# Compositional stability criteria based on cyclically neutral supply conditions

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# Compositional stability criteria based on cyclically neutral supply conditions

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**Abstract:** In this paper we consider stability of large scale interconnected nonlinear systems that satisfy a strict dissipativity property in terms of local storage and supply functions. Existing compositional stability criteria certify global stability by constructing a global Lyapunov function as the (weighted) sum of local storage functions. We generalize these results by unifying spatial composition, i.e., (weighted) sum of local supply functions is neutral, with temporal composition, i.e., (weighted) sum of supply functions over a time cycle is neutral. Two benchmark examples illustrate the benefits of the developed compositional stability criteria in terms of reducing conservatism and constrained distributed stabilization.



*Keywords:* Stability of nonlinear systems, Large scale interconnected systems, Dissipative systems, Dissipation inequalities, Scalable stability analysis.

## 1. INTRODUCTION

Dissipative dynamical systems and dissipation inequalities formulated in terms of a storage function $V$ and a supply function $s$ were originally introduced in (Willems, 1972), as a generalization of Lyapunov's inequality for open and interconnected systems, i.e.,

$$V(x(k+1)) - V(x(k)) \leq s(y(k), u(k), x(k)), \quad k \in \mathbb{N}.$$

Indeed, Lyapunov's inequality corresponds to a specific dissipation inequality, i.e.,

$$V(x(k+1)) - V(x(k)) \leq -\alpha(\|x(k)\|), \quad k \in \mathbb{N},$$

for a positive definite storage function $V$ and a class $\mathcal{K}_\infty$ real–valued function $\alpha$. Initially, there has been some interest in developing more general conditions than Lyapunov's inequality under which dissipativity or dissipation inequalities imply asymptotic stability. One such condition in terms of strongly passive supply functions was presented in (Hill and Moylan, 1976), which can be reformulated as

$$V(x(k+1)) - V(x(k)) \leq -\alpha(\|x(k)\|) + u(k)^\top y(k), \quad k \in \mathbb{N},$$

and corresponds to a strictly passive system. For more recent results we refer for example to (Arcak, 2007), the survey on dissipation inequalities (Ebenbauer et al., 2009), the overview of various dissipativity properties (Kottenstette and Antsaklis, 2010) and the books (Brogliato et al., 2007) and (van der Schaft, 2017).

One of the most important application of dissipation inequalities is structured stability analysis of large scale interconnected systems, see, e.g., (Willems, 1976; Moylan and Hill, 1978). The main idea around this approach, which is still largely adopted, is to construct a global Lyapunov function for interconnected systems by summing up over local storage functions that satisfy local dissipation inequalities. To illustrate this construction consider a finite number of interconnected systems with local state $x_i$ and global state $x$, which satisfy a local dissipation inequality for all $i$:

$$V_i(x_i(k+1)) - V_i(x_i(k)) \leq s_i(y_i(k), u_i(k), x(k)), \ k \in \mathbb{N}.$$

By defining the global storage $V = \sum_i V_i$ and supply $s = \sum_i s_i$ and by requiring that $s(y(k), u(k), x(k)) \leq 0$ (i.e., global neutral supply) or $s(y(k), u(k), x(k)) \leq -\alpha(\|x(k)\|)$ (i.e., global negative definite supply) stability and asymptotic stability, respectively, of the interconnected systems can be established from Lyapunov's theorem with $V$ as a global Lyapunov function (see, e.g., (Jokic and Lazar, 2009) for results in discrete–time). More recently, in (Arcak et al., 2016), a generalization of this stability criterion was derived, by defining the global storage as a weighted sum of local storage functions, i.e., $V = \sum_i \mu_i V_i$, $\mu_i > 0$ for all $i$, which yields a less conservative global neutral supply condition, since $s = \sum_i \mu_i s_i$.

In (Gielen and Lazar, 2015), a simple example of a globally exponentially stable linear interconnected system was given, which does not admit a set of structured dissipation inequalities/supply functions that satisfy Willem's global neutral supply condition. This is the case, for example, if all local systems have a positive supply for certain initial conditions. Hence, there is still room for less conservative stability criteria for interconnected dissipative systems.

To reduce conservatism, (Gielen and Lazar, 2015) introduced a new type of dissipation inequalities, i.e.,

$$V(x(k+M)) - \rho V(x(k)) \leq s(x(k)), \quad k \in \mathbb{N},$$

where $\rho < 1$ is a strictly positive real number and the integer $M \geq 1$ is referred to as a finite–step. If the supply function is non–positive, then $V$ becomes a non–monotonic, finite–step Lyapunov function from which asymptotic stability can be inferred. Structured finite–step dissipation inequalities together with a global neutral supply condition were used in (Gielen and Lazar, 2015) to obtain less conservative global stability criteria for interconnected nonlinear systems.

In (Lazar, 2021), it was proven that for a positive definite (storage) function $V$, finite–step Lyapunov inequalities,


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i.e.,

$$V(x(k+M)) - V(x(k)) \le -\alpha(\|x(k)\|), \quad k \in \mathbb{N},$$

and dissipation inequalities, i.e.,

$$V(x(k+1)) - V(x(k)) \le s(x(k)), \quad k \in \mathbb{N},$$

are equivalent if the supply function $s$ satisfies

$$\sum_{i=0}^{M-1} s(x(k+i)) \le -\alpha(\|x(k)\|).$$

The above cyclic condition requires that the sum of the supply function over a discrete–time cycle, i.e., the time interval $[k,\, k+M-1]$, is negative definite.

Motivated by the need of less conservative stability criteria, in this paper we exploit cyclically neutral supply conditions to derive generalized compositional stability criteria for large scale interconnected systems. The developed stability criteria unify spatial composition, i.e., (weighted) sum of local supply functions is neutral, with temporal composition, i.e., (weighted) sum of supply functions over a time cycle is neutral. Two benchmark examples illustrate the benefits of the developed compositional stability criteria, in terms of reducing conservatism and constrained distributed stabilization.

## 2. PRELIMINARIES

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ define $\Pi_{\ge c} := \{k \in \Pi \mid k \ge c\}$. For a vector $x \in \mathbb{R}^n$, $\|\cdot\|$ denotes an arbitrary $p$-norm, $p \in \mathbb{Z}_{\ge 1} \cup \infty$.

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and $\lim_{z\to\infty} \varphi(z) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ if for each fixed $k \in \mathbb{R}_+$, $\beta(\cdot, k) \in \mathcal{K}$ and for each fixed $z \in \mathbb{R}_{>0}$, $\beta(z, \cdot)$ is decreasing and $\lim_{k\to\infty} \beta(z, k) = 0$. For any $N \in \mathbb{N}_{\ge 1}$, let $\{\xi_1, \ldots, \xi_N\} \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_N}$ and define $\mathrm{col}(\xi_1, \ldots, \xi_N) := [\xi_1^\top, \ldots, \xi_N^\top]^\top$.

*Lemma 1.* Let $\xi := \mathrm{col}(\xi_1, \ldots, \xi_N)$ and $\alpha \in \mathcal{K}_\infty$. Then it holds that

$$\alpha\left(\frac{1}{N}\|\xi\|\right) \le \sum_{i=1}^{N} \alpha(\|\xi_i\|) \le N\alpha(\|\xi\|).$$

For a proof of Lemma 1 we refer to (Gielen and Lazar, 2015, proof of Theorem 10).

Consider a discrete–time dynamical system

$$\begin{aligned} x(k+1) &= f(x(k), u(k)) \\ y(k) &= h(x(k), u(k)), \end{aligned} \tag{1}$$

where $k \in \mathbb{N}$ and $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^o$ are suitable functions that are zero at zero. We assume that the origin is a stabilizable equilibrium for (1). The system variables are constrained to some sets with the origin in their interior, i.e. $(y, x, u) \in \mathbb{Y} \times \mathbb{X} \times \mathbb{U}$. Since we consider constraints, the following standing assumption is made to simplify the presentation of the results.

*Assumption 2.* For all $x \in \mathbb{X}$, there exists a $u := \kappa(x) \in \mathbb{U}$ with $\kappa(0) = 0$ such that $(h(x, u), f(x, u)) \in \mathbb{Y} \times \mathbb{X}$.

A control law $u(x(k)) := \kappa(x(k))$, $k \in \mathbb{N}$ is called admissible if it satisfies the properties of Assumption 2.

*Definition 3.* We call system (1) $\mathcal{KL}$–asymptotically stable in $\mathbb{X} \subseteq \mathbb{R}^n$, or shortly, AS($\mathbb{X}$), if there exists a $\mathcal{KL}$-function $\beta(\cdot, \cdot)$ such that for all $x(0) \in \mathbb{X}$ it holds that

$$\|x(k)\| \le \beta(\|x(0)\|, k), \quad \forall k \in \mathbb{N}_{\ge 1}.$$

*Definition 4.* A real–valued function $V : \mathbb{X} \subseteq \mathbb{R}^n \to \mathbb{R}_+$ is called positive definite (in $\mathbb{X}$) if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|), \quad \forall x \in \mathbb{X}. \tag{2}$$

For system (1) in closed–loop with an admissible state–feedback control law $u(k) := \kappa(x(k))$ consider the following discrete–time dissipation inequality

$$V(x(k+1)) - V(x(k)) \le -\alpha_3(\|x(k)\|) + s(y(k), u(k), x(k)), \tag{3}$$

where $V : \mathbb{R}^n \to \mathbb{R}_+$ is a storage function, $s : \mathbb{R}^o \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ is a supply function and $\alpha_3 \in \mathcal{K}_\infty$. In this paper we assume that $V$ is positive definite, $s$ is bounded on bounded sets and $s(0, 0, 0) = 0$.

*Definition 5.* We call system (1) in closed–loop with the state–feedback law $u(k) = \kappa(x(k))$ strictly dissipative in $\mathbb{X} \subseteq \mathbb{R}^n$ if there exists a pair of storage and supply functions $(V, s)$ such that the dissipation inequality (3) holds for all $x(0) \in \mathbb{X}$ and all $k \in \mathbb{N}$.

## 3. CYCLICALLY NEUTRAL SUPPLY CONDITIONS

In this section we will present a generalized condition on the supply function $s$ that implies asymptotic stability. To this end, the following assumption is instrumental.

*Assumption 6. Controlled $\mathcal{K}$–boundedness:* For the system (1) dynamics $f(\cdot, \cdot)$ and any admissible state–feedback control law $u(k) = \kappa(x(k))$ it holds that $\|f(x, \kappa(x))\| \le \sigma(\|x\|)$ for all $x \in \mathbb{X}$ and some $\sigma \in \mathcal{K}_\infty$.

Note that the above $\mathcal{K}$–boundedness property is a necessary condition for $\mathcal{KL}$–asymptotic stability, as shown in (Gielen and Lazar, 2015).

*Theorem 7.* Suppose that system (1) in closed–loop with an admissible control law $u(k) = \kappa(x(k))$ satisfies Assumption 6 and it is strictly dissipative in $\mathbb{X}$ with storage and supply function $(V, s)$, respectively. Suppose that there exist $M \in \mathbb{N}_{\ge 1}$ and $\{\mu_l\}_{l \in \mathbb{N}_{[0,M-1]}}$ with $\mu_l > 0$ for all $l \in \mathbb{N}_{[0,M-1]}$ such that

$$\sum_{l=0}^{M-1} \mu_l s(y(k+l), u(k+l), x(k+l)) \le 0, \quad \forall x(0) \in \mathbb{X}, \forall k \in \mathbb{N}. \tag{4}$$

Then system (3) is AS($\mathbb{X}$) in the sense of Definition 3.

**Proof.** For any $x(k) \in \mathbb{X}$, $k \in \mathbb{N}$, define the candidate Lyapunov function

$$\bar{V}(x(k)) := \sum_{l=0}^{M-1} \mu_l V(x(k+l))$$

and observe that it satisfies

$$\bar{V}(x(k+1)) - \bar{V}(x(k)) \le -\bar{\alpha}_3(\|x(k)\|),$$

where $\bar{\alpha}_3 := \mu_0 \alpha_3 \in \mathcal{K}_\infty$. The above property is obtained by multiplying the dissipation inequality (3) for $x(k+l)$ with $\mu_l$ for $l = 0, \ldots, M-1$, summing up and using (4) and $-\sum_{l=0}^{M-1} \mu_l \alpha_3(\|x(k+l)\|) \le -\mu_0 \alpha_3(\|x(k)\|)$. By definition it holds that $\bar{V}(x(k)) \ge \bar{\alpha}_1(\|x(k)\|)$ with $\bar{\alpha}_1 := \mu_0 \alpha_1$. To establish an upper bound on $\bar{V}$, first notice that due to Assumption 6, for any $x \in \mathbb{X}$ it holds that

$$V(f(x,\kappa(x))) \leq \alpha_2(\|f(x,\kappa(x))\|) \leq \alpha_2 \circ \sigma(\|x\|),$$

where $\alpha_2 \circ \sigma \in \mathcal{K}_\infty$. By exploiting this property repetitively for all $x(k+l)$, $l = 1, \ldots, M-1$ and since the weighted finite sum of $\mathcal{K}_\infty$ functions is a $\mathcal{K}_\infty$ function (for positive weights) we obtain that there exists a $\bar{\alpha}_2 \in \mathcal{K}_\infty$ such that $\bar{V}(x(k)) \leq \bar{\alpha}_2(\|x(k)\|)$. The claim then follows via the standard $\mathcal{KL}$–stability and Lyapunov arguments. □

*Remark 8.* For the particular weights $\mu_l = 1$ for all $l = 0, \ldots, M-1$, the cyclically neutral condition (4) becomes

$$\sum_{l=0}^{M-1} s(y(k+l), u(k+l), x(k+l)) \leq 0, \quad k \in \mathbb{N}, \tag{5}$$

which implies that the storage $V$ is a finite–step Lyapunov function, as shown in (Lazar, 2021). The generalized cyclically neutral supply condition (4) no longer implies that the storage $V$ is a finite–step Lyapunov function, which results in a significant decrease of conservatism, as shown in Section 5. Also, the fact that $\bar{V}$ defined in the proof of Theorem 7 is a Lyapunov function does not imply that the storage function $V$ is a Lyapunov function, except for the particular case when (4) holds with $M = 1$, i.e., $\bar{V} = \mu_0 V$. Since $\mu_0 > 0$ suffices for positive definiteness of $\bar{V}$, condition (4) can be further relaxed by allowing $\mu_l \geq 0$ for $l = 1, \ldots, M-1$.

*Remark 9.* For a smooth storage function $V$, a continuous–time version of (3) is provided by:

$$\dot{V}(x(t)) \leq -\alpha_3(\|x(t)\|) + s(y(t), u(t), x(t)), \quad t \in \mathbb{R}_+, \tag{6}$$

which further implies

$$V(x(t_2)) - V(x(t_1)) \leq -\alpha_3(\|x(t_1)\|) + \int_{t_1}^{t_2} s(y(\tau), u(\tau), x(\tau))\mathrm{d}\tau \tag{7}$$

for all $t_2 > t_1 \geq 0$ in $\mathbb{R}_+$. Then the cyclically neutral supply condition (5) translates into

$$\int_t^{t+M} s(y(\tau), u(\tau), x(\tau))\mathrm{d}\tau \leq 0, \quad t \in \mathbb{R}_+, \tag{8}$$

for some $M \in \mathbb{R}_{>0}$. Checking condition (8) requires knowledge of the system solution for a finite time, which can be computed based on linearized dynamics see, e.g., (Doban and Lazar, 2018, Section IV).

## 4. COMPOSITIONAL STABILITY CRITERIA BASED ON CYCLICALLY NEUTRAL SUPPLY CONDITIONS

Next we consider interconnected discrete–time systems

$$\begin{aligned} x_i(k+1) &= f_i(x_i(k), u_i(k), p_i(k)) \\ y_i(k) &= h_i(x_i(k), u_i(k)), \end{aligned} \tag{9}$$

where $k \in \mathbb{N}$, $f_i : \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \times \mathbb{R}^{q_i} \to \mathbb{R}^{n_i}$, $h_i : \mathbb{R}^{n_i} \times \mathbb{R}^{m_i} \to \mathbb{R}^{o_i}$ are suitable functions that are zero at zero. Here $i \in \mathcal{N}_{net} := \{1, \ldots, N_{net}\}$ is the index of a local dynamical system which is interconnected with other dynamical systems within a network described by an arbitrary connected graph, see, e.g., (Jokic and Lazar, 2009). $N_{net} \in \mathbb{N}_{\geq 1}$ represents the total number of systems in the network.

The vector $p_i(k)$ that collects all the interconnection variables for system $i$, i.e., all the variables shared with other systems in the network. For simplicity of exposition in this paper we assume that systems are interconnected via shared state variables. Then, if we define $\mathcal{N}_i$ as the set of indexes that correspond to the neighbors of system $i$, i.e., systems which share variables with system $i$, we obtain

$$p_i(k) := \mathrm{col}(\{x_j(k)\}_{j \in \mathcal{N}_i}).$$

We consider suitable local sets that constrain the variables of system $i$, i.e. $(y, x, u) \in \mathbb{Y}_i \times \mathbb{X}_i \times \mathbb{U}_i$ and admissible feedback laws $u_i(k) := \kappa_i(x_i(k), p_i(k))$ with $\kappa_i(0,0) = 0$. The complete network of dynamical systems can be represented as a global nonlinear system of the form (1), where $x(k) = \mathrm{col}(\{x_i(k)\}_{i \in \mathcal{N}_{net}})$, $y(k) = \mathrm{col}(\{y_i(k)\}_{i \in \mathcal{N}_{net}})$ and $u(k) = \kappa(x(k)) = \mathrm{col}(\{u_i(k)\}_{i \in \mathcal{N}_{net}})$, with $\kappa(x(k)) := \mathrm{col}(\{\kappa_i(\cdot,\cdot)\}_{i \in \mathcal{N}_{net}})$, and the functions $f_i, h_i$ are appropriately merged into the functions $f, h$.

For each system (9) in closed–loop with an admissible state–feedback control law $u_i(k) := \kappa_i(x_i(k), p_i(k))$ consider the following discrete–time dissipation inequality

$$\begin{aligned} &V_i(x_i(k+1)) - V_i(x_i(k)) \\ &\quad \leq -\alpha_{i,3}(\|x_i(k)\|) + s_i(y_i(k), u_i(k), p_i(k)), \end{aligned} \tag{10}$$

where $V_i : \mathbb{R}^{n_i} \to \mathbb{R}_+$ is a local positive definite storage function with corresponding $\alpha_{i,1}, \alpha_{i,2} \in \mathcal{K}_\infty$ bounds, $s_i : \mathbb{R}^{o_i} \times \mathbb{R}^{m_i} \times \mathbb{R}^{q_i} \to \mathbb{R}$ is a local supply function and $\alpha_{i,3} \in \mathcal{K}_\infty$.

*Definition 10.* We call system (9) in closed–loop with the state–feedback law $u_i(k) = \kappa_i(x_i(k), p_i(k))$ strictly dissipative in $\mathbb{X}_i \subseteq \mathbb{R}^{n_i}$ with storage and supply function $(V_i, s_i)$, respectively, if the dissipation inequality (10) holds for all $x_i(0) \in \mathbb{X}_i$, all $x_j(0) \in \mathbb{X}_j$, $j \in \mathcal{N}_i$ and all $k \in \mathbb{N}$.

Next, we are ready to state the first main result for inferring global stability of interconnected system from local dissipation inequalities.

*Theorem 11. Weighted–temporal–spatial composition* Suppose that: *(a)* for all $i \in \mathcal{N}_{net}$, local systems (9) in closed–loop with admissible control laws $u_i(k) = \kappa_i(x_i(k), p_i(k))$ are strictly dissipative in $\mathbb{X}_i$ with storage and supply $(V_i, s_i)$, respectively; *(b)* the corresponding closed–loop global dynamics (1) satisfy Assumption 6 and *(c)* there exist $M \in \mathbb{N}_{\geq 1}$, $\{\mu_l\}_{l \in \mathbb{N}_{[0,M-1]}}$ with $\mu_l > 0$ for all $l \in \mathbb{N}_{[0,M-1]}$, $\{\eta_i\}_{i \in \mathcal{N}_{net}}$ with $\eta_i > 0$ for all $i \in \mathcal{N}_{net}$ such that

$$\begin{aligned} &\sum_{l=0}^{M-1} \mu_l \sum_{i \in \mathcal{N}_{net}} \eta_i s_i(y_i(k+l), u_i(k+l), p_i(k+l)) = \\ &= \sum_{i \in \mathcal{N}_{net}} \eta_i \sum_{l=0}^{M-1} \mu_l s_i(y_i(k+l), u_i(k+l), p_i(k+l)) \leq 0 \end{aligned} \tag{11}$$

for all $x_i(0)$, $i \in \mathcal{N}_{net}$, and all $k \in \mathbb{N}$. Then the corresponding closed–loop global system (3) is AS($\mathbb{X}$) in the sense of Definition 3, with $\mathbb{X} := \mathbb{X}_1 \times \ldots \times \mathbb{X}_{N_{net}}$.

**Proof.** Define $V(x) := \sum_{i \in \mathcal{N}_{net}} \eta_i V_i(x)$. By multiplying (10) with $\eta_i > 0$, summing up over $i \in \mathcal{N}_{net}$ and utilizing Lemma 1 to construct a suitable $\alpha_3 \in \mathcal{K}_\infty$ function, we obtain

$$V(x(k+1)) - V(x(k)) \leq -\alpha_3(\|x(k)\|) + s(y(k), u(k), x(k)),$$

where

$$s(y(k), u(k), x(k)) := \sum_{i \in \mathcal{N}_{net}} \eta_i s_i(y_i(k), u_i(k), p_i(k)).$$

Since by Lemma 1 we can construct class $\mathcal{K}_\infty$ upper and lower bounds on $V(x)$ as a function of $\|x\|$ with $x$ the global state vector, it follows that the storage and supply pair $(V, s)$ defined above satisfy the global dissipation inequality (3). Since from (11) we have that

$$\sum_{l=0}^{M-1} \mu_l \sum_{i\in\mathcal{N}_{net}} \eta_i s_i(y_i(k+l), u_i(k+l), p_i(k+l)) = \sum_{l=0}^{M-1} \mu_l s(y(k+l), u(k+l), p(k+l)) \leq 0,$$

and thus, the global supply $s$ satisfies condition (4), the claim then follows from Theorem 7. □

*Corollary 12. Temporal–spatial composition* Suppose that the hypotheses $(a)$ and $(b)$ of Theorem 11 hold. Suppose that there exists $M \in \mathbb{N}_{\geq 1}$ such that

$$\sum_{l=0}^{M-1} \sum_{i\in\mathcal{N}_{net}} s_i(y_i(k+l), u_i(k+l), p_i(k+l)) \leq 0 \qquad (12)$$

for all $x_i(0)$, $i \in \mathcal{N}_{net}$ and all $k \in \mathbb{N}$. Then the corresponding closed–loop global system (3) is AS($\mathbb{X}$) in the sense of Definition 3, with $\mathbb{X} := \mathbb{X}_1 \times \ldots \times \mathbb{X}_{N_{net}}$.

The above result directly follows from Theorem 11, for the specific case of all weights equal to one. The merit of the particular (with respect to (11)) condition (12) is that the local dissipation inequalities (10) subject to (12) can be verified using linear matrix inequalities (LMIs) for linear interconnected systems and quadratic storage and supply.

### *4.1 Separable stability criteria for interconnected systems*

In the linear case, verifying the stability conditions (10)–(12) scales well due to the advances in semidefinite programming solvers, such as MOSEK (ApS, 2019). However, it is still of interest to obtain separable stability criteria, because this allows for a modular design or reconfiguration of networks. I.e., if a system is added or removed, it would be of practical interest to certify global stability from local neutral supply conditions rather than global neutral supply conditions. To this end, we need to structure Assumption 6 for local dynamics.

*Assumption 13.* For the local system (9) dynamics $f_i(\cdot,\cdot,\cdot)$ and any admissible state–feedback control law $u_i(k) = \kappa_i(x_i(k), p_i(k))$ it holds that $\|f_i(x_i, \kappa_i(x_i, p_i), p_i)\| \leq \sigma_i(\|x\|)$ with $\sigma_i \in \mathcal{K}_\infty$, for all $i \in \mathcal{N}_{net}$ and $x \in \mathbb{X}$.

*Theorem 14.* Suppose that: *(a)* for all $i \in \mathcal{N}_{net}$, local systems (9) in closed–loop with admissible control laws $u_i(k) = \kappa_i(x_i(k), p_i(k))$ are strictly dissipative in $\mathbb{X}_i$ with storage and supply $(V_i, s_i)$, respectively; *(b)* the corresponding closed–loop local dynamics (9) and global dynamics (1) satisfy Assumption 13 and Assumption 6, respectively, and *(c)* there exist $M_i \in \mathbb{N}_{\geq 1}$, $\{\mu_{i,l}\}_{l\in\mathbb{N}_{[0,M_i-1]}}$ with $\mu_{i,l} > 0$ for all $l \in \mathbb{N}_{[0,M_i-1]}$, $i \in \mathcal{N}_{net}$, such that

$$\sum_{l=0}^{M_i-1} \mu_{i,l} s_i(y_i(k+l), u_i(k+l), p_i(k+l)) \leq 0, \quad k \in \mathbb{N}, \quad (13)$$

for all $x_i(0)$, $i \in \mathcal{N}_{net}$ and all $k \in \mathbb{N}$. Then the corresponding closed–loop global system (1) is AS($\mathbb{X}$) in the sense of Definition 3, with $\mathbb{X} := \mathbb{X}_1 \times \ldots \times \mathbb{X}_{N_{net}}$.

**Proof.** The proof relies on defining local candidate Lyapunov functions $\bar{V}_i(x_i(k)) := \sum_{l=0}^{M_i-1} \mu_{i,l} V_i(x_i(k+l))$ and observing that they satisfy

$$\bar{V}_i(x_i(k+1)) - \bar{V}_i(x_i(k)) \leq -\mu_{i,0}\alpha_{i,3}(\|x_i(k)\|),$$

via the same reasoning used in the proof of Theorem 7. By definition, for all $i \in \mathcal{N}_{net}$ it holds that $\bar{V}_i(x(k)) \geq \bar{\alpha}_{i,1}(\|x_i(k)\|)$ with $\bar{\alpha}_{i,1} := \mu_{i,0}\alpha_{i,1}$, where the $\alpha_{i,1} \in \mathcal{K}_\infty$ is the lower bound on the local storage $V_i$. To establish an upper bound on $\bar{V}_i$, first notice that due to Assumption 13, for any $x \in \mathbb{X}$ it holds that

$$\begin{aligned} V_i(f_i(x_i, \kappa_i(x_i, p_i), p_i)) &\leq \alpha_{i,2}(\|f_i(x_i, \kappa_i(x_i, p_i), p_i)\|) \\ &\leq \alpha_{i,2} \circ \sigma_i(\|x\|), \end{aligned}$$

where $\alpha_{2,i} \circ \sigma_i \in \mathcal{K}_\infty$ and $\alpha_{2,i} \in \mathcal{K}_\infty$ is the upper bound on the local storage $V_i$. For all $i \in \mathcal{N}_{net}$ and all $x_i(k+l)$, $l = 0, \ldots, M_i - 2$, the above property yields:

$$\begin{aligned} &V_i(x_i(k+l+1)) \leq \\ &\quad \alpha_{i,2}(\|f_i(x_i(k+l), \kappa_i(x_i(k+l), p_i(k+l)), p_i(k+l))\|) \\ &\quad \leq \alpha_{i,2} \circ \sigma_i(\|x(k+l)\|) \leq \alpha_{i,2} \circ \sigma_i \circ \sigma^l(\|x(k)\|), \end{aligned}$$

where the last inequality follows from Assumption 6 and $\sigma^l$ denotes the $l$–times composition of the function $\sigma$. Since $\alpha_{i,2} \circ \sigma_i \circ \sigma^l \in \mathcal{K}_\infty$ for any $l = 0, \ldots, M_i - 2$, we have that $\bar{V}_i(x_i(k)) = \sum_{l=0}^{M_i-1} \mu_{i,l} V_i(x_i(k+l))$ is upper bounded by a finite sum of $\mathcal{K}_\infty$ functions of $\|x\|$, which in turn is a $\mathcal{K}_\infty$ function of $\|x\|$. Then from Lemma 1 we obtain that $V(x(k)) := \sum_{i\in\mathcal{N}_{net}} \bar{V}_i(x_i(k))$ is a global Lyapunov function for system (1) and the claim follows via standard $\mathcal{KL}$–stability and Lyapunov arguments. □

*Corollary 15.* Suppose that the hypotheses $(a)$ and $(b)$ of Theorem 14 hold. Suppose that there exist $M_i \in \mathbb{N}_{\geq 1}$ such that

$$\sum_{l=0}^{M_i-1} s_i(y_i(k+l), u_i(k+l), p_i(k+l)) \leq 0 \qquad (14)$$

for all $x_i(0)$, $i \in \mathcal{N}_{net}$ and all $k \in \mathbb{N}$. Then the corresponding closed–loop global system (1) is AS($\mathbb{X}$) in the sense of Definition 3, with $\mathbb{X} := \mathbb{X}_1 \times \ldots \times \mathbb{X}_{N_{net}}$.

The above result directly follows from Theorem 14, for the specific case of all weights equal to one.

*Remark 16.* The separable cyclically neutral supply conditions (13) can be verified as follows for linear interconnected systems. First, for given linear local state–feedback controllers (or for autonomous linear systems), quadratic storage and supply, the local dissipation inequalities (10) can be verified by solving an LMI. Then for any fixed $M_i$, verifying (13) for the resulting local supply functions is also an LMI. Condition (14) can be merged together with (10) as a single local LMI, but using unitary weights is conservative, as shown in the next section. In contrast, conditions (13) are feasible for both examples tested in this paper, which shows the importance of using non–unitary weights in cyclically neutral supply conditions.

## 5. ILLUSTRATIVE EXAMPLES

*Example 1* Consider first the simple, but insightful linear system example put forward in (Gielen and Lazar, 2015):

$$x(k+1) = Ax(k), \quad k \in \mathbb{N}, \quad A = \begin{pmatrix} 1 & -0.5 \\ 1 & 0 \end{pmatrix}.$$

This stable linear system can be written as the interconnection of an unstable and a stable linear system, i.e.,

$$x_1(k+1) = x_1(k) - 0.5x_2(k) = A_1x(k), \quad A_1 := (1\ -0.5)$$
$$x_2(k+1) = 0 * x_2(k) + x_1(k) = A_2x(k), \quad A_2 := (1\ 0).$$

Next, we define

$$V_i(x_i) = x_i^\top P_i x_i,\ Q_i = 0.01,\ s_i(x) = x^\top S_i x,\ i = 1, 2.$$

The dissipation inequalities (10) yield the following LMIs:

$$A_1^\top P_1 A_1 - \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix} \leq -\begin{pmatrix} Q_1 & 0 \\ 0 & 0 \end{pmatrix} + S_1,\ P_1 \geq 0.1, \quad (15a)$$
$$A_2^\top P_2 A_2 - \begin{pmatrix} 0 & 0 \\ 0 & P_2 \end{pmatrix} \leq -\begin{pmatrix} 0 & 0 \\ 0 & Q_2 \end{pmatrix} + S_2,\ P_2 \geq 0.1. \quad (15b)$$

For $M = 3$, the condition (12) yields the LMI:

$$(A^2)^\top(S_1 + S_2)A^2 + A^\top(S_1 + S_2)A + (S_1 + S_2) \leq 0. \quad (16)$$

The LMIs (15)-(16) were solved using YALMIP (Lofberg, 2004) and MOSEK (ApS, 2019), yielding

$$P_1 = 2.1028,\ S_1 = \begin{pmatrix} 0.1561 & -1.1394 \\ -1.1394 & 0.6232 \end{pmatrix},$$
$$P_2 = 1.5381,\ S_2 = \begin{pmatrix} 1.6842 & -0.0880 \\ -0.0880 & -1.4306 \end{pmatrix}, \quad (17)$$

which successfully certifies stability. In comparison, the neutral supply condition corresponding to (Willems, 1976) is $S_1 + S_2 \leq 0$. Solving (15) with this condition yields an LMI, which turned out infeasible. The weighted neutral supply condition corresponding to (Arcak et al., 2016) is $\eta_1 S_1 + \eta_2 S_2 \leq 0$, $\eta_1 > 0, \eta_2 > 0$. Solving (15) with this condition is a bilinear matrix inequality. Instead, we searched for weights $\eta_1 > 0, \eta_2 > 0$ that satisfy $\eta_1 S_1 + \eta_2 S_2 \leq 0$ for the $S_1, S_2$ in (17), which turned out infeasible.

Next, we test the separable stability conditions (14), which correspond to the case when the local storage $V_i$ is a finite–step Lyapunov function. This can be done by solving the LMIs (15a) and (15b) separately, each with the corresponding additional LMI:

$$(A^{M-1})^\top S_i A^{M-1} + \ldots + A^\top S_i A + S_i \leq 0,\ i = 1, 2.$$

These conditions remained infeasible despite increasing $M$. Next, we illustrate the effectiveness of the weighted separable stability conditions (13) inspired by (Arcak et al., 2016). For $M_i = 3$, $i = 1, 2$, these conditions yield the BMIs:

$$\mu_{i,2}(A^2)^\top S_i A^2 + \mu_{i,1} A^\top S_i A + \mu_{i,0} S_i \leq 0, \quad (18)$$

$\mu_{i,l} \geq 0.01$ for $l = 0, 1, 2$. To avoid solving BMIs, we first solve the LMIs (15a) and (15b) independently, while minimizing trace($S_i$), respectively, which yields:

$$P_1 = 0.1000,\ S_1 = \begin{pmatrix} 0.0100 & -0.0500 \\ -0.0500 & 0.0250 \end{pmatrix},$$
$$P_2 = 1.3071,\ S_2 = \begin{pmatrix} 1.3071 & 0 \\ 0 & -1.2971 \end{pmatrix}.$$

Then we can fix $S_i$ with the above values in (18), which results in LMIs, and yields the weights:

$$\mu_{1,0} = 0.9943,\ \mu_{1,1} = 0.8114,\ \mu_{1,2} = 2.1454,$$
$$\mu_{2,0} = 0.5698,\ \mu_{2,1} = 0.7725,\ \mu_{2,2} = 1.3695.$$

This successfully certifies stability in a separable way, i.e., LMIs for each subsystem are solved independently, without any coupling constraint.

The stability criteria (12) and (13) can be formulated via LMIs for general interconnected linear systems by using the approach in (Gielen and Lazar, 2015, Section 3.2); the complete derivations will be included in an extended version of this paper, due to space limitations.

*Example 2* Consider next the simplified dynamics (Dörfler et al., 2013), (Lazar, 2021), of angle ($x_{i,1} = \theta_i$) and frequency ($x_{i,2} = \omega_i$) deviations for 4 synchronous generators interconnected within a ring network:

$$x_i(k+1) = A_i x_i(k) + B_i u_i(k) + E_i \sum_{j \in \mathcal{N}_i} \sin(x_{i,1}(k) - x_{j,1}(k)), \quad (19)$$

where for all $i = 1, \ldots, 4$,

$$A_i = \begin{pmatrix} 1 & 31.4159 \\ 0 & 0.9990 \end{pmatrix},\ B_i = \begin{pmatrix} 0 \\ 0.01 \end{pmatrix},\ E_i = \begin{pmatrix} 0 \\ -0.005 \end{pmatrix},$$

$\mathcal{N}_1 = \mathcal{N}_3 = \{2, 4\}$ and $\mathcal{N}_2 = \mathcal{N}_4 = \{1, 3\}$. The generator dynamics are open–loop unstable.

Next, we consider the local storage functions $V_i(x_i) := x_i^\top Q x_i$ with $Q = \begin{pmatrix} 0.1 & 0 \\ 0 & 10 \end{pmatrix}$ for all $i$ and we construct for each generator a nonlinear model predictive controller (NMPC) that optimizes the achievable supply at each time instant $k \in \mathbb{N}$, while minimizing a cost function that sums up its local storage over a prediction horizon $N$, i.e.,

$$\min_{\mathbf{u}_i(k)} \sum_{j=0}^{N-1} V_i(x_i(j|k)) + \lambda_i s_i(k) \quad (20a)$$

subject to constraints:

$$V_i(x_i(1|k)) - V_i(x_i(0|k)) \leq -0.1 x_i^\top(k) Q x_i(k) + s_i(k) \quad (20b)$$
$$x_i(0|k) = x_i(k),$$
$$x_i(j+1|k) = f_i(x(j|k), u_i(j|k)),\ \forall j \in \mathbb{N}_{[0,N-2]}, \quad (20c)$$
$$(x_i(j+1|k), u_i(j|k)) \in \mathbb{X}_i \times \mathbb{U}_i,\ \forall j \in \mathbb{N}_{[0,N-2]}. \quad (20d)$$

Above, $\mathbf{u}_i(k) = \{u_i(0|k), \ldots, u_i(N-2|k)\}$ are the predicted control inputs and $\{x_i(1|k), \ldots, x_i(N-1|k)\}$ are the corresponding predicted states. As typically done in distributed NMPC, we assume that at each time instant $k$, each NMPC communicates with its neighbors and transmits a shifted sequence based on the optimal state trajectory computed at $k - 1$, i.e.,

$$\{x_i(k), x_i^*(2|k), \ldots, x_i^*(N-1|k), x_i^*(N-1|k)\}.$$

This information is required in (20c), where $f_i(\cdot,\cdot)$ corresponds to the right–hand side in (19). Because the first state in the shifted sequence is the actual measured state at time $k$, the prediction $x_i(1|k)$ will be exact for each NMPC and hence, constraint (20b) ensures local strict dissipativity along closed–loop trajectories with the dynamic supply $s_i(k)$. Notice that the dynamic supply weighted by $\lambda_i = 10$ for all $i$ is minimized in the cost (20a). For all $i$ we use the sets $\mathbb{U}_i := \{u_i \in \mathbb{R} : |u_i| \leq 5\}$ and $\mathbb{X}_i := \{x_i \in \mathbb{R}^2 : \|x_i\|_\infty \leq 10\}$. The initial conditions are $x_1(0) = x_3(0) = [0,\ -0.142]^\top$ and $x_2(0) = x_4(0) = [0,\ 0.142]^\top$, which puts the interconnected systems at the limit of the frequency deviations that can be handled under the given constraints.

In Figure 1 we plot the closed–loop trajectories for $N = 4$. Constraints are satisfied at all times and the distributed NMPC scheme successfully stabilizes the interconnected

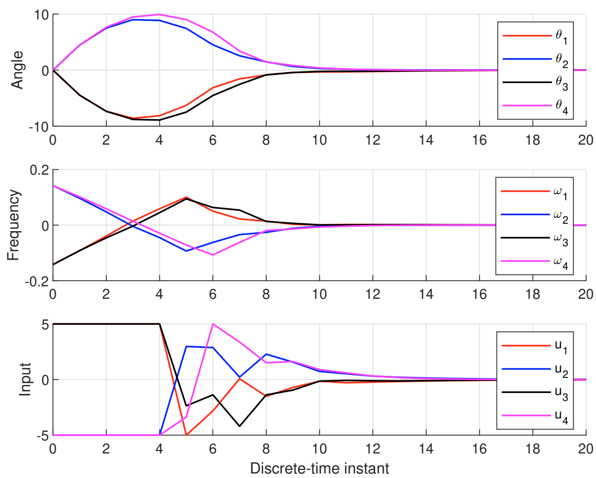


Fig. 1. *Example 2*: State and input trajectories, $N = 4$.

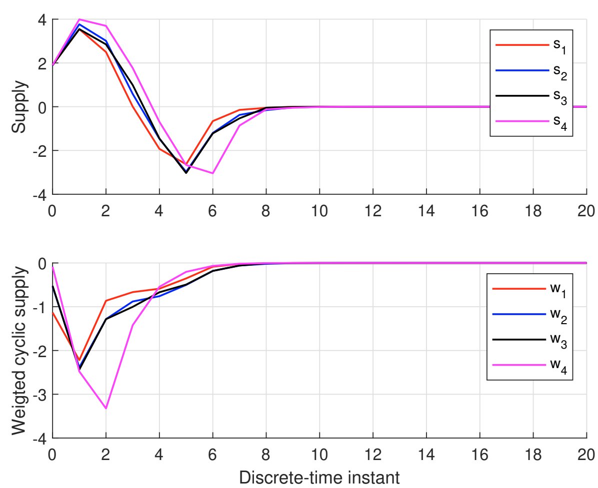


Fig. 2. *Example 2*: Supply trajectories.

generators. In Figure 2 we provide the corresponding dynamic supply trajectories and the weighted cyclic supplies $w_i(k) := \sum_{l=0}^{M_i-1} \mu_{i,l} s_i(k+l)$ with $M_i = 5$ for all $i$ and

$$\mu_{i,0} = \mu_{i,1} = \mu_{i,2} = \mu_{i,3} = 0.1,\ \mu_{i,4} = 1,\ \text{ for } i = 1,2,3,$$
$$\mu_{4,0} = \mu_{4,1} = 0.01,\ \mu_{4,2} = \mu_{4,3} = 0.1,\ \mu_{4,4} = 1.$$

Minimizing $\lambda_i s_i(k)$ has a damping effect on the closed–loop system trajectories and reduces the control effort, despite not penalizing the control inputs.

As visible in Figure 2, $s_i(k) > 0$ for all $i$ and $k = 0, 1, 2$ and thus, the conditions corresponding to (Willems, 1976), (Arcak et al., 2016) do not hold. Also, the cyclically neutral supply conditions (14), which require that each $V_i$ is a local finite–step Lyapunov function, do not hold even for $M_i = 10$. In constrast, the weighted cyclically neutral supply conditions (13) hold with $M_i = 5$ and the weights given above, as shown in the bottom plot in Figure 2.

## 6. CONCLUSIONS

In this paper we have considered stability of large scale interconnected nonlinear dissipative systems. We have developed generalized compositional stability criteria for interconnected dissipative systems by means of cyclically neutral supply conditions. The resulting stability criteria are less conservative and scale well with the network size. For future work is particularly of interest to utilize the developed stability criteria for solving the stability problems arising in current/future smart grids.

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