)}\dot{x}_\rho(t) &= -\mathrm{e}^{-\tau s}\mathbb{L}_\sigma^{(\tau)} x_\rho(t-\tau) + \mathrm{e}^{-\tau M}Vu(t) \\ y_\rho(t) &= W\mathrm{e}^{-\tau\Lambda}x_\rho(t) \end{aligned}$$

is an interpolant of the original data (1.1).

Again, the condition $\det(\tilde{s}\mathbb{L}^{(\tau)} - \mathrm{e}^{-\tau\tilde{s}}\mathbb{L}_\sigma^{(\tau)}) \neq 0$ can be relaxed by applying the truncated SVD as in Theorem 2.4.

3 A Loewner Framework for Systems with Delay

In this section, we aim at extending the Loewner matrix and shifted Loewner matrix to the time delay case. Suppose that $\mu_j + \mathrm{e}^{-\tau\mu_j} \neq \lambda_i + \mathrm{e}^{-\tau\lambda_i}$ for all $i, j = 1, \ldots, \rho$. Define the matrices $\mathbb{T}$ and $\mathbb{T}_\sigma$

$$\begin{aligned} [\mathbb{T}]_{i,j} &:= \frac{v_i r_j - \ell_i w_j}{\mu_i + \mathrm{e}^{-\tau\mu_i} - (\lambda_j + \mathrm{e}^{-\tau\lambda_j})} \qquad \text{and} \\ [\mathbb{T}_\sigma]_{i,j} &:= \frac{(\mu_i + \mathrm{e}^{-\tau\mu_i})v_i r_j - (\lambda_j + \mathrm{e}^{-\tau\lambda_j})\ell_i w_j}{\mu_i + \mathrm{e}^{-\tau\mu_i} - (\lambda_j + \mathrm{e}^{-\tau\lambda_j})}. \end{aligned} \tag{3.1}$$

Lemma 3.1 *Suppose that the denominators in* (3.1) *are not zero. Then the matrices* $\mathbb{T}$ *and* $\mathbb{T}_\sigma$ *are the unique solutions of the Sylvester equations*

$$\mathbb{T}(\Lambda + \mathrm{e}^{-\tau\Lambda}) - (M + \mathrm{e}^{-\tau M})\mathbb{T} = LW - VR, \tag{3.2}$$

$$\mathbb{T}_\sigma(\Lambda + \mathrm{e}^{-\tau\Lambda}) - (M + \mathrm{e}^{-\tau M})\mathbb{T}_\sigma = LW(\Lambda + \mathrm{e}^{-\tau\Lambda}) - (M + \mathrm{e}^{-\tau M})VR, \tag{3.3}$$

respectively.

Proof. It is well known that under the stated assumptions the Sylvester equations possess unique solutions (cf. [1]). Multiplication of (3.2) and (3.3) from the left by e_i^T and from the right by e_j yields the matrices as in (3.1). □

Similar as in [16], we get the following identities.

Corollary 3.2 *Suppose that the denominators in* (3.1) *are not zero. Then we have*

$$\mathbb{T}_\sigma - \mathbb{T}(\Lambda + \mathrm{e}^{-\tau\Lambda}) = VR \qquad \text{and} \qquad \mathbb{T}_\sigma - (M + \mathrm{e}^{-\tau M})\mathbb{T} = LW.$$

Proof. Multiplying (3.2) by $(\Lambda + \mathrm{e}^{-\tau\Lambda})$ from the right and subtracting it from (3.3) yields

$$\begin{aligned}\left(\mathbb{T}_\sigma - \mathbb{T}(\Lambda + \mathrm{e}^{-\tau\Lambda}) - VR\right)(\Lambda + \mathrm{e}^{-\tau\Lambda})& \\ - (M + \mathrm{e}^{-\tau M})\left(\mathbb{T}_\sigma - \mathbb{T}(\Lambda + \mathrm{e}^{-\tau\Lambda}) - VR\right) &= 0.\end{aligned} \tag{3.4}$$

Since $\Lambda + \mathrm{e}^{-\tau\Lambda}$ and $M + \mathrm{e}^{-\tau M}$ have no eigenvalues in common (by assumption), the solution of the Sylvester equation (3.4) is zero, which yields the first equality. By the same argument, we get the second identity if we multiply (3.2) by $(M + \mathrm{e}^{-\tau M})$ from the left and subtract it from (3.3). □

Theorem 3.3 *Suppose that the denominators in* (3.1) *are not zero and*

$$\det\left(\mathbb{T}_\sigma - (\tilde{s} + \mathrm{e}^{-\tau\tilde{s}})\mathbb{T}\right) \neq 0 \qquad \text{for all } \tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}. \tag{3.5}$$

Then the delay descriptor system given by $\Sigma_\rho = (-\mathbb{T}, -\mathbb{T}_\sigma, \mathbb{T}, V, W)$ *with associated transfer function* $H_\rho(s) = W\left(\mathbb{T}_\sigma - s\mathbb{T} - \mathrm{e}^{-\tau s}\mathbb{T}\right)^{-1} V$ *interpolates the data, i. e.,*

$$H_\rho(\lambda_i)r_i = w_i \qquad \text{and} \qquad \ell_i H_\rho(\mu_i) = v_i \qquad \text{for } i = 1, \ldots, \rho.$$

Moreover, the pencil of the associated DAE of Σ_ρ *is regular.*

Proof. Multiply (3.2) by $(s + \mathrm{e}^{-\tau s})$ and subtract it from (3.3) to obtain

$$\begin{aligned}&\left(\mathbb{T}_\sigma - \mathbb{T}(s + \mathrm{e}^{-\tau s})\right)(\Lambda + \mathrm{e}^{-\tau\Lambda}) - (M + \mathrm{e}^{-\tau M})\left(\mathbb{T}_\sigma - \mathbb{T}(s + \mathrm{e}^{-\tau s})\right) = \\ &LW\left((\Lambda + \mathrm{e}^{-\tau\Lambda}) - (s + \mathrm{e}^{-\tau s})I_\rho\right) - \left(M + \mathrm{e}^{-\tau M} - (s + \mathrm{e}^{-\tau s})I_\rho\right)VR.\end{aligned} \tag{3.6}$$

Multiply (3.6) from the right by the i-th unit vector e_i to obtain

$$\begin{aligned}&\left(\mathbb{T}_\sigma - \mathbb{T}(s + \mathrm{e}^{-\tau s})\right)(\lambda_i + \mathrm{e}^{-\tau\lambda_i})e_i - (M + \mathrm{e}^{-\tau M})\left(\mathbb{T}_\sigma - \mathbb{T}(s + \mathrm{e}^{-\tau s})\right)e_i = \\ &\qquad LW\left((\lambda_i + \mathrm{e}^{-\tau\lambda_i}) - (s + \mathrm{e}^{-\tau s})\right)e_i - \left(M + \mathrm{e}^{-\tau M} - (s + \mathrm{e}^{-\tau s})I_\rho\right)Vr_i,\end{aligned}$$

which is equivalent to

$$\left(\left(\lambda_i + \mathrm{e}^{-\tau\lambda_i}\right) I_\rho - \left(M + \mathrm{e}^{-\tau M}\right)\right) \left(\mathbb{T}_\sigma - \mathbb{T}(s + \mathrm{e}^{-\tau s})\right) e_i = \\ LW \left((\lambda_i + \mathrm{e}^{-\tau\lambda_i}) - (s + \mathrm{e}^{-\tau s})\right) e_i - \left(M + \mathrm{e}^{-\tau M} - (s + \mathrm{e}^{-\tau s}) I_\rho\right) V r_i.$$

Setting $s = \lambda_i$ implies

$$\left(\left(\lambda_i + \mathrm{e}^{-\tau\lambda_i}\right) I_\rho - \left(M + \mathrm{e}^{-\tau M}\right)\right) \left(\mathbb{T}_\sigma - \mathbb{T}(\lambda_i + \mathrm{e}^{-\tau\lambda_i})\right) e_i \\ = \left((\lambda_i + \mathrm{e}^{-\tau\lambda_i}) I_\rho - (M + \mathrm{e}^{-\tau M})\right) V r_i. \quad (3.7)$$

By assumption, the matrix $(\lambda_i + \mathrm{e}^{-\tau\lambda_i}) I_\rho - (M + \mathrm{e}^{-\tau M})$ is nonsingular. In particular, (3.7) is equivalent to

$$\left(\mathbb{T}_\sigma - \mathbb{T}(\lambda_i + \mathrm{e}^{-\tau\lambda_i})\right) e_i = V r_i$$

and assumption (3.5) implies

$$e_i = \left(\mathbb{T}_\sigma - \mathbb{T}(\lambda_i + \mathrm{e}^{-\tau\lambda_i})\right)^{-1} V r_i.$$

Multiplication by W from the left yields

$$H_\rho(\lambda_i) r_i = w_i = W e_i = W \left(\mathbb{T}_\sigma - \mathbb{T}(\lambda_i + \mathrm{e}^{-\tau\lambda_i})\right)^{-1} V r_i.$$

The proof for the left interpolation conditions works analogously by multiplying (3.6) from the left by e_i^T and setting $s = \mu_i$. The regularity of the pencil of the associated DAE follows directly from (3.5). □

Remark 3.4 Theorem 3.3 is a generalization of Theorem 2.3 in the following sense. If $\tau = 0$, then system (1.3) reduces to the descriptor system

$$E_\rho \dot{x}_\rho(t) = A_\rho x_\rho(t) + B_\rho u(t), \\ y_\rho(t) = C_\rho x_\rho(t)$$

with $A_\rho = A_{1,\rho} + A_{2,\rho}$ and the realization via the matrices $\mathbb{T}$ and $\mathbb{T}_\sigma$ yields

$$[E_\rho]_{i,j} = -[\mathbb{T}]_{i,j} = -\frac{v_i r_j - \ell_i w_j}{\mu_i + 1 - (\lambda_j + 1)} = -[\mathbb{L}]_{i,j}\,, \\ [A_\rho]_{i,j} = [\mathbb{T}]_{i,j} - [\mathbb{T}_\sigma]_{i,j} = \frac{v_i r_j - \ell_i w_j}{\mu_i - \lambda_j} - \frac{(\mu_i + 1) v_i r_j - (\lambda_j + 1) \ell_i w_j}{\mu_i - \lambda_j} = -[\mathbb{L}_\sigma]_{i,j}\,,$$

which are the Loewner matrix and shifted Loewner matrix, respectively.

Remark 3.5 The realization given in Theorem 3.3 is restrictive in the sense that E_ρ and $A_{2,\rho}$ coincide apart from a constant factor. Thus, there are delay descriptor systems whose dynamical behavior is not captured accurately by this approach. To circumvent this restriction, we consider a more general framework in the next section.

4 Moment Matching for Linear Delay Descriptor Systems

Subsequently, we present a general framework for matching the moments of a linear time-invariant time-delay descriptor system of the form

$$\begin{aligned} E\dot{x}(t) &= A_1x(t) + A_2x(t-\tau) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \tag{4.1}$$

with state-space dimension n, input dimension m, output dimension p, and matrices E, A_1, A_2, B, and C of appropriate size. The transfer function of the system is given by

$$H(s) = C\left(sE - A_1 - \mathrm{e}^{-\tau s}A_2\right)^{-1}B$$

and we assume that $\tilde{s}E - A_1 - \mathrm{e}^{-\tau\tilde{s}}A_2$ is nonsingular for some $\tilde{s} \in \mathbb{C}$. We generalize the moment matching approach as introduced in [4] to delay DAE systems of the form (4.1). A similar method for delay differential equations is derived in [19].

Moment matching is a technique to reduce a large-dimensional system with given state-space representation by interpolation of the transfer function and its derivatives. In the next section, we revert this process to derive conditions for a data-driven realization. In the data-driven approach, we assume that we do not have information about the derivatives of the transfer function. This is the reason for only considering 0-moments (see Definition 4.1 below) and ignoring moments of higher order. In the case that we have multiple inputs and multiple outputs (MIMO) the transfer function H is a matrix function we interpolate along interpolation directions. This is known as tangential interpolation [10].

Definition 4.1 (0-moment) *Consider $\lambda_i, \mu_i \in \mathbb{C}$ in the resolvent set of $sE - A_1 - \mathrm{e}^{-\tau s}A_2$. The right 0-moment $\underline{\eta}(\lambda_i, r_i)$ of the system (4.1) at λ_i in direction r_i is defined as*

$$\underline{\eta}(\lambda_i, r_i) := C\left(\lambda_i E - A_1 - A_2\mathrm{e}^{-\tau\lambda_i}\right)^{-1}Br_i = H(\lambda_i)r_i.$$

The left 0-moment $\overline{\eta}(\mu_i, \ell_i)$ of the system (4.1) at μ_i in direction ℓ_i is defined as

$$\overline{\eta}(\mu_i, \ell_i) := \ell_i C\left(\mu_i E - A_1 - A_2\mathrm{e}^{-\tau\mu_i}\right)^{-1}B = \ell_i H(\mu_i).$$

Note that the interpolation data (1.1) consists of the interpolation points λ_i and μ_i, the directions r_i and ℓ_i, and the right and left 0-moments. An equivalent notion of moment is presented in the following proposition.

Proposition 4.2 *Consider system (4.1) and complex numbers $\lambda_i, \mu_i \in \mathbb{C}$ in the resolvent set of $sE - A_1 - \mathrm{e}^{-\tau s}A_2$.*

1. *The right 0-moment of system (4.1) at λ_i in direction r_i is given by*

$$\underline{\eta}(\lambda_i, r_i) = C\pi_i,$$

where π_i is the unique solution of the linear algebraic system

$$\lambda_i E\pi_i = A_1\pi_i + A_2\pi_i\mathrm{e}^{-\tau\lambda_i} + Br_i. \tag{4.2}$$

2. *The left* 0*-moment of system (4.1) at* μ_i *in direction* ℓ_i *is given by*

$$\overline{\eta}\left(\mu_i, \ell_i\right) = \psi_i B,$$

where ψ_i *is the unique solution of the linear algebraic system*

$$\mu_i \psi_i E = \psi_i A_1 + \psi_i A_2 \mathrm{e}^{-\tau \mu_i} + \ell_i C. \tag{4.3}$$

Proof. Since λ_i is in the resolvent set, there exists a unique solution of (4.2) given by $\pi_i = \left(\lambda_i E - A_1 - A_2 \mathrm{e}^{-\tau\lambda_i}\right)^{-1} B r_i$ and hence $C\pi_i = \underline{\eta}\left(\lambda_i, r_i\right)$. The second statement follows analogously. □

Remark 4.3 As a direct consequence from Proposition 4.2, the set of right 0-moments corresponding to $\{\lambda_1, \dots, \lambda_\rho\}$ and $\{r_1, \dots, r_\rho\}$ is given by $C\Pi$, where Π is the unique solution of the matrix equation

$$E\Pi\Lambda = A_1\Pi + A_2\Pi\mathrm{e}^{-\tau\Lambda} + BR, \tag{4.4}$$

where Λ and R are defined as in section 2. Analogously, the set of left 0-moments corresponding to $\{\mu_1, \dots, \mu_\rho\}$ and $\{\ell_1, \dots, \ell_\rho\}$ is given by ΨB, where Ψ is the unique solution of the matrix equation

$$M\Psi E = \Psi A_1 + \mathrm{e}^{-\tau M}\Psi A_2 + LC. \tag{4.5}$$

With the help of (4.4) and (4.5), we can derive families of reduced order models achieving moment matching at the given interpolation points.

Theorem 4.4 *Consider the time-delay descriptor system* Σ_ρ *given in* (1.3) *and let right and left interpolation data of the transfer function of* (4.1) *be given as in* (1.1)*, where* $C\Pi = W$ *and* $\Psi B = V$ *(see Proposition 4.2 and Remark 4.3). Furthermore, assume that the matrix* $\tilde{s}E_\rho - A_{1,\rho} - \mathrm{e}^{-\tau\tilde{s}}A_{2,\rho}$ *is nonsingular for all* $\tilde{s} \in \{\lambda_1, \dots, \lambda_\rho\} \cup \{\mu_1, \dots, \mu_\rho\}$.

1. *System* (1.3) *matches the right* 0*-moments* $C\Pi$ *if and only if*

$$C\Pi = C_\rho \underline{P},$$

where Π *is the unique solution of* (4.4) *and* $\underline{P}$ *is the unique solution of the matrix equation*

$$E_\rho \underline{P}\Lambda = A_{1,\rho}\underline{P} + A_{2,\rho}\underline{P}\mathrm{e}^{-\tau\Lambda} + B_\rho R. \tag{4.6}$$

2. *System* (1.3) *matches the left* 0*-moments* ΨB *if and only if*

$$\Psi B = \overline{P} B_\rho,$$

where Ψ *is the unique solution of* (4.5) *and* $\overline{P}$ *is the unique solution of the matrix equation*

$$M\overline{P}E_\rho = \overline{P}A_{1,\rho} + \mathrm{e}^{-\tau M}\overline{P}A_{2,\rho} + LC_\rho. \tag{4.7}$$

Proof. The claim is a straightforward consequence of the notion of moments as introduced in Proposition 4.2 and Remark 4.3. Following this, the right 0-moments of system (1.3) are given by $C_\rho \underline{P}$, where $\underline{P}$ is the unique solution of the matrix equation (4.6). Likewise, the right 0-moments of the original system (4.1) are given by $C\Pi$, where Π is the unique solution of (4.4). Consequently, the right moments are matched if and only if $C\Pi = C_\rho \underline{P}$. The matching of the left 0-moments is proven analogously. □

Corollary 4.5

1. *The family of reduced order models*

$$\begin{aligned} E_\rho \dot{x}_\rho(t) &= (E_\rho \Lambda - A_{2,\rho} \mathrm{e}^{-\tau\Lambda} - B_\rho R) x_\rho(t) + A_{2,\rho} x_\rho(t-\tau) + B_\rho u(t), \\ y_\rho(t) &= C\Pi x_\rho(t) \end{aligned} \tag{4.8}$$

   matches the right 0*-moments* $C\Pi$ *if* $\tilde{s}E_\rho - (E_\rho\Lambda - A_{2,\rho}\mathrm{e}^{-\tau\Lambda} - B_\rho R) - \mathrm{e}^{-\tau\tilde{s}}A_{2,\rho}$ *is nonsingular for all* $\tilde{s} \in \{\lambda_1, \ldots, \lambda_\rho\}$.

2. *The family of reduced order models*

$$\begin{aligned} E_\rho \dot{x}_\rho(t) &= (ME_\rho - \mathrm{e}^{-\tau M} A_{2,\rho} - LC_\rho) x_\rho(t) + A_{2,\rho} x_\rho(t-\tau) + \Psi B u(t), \\ y_\rho(t) &= C_\rho x_\rho(t) \end{aligned} \tag{4.9}$$

   matches the left 0*-moments* ΨB *if* $\tilde{s}E_\rho - (ME_\rho - \mathrm{e}^{-\tau M}A_{2,\rho} - LC_\rho) - \mathrm{e}^{-\tau\tilde{s}}A_{2,\rho}$ *is nonsingular for all* $\tilde{s} \in \{\mu_1, \ldots, \mu_\rho\}$.

Proof. The assertions follow directly from Theorem 4.4 by setting $\overline{P} = I_\rho = \underline{P}$. This leads to $A_{1,\rho} = E_\rho\Lambda - A_{2,\rho}\mathrm{e}^{-\tau\Lambda} - B_\rho R$ and $C_\rho = C\Pi$ for the family matching the right 0-moments and $A_{1,\rho} = ME_\rho - \mathrm{e}^{-\tau M}A_{2,\rho} - LC_\rho$ and $B_\rho = \Psi B$ for the family matching the left 0-moments. □

Remark 4.6 Note that the families of reduced order models (4.8) and (4.9) are parameterized by E_ρ, $A_{2,\rho}$, B_ρ, and C_ρ. These parameters are only restricted by the generic conditions stated in Corollary 4.5. This freedom may be exploited in order to tailor the reduced order model to additional requirements, e. g., preserving structures or properties of the original system (1.3).

To find a family of reduced order models that matches both, the left and right 0-moments, we compare the coefficient matrices of (4.8) and (4.9). This results in the following theorem.

Theorem 4.7 *The family of reduced order models*

$$\begin{aligned} E_\rho \dot{x}_\rho(t) &= \left(E_\rho \Lambda - A_{2,\rho} \mathrm{e}^{-\tau\Lambda} - \Psi B R\right) x_\rho(t) + A_{2,\rho} x_\rho(t-\tau) + \Psi B u(t) \\ y_\rho(t) &= C\Pi x_\rho(t) \end{aligned} \tag{4.10}$$

matches the 0*-moments* ΨB *and* $C\Pi$ *if the matrix* $\tilde{s}E_\rho - \left(E_\rho\Lambda - A_{2,\rho}\mathrm{e}^{-\tau\Lambda} - \Psi BR\right) - \mathrm{e}^{-\tau\tilde{s}}A_{2,\rho}$ *is nonsingular for all* $\tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}$ *and the Sylvester equation*

$$E_\rho\Lambda - ME_\rho + \mathrm{e}^{-\tau M}A_{2,\rho} - A_{2,\rho}\mathrm{e}^{-\tau\Lambda} = \Psi BR - LC\Pi \tag{4.11}$$

is satisfied.

A more general version of Theorem 4.7 may be derived. Instead of coinciding coefficients, it is sufficient to require that the family (4.8) also matches the left 0-moments of (4.9) or that the family (4.9) also matches the right 0-moments of (4.8). This leads to an additional degree of freedom, which we omit for the sake of clarity.

The Sylvester equation (4.11) does not depend on the system matrices E, A_1, A_2, B and C explicitly, since we have the identities $C\Pi = W$ and $\Psi B = V$. Thus, Theorem 4.7 allows us to construct reduced order models from data only. The obtained realization (4.10) is parameterized by the matrices E_ρ and $A_{2,\rho}$, which are coupled via (4.11).

Corollary 4.8 *For the special case $A_{2,\rho} = -E_\rho$ the unique solution of the Sylvester equation* (4.11) *is given by the matrix $E_\rho = -\mathbb{T}$. Moreover, from Lemma 3.2 we have*

$$A_\rho = \left(E_\rho \Lambda - A_{2,\rho}\mathrm{e}^{-\tau\Lambda} - \Psi BR\right) = E_\rho\left(\Lambda + \mathrm{e}^{-\tau\Lambda}\right) - VR = -\mathbb{T}_\sigma.$$

Hence, for the particular case $A_{2,\rho} = -E_\rho$, system (4.10) *is equivalent to the realization in Theorem 3.3.*

Remark 4.9 By solving the equations (4.4) and (4.5) for BR and LC, respectively, and substituting the obtained expressions into the equations (4.10) and (4.11), it is easy to show that a reduced order model matching the left and right 0-moments is given by the projection

$$E_\rho = \Psi E\Pi, \qquad A_{1,\rho} = \Psi A_1 \Pi, \qquad A_{2,\rho} = \Psi A_2 \Pi, \qquad B_\rho = \Psi B, \qquad \text{and} \qquad C_\rho = C\Pi.$$

Example 4.10 Consider the system

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} &= \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t-\pi) \\ x_2(t-\pi) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \\ y(t) &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \end{aligned} \tag{4.12}$$

with transfer function $H(s) = -(1 + \mathrm{e}^{-\pi s})^{-1}$. We choose $\rho = 1$, $\lambda = 0$, and $\mu = \frac{1}{2}$. Since (4.12) is a single-input single-output (SISO) system, we set $\ell = r = 1$ for the tangential directions. The projection matrices Ψ and Π are given by

$$\Psi = \begin{bmatrix} 0 & -(1 + \mathrm{e}^{-\frac{\tau}{2}})^{-1} \end{bmatrix} \qquad \text{and} \qquad \Pi = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

and the reduced order model is given by

$$\begin{aligned} 0 &= \frac{1}{2}\left(1 + \mathrm{e}^{-\frac{\tau}{2}}\right)^{-1} x_\rho(t) + \frac{1}{2}\left(1 + \mathrm{e}^{-\frac{\tau}{2}}\right)^{-1} x_\rho(t-\tau) - \left(1 + \mathrm{e}^{-\frac{\tau}{2}}\right)^{-1} u(t), \\ y_\rho(t) &= -\frac{1}{2} x_\rho(t). \end{aligned}$$

Simple calculations show that the left and right 0-moments are matched by the transfer function of the reduced order model.

Remark 4.11 In [18] the authors consider the special case $A_{1,\rho} = 0$ (cf. Theorem 2.5 above). In terms of Corollary 4.5 and Theorem 4.7, this implies that E_ρ must satisfy the Sylvester equation

$$E_\rho \Lambda \mathrm{e}^{\tau\Lambda} - \mathrm{e}^{\tau M} M E_\rho = VR\mathrm{e}^{\tau\Lambda} - \mathrm{e}^{\tau M} LW$$

and $A_{2,\rho}$ is given by

$$A_{2,\rho} = \left(E_\rho \Lambda - VR\right)\mathrm{e}^{\tau\Lambda} = \mathrm{e}^{\tau M}\left(E_\rho - LW\right).$$

Consequently, the result obtained in [18] is a special case of the realization presented in Theorem 4.7.

5 Generalized Delay Loewner Framework

Recall that the reduced models (4.10) allow for choosing either E_ρ or $A_{2,\rho}$. Inspired by the mass- and stiffness-proportional damping, normally referred to as Rayleigh damping [8] in mechanical systems, we make the ansatz $A_{2,\rho} = \alpha E_\rho + \beta A_{1,\rho}$ with some constants α and β. For the special setting $\alpha = -1$ and $\beta = 0$, we recover the framework presented in section 3.

Lemma 5.1 *Suppose that $1 + \beta \mathrm{e}^{-\tau \tilde{s}} \neq 0$ for all $\tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}$. The reduced order model*

$$\begin{aligned} E_\rho \dot{x}_\rho(t) &= A_{1,\rho} x_\rho(t) + (\alpha E_\rho + \beta A_{1,\rho})\, x_\rho(t-\tau) + B_\rho u(t), \\ y_\rho(t) &= C_\rho x_\rho(t) \end{aligned} \tag{5.1}$$

matches the 0-moments ΨB and $C\Pi$ if the matrix $\tilde{s} E_\rho - A_{1,\rho} - \mathrm{e}^{-\tau \tilde{s}} (\alpha E_\rho + \beta A_{1,\rho})$ is non-singular for all $\tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}$,

$$\begin{aligned} C_\rho &= C\Pi, \qquad B_\rho = \Psi B, \\ A_{1,\rho} &= \left(E_\rho \left(\Lambda - \alpha \mathrm{e}^{-\tau \Lambda}\right) - B_\rho R\right) \left(I_\rho + \beta \mathrm{e}^{-\tau\Lambda}\right)^{-1}, \end{aligned}$$

and E_ρ satisfies the Sylvester equation

$$\begin{aligned} E_\rho(\Lambda - \alpha \mathrm{e}^{-\tau\Lambda})(I_\rho + \beta \mathrm{e}^{-\tau\Lambda})^{-1} - (I_\rho + \beta \mathrm{e}^{-\tau M})^{-1}(M - \alpha \mathrm{e}^{-\tau M}) E_\rho = \\ B_\rho R (I_\rho + \beta \mathrm{e}^{-\tau\Lambda})^{-1} - (I_\rho + \beta \mathrm{e}^{-\tau M})^{-1} L C_\rho. \end{aligned} \tag{5.2}$$

Proof. For the right 0-moments, from Corollary 4.5 we get the conditions $C_\rho = C\Pi$ and

$$A_{1,\rho} = E_\rho \Lambda - A_{2,\rho} \mathrm{e}^{-\tau\Lambda} - B_\rho R = E_\rho \Lambda - (\alpha E_\rho + \beta A_{1,\rho}) \mathrm{e}^{-\tau\Lambda} - B_\rho R,$$

which is equivalent to

$$A_{1,\rho} = \left(E_\rho(\Lambda - \alpha \mathrm{e}^{-\tau\Lambda}) - B_\rho R\right)(I_\rho + \beta \mathrm{e}^{-\tau\Lambda})^{-1}. \tag{5.3}$$

Analogously, to match the left 0-moments, the condition $B_\rho = \Psi B$ must be satisfied and the matrix $A_{1,\rho}$ is given by

$$A_{1,\rho} = (I_\rho + \beta \mathrm{e}^{-\tau M})^{-1} \left((M - \alpha \mathrm{e}^{-\tau M}) E_\rho - L C_\rho\right). \tag{5.4}$$

By comparison of the coefficients (as in the proof of Theorem 4.7), we deduce that (5.3) and (5.4) must coincide, which yields the Sylvester equation (5.2). □

Corollary 5.2 *Suppose that the assumptions of Lemma 5.1 are satisfied and the matrices*

$$(\Lambda - \alpha \mathrm{e}^{-\tau\Lambda})(I_\rho + \beta \mathrm{e}^{-\tau\Lambda})^{-1} \qquad \text{and} \qquad (I_\rho + \beta \mathrm{e}^{-\tau M})^{-1}(M - \alpha \mathrm{e}^{-\tau M})$$

have no common eigenvalues. Then the unique solution of the Sylvester equation (5.2) is given by

$$[E_\rho]_{i,j} = \frac{\dfrac{v_i r_j}{1 + \beta \mathrm{e}^{-\tau\lambda_j}} - \dfrac{\ell_i w_j}{1 + \beta \mathrm{e}^{-\tau\mu_i}}}{\dfrac{\lambda_j - \alpha \mathrm{e}^{-\tau\lambda_j}}{1 + \beta \mathrm{e}^{-\tau\lambda_j}} - \dfrac{\mu_i - \alpha \mathrm{e}^{-\tau\mu_i}}{1 + \beta \mathrm{e}^{-\tau\mu_i}}}. \tag{5.5}$$

Proof. Using the identities

$$C_\rho = C\Pi = W \qquad \text{and} \qquad B_\rho = \Psi B = V$$

the result follows by multiplying (5.2) from the left by e_i^T and from the right by e_j. ∎

Lemma 5.3 *Under the assumptions of Corollary 5.2, the matrix $A_{1,\rho}$ from Lemma 5.1 satisfies the Sylvester equation*

$$\begin{aligned} A_{1,\rho}(\Lambda - \alpha \mathrm{e}^{-\tau\Lambda})(I_\rho + \beta \mathrm{e}^{-\tau\Lambda})^{-1} - (I_\rho + \beta \mathrm{e}^{-\tau M})^{-1}(M - \alpha \mathrm{e}^{-\tau M})A_{1,\rho} = \\ (I_\rho + \beta \mathrm{e}^{-\tau M})^{-1}(M - \alpha \mathrm{e}^{-\tau M})VR(I_\rho + \beta \mathrm{e}^{-\tau\Lambda})^{-1} \\ -(I_\rho + \beta \mathrm{e}^{-\tau M})^{-1}LW(\Lambda - \alpha \mathrm{e}^{-\tau\Lambda})(I_\rho + \beta \mathrm{e}^{-\tau\Lambda})^{-1} \end{aligned} \tag{5.6}$$

and is given by

$$[A_{1,\rho}]_{i,j} = \frac{\dfrac{\mu_i - \alpha \mathrm{e}^{-\tau\mu_i}}{1 + \beta \mathrm{e}^{-\tau\mu_i}} \dfrac{v_i r_j}{1 + \beta \mathrm{e}^{-\tau\lambda_j}} - \dfrac{\ell_i w_j}{1 + \beta \mathrm{e}^{-\tau\mu_i}} \dfrac{\lambda_j - \alpha \mathrm{e}^{-\tau\lambda_j}}{1 + \beta \mathrm{e}^{-\tau\lambda_j}}}{\dfrac{\lambda_j - \alpha \mathrm{e}^{-\tau\lambda_j}}{1 + \beta \mathrm{e}^{-\tau\lambda_j}} - \dfrac{\mu_i - \alpha \mathrm{e}^{-\tau\mu_i}}{1 + \beta \mathrm{e}^{-\tau\mu_i}}}. \tag{5.7}$$

Proof. The proof proceeds analogously to the proof of Lemma 3.2. Multiplying (5.2) with $(\Lambda - \alpha \mathrm{e}^{-\tau\Lambda})(I + \beta \mathrm{e}^{-\tau\Lambda})^{-1}$ and subtracting it from (5.6) results in a Sylvester equation with a unique solution that yields (5.4). The result for equation (5.3) follows likewise. Multiplying (5.6) by e_i^T from the left and e_j from the right yields the representation (5.7). ∎

Summarizing the previous discussion, we have shown the following theorem.

Theorem 5.4 *Let*

$$\det\left(\left(\tilde{s} - \alpha \mathrm{e}^{-\tau\tilde{s}}\right) E_\rho - \left(1 + \beta \mathrm{e}^{-\tau\tilde{s}}\right) A_{1,\rho}\right) \neq 0 \qquad \textit{for all } \tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}. \tag{5.8}$$

Then, under the assumptions of Corollary 5.2, the system

$$\begin{aligned} E_\rho \dot{x}_\rho(t) &= A_{1,\rho} x_\rho(t) + (\alpha E_\rho + \beta A_{1,\rho}) x_\rho(t - \tau) + V u(t), \\ y_\rho(t) &= W x_\rho(t) \end{aligned}$$

with E_ρ and $A_{1,\rho}$ as in (5.5) *and* (5.7)*, respectively, is an interpolant of the data, i. e., its transfer function*

$$H_\rho(s) = W\left(\left(s - \alpha \mathrm{e}^{-\tau s}\right) E_\rho - \left(1 + \beta \mathrm{e}^{-\tau s}\right) A_{1,\rho}\right)^{-1} V$$

interpolates the data (1.1)*. Moreover, the pencil of the associated DAE is regular.*

Remark 5.5 Following the idea of [18], we can (formally) derive the same results by only employing the Loewner realization theorem (Theorem 2.3). Using the proportional ansatz $A_{2,\rho} = \alpha E_\rho + \beta A_{1,\rho}$ with some constants α and β, we can rewrite the transfer function as

$$\begin{aligned} H(s) &= C_\rho \left(sE_\rho - A_{1,\rho} - \mathrm{e}^{-\tau s}\left(\alpha E_\rho + \beta A_{1,\rho}\right)\right)^{-1} B_\rho \\ &= C_\rho \left(\frac{s - \alpha \mathrm{e}^{-\tau s}}{1 + \beta \mathrm{e}^{-\tau s}} E_\rho - A_{1,\rho}\right)^{-1} B_\rho \frac{1}{1 + \beta \mathrm{e}^{-\tau s}} \\ &= G(f(s)) \frac{1}{1 + \beta \mathrm{e}^{-\tau s}}, \end{aligned}$$

where G is the transfer function of a generalized state-space system without delay and f is given by $f(s) = \frac{s-\alpha \mathrm{e}^{-\tau s}}{1+\beta \mathrm{e}^{-\tau s}}$. The Loewner matrices are then constructed for the transfer function $G(s)$ with transformed data $(f(\lambda_i), r_i(1+\beta \mathrm{e}^{-\tau\lambda_i})^{-1}, w_i)$ and $(f(\mu_i), (1+\beta \mathrm{e}^{-\tau\mu_i})^{-1}\ell_i, v_i)$. Note that this allows to efficiently implement the system matrices similar as in Remark 2.2.

With Remark 5.5 we can immediately relax assumption (5.8) in Theorem 5.4.

Theorem 5.6 *Suppose that the assumptions of Corollary 5.2 are satisfied and that*

$$\operatorname{rank}\left(\frac{\tilde{s}-\alpha \mathrm{e}^{-\tau\tilde{s}}}{1+\beta \mathrm{e}^{-\tau\tilde{s}}} E_\rho - A_{1,\rho}\right) = \operatorname{rank}\begin{bmatrix} E_\rho & A_{1,\rho}\end{bmatrix} = \operatorname{rank}\begin{bmatrix} E_\rho \\ A_{1,\rho}\end{bmatrix} =: r$$

holds for all $\tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}$. Then, an interpolant of the data (1.1) *is given by the system*

$$\begin{aligned} -Y^* E_\rho X \dot{x}_r(t) &= -Y^* A_{1,\rho} X x_r(t) + Y^*(\alpha E_\rho + \beta A_{1,\rho}) X x_r(t-\tau) + Y^* V u(t), \\ y_r(t) &= W X x_r(t), \end{aligned} \tag{5.9}$$

where $Y \in \mathbb{C}^{\rho,r}$ and $X \in \mathbb{C}^{\rho,r}$ are the orthogonal factors of the truncated SVD

$$\frac{\tilde{s}-\alpha \mathrm{e}^{-\tau\tilde{s}}}{1+\beta \mathrm{e}^{-\tau\tilde{s}}}\mathbb{L} - \mathbb{L}_\sigma = Y \Sigma X^*$$

for any $\tilde{s} \in \{\lambda_i\} \cup \{\mu_i\}$, where $\Sigma \in \mathbb{C}^{r,r}$ is positive definite and diagonal.

Proof. The proof follows directly from Remark 5.5 and Theorem 2.4. □

6 Delay Reconstruction

So far, we have assumed that the delay $\tau \geq 0$ is known, which in general is not the case. In practical applications, e. g. neuronal processing [20], only a range for the delay time is known from experiments. Checking the moment matching conditions as stated in Theorem 4.4 implies that the reduced order model matches the moments V and W for any choice of the delay time. We propose the following strategy to recover a delay term $\tau_{\kappa,\rho} \approx \tau$. For some $\kappa < \rho$, split the interpolation data (1.1) as

$$\begin{aligned} &\{(\lambda_i, r_i, w_i) \mid \lambda_i \in \mathbb{C}, r_i \in \mathbb{C}^m, w_i \in \mathbb{C}^p, i = 1, \ldots, \kappa, \kappa+1, \ldots, \rho\} \qquad \text{and} \\ &\{(\mu_i, \ell_i, v_i) \mid \mu_i \in \mathbb{C}, \ell_i^T \in \mathbb{C}^p, v_i^T \in \mathbb{C}^m, i = 1, \ldots, \kappa, \kappa+1, \ldots, \rho\} \end{aligned} \tag{6.1}$$

and construct the matrices $E_\kappa(\tau)$, $A_{1,\kappa}(\tau)$, $A_{2,\kappa}(\tau)$, B_κ, and C_κ only from the data (λ_i, r_i, w_i) and (μ_i, ℓ_i, v_i) for $i = 1, \ldots, \kappa$. Note that the transfer function

$$H_\kappa(s, \tau) = C_\kappa(sE_\kappa(\tau) - A_{1,\kappa}(\tau) - \mathrm{e}^{-\tau s} A_{2,\kappa}(\tau))^{-1} B_\kappa$$

interpolates the data independently from τ. The remaining data (λ_i, r_i, w_i) and (μ_i, ℓ_i, v_i) for $i = \kappa+1, \ldots, \rho$, in the following called *test data*, is used to fit the transfer function in a least-squares sense, i. e., we solve the minimization problem

$$\min_{\tau \in [\tau_-, \tau_+]} J_{\kappa,\rho}(\tau) := \frac{1}{2}\sum_{i=\kappa+1}^{\rho} \left(\|H_\kappa(\lambda_i, \tau) r_i - w_i\|^2 + \|\ell_i H_\kappa(\mu_i, \tau) - v_i\|^2\right) \tag{6.2}$$

over all delay times τ in a realistic range $[\tau_-, \tau_+]$ and denote the minimizer by $\tau_{\kappa,\rho}$. With the 'optimal' delay $\tau_{\kappa,\rho}$, we can rebuild the realization with the complete data set as in Theorem 5.6.

Note that the cost functional $J_{\kappa,\rho}$ is of nonconvex type and gradient-based optimization might yield a local instead of the global minimum. Hence, either a good choice of the initial value for the optimization is required, or global optimization methods like evolutionary algorithms (see [21] and the references within) can be employed.

In the previous section we have introduced the additional parameters α and β. These parameters are degrees of freedom, which can be used to tailor the realization to additional data. Hence, the strategy used to find the delay time may be extended to determine good values for α and β. More precisely, the cost functional $J_{\kappa,\rho}$ is given by

$$J_{\kappa,\rho}(\tau,\alpha,\beta) := \frac{1}{2}\sum_{i=\kappa+1}^{\rho}\left(\|H_\kappa(\lambda_i,\tau,\alpha,\beta)r_i - w_i\|^2 + \|\ell_i H_\kappa(\mu_i,\tau,\alpha,\beta) - v_i\|^2\right)$$

and the minimization in (6.2) can be performed over τ, α, and β simultaneously. Note that in contrast to the delay time τ, there are in general no a priori known ranges for the parameters α and β. The choice of good upper and lower bounds for α and β is not within the scope of this paper. Instead, we propose to optimize over different intervals, which may be picked randomly, and then use the result that yields the smallest value of $J_{\kappa,\rho}$.

7 Examples

In all subsequent examples, we denote the transfer function of the original model with H. The transfer function of the Loewner realization (Theorem 2.3) is denoted by H_L. By H_T we denote the transfer function based on transformed data from [18] (Theorem 2.5) and the generalized delay Loewner realization (Theorem 5.4) is referred to as H_D. For details of the

Table 7.1: Transfer functions for the different approaches

Transfer Function	**Reference**
$H(s)$	original model
$H_L(s) = W(\mathbb{L}_\sigma - s\mathbb{L})^{-1}V$	[16]
$H_T(s) = W\mathrm{e}^{-\tau\Lambda}\left(\mathrm{e}^{-\tau s}\mathbb{L}_\sigma^{(\tau)} - s\mathbb{L}^{(\tau)}\right)^{-1}\mathrm{e}^{-\tau M}V$	[18]
$H_D(s) = W\left(sE_\rho - A_{1,\rho} - \mathrm{e}^{-\tau s}\left(\alpha E_\rho + \beta A_{1,\rho}\right)\right)^{-1}V$	Theorem 5.4

transfer functions, see Table 7.1. Whenever necessary, the redundant parts in the realizations are truncated by means of the truncated SVD as outlined in Theorem 2.4 and Theorem 5.6.

Example 7.1 Consider again the system from Example 4.10 given by

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}1 & \frac{1}{2}\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0 & \frac{1}{2}\\ 0 & 1\end{bmatrix}\begin{bmatrix}x_1(t-\pi)\\ x_2(t-\pi)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t),$$
$$y(t) = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}$$

together with the interpolation data $\lambda = 0$ and $\mu = \frac{1}{2}$. Recall that the transfer function H is given by $H(s) = -(1 + \mathrm{e}^{-\pi s})^{-1}$ and note that $A_2 = A_1 - E$, i. e., the considered system exhibits the structure $A_2 = \alpha E + \beta A_1$. Comparison of the transfer functions

$$H_L(s) = -\frac{1}{s\pi(1 - \frac{1}{e}) + 2}, \quad H_D(s) = -\frac{1}{1 + \mathrm{e}^{-\pi s}}, \quad \text{and} \quad H_T(s) = -\frac{1}{s\pi(1 - \frac{1}{e}) + 2\mathrm{e}^{-\pi s}}$$

of the different realizations shows that the transfer function H_D is the only one matching the full model exactly. This is due to the structure of the system, which can neither be captured by H_L nor by H_T.

Example 7.2 For Example 7.1 we employ the methodology of section 6 to recover the delay time τ. We use the additional interpolation point $\lambda = 10$. For the minimization of the cost functional (6.2), we apply the optimizer `fmincon` (MATLAB version R2015a with standard settings) with initial value $\tau_0 = 5$ to recover the original delay $\tau = \pi$ up to 8 decimal places. This example fortifies the approach introduced in section 6.

Example 7.3 We test our approach with the delay model from [5] given by the $n \times n$ matrices

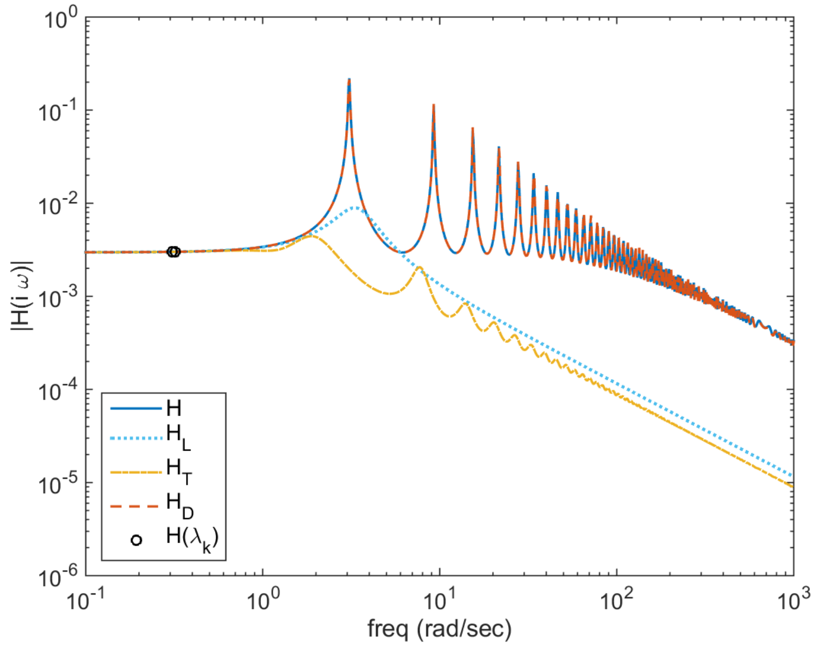
$$E = \theta I_n + T, \qquad A_1 = \frac{1}{\tau}\left(\frac{1}{\zeta} + 1\right)(T - \theta I_n), \qquad A_2 = \frac{1}{\tau}\left(\frac{1}{\zeta} - 1\right)(T - \theta I_n),$$

where T is an $n \times n$ matrix with ones on the sub- and superdiagonal, in the $(1, 1)$, and in the (n, n) position and zeros everywhere else. We choose $n = 500$, $\tau = 1$, $\zeta = 0.01$, and $\theta = 5$. The input matrix $B \in \mathbb{R}^n$ has ones in the first two components and zeros everywhere else and we choose $C = B^T$. Note that the matrices A_1 and A_2 satisfy the relation $A_2 = (\frac{1}{\zeta}+1)/(\frac{1}{\zeta}-1)A_1$ and hence feature the structure $A_2 = \alpha E + \beta A_1$. We pick $\rho = 10$ random interpolation points λ_k on the imaginary axis between $10^{-1}\imath$ and $10^3\imath$ together with their complex conjugates μ_k. The realizations based on the data from $k = 1, 2$ (i. e., $\kappa = 2$ in (6.1)) and exact parameters τ, α, and β are displayed in Figure 7.1. Clearly, only our framework captures the transfer function accurately. In contrast, the Loewner realization (H_L) and the realization based on transformed data (H_T) yield poor approximations that do not capture one single peak of the original model.

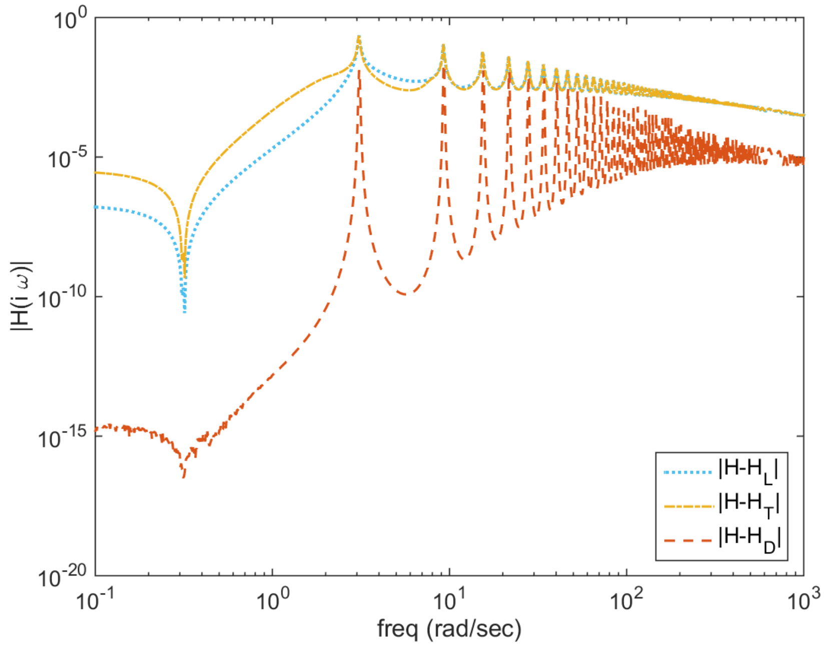
Next, the test data is used to find estimates for the parameters τ, α, and β from the data by means of minimizing the cost functional (6.2) over the parameter set. The optimization is performed via the global optimization algorithm `particleswarm` of the global optimization toolbox [15]. This algorithm evaluates the cost functional in (6.2) at a collection of sample points called particles at each steps. After this evaluation, every particle is moved in the parameter range and reevaluated. This process continues until some stopping criterion is satisfied. In this case, the algorithm terminated with

$$\tau_{\kappa,\rho} = 0.999367, \qquad \alpha_{\kappa,\rho} = 0.652009, \qquad \text{and} \qquad \beta_{\kappa,\rho} = 0.995628, \tag{7.1}$$

i. e., the delay is captured accurately.



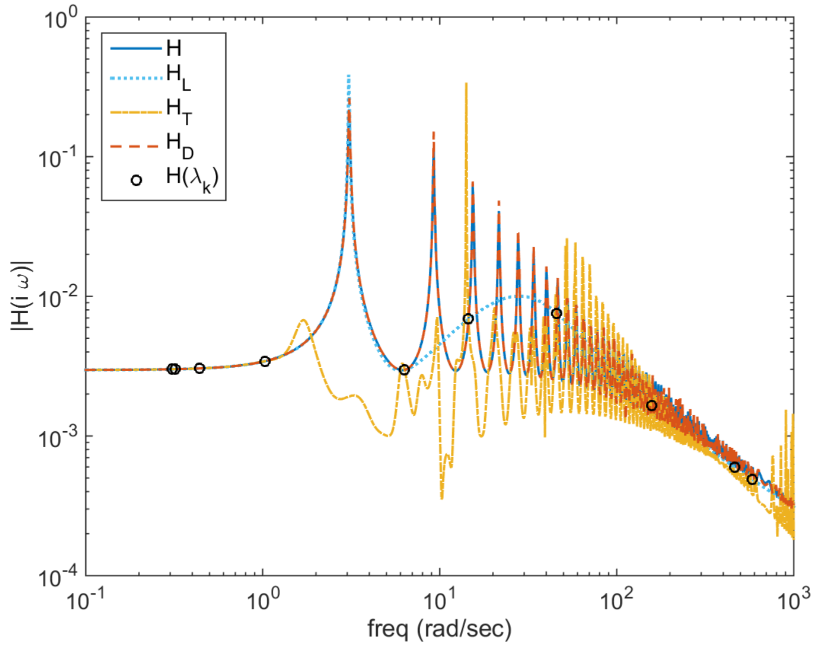
(a) Bode plot of H, H_L, H_D, and H_T.



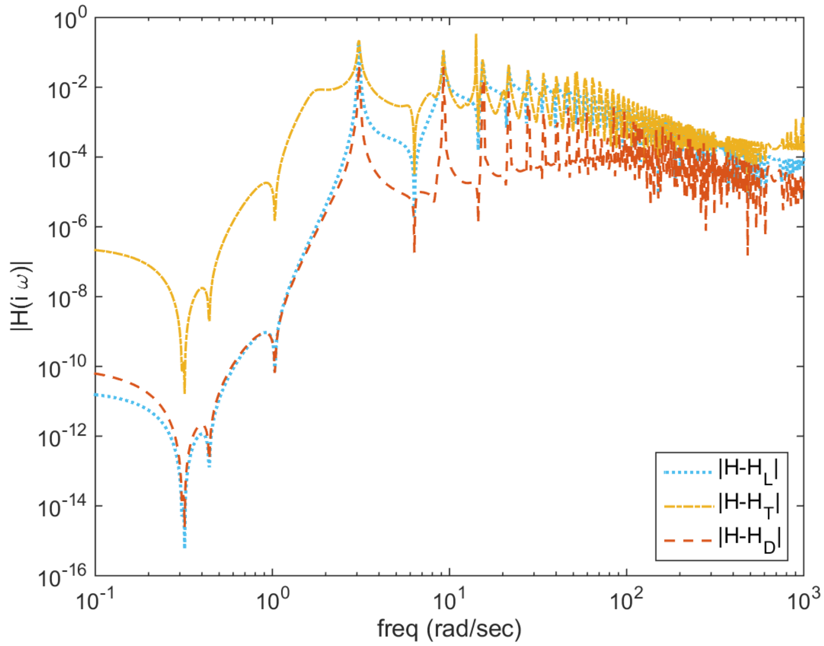
(b) Error plot for H_L, H_D, and H_T.

Figure 7.1: Example 7.3 - Realizations for $\kappa = 2$ and exact parameters τ, α, and β.

We illustrate the transfer functions of the realizations based on all interpolation points and the parameters (7.1) in Figure 7.2. As before, the realization obtained with our methodology is a very good approximation of the original model. It is worth to emphasize that this realization is purely data-driven, i. e., is constructed from data only. Again, only our realization approximates the qualitative behavior of the original model reasonably well.



(a) Bode plot of H, H_L, H_D, and H_T.



(b) Error plot for H_L, H_D, and H_T.

Figure 7.2: Example 7.3 - Realizations for all interpolation points and parameters (7.1).

Example 7.4 A heated rod with distributed control and homogeneous Dirichlet boundary conditions, which is cooled by delayed feedback, can be modeled (cf. [7, 17]) via the one-dimensional partial differential equation

$$\begin{aligned} \frac{\partial v(\xi,t)}{\partial t} &= \frac{\partial^2 v(\xi,t)}{\partial \xi^2} + a_1(\xi)v(\xi,t) + a_2(\xi)v(\xi,t-1) + u(t) \qquad &&\text{in } (0,\pi)\times(0,T], \\ v(0,t) &= v(\pi,t) = 0 &&\text{in } [0,T]. \end{aligned} \tag{7.2}$$

Discretization of (7.2) via centered finite differences with step size $h = \frac{\pi}{n+1}$ yields a system

$$\begin{aligned} \dot{x}(t) &= (L_n + A_{1,n})x(t) + A_{2,n}x(t-1) + Bu(t), \\ y(t) &= Cx(t), \end{aligned}$$

where $L_n \in \mathbb{R}^{n,n}$ is the discrete Laplacian and $A_{1,n}, A_{2,n} \in \mathbb{R}^{n,n}$ are discrete approximations of the functions a_1 and a_2, respectively. The input matrix $B \in \mathbb{R}^n$ is a vector of ones. As output we use the average temperature of the rod, i. e, $C = \frac{1}{\|B\|}B^T$. For our tests we use $n = 100$, $\kappa = 3$, and $\rho = 8$ random interpolation points λ_k on the imaginary axis between $10^{-1}\imath$ and $10^2\imath$ together with their complex conjugates μ_k.

We distinguish two cases. First, we set $a_1 \equiv 1 \equiv a_2$, which yields $E = A_2$, i. e., resulting in the structure $A_2 = \alpha E + \beta A_1$. The computed estimates for τ, α, and β are

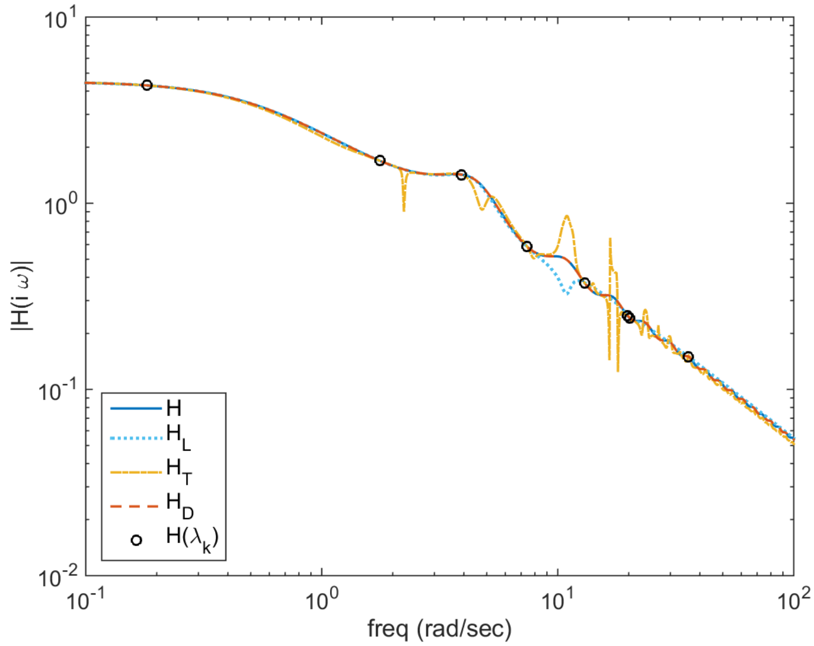
$$\tau_{\kappa,\rho} = 1.000196, \qquad \alpha_{\kappa,\rho} = 1.000384, \qquad \beta_{\kappa,\rho} = -0.002923 \tag{7.3}$$

and the results are illustrated in Figure 7.3. As before, the generalized Loewner realization outperforms the other approaches. In the original model from [7, 17], the coefficients a_1 and a_2 are chosen as

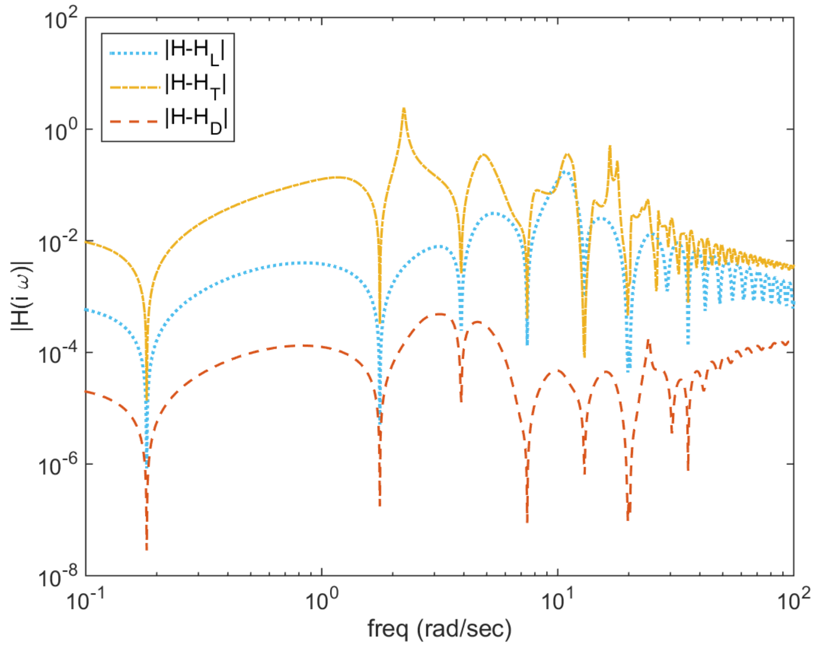
$$a_1(\xi) = -2\sin(\xi) \qquad \text{and} \qquad a_2(\xi) = 2\sin(\xi).$$

Note that in this case A_2 is not a linear combination of E and A_1. Comparing the results depicted in Figure 7.4, the generalized delay Loewner approach still captures the qualitative behavior of the full model and is the best approximation in terms of the maximal error within the considered range of frequencies.

It is worth to note that the performance of the three approaches depends on the choice of interpolation points. For each method we found at least one selection of interpolation points, where this method outperforms the others. Qualitatively, only our ansatz captures the main features of the full model for a small number of interpolation points in all our tests.

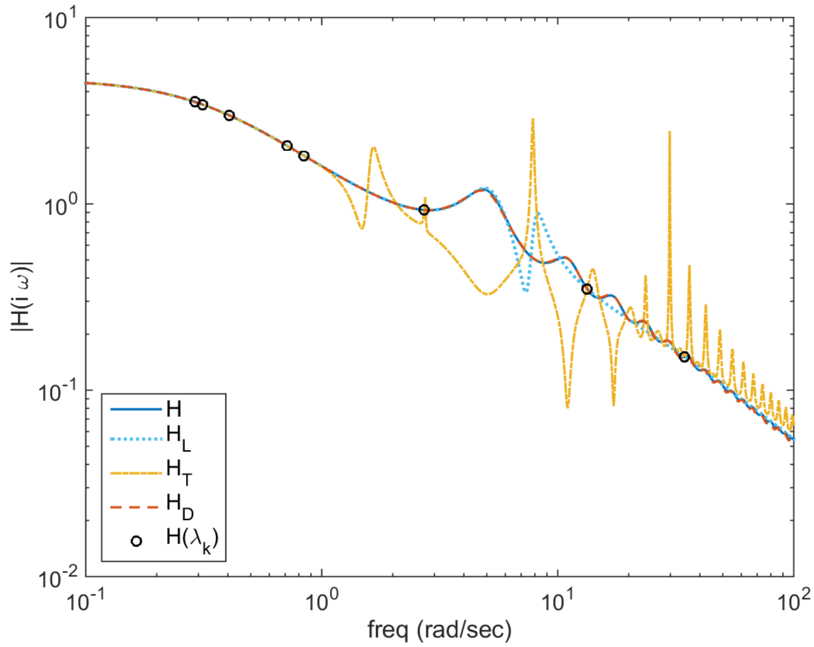


(a) Bode plot of H, H_L, H_D, and H_T.

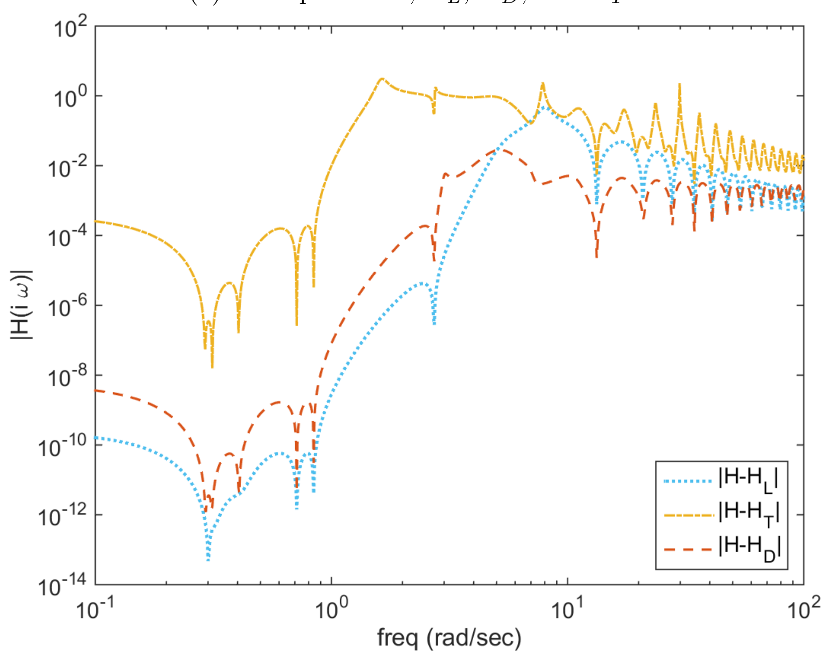


(b) Error plot for H_L, H_D, and H_T.

Figure 7.3: Example 7.4 - Realizations for $a_1 \equiv 1 \equiv a_2$ and parameters (7.3).



(a) Bode plot of H, H_L, H_D, and H_T.



(b) Error plot for H_L, H_D, and H_T.

Figure 7.4: Example 7.4 - Realizations for $a_1(\xi) = -2\sin(\xi)$, $a_2(\xi) = 2\sin(\xi)$ and recovered $\tau_{\kappa,\rho} = 1.006960$.

8 Conclusions

We have extended the Loewner realization [16] and the moment matching framework [4] to descriptor systems with internal delay. We used the results obtained for moment matching to construct a realization directly from measurements of the transfer function. The system matrices can be assembled efficiently by matrix-matrix operations of size $\rho \times \rho$, where 2ρ is the number of interpolation points. The internal delay is estimated by solving a least-square optimization over some sample data. Examples show that our approach produces a low-order model that captures the dynamics of the full model very accurately. Also, the delay parameter τ is recovered almost exactly. Comparing our ansatz with the common Loewner realization and an extension introduced in [18] reveals the necessity for preserving the delay structure. Consequently, our approach yields better approximations of the transfer function, in particular if the state dimension of the realization is small. Open problems are the optimal choice of the interpolation points λ_i and μ_i and estimators for the output error $\|y - y_\rho\|$, where y and y_ρ denote the outputs of the original model and the realization, respectively. In addition, the proportional ansatz $A_{2,\rho} = \alpha E_\rho + \beta A_{1,\rho}$ seems rather restrictive compared to the overall degrees of freedom and further approaches are to be investigated. Moreover, an extension to multiple delays and more general structures of the transfer function is currently under investigation.

References

[1] A. C. Antoulas. *Approximation of large-scale dynamical systems.* Advances in Design and Control. Society for Industrial and Applied Mathematics, Philadelphia, PA, USA, 2005.

[2] A. C. Antoulas, S. Lefteriu, and A. C. Ionita. A tutorial introduction to the Loewner framework for model reduction. In P. Benner, A. Cohen, M. Ohlberger, and K. Willcox, editors, *Model Reduction and Approximation for Complex Systems*, ISNM Series. Birkhäuser, 2015.

[3] A. C. Antoulas, D. C. Sorensen, and S. Gugercin. A survey of model reduction methods for large-scale systems. *Contemp. Math.*, 280:193–219, 2001.

[4] A. Astolfi. Model reduction by moment matching for linear and nonlinear systems. *IEEE Trans. Automat. Control*, 55(10):2321–2336, 2010.

[5] C. Beattie and S. Gugercin. Interpolatory projection methods for structure-preserving model reduction. *Syst. Control Lett.*, 58(3):225–232, 2009.

[6] A. Bellen and M. Zennaro. *Numerical methods for delay differential equations.* Oxford University Press, 2003.

[7] D. Breda, S. Maset, and R. Vermiglio. Numerical approximation of characteristic values of partial retarded functional differential equations. *Numer. Math.*, 113(2):181–242, 2009.

[8] R. W. Clough and J. Penzien. *Dynamics of Structures.* McGraw-Hill, third edition, 2003.

[9] T. Erneux. *Applied Delay Differential Equations.* Surveys and Tutorials in the Applied Mathematical Sciences. Springer New York, 2009.

[10] K. Gallivan, A. Vandendorpe, and P. Van Dooren. Model reduction of MIMO systems via tangential interpolation. *SIAM J. Matrix Anal. Appl.*, 26(2):328–349, 2004.

[11] G. H. Golub and C. F. Van Loan. *Matrix Computations*. Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, Maryland, fourth edition, 2013.

[12] P. Ha and V. Mehrmann. Analysis and numerical solution of linear delay differential-algebraic equations. Preprint 42–2014, Institut für Mathematik, TU Berlin, 2014. *http://www.math.tu-berlin.de/preprints/*. Accepted for publication in *BIT*, 2015.

[13] J. K. Hale and S. M. Verduyn Lunel. *Introduction to Functional Differential Equations*. Number 99 in Applied Mathematical Sciences. Springer, New York, 1993.

[14] P. Kunkel and V. Mehrmann. *Differential-Algebraic Equations. Analysis and Numerical Solution*. European Mathematical Society, 2006.

[15] MATLAB. *Global Optimization Toolbox User's Guide, Version R2015a*. Natick, Massachusetts, 2015.

[16] A. J. Mayo and A. C. Antoulas. A framework for the solution of the generalized realization problem. *Linear Algebra Appl.*, 425(2-3):634–662, 2007.

[17] W. Michiels, E. Jarlebring, and K. Meerbergen. Krylov-based model order reduction of time-delay systems. *SIAM J. Matrix Anal. Appl.*, 32(4):1399–1421, 2011.

[18] I. Pontes Duff, C. Poussot-Vassal, and C. Seren. Realization independent single time-delay dynamical model interpolation and $\mathcal{H}_2$-optimal approximation, 2015. arXiv Preprint: 1504.06457v1.

[19] G. Scarciotti and A. Astolfi. Model reduction by moment matching for linear time-delay systems. In *Proceedings of the 19th IFAC World Congress*, pages 3642 – 3647, 2014.

[20] M. J. Tovée. Neuronal processing. How fast is the speed of thought? *Curr. Biol.*, 4(12):1125–1127, 1994.

[21] X. Yu and M. Gen. *Introduction to Evolutionary Algorithms*. Decision Engineering. Springer London, 2010.