

# Symmetric wide-matrix varieties and powers of GL-varieties

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# Symmetric wide-matrix varieties and powers of GL-varieties

PROEFSCHRIFT

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# Summary

The aim of this thesis Symmetric Wide-matrix Varieties and Powers of GL-varieties is to study (sequences of) algebraic varieties defined by systems of polynomial equations with a large symmetry group. In the limit, they become infinite-dimensional with an infinite symmetry group and are amenable to techniques exploiting this symmetry.

The thesis is composed of four chapters, the first of which provides an introduction with essential definitions. Each of the remaining chapters, based on journal articles, covers a distinct topic related to the central theme of the thesis.

In Chapter 2 and Chapter 3, the focus is on sequences of varieties of matrices where the number of rows is fixed, but the number of columns increases. More specifically, Chapter 2 concentrates on counting the number of irreducible components of varieties within such a sequence, up to symmetry, while Chapter 3 establishes the descending chain property of their image closures under polynomial maps that respect symmetry. The final chapter, Chapter 4, focuses on the stabilization of infinite-dimensional varieties that are stable under coordinate permutations and linear coordinate transformations.



# Notation

$\mathbb{N}$	$\{1, 2, 3, \dots\}$
$\mathbb{Z}_{\geq 0}$	$\{0, 1, 2, \dots\}$
$\mathbb{Z}$	$\{0, \pm 1, \pm 2, \dots\}$
$[n]$	$\{1, 2, \dots, n\}$ for $n \in \mathbb{Z}_{\geq 0}$
$\text{Sym}(n)$	the symmetric group on $n$ letters
$\text{Sym}$	$\text{Sym}(\infty) = \cup_{n \geq 1} \text{Sym}(n)$
$V^*$	dual of a vector space $V$
$\text{GL}(V)$	the group of invertible linear maps $V \rightarrow V$
$\text{GL}_n$	$\text{GL}(K^n)$ for a field $K$
$\text{GL}$	$\text{GL}_\infty = \cup_{n \geq 1} \text{GL}_n$
$E_{i,j}$	$n \times n$ -matrix where the entry in position $(i, j)$ equals one and the other entries are zeroes



# Chapter 1

## Preliminaries

### 1.1 Introduction

In algebraic geometry, infinite-dimensional varieties are difficult to understand. But many of them that are equipped with an action of a large group are well behaved up to that action. This lays a foundation for equivariant commutative algebra and equivariant algebraic geometry: the study of commutative rings and varieties equipped with group action.

For instance, varieties of  $c \times \mathbb{N}$ -matrices that are stable under the action of the infinite symmetric group  $\text{Sym}(\mathbb{N})$  by permuting columns are equivariantly Noetherian (Noetherian up to the  $\text{Sym}(\mathbb{N})$  action): every descending chain

$$X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n \supseteq \cdots$$

of Zariski closed subsets, each stable under the column permutations, stabilizes, which means that  $X_n = X_{n+1} = \cdots$  for  $n$  sufficiently large.

There are two general problems in the equivariant world: one is to investigate equivariantly Noetherian spaces taking into account the group action and the other is to discover spaces that are equivariantly Noetherian or construct new equivariantly Noetherian spaces from existing ones.

#### 1.1.1 Main results

This thesis demonstrates the following results in relation to the aforementioned problems.

**Theorem 1.1.1.1.** *The image closures of equivariantly Noetherian varieties of  $c \times \mathbb{N}$ -matrices under  $\mathrm{Sym}(\mathbb{N})$ -equivariant polynomial maps (maps respecting the  $\mathrm{Sym}(\mathbb{N})$ -action) in the space of  $\mathbb{N} \times \cdots \times \mathbb{N}$ -tensors are equivariantly Noetherian. Moreover, they are defined, as a set, by  $\mathrm{Sym}(\mathbb{N})$ -orbits of finitely many equations.*

A natural approach to examine a variety  $X$  of  $c \times \mathbb{N}$ -matrices is by studying a sequence of varieties  $X_n$  of truncated matrices:

$$X_n := X \cap \mathbb{A}^{c \times n}$$

where  $\mathbb{A}^{c \times n}$  is the space of all  $c \times n$ -matrices. Such a sequence of varieties is called a wide-matrix variety. Each  $X_n$  is stable under the group  $\mathrm{Sym}(n)$  of column permutations and the action of  $\mathrm{Sym}(n)$  on  $X_n$  induces an action of  $\mathrm{Sym}(n)$  on the set  $\mathcal{C}(X_n)$  of irreducible components (irreducible subsets that are maximal under inclusion) of  $X_n$ . Note that the set  $\mathcal{C}(X_n)$  is finite as  $X_n$  is a finite-dimensional variety.

Recall that a quasipolynomial is a function  $f : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$  of the form

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_0(n)$$

where each  $c_i : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}$  is periodic with integral period.

**Theorem 1.1.1.2.** *There exists a quasipolynomial  $f$  and a natural number  $N$  such that the number of  $\mathrm{Sym}(n)$ -orbits on  $\mathcal{C}(X_n)$  equals  $f(n)$  for all  $n \geq N$ .*

**Theorem 1.1.1.3.** *Let  $Z$  be a variety that is Noetherian up to an action of the infinite general linear group  $\mathrm{GL}$ . Then the product variety  $Z^{\mathbb{N}}$  is Noetherian up to the natural action of the product group  $\mathrm{Sym}(\mathbb{N}) \times \mathrm{GL}$ .*

Theorem 1.1.1.1 is an extension to a result by Draisma, Eggermont, Krone, and Leykin in [DEKL16]. They proved that the image closures of wide-matrix varieties under monomial maps satisfy the descending chain condition. This work has been generalized to image closures under polynomial maps. However, the result is topological rather than ring theoretic.

Theorem 1.1.1.2 is inspired by a recent work of Le, Nagel, Nguyen, and Römer in [LNNR19; LN22]. They proved and conjectured many nice results about homological invariants of wide-matrix varieties.

Theorem 1.1.1.3 is motivated by recent works on  $\mathrm{Sym}(\mathbb{N})$ -stable varieties [AH07; HS12; Dra14; NS21b] and on  $\mathrm{GL}$ -stable varieties [DES17; Dra19; BDES22]. This work gives a common generalization to the above works, offering a positive outlook for a future in which the  $\mathrm{Sym}(\mathbb{N})$ -world and the  $\mathrm{GL}$ -world can be effectively combined to advance equivariant algebraic geometry.

### 1.1.2 Relation to existing literature

Noetherianity up to symmetry was first studied by Cohen in [Coh67; Coh87]. He proved that the (non-Noetherian) polynomial ring  $K[x_1, x_2, x_3, \dots]$  in infinitely many variables is Noetherian up to the permutation of variables as well as a generalization to finitely many rows of variables. The same was re-discovered in [AH07; HS12]. In the later article, Hillar and Sullivant used this to prove the Independent Set Theorem which is an important result in algebraic statistics. A further application of this result is due to Draisma [Dra10]. He proved that the polynomial equations for the  $k$ -factor model and for the chirality varieties are finitely characterizable.

Many challenging questions about Sym-invariant chains of ideals and varieties have been answered using Noetherianity up to symmetry. For example, the bivariate Hilbert series for such a chain of ideals is a rational function [NR17; KLS17; GN18]. As a consequence of the rationality result codimension (respectively multiplicity) of varieties appearing in a wide-matrix variety grows eventually linearly (respectively exponentially); see [NR17; LNNR19]. Furthermore, asymptotic behavior of projective dimension, Betti numbers, and Castelnuovo-Mumford regularity along such chains have been studied in [LN22; Bie+20]. A very nice survey is [JLR20] which also contains a number of open problems. Further geometry of these varieties has been studied in [NS21b; NS21a; KR22; KVR22].

Returning to the general linear group: A fundamental property, the descending chain property, of varieties acted upon by the infinite general linear group was proved by Draisma in [Dra19] (for polynomial representations) and by Eggermont and Snowden in [ES21] (for algebraic representations). From the former article, this property was used in [DLL19; ESS19a; ESS19b] to give a new proof of a very beautiful conjecture, Stillman’s conjecture, which is a motivation for the growing literature on equivariant commutative algebra. Further applications of GL-varieties are [BDE19; DES17]. For streamlined literature on GL-varieties see [BDES22] where many foundational results about them have been established.

Inspired by literature on **FI**-modules [CEF15; NR19] this work uses the language of categories and functors. We think that this is a useful and convenient language to prove the above results. In this setup, a wide-matrix variety is an **FI**<sup>op</sup>-scheme: a contravariant functor from the category **FI** of finite sets with injections to the category of affine schemes.

### 1.1.3 Organization of this thesis

The remainder of this chapter discusses direct and inverse limits, Noetherianity up to the action of a group, **FI**-algebras, and **FI<sup>op</sup>**-schemes. These concepts have been used throughout this work. The remaining chapters are based on my work with my advisors and research colleagues during my Ph.D. studies at the Eindhoven University of Technology. Chapter 2 is about components of wide-matrix varieties and is based on the article [DEF22] with Jan Draisma and Rob Eggermont. Chapter 3 is about the image closures of wide-matrix varieties and is based on the article [DEFM22] with Jan Draisma, Rob Eggermont, and Leandro Meier. Chapter 4, the last chapter, is devoted to studying powers of GL-varieties and is based on the article [Chi+22] with Christopher H. Chiu, Alessandro Danelon, Jan Draisma, and Rob Eggermont.

## 1.2 Limits

### 1.2.1 Direct limit

Direct limit is a way of constructing a larger object from a sequence of smaller objects that are combined in a particular manner. In this section, the objects are defined as sets with a specific algebraic structure, such as groups, rings, and algebras (over a fixed ring). The process of combining them is established by a system of morphisms (in the relevant category) between these smaller objects.

**Definition 1.2.1.1.** Let  $(A_n)_{n \in \mathbb{N}}$  be a sequence of objects in some category and  $\varphi_{mn} : A_m \rightarrow A_n$  be a morphism for all  $m \leq n$  with the following properties:

1.  $\varphi_{nn}$  is the identity of  $A_n$ , and
2.  $\varphi_{ln} = \varphi_{mn} \circ \varphi_{lm}$  for all  $l \leq m \leq n$ .

The pair  $((A_n)_n, (\varphi_{mn})_{m \leq n})$  is called a direct system. ◇

**Definition 1.2.1.2.** The direct limit, also called a colimit, of the direct system  $((A_n)_n, (\varphi_{mn})_{m \leq n})$  is denoted by  $\varinjlim A_n$  and is defined as follows. Its underlying set is the disjoint union of the  $A_n$ 's modulo the following equivalence relation  $\sim$ : if  $a_l \in A_l$  and  $a_m \in A_m$ , then  $a_l \sim a_m$  if and only if there is some  $n$  with  $l \leq n$  and  $m \leq n$  and such that  $\varphi_{ln}(a_l) = \varphi_{mn}(a_m)$ . ◇

Intuitively, two elements in the disjoint union are equivalent if and only if they eventually become equal in the direct system.

Denote by  $\text{Sym}(n)$  the symmetric group on  $n$  letters. Let  $\varphi_{n,n+1} : \text{Sym}(n) \rightarrow \text{Sym}(n+1)$  be the inclusion map by regarding  $\text{Sym}(n)$  as stabilizer of  $n+1$  in  $\text{Sym}(n+1)$ .

**Definition 1.2.1.3.** Denote by  $\text{Sym}(\infty)$  (or simply  $\text{Sym}$ ) the direct limit of the direct system  $(\text{Sym}(n), \varphi_{mn})$  where  $\varphi_{mn} : \text{Sym}(m) \rightarrow \text{Sym}(n)$  is the composition of the inclusion maps.  $\diamond$

Denote by  $\text{GL}_n$  the general linear group of  $n \times n$  invertible matrices over some field. Let  $\varphi_{n,n+1} : \text{GL}_n \rightarrow \text{GL}_{n+1}$  be the natural inclusion map  $g \mapsto \begin{bmatrix} g & 0 \\ 0 & 1 \end{bmatrix}$ .

**Definition 1.2.1.4.** Denote by  $\text{GL}_\infty$  (or simply  $\text{GL}$ ) the direct limit of the direct system  $(\text{GL}_n, \varphi_{mn})$  where  $\varphi_{mn} : \text{GL}_m \rightarrow \text{GL}_n$  is the composition of the inclusion maps.  $\diamond$

## 1.2.2 Inverse limit of topological spaces

In this section, we define the inverse limit of topological spaces which is a construction that allows us to glue together a sequence of topological spaces. The precise gluing process is specified by continuous maps between the topological spaces.

**Definition 1.2.2.1.** Let  $(X_n)_{n \in \mathbb{N}}$  be a family of topological spaces and  $f_{mn} : X_n \rightarrow X_m$  be a continuous map for all  $m \leq n$  (note the order) with the following properties:

1.  $f_{nn}$  is the identity of  $X_n$ , and
2.  $f_{ln} = f_{lm} \circ f_{mn}$  for all  $l \leq m \leq n$ .

The pair  $((X_n)_n, (f_{mn})_{m \leq n})$  is called an inverse system.  $\diamond$

**Definition 1.2.2.2.** The inverse limit of the inverse system  $((X_n)_n, (f_{mn})_{m \leq n})$  is defined as

$$X_\infty := \varprojlim X_n = \{(x_n) \in \prod_{n \in \mathbb{N}} X_n \mid f_{n,n+1}(x_{n+1}) = x_n\}.$$

$X_\infty$  is equipped with the subspace topology, where  $\prod_{n \in \mathbb{N}} X_n$  has the product topology. We call it inverse limit topology.  $\diamond$

Denote by  $\pi_m$  the natural projection map  $\prod_{n \in \mathbb{N}} X_n \rightarrow X_m$  as well as its restriction  $X_\infty \rightarrow X_m$ . These mappings  $\pi_m$  are continuous and satisfy the following compatibility condition:

$$\begin{array}{ccc} & X_\infty & \\ \pi_n \swarrow & & \searrow \pi_m \\ X_n & \xrightarrow{f_{mn}} & X_m \end{array}$$

$f_{mn} \circ \pi_n = \pi_m$  whenever  $m \leq n$ .

**Remark 1.2.2.3.** The inverse limit topology  $X_\infty$  satisfies the following universal property: whenever  $Y$  is a topological space and for each  $n \in \mathbb{N}$ ,  $\psi_n : Y \rightarrow X_n$  is a continuous map satisfying the compatibility condition  $f_{mn} \circ \psi_n = \psi_m$  whenever  $m \leq n$ ,

$$\begin{array}{ccc} & Y & \\ \psi_n \swarrow & \downarrow h & \searrow \psi_m \\ & X_\infty & \\ \pi_n \swarrow & & \searrow \pi_m \\ X_n & \xrightarrow{f_{mn}} & X_m \end{array}$$

then there is a unique continuous map  $h : Y \rightarrow X_\infty$  such that  $\pi_n \circ h = \psi_n$  for all  $n \in \mathbb{N}$ .  $\diamond$

### 1.3 Noetherianity up to the action of a group

Noetherianity is a desirable property but not all spaces possess this property. However, it has been observed that many non-Noetherian rings and topological spaces can exhibit Noetherian-like behavior under the action of a large group. Specifically, these objects are Noetherian up to a group action in the following sense.

**Definition 1.3.0.1.** Let  $X$  be a topological space equipped with an action of a group  $G$  by means of continuous maps.

1. A closed subset  $Y$  of  $X$  is called a  $G$ -stable closed subset of  $X$  if  $g \cdot y \in Y$  for all  $g \in G$  and  $y \in Y$ .

2. The space  $X$  is said to be  $G$ -Noetherian or Noetherian up to the action of  $G$  (or equivariantly Noetherian when  $G$  is clear from the context) if it satisfies the descending chain condition on  $G$ -stable closed subsets: every descending chain

$$X_1 \supseteq X_2 \supseteq \cdots \supseteq X_n \supseteq \cdots$$

of  $G$ -stable closed subsets stabilizes, which means that  $X_n = X_{n+1} = \cdots$  for  $n$  sufficiently large..

◇

**Definition 1.3.0.2.** Let  $R$  be a commutative ring equipped with an action of a group  $G$  by means of ring homomorphisms.

1. An ideal  $I$  of the ring  $R$  is said to be a  $G$ -stable ideal if  $g \cdot f \in I$  for all  $g \in G$  and  $f \in I$ .
2. The ring  $R$  is said to be  $G$ -Noetherian or Noetherian up to the action of  $G$  if it satisfies the ascending chain condition on  $G$ -stable ideals, or equivalently, if every  $G$ -stable ideal of  $R$  is generated by  $G$ -orbits of finitely many elements of  $R$ .

◇

It is easy to show that if  $R$  is  $G$ -Noetherian ring, then the spectrum  $\text{Spec } R$  of  $R$  equipped with the Zariski topology is  $G$ -Noetherian with respect to the induced action of  $G$  on  $\text{Spec } R$ .

Since every Noetherian ring satisfies the ascending chain condition on ideals ( $G$ -stable or not), every Noetherian ring is  $G$ -Noetherian and also the spectrum of a Noetherian ring is  $G$ -Noetherian.

Presented below is a noteworthy theorem that provides an example of a nontrivial  $\text{Sym}(\infty)$ -Noetherian ring.

**Theorem 1.3.0.3.** [Coh87; HS12] Fix a positive integer  $c$  and a Noetherian ring  $K$ . The polynomial ring

$$K[X_{c \times \mathbb{N}}] = K[x_{i,j} \mid i \in [c], j \in \mathbb{N}]$$

with the action of the symmetric group  $\text{Sym}(\infty)$  defined by  $\sigma \cdot x_{i,j} = x_{i,\sigma(j)}$  is  $\text{Sym}(\infty)$ -Noetherian.

**Corollary 1.3.0.4.** The space  $\text{Spec } K[X_{c \times \mathbb{N}}]$  of  $c \times \mathbb{N}$ -matrices under the action of the symmetric group  $\text{Sym}(\infty)$  by permuting the columns is topologically  $\text{Sym}(\infty)$ -Noetherian.

**Example 1.3.0.5.** [HS12, Example 3.8.] The polynomial ring

$$K[X_{\mathbb{N} \times \mathbb{N}}] = K[x_{i,j} \mid (i,j) \in \mathbb{N} \times \mathbb{N}]$$

under the action of  $\text{Sym}(\infty)$  defined by  $\sigma \cdot x_{i,j} = x_{\sigma(i),\sigma(j)}$  is not  $\text{Sym}(\infty)$ -Noetherian (not even  $\text{Sym}(\infty) \times \text{Sym}(\infty)$ -Noetherian with respect to the action defined by  $(\tau, \sigma) \cdot x_{i,j} = x_{\tau(i),\sigma(j)}$ ). Indeed, for  $l \geq 3$  denote by  $M_l$  the monomial  $x_{1,1}x_{1,2}x_{2,2}x_{2,3} \cdots x_{l,l}x_{l,1}$  and for  $n \geq 3$  denote by  $I_n$  the ideal generated by  $\text{Sym}(\infty)$ -orbits of monomials  $M_3, M_4, \dots, M_n$ . Then, the following chain

$$I_3 \subset I_4 \subset I_5 \subset \cdots$$

of  $\text{Sym}(\infty)$ -stable ideals of  $K[X_{\mathbb{N} \times \mathbb{N}}]$  does not stabilize.  $\diamond$

**Remark 1.3.0.6.** The above example also shows that the space  $\text{Spec } K[X_{\mathbb{N} \times \mathbb{N}}]$  of  $\mathbb{N} \times \mathbb{N}$ -matrices equipped with the natural action of  $\text{Sym}(\infty)$  is not equivariantly Noetherian (not even  $\text{Sym}(\infty) \times \text{Sym}(\infty)$ -Noetherian).  $\diamond$

We close this section by stating a general problem which is the main motivation for this work.

**Problem 1.3.0.7.** *Study the  $\text{Sym}(\infty)$ -stable ideals of  $K[X_{c \times \mathbb{N}}]$  and the  $\text{Sym}(\infty)$ -stable varieties of  $\text{Spec } K[X_{c \times \mathbb{N}}]$ .*

## 1.4 FI-algebras and $\mathbf{FI}^{\text{op}}$ -schemes

In this section, the central focus will be on  $\mathbf{FI}$ -algebras and  $\mathbf{FI}^{\text{op}}$ -schemes. They form the backbone of our work and will be thoroughly explored and analyzed. Our main references for this section are [NR19; DEF22], which provide comprehensive coverage of these topics.

### 1.4.1 The category $\mathbf{FI}$

**Definition 1.4.1.1.** Denote by  $\mathbf{FI}$  the category whose objects are finite sets and whose morphisms are injective maps.  $\mathbf{FI}^{\text{op}}$  denotes its opposite category.  $\diamond$

Denote by  $\mathbb{Z}_{\geq 0}$  and  $\mathbb{N}$  the set of nonnegative integers and positive integers, respectively. The category  $\mathbf{FI}$  is equivalent to the category with objects  $[n] := \{1, 2, \dots, n\}$  for  $n \in \mathbb{Z}_{\geq 0}$  (by convention  $[0] := \emptyset$ ) and morphisms from  $[m]$  to  $[n]$  being the injective maps  $[m] \hookrightarrow [n]$ .

## 1.4.2 The category of $\mathbf{FI}^{\text{op}}$ -schemes

Let  $K$  be a commutative ring with unity. Denote by  $\mathbf{Alg}_K$  the category of commutative, unital  $K$ -algebras  $M$  whose morphisms are unital ring homomorphisms  $M \rightarrow N$  compatible with the homomorphisms from  $K$  into them.

**Definition 1.4.2.1.** An  $\mathbf{FI}$ -algebra over  $K$  is a covariant functor from  $\mathbf{FI}$  to the category  $\mathbf{Alg}_K$  of  $K$ -algebras.  $\diamond$

For a finite set  $S \in \mathbf{FI}$  we denote the  $K$ -algebra  $A(S)$  by  $A_S$  and we write  $A_n$  for  $A_{[n]}$ .

**Example 1.4.2.2.** Let  $L$  be an algebra over  $K$ , then the constant functor  $\mathbf{FI} \rightarrow \mathbf{Alg}_K$  assigning  $L$  to every finite set and assigning  $\text{id}_L$  to every injection is an  $\mathbf{FI}$ -algebra.  $\diamond$

**Example 1.4.2.3.** The functor  $P : \mathbf{FI} \rightarrow \mathbf{Alg}_K$  that assigns to each finite set  $S$  the polynomial ring  $P_S := K[x_i \mid i \in S]$  and assigns to each injection  $\sigma : S \rightarrow T$  the  $K$ -algebra homomorphism  $P(\sigma) : P_S \rightarrow P_T$  determined by  $x_i \mapsto x_{\sigma(i)}$  is an  $\mathbf{FI}$ -algebra.  $\diamond$

**Example 1.4.2.4.** The functor  $Q : \mathbf{FI} \rightarrow \mathbf{Alg}_K$  that assigns to each finite set  $S$  the polynomial ring  $Q_S := K[y_{i,j} \mid i, j \in S]$  and assigns to each injection  $\sigma : S \rightarrow T$  the  $K$ -algebra homomorphism  $Q(\sigma) : Q_S \rightarrow Q_T$  determined by  $y_{i,j} \mapsto y_{\sigma(i), \sigma(j)}$  is an  $\mathbf{FI}$ -algebra.  $\diamond$

**Definition 1.4.2.5.** An affine  $\mathbf{FI}^{\text{op}}$ -schemes over  $K$  is a contravariant functor from  $\mathbf{FI}$  to the category of affine schemes over  $K$ .  $\diamond$

For a finite set  $S \in \mathbf{FI}$  we denote the scheme  $X(S)$  by  $X_S$  and we write  $X_n$  for  $X_{[n]}$ . As we will only consider affine  $\mathbf{FI}^{\text{op}}$ -schemes, we will usually drop the adjective “affine”.

**Example 1.4.2.6.** The contravariant functor from  $\mathbf{FI}$  to the category of affine schemes over  $K$  that assigns to each finite set  $S$  the spectrum  $\text{Spec } P_S = \text{Spec } K[x_i \mid i \in S]$  and assigns to each injection  $\sigma : S \rightarrow T$  the morphism  $\text{Spec } P(\sigma) : \text{Spec } P_T \rightarrow \text{Spec } P_S$  dual to the  $K$ -algebra homomorphism  $P(\sigma) : P_S \rightarrow P_T$  is an  $\mathbf{FI}^{\text{op}}$ -scheme.  $\diamond$

**Definition 1.4.2.7.** Given an  $\mathbf{FI}$ -algebra  $A$  over  $K$ , the spectrum of  $A$  denoted by  $\text{Spec}(A)$  is the  $\mathbf{FI}^{\text{op}}$ -scheme that maps a finite set  $S$  to  $\text{Spec}(A_S)$ . Dually, given an  $\mathbf{FI}^{\text{op}}$ -scheme  $X$ , the coordinate ring of  $X$  denoted by  $K[X]$  is the  $\mathbf{FI}$ -algebra that maps  $S$  to  $K[X_S]$  the coordinate ring of the scheme  $X_S$ .  $\diamond$

**Remark 1.4.2.8.** If  $A$  is an **FI**-algebra over  $K$ , then  $A$  is also an **FI**-algebra over  $A_0$ : for each finite set  $S$  the unique inclusion  $\emptyset \rightarrow S$  yields an algebra homomorphism  $A_0 \rightarrow A_S$ , and functoriality implies that the  $K$ -algebra homomorphisms  $A_S \rightarrow A_T$  corresponding to injections  $S \rightarrow T$  are compatible with the  $A_0$ -algebra structure.  $\diamond$

**Definition 1.4.2.9.** An **FI**-ideal  $I$  of an **FI**-algebra  $A$  assigns to each finite set  $S$  an ideal  $I(S)$  of  $A(S)$ , in such a manner that the algebra homomorphism  $A(\pi) : A(S) \rightarrow A(T)$  corresponding to any  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  maps  $I(S)$  into  $I(T)$ . A closed subscheme  $Z$  of an **FI**<sup>op</sup>-scheme  $X$  is a subfunctor of  $X$  such that, for each finite set  $S$ ,  $Z(S)$  is a closed subscheme of  $X(S)$ .  $\diamond$

Every ideal  $I$  of an **FI**-algebra  $A$  gives rise to a closed **FI**<sup>op</sup>-subscheme  $\text{Spec}(A/I)$  of the **FI**<sup>op</sup>-scheme  $\text{Spec } A$  defined by  $\text{Spec}(A/I)_S := \text{Spec}(A_S/I_S)$ .

A morphism  $A \rightarrow B$  of **FI**-algebras over  $K$  is a natural transformation of functors: it consists of a  $K$ -algebra homomorphism  $\varphi(S) : A_S \rightarrow B_S$  for each  $S \in \mathbf{FI}$  such that for all  $S, T \in \mathbf{FI}$  and for each injection  $\sigma$  from  $S$  to  $T$ , the following diagram commutes

$$\begin{array}{ccc} A_S & \xrightarrow{\varphi(S)} & B_S \\ A(\sigma) \downarrow & & \downarrow B(\sigma) \\ A_T & \xrightarrow{\varphi(T)} & B_T. \end{array}$$

Morphisms of **FI**<sup>op</sup>-schemes over  $K$  are defined dually. The category of **FI**-algebras is equivalent to the category of **FI**<sup>op</sup>-schemes. The equivalence is given by the functor that associates to an **FI**-algebra  $A$  the spectrum  $\text{Spec } A$  of  $A$ .

Let  $A$  be an **FI**-algebra, and consider any finite set  $S \in \mathbf{FI}$ . In this context, the symmetric group  $\text{Sym}(S)$  acts from the left on  $A(S)$ , where  $\text{Sym}(S) = \text{Hom}_{\mathbf{FI}}(S, S)$ . Specifically, each  $\pi$  in  $\text{Hom}_{\mathbf{FI}}(S, S)$  yields a  $K$ -algebra homomorphism  $A(\pi) : A(S) \rightarrow A(S)$ . It follows from the functorial axioms of  $A$  that  $(\pi, a) \mapsto A(\pi)(a)$  represents a left action of  $\text{Sym}(S)$  by  $K$ -algebra automorphisms on  $A(S)$ .

Similarly, given an **FI**<sup>op</sup>-scheme  $X$ , for each finite set  $S \in \mathbf{FI}$  the symmetric group  $\text{Sym}(S)$  acts on  $X_S$  by automorphisms of  $K$ -schemes. When acting on points of  $X_S$  with values in a  $K$ -algebra  $L$ , i.e., on the set of  $K$ -algebra homomorphisms  $K[X_S] \rightarrow L$ , this is a naturally a *right* action, reflecting the fact that  $X$  is a contravariant functor.

**Remark 1.4.2.10.** The symmetric group  $\text{Sym}(n)$  acts on  $A_n$  by  $K$ -algebra automorphisms, and the map  $A(\iota) : A_n \rightarrow A_{n+1}$ , where  $\iota : [n] \rightarrow [n+1]$  is

the standard inclusion, is a  $\text{Sym}(n)$ -equivariant  $K$ -algebra homomorphism, if  $\text{Sym}(n)$  is regarded as the subgroup of  $\text{Sym}(n+1)$  consisting of all permutations that fix  $n+1$ . Conversely, from the data (for all  $n$ ) of  $A_n$ , the action of  $\text{Sym}(n)$  on  $A_n$ , and the  $\text{Sym}(n)$ -equivariant map  $A_n \rightarrow A_{n+1}$  the **FI**-algebra  $A$  can be recovered up to isomorphism. This gives another, more concrete picture of **FI**-algebras. However, the definition of **FI**-algebras as a functor from **FI** to  $K$ -algebras is more elegant and, as we will see, often more convenient.  $\diamond$

The tensor product  $A \otimes B$  of **FI**-algebras  $A$  and  $B$  over  $K$  is defined in the obvious way: for an object  $S \in \mathbf{FI}$ ,

$$(A \otimes B)_S := A_S \otimes_K B_S$$

and for an injection  $\sigma : S \rightarrow T$  the map  $(A \otimes B)(\sigma)$  is determined by,

$$(A \otimes B)(\sigma)(a \otimes b) = A(\sigma)(a) \otimes B(\sigma)(b)$$

where  $a \in A_S$  and  $b \in B_S$ .

The direct sum  $A \oplus B$  of **FI**-algebras  $A$  and  $B$  over  $K$  is defined in a similar way.

### 1.4.3 Wide-matrix spaces

Recall that  $P$  denotes the **FI**-algebra that maps a finite set  $S$  to the polynomial ring  $P_S := K[x_i \mid i \in S]$ .

**Definition 1.4.3.1.** For a positive integer  $c \in \mathbb{N}$ , the tensor product  $P^{\otimes c}$  of the **FI**-algebra  $P$  is isomorphic to, and will be identified with, the **FI**-algebra that maps an object  $S \in \mathbf{FI}$  to

$$P_S^{\otimes c} := K[x_{i,j} \mid 1 \leq i \leq c, j \in S].$$

The **FI**<sup>op</sup>-scheme  $\text{Spec } P^{\otimes c}$  over  $K$  is called a wide-matrix space. It is denoted by  $\text{Mat}_{c,K}$ .  $\diamond$

If  $L$  is a  $K$ -algebra, then the *set of  $L$ -points*  $\text{Mat}_{c,K}(L)$  is the contravariant functor from **FI** to sets that assigns to  $S$  the set  $L^{c \times S}$  of  $c \times S$ -matrices over  $L$ , and to a morphism  $\pi : S \rightarrow T$  the map  $L^{c \times T} = (L^c)^T \rightarrow (L^c)^S = L^{c \times S}$  where the middle map is composition with  $\pi$ .

### 1.4.4 Width-d $\mathbf{FI}^{\text{op}}$ -schemes

Width-one  $\mathbf{FI}^{\text{op}}$ -schemes of finite type are nicely behaved, as we will see, and are the main characters of this work.

**Definition 1.4.4.1.** Let  $A$  be an  $\mathbf{FI}$ -algebra over a ring  $K$ . Let  $d$  be a nonnegative integer.

1. A subalgebra  $B$  of  $A$  is a functor from  $\mathbf{FI}$  to  $K\text{-}\mathbf{Alg}$  such that  $B_S$  is a subalgebra of  $A_S$  for each finite set  $S$  and  $B(\sigma)$  is the restriction of the map  $A(\sigma)$  to  $B_S$  for each injection  $\sigma : S \rightarrow T$ .
2.  $A$  is said to be finitely generated if there are finitely many objects  $S_i \in \mathbf{FI}$  and a finite set  $D \subset \bigcup A_{S_i}$  which is not contained in any proper subalgebra of  $A$ . If such  $D$  exists, we say that  $A$  is generated by  $D$ .

◇

**Definition 1.4.4.2.** For a finite set  $S \in \mathbf{FI}$  and an element  $a \in A_S$ , the minimal  $n$  such that  $a$  lies in  $A(\pi)(A_n)$  for an injection  $\pi : [n] \rightarrow S$ , is called the width of  $a$ . It is denoted by  $w(a)$ .

◇

**Example 1.4.4.3.** The element  $x_3 + x_5^2 \in P([5])$  has width 2 because  $x_3 + x_5^2 = P(\sigma)(x_1 + x_2^2)$  where  $\sigma : [2] \rightarrow [5]$  is an injection mapping  $1 \mapsto 3$  and  $2 \mapsto 5$ .

◇

**Definition 1.4.4.4.** Let  $A$  be an  $\mathbf{FI}$ -algebra.

1.  $A$  is said to be generated in width  $\leq d$  if there exists a collection  $(a_i \in A_{S_i})_{i \in I}$  of elements of width  $\leq d$  that generates  $A$ .
2.  $A$  is said to be finitely generated in width  $\leq d$  if  $A$  is finitely generated and generated in width  $\leq d$ .

◇

**Remark 1.4.4.5.** The definition finitely generated in width  $\leq d$  is equivalent to the condition that  $A$  is generated by a finite collection of elements  $a_i \in A_{S_i}$  of width  $\leq d$ .

◇

**Example 1.4.4.6.** The  $\mathbf{FI}$ -algebra  $P$  that assigns  $S \mapsto K[x_i \mid i \in S]$  is finitely generated in width  $\leq 1$ , namely, by the element  $x_1 \in P([1])$ .

◇

**Example 1.4.4.7.** The  $\mathbf{FI}$ -algebra  $Q$  that maps  $S \rightarrow K[y_{i,j} \mid i, j \in S]$  is generated by two elements  $y_{1,1}$  and  $y_{1,2}$  having width 1 and 2 respectively. So it is finitely generated in width  $\leq 2$  (but not in width  $\leq 1$ ).

◇

**Definition 1.4.4.8.** The spectrum of a finitely generated **FI**-algebra over  $K$  is called an **FI**<sup>op</sup>-scheme of finite type over  $K$ .  $\diamond$

**Definition 1.4.4.9.** The spectrum of an **FI**-algebra over  $K$  that is finitely generated in width  $\leq d$  is called a width- $d$  **FI**<sup>op</sup>-scheme of finite type over  $K$ .  $\diamond$

**Remark 1.4.4.10.** The class of **FI**-algebras that are finitely generated in width at most 1 is closed under taking finite direct sums and tensor products over  $K$ . Dually, the corresponding class of width-one **FI**<sup>op</sup>-schemes of finite type is closed under disjoint unions and Cartesian products.  $\diamond$

For later use we record the following lemma.

**Lemma 1.4.4.11.** *Let  $A$  be an **FI**-algebra over  $K$  that is finitely generated in width  $\leq 1$ , and let  $X = \text{Spec}(A)$  be the corresponding width-one **FI**<sup>op</sup>-scheme over  $K$ . If we set  $Z := X_1$ , then for each  $S \in \mathbf{FI}$  the map  $X_S \rightarrow \prod_{j \in S} X_{\{j\}} \cong Z^S$ , where the product is over  $\text{Spec}(A_0)$ , is a closed embedding. Furthermore, the **FI**<sup>op</sup>-scheme  $Z^S$  is isomorphic to a closed subset of  $\text{Mat}_{c, A_0}$  for some  $c$ .*

*Proof.* Dually, we need to show that the map  $\bigotimes_{j \in S} A_{\{j\}} \rightarrow A_S$ , where the tensor product is over  $A_0$  and where  $A_{\{j\}} \rightarrow A_S$  comes from the inclusion  $\{j\} \rightarrow S$ , is surjective. This follows from the fact that  $A$  is generated in width at most 1. For the last statement, note that  $A_1$  is finitely generated as a  $K$ -algebra, hence *a fortiori* as an  $A_0$ -algebra. If  $A_1$  is generated by  $c$  elements over  $A_0$ , then  $A$  is a quotient of  $P_{A_0}^{\otimes c}$ ,  $Z$  is a closed **FI**<sup>op</sup>-subscheme of the  $c$ -dimensional affine space  $\mathbb{A}_{A_0}^c$  over  $A_0$ , and  $S \mapsto Z^S$  a closed **FI**<sup>op</sup>-subscheme of  $\text{Mat}_{c, A_0}$ .  $\square$

## 1.4.5 Limit of **FI**<sup>op</sup>-schemes

Limits are a natural way to connect **FI**-algebras and **FI**<sup>op</sup>-schemes with infinite-dimensional objects that are acted upon by the infinite symmetric group.

Denote by  $\iota_{m,n} \in \text{Hom}_{\mathbf{FI}}([m], [n])$  the inclusion map  $[m] \rightarrow [n]$ ,  $j \mapsto j$  for  $m \leq n$ .

**Definition 1.4.5.1.** Given any **FI**-algebra  $A$  over  $K$ , define its direct limit

$$A_\infty = \varinjlim A_n$$

using the direct system  $((A_n)_{n \in \mathbb{N}}, (A(\iota_{m,n}))_{m \leq n})$ .

Dually, the inverse limit of an  $\mathbf{FI}^{\text{op}}$ -scheme  $X$  over  $K$  is defined as

$$X_\infty = \varprojlim X_n$$

using the inverse system  $((X_n)_{n \in \mathbb{N}}, (X(\iota_{m,n}))_{m \leq n})$ .  $\diamond$

**Remark 1.4.5.2.** Each  $X_n$  is equipped with the Zariski topology so  $X_\infty$  has the inverse limit topology.  $\diamond$

By functoriality, each  $A_n$  is acted upon by the symmetric group  $\text{Sym}(n)$ , the map  $A(\iota_{n,n+1})$  is  $\text{Sym}(n)$ -equivariant if  $\text{Sym}(n)$  is embedded into  $\text{Sym}(n+1)$  as the stabilizer of  $\{n+1\}$ , and hence the direct limit  $A_\infty$  is acted upon by the group  $\text{Sym}(\infty)$ . Dually,  $\text{Sym}(\infty)$ -acts on the inverse limit  $X_\infty$  of an  $\mathbf{FI}^{\text{op}}$ -scheme  $X$  over  $K$ .

Given an ideal  $I$  of  $A$ , the direct limit  $I_\infty := \varinjlim I_n$  is a  $\text{Sym}(\infty)$ -stable ideal of  $A_\infty$ . Conversely, given a  $\text{Sym}(\infty)$ -stable ideal  $J$  of  $A_\infty$ , then for any finite set  $S \in \mathbf{FI}$  and any bijective map  $\pi : [n] \rightarrow S$ , set  $I(S) := A(\pi)(J_n)$ , where  $J_n$  is the preimage of  $J$  in  $A_n$ .

**Example 1.4.5.3.** The direct limit  $(P^{\otimes c})_\infty$  of the  $\mathbf{FI}$ -algebra  $P^{\otimes c}$  that maps  $S \mapsto K[x_{i,j} \mid i \in [c] \text{ and } j \in S]$  is naturally isomorphic to the polynomial ring

$$K[x_{i,j} : 1 \leq i \leq c, j \in \mathbb{N}]$$

and admits a  $\text{Sym}(\infty)$ -action induced by  $\sigma(x_{i,j}) = x_{i,\sigma(j)}$  for any  $\sigma \in \text{Sym}(\infty)$ .  $\diamond$

**Example 1.4.5.4.** The direct limit  $Q_\infty$  of the  $\mathbf{FI}$ -algebra  $Q$  that maps  $S \mapsto K[y_{i,j} \mid i, j \in S]$  is naturally isomorphic to the polynomial ring

$$K[y_{i,j} : 1 \leq i \leq c, j \in \mathbb{N}]$$

and admits a  $\text{Sym}(\infty)$ -action induced by  $\sigma(y_{i,j}) = y_{\sigma(i),\sigma(j)}$  for any  $\sigma \in \text{Sym}(\infty)$ .  $\diamond$

### 1.4.6 Noetherianity of $\mathbf{FI}^{\text{op}}$ -schemes

We close this section by describing a very useful property, the descending chain property, of width-one  $\mathbf{FI}^{\text{op}}$ -schemes.

**Definition 1.4.6.1.** An  $\mathbf{FI}$ -algebra  $A$  is called Noetherian if it satisfies the ascending chain condition on ideals, i.e., if

$$I_1 \subseteq I_2 \subseteq \dots$$

is an ascending chain of ideals in  $A$ , then there exists a positive integer  $m$  such that  $I_m = I_{m+1} = \dots$ .  $\diamond$

It is easy to show that the direct limit of a Noetherian **FI**-algebra is  $\text{Sym}(\infty)$ -Noetherian.

**Definition 1.4.6.2.** An **FI**<sup>op</sup>-scheme  $X$  is called Noetherian if it satisfies the descending chain condition on closed **FI**<sup>op</sup>-subschemes, and topologically Noetherian if it satisfies the descending chain condition on closed and reduced **FI**<sup>op</sup>-subschemes: for each descending chain

$$X_1 \supseteq X_2 \supseteq \dots$$

of closed and reduced **FI**<sup>op</sup>-subschemes  $X_i \subseteq X$ , there exists an integer  $n$  such that  $X_n = X_{n+1} = \dots$ .  $\diamond$

Again the inverse limit of a topologically Noetherian **FI**<sup>op</sup>-scheme is topologically  $\text{Sym}$ -Noetherian. From now on we will use the word “closed subset” to mean “reduced subschemes”.

The following theorem and corollary are of fundamental importance for the rest of the thesis.

**Theorem 1.4.6.3.** [NR19, Theorem 6.15] *Let  $K$  be a Noetherian ring. Then every **FI**-algebra  $A$  over  $K$  that is finitely generated in width  $\leq 1$  is Noetherian. In particular, the **FI**-algebra  $P^{\otimes c}$  over a Noetherian ring  $K$  is Noetherian.*

**Corollary 1.4.6.4.** *Any width-one **FI**<sup>op</sup>-scheme of finite type over a Noetherian ring  $K$  is topologically Noetherian. In particular, the wide-matrix space  $\text{Spec } P^{\otimes c}$  over a Noetherian ring  $K$  is topologically Noetherian.*

**Example 1.4.6.5.** The space of  $c \times \mathbb{N}$ -matrices is  $\text{Sym}(\infty)$ -Noetherian up to column permutations being the inverse limit of the **FI**<sup>op</sup>-scheme  $\text{Spec } P^{\otimes c}$ .  $\diamond$

**Lemma 1.4.6.6.** *Let  $A$  be a Noetherian **FI**-algebra and  $I$  be an ideal of  $A$ . Then the quotient **FI**-algebra  $A/I$  defined by  $(A/I)_S = A_S/I_S$  for all  $S \in \mathbf{FI}$  is Noetherian.*

*Proof.* Any ascending chain of ideals in the quotient  $A/I$  lifts up to an ascending chain of ideals in  $A$ . The latter chain stabilizes so is the former.  $\square$

The above lemma shows that closed **FI**<sup>op</sup>-subschemes of a topological Noetherian **FI**<sup>op</sup>-scheme are topologically Noetherian.

We record the following properties of topologically Noetherian **FI**<sup>op</sup>-schemes.

**Lemma 1.4.6.7.** [DK14] *Let  $X$ ,  $Y$  and  $Z$  be **FI**<sup>op</sup>-schemes over a ring  $K$ . Suppose, moreover, that  $X$  and  $Y$  are topologically Noetherian.*

- Any closed subset of a topologically Noetherian  $\mathbf{FI}^{\mathbf{op}}$ -scheme is topologically Noetherian.
- The disjoint union  $X \sqcup Y$  defined by  $(X \sqcup Y)_S = X_S \sqcup Y_S$  of  $X$  and  $Y$  is topologically Noetherian with respect to the disjoint union topology.
- Let  $\psi : X \rightarrow Z$  be a morphism of  $\mathbf{FI}^{\mathbf{op}}$ -schemes. Then the image  $\mathrm{Im}(\psi)$  is topologically Noetherian with respect to the topology induced from  $Z$ .

# Chapter 2

## Components of symmetric wide-matrix varieties

This chapter is based on the paper [DEF22] with Jan Draisma and Rob Eggermont.

### 2.1 Introduction

#### 2.1.1 Main result

Let  $K$  be a Noetherian ring (commutative with 1), let  $c \in \mathbb{Z}_{\geq 0}$ , and, for all  $n \in \mathbb{Z}_{\geq 0}$ , let  $I_n$  be an ideal in the polynomial ring  $A_n := K[x_{i,j} \mid i \in [c], j \in [n]]$  such that the following two conditions are satisfied:

1.  $I_n$  is preserved by the (left) action of the symmetric group  $\text{Sym}([n])$  on  $A_n$  via  $K$ -algebra automorphisms determined by  $\pi x_{i,j} = x_{i,\pi(j)}$ ; and
2.  $I_n \subseteq I_{n+1}$ .

Dually, let  $X_n$  be the prime spectrum of  $A_n/I_n$ , a closed subscheme of  $\text{Spec}(A_n)$ . Then the two conditions above express that

1.  $X_n$  is preserved by the induced action of  $\text{Sym}([n])$  on  $\text{Spec}(A_n)$ ; and
2. the projection  $\text{Spec}(A_{n+1}) \rightarrow \text{Spec}(A_n)$  dual to the inclusion  $A_n \rightarrow A_{n+1}$  maps  $X_{n+1}$  into  $X_n$ .

Such a sequence  $(X_n)_n$  of schemes of matrices is called a *width-one  $\mathbf{FI}^{\text{op}}$ -scheme* of finite type over  $K$  (see Section 1.4 for a more convenient, functorial definition) or, more informally, a *symmetric wide-matrix scheme*, where the adjective *wide* refers to the fact that  $c$  is constant and we are interested in the case where  $n \gg 0$ ; for brevity, we will usually drop the adjective *symmetric*.

Recall that a *quasipolynomial* is a function  $f : \mathbb{Z} \rightarrow \mathbb{R}$  of the form

$$f(n) = c_d(n)n^d + c_{d-1}(n)n^{d-1} + \cdots + c_0(n)$$

where each  $c_i : \mathbb{Z} \rightarrow \mathbb{R}$  is periodic with integral period. Equivalently,  $f$  is a quasipolynomial if and only if there exist an  $N$  and polynomials  $f_0, \dots, f_{N-1}$  such that  $f(n) = f_i(n)$  whenever  $n \equiv i$  modulo  $N$ .

**Theorem 2.1.1.1** (Main Theorem). *Let  $(X_n)_n$  be a width-one  $\mathbf{FI}^{\text{op}}$ -scheme of finite type over a Noetherian ring  $K$ . Then the action of  $\text{Sym}([n])$  on  $X_n$  induces an action of  $\text{Sym}([n])$  on the set  $\mathcal{C}(X_n)$  of irreducible components of  $X_n$ , and there exists a quasipolynomial  $f : \mathbb{Z} \rightarrow \mathbb{R}$  and a natural number  $n_0 \in \mathbb{Z}_{\geq 0}$  such that the number  $|\mathcal{C}(X_n)/\text{Sym}([n])|$  of  $\text{Sym}([n])$ -orbits on  $\mathcal{C}(X_n)$  equals  $f(n)$  for all  $n \geq n_0$ .*

## 2.1.2 Examples

We illustrate the Main Theorem with a number of examples. We recall that the irreducible components of  $X_n$  are in one-to-one correspondence with the inclusion-wise minimal prime ideals of  $A_n$  that contain  $I_n$ .

**Example 2.1.2.1.** Let  $K$  be a domain, take  $c = 1$ , write  $x_j$  instead of  $x_{1,j}$ , and let  $I_n$  be the ideal generated by all monomials  $x_i x_j x_k$  with  $i, j, k \in [n]$  distinct. Clearly, the sequence  $(I_n)_n$  satisfies the conditions (1) and (2) above. A prime ideal containing  $I_n$  contains at least one variable from each triple of distinct variables. Hence the minimal prime ideals containing  $I_n$  are the ideals  $I_S := (\{x_i \mid i \in S\})$  where  $S \subseteq [n]$  is a set of cardinality  $n - 2$ ; the corresponding subscheme is the coordinate plane corresponding to the coordinates not labeled by  $S$ . Hence  $X_n$  has  $\binom{n}{n-2} = \binom{n}{2}$  irreducible components, which form a single orbit under the symmetric group  $\text{Sym}([n])$ . The quasipolynomial from the Main Theorem is 1.  $\diamond$

**Example 2.1.2.2.** Set  $K := \mathbb{C}$ , let  $d \in \mathbb{Z}_{\geq 0}$ , take  $c = 1$ , and let  $I_n$  be the ideal generated by all polynomials  $x_i^d - 1$  with  $i \in [n]$ . The irreducible components of  $X_n$  are the points  $(\zeta_1, \dots, \zeta_n)$  where each  $\zeta_i$  is an  $d$ -th root of unity. Thus  $X_n$  has  $d^n$  irreducible components, and these form  $\binom{n+d-1}{d-1}$  orbits under the

group  $\text{Sym}([n])$ , each of which has a unique representative of the form

$$(1, \dots, 1, e^{2\pi i/d}, \dots, e^{2\pi i/d}, e^{2 \cdot 2\pi i/d}, \dots, e^{2 \cdot 2\pi i/d}, \dots, e^{(d-1) \cdot 2\pi i/d}, \dots, e^{(d-1) \cdot 2\pi i/d}),$$

where the numbers of occurrences of  $e^{j \cdot 2\pi i/d}$ ,  $j = 0, \dots, d-1$  are arbitrary nonnegative integers whose sum is  $n$ .  $\diamond$

**Example 2.1.2.3.** Set  $K := \mathbb{C}(t)$ , where  $t$  is a variable, let  $c = 1$ , and let  $I_n$  be the ideal generated by all polynomials of the form  $x_i^2 - t$  with  $i \in [n]$ . Then  $I_1$  is a prime ideal, but for  $n \geq 2$  and two distinct  $i, j \in [n]$  any prime ideal  $P$  containing  $I_n$  also contains

$$(x_i^2 - t) - (x_j^2 - t) = (x_i - x_j)(x_i + x_j)$$

and hence either  $x_i - x_j$  or  $x_i + x_j$ . Hence, modulo  $P$ , the variables  $x_1, \dots, x_n$  can be partitioned into two subsets: within each of these sets, all variables are equal modulo  $P$ , and they are minus the variables in the other set, again modulo  $P$ .

Conversely, if  $A, B \subseteq [n]$  are disjoint sets with  $A \cup B = [n]$ , then the ideal  $P_{A,B}$  generated by the polynomials  $x_i - x_j$  with  $(i, j) \in (A \times A) \cup (B \times B)$ , the polynomials  $x_i + x_j$  with  $(i, j) \in A \times B$ , and the polynomial  $x_1^2 - t$  is a minimal prime ideal over  $I_n$ . By the above, these are all minimal primes over  $I_n$ . Note that  $P_{A,B} = P_{C,D}$  if and only if  $\{A, B\} = \{C, D\}$ , so there is a bijection between unordered partitions of  $[n]$  into two parts (one of which may be empty).

The number of unordered partitions of  $[n]$  is  $2^{n-1}$ . The action of  $\text{Sym}([n])$  on minimal primes over  $I_n$  corresponds to the natural action of  $\text{Sym}([n])$  on unordered partitions of  $[n]$ . The number of  $\text{Sym}([n])$ -orbits on the latter is  $\lfloor n/2 \rfloor + 1$ —indeed,  $\{A, B\}, \{\tilde{A}, \tilde{B}\}$  are in the same  $\text{Sym}([n])$ -orbit if and only if  $\min\{|A|, |B|\} = \min\{|\tilde{A}|, |\tilde{B}|\}$ , and this number takes any of the values in  $\{0, \dots, \lfloor n/2 \rfloor\}$ . Here the quasipolynomial is the function  $f(n) = \lfloor n/2 \rfloor + 1$ , and it holds for all  $n \geq 0$ .  $\diamond$

**Example 2.1.2.4.** Set  $K := \mathbb{C}$ ,  $c := 1$ , let  $d \in \mathbb{Z}_{\geq 1}$ , and let  $I_n$  be the ideal generated by all differences  $x_i^d - x_j^d$  with  $i, j \in [n]$ . In this case, each prime ideal  $P$  containing  $I_n$  also contains, for each  $i \neq j$ , a polynomial of the form  $x_i - \zeta x_j$ , where  $\zeta$  is a  $d$ -th root of unity. Hence the variables can be partitioned into  $d$  sets, where modulo  $P$  the variables in each set are  $e^{2\pi i/d}$  times the variables in the previous set.

Conversely, let  $A = (A_0, \dots, A_{d-1})$  be an ordered partition of  $[n]$ : a sequence of disjoint, potentially empty subsets of  $[n]$  whose union is  $[n]$ . Then the ideal  $P_A$  generated by the polynomials  $x_j - e^{2b\pi i/d} x_l$  whenever  $l$  lies in some  $A_a$  and  $j$  lies in  $A_{a+b}$  (indices modulo  $d$ ) is a minimal prime over  $I_n$ , and by the above, all primes arise in this manner. Furthermore,  $P_A = P_B$  if and only if the sequence  $B$  arises from  $A$  by a cyclic permutation, i.e., by adding an element of  $\mathbb{Z}/d\mathbb{Z}$  to the indices.

Hence the irreducible components of the scheme defined by  $I_n$  correspond bijectively to orbits of ordered partitions of  $[n]$  into  $d$  parts under the action of  $\mathbb{Z}/d\mathbb{Z}$  by rotation of the parts. The number of  $\text{Sym}([n])$ -orbits on such components is therefore equal to the number of  $(\mathbb{Z}/d\mathbb{Z}) \times \text{Sym}([n])$ -orbits on ordered partitions. Modding out  $\text{Sym}([n])$  first, what remains is to count the  $\mathbb{Z}/d\mathbb{Z}$ -orbits on ordered *integer* partitions of  $n$  into  $d$  nonnegative parts. This is done using the orbit-counting lemma (due to Cauchy, Frobenius, and not Burnside) for  $\mathbb{Z}/d\mathbb{Z}$ : for  $e \in \mathbb{Z}/d\mathbb{Z}$  define

$$f(e) := \gcd(d, e) \in \{1, \dots, d\}.$$

Then rotation by  $e$  on such integer partitions has the same fixed points as rotation by  $f(e)$ . This number is 0 if  $n$  is not divisible by  $d/f(e)$ , and equal to  $\binom{n/(d/f(e)) + f(e) - 1}{f(e) - 1}$  otherwise: the first  $f(e)$  positions in the partition can be filled arbitrarily with nonnegative integers whose sum is  $n/(d/(f(e)))$ , and this determines the partition fixed under rotating over  $f(e)$ . Thus the number of  $\text{Sym}([n])$ -orbits on components equals

$$\frac{1}{d} \sum_{e \in \mathbb{Z}/d\mathbb{Z}: (d/f(e)) | n} \binom{n/(d/f(e)) + f(e) - 1}{f(e) - 1},$$

which is, indeed, a quasipolynomial in  $n$ . ◇

These examples illustrate different aspects of the proof of Theorem 2.1.1.1. First, the projection  $X_{n+1} \rightarrow X_n$  maps each irreducible component of  $X_{n+1}$  into some component of  $X_n$ , but not necessarily *onto* some such component: in Example 2.1.2.1, the coordinate planes involving the variable  $x_{n+1}$  are mapped onto coordinate *lines* rather than planes. However, “most” coordinate planes are mapped onto coordinate planes. We will capture these relations between components of the  $X_n$  as  $n$  varies by the so-called *component functor* (see Section 2.4), which is a contravariant functor from **FI** to the category **PF** of finite sets with partially defined maps. This functor plays a fundamental role in the proof of Theorem 2.1.1.1, and also yields a more detailed picture of the components of the  $X_n$  for varying  $n$ .

Second, Example 2.1.2.2 illustrates that, while the number of components of  $X_n$  can grow exponentially with  $n$ , the number of orbits is upper-bounded by a polynomial.

Third, in Example 2.1.2.3 we see that, if we adjoin  $\sqrt{t}$  to the ground field  $K$ , then the example reduces to a variation on Example 2.1.2.2: there are  $2^n$  components and  $n+1$  orbits on components. This suggests that the quasipolynomiality in the Main Theorem is due to the action of a Galois group. We will see that this is, indeed, the case when the wide-matrix scheme is of *product type*; see §2.5.4.

Finally, Example 2.1.2.4 shows that even when  $K$  is an algebraically closed field, quasipolynomiality (rather than polynomiality) occurs. In part, this is because we will have to work over larger base fields that are transcendental extensions of the ground field  $K$ ; and in part, it is because Galois groups are not the sole reason for quasipolynomiality: in §2.5.7, we will replace the orbit-counting via Galois groups to orbit-counting via certain groupoids.

### 2.1.3 Applications

In this subsection, we give two interesting applications of our techniques.

**Corollary 2.1.3.1.** *Let  $K$  be a field, let  $S \subseteq K$  be a finite subset, and let  $k$  be a natural number. For every  $n \in \mathbb{Z}_{\geq 0}$ , define*

$$M_n := \{A \in S^{n \times n} \subseteq K^{n \times n} \mid \text{rk}(A) = k\},$$

*the set of all rank- $k$  matrices all of whose entries are in  $S$ . Let  $\text{Sym}([n])$  act by simultaneous row and column permutations on  $M_n$ . Then  $|M_n/\text{Sym}([n])|$  is a quasipolynomial in  $n$  for  $n \gg 0$ .*

*Proof.* Consider the morphism  $\varphi_n : \mathbb{A}_K^{k \times n} \times \mathbb{A}_K^{k \times n} \rightarrow \mathbb{A}_K^{n \times n}$  given by  $(A, B) \mapsto A^T \cdot B$ . For each  $C \in M_n$ , the closed subscheme  $\varphi_n^{-1}(C)$  is irreducible—indeed, for any field extension  $L$  of  $K$ , its  $L$ -points form an orbit under the action of the group  $\text{GL}_k(L)$  acting via  $g(A, B) = ((g^{-1})^T A, gB)$ , and irreducibility follows from irreducibility of the group scheme  $\text{GL}_k$ . Hence  $(X_n)_n$  defined by  $X_n := \varphi_n^{-1}(M_n)$  is a wide-matrix scheme with  $c = 2k$  whose irreducible components are in  $\text{Sym}(n)$ -equivariant one-to-one correspondence with the points of  $M_n$ . The corollary follows from the Main Theorem to  $(X_n)_n$ .  $\square$

**Remark 2.1.3.2.** The same argument works for *symmetric* rank- $k$  matrices and for *skew-symmetric* rank- $k$  matrices. More generally, by a similar argument: if  $Y_n \subseteq \mathbb{A}_K^{n \times n}$  is a closed subscheme of the variety of rank- $\leq k$  matrices

such that  $Y_n$  is preserved under the action of  $\text{Sym}([n])$  by conjugation and such that forgetting the last row and column maps  $Y_{n+1}$  to  $Y_n$ , then, too, the number of orbits of  $\text{Sym}([n])$  on irreducible components of  $Y_n$  is a quasipolynomial in  $n$  for  $n \gg 0$ .  $\diamond$

**Example 2.1.3.3.** Consider  $K = \mathbb{Q}$  and let  $M_{k,n}$  be the set of *symmetric*  $n \times n$ -matrices with entries in  $\{0, 1\}$  of rank precisely  $k$ . Then:

- $M_{0,n}$  consists of the zero matrix only, so there is a single  $S_n$ -orbit.
- Each  $S_n$ -orbit in  $M_{1,n}$  has a unique representative of the form

$$\begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix}$$

where  $J$  is an  $m \times m$ -matrix ( $1 \leq m \leq n$ ) with all ones, and the zeros are block matrices of appropriate sizes. Hence there are  $n$  orbits.

- There are three types of  $S_n$ -orbits in  $M_{2,n}$ , with representatives

$$\begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & J & 0 \\ J^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} J_1 & J_2 & 0 \\ J_2^T & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where, in the first case,  $J_1, J_2$  are all-one matrices of formats  $n_1 \times n_1, n_2 \times n_2$  with  $1 \leq n_1 \leq n_2$  and  $n_1 + n_2 \leq n$ ; in the second case,  $J$  is an all-one  $n_1 \times n_2$ -matrix with  $1 \leq n_1 \leq n_2$  and  $n_1 + n_2 \leq n$ ; and in the third case,  $J_1$  is an all-one  $n_1 \times n_1$ -matrix and  $J_2$  is an all-one  $n_1 \times n_2$ -matrix with  $1 \leq n_1, n_2$  and  $n_1 + n_2 \leq n$ .

In the last case, the number of pairs  $(n_1, n_2)$  is  $\binom{n}{2}$ . In the first and second cases, we have to count pairs  $(n_1, n_2)$  with  $1 \leq n_1 \leq n_2$  and  $n_1 + n_2 \leq n$ . This number equals

$$\sum_{n_1=1}^{\lfloor n/2 \rfloor} (n - 2n_1 + 1) = \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil.$$

Summarising, the number of  $S_n$ -orbits on  $M_{2,n}$  equals  $2 \cdot \lfloor n/2 \rfloor \cdot \lceil n/2 \rceil + \binom{n}{2}$ , clearly a quasipolynomial in  $n$ .  $\diamond$

For the next application, recall that a *linear code* of length  $n$  and dimension  $m$  over  $\mathbb{F}_q$  is a linear subspace of  $\mathbb{F}_q^n$  of dimension  $m$ . *Puncturing* such a code means deleting a coordinate; we only allow this when the dimension of the code does not drop. There is a natural notion of isomorphism of linear codes, involving permuting and scaling coordinates as well as applying an automorphism of  $\mathbb{F}_q$ ; see Example 2.5.6.5.

**Theorem 2.1.3.4.** *Fix a finite field  $\mathbb{F}_q$  and a natural number  $m$ . Let  $\mathcal{C}$  be a family of isomorphism classes of  $m$ -dimensional codes, of varying lengths, that are preserved under puncturing. Then for  $n \gg 0$  the number of length- $n$  elements in  $\mathcal{C}$  is a quasipolynomial in  $n$ .*

This theorem is not a direct consequence of our Main Theorem, but rather than one of the tools that we develop for the Main Theorem, see §2.5.6 and §2.5.7.

## 2.1.4 Organisation of this chapter

This chapter is organized as follows. In section 2, we introduce certain classes of width-one  $\mathbf{FI}^{\mathbf{OP}}$ -schemes of finite type. Such as nice  $\mathbf{FI}^{\mathbf{OP}}$ -schemes and reduced  $\mathbf{FI}^{\mathbf{OP}}$ -schemes. These schemes are very useful for counting irreducible components of the scheme  $X_n$  for large values of  $n$ . We also discuss the shifting of  $\mathbf{FI}^{\mathbf{OP}}$ -schemes which is a key ingredient for the Shift Theorem 2.3.1.1.

In Section 3, we prove that any width-one  $\mathbf{FI}^{\mathbf{OP}}$ -scheme of finite type is of *product type* after a suitable shift and a suitable localisation (see the Shift Theorem 2.3.1.1 and Proposition 2.3.3.1). The shifting technique is also used by Draisma in his work on polynomial functors [Dra19], and the Shift Theorem is reminiscent of and was inspired by, the Shift Theorem in [BDES22]. It is the strongest new structural result that we prove about wide-matrix schemes.

In Section 4, for an  $\mathbf{FI}^{\mathbf{OP}}$ -scheme  $X$ , we define the component functor  $\mathcal{C}_X$  from  $\mathbf{FI}$  to the category  $\mathbf{PF}$  of finite sets with partially defined maps. The component functor is one of the most important notions of this chapter, and in the remainder of the chapter, we obtain an almost complete combinatorial description of  $\mathcal{C}_X$  in the case where  $X$  is a width-one  $\mathbf{FI}^{\mathbf{OP}}$ -scheme.

Section 5 is devoted to the proof of the Main Theorem. In three steps, we construct more and more refined combinatorial models for  $\mathcal{C}_X$ . The first ones, called *elementary model functors*, allow us to prove the Main Theorem when  $X$  is of product type (see §2.5.4). By the Shift Theorem this situation is always attained by a shift and a localisation, and to undo the simplifications caused by that shift and localisation, we need the two more complicated combinatorial models dubbed *model functors* (see §2.5.6) and *pre-component functors* (see

§2.5.8). A major generalization in the step from elementary model functors to model functors is that we pass from counting orbits under a finite group—in the application to  $\mathcal{C}_X$ , this is the image of a Galois group—which is part of the defining data of an elementary model functor, to counting orbits under a finite groupoid which emerges by itself from the defining data of a model functor. The proof of Theorem 2.5.7.1 that model functors have a quasipolynomial count is entirely elementary, but very subtle. A direct application is Theorem 2.1.3.4.

In comparison, the step from model functors to pre-component functors is conceptually small. In §2.5.9 we prove that pre-component functors always have a quasipolynomial count, and in §2.5.10 we establish that the component functor of a wide-matrix scheme satisfies the properties Compatibilities (1)–(3) of a pre-component functor. This, then, completes the proof of the Main Theorem.

## 2.2 Width-one $\mathbf{FI}^{\text{op}}$ -schemes

Recall from Theorem 1.4.6.3 that any width-one  $\mathbf{FI}^{\text{op}}$ -scheme of finite type over a Noetherian ring  $K$  is topologically Noetherian, i.e., it satisfies the descending chain condition on closed  $\mathbf{FI}^{\text{op}}$ -subschemes.

### 2.2.1 Nice width-one $\mathbf{FI}^{\text{op}}$ -schemes

The following consequence of Noetherianity will be useful to us: it implies that when we are interested in the tail of a width-one  $\mathbf{FI}^{\text{op}}$ -scheme  $X$  of finite type over a Noetherian ring  $K$ , i.e., in  $X([n])$  for  $n \gg 0$ , then we may without loss of generality assume that the map  $X([n+1]) \rightarrow X([n])$  dual to the inclusion  $[n] \rightarrow [n+1]$  is dominant for all  $n$ .

**Proposition 2.2.1.1.** *Let  $K$  be a Noetherian ring and let  $X = \text{Spec}(B)$  be a width-one scheme of finite type over  $K$ . Then there exists an  $n_0 \in \mathbb{Z}_{\geq 0}$  such that for all  $S, T \in \mathbf{FI}$  with  $n_0 \leq |S| \leq |T|$  and all  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$ , the homomorphism  $B(\pi) : B(S) \rightarrow B(T)$  is injective. Define*

$$B'(S) := \begin{cases} B(S) & \text{if } |S| \geq n_0, \text{ and} \\ B(S)/\ker(B(\sigma)) & \text{if } |S| \leq n_0 \end{cases}$$

where  $\sigma$  is any chosen element of  $\text{Hom}_{\mathbf{FI}}(S, [n_0])$  (the result doesn't depend on  $\sigma$ ). For any  $S, T \in \mathbf{FI}$  and  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  the  $K$ -algebra homomorphism  $B(\pi) : B(S) \rightarrow B(T)$  induces a well-defined  $K$ -algebra homomorphism  $B'(\pi) : B'(S) \rightarrow B'(T)$ , and thus  $B'$  becomes an  $\mathbf{FI}$ -algebra over  $K$ , finitely generated

in width  $\leq 1$ , with the property that  $B'(\pi)$  is injective for all  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$ . Set  $X' := \text{Spec}(B')$ ; then  $X'(\pi) : X'(T) \rightarrow X'(S)$  is dominant for all  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$ .

*Proof.* First we show that for all  $S, T \in \mathbf{FI}$  we have

$$\ker B(\pi) = \ker B(\sigma) \text{ for all } \pi, \sigma \in \text{Hom}_{\mathbf{FI}}(S, T).$$

For any two injections  $\pi, \sigma : S \rightarrow T$  there exists a permutation  $\tau$  of  $T$  such that  $\pi = \tau \circ \sigma$ . For  $f \in \ker B(\sigma)$  we have  $B(\pi)(f) = B(\tau \circ \sigma)(f) = B(\tau)(B(\sigma)(f)) = B(\tau)(0) = 0$ . This implies that  $\ker B(\sigma) \subset \ker B(\pi)$ . By symmetry, also the reverse inclusion holds. In particular, this shows that  $\ker B(\sigma)$  is independent of the choice of  $\sigma$  and it is stable under the action of the group  $\text{Sym}(S)$ .

Now suppose that the first claim of the proposition is not true, that is, there does not exist such an  $n_0$ . Then there exists a strictly increasing sequence of positive integers  $(m_i)_i$  and injections  $\pi_i : [m_i] \rightarrow [m_i + 1]$  such that for all  $i$ ,  $\ker B(\pi_i)$  is not trivial. Let  $I_i$  be the  $\mathbf{FI}$ -ideal in  $B$  generated by  $\bigcup_{j=1}^i \ker B(\pi_j)$ ; by the first paragraph,  $I_i(T) = \{0\}$  for all  $T$  with  $|T| > m_i$ . Hence the sequence  $(I_i)_i$  is a strictly increasing chain of  $\mathbf{FI}$ -ideals of  $B$ ; this is a contradiction to the fact that  $B$  is Noetherian.

Let  $S, T \in \mathbf{FI}$  and let  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$ . If  $|S| \geq n_0$ , then  $B'(S) = B(S)$  and it is immediate that  $B(\pi) : B(S) \rightarrow B(T)$  induces a  $K$ -algebra homomorphism  $B'(S) \rightarrow B'(T)$ . Otherwise, let  $\sigma : S \rightarrow [n_0]$  be an injection, so that  $B'(S) = B(S)/\ker B(\sigma)$ . If  $|T| \geq n_0$ , then  $\pi$  factors via  $\sigma$  and it follows that

$$\ker B(\pi) \supseteq \ker B(\sigma);$$

again,  $B(\pi)$  induces a map  $B'(S) \rightarrow B'(T) = B(T)$ . Finally, if also  $|T| \leq n_0$ , then let  $\iota : T \rightarrow [n_0]$  be an injection. Replace  $\sigma$  by  $\iota \circ \pi$ , another injection  $S \rightarrow [n_0]$ . Then  $\ker B(\sigma) = \ker(B(\iota) \circ B(\pi))$  by the first paragraph, and hence  $B(\pi)$  maps  $\ker B(\sigma)$  into  $\ker B(\iota)$ , so that, once more, it induces a map  $B'(S) \rightarrow B'(T)$ .

The check that  $B'$  is an  $\mathbf{FI}$ -algebra over  $K$  finitely generated in width  $\leq 1$  is straightforward, and the check that each  $B'(\pi)$  is injective follows from a similar analysis to that in the previous paragraph. The final statement is standard: injective  $K$ -algebra homomorphisms yield dominant morphisms.  $\square$

**Definition 2.2.1.2.** We call an  $\mathbf{FI}$ -algebra  $B$  over a ring  $K$  *nice* if for all  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  the map  $B(\pi) : B(S) \rightarrow B(T)$  is injective; also its spectrum is then called nice. Proposition 2.2.1.1 says that if  $K$  is Noetherian, then any

width-one **FI**<sup>op</sup>-scheme of finite type over  $K$  agrees with a nice scheme for sufficiently large  $S$ .  $\diamond$

**Lemma 2.2.1.3.** *Let  $B$  be a nice **FI**-algebra over  $K$  and let  $h \in K$ . Then  $B[1/h]$  is a nice **FI**-algebra over  $K[1/h]$ .*

*Proof.* For each  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$ ,  $B[1/h](\pi)$  is the  $K[1/h]$ -algebra homomorphism  $B(S)[1/h] \rightarrow B(T)[1/h]$  obtained by the  $K$ -algebra homomorphism  $B(S) \rightarrow B(T)$  by localisation. By assumption, the latter is injective. Hence, since localisation is an exact functor from  $K$ -modules to  $K[1/h]$ -modules, so is the former.  $\square$

## 2.2.2 Reduced **FI**<sup>op</sup>-schemes

**Definition 2.2.2.1.** The **FI**-algebra  $B$  over  $K$  is called *reduced* if  $B(S)$  has no nonzero nilpotent elements for any  $S \in \mathbf{FI}$ . Then also  $X = \text{Spec}(B)$  is called reduced.  $\diamond$

The following lemma is immediate.

**Lemma 2.2.2.2.** *Let  $B$  be an **FI**-algebra over  $K$  and for each  $S \in \mathbf{FI}$  let  $B^{\text{red}}(S)$  be the quotient of  $B(S)$  by the ideal of nilpotent elements. Then for  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  the homomorphism  $B(\pi)$  induces a homomorphism  $B^{\text{red}}(S) \rightarrow B^{\text{red}}(T)$ , and this makes  $B^{\text{red}}$  into a reduced **FI**-algebra over  $K$ . Furthermore, if  $B$  is finitely generated in width  $\leq 1$ , then so is  $B^{\text{red}}$ .  $\square$*

It follows that to prove our Main Theorem, we may always assume that  $X$  is reduced.

## 2.2.3 Shifting

The idea of shifting an **FI**-structure over a finite set goes back to [CEF15]. Draisma also used it in his work on topological Noetherianity of polynomial functors [Dra19], except that there, one shifts over a vector space.

**Definition 2.2.3.1.** Let  $S_0$  be a finite set. Then  $\text{Sh}_{S_0} : \mathbf{FI} \rightarrow \mathbf{FI}$  is the functor that sends  $S$  to the disjoint union  $S_0 \sqcup S$  and  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  to  $\text{Sh}_{S_0} \pi : S_0 \sqcup S \rightarrow S_0 \sqcup T$  that is the identity on  $S_0$  and equal to  $\pi$  on  $S$ . For an **FI**-algebra  $B$  over  $K$  we write

$$\text{Sh}_{S_0} B := B \circ \text{Sh}_{S_0}$$

and for the  $\mathbf{FI}^{\mathbf{OP}}$ -scheme  $X = \mathrm{Spec}(B)$  over  $K$  we write

$$\mathrm{Sh}_{S_0} X := X \circ \mathrm{Sh}_{S_0} = \mathrm{Spec}(\mathrm{Sh}_{S_0} B).$$

Furthermore, for a homomorphism  $\varphi : B \rightarrow R$  of  $\mathbf{FI}$ -algebras over  $K$ , we write  $\mathrm{Sh}_{S_0} \varphi$  for the morphism  $\mathrm{Sh}_{S_0} B \rightarrow \mathrm{Sh}_{S_0} R$  that sends  $S$  to  $\varphi(S_0 \sqcup S)$ , and similarly for morphisms of  $\mathbf{FI}^{\mathbf{OP}}$ -schemes. A straightforward check shows that  $\mathrm{Sh}_{S_0}$  is a covariant functor from  $\mathbf{FI}$ -algebras over  $K$  into itself and from  $\mathbf{FI}^{\mathbf{OP}}$ -schemes over  $K$  into itself.  $\diamond$

If  $B$  is finitely generated in width  $\leq d$ , then so is  $\mathrm{Sh}_{S_0} B$ ; and hence, if  $X$  is a width- $d$   $\mathbf{FI}^{\mathbf{OP}}$ -scheme of finite type over  $K$ , then so is  $\mathrm{Sh}_{S_0} X$ .

For future use, we note that  $\mathrm{Sh}_{S_0}(\mathrm{Sh}_{S_1} B)$  is canonically isomorphic to  $\mathrm{Sh}_{S_0 \sqcup S_1} B$ , and similarly for  $\mathbf{FI}^{\mathbf{OP}}$ -schemes. Furthermore, shifting preserves reducedness and niceness.

## Base change

**Definition 2.2.3.2.** If  $A$  is an  $\mathbf{FI}$ -algebra over a ring  $K$ , and  $L$  is a  $K$ -algebra, then we obtain an  $\mathbf{FI}$ -algebra  $A_L$  over  $L$  by setting  $S \mapsto L \otimes_K A_S$ . In the special case where  $L$  is the localisation  $K[1/h]$  for some  $h \in K$ , we also write  $A[1/h]$  for  $A_L$ .

Dually, if  $X = \mathrm{Spec}(A)$  is the associated  $\mathbf{FI}^{\mathbf{OP}}$ -scheme, then we write  $X_L = \mathrm{Spec}(A_L)$  for the base change, and  $X[1/h]$  if  $L = K[1/h]$ .  $\diamond$

**Remark 2.2.3.3.** If  $B' := \mathrm{Sh}_{S_0} B$ , then  $B'$  is naturally an  $\mathbf{FI}$ -algebra over  $B'_0 = B(S_0)$  (see Remark 1.4.2.8). Thus shifting naturally leads to a change of base ring—informally, by shifting we “move some functions into the constants”. For an  $f \in B(S_0)$ , its image in  $B(S_0 \sqcup S)$  under  $B(\iota)$ , where  $\iota$  is the natural injection  $S_0 \rightarrow S_0 \sqcup S$ , will also be denoted simply by  $f$ . This is a slight abuse of notation, especially as  $B(\iota)$  needs not to be an injection if  $B$  is not nice, but this will not lead to confusion.

In the interpretation from Remark 1.4.2.10 of  $\mathbf{FI}$ -algebras consisting of algebras acted upon by  $\mathrm{Sym}(n)$  with suitable maps between them, one may model shifting by restricting the action to the subgroup of  $\mathrm{Sym}(n)$  that fixes the numbers 1 up to  $n_0 := |S_0|$ . We will, however, not explicitly use this model.  $\diamond$

## 2.3 The Shift Theorem

### 2.3.1 Formulation of the Shift Theorem

Recall from Lemma 1.4.4.11 that a width-one  $\mathbf{FI}^{\text{op}}$ -scheme  $X = \text{Spec}(B)$  of finite type over a ring  $K$  is a closed  $\mathbf{FI}^{\text{op}}$ -subscheme of  $S \mapsto Z^S$ , where  $Z = X([1])$  and where the product is over  $B_0$ . In this section, we establish the fundamental result that in fact, after a suitable shift and localisation,  $X$  becomes *equal* to such a product.

**Theorem 2.3.1.1** (Shift Theorem). *Let  $B$  be a reduced and nice  $\mathbf{FI}$ -algebra that is finitely generated in width  $\leq 1$  over a ring  $K$ , assume that  $1 \neq 0$  in  $B_0$ , and set  $X := \text{Spec}(B)$ . Then there exists a finite set  $S_0 \in \mathbf{FI}$  and a nonzero element  $h \in B(S_0)$  such that  $X' := (\text{Sh}_{S_0} X)[1/h]$  is isomorphic to  $S \mapsto Z^S$ , where  $Z = X'([1])$  and where the product is over  $B(S_0)[1/h]$ .*

### 2.3.2 Shift-and-localise

Before proving the Shift Theorem, we establish that shifting and localisation commute in a suitable sense.

**Lemma 2.3.2.1.** *Let  $B$  be a reduced  $\mathbf{FI}$ -algebra over  $K$ ,  $S_0$  and  $S_1$  be finite sets,  $h_0 \in B(S_0)$  nonzero,  $B' := (\text{Sh}_{S_0} B)[1/h_0]$ ,  $h_1 \in B'(S_1)$  nonzero, and  $B'' := (\text{Sh}_{S_1} B')[1/h_1]$ . Then there exists a nonzero  $h \in B(S_0 \sqcup S_1)$  such that  $(\text{Sh}_{S_0 \sqcup S_1} B)[1/h] \cong B''$  as  $\mathbf{FI}$ -algebras over  $K$ .*

*Proof.* By multiplying  $h_1$  with a suitable power of the image of  $h_0$  in  $B'(S_1)$ , we achieve that  $h_1$  lies in the image of  $B(S_0 \sqcup S_1)$  in  $B(S_0 \sqcup S_1)[1/h_0] = B'(S_1)$ . Let  $\tilde{h}_1$  be an element of  $B(S_0 \sqcup S_1)$  mapping to  $h_1$ . Then, by a straightforward computation,

$$h := h_0 \tilde{h}_1 \in B(S_0 \sqcup S_1)$$

does the trick. □

### 2.3.3 Proof of the Shift Theorem

*Proof.* By Lemma 1.4.4.11,  $X$  is (isomorphic to) a closed  $\mathbf{FI}^{\text{op}}$ -subscheme of  $\text{Mat}_{c, B_0}$  for some  $c$ . Let  $R : S \rightarrow B_0[x_{ij} \mid i \in [c], j \in S]$  be the coordinate ring of the latter wide-matrix space, and let  $I$  be the ideal of  $X$  in  $R$ .

Fix any monomial order on  $\mathbb{Z}_{\geq 0}^c$ . We will use this order to compare monomials in the variables  $x_{1j}, \dots, x_{cj}$  for any  $j$ .

Elements of  $I([1])$  are  $B_0$ -linear combinations of monomials  $x_{1,1}^{\alpha_1} \cdots x_{c,1}^{\alpha_c}$  with  $\alpha \in \mathbb{Z}_{\geq 0}^c$ . Let  $M \subseteq \mathbb{Z}_{\geq 0}^c$  be the set of (exponent vectors of) leading monomials of *monic* elements of  $I([1])$ . By Dickson's lemma, there exist finitely many monic elements  $f_1, \dots, f_k \in I([1])$  whose leading monomials  $u_1, \dots, u_k$  generate  $M$ .

Now there are two possibilities. Either for every  $n \in \mathbb{Z}_{\geq 1}$  and every nonzero  $f \in I([n]) \subseteq R([n])$ , some monomial in  $f$  is divisible by  $R(\pi)u_i$  for some  $i \in [k]$  and some  $\pi \in \text{Hom}_{\mathbf{FI}}([1], [n])$ —or not. In the former case, using the  $f_i$  are monic, we can do division with the remainder by the  $R(\pi)f_i$  until the remainder is zero, and it follows that  $f_1, \dots, f_k \in I([1])$  generate the  $\mathbf{FI}$ -ideal  $I$ . Then  $X$  itself is a product as desired—indeed, by Lemma 1.4.4.11,  $X$  is a closed  $\mathbf{FI}^{\text{op}}$ -subscheme of the  $\mathbf{FI}^{\text{op}}$ -scheme  $S \mapsto X([1])^S$ , and the fact that the  $\mathbf{FI}$ -ideal of  $X$  is generated by  $I([1])$  implies that the corresponding closed embedding is an isomorphism. Hence in this case we can take  $S_0 := \emptyset$  and  $h := 1 \neq 0 \in B_0$ .

In the latter case, let  $n_0$  be minimal such that there exists a nonzero  $f \in I([n_0])$  none of whose terms are divisible by any  $R(\pi)u_i$ . Regard  $f$  as a polynomial in  $x_{1,n_0}, \dots, x_{c,n_0}$  with coefficients in  $R([n_0 - 1])$ , let  $u = x_{1,n_0}^{\alpha_1} \cdots x_{c,n_0}^{\alpha_c}$  be the leading monomial of  $f$ , and let  $\tilde{h} \in R([n_0 - 1])$  be the coefficient of  $u$  in  $f$ . Now  $\tilde{h} \notin I([n_0 - 1])$  by minimality of  $n_0$  and the fact that no term in  $\tilde{h}$  is divisible by any  $R(\pi)u_i$  with  $i \in [k]$  and  $\pi \in \text{Hom}_{\mathbf{FI}}([1], [n - 1])$ —indeed, such a term, multiplied with  $u$ , would yield a term in  $f$  with the same property. Set  $S_0 := [n_0 - 1]$  and let  $h$  be the image of  $\tilde{h}$  in  $B(S_0)$ ; this is nonzero by construction.

Now set  $B' := (\text{Sh}_{S_0} B)[1/h]$  and  $X' := \text{Spec}(B')$ , and note that  $1 \neq 0$  in  $B'_0$ . Then  $X'$  is a closed  $\mathbf{FI}^{\text{op}}$ -subscheme of  $\text{Mat}_{c, B'_0}$ , and we claim that if we construct  $M' \subseteq \mathbb{Z}_{\geq 0}^c$  for  $X'$  in the same manner as we constructed  $M$  for  $X$ , then  $M' \supsetneq M$ . Indeed, if  $\iota : [1] \rightarrow S_0 \sqcup [1]$  is the natural inclusion, then  $R(\iota)$  maps  $f_i$  to an element in the ideal of  $X(S_0 \sqcup [1])$  with the same leading monomial  $u_i$ , and this maps to an element of the ideal of  $X'([1])$  with that same leading monomial. This shows that  $M' \supseteq M$ . Furthermore, via the bijection  $\tau : [n_0] \rightarrow S_0 \sqcup [1]$  that is the identity on  $S_0 = [n_0 - 1]$  and sends  $n_0$  to 1 we obtain another element  $R(\tau)f$  in the ideal of  $X(S_0 \sqcup [1])$ , whose image in the ideal of  $X'([1])$  has an invertible leading coefficient (namely,  $h$ ) and leading monomial  $x_{1,1}^{\alpha_1} \cdots x_{c,1}^{\alpha_c}$ . We thus find that  $\alpha \in M'$ , while  $\alpha \notin M$  by construction.

The fact that  $B$  is nice and reduced implies that so is  $B'$ . Hence we can continue in the same manner with  $B'$ . By Dickson's lemma, the set  $M$  can strictly increase only finitely many times. Hence after finitely many shift-and-localise steps, we reach the former case, where we know that  $X$  is a product.

Finally, we invoke Lemma 2.3.2.1 to conclude that this finite sequence of shift-and-localise steps can be turned into a single shift followed by a single localisation inverting a nonzero element.  $\square$

We will use the following strengthening of the Shift Theorem in the case where  $K$  is Noetherian.

**Proposition 2.3.3.1.** *In the setting of the Shift Theorem, if we further assume that  $K$  is Noetherian, then there exists a nonzero  $h' \in B'_0$  such that  $B'' := B'[1/h']$  and  $X'' := \text{Spec}(B'')$  have the following properties:*

1. *like in the Shift Theorem,  $X''$  is isomorphic to  $S \mapsto V^S$  where  $V := X''([1])$  and where the product is over  $B''_0$ ;*
2.  *$B''_0$  is a domain; and*
3. *for each  $S \in \mathbf{FI}$ , every irreducible component of  $V^S$  maps dominantly into  $\text{Spec}(B''_0)$ .*

*Proof.* The  $\mathbf{FI}^{\text{op}}$ -scheme  $X' = \text{Spec}(B')$  from the Shift Theorem maps  $S$  to  $Z^S$ , where  $Z = X'([1])$  and where the product is over  $B'_0$ . By construction,  $B'$  is reduced, nice, and  $1 \neq 0$  in  $B'_0$ . Any localisation by a nonzero  $h' \in B'_0$  satisfies (1). We will now construct  $h'$  so as to satisfy (2) and (3).

As  $K$  is Noetherian and  $B'_0$  is a finitely generated  $K$ -algebra,  $B'_0$  is Noetherian. Hence  $\text{Spec}(B'_0)$  is the union of finitely many irreducible components; let  $C$  be one of them. Then there exists a nonzero  $h_1 \in B'_0$  that vanishes identically on all other irreducible components of  $\text{Spec}(B'_0)$ . Now  $B'_0[1/h_1]$  is a domain, namely, the coordinate ring of  $C[1/h_1]$ .

Furthermore,  $B'_1[1/h_1]$  is a finitely generated  $B'_0[1/h_1]$ -algebra and by generic freeness [Eis95, Theorem 14.4], there exists a nonzero  $h_2 \in B'_0[1/h_1]$  such that  $B'_1[1/h_1][1/h_2]$  is a free  $B'_0[1/h_1][1/h_2]$ -module. After multiplying with a power of (the image of)  $h_1$ , we may assume that  $h_2$  the image of some  $\tilde{h}_2 \in B_0$ . Then set  $h' := h_1 \tilde{h}_2$ .

Set  $B'' := B'[1/h']$  and  $X'' := \text{Spec}(B'') = X'[1/h']$ . Now  $B''_0$  is a localisation of the domain  $B'[1/h_1]$ , hence a domain, so (2) holds.

Furthermore, for every  $S \in \mathbf{FI}$ ,  $X''(S)$  is the product over  $B''_0 = B'[1/h']$  of  $|S|$  copies of  $V := X''([1])$ . Its coordinate ring  $B''(S)$  is then a tensor product over  $B''_0$  of  $|S|$  copies of the free  $B''_0$ -module  $B''_1$ , and hence  $B''(S)$  is itself a free  $B''_0$ -module. Furthermore, again since niceness is preserved, the map  $B''_0 \rightarrow B''(S)$  is injective. Then, by the going-down theorem for flat extensions [Eis95, Lemma 10.11], every minimal prime ideal of  $B''(S)$  intersects  $B''_0$  in the zero ideal, so that every irreducible component of  $X''(S)$  maps onto  $\text{Spec}(B''_0)$ , as desired.  $\square$

**Definition 2.3.3.2.** Let  $L$  be a Noetherian domain,  $Q \supseteq L$  a ring extension such that  $Q$  is a finitely generated  $L$ -algebra and free as an  $L$ -module. Set  $Z := \text{Spec}(Q)$ . Then the  $\mathbf{FI}^{\text{op}}$ -scheme over  $L$  defined by  $S \mapsto Z^S$ , where the product is over  $L$ , is said to be *of product type*. As we have seen above, each irreducible component of  $Z^S$  then maps dominantly into  $\text{Spec } L$ .  $\diamond$

In Section 2.5 we will establish our Main Theorem for  $\mathbf{FI}^{\text{op}}$ -schemes of product type and then relate the general case to the product case via the Shift Theorem.

**Example 2.3.3.3.** To illustrate the Shift Theorem and Proposition 2.3.3.1 we analyse [KR22, Example 3.20] in the case of curves. In our notation, let  $X_d(S)$  be the reduced, closed subvariety of  $\text{Mat}_{2,\mathbb{C}}(S)$  consisting of all  $S$ -tuples of points  $(x_i, y_i)$  for which there exists a nonzero degree- $\leq d$  polynomial  $p \in \mathbb{C}[x, y]$  with  $p(x_i, y_i) = 0$  for all  $i \in S$ . It is proved in [KR22] that  $X_d(S)$  is an irreducible variety for all  $d \geq 1$  and all  $S$ , so it is not particularly interesting from the perspective of counting components. However, it *is* interesting from the perspective of the Shift Theorem. Take  $n_0 := \dim \mathbb{C}[x, y]_{\leq d} - 1$ , so that through  $n_0$  points  $(x_i, y_i)$ ,  $i = 1, \dots, n_0$  in general position goes a unique plane curve  $C$  of degree  $\leq d$ . The coefficients of the corresponding polynomial  $p$  are rational functions of the  $(x_i, y_i)$  with  $i \in [n_0]$ . Take for  $h$  a common multiple of the denominators of these rational functions, so that  $C$  is a curve over the ring  $\mathbb{C}[x_1, y_1, \dots, x_{n_0}, y_{n_0}][1/h] =: B'_0$ . Then  $X' = (\text{Sh}_{[n_0]} X)[1/h]$  is the  $\mathbf{FI}^{\text{op}}$ -variety that maps  $S$  to  $C^S$ , where the product is over  $B'_0$ .  $\diamond$

## 2.4 The component functor

To establish the Main Theorem, we will analyse the functor that assigns to a finite set  $S$  the set of components of  $X(S)$ . This functor takes values in another category called  $\mathbf{PF}$ .

### 2.4.1 Contravariant functors $\mathbf{FI} \rightarrow \mathbf{PF}$

**Definition 2.4.1.1.** Let  $\mathbf{PF}$  be the category whose objects are finite sets and whose morphisms  $T \rightarrow S$  are partially defined maps from  $T$  to  $S$ , i.e., maps  $\pi$  into  $S$  whose domain  $\text{dom}(\pi)$  is a subset of  $T$ . If  $\pi : T \rightarrow S$  and  $\sigma : S \rightarrow U$  are morphisms in this category, then  $\sigma \circ \pi$  is defined precisely at those  $i \in T$  for which  $i \in \text{dom}(\pi)$  and  $\pi(i) \in \text{dom}(\sigma)$ ; and  $\sigma \circ \pi$  takes the value  $\sigma(\pi(i))$  there.  $\diamond$

We are interested in contravariant functors  $\mathcal{F} : \mathbf{FI} \rightarrow \mathbf{PF}$  and morphisms between these.

**Definition 2.4.1.2.** A *morphism* from a contravariant functor  $\mathcal{F} : \mathbf{FI} \rightarrow \mathbf{PF}$  to another such functor  $\mathcal{F}'$  is a collection of everywhere defined maps  $(\Psi(S) : \mathcal{F}(S) \rightarrow \mathcal{F}'(S))_{S \in \mathbf{FI}}$  such that for all  $S, T \in \mathbf{FI}$  and  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  the diagram

$$\begin{array}{ccc} \mathcal{F}(T) & \xrightarrow{\Psi(T)} & \mathcal{F}'(T) \\ \mathcal{F}(\pi) \downarrow & & \downarrow \mathcal{F}'(\pi) \\ \mathcal{F}(S) & \xrightarrow{\Psi(S)} & \mathcal{F}'(S) \end{array}$$

commutes in the following sense: if the leftmost map  $\mathcal{F}(\pi)$  is defined at some  $f \in \mathcal{F}(T)$ , then the rightmost map  $\mathcal{F}'(\pi)$  is defined at  $\Psi(T)(f)$ , and we have

$$\mathcal{F}'(\pi)(\Psi(T)(f)) = \Psi(S)(\mathcal{F}(\pi)(f)).$$

The morphism is called *injective/surjective* if each  $\Psi(S)$  is injective/surjective, and an *isomorphism* if each  $\Psi(S)$  is bijective and moreover  $\mathcal{F}'(\pi)$  is defined *precisely* at all  $f' \in \mathcal{F}'(T)$  such that  $\mathcal{F}(\pi)$  is defined at  $\Psi(T)^{-1}(f')$ .  $\diamond$

Note that our morphisms  $\mathcal{F} \rightarrow \mathcal{F}'$  are not precisely natural transformations, since we do not require that the diagram above commutes as a diagram of partially defined maps: we allow the partially defined map  $\mathcal{F}'(\pi) \circ \psi(T)$  to have a larger domain than  $\psi(S) \circ \mathcal{F}(\pi)$ .

## 2.4.2 The component functor of an $\mathbf{FI}^{\text{op}}$ -scheme

**Definition 2.4.2.1.** Let  $B$  be a finitely generated  $\mathbf{FI}$ -algebra over a Noetherian ring  $K$ , so that  $X = \text{Spec}(B)$  is an  $\mathbf{FI}^{\text{op}}$ -scheme of finite type over  $K$ . We define the contravariant functor  $\mathcal{C}_X : \mathbf{FI} \rightarrow \mathbf{PF}$  on objects by

$$\mathcal{C}_X(S) = \{\text{the irreducible components of } X(S)\}$$

and on morphisms  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  as follows:  $\mathcal{C}_X(\pi)$  is defined at some component  $C \in \mathcal{C}_X(T)$  if (and only if)  $X(\pi) : X(T) \rightarrow X(S)$  maps  $C$  dominantly into a component of  $X(S)$ . The functor  $\mathcal{C}_X$  is called the *component functor* of  $X$ .  $\diamond$

Note that the condition that  $K$  is Noetherian and  $B$  is finitely generated implies that, indeed,  $\mathcal{C}_X(S)$  is a finite set for each  $S$ .

**Example 2.4.2.2.** In Example 2.1.2.1,  $\mathcal{C}_X$  is isomorphic to the functor  $\mathbf{FI} \rightarrow \mathbf{PF}$  that assigns to the set  $S$  the set  $\binom{S}{2}$  of two-element subsets and to  $\pi : S \rightarrow T$  the partially defined map  $\binom{T}{2} \rightarrow \binom{S}{2}$  that sends  $\{i, j\}$  to  $\{\pi^{-1}(i), \pi^{-1}(j)\}$  whenever this is defined.  $\diamond$

In the definition of the component functor we have not assumed that  $B$  is generated in width  $\leq 1$ , and indeed larger  $\mathbf{FI}$ -algebras also yield interesting examples.

**Example 2.4.2.3.** Let  $K$  be a field and let  $R$  be the  $\mathbf{FI}$ -algebra that assigns to  $S$  the ring

$$R(S) = K[x_{i,j} \mid i, j \in S] / (\{x_{i,j} - x_{j,i} \mid i, j \in S\})$$

and to a morphism  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  the  $K$ -algebra homomorphism determined by  $x_{i,j} \mapsto x_{\pi(i), \pi(j)}$ . This  $\mathbf{FI}$ -algebra is generated in width 2 by the two elements  $x_{1,1}, x_{1,2} \in R([2])$ .

It is well known that this  $\mathbf{FI}$ -algebra is *not* Noetherian; the following example is closely related to [HS12, Example 3.8]. Let  $I_d(S)$  be the ideal generated by all *cycle monomials* of the form  $x_{i_1, i_2} x_{i_2, i_3} \cdots x_{i_k, i_1}$  where  $3 \leq k \leq d$  and  $i_1, \dots, i_k$  are distinct. Then  $I_3 \subsetneq I_4 \subsetneq \dots$  is an infinite strictly increasing chain of ideals in  $R$ . Let  $I_\infty$  be their union, and  $X = \text{Spec}(R/I_\infty)$ . A prime ideal  $P$  in  $R(S)$  containing  $I_\infty(S)$  contains at least one variable from every cycle of length at least 3, so the edges  $\{i, j\}$  corresponding to variables  $x_{i,j}$  with  $i \neq j$  that are *not* in  $P$  form a forest with vertex set  $S$ . Every forest is contained in a tree with vertex set  $S$ . Correspondingly, every such tree  $T$  gives rise to a minimal prime ideal containing  $I_\infty(S)$ , namely the ideal generated by all  $x_{i,j}$  with  $\{i, j\}$  not an edge in  $T$ .

It follows that the minimal prime ideals of  $R(S)/I_\infty(S)$  are in bijection to the trees with vertex set  $S$ . Recall that, by Cayley's formula, this number of trees is  $n^{n-2}$  when  $n := |S| \geq 2$ . In particular, the number of  $\text{Sym}([n])$ -orbits is at least  $(n^{n-2})/n!$  and hence superpolynomial in  $n$ ; this shows that in the Main Theorem, the width-one condition cannot be dropped.

Furthermore, given a  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$ ,  $\mathcal{C}_X(\pi)$  is defined on trees  $\Delta$  with vertex set  $T$  as follows. If the induced subgraph of  $\Delta$  on  $\pi(S)$  is connected (and hence a tree), then  $\mathcal{C}_X(\pi)(\Delta)$  is that tree but with the label  $j \in \pi(S)$  replaced by  $\pi^{-1}(j)$ . Otherwise,  $\mathcal{C}_X(\pi)$  is not defined at  $\Delta$ .  $\diamond$

### 2.4.3 The underlying species

In our proof of the Main Theorem, we will give a fairly complete picture of the component functor of width-one  $\mathbf{FI}^{\text{op}}$ -schemes, at least on sets  $S \in \mathbf{FI}$

with  $|S| \gg 0$ . The first observation is that for *any* contravariant functor  $\mathcal{F} : \mathbf{FI} \rightarrow \mathbf{PF}$  and any  $\pi \in \text{End}_{\mathbf{FI}}(S) = \text{Sym}(S)$ ,  $\mathcal{F}(\pi)$  is defined everywhere on  $\mathcal{F}(S)$ , and a bijection there. After all, by the properties of a contravariant functor  $\text{id}_{\mathcal{F}(S)} = \mathcal{F}(\pi \circ \pi^{-1}) = \mathcal{F}(\pi^{-1}) \circ \mathcal{F}(\pi)$ . It follows that the functor from the category of finite sets with bijections to itself that sends  $S$  to  $\mathcal{F}(S)$  and  $\pi$  to  $\mathcal{F}(\pi)^{-1}$  is a covariant functor and hence a *species* in the sense of [Joy81]; we call this the *underlying species* of the  $\mathcal{F}$ . For the Main Theorem, it would suffice to know the underlying species of the component functor  $\mathcal{C}_X$  of  $X$ . However, to understand this species, we will also need to have some information on the partially defined maps  $\mathcal{C}_X(\pi)$  where  $\pi : S \rightarrow T$  is *not* a bijection.

#### 2.4.4 A property in width one

The second observation on component functors concerns width-one  $\mathbf{FI}^{\text{op}}$ -schemes.

**Lemma 2.4.4.1.** *Suppose that  $X$  is a width-one  $\mathbf{FI}^{\text{op}}$ -scheme of finite type over a Noetherian ring  $K$ . Then there exists an  $n_0$  such that for all  $S, T \in \mathbf{FI}$  with  $n_0 \leq |S| \leq |T|$  and all  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$ , the partially defined map  $\mathcal{C}_X(\pi)$  is surjective.*

*Proof.* Take the  $n_0$  from Proposition 2.2.1.1, so that  $X(\pi) : X_T \rightarrow X_S$  is dominant for all  $S, T \in \mathbf{FI}$  with  $n_0 \leq |S| \leq |T|$ . Then for each component of  $X_S$  there must be some component of  $X_T$  mapping dominantly into it.  $\square$

### 2.5 Proof of the main theorem

In this section, which takes up the remainder of the chapter, we establish the Main Theorem. To do so, on the one hand we develop purely combinatorial tools (see §§2.5.2, 2.5.3, 2.5.5, 2.5.6, 2.5.7, 2.5.8, 2.5.9) and on the other hand we establish algebraic results relating the component functors of width-1  $\mathbf{FI}^{\text{op}}$ -schemes to those combinatorial tools (see §§2.5.1, 2.5.4, 2.5.10). Finally, all is combined in §2.5.11 to establish the Main Theorem. We would like to highlight §2.5.7, where from a so-called model functor we extract certain groupoids acting on unions of rational cones, after which we use an orbit-counting lemma for groupoids from §2.5.5 to establish quasipolynomiality in that crucial case.

### 2.5.1 The component functor of a wide-matrix space

Let  $L$  be a finitely generated  $K$ -algebra, where  $K$  is a Noetherian ring, and  $c \in \mathbb{Z}_{\geq 0}$ . For each finite set  $S \in \mathbf{FI}$  we have a natural morphism  $\mathrm{Mat}_{c,L}(S) \rightarrow \mathrm{Spec}(L)$  (corresponding to the natural embedding  $L \rightarrow L[\mathrm{Mat}_{c,L}(S)]$ ), and the preimages of the irreducible components of  $L$  are the irreducible components of  $\mathrm{Mat}_{c,L}(S)$ . This establishes the following.

**Lemma 2.5.1.1.** *Let  $k$  be the number of minimal prime ideals of  $L$ . The component functor of  $\mathrm{Mat}_{c,L}$  is isomorphic to the functor that assigns the set  $[k]$  to each  $S \in \mathbf{FI}$  and the identity on  $[k]$  to each  $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$ . In particular, the number of  $\mathrm{Sym}([n])$ -orbits on  $\mathcal{C}_{\mathrm{Mat}_{c,L}}([n])$  is equal to  $k$ .  $\square$*

### 2.5.2 Elementary model functors

We will construct a class of contravariant functors  $\mathbf{FI} \rightarrow \mathbf{PF}$  from which, as we will see, the component functor of a width-one  $\mathbf{FI}^{\mathrm{op}}$ -scheme of finite type over a Noetherian ring is built up in a suitable sense.

**Definition 2.5.2.1.** Let  $k \in \mathbb{Z}_{\geq 0}$ , let  $G$  be a subgroup of  $\mathrm{Sym}([k])$ , and, for  $S \in \mathbf{FI}$ , let  $\mathcal{E}(S)$  be a subset of  $[k]^S$  that is preserved under the diagonal action of  $G$  on  $[k]^S$ . Assume, furthermore, that for all  $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$  the map  $[k]^T \rightarrow [k]^S$ ,  $\alpha \mapsto \alpha \circ \pi$  maps  $\mathcal{E}(T)$  into  $\mathcal{E}(S)$ . Then the contravariant functor  $\mathbf{FI} \rightarrow \mathbf{PF}$  that sends  $S$  to  $\mathcal{E}(S)/G$  and  $\pi \in \mathrm{Hom}_{\mathbf{FI}}(S, T)$  to the (everywhere defined) map

$$\mathcal{E}(T)/G \rightarrow \mathcal{E}(S)/G, \quad G \cdot \alpha \mapsto G \cdot (\alpha \circ \pi)$$

is called an *elementary model functor*  $\mathbf{FI} \rightarrow \mathbf{PF}$ . Note that the latter map is well-defined as  $G$  acts diagonally.  $\diamond$

### 2.5.3 A first quasipolynomial count

**Proposition 2.5.3.1.** *Let  $k \in \mathbb{Z}_{\geq 0}$ ,  $G$  a subgroup of  $\mathrm{Sym}([k])$  and let  $M$  be a  $G$ -stable downward closed subset of  $\mathbb{Z}_{\geq 0}^k$ ; that is, for all  $\beta \in M$  and  $g \in G$  we have  $g\beta \in M$ , and for all  $\beta \in M$  and  $j \in [k]$  with  $\beta_j > 0$  we have  $\beta - e_j \in M$ , where  $e_j$  is the  $j$ -th standard basis vector in  $\mathbb{Z}^k$ . Then there exists a quasipolynomial  $f$  such that for  $n \gg 0$  the number of  $G$ -orbits on the set  $M_n$  of elements  $\beta \in M$  of total degree  $|\beta| := \sum_j \beta_j$  equal to  $n$  equals  $f(n)$ .*

*Proof.* By the orbit-counting lemma the number of orbits equals

$$\frac{1}{|G|} \sum_{g \in G} |M_n^g|$$

where  $M_n^g = \{\alpha \in M_n \mid g\alpha = \alpha\}$ . So it suffices to prove that each of the summands is a quasipolynomial for  $n \gg 0$ .

The set  $M$  has a so-called *Stanley decomposition* [Sta82]

$$M = \bigsqcup_{i=1}^d (\alpha_i + \mathbb{Z}_{\geq 0}^{I_i})$$

for suitable subsets  $I_i \subseteq [k]$ . Call the  $i$ -th term  $N(i)$ . Then, for each  $g \in G$ ,  $N(i)^g$  is the set of nonnegative integers points in a certain rational polyhedron, and its elements of degree  $n \gg 0$  are counted by a quasipolynomial by [Sta97, Theorem 4.5.11 and Proposition 4.4.1].  $\square$

An immediate consequence is the following.

**Corollary 2.5.3.2.** *Let  $S \mapsto \mathcal{E}(S)/G \subseteq [k]^S/G$  be an elementary model functor. Then  $|(\mathcal{E}([n])/G)/\text{Sym}([n])|$  equals some quasipolynomial in  $n$ , for all  $n \gg 0$ .*

*Proof.* Define a map  $\mathcal{E}([n]) \rightarrow \mathbb{Z}_{\geq 0}^k$  by sending the vector  $\alpha$  to its *count vector*  $\beta$ , i.e., the vector in which  $\beta_j$  is the number of  $l \in [n]$  with  $\alpha_l = j$ . Note that this map is  $G$ -equivariant, so the image  $M_n$  is  $G$ -stable; and that the fibres are precisely the  $\text{Sym}([n])$ -orbits. Furthermore, the fact that  $\mathcal{E}$  is a model functor implies that the union  $M = \bigcup_n M_n$  is downward closed. Now apply Proposition 2.5.3.1.  $\square$

## 2.5.4 FI<sup>op</sup>-schemes of product type

Elementary model functors are combinatorial models for the component functor of **FI<sup>op</sup>**-schemes of product type in the sense of Definition 2.3.3.2, as follows.

**Proposition 2.5.4.1.** *Let  $L$  be a Noetherian domain and let  $X$  be a width-one **FI<sup>op</sup>**-scheme of product type over  $L$ . Then the component functor  $\mathcal{C}_X$  is isomorphic to an elementary model functor.*

Before we prove this result, we show that it holds in Example 2.1.2.3.

**Example 2.5.4.2.** Let  $X(S)$  be the closed subscheme of  $\mathbb{A}_{\mathbb{C}(t)}^S$  defined by the equations  $x_i^2 - t$  for all  $i \in S$ . We claim that this is of product type. First,  $X(S) = Z^S$  where  $Z$  is the subscheme of  $\mathbb{A}_{\mathbb{C}(t)}^1$  defined by  $x^2 - t$ , and where the product is over  $M := \mathbb{C}(t)$ . Second, to determine the irreducible components of  $X(S)$ , we extend scalars to a separable closure  $\overline{M}$  of  $M$ , which in particular

contains a  $\sqrt{t}$ . Then  $X_{\overline{M}}(S)$  is just  $\{\pm\sqrt{t}\}^S$ , each point of which maps onto  $\text{Spec}(M)$ . Thus  $X$  is of product type as claimed. The irreducible components of  $X(S)$  are orbits of irreducible components of  $X_{\overline{M}}(S)$  under the Galois group, which acts diagonally on  $\{\pm\sqrt{t}\}^S$  by swapping  $\sqrt{t}$  and  $-\sqrt{t}$ . Thus  $\mathcal{C}_X$  is isomorphic to the elementary model functor that maps  $S$  to  $\{1, 2\}^S / \text{Sym}([2])$ .  $\diamond$

*Proof.* By assumption,  $X(S) = Z^S$ , where  $Z$  is a fixed affine scheme over  $L$ , and each irreducible component of  $X(S)$  maps dominantly into  $\text{Spec}(L)$ . Let  $M$  be the fraction field of  $L$ , and let  $X_M$  be the base change of  $X$  to  $M$ . Since each irreducible component of  $X(S)$  maps dominantly into  $\text{Spec}(L)$ , basic properties of localisation imply that the morphism  $X_M(S) \rightarrow X(S)$  is a bijection at the level of irreducible components. Furthermore, taking these bijections for all  $S$ , we obtain an isomorphism  $\mathcal{C}_{X_M} \rightarrow \mathcal{C}_X$  of contravariant functors **FI**  $\rightarrow$  **PF**.

Let  $\overline{M}$  be a separable closure of  $M$  and let  $X_{\overline{M}}$  be the base change of  $X_M$  to  $\overline{M}$ . For each  $S \in \mathbf{FI}$ , the morphism  $X_{\overline{M}}(S) \rightarrow X_M(S)$  induces a surjection  $\mathcal{C}_{X_{\overline{M}}}(S) \rightarrow \mathcal{C}_{X_M}(S)$ , and the fibers are precisely the orbits of the Galois group  $\text{Gal}(\overline{M} : M)$  on  $\mathcal{C}_{X_{\overline{M}}}(S)$  (see [Sta20, Tag 0364]). In other words,  $\mathcal{C}_{X_M}(S)$  has a canonical bijection to the group  $\mathcal{C}_{X_{\overline{M}}}(S) / \text{Gal}(\overline{M} : M)$ . To complete the proof, we need to analyse the component functor of  $X_{\overline{M}}$ .

To this end, let  $Z_1, \dots, Z_k$  be the irreducible components of the base change  $Z_{\overline{M}}$ . Then

$$X_{\overline{M}}(S) = Z_{\overline{M}}^S = \bigcup_{\alpha \in [k]^S} \prod_{i \in S} Z_i^{\alpha_i}$$

where each product over  $i \in S$  is a product of irreducible varieties over the separably closed field  $\overline{M}$ , and hence irreducible. To construct our component functor, we just set  $\mathcal{E}(S) := [k]^S$ .

Finally, let  $G$  be the image of  $\text{Gal}(\overline{M} : M)$  in  $\text{Sym}([k])$  through its action on the irreducible components  $Z_1, \dots, Z_k$  of  $Z$ . Then the (image of the) action of  $\text{Gal}(\overline{M} : M)$  on irreducible components of  $X_{\overline{M}}(S)$  corresponds precisely to the (image of the) diagonal action of  $G$  on  $\mathcal{E}(S)$ , and hence the orbit space  $\mathcal{E}(S)/G$  is in bijection with the irreducible components of  $X_M(S)$ . This bijection, taken for all  $S$ , is an isomorphism from the elementary model functor given by  $\mathcal{E}(S) = [k]^S$  and the group  $G \subseteq \text{Sym}([k])$ .  $\square$

**Remark 2.5.4.3.** Note that the elementary model functors coming from **FI**<sup>pp</sup>-schemes of product type all have  $\mathcal{E}(S) = [k]^S$  rather than just  $\mathcal{E}(S) \subseteq [k]^S$ . The set of count vectors is therefore all of  $\mathbb{Z}_{\geq 0}^k$ . However, in our proof of the Main Theorem we will need to do induction over the poset of downward

closed subsets of  $\mathbb{Z}_{\geq 0}^k$ ; this requires the greater generality in the definition of elementary model functors.  $\diamond$

**Corollary 2.5.4.4.** *The Main Theorem holds for affine  $\mathbf{FI}^{\text{op}}$ -schemes of product type over some Noetherian domain.*

*Proof.* This is an immediate corollary of Proposition 2.5.4.1 and Corollary 2.5.3.2.  $\square$

We are now in a position to prove the following result, most of which also follows from combining results from [LNNR19] (linearity of codimension) and [NR17] (the form of the Hilbert function).

**Theorem 2.5.4.5.** *Let  $X$  be an affine width-one  $\mathbf{FI}^{\text{op}}$ -scheme  $X$  of finite type over a Noetherian ring  $K$ . Assume that  $X([n])$  is not the empty scheme for any  $n$ . Then for  $n \gg 0$  the Krull dimension of  $X([n])$  is eventually equal to an affine-linear polynomial in  $n$ , and the number of irreducible components  $|\mathcal{C}_X([n])|$  is bounded from above by  $c^n$  for some constant  $c \geq 1$ .*

*Proof.* By Lemma 2.2.2.2 we may assume that  $X = \text{Spec}(B)$  is reduced, and by Proposition 2.2.1.1 we may assume that  $X$  is nice. Since  $X([n])$  is not the empty scheme for any  $n$ , we have  $1 \neq 0$  in  $B_0$ . By the Shift Theorem 2.3.1.1 and Proposition 2.3.3.1 there exists an  $S_0 \in \mathbf{FI}$  and a nonzero  $h \in B(S_0)$  such that  $X' := (\text{Sh}_{S_0} X)[1/h]$  is of product type; in particular, it sends  $S \rightarrow Z^S$  for some reduced scheme  $Z$  of finite type over  $L := B(S_0)[1/h]$ .

Let  $Y$  be the closed  $\mathbf{FI}^{\text{op}}$ -subscheme of  $X$  defined by the vanishing of  $h$ . For any  $S \in \mathbf{FI}$ ,  $X(S_0 \sqcup S)$  is the union of  $Y(S_0 \sqcup S)$  and all  $X(\pi)X'(S)$  where  $\pi$  ranges over the finite set  $\text{Sym}(S_0 \sqcup S)$ . Therefore,

$$\dim X(S_0 \sqcup S) = \max\{\dim(Y(S_0 \sqcup S)), \dim(X'(S))\}.$$

By Noetherian induction using Theorem 1.4.6.3, we may assume that the theorem holds for  $Y$ . On the other hand, we have  $\dim(X'(S)) = \dim(L) + |S| \cdot \dim(Z)$ . We conclude that, for  $n \gg 0$ ,  $\dim(X([n]))$  is a maximum of two affine-linear functions of  $n$ , hence itself an affine-linear function of  $n$ . Similarly, to bound  $|\mathcal{C}_X(S_0 \sqcup S)|$  we claim that

$$\begin{aligned} |\mathcal{C}_X(S_0 \sqcup S)| &\leq |\mathcal{C}_Y(S_0 \sqcup S)| + (|S| + |S_0|)(|S| + |S_0| - 1) \\ &\quad \cdots (|S| + 1)|\mathcal{C}_{X'}(S)|. \end{aligned}$$

Indeed, if  $C$  is an irreducible component of  $X(S_0 \sqcup S)$ , then either  $C$  is contained in  $Y(S_0 \sqcup S)$  (and then a component there) or else there exists an

injection  $\pi : S_0 \rightarrow S_0 \sqcup S$  such that  $B(\pi)h$  is not identically zero on  $C$ . In the latter case, let  $\sigma \in \text{Sym}(S_0 \sqcup S)$  be any element with  $\sigma \circ \pi = \text{id}_{S_0}$ . Then  $B(\sigma)B(\pi)h = h$ , and hence  $C = X(\sigma)C'$  for a component  $C'$  of  $X(S_0 \sqcup S)$  on which  $h$  is nonzero. These components correspond bijectively to components of  $X'(S)$ . This explains the second term, where the first  $|S_0|$  factors count the number of possibilities for  $\pi$ .

Now the first term is bounded by an exponential function of  $|S_0| + |S|$  by the induction hypothesis, and the second term is bounded by an exponential function by the proof of Proposition 2.5.4.1. Hence so is the sum.  $\square$

### 2.5.5 The orbit-counting lemma for groupoids

It turns out that in the general case of the Main Theorem, the (Galois) group that featured in the proof of Corollary 2.5.4.4, is replaced by a suitable *groupoid*. We briefly recall the relevant set-up.

Let  $G$  be a finite groupoid, that is, a category whose class of objects is a finite set  $Q$  and in which for any  $p, q \in Q$  the set  $G(p, q) := \text{Hom}(p, q)$  is a finite set all of whose elements are isomorphisms. Rather than homomorphisms or isomorphisms, we will call these elements *arrows*.

For a groupoid to act on a finite set  $X$ , one first specifies an *anchor map*  $a : X \rightarrow Q$ . For  $p \in Q$ , set

$$X_p := a^{-1}(p).$$

Then, an action of  $G$  on  $X$  consists of the data of a map  $\varphi(g) : X_p \rightarrow X_q$  for each homomorphism  $g : p \rightarrow q$ , subject to the conditions that  $\varphi(\text{id}_p) = \text{id}_{X_p}$  and  $\varphi(h \circ g) = \varphi(h) \circ \varphi(g)$  for any two arrows  $g : p \rightarrow q$  and  $h : q \rightarrow r$ . We often write  $gx$  instead of  $\varphi(g)(x)$ .

Write  $G(p) := \bigcup_{q \in Q} G(p, q)$  for the set of arrows from  $p$ . For  $x \in X_p$  we have a map  $G(p) \rightarrow X, g \mapsto gx$ . The image of this map is called the *orbit* of  $x$  and denoted  $G(p)x$ . On the other hand, we write

$$G(p, p)_x := \{g \in G(p, p) \mid gx = x\},$$

the stabiliser of  $x$  in  $G(p, p)$ , which is a subgroup of the group  $G(p, p)$ . Observe that the map  $G(p) \rightarrow G(p)x$  yields a bijection  $G(p)/G(p, p)_x \rightarrow G(p)x$ ; here  $G(p, p)_x$  acts freely on  $G(p)$  by precomposition, so that  $|G(p)x| = |G(p)|/|G(p, p)_x|$ . Furthermore, for every element  $y \in G(p)x$  we have  $|G(a(y))| = |G(p)|$  and  $|G(a(y), a(y))_y| = |G(p, p)_x|$ .

Finally, for  $g \in G(p, p)$  we write  $X_p^g$  for the set of elements  $x \in X_p$  with  $gx = x$ . The following is a generalization of the orbit-counting lemma for groups.

**Lemma 2.5.5.1.** *The number of orbits of  $G$  on  $X$  equals*

$$\sum_{p \in Q} \frac{1}{|G(p)|} \sum_{g \in G(p,p)} |X_p^g|.$$

*Proof.* We count the triples  $(p, g, x)$  with  $p \in Q$  and  $x \in X_p$  and  $g \in G(p, p)$  with  $gx = x$  and  $|G(p)| = N$  in two different ways. If we first fix  $x$ , then we are forced to take  $p := a(x)$ , and we obtain

$$\begin{aligned} & \sum_{x \in X: |G(a(x))| = N} |G(a(x), a(x))_x| \\ &= \sum_{x \in X: |G(a(x))| = N} |G(a(x))| / |G(a(x))x| \\ &= \sum_{x \in X: |G(a(x))| = N} N / |G(a(x))x|. \end{aligned}$$

This is  $N$  times the number of orbits of  $G$  on the set of  $x$  with  $|G(a(x))| = N$ .

On the other hand, if we first fix  $p$  with  $|G(p)| = N$  and  $g \in G(p, p)$ , then we find

$$\sum_{p \in Q: |G(p)| = N} \sum_{g \in G(p,p)} |X_p^g|.$$

Hence the number of orbits of  $G$  on the set of  $x$  with  $|G(a(x))| = N$  equals

$$\frac{1}{N} \sum_{p \in Q: |G(p)| = N} \sum_{g \in G(p,p)} |X_p^g|.$$

Now sum over all possible values of  $N$  to obtain the formula in the lemma.  $\square$

## 2.5.6 Model functors

Elementary model functors are special cases of a more general class of functors  $\mathbf{FI} \rightarrow \mathbf{PF}$ , which we call *model functors*. Their construction is motivated by Theorem 2.3.1.1 and Proposition 2.5.4.1, as we will see below.

Fix  $S_0 \in \mathbf{FI}$ , a  $k \in \mathbb{Z}_{\geq 0}$ , and a subfunctor  $\mathcal{E} : \mathbf{FI} \rightarrow \mathbf{PF}$  of the functor  $S \mapsto [k]^S$ , where we require that for all  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$ ,  $\mathcal{E}(\pi)$  is defined everywhere. Hence, by passing to count vectors as in §2.5.3,  $\mathcal{E}$  is uniquely determined by a downward closed subset of  $\mathbb{Z}_{\geq 0}^k$ .

Then we define a new functor  $\mathcal{F} : \mathbf{FI} \rightarrow \mathbf{PF}$  on objects by

$$\mathcal{F}(S) = \{(\sigma, \alpha) \mid \sigma \in \text{Hom}_{\mathbf{FI}}(S_0, S), \alpha \in \mathcal{E}(S \setminus \text{im}(\sigma))\}$$

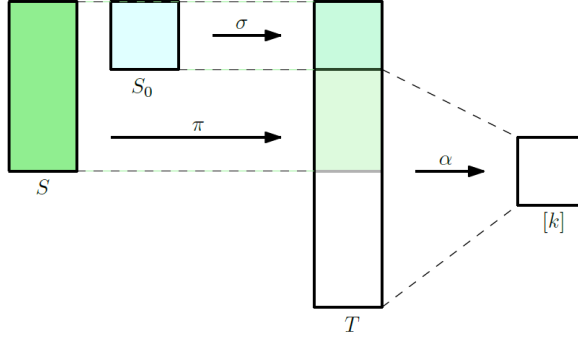


Figure 2.1: The construction of  $\mathcal{F}(\pi)$  in Definition 2.5.6.1.

and on a morphism  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  as follows: for  $(\sigma, \alpha) \in \mathcal{F}(T)$  we set

$$\mathcal{F}(\pi)((\sigma, \alpha)) := \begin{cases} \text{undefined if } \text{im}(\sigma) \not\subseteq \text{im}(\pi); \text{ and} \\ (\sigma', \alpha \circ \pi|_{S \setminus \text{im}(\sigma')}) \text{ where } \sigma' := \pi^{-1} \circ \sigma \text{ otherwise.} \end{cases}$$

Figure 2.1 depicts all relevant maps. At the level of species (so remembering the maps  $\mathcal{F}(\pi)$  only when  $\pi$  is a bijection), this is an instance of a well-known construction:  $\mathcal{F}$  is the product of the species that maps  $S$  to its set of bijections  $S_0 \rightarrow S$  and the species that maps  $S$  to  $\mathcal{E}(S)$ .

Next let  $\sim_S$  be an equivalence relation on  $\mathcal{F}(S)$  for each  $S$ , and assume that these relations satisfy the following three axioms:

**Axiom (1)** if  $(\sigma, \alpha) \sim_T (\sigma', \alpha')$  and  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  has  $\text{im}(\pi) \supseteq \text{im}(\sigma) \cup \text{im}(\sigma')$ , then  $\mathcal{F}(\pi)((\sigma, \alpha)) \sim_S \mathcal{F}(\pi)((\sigma', \alpha'))$ ;

**Axiom (2)** conversely, if the pairs  $(\sigma, \alpha) \in \mathcal{F}(T)$ ,  $(\sigma'', \alpha'') \in \mathcal{F}(S)$ , and the map  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  satisfy  $\text{im}(\pi) \supseteq \text{im}(\sigma)$  and  $\mathcal{F}(\pi)((\sigma, \alpha)) \sim_S (\sigma'', \alpha'')$ , then there exists a pair  $(\sigma', \alpha') \in \mathcal{F}(T)$  with  $\text{im}(\sigma') \subseteq \text{im}(\pi)$  such that  $(\sigma', \alpha') \sim_T (\alpha, \sigma)$  and  $\mathcal{F}(\pi)((\sigma', \alpha')) = (\sigma'', \alpha'')$ ; and

**Axiom (3)** if  $(\sigma, \alpha) \sim_T (\sigma', \alpha')$  and  $i, j \in T \setminus (\text{im}(\sigma) \cup \text{im}(\sigma'))$ , then  $\alpha(i) = \alpha(j) \Leftrightarrow \alpha'(i) = \alpha'(j)$ .

The first axiom ensures that  $\mathcal{F}/\sim: S \mapsto \mathcal{F}(S)/\sim_S$  is a functor  $\mathbf{FI} \rightarrow \mathbf{PF}$  that comes with a canonical surjective morphism  $\mathcal{F} \rightarrow \mathcal{F}/\sim$  in the sense of Definition 2.4.1.2; in particular, this implies that  $\sim_S$  is preserved under the

symmetric group  $\text{Sym}(S)$  acting on  $\mathcal{F}(S)$ . The second and third axioms will be crucial in §2.5.7.

**Definition 2.5.6.1.** A functor of the form  $S \mapsto \mathcal{F}(S)/\sim_S$  as constructed above is called a *model functor*.  $\diamond$

**Remark 2.5.6.2.** Each elementary model functor  $S \mapsto \mathcal{E}(S)/G$  is isomorphic to a model functor with  $S_0 = \emptyset$  (so that we may leave out the  $\sigma$ s from the pairs) and  $\alpha \sim_S \alpha'$  if and only if  $\alpha' \in G\alpha$ . We will see that, conversely, a model functor gives rise to certain groupoids that play the role of  $G$ .  $\diamond$

We revisit Example 2.1.2.4 from the perspective of model functors.

**Example 2.5.6.3.** Let  $X(S)$  be the subscheme of  $\mathbb{A}_{\mathbb{C}}^S$  defined by the equations  $x_j^d - x_l^d$  for all  $j, l \in S$ . Set  $\zeta := e^{2\pi i/d}$ . If we fix a  $j_0 \in S$ , there is a bijection between irreducible components of  $X(S)$  and elements of  $(\mathbb{Z}/d\mathbb{Z})^{S \setminus \{j_0\}}$ , where the component corresponding to  $\alpha \in (\mathbb{Z}/d\mathbb{Z})^{S \setminus \{j_0\}}$  is that in which each  $x_j, j \neq j_0$ , equals  $\zeta^{\alpha_j} x_{j_0}$ . By identifying  $\mathbb{Z}/d\mathbb{Z}$  with  $[d]$  in the natural manner and regarding  $j_0$  as the image of a  $\sigma \in \text{Hom}_{\mathbf{FI}}(S_0, S)$  where  $S_0$  is a singleton, we obtain a surjection from the functor

$$\mathcal{F} : S \mapsto \{(\sigma, \alpha) \mid \sigma \in \text{Hom}_{\mathbf{FI}}(S_0, S), \alpha \in [d]^{S \setminus \text{im}(\sigma)}\}$$

to the component functor  $\mathcal{C}_X$ . A pair  $(\sigma, \alpha)$  is mapped to the same component as a pair  $(\sigma', \alpha')$  if and only if either  $(\sigma, \alpha) = (\sigma', \alpha')$  or else  $\sigma, \sigma'$  have distinct images  $j_0, j'_0$  and for all  $j \in S \setminus \{j_0, j'_0\}$  we have  $\alpha'(j) - \alpha(j_0) = \alpha(j)$  and  $\alpha(j) - \alpha(j'_0) = \alpha'(j)$  (both in  $\mathbb{Z}/d\mathbb{Z}$ ). This defines an equivalence relation  $\sim_S$  satisfying Axioms (1)–(3), and  $\mathcal{C}_X$  is isomorphic to the model functor  $S \mapsto \mathcal{F}(S)/\sim_S$ .  $\diamond$

**Remark 2.5.6.4.** Informally, we think of  $(\sigma, \alpha)$  as a word in  $[k]^S$  in which the letters corresponding to  $\text{im}(\sigma) \subseteq S$  are concealed. If  $(\sigma', \alpha') \sim_S (\sigma, \alpha)$ , then  $(\sigma', \alpha')$  corresponds to a different word in which the letters corresponding to  $\text{im}(\sigma')$  are concealed. Outside of  $\text{im}(\sigma) \cup \text{im}(\sigma')$ , by Axiom (3), the two words are equal up to some permutation of  $[k]$ . Axioms (1) and (2) simply ask that these equivalent words behave well with respect to  $\mathbf{FI}$ -morphisms. In the next section, we will roughly speaking attempt to “discover” the information concealed in  $\text{im}(\sigma)$  by looking at equivalent pairs  $(\sigma', \alpha')$  where  $\text{im}(\sigma')$  does not contain  $\text{im}(\sigma)$ .  $\diamond$

**Example 2.5.6.5.** Fix a finite field  $\mathbb{F}_q$  and a natural number  $m$ . Let  $S$  be a finite set. On the set of rank- $m$  matrices in  $\mathbb{F}_q^{m \times S}$  act four groups:

1.  $\mathrm{GL}_m(\mathbb{F}_q)$  by row operations;
2.  $\mathrm{Sym}(S)$  by permuting columns;
3.  $(\mathbb{F}_q^*)^S$  by scaling columns; and
4.  $\mathrm{Aut}(\mathbb{F}_q)$  by acting on all coordinates.

Modding out only  $\mathrm{GL}_m(\mathbb{F}_q)$ , we get the set of  $m$ -dimensional codes in  $\mathbb{F}_q^S$ , and two codes are called isomorphic if they are in the same orbit under  $\mathrm{Sym}(S) \ltimes G(S)$ , where  $G(S) = \mathrm{Aut}(\mathbb{F}_q) \ltimes (\mathbb{F}_q^*)^S$ .

Let  $\mathcal{N}$  be a functor  $\mathbf{FI} \rightarrow \mathbf{PF}$  that assigns to  $S$  a set of  $m$ -dimensional codes in  $\mathbb{F}_q^S$  and to an injective map  $\pi : S \rightarrow T$  the map that sends a code  $C \in \mathcal{N}(T)$  to the code

$$\{v \circ \pi \mid v \in C\} \subseteq \mathbb{F}_q^S,$$

provided that this linear space still has dimension  $m$ . In particular, we require that the code above is then an element of  $\mathcal{N}(S)$ . This implies that  $\mathcal{N}(S)$  is preserved under  $\mathrm{Sym}(S)$  and that  $\mathcal{N}$  is closed under puncturing. Furthermore, we assume that  $\mathcal{N}(S)$  is preserved under  $G(S)$ . To prove Theorem 2.1.3.4, we need to count the orbits of  $\mathrm{Sym}(S) \ltimes G(S)$  on  $\mathcal{N}(S)$ ; this is the same as the number of orbits of  $\mathrm{Sym}(S)$  on  $\mathcal{M}(S) := \mathcal{N}(S)/G(S)$ . We will informally call the elements of  $\mathcal{M}(S)$  codes, as well. This  $\mathcal{M}$  is another functor  $\mathbf{FI} \rightarrow \mathbf{PF}$ , and we claim that it is isomorphic to a model functor.

To see this, let  $k := |\mathbb{F}_q^m / \mathbb{F}_q^*| = 1 + (q^m - 1)/(q - 1)$  and fix any bijection between  $[k]$  and  $\mathbb{F}_q^m / \mathbb{F}_q^*$  by which we identify these two sets. Also set  $S_0 := [m]$ . Now define

$$\begin{aligned} \mathcal{E}(S) := \{ & \alpha \in ((\mathbb{F}_q)^m / \mathbb{F}_q^*)^S \mid \text{the row space of} \\ & \text{the matrix } (I|\alpha) \in (\mathbb{F}_q)^{[m] \times (S_0 \sqcup S)} \text{ is in } \mathcal{N}(S_0 \sqcup S) \}, \end{aligned}$$

where  $(I|\alpha)$  is the  $[m] \times (S_0 \sqcup S)$ -matrix which in the  $[m] \times S_0$ -block has the identity matrix  $I$  and in the  $[m] \times S$ -block has the matrix  $\alpha$ . Furthermore, define  $\mathcal{F}(S)$  as the set of pairs  $(\sigma, \alpha)$  with  $\sigma \in \mathrm{Hom}_{\mathbf{FI}}(S_0, S)$  and  $\alpha \in \mathcal{E}(S \setminus \mathrm{im}(\sigma))$ . We have a surjective morphism  $\Psi : \mathcal{F} \rightarrow \mathcal{M}$  in the sense of Definition 2.4.1.2 that sends  $(\sigma, \alpha)$  to the orbit under  $G(S)$  of the row space of  $(I|\alpha) \in \mathbb{F}_q^{m \times S}$ , where now  $I \in \mathbb{F}_q^{[m] \times \mathrm{im}(\sigma)}$  is the permutation matrix with entries  $\delta_{i, \sigma^{-1}(j)}$ .

We define  $\sim_S$  on  $\mathcal{F}(S)$  by  $(\sigma, \alpha) \sim (\sigma', \alpha')$  if and only if  $\Psi((\sigma, \alpha)) = \Psi((\sigma', \alpha'))$ . We need to show that this satisfies Axioms (1), (2), and (3). Axiom (1) follows from the fact that if  $(\sigma, \alpha), (\sigma', \alpha')$  represent the same  $(G(T)$ -orbit of) code(s), then the same is true after puncturing this code in a coordinate

outside  $\text{im}(\sigma) \cup \text{im}(\sigma')$ . To see Axiom (2), we puncture a code  $C$  represented by  $(\sigma, \alpha)$  in a coordinate  $i \in T \setminus \text{im}(\sigma)$ , and assume that the resulting code  $C'$ , represented by  $(\sigma, \alpha|_{T \setminus \{i\}})$ , is also represented by some other pair  $(\sigma'', \alpha'') \in \mathcal{F}(T \setminus \{i\})$ . Then the projection of  $C'$  to  $\mathbb{F}_q^{\text{im}(\sigma')}$  is surjective, and hence the same holds for  $C$ , so that  $C$  is also represented by a pair of the form  $(\sigma'', \alpha')$ . Finally, Axiom (3) says that if in the generator matrix  $(I|\alpha)$  for the code  $C$  represented by  $(\sigma, \alpha)$  the columns labelled  $i, j \in T \setminus \text{im}(\sigma)$  are parallel, then the same holds for any other pair  $(\sigma', \alpha')$  that represents  $C$  and satisfies  $i, j \in T \setminus \text{im}(\sigma')$ —the main point here is that parallelness is preserved under field automorphisms.  $\diamond$

## 2.5.7 A second quasi-polynomial count

We use the notation from §2.5.6. So  $S \mapsto \mathcal{F}(S)/\sim_S$  is a model functor, where  $\mathcal{F}(S)$  is the set of pairs  $(\sigma, \alpha)$  with  $\sigma \in \text{Hom}_{\mathbf{FI}}(S_0, S)$  and  $\alpha \in \mathcal{E}(S \setminus S_0) \subseteq [k]^{S \setminus S_0}$ , and where  $\sim_S$  is an equivalence relation on  $\mathcal{F}(S)$  that satisfies Axioms (1),(2),(3) for model functors. The following theorem and its proof are elementary, but quite subtle—indeed, this is probably the most intricate part of the chapter.

**Theorem 2.5.7.1.** *Let  $S \mapsto \mathcal{F}(S)/\sim_S$  be a model functor  $\mathbf{FI} \rightarrow \mathbf{PF}$ . Then there exists a quasipolynomial  $f$  such that the number of  $\text{Sym}([n])$ -orbits on  $\mathcal{F}([n])/\sim_{[n]}$  equals  $f(n)$  for all  $n \gg 0$ .*

This immediately implies Theorem 2.1.3.4.

*Proof of Theorem 2.1.3.4.* By Example 2.5.6.5, the number of length- $n$  elements in  $\mathcal{C}$  is the number of  $\text{Sym}([n])$ -orbits on  $\mathcal{F}([n])/\sim_{[n]}$  for a suitable model functor. The result now follows from Theorem 2.5.7.1.  $\square$

To prove Theorem 2.5.7.1, we introduce the notion of sub-model functor.

**Definition 2.5.7.2.** Suppose that we have, for each  $S \in \mathbf{FI}$ , a subset  $\mathcal{E}'(S) \subseteq \mathcal{E}(S)$  such that, first, for all  $\pi \in \text{Hom}_{\mathbf{FI}(S,T)}$  the pull-back map  $[k]^T \rightarrow [k]^S$  that maps  $\mathcal{E}(T)$  into  $\mathcal{E}(S)$  also maps  $\mathcal{E}'(T)$  into  $\mathcal{E}'(S)$ ; and second, for all  $(\sigma, \alpha) \sim_S (\sigma', \alpha') \in \mathcal{F}(S)$  with  $\alpha \in \mathcal{E}'(S \setminus \text{im}(\sigma))$ , we also have  $\alpha' \in \mathcal{E}'(S \setminus \text{im}(\sigma'))$ . Then  $S \mapsto \mathcal{F}'(S)/\sim_S$ , where

$$\mathcal{F}'(S) := \{(\sigma, \alpha) \in \mathcal{F}(S) \mid \alpha \in \mathcal{E}'(S)\}$$

and where  $\sim_S$  stands for the restriction of  $\sim_S$  to  $\mathcal{F}'(S)$  is a model functor called a *sub-model functor* of  $\mathcal{F}$ .  $\diamond$

*Proof of Theorem 2.5.7.1.* The proof of this theorem will take up the remainder of this subsection. This will involve an induction hypothesis for a sub-model functor  $\mathcal{F}'$  of  $\mathcal{F}$  and the construction of a certain groupoid for the complement  $\mathcal{F} \setminus \mathcal{F}'$ .

Let  $M \subseteq \mathbb{Z}_{\geq 0}^k$  be the downward-closed set consisting of all the count vectors of elements in  $\mathcal{E}(S)$  for  $S$  running over **FI**. Since, by Dickson's lemma, the set of downward-closed sets in  $\mathbb{Z}_{\geq 0}^k$  satisfies the descending chain property, we may assume that the theorem holds for all model functors whose corresponding downward set is strictly contained in  $M$ .

Our goal is now to find  $\mathcal{E}'(S) \subseteq \mathcal{E}(S)$  that defines a sub-model functor  $\mathcal{F}'$  of  $\mathcal{F}$  such that the downward closed set  $M'$  of  $\mathcal{E}'$  is strictly contained in  $M$ . Then we have

$$|\mathcal{F}(S)/\sim_S| = |\mathcal{F}'(S)/\sim_S| + |(\mathcal{F}(S) \setminus \mathcal{F}'(S))/\sim_S|$$

and this equality continues to hold if we mod out the action of  $\text{Sym}(S)$  on the three sets in question. Hence by the induction hypothesis we are done if we can show that the number of  $\text{Sym}(S)$ -orbits on  $(\mathcal{F}(S) \setminus \mathcal{F}'(S))/\sim_S$  grows quasipolynomially in  $|S|$  for  $|S| \gg 0$ . In this induction argument, we may of course assume that  $M$  is not empty—otherwise, the quasipolynomial 0 will do.

To construct  $\mathcal{E}'$  we proceed as follows. Let

$$A := \{I \subseteq [k] \mid \exists v \in M : v + \mathbb{Z}_{\geq 0}^I \subseteq M\};$$

here, and in the rest of the chapter, we identify  $\mathbb{Z}_{\geq 0}^I$  with  $\mathbb{Z}_{\geq 0}^I \times \{0\}^{[k] \setminus I} \subseteq \mathbb{Z}_{\geq 0}^k$ . Note that  $A$  is nonempty because  $M$  is. Let  $I$  be an inclusion-wise maximal element of  $A$  and set

$$d := \max \left\{ \sum_{l \in [k] \setminus I} v(l) \mid v + \mathbb{Z}_{\geq 0}^I \subseteq M \right\}.$$

This is well-defined, since if the sum of the entries in such  $v$  at positions outside  $I$  were unbounded, then  $I$  would be contained in a strictly larger element of  $A$ . Choose  $v \in M$  such that  $v + \mathbb{Z}_{\geq 0}^I \subseteq M$  and  $\sum_{l \in [k] \setminus I} v(l) = d$ .

**Lemma 2.5.7.3.** *For  $j \in \mathbb{Z}_{\geq 0}$ , define  $v_j := v + j \cdot (\sum_{i \in I} e_i) \in M$ . There exists a  $j \in \mathbb{Z}_{\geq 0}$  such that the vectors in  $M$  that are componentwise  $\geq v_j$  are precisely the vectors in  $v_j + \mathbb{Z}_{\geq 0}^I$ ; this then also holds for all larger values of  $j$ .*

*Proof.* Suppose that for every  $j \in \mathbb{Z}_{\geq 0}$  there is a  $w_j \in M \setminus (v_j + \mathbb{Z}_{\geq 0}^I)$  that is componentwise  $\geq v_j$ . Then by Dickson's lemma the sequence

$$w_1|_{[k] \setminus I}, w_2|_{[k] \setminus I}, w_3|_{[k] \setminus I}, \dots$$

would contain an infinite subsequence, labelled by  $i_1 < i_2 < \dots$ , that weakly increases componentwise. It follows that  $w_{i_1} + \mathbb{Z}_{\geq 0}^I \subseteq M$  because  $M$  is downward closed and the entries of  $w_{i_j}$  labelled by  $I$  diverge to infinity. Furthermore, by the choice of  $w_{i_1}$ , the sum  $\sum_{l \in [k] \setminus I} w_{i_1}(l)$  is strictly larger than  $d$ , a contradiction.  $\square$

From now on, we will make the following assumption on  $v$ :

*The entries  $v(l)$  are  $\gg 0$  for all  $l \in I$ .*

In particular, after replacing  $v$  with the  $v_j$  from the lemma, this implies that the vectors in  $M$  that are componentwise  $\geq v$  are precisely the vectors in  $v + \mathbb{Z}_{\geq 0}^I$ . In the course of our reasoning, we will need further assumptions on how large the  $v(l)$  with  $l \in I$  are—to avoid technicalities, however, we make no attempt to specify a precise lower bound that works.

The set  $I$  is now uniquely determined by  $v$  as the set of all positions  $l \in [k]$  where  $v(l)$  is very large. We call it the *frequent set* of  $v$ .

Write  $v = v_0 + v_1$  with  $v_0 \in \mathbb{Z}_{\geq 0}^{[k] \setminus I}$  and  $v_1 \in \mathbb{Z}_{\geq 0}^I$ . We now define  $\mathcal{F}' : \mathbf{FI} \rightarrow \mathbf{PF}$  by setting  $\mathcal{F}'(S)$  to be the set of pairs  $(\sigma, \alpha) \in \mathcal{F}(S)$  for which there exists *no* pair  $(\sigma', \alpha') \sim_S (\sigma, \alpha)$  such that the count vector of  $\alpha'$  is in  $\tilde{v} + \mathbb{Z}_{\geq 0}^I$ , where  $\tilde{v} := v + v_1 = v_0 + 2v_1$ . Notice the factor 2; the relevance of this will become clear towards the end of the proof.

**Lemma 2.5.7.4.** *The association  $S \mapsto \mathcal{F}'(S)/\sim_S$  is a sub-model functor of  $\mathcal{F}$ .*

*Proof.* By definition,  $\mathcal{F}'(S)$  is a union of  $\sim_S$ -equivalence classes, and uniquely determined by a subset  $\mathcal{E}'(S) \subseteq \mathcal{E}(S)$  of allowed second components  $\alpha$ . So we need only prove that  $\mathcal{F}'$  is preserved under morphisms.

Hence let  $(\sigma, \alpha) \in \mathcal{F}'(T)$ , let  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  satisfy  $\text{im}(\pi) \supseteq \text{im}(\sigma)$ , and consider  $(\tilde{\sigma}, \tilde{\alpha}) := \mathcal{F}(\pi)((\sigma, \alpha))$ . If  $(\tilde{\sigma}, \tilde{\alpha}) \sim_S (\sigma'', \alpha'')$  where  $\alpha''$  has a count vector in  $\tilde{v} + \mathbb{Z}_{\geq 0}^I$ , then by Axiom (2) for model functors there exists a pair  $(\sigma', \alpha') \sim_T (\sigma, \alpha)$  with  $\mathcal{F}(\pi)((\sigma', \alpha')) = (\sigma'', \alpha'')$ . This means that the count vector of  $\alpha'$  is an element of  $M$  that is componentwise greater than or equal to the count vector of  $\alpha''$  and hence, by the choice of  $\tilde{v}$ , the count vector of  $\alpha'$  lies in  $\tilde{v} + \mathbb{Z}_{\geq 0}^I$ . This contradicts the fact that  $(\sigma, \alpha) \in \mathcal{F}'(T)$ . Therefore,  $\mathcal{F}(\pi)((\sigma, \alpha)) \in \mathcal{F}'(S)$ .  $\square$

Furthermore, we observe that the downward closed subset  $M'$  corresponding to  $\mathcal{E}'$  is strictly contained in  $M$ , since it does not contain  $\tilde{v} \in M$ . Hence the induction hypothesis applies to  $\mathcal{F}'$ .

Our task is therefore reduced to counting the  $\text{Sym}(S)$ -orbits on the set of  $\sim_S$ -equivalence classes on  $\tilde{\mathcal{F}}(S) := \mathcal{F}(S) \setminus \mathcal{F}'(S)$ . Note that  $\tilde{\mathcal{F}}$  is not a sub-model functor of  $\mathcal{F}$ ; rather,  $\tilde{\mathcal{F}}(S)$  consists of all pairs  $(\sigma, \alpha)$  that are equivalent to some pair  $(\sigma', \alpha')$  where  $\alpha'$  has a count vector in  $\tilde{v} + \mathbb{Z}_{\geq 0}^I$ . Before counting these, we will work for a while with the larger set  $\mathcal{F}''(S) \supseteq \tilde{\mathcal{F}}(S)$  consisting of all pairs  $(\sigma, \alpha) \in \mathcal{F}(S)$  that are equivalent to some pair  $(\sigma', \alpha')$  where the count vector of  $\alpha'$  is in  $v + \mathbb{Z}_{\geq 0}^I \supseteq \tilde{v} + \mathbb{Z}_{\geq 0}^I$ .

For the time being, fix a pair  $(\sigma, \alpha) \in \mathcal{F}''(T)$  where  $\alpha$  has count vector in  $v + \mathbb{Z}_{\geq 0}^I$ . This implies that all elements in  $I$  occur very frequently among the entries of  $\alpha$ , while all elements in  $[k] \setminus I$  occur very infrequently; we call  $I$  the *frequent set* of  $\alpha$  and of the pair  $(\sigma, \alpha)$ . Now let  $(\sigma', \alpha') \sim_T (\sigma, \alpha)$ . Since, by Axiom (3) for model functors, the equality patterns of  $\alpha'$  and  $\alpha$  agree on  $T \setminus (\text{im}(\sigma) \cup \text{im}(\sigma'))$ , there exists a set  $I' \subseteq [k]$  of the same cardinality as  $I$ , and a bijection  $g = g((\sigma', \alpha'), (\sigma, \alpha)) : I \rightarrow I'$  such that, for  $i \in T \setminus (\text{im}(\sigma) \cup \text{im}(\sigma'))$ , we have  $\alpha(i) = l \in I$  if and only if  $\alpha'(i) = g(l)$ . In particular,  $\alpha'$ , too, has a distinguished set  $I' \subseteq [k]$  of elements that occur very frequently among its entries, while the complement occurs very infrequently. We call  $I'$  the frequent set of  $\alpha'$ .

Furthermore, if also  $(\sigma'', \alpha'') \sim_T (\sigma, \alpha)$ , then we have

$$g((\sigma'', \alpha''), (\sigma', \alpha')) \circ g((\sigma', \alpha'), (\sigma, \alpha)) = g((\sigma'', \alpha''), (\sigma, \alpha)) \quad (2.1)$$

as a map from  $I$  to the frequent set  $I''$  of  $\alpha''$ , and we have

$$g((\sigma, \alpha), (\sigma, \alpha)) = \text{id}_I. \quad (2.2)$$

Still using elements from the  $\sim_T$ -equivalence class of  $(\sigma, \alpha)$ , we define a relation  $\equiv$  on  $T$  as follows: first,  $\equiv$  is reflexive, and second, for  $i \neq j$  we have  $i \equiv j$  if and only if there exists  $(\sigma', \alpha') \sim_T (\sigma, \alpha)$  with  $i, j \notin \text{im}(\sigma')$  and  $\alpha'(i) = \alpha'(j) \in I' := g((\sigma', \sigma'), (\sigma, \sigma))I$ .

**Lemma 2.5.7.5.** *If  $i \equiv j$  then  $(\sigma, \alpha) \sim_T \mathcal{F}((i \ j))(\sigma, \alpha)$ , where  $(i \ j)$  is the transposition of  $i$  and  $j$ .*

*Proof.* If  $i = j$ , then the statement is obvious. Otherwise, there exists a pair  $(\sigma', \alpha') \sim_T (\sigma, \alpha)$  such that  $\alpha'$  is defined at  $i$  and  $j$  and takes the same value  $l$  in the frequent set of  $\alpha'$ . Then  $\mathcal{F}((i \ j))(\sigma', \alpha') = (\sigma', \alpha')$  and by Axiom (1),  $\mathcal{F}((i \ j))(\sigma, \alpha) \sim_T \mathcal{F}((i \ j))(\sigma', \alpha') = (\sigma', \alpha') \sim_T (\sigma, \alpha)$ , as desired.  $\square$

**Lemma 2.5.7.6.** *The relation  $\equiv$  is an equivalence relation on  $T$ .*

*Proof.* First note that—using Axiom (3) for model functors—if  $i \neq j$  satisfy  $i \equiv j$ , then in fact for *all*  $(\sigma', \alpha') \sim_T (\sigma, \alpha)$  with  $i, j \in T \setminus \text{im}(\sigma')$  we have  $\alpha'(i) = \alpha'(j) \in I'$ . Since  $\equiv$  is reflexive and symmetric by definition, we only need to show transitivity. For this, assume that  $i \equiv j$  and  $j \equiv h$ , where we may assume that  $i, j, h$  are all distinct. Let  $(\sigma', \alpha') \sim_T (\sigma, \alpha)$  be such that  $\alpha'(j) = \alpha'(h) \in I'$ . Now if  $\alpha'$  is defined at  $i$ , then  $i \equiv j$  implies that also  $\alpha'(i) = \alpha'(j)$  so that  $i \equiv h$ . Assume that  $\alpha'$  is not defined at  $i$ . Then let  $i' \in T \setminus \{i, j, h\}$  be a position where  $\alpha'$  is defined and such that  $i' \equiv i$ —this exists, because there exists a pair  $(\sigma'', \alpha'') \sim_T (\sigma, \alpha)$  for which  $\alpha''$  is defined at  $i$  (and defined and equal at  $j$ ); now set  $l := \alpha''(i)$ , an element in the frequent set of  $\alpha''$ , and take for  $i'$  any element from  $((\alpha'')^{-1}(l)) \setminus \{i, j, h\}$ . Then by Lemma 2.5.7.5 we have  $(\sigma', \alpha') \sim_T \mathcal{F}((i \ i'))(\sigma', \alpha')$  and the latter element is defined at  $i, j, h$ . This proves transitivity.  $\square$

So for all elements  $(\sigma', \alpha')$  in the  $\sim_T$ -equivalence class of  $(\sigma, \alpha)$  we have the same, well-defined equivalence relation  $\equiv$  on  $T$ . Let  $T_1 \subseteq T$  be the set of elements that form a singleton class; we call  $T_1$  the *core* of (the  $\sim_T$ -equivalence class of)  $(\sigma, \alpha)$ . Note that  $T_1 = T_{10} \sqcup T_{11}$  where  $T_{10}$  is the set of those positions in  $T$  that are in  $\text{im}(\sigma')$  for *all*  $(\sigma', \alpha') \sim_T (\sigma, \alpha)$  and  $T_{11}$  is the set of elements that are in  $(\alpha')^{-1}([k] \setminus I')$  for some  $(\sigma', \alpha') \sim_T (\sigma, \alpha)$  and  $I' = g((\sigma', \alpha'), (\sigma, \alpha))I$ . We have  $T_{10} \subseteq \text{im}(\sigma)$  and also  $T_{11} \subseteq \text{im}(\sigma) \cup \alpha^{-1}([k] \setminus I)$ ; in particular,  $|T_1|$  is bounded from above by  $|S_0| + |\alpha^{-1}([k] \setminus I)| = |S_0| + d$ , where  $d$  was the number used in the construction of  $v \in M$ .

The same reasoning applies to *any* element  $(\sigma, \alpha) \in \mathcal{F}''(T)$ : it unambiguously determines a subset  $J \subseteq [k]$  (the frequent set of  $\alpha$ ) of cardinality  $|J| = |I|$  and a subset  $T_1 \subseteq T$  of some bounded size (the core of the pair), as well as a surjection  $\tau : T \setminus T_1 \rightarrow J$  defined by  $\tau(i) = l$  if and only if there exists a  $(\alpha', \sigma') \sim_T (\alpha, \sigma)$  with  $\alpha'$  defined at  $i$  and  $\alpha'(i) = g((\alpha', \sigma'), (\alpha, \sigma))(l)$ ; and this surjection only has large fibres. Furthermore, passing to another element of the  $\sim_T$ -equivalence class, the core  $T_1$  remains the same,  $J$  is acted upon by a bijection  $g$  to yield a  $J'$ , and  $\tau$  is composed with that same bijection. We now determine how certain morphisms transform the data  $J, T_1, \tau$ .

**Lemma 2.5.7.7.** *Let  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  be such that  $\text{im}(\pi)$  contains the core  $T_1 \subseteq T$  of (the  $\sim_T$ -equivalence class of)  $(\sigma, \alpha)$ , and assume that  $(\tilde{\sigma}, \tilde{\alpha}) := \mathcal{F}(\pi)((\sigma, \alpha))$  lies in  $\mathcal{F}''(S)$ . Then the frequent set of  $\tilde{\alpha}$  equals the frequent set  $J$  of  $\alpha$ , the core  $S_1 \subseteq S$  of  $(\tilde{\sigma}, \tilde{\alpha})$  equals  $\pi^{-1}(T_1)$ , and the surjection  $\tilde{\tau} : S \setminus S_1 \rightarrow J$  determined by  $(\tilde{\sigma}, \tilde{\alpha})$  is the map  $\tau \circ (\pi|_{S \setminus S_1})$  where  $\tau : T \setminus T_1 \rightarrow J$  is the surjection determined by  $(\sigma, \alpha)$ .*

*Proof.* That the frequent set  $J$  remains unchanged is immediate: the elements that appear frequently in  $\tilde{\alpha}$  also appear frequently in  $\alpha$  and vice versa.

For the statement about the core, it suffices to show that distinct  $i, j \in S$  satisfy  $i \equiv j$  in the equivalence relation on  $S$  defined by  $(\tilde{\sigma}, \tilde{\alpha})$  if and only if  $\pi(i) \equiv \pi(j)$  in the equivalence relation on  $T$  defined by  $(\sigma, \alpha)$ .

Let  $i, j \in S$  be distinct and assume  $i \equiv j$ , so that there exists a pair  $(\tilde{\sigma}', \tilde{\alpha}') \sim_S (\tilde{\sigma}, \tilde{\alpha})$  with  $\tilde{\alpha}'(i) = \tilde{\alpha}'(j) \in J$ . By Axiom (2) there exists a pair  $(\sigma', \alpha') \sim_T (\sigma, \alpha)$  with  $\mathcal{F}(\pi)((\sigma', \alpha')) = (\tilde{\sigma}', \tilde{\alpha}')$ , and we find that  $\alpha'(\pi(i)) = \alpha'(\pi(j)) \in J$ , so  $\pi(i) \equiv \pi(j)$ .

Conversely, let  $i, j \in S$  be distinct and assume that  $\pi(i) \equiv \pi(j)$ , so there exists a pair  $(\sigma', \alpha') \sim_T (\sigma, \alpha)$  with  $\alpha'(\pi(i)) = \alpha'(\pi(j)) \in J$ . Since  $\text{im}(\pi)$  contains  $T_1$ , it contains all elements of  $\text{im}(\sigma') \cap T_1$ . Using Lemma 2.5.7.5 we may apply transpositions  $\mathcal{F}((h \ h'))$  to  $(\sigma', \alpha')$  for all  $h \in \text{im}(\sigma) \setminus \text{im}(\pi) \subseteq T \setminus T_1$ , where the  $h' \equiv h$  are all chosen distinct, disjoint from  $\text{im}(\sigma)$  and from  $\{i, j\}$ , and inside  $\text{im}(\pi) \setminus T_1$ , to arrive at a  $(\sigma'', \alpha'') \sim_T (\sigma', \alpha')$  still satisfying  $\alpha''(\pi(i)) = \alpha''(\pi(j))$  and now also satisfying  $\text{im}(\sigma'') \subseteq \text{im}(\pi)$ . So we may apply  $\mathcal{F}(\pi)$  to  $(\sigma'', \alpha'')$  and find that  $i \equiv j$ .

For the statement about  $\tilde{\tau}$  let  $i \in S \setminus S_1 = S \setminus \pi^{-1}(T_1)$ . Then there exists a  $(\tilde{\sigma}', \tilde{\alpha}') \sim_S (\tilde{\sigma}, \tilde{\alpha})$  such that  $\tilde{\alpha}'$  is defined at  $i$ . By Axiom (2) there exists a  $(\sigma', \alpha') \sim_T (\sigma, \alpha)$  with  $\mathcal{F}(\pi)((\sigma', \alpha')) = (\tilde{\sigma}', \tilde{\alpha}')$ . In particular,  $\alpha'$  is defined at  $\pi(i)$  and we have  $\alpha'(\pi(i)) = \tilde{\alpha}'(i)$ , so that  $\tilde{\tau}(i) = \tau(\pi(i))$ , as desired.  $\square$

A special case of the lemma is that where  $S = T$ , and we find that  $\text{Sym}(S)$  acts in the expected manner on the data consisting of the frequent set (namely, trivially) and on the core  $S_1$  and the map  $\tau$ . The cardinality of the core is an invariant under this action, and also preserved under the more general morphisms of Lemma 2.5.7.7.

The core is a finite subset of cardinality at most  $|S_0| + d$ . For each  $e \in \{0, \dots, |S_0| + d\}$  let  $\mathcal{F}_e''(S)$  be the set of elements in  $\mathcal{F}''(S)$  with a core of cardinality  $e$ , and set  $\tilde{\mathcal{F}}_e(S) := \tilde{\mathcal{F}}(S) \cap \mathcal{F}_e''(S)$ . We are done once we establish that for each  $e$  the set of  $\text{Sym}(S)$ -orbits on  $\tilde{\mathcal{F}}_e(S)/\sim_S$  is quasipolynomial in  $|S|$  for  $|S| \gg 0$ .

We will decouple the core from the rest of  $S$ . A first justification for this is the following lemma.

**Lemma 2.5.7.8.** *The number of  $\text{Sym}([e] \sqcup S)$ -orbits on  $A := \tilde{\mathcal{F}}_e([e] \sqcup S)/\sim_{[e] \sqcup S}$  equals the number of  $\text{Sym}([e]) \times \text{Sym}(S)$ -orbits on the set  $B$  of elements in  $\tilde{\mathcal{F}}_e([e] \sqcup S)/\sim_{[e] \sqcup S}$  with core equal to  $[e]$ .*

*Proof.* The inclusion map  $B \rightarrow A$  induces a map  $B/(\text{Sym}([e]) \times \text{Sym}(S)) \rightarrow A/\text{Sym}([e] \sqcup S)$ . This map is surjective since the core of any element in

$A$  can be moved into  $[e]$  by an element of  $\text{Sym}([e] \sqcup S)$ . To see that it is also injective, note that if  $\pi \in \text{Sym}([e] \sqcup S)$  and  $(\sigma, \alpha), (\sigma', \alpha') \in B$  satisfy  $\mathcal{F}(\pi)((\sigma, \alpha)) = (\sigma', \alpha')$ , then  $\pi$  must preserve the common core  $[e]$  of both tuples, hence  $\pi \in \text{Sym}([e]) \times \text{Sym}(S)$ .  $\square$

Consider a pair  $(\sigma, \alpha) \in \mathcal{F}_e''([e] \sqcup T)$  with core equal to  $[e]$ . To such a pair we associate the quintuple  $(J, \tau, \sigma_0, \sigma_1, \bar{\alpha})$  determined by:

1. the frequent set  $J$  of  $\alpha$ ;
2. the surjection  $\tau : T \rightarrow J$ ;
3. the partially defined map  $\sigma_0 : S_0 \rightarrow [e]$ , which is the restriction of  $\sigma$  to  $\sigma^{-1}([e])$ ;
4. the map  $\sigma_1 : S_0 \setminus \text{dom}(\sigma_0) \rightarrow J$  defined by  $\sigma_1(i) = \tau(\sigma(i))$ .
5. the restriction  $\bar{\alpha}$  of  $\alpha$  to  $[e] \setminus \text{im}(\sigma)$ , which takes values in  $[k] \setminus J$ .

The quintuple remembers everything about the pair  $(\sigma, \alpha)$  *except* the exact values (in  $T$ ) of  $\sigma$  on  $S_0 \setminus \text{dom}(\sigma_0)$ : of these values, only their equivalence classes under  $\equiv$  classes are remembered; these can be read off from  $\sigma_1$  (and  $\tau$ ). If another pair  $(\sigma', \alpha') \in \mathcal{F}_e''([e] \sqcup T)$  with core equal to  $[e]$  yields the same quintuple, then  $(\sigma', \alpha')$  differs from  $(\sigma, \alpha)$  by a permutation of  $T$  that permutes elements within their equivalence classes under  $\equiv$ . Hence, by repeatedly applying Lemma 2.5.7.5, we find that  $(\sigma', \alpha') \sim_{[e] \sqcup T} (\sigma, \alpha)$ .

We will use the notation  $\sim_{[e] \sqcup T}$  for the induced equivalence relation on such quintuples. Note that  $\text{Sym}([e])$  fixes  $(J$  and)  $\tau$ , whereas  $\text{Sym}(T)$  fixes  $(J$  and)  $\sigma_0, \sigma_1, \bar{\alpha}$ . Also note that all components of the quintuple except  $\tau$  can take only finitely many different values as  $T$  varies. We call  $(J, \sigma_0, \sigma_1, \bar{\alpha})$  the quadruple determined by the quintuple (and *a fortiori* also determined by the pair  $(\sigma, \alpha)$ ).

We now come to the central tool for proving quasipolynomiality; note that now we work with  $\tilde{\mathcal{F}}$  instead of  $\mathcal{F}''$ —recall that  $\tilde{\mathcal{F}}$  and its complement  $\mathcal{F}'$  were defined using  $\tilde{v} = v + v_1 = v_0 + 2v_1$ , while  $\mathcal{F}'' \supseteq \tilde{\mathcal{F}}$  was defined using  $v$ .

**Definition 2.5.7.9.** Let  $G$  be the finite directed graphs whose vertices are all quadruples of pairs in  $\tilde{\mathcal{F}}_e([e] \sqcup T)$  with core  $[e]$ , where  $T$  runs through **FI**, and whose arrows from one quadruple  $(J, \sigma_0, \sigma_1, \bar{\alpha})$  to  $(J', \sigma'_0, \sigma'_1, \bar{\alpha}')$  are all bijections  $g((\sigma', \alpha'), (\sigma, \alpha)) : J \rightarrow J'$  coming from pairs  $(\sigma', \alpha') \sim_{[e] \sqcup T} (\sigma, \alpha)$  in  $\tilde{\mathcal{F}}_e([e] \sqcup T)$  with core  $[e]$  and with the prescribed quadruples.  $\diamond$

Typically, the same bijection  $g$  will arise from more than one pair of pairs with the prescribed quadruples; it then only appears once as an arrow  $g$  between those quadruples. The following proposition will be used below to establish that  $G$  is, in fact, a groupoid.

**Proposition 2.5.7.10.** *Let  $(J, \tau, \sigma_0, \sigma_1, \bar{\alpha})$  be the quintuple of a pair  $(\sigma, \alpha) \in \tilde{\mathcal{F}}_e([e] \sqcup T)$  with core  $[e]$  and let  $g$  be an arrow in  $G$  from  $(J, \sigma_0, \sigma_1, \bar{\alpha})$  to  $(J', \sigma'_0, \sigma'_1, \bar{\alpha}')$ . Then  $(J', g \circ \tau, \sigma'_0, \sigma'_1, \bar{\alpha}')$  is also the quintuple of some pair in  $\tilde{\mathcal{F}}_e([e] \sqcup T)$  with core  $[e]$ , and that quintuple is  $\sim_{[e] \sqcup T}$ -equivalent to the original quintuple. Furthermore, all quintuples equivalent to the original quintuple arise in this manner.*

*Proof.* The last statement is immediate: such an equivalent quintuple comes from an equivalent pair  $(\sigma', \alpha') \sim_{[e] \sqcup T} (\sigma, \alpha)$ , and for the arrow we can take  $g = g((\sigma', \alpha'), (\sigma, \alpha))$ .

For the first statement, let  $(\tilde{\sigma}, \tilde{\alpha}) \sim_{[e] \sqcup S} (\tilde{\sigma}', \tilde{\alpha}')$  be pairs with the given quadruples such that  $g = g((\tilde{\sigma}', \tilde{\alpha}'), (\tilde{\sigma}, \tilde{\alpha}))$ .

We will replace these equivalent pairs by smaller pairs, the first of which we can relate to  $(\sigma, \alpha)$  so as to apply Axiom (2). The details are as follows.

For each  $l \in J$  let  $m_l$  be the minimum of  $|\alpha^{-1}(l)|$  and  $|\tilde{\alpha}^{-1}(l)|$  and set  $n_l := |\sigma_1^{-1}(l)|$ . Define an injection  $\pi$  from  $[e] \sqcup \bigsqcup_{l \in J} ([m_l] \sqcup [n_l])$  to  $[e] \sqcup S$  that is the identity on  $[e]$ , sends each  $[m_l]$  injectively to  $m_l$  elements in  $\alpha^{-1}(l) \subseteq S$  and sends each  $[n_l]$  bijectively to the elements in  $\text{im}(\tilde{\sigma}) \cap S$  where  $\tau$  takes the value  $l$ . This construction ensures that  $\mathcal{F}(\pi)$  is defined at  $(\tilde{\sigma}, \tilde{\alpha})$ . It might *a priori* not be defined at  $(\tilde{\sigma}', \tilde{\alpha}')$ , because  $\text{im}(\tilde{\sigma}') \cap S$  might not be contained in  $\text{im}(\pi)$ . But if it is not, then using Lemma 2.5.7.5 we can replace  $(\tilde{\sigma}', \tilde{\alpha}')$  by a  $\sim_S$ -equivalent pair, with the same quadruple and with the same bijection  $g : J \rightarrow J'$ , such that  $\mathcal{F}(\pi)$  is defined at  $(\tilde{\sigma}', \tilde{\alpha}')$ . Now replace  $(\tilde{\sigma}, \tilde{\alpha})$  and  $(\tilde{\sigma}', \tilde{\alpha}')$  by their images under  $\mathcal{F}(\pi)$ .

The point of distinguishing  $\tilde{\mathcal{F}}$  and  $\mathcal{F}''$  is that these images may not be in  $\tilde{\mathcal{F}}$ . The reason is that while  $\tilde{v} + \mathbb{Z}_{\geq 0}^I$  is closed under the map that sends a pair  $w_1, w_2$  of vectors to the componentwise minimum vector  $\min(w_1, w_2)$  defined by  $\min(w_1, w_2)(l) := \min(w_1(l), w_2(l))$ , the count vectors of pairs in  $\tilde{\mathcal{F}}$  with a fixed frequent set  $J$  may not quite be closed under componentwise minimum: the number of entries that  $\alpha$  has in the frequent set is not quite invariant under the equivalence relations  $\sim$ , although it is up to a bounded difference. This is remedied by allowing the componentwise minimum to have a count vector in the bigger set  $v + \mathbb{Z}_{\geq 0}^I$ .

Hence the new  $(\tilde{\sigma}, \tilde{\alpha})$  and  $(\tilde{\sigma}', \tilde{\alpha}')$  are in  $\mathcal{F}''([e] \sqcup \bigsqcup_{l \in J} ([m_l] \sqcup [n_l]))$  and are related by the same bijection  $g$ . Now construct similarly an injection  $\pi' :$

$[e] \sqcup \bigsqcup_{l \in J} ([m_l] \sqcup [n_l]) \rightarrow [e] \sqcup T$  which is the identity on  $[e]$ , sends  $[m_l]$  injectively into the set  $\alpha^{-1}(l)$ , and sends  $[n_l]$  bijectively to the set of elements in  $\text{im}(\sigma) \cap T$  where  $\tau$  takes the value  $l$ . Then  $(\tilde{\sigma}, \tilde{\alpha}) = \mathcal{F}(\pi')((\sigma, \alpha))$ . Now apply Axiom (2) to find that there exists a pair  $(\sigma', \alpha') \sim_{[e] \sqcup T} (\sigma, \alpha)$  such that  $\mathcal{F}(\pi')((\sigma', \alpha')) = (\tilde{\sigma}', \tilde{\alpha}')$ . The pair  $(\sigma', \alpha')$  has the required quintuple. Moreover, since the pair is  $\sim_{[e] \sqcup T}$ -equivalent to  $(\sigma, \alpha)$ , so are their quintuples.  $\square$

**Proposition 2.5.7.11.** *The finite graph  $G$  is a groupoid with objects its vertices (quadruples), arrows as in Definition 2.5.7.9, and composition maps given by composing the bijections  $g$ .*

Before proving this in general, we revisit Example 2.5.6.3.

**Example 2.5.7.12.** Consider the functor  $\mathcal{F} : \mathbf{FI} \rightarrow \mathbf{PF}$  that maps  $S$  to

$$\mathcal{F}(S) = \{(\sigma, \alpha) \mid \sigma \in \text{Hom}_{\mathbf{FI}}(S_0, S), \alpha \in [d]^{S \setminus \text{im}(\sigma)}\}$$

where  $S_0$  is a singleton and where we identify  $[d]$  with  $\mathbb{Z}/d\mathbb{Z}$  via  $a \mapsto a + d\mathbb{Z}$ . Let  $\sim_S$  be as in Example 2.5.6.3, i.e.  $(\sigma, \alpha) \sim (\sigma', \alpha')$  if and only if either the pairs are equal, or else  $\{j_0\} := \text{im}(\sigma) \neq \text{im}(\sigma') =: \{j'_0\}$  and for all  $j \in S \setminus \{j_0, j'_0\}$  we have  $\alpha'(j) - \alpha'(j_0) = \alpha(j)$  and  $\alpha(j) - \alpha(j_0) = \alpha'(j)$  in  $\mathbb{Z}/d\mathbb{Z}$ .

In this case,  $I = [d]$ , and  $v \in [d]^I$  is any vector containing sufficiently many copies of each element of  $[d]$ . Fix a pair  $(\sigma, \alpha) \in \mathcal{F}''(T)$  where  $\alpha$  has count vector in  $v + \mathbb{Z}_{\geq 0}^I$ . Set  $\{j_0\} := \text{im}(\sigma)$ . The frequent set of  $\alpha$  is  $J = I = [d]$ , and all equivalence classes in  $T$  under  $\equiv$ , except possibly for that of  $j_0$ , are large, since the fibres of  $\alpha$  are large.

Pick a  $j'_0 \neq j_0$ , let  $\sigma'$  be the map  $S_0 \rightarrow T$  with image  $\{j'_0\}$ , and define  $\alpha' : T \setminus \{j'_0\} \rightarrow [d]$  via  $\alpha'(j) = \alpha(j)$  if  $j \neq j_0, j'_0$  and  $\alpha'(j_0) = d$  (which is identified with  $0 + d\mathbb{Z}$ ). Then it follows that  $(\sigma, \alpha) \sim_T (\sigma', \alpha')$ . Since  $\alpha'$  is defined at  $j_0$  and takes the same value  $d$  at least once more (in fact, many times), we conclude that the equivalence class of  $j_0$  under  $\equiv$  is not a singleton, either. Hence the core of  $(\sigma, \alpha)$  is empty.

We can now complete the quintuple of  $(\alpha, \sigma)$ : the partially defined map  $\sigma_0$  from  $S_0$  to the core  $\emptyset$  is the only such map, the surjection  $\tau : T \rightarrow J$  equals  $\alpha$  on  $T \setminus \{j_0\}$  and  $d$  on  $j_0$ , the map  $\sigma_1$  maps the singleton  $S_0$  to  $d$ , and  $\bar{\alpha}$  is the empty map  $\emptyset \rightarrow \emptyset$ . Note that the quadruple  $(J, \sigma_0, \sigma_1, \bar{\alpha})$  does not depend on the particular choice of  $(\sigma, \alpha)$ , so the groupoid  $G$  has a single object.

To determine all arrows in  $G$ , suppose that  $(\sigma, \alpha) \sim_T (\sigma'', \alpha'')$ , set  $\{j''_0\} := \text{im}(\sigma'')$ , and assume that  $j''_0 \neq j_0$  (otherwise we get the identity arrow). Then

for any  $j \in T'' := T \setminus \{j_0, j_0''\}$ , we have

$$\begin{aligned}\alpha''(j) - \alpha(j) &= (\alpha''(j) - \alpha''(j_0)) + (\alpha''(j_0) - \alpha(j)) \\ &= \alpha(j) + (\alpha''(j_0) - \alpha(j)) \\ &= \alpha''(j_0)\end{aligned}$$

so that  $\alpha''|_{T''}$  is obtained from  $\alpha|_{T''}$  by a coordinate-wise shift over the constant  $\alpha''(j_0) =: a$ . The map  $g_{(\sigma, \alpha), (\sigma'', \alpha'')} : [d] \rightarrow [d]$  is adding  $a$  (modulo  $d$ ). Conversely, for every choice of  $\alpha''(j_0) \in [d] = \mathbb{Z}/d\mathbb{Z}$ , there is a unique pair  $(\sigma'', \alpha'')$  equivalent to  $(\sigma, \alpha)$ . We conclude that the groupoid  $G$  is just (isomorphic to) the group  $\mathbb{Z}/d\mathbb{Z}$ .  $\diamond$

*Proof.* That  $G$  has identity arrows follows from (2.2), and that all arrows are invertible follows from (2.1) combined with (2.2). It remains to check that composition is well-defined. So let  $g_1$  be an arrow in  $G$  from a vertex  $q := (J, \sigma_0, \sigma_1, \bar{\alpha})$  to a vertex  $q' := (J', \sigma'_0, \sigma'_1, \bar{\alpha}')$ , and let  $g_2$  be an arrow in  $G$  from  $q'' := (J'', \sigma''_0, \sigma''_1, \bar{\alpha}'')$  to  $q$ . We need to show that  $g_1 \circ g_2$  is an arrow in  $G$  from  $q''$  to  $q'$ . Now the existence of the arrow  $g_2$  means that there are a finite set  $T$  and pairs  $(\sigma, \alpha), (\sigma'', \alpha'') \in \tilde{F}_e([e] \sqcup T)$  with core  $[e]$  and quadruples  $q, q''$ , respectively, such that  $g_2 = g((\sigma, \alpha), (\sigma'', \alpha''))$ .

Let  $(J, \tau, \sigma_0, \sigma_1, \bar{\alpha})$  be the quintuple of  $(\sigma, \alpha)$ . By Proposition 2.5.7.10,  $(J', g_1 \circ \tau, \sigma'_0, \sigma'_1, \bar{\alpha}')$  is the quintuple of another pair  $(\sigma', \alpha') \in \tilde{\mathcal{F}}_e([e] \sqcup T)$  with core  $[e]$  and  $(\sigma', \alpha') \simeq_{[e] \sqcup T} (\sigma, \alpha)$ . Then  $g_1 = g((\sigma', \alpha'), (\sigma, \alpha))$ , and  $g_1 \circ g_2 = g((\sigma', \alpha'), (\sigma'', \alpha''))$  is an arrow from  $q''$  to  $q'$ , as desired.  $\square$

Consider the class of quintuples arising from pairs  $(\sigma', \alpha') \in \tilde{\mathcal{F}}([e] \sqcup T), T \in \mathbf{FI}$  with core  $[e]$ . This class of quintuples comes with a natural anchor map to the objects of  $G$ , namely, the map that forgets  $\tau$ . The following says that equivalence classes of quintuples are precisely orbits under  $G$ .

**Corollary 2.5.7.13.** *The groupoid  $G$  acts on the set of quintuples of pairs in  $\tilde{\mathcal{F}}([e] \sqcup T)$  with core  $[e]$ , for  $T$  varying through  $\mathbf{FI}$ , via the anchor map that sends a quintuple to its corresponding quadruple. The orbits of this action are precisely the  $\sim_{[e] \sqcup T}$ -equivalence classes. The action of  $G$  commutes with the action of  $\text{Sym}(T)$ .*

*Proof.* The action of an arrow  $g : (J, \sigma_0, \sigma_1, \bar{\alpha}) \rightarrow (J', \sigma'_0, \sigma'_1, \bar{\alpha}')$  on a quintuple  $(J, \tau, \sigma_0, \sigma_1, \bar{\alpha})$  yields the quintuple  $(J', g \circ \tau, \sigma'_0, \sigma'_1, \bar{\alpha}')$ , and the axioms for a groupoid action are readily verified. The action commutes with  $\pi \in \text{Sym}(T)$  because  $(g \circ \tau) \circ \pi = g \circ (\tau \circ \pi)$ . That the  $G$ -orbits on quintuples coming from pairs  $(\sigma, \alpha) \in \tilde{\mathcal{F}}([e] \sqcup T)$  with core  $[e]$

are precisely the  $\sim_{[e] \sqcup T}$ -equivalence classes follows from the last statement of Proposition 2.5.7.10.  $\square$

Recall that, by Lemma 2.5.7.8, we need to count the elements of  $\tilde{\mathcal{F}}([e] \sqcup T)$  with core  $[e]$  up to  $\sim_{[e] \sqcup T}$  as well as up to the action of  $\text{Sym}([e]) \times \text{Sym}(T)$ . Modding out the action of  $\text{Sym}(T)$  is now straightforward: it just consists of replacing  $\tau : T \rightarrow J$  by its count vector  $u$  in  $\mathbb{Z}_{\geq 0}^J$ . Thus now the groupoid  $G$  acts on quintuples  $(J, u, \sigma_0, \sigma_1, \bar{\alpha})$  coming from pairs in  $\tilde{\mathcal{F}}_e([e] \sqcup T)$  with core  $[e]$ , where  $u$  is a vector in  $\mathbb{Z}_{\geq 0}^J$ . Also the group  $\text{Sym}([e])$  acts on such quintuples, indeed, it acts on the corresponding quadruples and fixes  $u$ . We need to count the quintuples up to the action of  $G$  and  $\text{Sym}([e])$ . To do so, we first observe that  $\text{Sym}([e])$  acts by automorphisms of  $G$  on the objects of  $G$ : if there is an arrow  $g : q \rightarrow q'$ , then there is also an arrow  $\pi(q) \rightarrow \pi(q')$  with the same label  $g$ , for each  $\pi \in \text{Sym}([e])$ . To stress that this arrow has a different source and target, we write  $\pi(g : q \rightarrow q')\pi^{-1}$  for that arrow  $\pi(q) \rightarrow \pi(q')$ .

We now combine these actions into that of a larger groupoid  $\tilde{G}$  with the same ground set as  $G$  and with arrows  $q \rightarrow q''$  all pairs  $(\pi, g : q \rightarrow q')$  where  $g$  is an arrow from the quadruple  $q$  to some quadruple  $q'$  and  $\pi \in \text{Sym}([e])$  maps  $q'$  to  $q''$ . The composition  $(\pi', g') \circ (\pi, g)$ , where  $g' : q'' \rightarrow q'''$  and  $\pi'(q''') = q''''$ , is defined as  $(\pi' \circ \pi, \pi^{-1}(g' : q'' \rightarrow q''')\pi \circ g)$ . It is straightforward to see that  $\tilde{G}$  is, indeed, a finite groupoid acting on quintuples. Now we are left to count orbits of  $\tilde{G}$  on quintuples.

**Proposition 2.5.7.14.** *There exists a quasipolynomial  $f$  such that, for  $n \gg 0$ , the number of orbits of  $\tilde{G}$  on quintuples  $(J, u \in \mathbb{Z}_{\geq 0}^J, \sigma_0, \sigma_1, \bar{\alpha})$  arising from pairs  $(\sigma, \alpha) \in \tilde{\mathcal{F}}_e([e] \sqcup [n])$  with core  $[e]$  equals  $f(n)$ .*

*Proof.* Applying the orbit-counting lemma for groupoids (Lemma 2.5.5.1), it suffices to show that for each object of  $\tilde{G}$ , which is a quadruple  $q = (J, \sigma_0, \sigma_1, \bar{\alpha})$ , and for each arrow  $(\pi, g) : q \rightarrow q$  in  $\tilde{G}$ , the number of fixed points of  $(\pi, g)$  on quintuples with the given quadruple  $q$  is quasipolynomial for  $n \gg 0$ . Such a quintuple is fixed if and only if the count vector  $u \in \mathbb{Z}_{\geq 0}^J$  is fixed by  $g$ —indeed,  $\pi$  acts trivially on the count vector.

Now we figure out the structure of the set of count vectors arising from such quintuples. First, if  $u$  is the count vector of a quintuple (with the fixed quadruple  $q$ ) arising from  $(\sigma, \alpha) \in \tilde{\mathcal{F}}_e([e] \sqcup [n])$ , and if  $l \in J$  and  $i \in [n]$  such that  $\alpha(i) = l$ , then  $u - e_l$  is the count vector of the quintuple arising from the pair  $(\tilde{\sigma}, \tilde{\alpha}) := \mathcal{F}(\pi)((\sigma, \alpha))$ , where  $\pi : [e] \sqcup [n-1] \rightarrow [e] \sqcup [n]$  is the identity on  $[e]$  and increasing from  $[n-1] \rightarrow [n]$  and does not hit  $i$ . This suggest that the

set of count vectors that we are considering is downward closed. However, it may be that  $(\tilde{\sigma}, \tilde{\alpha})$  is in  $\mathcal{F}''([e] \sqcup [n-1]) \setminus \tilde{\mathcal{F}}([e] \sqcup [n-1])$ .

On the other hand, if we had started with  $(\sigma, \alpha) \in \mathcal{F}''([e] \sqcup [n]) \setminus \tilde{\mathcal{F}}([e] \sqcup [n])$  and applied such an element  $\mathcal{F}(\pi)$  to it, the result would not have been an element of  $\tilde{\mathcal{F}}([e] \sqcup [n-1])$ .

This shows that the set of relevant count vectors is the difference  $N \setminus N'$ , where  $N$  and  $N'$  are downward closed sets in  $\mathbb{Z}_{\geq 0}^J$ . Hence, using the Stanley decomposition as in the proof of Proposition 2.5.3.1, the fixed points  $u$  that we are counting are the lattice points in a finite disjoint union of rational cones, each given as the intersection of a linear space (the eigenspace of  $g$  in  $\mathbb{R}^J$  with eigenvalue 1) and a finite union of sets of the form  $u + \mathbb{Z}_{\geq 0}^{J'}$  with  $J' \subseteq J$ . The number of such  $u$  is a quasipolynomial in  $n = \sum_l u(l)$  for  $n \gg 0$  for each of the finitely many choices of  $(\pi, g)$ , hence so is their sum.  $\square$

This completes the proof of Theorem 2.5.7.1.  $\square$

## 2.5.8 Pre-component functors

We will see that, if  $X$  is a width-one **FI**-scheme of finite type over a Noetherian ring  $K$ , then we can cover  $X(S)$  by means of closed, irreducible subsets parameterised by a finite number of model functors evaluated at  $S$ . Then, of course, the irreducible components of  $X(S)$  are among these. But to ensure that we are neither double-counting components of  $X(S)$  nor counting closed subsets that are strictly contained in components, we need to keep track of the inclusions among these closed subsets. At the combinatorial level, this is done using compatible quasi-orders.

**Definition 2.5.8.1.** Let  $a \in \mathbb{Z}_{\geq 0}$ , for each  $b \in [a]$  let  $k_b \in \mathbb{Z}_{\geq 0}$  and let  $S \mapsto \mathcal{F}_b(S)/\sim_{b,S}$  be a model functor, where  $\mathcal{F}_b(S)$  is the set of pairs  $(\sigma, \alpha)$  with  $\sigma : S_{0,b} \rightarrow S$  and  $\alpha \in \mathcal{E}_b(S) \subseteq [k_b]^S$ .

Suppose that we are given, for each  $S \in \mathbf{FI}$ , a quasi-order  $\preceq_S$  (i.e., a reflexive and transitive relation) on the disjoint union

$$\mathcal{F}(S) := \bigsqcup_{b=1}^a \mathcal{F}_b(S).$$

This collection of quasi-orders is called *compatible* if the following properties are satisfied:

**Compatibility (1)** whenever  $\mathcal{F}_b(S) \ni (\sigma, \alpha) \preceq_S (\sigma', \alpha') \in \mathcal{F}_{b'}(S)$ , we have  $b \leq b'$ ;

**Compatibility (2)** the restriction of  $\preceq_S$  to each  $\mathcal{F}_b(S)$  equals the equivalence relation  $\sim_{b,S}$ ; and

**Compatibility (3)** for all  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  and  $(\sigma, \alpha) \in \mathcal{F}_b(T)$  and  $(\sigma', \alpha') \in \mathcal{F}_{b'}(T)$  with  $\text{im}(\pi) \supseteq \text{im}(\sigma) \cup \text{im}(\sigma')$  we have

$$(\sigma, \alpha) \preceq_T (\sigma', \alpha') \Rightarrow \mathcal{F}_b(\pi)((\sigma, \alpha)) \preceq_S \mathcal{F}_{b'}(\pi)((\sigma', \alpha')). \quad \diamond$$

Note that the condition that  $\text{im}(\pi)$  contains both  $\text{im}(\sigma)$  and  $\text{im}(\sigma')$  implies that  $\mathcal{F}_b(\pi)((\sigma, \alpha))$  and  $\mathcal{F}_{b'}(\pi)((\sigma', \alpha'))$  are both defined.

**Definition 2.5.8.2.** In the setting of the previous definition, we introduce an equivalence relation  $\sim_S$  on  $\mathcal{F}(S)$  by  $(\sigma, \alpha) \sim_S (\sigma', \alpha')$  if both  $(\sigma, \alpha) \preceq_S (\sigma', \alpha')$  and  $(\sigma', \alpha') \preceq_S (\sigma, \alpha)$ . Note that, by the first and second axioms for pre-component functors,  $(\sigma, \alpha) \in \mathcal{F}_b(S)$  is  $\sim_S$ -equivalent to  $(\sigma', \alpha') \in \mathcal{F}_{b'}(S)$  if and only if  $b = b'$  and  $(\sigma, \alpha) \sim_{b,S} (\sigma', \alpha')$ .

The quasi-order  $\preceq_S$  induces a partial order on the set

$$\mathcal{F}(S)/\sim_S = \bigsqcup_b (\mathcal{F}_b(S)/\sim_{b,S})$$

of equivalence classes. We define the functor  $\mathcal{C} : \mathbf{FI} \rightarrow \mathbf{PF}$  on objects by

$$\mathcal{C}(S) := \{\text{the maximal elements of } \mathcal{F}(S)/\sim_S\}$$

and on a morphism  $\pi : S \rightarrow T$  as follows. Let  $c \in \mathcal{C}(T)$  be a maximal equivalence class. If  $c$  contains some element  $(\sigma, \alpha) \in \mathcal{F}_b(T)$  at which  $\mathcal{F}_b(\pi)$  is defined (i.e., with  $\text{im}(\sigma) \subseteq \text{im}(\pi)$ ), and if, moreover,  $c' := [\mathcal{F}_b(\pi)((\sigma, \alpha))]\sim_S$  is also maximal in  $\mathcal{F}(S)/\sim_S$ , then we set  $\mathcal{C}(\pi)(c) := c' \in \mathcal{C}(S)$ . By the axioms for compatible quasi-orders, this is independent of the choice of the representative  $(\sigma, \alpha)$  of  $c$ —subject to the requirement that  $\mathcal{F}_i(\pi)$  be defined at that representative.

A functor  $\mathcal{C} : \mathbf{FI} \rightarrow \mathbf{PF}$  obtained in this manner is called a *pre-component functor*.  $\diamond$

## 2.5.9 The final quasipolynomial count

We retain the notation from §2.5.8: for each  $b \in [a]$  we have a model functor  $S \mapsto \mathcal{F}_b(S)/\sim_{b,S}$ , and on the disjoint unions  $\mathcal{F}(S) := \bigsqcup_b \mathcal{F}_b(S)$  (for  $S \in \mathbf{FI}$ ) we have compatible quasi-orders  $\preceq_S$ .

**Theorem 2.5.9.1.** *Let  $\mathcal{C}$  be the pre-component functor corresponding to the data above. Then the number of  $\text{Sym}(S)$ -orbits on  $\mathcal{C}(S)$  is a quasipolynomial in  $|S|$  for all  $S$  with  $|S| \gg 0$ .*

The proof requires the proof technique used in §2.5.7, and takes up the rest of this subsection.

*Proof.* By Theorem 2.5.7.1, we know that the number of  $\text{Sym}(S)$ -orbits on the disjoint union of model functors  $\bigsqcup_b (\mathcal{F}_b(S)/\sim_{b,S})$  is eventually quasipolynomial in  $|S|$ . From this disjoint union we will remove, for each  $b \in [a]$ , the  $\sim_{b,S}$ -classes of pairs  $(\sigma, \alpha)$  for which there exists a  $b' > b$  and a  $(\sigma', \alpha') \in \mathcal{F}_{b'}(S)$  with  $(\sigma, \alpha) \preceq_S (\sigma', \alpha')$ . We will see that, for a fixed  $b \in [a]$ , the  $\text{Sym}(S)$ -orbits on these deletions are also counted by a quasipolynomial.

To this end, fix  $b \in [a]$  and let  $M \subseteq \mathbb{Z}_{\geq 0}^{k_b}$  be the downward closed subset corresponding to  $\mathcal{E}_b$ . From §2.5.7 we recall that, to prove that the number of  $\text{Sym}(S)$ -orbits on  $\mathcal{F}_b(S)/\sim_{b,S}$  is quasipolynomial in  $|S|$  for  $|S| \gg 0$ , we performed induction on  $M$  using Dickson's lemma. We do the same here.

**Lemma 2.5.9.2.** *Fix  $b \in [a]$ . For  $|S| \gg 0$ , the number of  $\text{Sym}(S)$ -orbits on pairs  $(\sigma, \alpha) \in \mathcal{F}_b(S)$  that are  $\preceq$  some pair  $(\sigma', \alpha') \in \mathcal{F}_{b'}(S)$  for some  $b' > b$  is quasipolynomial in  $|S|$ .*

*Proof.* We construct the sub-functor  $\mathcal{E}'_b \subseteq \mathcal{E}_b$  and the corresponding sub-functor  $\mathcal{F}'_b \subseteq \mathcal{F}_b$  as in §2.5.7, as well as the difference  $\tilde{\mathcal{F}}_b(S) := \mathcal{F}_b(S) \setminus \mathcal{F}'_b(S)$ . By induction on  $M$  using Dickson's lemma, we may assume that the lemma holds for the sub-model functor  $S \mapsto \mathcal{F}'_b(S)/\sim_{b,S}$ . On the other hand, in §2.5.7 we counted the  $\text{Sym}(S)$ -orbits on  $\tilde{\mathcal{F}}_b(S)/\sim_{b,S}$ , for all  $S$ , as a finite sum of orbit counts of certain finite groupoids. More precisely, for finitely many values of a nonnegative integer  $e$ , we there considered the pairs  $(\alpha, \sigma) \in \tilde{\mathcal{F}}_b([e] \sqcup S)$  with core equal to  $[e]$ . Each of these pairs gives rise to a quintuple  $(J, u \in \mathbb{Z}_{\geq 0}^J, \sigma_0, \sigma_1, \bar{\alpha})$  where  $J \subseteq [k_b]$  is the frequent set of  $\alpha$ , and we showed that the  $\text{Sym}(S)$ -orbits on  $\sim_{b,[e] \sqcup S}$ -equivalence classes of such pairs  $(\alpha, \sigma)$  are in bijection with the orbits of a certain groupoid on the sum- $|S|$  level set of a difference  $N \setminus N'$  where  $N, N' \subseteq \mathbb{Z}_{\geq 0}^{k_b}$  are downward closed; this difference is where the count vector  $u \in \mathbb{Z}_{\geq 0}^J \subseteq \mathbb{Z}_{\geq 0}^{k_b}$  lives.

Now suppose that  $(\sigma, \alpha) \preceq_{[e] \sqcup S} (\sigma', \alpha')$  for some  $(\sigma', \alpha') \in \mathcal{F}_{b'}([e] \sqcup S)$  with  $b' > b$ , and let  $i \in S$  be such that  $l := \alpha(i) \in J$ . If  $i$  happens to be in  $\text{im}(\sigma) \cup \text{im}(\sigma')$ , then choose  $j \in S \setminus (\text{im}(\sigma) \cup \text{im}(\sigma'))$  such that  $\alpha(i) = \alpha(j) = l$ —this can be done since  $l$  appears frequently in  $\alpha$  (although, to be precise, when choosing the vector  $\tilde{v}$  in the construction of  $\tilde{F}_b$ , we now have to make sure that its large entries values are large even compared to the finitely many numbers  $k_{b'}, b' \in [a]$ ).

By Lemma 2.5.7.5,  $(\sigma, \alpha) \sim_{b,[e] \sqcup S} (\tilde{\sigma}, \tilde{\alpha}) := \mathcal{F}_b((i \ j))((\sigma, \alpha))$ , and the axioms for compatible quasi-orders imply that  $(\tilde{\sigma}, \tilde{\alpha})$

$\preceq_{[e] \sqcup S} (\tilde{\sigma}', \tilde{\alpha}') := \mathcal{F}_{b'}((i \ j))((\sigma', \alpha'))$ . Moreover, we have achieved that  $i \in S \setminus (\text{im}(\tilde{\sigma}) \cup \text{im}(\tilde{\sigma}'))$ .

Now let  $\pi : [e] \sqcup S \setminus \{i\} \rightarrow [e] \sqcup S$  be the inclusion map. Then  $\mathcal{F}_b(\pi)$  and  $\mathcal{F}_{b'}(\pi)$  are defined at  $(\tilde{\sigma}, \tilde{\alpha})$  and  $(\tilde{\sigma}', \tilde{\alpha}')$ , respectively, and we have

$$\mathcal{F}_b(\pi)((\tilde{\sigma}, \tilde{\alpha})) \preceq_{[e] \sqcup S \setminus \{i\}} \mathcal{F}_{b'}(\pi)((\tilde{\sigma}', \tilde{\alpha}')).$$

The count vector in  $\mathbb{Z}_{\geq 0}^J$  of the left-hand side is just  $u - e_i$ .

We conclude that the set of count vectors of quintuples corresponding to pairs in  $\tilde{\mathcal{F}}_{b,e}([e] \sqcup S)$  with core  $[e]$  whose  $\sim_{[e] \sqcup S}$ -equivalence class is not maximal in  $\tilde{F}([e] \sqcup S) / \sim_{[e] \sqcup S}$  is downward-closed, or more precisely the intersection of a downward-closed set with the earlier difference  $N \setminus N'$  of downward-closed sets, hence again a difference of downward-closed subsets of  $\mathbb{Z}_{\geq 0}^{k_b}$ . The same orbit-counting argument with groupoids as in §2.5.7 applies, and shows that the number of  $\text{Sym}(S)$ -orbits on pairs in  $\tilde{\mathcal{F}}(S)$  that are not maximal in  $\preceq$  is a quasipolynomial.  $\square$

This concludes the proof of Theorem 2.5.9.1.  $\square$

## 2.5.10 Component functors “are” pre-component functors

We are now ready to establish our main result on component functors of width-one  $\mathbf{FI}^{\text{op}}$ -schemes.

**Theorem 2.5.10.1.** *Let  $X$  be a reduced and nice width-one affine  $\mathbf{FI}^{\text{op}}$ -scheme of finite type over a Noetherian ring  $K$ . Then there exists a pre-component functor  $\mathcal{C}$  and a morphism  $\varphi : \mathcal{C} \rightarrow \mathcal{C}_X$  such that  $\varphi$  is an isomorphism at the level of species.*

More precisely, there exist an  $a \in \mathbb{Z}_{\geq 0}$ , nonnegative integers  $k_b \in \mathbb{Z}_{\geq 0}$  for  $b \in [a]$ , model functors  $S \mapsto \mathcal{F}_b(S) / \sim_{b,S}$ , a collection of compatible quasi-orders  $\preceq_S$  on  $\bigsqcup_b \mathcal{F}_b(S)$  for each  $S \in \mathbf{FI}$ , and maps  $\varphi_b$  that assign to any  $(\sigma, \alpha) \in \mathcal{F}_b(S)$  an irreducible, locally closed subset  $\varphi_b((\sigma, \alpha))$  of  $X(S)$  such that, for all  $S$  and  $T$ ,

1.  $X(S)$  is the union of the sets  $\varphi_b((\sigma, \alpha))$  for  $b \in [a]$  and  $(\sigma, \alpha) \in \mathcal{F}_b(S)$ ;
2. two locally closed sets  $\varphi_b((\sigma, \alpha))$  and  $\varphi_{b'}((\sigma', \alpha'))$  with  $(\sigma, \alpha) \in \mathcal{F}_b(S)$  and  $(\sigma', \alpha') \in \mathcal{F}_{b'}(S)$  are either equal or disjoint;

3.  $\mathcal{F}_b(S) \ni (\sigma, \alpha) \preceq_S (\sigma', \alpha') \in \mathcal{F}_{b'}(S)$  holds if and only if  $\varphi_b((\sigma, \alpha))$  is contained in the Zariski closure of  $\varphi_{b'}((\sigma', \alpha'))$ ; and
4. if  $(\sigma, \alpha) \in \mathcal{F}_b(T)$  and  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  are such that  $\text{im}(\pi) \supseteq \text{im}(\sigma)$ , so that  $(\sigma', \alpha') := \mathcal{F}_b(\pi)((\sigma, \alpha))$  is defined, then the map  $X(\pi)$  maps  $\varphi_b((\sigma, \alpha))$  dominantly into  $\varphi_b((\sigma', \alpha'))$ .

The pre-component functor  $\mathcal{C}$  then assigns to  $S$  the equivalence classes of the maximal elements of  $\preceq_S$ , and the morphism  $\varphi$  is given by restricting the  $\varphi_b$  to these maximal elements and taking the Zariski closure.

The proof of this theorem takes up the rest of this subsection.

*Proof.* Let  $X = \text{Spec}(B)$  be a width-one  $\mathbf{FI}^{\text{op}}$ -scheme of finite type over a Noetherian ring  $K$ . We assume that  $X$  is reduced and nice. By the Shift Theorem and Proposition 2.3.3.1 there exist  $S_0 \in \mathbf{FI}$  and  $h \in B(S_0)$  such that  $X' := \text{Sh}_{S_0} X[1/h] = \text{Spec}(B')$  is of product type in the sense of Definition 2.3.3.2. In particular,  $B'_0 = B(S_0)[1/h]$  is a domain,  $X'$  is isomorphic to  $S \mapsto Z^S$  where  $Z := X'([1])$ , and for each  $S \in \mathbf{FI}$ , each irreducible component of  $Z^S$  maps dominantly into  $\text{Spec}(B'_0)$ .

For an  $S \in \mathbf{FI}$ , let  $Z(S)$  be the open subset of  $X(S)$  defined by

$$Z(S) := \{p \in X(S) \mid \exists \sigma : S_0 \rightarrow S : (\sigma h)(p) \neq 0\}.$$

Let  $Y(S) := X(S) \setminus Z(S)$ . Note that  $Y \subsetneq X$  is a proper closed  $\mathbf{FI}^{\text{op}}$ -subscheme of  $X(S)$ , but  $Z(S)$  is not (quite) functorial in  $S$ : for  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  and  $p \in Z(T)$  it might happen that  $X(\pi)(p)$  lies in  $Y(S)$  rather than in  $Z(S)$ .

However, if  $\sigma \in \text{Hom}_{\mathbf{FI}}(S_0, T)$  and  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  satisfy  $\text{im}(\sigma) \subseteq \text{im}(\pi)$ , then  $X(\pi)$  maps the points of  $Z(T)$  where  $\sigma(h)$  is nonzero to points of  $Z(S)$  where  $(\pi^{-1}\sigma)(h)$  is nonzero. Consequently, in spite of the fact that  $Z$  is not an  $\mathbf{FI}^{\text{op}}$ -scheme, as in Definition 2.4.2.1 we associate to  $Z$  the component functor  $\mathcal{C}_Z : \mathbf{FI} \rightarrow \mathbf{PF}$  that assigns to  $S$  the set of irreducible components of  $Z(S)$  and to  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  the partially defined map  $\mathcal{C}_Z(\pi) : \mathcal{C}_Z(T) \rightarrow \mathcal{C}_Z(S)$  that at a component  $c \in \mathcal{C}_Z(T)$  is defined and takes the value  $c' \in \mathcal{C}_Z(S)$  if and only if  $X(\pi)$  maps  $c$  dominantly into  $c'$ .

Now  $\mathcal{C}_X(S)$  is the union of  $\mathcal{C}_Z(S)$  and the components in  $\mathcal{C}_Y(S)$  that are not contained in the closure in  $X(S)$  of any component of  $Z(S)$ .

By Noetherian induction using Theorem 1.4.6.3, we may assume that Theorem 2.5.10.1 holds for  $Y \subsetneq X$ , that is, there exists a morphism  $\varphi$  from a pre-component functor  $\mathcal{C}_1$  to  $\mathcal{C}_Y$  that is an isomorphism at the level of species, and  $\mathcal{C}_1$  is constructed from model functors  $S \mapsto \mathcal{F}_b(S)/\sim_{b,S}, b \in [a]$  and a collection of compatible quasi-orders  $\preceq_S, S \in \mathbf{FI}$ , while  $\varphi$  is constructed from

maps  $\varphi_b$  that assign (disjoint or equal) locally closed subsets in  $Y(S)$  to pairs  $(\sigma, \alpha) \in \mathcal{F}_b(S)$ ,  $b \in [a]$ .

We now construct a model functor  $\mathcal{C}_2$  and a morphism  $\mathcal{C}_2 \rightarrow \mathcal{C}_Z$  that is an isomorphism at the level of species.

Let  $L$  be the fraction field of  $B'_0$  and let  $X'_L$  be the base change of  $X'$  to  $L$ . Since every irreducible component of  $X'(S)$  maps dominantly into  $\text{Spec}(B'_0)$ , the morphism  $\mathcal{C}_{X'_L} \rightarrow \mathcal{C}_{X'}$  is an isomorphism. So for each  $S \in \mathbf{FI}$ , the irreducible components of  $X'_L(S)$  are in bijection with the irreducible components of  $X(S_0 \sqcup S)[1/h] \subseteq Z(S_0 \sqcup S)$ . Consequently, for each injection  $\sigma : S_0 \rightarrow S$  we now have an injective map

$$\{\text{irreducible components of } X'_L(S \setminus \text{im}(\sigma))\} \rightarrow \mathcal{C}_Z(S), \quad (2.3)$$

$$c \mapsto X(\tilde{\sigma})(c)$$

where  $\tilde{\sigma} : S \rightarrow S_0 \sqcup (S \setminus \text{im}(\sigma))$  is the bijection that equals  $\sigma^{-1}$  on  $\text{im}(\sigma)$  and the identity on  $S \setminus \text{im}(\sigma)$ . The image of this injective map consists precisely of the irreducible components of  $Z(S)$  on which  $\sigma(h)$  is not identically zero. As we vary  $\sigma$ , we thus obtain all irreducible components of  $Z(S)$ , indeed typically multiple times.

Let  $\bar{L}$  be a separable closure of  $L$ , and let  $X'_{\bar{L}}$  be the base change of  $X'$  to  $\bar{L}$ . Now we have a surjective morphism  $\mathcal{C}_{X'_{\bar{L}}} \rightarrow \mathcal{C}_{X'_L}$  whose fibres are Galois orbits. More precisely, let  $C_1, \dots, C_k$  be the irreducible components of  $X'_{\bar{L}}([1]) = Z_{\bar{L}}$ , and let  $G$  be the image of the Galois group  $\text{Gal}(\bar{L}/L)$  in  $\text{Sym}([k])$  through its action on the components  $C_1, \dots, C_k$ . Then, as we have seen in §2.5.4,  $\mathcal{C}_{X'_{\bar{L}}}$  is isomorphic to the functor  $S \mapsto \mathcal{E}(S) := [k]^S$ . Furthermore,  $\mathcal{C}_{X'} \cong \mathcal{C}_{X'_L}$  is isomorphic to the elementary model functor  $S \mapsto \mathcal{E}(S)/G$ .

Now we construct the functor  $\mathcal{F} : \mathbf{FI} \rightarrow \mathbf{PF}$  by

$$\mathcal{F}(S) := \{(\sigma, \alpha) \mid \sigma : S_0 \rightarrow S, \alpha \in \mathcal{E}(S \setminus \text{im}(\sigma))\}.$$

Then, by the above, we have a surjective morphism

$$\Psi : \mathcal{F} \rightarrow \mathcal{C}_Z;$$

concretely,  $\Psi(S)$  takes  $(\sigma, \alpha) \in \mathcal{F}(S)$ , computes the component of  $X'_L(S \setminus \text{im}(\sigma))$  corresponding to  $\alpha$ , its image in  $\mathcal{C}_{X'_L}(S \setminus \text{im}(\sigma))$  (modding out the Galois group), and then applies the map (2.3). To simplify notation, we will write  $\Psi((\sigma, \alpha))$  instead of  $\Psi(S)((\sigma, \alpha))$ .

This surjection is by no means a bijection: even ignoring the Galois groups for a moment, on the left we have pairs of a component in  $Z(S)$  and a specified

$\sigma : S_0 \rightarrow S$  such that  $\sigma(h)$  is not identically zero on that component; and on the right we just have components of  $Z(S)$ . A single component of  $Z(S)$  may admit many different such maps  $\sigma$ . We therefore introduce an equivalence relation  $\sim_S$  on  $\mathcal{F}(S)$  by  $(\sigma, \alpha) \sim_S (\sigma', \alpha') :\Leftrightarrow \Psi((\sigma, \alpha)) = \Psi((\sigma', \alpha'))$ . In text we will sometimes oppress the  $S$  and say that  $(\sigma, \alpha)$  is *equivalent* to  $(\sigma', \alpha')$ .

We will prove that the equivalence relation  $\sim_S$  satisfies the axioms in the definition of a model functor (§2.5.6).

**Lemma 2.5.10.2.** *Suppose that  $\mathcal{F}(T) \ni (\sigma, \alpha) \sim_T (\sigma', \alpha') \in \mathcal{F}(T)$  and let  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  with  $\text{im}(\pi) \supseteq \text{im}(\sigma) \cup \text{im}(\sigma')$ . Then*

$$\mathcal{F}(\pi)((\sigma, \alpha)) \sim_S \mathcal{F}(\pi)((\sigma', \alpha'))$$

*holds.*

*Proof.* The assumptions assert that  $\Psi((\sigma, \alpha)) = \Psi((\sigma', \alpha'))$  and that  $\mathcal{C}_Z(\pi)$  is defined at this component of  $Z(T)$ . By construction, we have

$$\begin{aligned} \Psi(\mathcal{F}(\pi)((\sigma, \alpha))) &= \mathcal{C}_Z(\pi)(\Psi((\sigma, \alpha))) \\ &= \mathcal{C}_Z(\pi)(\Psi((\sigma', \alpha'))) \\ &= \Psi(\mathcal{F}(\pi)((\sigma', \alpha'))), \end{aligned}$$

so that  $\mathcal{F}(\pi)((\sigma, \alpha)) \sim_S \mathcal{F}(\pi)((\sigma', \alpha'))$ , as desired.  $\square$

This lemma establishes Axiom (1) for the equivalence relations  $\sim_S$ . We continue with Axiom (2).

**Lemma 2.5.10.3.** *Let  $(\sigma, \alpha) \in \mathcal{F}(T)$  and let  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$  satisfy  $\text{im}(\pi) \supseteq \text{im}(\sigma)$ . Assume that  $\mathcal{F}(\pi)((\sigma, \alpha)) \sim_S (\sigma'', \alpha'') \in \mathcal{F}(S)$ . Set  $\sigma' := \pi \circ \sigma''$ . Then there exists an  $\alpha' \in \mathcal{E}(T)$  such that  $(\sigma', \alpha')$  lies in  $\mathcal{F}(T)$ , is  $\sim_T$ -equivalent to  $(\sigma, \alpha)$ , and satisfies  $\mathcal{F}(\pi)((\sigma', \alpha')) = (\sigma'', \alpha'')$ .*

*Proof.* Let  $C := \Psi((\sigma, \alpha))$  be the corresponding component of  $Z(T)$  and  $D$  its image in  $Z(S)$  under  $X(\pi)$ . The fact that  $\mathcal{F}(\pi)((\sigma, \alpha)) \sim_S (\sigma'', \alpha'') \in \mathcal{F}(S)$  implies that  $\sigma''(h)$  is not identically zero on  $D$ . Then  $\sigma'(h) = \pi(\sigma''(h))$  is not identically zero on  $C$ , and hence  $C = \Psi((\sigma', \tilde{\alpha}'))$  for a suitable  $\tilde{\alpha}' \in \mathcal{E}(T)$ . Then  $\mathcal{F}(\pi)((\sigma', \tilde{\alpha}')) = (\sigma'', \tilde{\alpha}'')$  for  $\tilde{\alpha}'' := \tilde{\alpha}' \circ \pi|_{S \setminus \text{im}(\sigma'')}$  and we have  $(\sigma'', \tilde{\alpha}'') \sim_S (\sigma'', \alpha'')$ . Since the first component  $\sigma''$  of these pairs is the same, it follows that  $\alpha'' = g\tilde{\alpha}''$  for some  $g \in G$ . Now set  $\alpha' := g\tilde{\alpha}'$ . As the action of  $G$  commutes with  $\mathcal{F}(\pi)$ , we have  $(\sigma'', \alpha'') = \mathcal{F}(\pi)((\sigma', \alpha'))$ , and since  $\Psi((\sigma', \alpha')) = C$ , we have  $(\sigma', \alpha') \sim_T (\sigma, \alpha)$ .  $\square$

Next, we establish Axiom (3) for model functors.

**Lemma 2.5.10.4.** *Assume that  $(\sigma, \alpha) \in \mathcal{F}(T)$  is  $\sim_T$ -equivalent to  $(\sigma', \alpha') \in \mathcal{F}(T)$ . Then for all  $i, j \in T \setminus (\text{im}(\sigma) \cup \text{im}(\sigma'))$  we have*

$$\alpha(i) = \alpha(j) \Leftrightarrow \alpha'(i) = \alpha'(j).$$

*Proof.* Set  $C := \Psi((\sigma, \alpha)) = \Psi((\sigma', \alpha'))$ , an irreducible component of  $Z(T)$ . Set  $D := X(\text{im}(\sigma))[1/(\sigma h)]$ , and note that the inclusion  $\text{im}(\sigma) \rightarrow T$  yields a dominant morphism  $C \rightarrow D$ . Similarly, the inclusion  $\text{im}(\sigma') \rightarrow T$  yields a dominant morphism  $C \rightarrow D' := X(\text{im}(\sigma'))[1/(\sigma' h)]$ .

Furthermore, let  $\tilde{D}$  the image of  $C$  in

$$X(\text{im}(\sigma) \cup \text{im}(\sigma'))[1/(\sigma h), 1/(\sigma' h)]$$

under the morphism coming from the inclusion  $\text{im}(\sigma) \cup \text{im}(\sigma') \rightarrow T$ . Note that  $X(\sigma)$  maps  $D$  isomorphically onto  $\text{Spec}(B'_0)$  and  $X(\sigma')$  maps  $D'$  isomorphically onto  $\text{Spec}(B'_0)$ , and that  $\tilde{D}$  maps dominantly into  $D$  and into  $D'$ .

Now let  $M$  be the field of rational functions on  $D$ , and  $M'$  the field of rational functions on  $D'$ . Note that  $\sigma$  and  $\sigma'$  give rise to isomorphisms from  $L$  to  $M$  and  $M'$ , respectively. We extend these isomorphisms to isomorphisms from  $\bar{L}$  to separable closures  $\bar{M}$  and  $\bar{M}'$  of  $M, M'$ . This yields isomorphisms from  $\text{Gal}(\bar{L}/L)$  to the Galois groups  $\text{Gal}(\bar{M}/M)$  and  $\text{Gal}(\bar{M}'/M')$ .

The base change  $C_{\bar{M}}$  equals

$$\bigcup_{\beta \in G\alpha} \prod_{i \in T \setminus \text{im}(\sigma)} C_{\beta_i};$$

here the product is over  $\bar{M}$  and  $C_{\beta_i}$  is regarded as a variety over  $\bar{M}$  via the isomorphism  $\bar{L} \rightarrow \bar{M}$ ; and similarly for  $(\sigma', \alpha')$ . The base change  $\tilde{D}_{\bar{M}}$  splits as a similar union of products, but now over  $\beta \in G\alpha|_{\text{im}(\sigma') \setminus \text{im}(\sigma)}$ .

Let  $\tilde{M} \supseteq M \cup M'$  be the field of rational functions of  $\tilde{D}$ , and let  $\bar{\tilde{M}} \supseteq \bar{M} \cup \bar{M}'$  be a separable closure of  $\tilde{M}$ . The components of the base change  $C_{\bar{\tilde{M}}}$  can then be computed in two different ways: by first doing a base change to  $\bar{M}$  or by first doing a base change to  $\bar{M}'$ . For the first route, we have to analyse what happens to the product over  $\bar{M}$

$$\prod_{i \in T \setminus \text{im}(\sigma)} C_{\beta_i}$$

when doing a base change to  $\bar{\tilde{M}}$ . Write this product as  $V \times_{\bar{M}} W$ , where  $V$  is the product over all  $i \in \text{im}(\sigma') \setminus \text{im}(\sigma)$ , and hence an irreducible component of  $\tilde{D}_{\bar{M}}$ ;

and  $W$  is the product over all  $i \in T \setminus (\text{im}(\sigma) \cup \text{im}(\sigma'))$ . The function field  $\tilde{M}$  of the irreducible  $M$ -scheme  $\tilde{D}$  embeds into the function field of any component of  $\tilde{D}_{\tilde{M}}$ , hence in particular into  $\tilde{M}(V)$ , and this field extension is algebraic, so that  $\tilde{M} \cong \overline{\tilde{M}(V)}$ . So the base change to  $\tilde{M}$  of the product  $V \times_{\tilde{M}} W$  is just the base change of  $W$  over the separably closed field  $\tilde{M}$  with the field extension  $\overline{\tilde{M}(V)}$  and hence still irreducible (e.g. by [Sta20, Tag 020J]).

Summarising, we obtain a bijection from the irreducible components of  $C_{\tilde{M}}$ , which are labelled by elements of  $G\alpha$ , to the irreducible components of  $C_{\tilde{M}'}^{\sim}$ , and that bijection is evidently  $\text{Sym}(T \setminus (\text{im}(\sigma) \cup \text{im}(\sigma')))$ -equivariant. As a consequence, for  $i, j \in T \setminus (\text{im}(\sigma) \cup \text{im}(\sigma'))$  we have  $\alpha(i) = \alpha(j)$  if and only if the transposition  $(i \ j)$  preserves some (and then each) component in  $C_{\tilde{M}}$ , if and only if that transposition preserves some (and then each) component in  $C_{\tilde{M}'}^{\sim}$ .

Similarly, we obtain a bijection from the irreducible components of  $C_{\tilde{M}'}$ , which are labelled by elements of  $G\alpha'$ , to the irreducible components of  $C_{\tilde{M}}^{\sim}$ , with the same remark about compatibility with the transposition  $(i \ j)$ . Combining these results, we find that for  $i, j \in T \setminus (\text{im}(\sigma) \cup \text{im}(\sigma'))$  we have  $\alpha(i) = \alpha(j)$  if and only if  $\alpha'(i) = \alpha'(j)$ .  $\square$

We have now concluded the proof that  $\mathcal{C}_2 : S \mapsto \mathcal{F}(S)/\sim_S$  is a model functor; and by construction, the map  $\mathcal{C}_2 \rightarrow \mathcal{C}_Z$  induced by  $\Psi$  is a morphism that is an isomorphism at the level of species.

Finally, we combine the pre-component functor  $\mathcal{C}_1$  (mapping onto  $\mathcal{C}_Y$ ) and the model functor  $\mathcal{C}_2$  (mapping onto  $\mathcal{C}_Z$ ) as follows. Recall that  $\mathcal{C}_1$  is constructed by taking the equivalence classes of maximal elements in  $\bigsqcup_{b \in [a]} \mathcal{F}_b(S)$ . We now set  $\mathcal{F}_{a+1} := \mathcal{F}(S)$ , and we extend  $\preceq_S$  from  $\bigsqcup_{b \in [a]} \mathcal{F}_b(S)$  to  $\bigsqcup_{b \in [a+1]} \mathcal{F}_b(S)$  by setting it equal to  $\sim_S$  on  $\mathcal{F}_{a+1}(S) = \mathcal{F}(S)$ , and for  $(\sigma, \alpha) \in \mathcal{F}_b(S)$  with  $b \leq a$  and  $(\sigma', \alpha') \in \mathcal{F}_{a+1}(S)$  we set  $(\sigma, \alpha) \preceq_S (\sigma', \alpha')$  if and only if the irreducible locally closed subset  $\varphi_b((\sigma, \alpha))$  of  $Y(S)$  is contained in the Zariski closure of the irreducible component  $\varphi_{a+1}((\sigma', \alpha')) := \Psi((\sigma', \alpha'))$  of  $Z(S)$ . Properties (1)–(4) in Theorem 2.5.10.1 are then straightforward, and so are the properties Compatibility (1) and (2) from the definition of pre-component functors. For instance, Compatibility (1) follows from the fact that no component of  $Z(S)$  can be contained in  $Y(S)$ .

Regarding Compatibility (3): for  $b = b' = a + 1$  this is just Axiom (1), verified in Lemma 2.5.10.2; and for  $b \leq a, b' = a + 1$  it follows from the simple fact that if the irreducible locally closed set  $\varphi_b((\sigma, \alpha))$  in  $Y(T)$  is contained in the Zariski closure of the component  $\Psi((\sigma', \alpha'))$  of  $Z(T)$ , then the same holds for their projections in  $Y(S)$  and  $Z(S)$ , respectively, along the map  $X(\pi)$ .

With this quasi-order, the pre-component functor  $\mathcal{C}_1$  and the model functor  $\mathcal{C}_2$  are combined into a pre-component functor  $\mathcal{C}_3$  with an obvious morphism to  $\mathcal{C}_X$  that is an isomorphism at the level of species.  $\square$

## 2.5.11 Proof of the main theorem

*Proof of Theorem 2.1.1.1.* Before the Main Theorem we introduced **FI**<sup>op</sup>-schemes slightly differently than we do in §1.4, but the two definitions are equivalent via Remark 1.4.2.10. So we may assume that  $X$  is a width-one **FI**<sup>op</sup>-scheme of finite type over a Noetherian ring  $K$  in the sense of §1.4. Furthermore, since the Main Theorem only makes a statement about the underlying topological space of  $X([n])$  for  $n \gg 0$ , we may assume that  $X$  is both nice and reduced.

By Theorem 2.5.10.1, the component functor  $\mathcal{C}_X$  of  $X$  is, at the level of species, isomorphic to a pre-component functor  $\mathcal{C}$ . In particular, for all  $S$ , the number of  $\mathrm{Sym}(S)$ -orbits on  $\mathcal{C}_X(S)$  equals the number of  $\mathrm{Sym}(S)$ -orbits on  $\mathcal{C}(S)$ . By Theorem 2.5.9.1, this number is a quasipolynomial in  $|S|$  for all sufficiently large  $S$ . This proves the Main Theorem and concludes the chapter.  $\square$

# Chapter 3

## Image closure of symmetric wide-matrix varieties

This chapter is based on the paper [DEFM22] with Jan Draisma, Rob Eggermont, and Leandro Meier.

### 3.1 Introduction

Recall from Section 1.3 that the space of  $k \times \mathbb{N}$ -matrices equipped with the natural action of  $\mathrm{Sym}(\mathbb{N})$  is topologically  $\mathrm{Sym}(\mathbb{N})$ -Noetherian but the space of  $\mathbb{N} \times \mathbb{N}$ -matrices equipped with the natural action of  $\mathrm{Sym}(\mathbb{N})$ , or even with the action of  $\mathrm{Sym}(\mathbb{N}) \times \mathrm{Sym}(\mathbb{N})$ , is not equivariantly Noetherian, and it gets only worse for higher-dimensional tensors. This is problematic because many varieties of relevance to applications, such as the  $k$ -factor model [Dra10] and hierarchical models [HS12] naturally live in matrix or tensor spaces and are preserved by (copies of) the symmetric group.

Hence it is interesting to find  $\mathrm{Sym}(\mathbb{N})$ -stable subvarieties of  $\mathbb{N} \times \mathbb{N}$ -matrices, or  $\mathbb{N} \times \cdots \times \mathbb{N}$ -tensors, that are defined by finitely many orbits of equations and  $\mathrm{Sym}(\mathbb{N})$ -Noetherian. In this chapter, we study such subvarieties that arise as image closures of  $\mathrm{Sym}(\mathbb{N})$ -equivariant polynomial maps from the space of  $k \times \mathbb{N}$ -matrices, and we show that these are always defined, set-theoretically, by finitely many  $\mathrm{Sym}(\mathbb{N})$ -orbits of equations, as well as themselves topologically  $\mathrm{Sym}(\mathbb{N})$ -Noetherian.

### 3.1.1 Main theorem

Recall from Chapter 1 that an **FI**-algebra  $A$  is *finitely generated in width at most 1* if  $A$  is generated by  $A(\emptyset)$  and  $A(\{1\})$ , and both of these are finitely generated. This is equivalent to the statement that  $A$  can be realised as a quotient (in the category of **FI**-algebras over  $K$ ) of an algebra of the form

$$S \mapsto K[x_{ij} \mid i \in [c], j \in S];$$

see 1.4.4.11. Geometrically,  $\mathrm{Spec}(A)$  is then an **FI**<sup>op</sup>-subscheme over  $K$  of the scheme that sends  $S$  to  $k \times S$ -matrices.

**Theorem 3.1.1.1** (Main Theorem). *Let  $\varphi$  be a morphism of **FI**-algebras  $B \rightarrow A$ , where  $A$  and  $B$  both are finitely generated **FI**-algebras over a Noetherian ring  $K$ . Suppose, moreover, that  $A$  is generated in width at most 1. Then the Zariski closure  $\overline{\mathrm{Im}(\varphi^*)}$  of the image of  $\mathrm{Spec}(A)$  under the dual map  $\varphi^*$  is defined, as a set, by finitely many elements in  $B$ . Equivalently, there exists a finitely generated **FI**-ideal  $I$  in  $B$  such that the radical of  $I$  is equal to the radical of the kernel  $\mathrm{Ker} \varphi$ .*

*Moreover,  $\overline{\mathrm{Im}(\varphi^*)}$  is topologically Noetherian, that is, it satisfies the descending chain condition on reduced closed subschemes.*

An immediate consequence of the Main Theorem is the following.

**Corollary 3.1.1.2.** *For any  $\mathrm{Sym}(\mathbb{N})$ -equivariant morphism  $\varphi$  of  $K$ -schemes from the scheme of  $k \times \mathbb{N}$ -matrices to the scheme of  $\mathbb{N} \times \cdots \times \mathbb{N}$ -tensors, the image closure  $\overline{\mathrm{Im}(\varphi)}$  of  $\varphi$  is defined, set-theoretically, by finitely many  $\mathrm{Sym}(\mathbb{N})$ -orbits of equations. Furthermore, it is topologically  $\mathrm{Sym}(\mathbb{N})$ -Noetherian: any descending chain*

$$\overline{\mathrm{im}(\varphi)} \supseteq X_1 \supseteq X_2 \supseteq \cdots$$

*of reduced, closed subschemes stabilizes.*

We do not know whether  $\ker \varphi$  itself is necessarily finitely generated.

**Conjecture 3.1.1.3.** *Let  $\varphi$  be the **FI**-algebra homomorphism defined in the Main Theorem 3.1.1.1. Then the **FI**-ideal  $\mathrm{Ker} \varphi$  is finitely generated.*

A particular case of the above conjecture was stated in [AH07, Conjecture 5.10] and proved in [DEKL16] when generators of the **FI**-algebra  $B$  are mapped under  $\varphi$  to monomials in  $A$ .

On the other hand, in the setting of the Main Theorem, there are examples where  $\overline{\mathrm{Im}(\varphi^*)}$  is not scheme-theoretically Noetherian, i.e., has an infinite

strictly descending chain of closed (but non-reduced)  $\mathbf{FI}^{\text{op}}$ -subschemes: in [DKK, Theorem 43] it is proved that the cone over the Grassmannian of  $2 \times S$ -spaces, with  $S \in \mathbf{FI}$ , has this bad behavior in characteristic 2 (but not in characteristic zero due to the existence of Reynolds operators, see [Dra10]).

To illustrate the strength of our Main Theorem, we explain how it immediately implies the main result in [Dra10].

**Example 3.1.1.4.** In [DSS06] the question was raised whether the ideal of polynomial relations among the entries of a matrix of the form  $\Sigma = AA^T + D$ , with  $A$  running through  $\mathbb{R}^{n \times k}$  and  $D$  running through the (positive-definite) diagonal matrices, is generated by a finite number of  $\text{Sym}([n])$ -orbits for  $n \rightarrow \infty$ . The set of all such matrices is the Gaussian  $k$ -factor model. This question remains open to this date. However, in [BD11] an affirmative answer was given for  $k = 2$ , and in [Dra10] an affirmative answer for a set-theoretical version was established for arbitrary  $k$ . The latter also follows from the Main Theorem, because the map that sends the matrix entry  $\sigma_{ij}$  to the entry  $\sum_{l=1}^k a_{il}a_{jl} + \delta_{ij} \cdot d_{ii}$  is a homomorphism from the  $\mathbf{FI}$ -algebra  $S \mapsto \mathbb{R}[\sigma_{ij} \mid i, j \in S]$  to the  $\mathbf{FI}$ -algebra  $S \mapsto \mathbb{R}[a_{il}, d_{ii} \mid l \in [k], i \in S]$ , and the latter is generated in width  $\leq 1$ .  $\diamond$

Recently our Main Theorem 3.1.1.1 has been used by Yulia Alexandr, Joe Kileel, and Bernd Sturmfels in [AKS23, Theorem 10] to prove the set-theoretic finiteness for moment varieties of conditionally independent mixture distributions on  $\mathbb{R}^n$ .

## 3.1.2 Organisation of this chapter

In Section 2 we recall elementary properties of finitely generated  $\mathbf{FI}$ -algebras. Section 3 is devoted to the reduction of the Main Theorem 3.1.1.1 to the case of free  $\mathbf{FI}$ -algebras. In Section 4 we prove our Main Theorem. To this end, we prove that the image of the map  $\varphi^*$  lies in the closed subscheme of  $\text{Spec}(B)$  defined by off-diagonal  $(l+1) \times (l+1)$ -subdeterminants of flattenings, see Lemma 3.4.2.5. This subscheme is defined by finitely many equations (Lemma 3.4.2.4) and is contained in the image of a topologically Noetherian space, see Proposition 3.4.2.10 and Lemma 3.4.2.8.

A result of potentially independent interest is the following tensor completion result: a tensor labeled by the tuples in  $S \times \cdots \times S$  in which no entry appears more than once (an *off-diagonal tensor*) and of which all the meaningful  $(l+1) \times (l+1)$ -subdeterminants of flattenings are zero, can be completed to a full tensor (including the diagonal) of some bounded rank; see Proposition 3.4.2.10.

## 3.2 Finitely generated FI-algebras

The following (class of) **FI**-algebra  $B_d$  is a building block for **FI**-algebras that are finitely generated in width  $\leq d$ .

**Definition 3.2.0.1.** For a non-negative integer  $d$ , denote by  $B_d$  the **FI**-algebra over a ring  $K$  that maps a finite set  $S$  to the  $K$ -algebra

$$B_d(S) = K[y_{i_1, i_2, \dots, i_d} : i_1, i_2, \dots, i_d \in S \text{ and } i_j \neq i_l \text{ when } j \neq l]$$

and for an injection  $\sigma : S \rightarrow T$ , the  $K$ -algebra homomorphism  $B_d(\sigma)$  is determined by  $B_d(\sigma)(y_{i_1, i_2, \dots, i_d}) := y_{\sigma(i_1), \sigma(i_2), \dots, \sigma(i_d)}$ . It is easy to see that  $B_d$  is generated by the single element  $y_{1, 2, \dots, d} \in B_d([d])$ , of width  $d$ .

**Remark 3.2.0.2.** 1. We think of the elements of  $\text{Spec } B_d$  as  $S \times S \times \dots \times S$ -tensors of which only the entries outside the big diagonal  $\{(i_1, i_2, \dots, i_d) : \exists j < l, i_j = i_l\}$  are given. We call these tensors “off-diagonal tensors”.

2. It is easy to show that the tensor product of such algebras is again finitely generated in width at most the largest of the relevant  $d$ .

3. For fixed non-negative integers  $d, k_0, k_1, k_2, \dots, k_d$ ,  $B[k_0, k_1, k_2, \dots, k_d]$  denotes the **FI**-algebra  $\bigotimes_{i=0}^d B_i^{\otimes k_i}$ , which is finitely generated in width at most  $d$ .

◇

We record the following lemmas for later use.

**Lemma 3.2.0.3.** *Let  $A$  be an **FI**-algebra. Then any homomorphism  $\varphi : B_d \rightarrow A$  of **FI**-algebras is completely determined by the image of  $y_{1, 2, \dots, d} \in B_d([d])$ . Moreover,  $B_d$  has the following universal property: if  $a \in A([d])$  is any element of  $A$ , then there exists a unique homomorphism  $\varphi : B_d \rightarrow A$  such that  $\varphi(y_{1, 2, \dots, d}) = a$ .*

*Proof.* Let  $y_{i_1, i_2, \dots, i_d}$  be a variable in  $B_d(S)$  where  $i_1, i_2, \dots, i_d \in S$ . Let  $\sigma : [d] \rightarrow S$  be the injection that sends  $j$  to  $i_j$ . Then we have

$$\varphi(y_{i_1, i_2, \dots, i_d}) = \varphi(B_d(\sigma)y_{1, 2, \dots, d}) = A(\sigma)\varphi(y_{1, 2, \dots, d}).$$

Thus the image of  $y_{1, 2, \dots, d}$  completely determines the map  $\varphi$ .

To prove the last statement, for  $i_1, \dots, i_d \in S$  distinct, define  $\varphi(y_{i_1, i_2, \dots, i_d}) := A(\sigma)a$ , where  $\sigma : [d] \rightarrow S$  is the map that sends  $j$  to  $i_j$ , and extend this to a homomorphism  $B_d(S) \rightarrow A(S)$  in the unique manner. A straightforward computation shows that this does, indeed, define an **FI**-algebra homomorphism  $B \rightarrow A$  with  $\varphi(y_{1, \dots, d}) = a$ . □

**Remark 3.2.0.4.** Lemma 3.2.0.3 implies that the **FI**-algebra  $B[k_0, k_1, k_2, \dots, k_d]$  has the same universal property when  $a$  is replaced by a tuple in  $\prod_{e=0}^d A([e])^{k_e}$ .  $\diamond$

**Lemma 3.2.0.5.** *Let  $A$  be an **FI**-algebra that is finitely generated in width at most  $d$ . Then there exists a surjective homomorphism  $\varphi : B[k_0, k_1, k_2, \dots, k_d] \rightarrow A$  for some non-negative integers  $k_0, k_1, k_2, \dots, k_d$ .*

*Proof.* Let  $k_i$  be the number of generators of width  $i$ . Assume  $h_1^{(i)}, h_2^{(i)}, \dots, h_{k_i}^{(i)} \in A([i])$  are generators of width  $i$ . Consider the unique **FI**-algebra homomorphism  $\varphi_i : B_i^{\otimes k_i} \rightarrow A$  that sends  $y_{1,2,\dots,i}^{(l)} \in B_i([i])^{\otimes k_i}$  to  $h_l^{(i)}$  for  $l = 1, 2, \dots, k_i$ . Then  $\varphi_i$  is surjective by construction. Then the tensor map

$$\bigotimes_{i=0}^d \varphi_i : B[k_0, k_1, k_2, \dots, k_d] \rightarrow A$$

is the required surjective morphism.  $\square$

**Remark 3.2.0.6.** Dually,  $\text{Spec}(A)$  embeds in a finite product of affine spaces of off-diagonal tensors.  $\diamond$

Recall from Section 2.2.3 that  $\text{Sh}_{T_0} A$  denotes the shift over a finite set  $T_0$  of an **FI**-algebra  $A$  and  $\text{Sh}_{T_0} X$  denotes the **FI**<sup>op</sup>-scheme  $\text{Spec}(\text{Sh}_{T_0} A)$ . Furthermore,  $\text{Sh}_{T_0} \varphi$  denotes the morphism  $\text{Sh}_{T_0} A \rightarrow \text{Sh}_{T_0} B$  that sends  $T$  to  $\varphi(T_0 \sqcup T)$  for a morphism  $\varphi : A \rightarrow B$  of **FI**-algebras.

**Lemma 3.2.0.7.** *For a fixed finite set  $T_0$ ,  $\text{Sh}_{T_0} B[k_0, k_1, k_2, \dots, k_d]$  is isomorphic to  $B[k'_0, k'_1, k'_2, \dots, k'_{d-1}, k_d]$  for some non-negative integers  $k'_0, k'_1, \dots, k'_{d-1}$ .*

*Proof.* We have  $\text{Sh}_{T_0}(A \otimes C) \cong (\text{Sh}_{T_0} A) \otimes (\text{Sh}_{T_0} C)$ , for any **FI**-algebras  $A$  and  $C$ . So it suffices to show that  $B_d$  is isomorphic to  $B[k'_0, k'_1, \dots, k'_{d-1}, 1]$ . Note that

$$\begin{aligned} \text{Sh}_{T_0} B_d(T) &= K[y_{i_1, i_2, \dots, i_d} : i_1, i_2, \dots, i_d \in T_0 \sqcup T \text{ and } i_l \neq i_m \text{ when } l \neq m] \\ &\cong \left( \bigotimes_{e=0}^{d-1} B_e(T)^{\otimes \binom{d}{e} \times |T_0| \times (|T_0|-1) \times \dots \times (|T_0|-d+e+1)} \right) \otimes (B_d(T)), \end{aligned}$$

where the exponent counts the number of ways of filling  $d - e$  of the  $d$  index positions with elements of  $T_0$ . Thus  $\text{Sh}_{T_0} B_d \cong B[k'_0, k'_1, k'_2, \dots, k'_{d-1}, 1]$ .  $\square$

### 3.3 Reduction to the case of free FI-algebras

This section is devoted to the reduction of our main theorem to the case of free **FI**-algebras.

#### 3.3.1 Making $B$ free

Here we introduce the following useful notations. Let  $A$  be an **FI**-algebra. By  $f \in A$  we mean that there exists  $S \in \mathbf{FI}$  such that  $f \in A(S)$ . Similarly, by  $p \in \text{Spec } A$  we mean that there exists  $T \in \mathbf{FI}$  such that  $p \in \text{Spec } A(T)$ . By  $f(p) = 0$  we mean that for all  $\sigma : S \rightarrow T$  injections,  $(\sigma f)(p) = 0$ .

**Proposition 3.3.1.1.** *Let  $\pi : C \rightarrow B$  and  $\varphi : B \rightarrow A$  be homomorphisms of FI-algebras, where  $\pi$  is surjective. If the image closure  $\overline{\text{Im}(\pi^* \circ \varphi^*)}$  of  $\pi^* \circ \varphi^* : \text{Spec } A \rightarrow \text{Spec } C$  is defined by finitely many elements in  $C$ , then the image closure  $\overline{\text{Im}(\varphi^*)}$  of  $\varphi^* : \text{Spec } A \rightarrow \text{Spec } B$  is also defined by finitely many elements in  $B$ . If, moreover,  $\overline{\text{Im}(\pi^* \circ \varphi^*)}$  is topologically Noetherian, then  $\overline{\text{Im}(\varphi^*)}$  is also topologically Noetherian.*

*Proof.* Suppose  $f_1, f_2, \dots, f_m \in C$  define the image closure  $\overline{\text{Im}(\pi^* \circ \varphi^*)}$ . We claim that their images  $\pi f_1, \pi f_2, \dots, \pi f_m \in B$  define the image closure  $\overline{\text{Im}(\varphi^*)}$ . First note that  $\overline{\text{Im}(\pi^* \circ \varphi^*)} = \pi^*(\overline{\text{Im}(\varphi^*)})$  as  $\pi^*$  is a closed embedding. Let  $p \in \overline{\text{Im}(\varphi^*)}$ , then  $\pi f_i(p) = f_i(\pi^* p) = 0$ . Thus for all  $i = 1, 2, \dots, m$  we have that  $\pi f_i$  vanishes on  $\overline{\text{Im}(\varphi^*)}$ .

Conversely, pick  $p \in \text{Spec } B$  such that  $p$  is in the closed subset defined by  $\pi f_1, \pi f_2, \dots, \pi f_m$ , then  $\pi^* p$  lies in the variety defined by  $f_1, f_2, \dots, f_m$ , that is, by our supposition,  $\pi^* p \in \overline{\text{Im}(\pi^* \circ \varphi^*)}$ . Now let  $g$  be an element of  $\text{Ker}(\varphi)$ . By the surjectivity of  $\pi$ , there exists  $g' \in C$  such that  $\pi g' = g$ , which implies that  $g' \in \text{Ker}(\varphi \circ \pi)$ . Hence  $g'$  vanishes on  $\pi^*(p)$ . So  $g(p) = \pi g'(p) = g'(\pi^*(p)) = 0$ . Hence  $p \in \overline{\text{Im}(\varphi^*)}$ .

The last statement is straightforward. Every descending chain of closed subsets in  $\overline{\text{Im}(\varphi^*)}$  maps under  $\pi^*$  to a descending chain of closed subsets in  $\overline{\text{Im}(\pi^* \circ \varphi^*)}$  because  $\pi^*$  is a closed embedding. The latter chain stabilizes, and when it does, so does the former chain. This completes the proof.  $\square$

**Remark 3.3.1.2.** Let  $B$  be an **FI**-algebra that is finitely generated in width at most  $d$  and  $\varphi : B \rightarrow A$  be a homomorphism of **FI**-algebras. Then, by Lemma 3.2.0.5, there exists a surjective map  $\pi$  from  $B[k_0, k_1, \dots, k_d]$  to  $B$ . Thus Proposition 3.3.1.1 implies that it suffices to prove the main theorem for the **FI**-algebra  $B[k_0, k_1, \dots, k_d]$  instead of  $B$ , i.e. without loss of generality, in the main theorem, we can assume that  $B$  is free.  $\diamond$

### 3.3.2 Making $A$ free, as well

**Proposition 3.3.2.1.** *Let  $\varphi : B \rightarrow A$ ,  $\psi : B \rightarrow A'$  and  $\pi : A' \rightarrow A$  be homomorphisms of finitely generated **FI**-algebras such that  $\pi \circ \psi = \varphi$ . If the closure  $\overline{\text{Im}(\psi^*)}$  of the image of  $\psi^*$  is topologically Noetherian, then the closure  $\overline{\text{Im}(\varphi^*)}$  of the image of  $\varphi^*$  is also topologically Noetherian. If, moreover,  $\overline{\text{Im}(\psi^*)}$  is defined by finitely many elements in  $B$ , then  $\overline{\text{Im}(\varphi^*)}$  is also defined by finitely many elements in  $B$ .*

*Proof.* The first statement is a straightforward implication of the fact that a closed subspace of a topologically Noetherian space is topologically Noetherian. Since  $\text{Ker}(\psi)$  is contained in  $\text{Ker}(\varphi)$ , we have

$$\overline{\text{Im}(\varphi^*)} = V_{\text{Spec } B}(\text{Ker}(\varphi)) = V_{\text{Spec } B}(\text{Ker}(\pi \circ \psi))$$

is a closed subset of  $V_{\text{Spec } B}(\text{Ker}(\psi)) = \overline{\text{Im}(\psi^*)}$ . The latter is topologically Noetherian, hence so is the former.

For the last statement, observe that  $\overline{\text{Im}(\varphi^*)}$ , being a closed subset of a topologically Noetherian space  $\overline{\text{Im}(\psi^*)}$ , is defined by finitely many elements in the coordinate ring  $K[\overline{\text{Im}(\psi^*)}] = B/I$ , where  $I = I(\overline{\text{Im}(\psi^*)})$  is an ideal generated by finitely many elements in  $B$ . This implies that  $\overline{\text{Im}(\varphi^*)}$  is defined by finitely many elements in  $B$ .  $\square$

**Remark 3.3.2.2.** Let  $A$  be an **FI**-algebra that is finitely generated in width at most 1 and let  $\varphi : B[k_0, k_1, \dots, k_d] \rightarrow A$  be a morphism. Then by Lemma 3.2.0.5 there is a surjective map  $\pi : B[k'_0, k'_1] \rightarrow A$ . By universality of  $B[k_0, k_1, \dots, k_d]$  (see Remark 3.2.0.4), there exists (not necessarily unique) map  $\psi : B[k_0, k_1, \dots, k_d] \rightarrow B[k'_0, k'_1]$  such that  $\varphi = \pi \circ \psi$ . Thus Proposition 3.3.2.1 implies that it suffices to prove the main theorem for  $B[k'_0, k'_1]$  instead of  $A$  i.e. without loss of generality, in the main theorem, we can assume that  $A$  is free.  $\diamond$

## 3.4 Proof of the main theorem

### 3.4.1 Flattening

For a more detailed description of flattening see [DK14].

**Definition 3.4.1.1.** Let  $V = \bigotimes_{j=1}^n V_j$  be a tensor product of vector spaces  $V_j$  over a field  $L$ . Then for each  $i \in [n]$ , there is a natural isomorphism

$$b_i : V \rightarrow \left( \bigotimes_{j \neq i} V_j \right) \otimes V_i.$$

For a  $t \in V$  the image  $b_i(t)$  is a 2-tensor called a *flattening* of  $t$ .  $\diamond$

A tensor has rank 1 if it can be written as a tensor product of vectors  $v_i \in V_i$ . A tensor  $t$  has rank  $l$  if  $l$  is the minimum number of rank 1 tensors that sum to  $t$ . For 2-tensors, the tensor rank is equal to the ordinary matrix rank.

**Remark 3.4.1.2.** Flattening does not increase the rank of a tensor, i.e.  $rk(b_i(t)) \leq rk(t)$ .  $\diamond$

### 3.4.2 Tensors

Define an **FI**-algebra  $C_d$  over a ring  $K$  that sends a finite set  $S$  to the  $K$ -algebra,

$$C_d(S) = K[y_{i_1, i_2, \dots, i_d} : i_1, i_2, \dots, i_d \in S],$$

and an injection  $\sigma : S \rightarrow T$  to  $C_d(\sigma) : C_d(S) \rightarrow C_d(T)$  maps  $y_{i_1, i_2, \dots, i_d}$  to  $y_{\sigma(i_1), \sigma(i_2), \dots, \sigma(i_d)}$ .

**Remark 3.4.2.1.** For a finite set  $S$ ,  $\text{Spec } C_d(S)$  is the set of  $S \times S \times \dots \times S$ -tensors, now including entries on the big diagonal.  $\diamond$

There is a natural inclusion  $\iota : B_d \rightarrow C_d$ . Dually, we have the projection map  $\iota^* : \text{Spec } C_d \rightarrow \text{Spec } B_d$ . For a tensor  $t \in \text{Spec } C_d(S)$  we think of the flattening  $b_i(t)$  as a matrix whose rows and columns are labeled by tuples in  $S^{[d] \setminus \{i\}}$  and elements in  $S$  respectively.

**Definition 3.4.2.2.** A tuple  $(x_i)_{i \in [d]}$  in  $S^d$  is called a distinct value tuple if  $x_i \neq x_j$  when  $i \neq j$ . Denote by  $DS^d$  the set of distinct value tuples in  $S^d$ .  $\diamond$

**Definition 3.4.2.3.** An off-diagonal  $l \times l$  sub-matrix of  $b_i(t)$  is a  $u_1 \times u_2$ -sub-matrix where  $u_1 \subset DS^{d-1}$ ,  $u_2 \subset S$  such that  $u_1 \times u_2 \subset DS^d$  and  $|u_1| = |u_2| = l$ . By an off-diagonal  $l \times l$ -sub-determinant of  $b_i(t)$ , we mean the determinant of an off-diagonal  $l \times l$ -sub-matrix of  $b_i(t)$ .  $\diamond$

Fix non-negative integers  $l$  and  $d$ . Let  $Z_{d,l}$  be the subset of  $\text{Spec } B_d$  defined as follows: for all  $S \in \mathbf{FI}$ ,  $Z_{d,l}(S)$  is the variety defined by all off-diagonal  $(l+1) \times (l+1)$ -sub-determinants of  $\flat_i(y)$ , for all  $i \in [d]$  and  $y \in \text{Spec } B_d(S)$ . The following lemma shows that  $Z_{d,l}$  is defined by finitely many equations in  $B_d$ .

**Lemma 3.4.2.4.** *The subset  $Z_{d,l}$  of  $\text{Spec } B_d$  is defined by finitely many equations in  $B_d$ .*

*Proof.* Let  $S$  be a finite set, let  $y \in \text{Spec } B_d(S)$  and  $i \in [d]$  be arbitrary. Let  $h$  be an arbitrary off-diagonal  $(l+1) \times (l+1)$ -subdeterminant of the flattening  $\flat_i(y)$ . The subdeterminant  $h$  is given by indices  $i_1, \dots, i_{l+1} \in S$  and distinct value tuples  $\alpha_1, \dots, \alpha_{l+1} \in DS^{d-1}$  such that none of the indices  $i_j$  occur in any of the  $\alpha_k$  and such that all the  $\alpha_k$  are distinct.

Let  $S'$  be the subset of  $S$  consisting of all the entries of the  $i_j$  and  $\alpha_k$ . Then  $|S'| \leq (d-1) \cdot (l+1) + l+1 = d \cdot (l+1)$ . Set  $n = |S'|$  and choose a bijection  $\rho: [n] \rightarrow S'$ . Define  $\alpha'_j = \rho^{-1} \circ \alpha_j$  and  $i'_j = \rho^{-1}(i_j)$  for all  $j \in [l+1]$ , where we consider the  $\alpha_j$  as maps from  $[d-1]$  to  $S'$ .

Then  $h = B_d(\rho) h'$ , where  $h'$  is the off-diagonal subdeterminant of the flattening  $\flat_i \left( (y_\beta)_{\beta \in D[n]^{d-1}} \right)$  given by columns  $i'_1, \dots, i'_{l+1}$  and rows  $\alpha'_1, \dots, \alpha'_{l+1}$ . Since  $n$  can take on at most  $d \cdot (l+1)$  values, this construction yields finitely many polynomials  $h'$ . These polynomials define  $Z_{d,l}$ .  $\square$

**Lemma 3.4.2.5.** *Let  $\varphi: B_d \rightarrow B_1^{\otimes k}$  be an  $\mathbf{FI}$ -algebra homomorphism. Then  $\overline{\text{Im}(\varphi^*)} \subset Z_{d,l}$  for some  $l$ .*

*Proof.* By Lemma 3.2.0.3, the map  $\varphi$  is determined by the image of  $y_{1,2,\dots,d} \in B_d([d])$ . First we suppose that  $y_{1,2,\dots,d} \in B_d([d])$  is mapped under  $\varphi$  to a monomial  $M = \prod_{i \in [k], j \in [d]} x_{i,j}^{\alpha_{i,j}} \in B_1^{\otimes k}([d])$ . Dually,  $a = (a_{i,j})_{i \in [k], j \in S} \in \text{Spec } B_1(S)^{\otimes k}$  is mapped to the projection (via  $\iota^*$ ) of the following tensor:

$$\begin{aligned} (a_{j_1, j_2, \dots, j_d} &= \prod_{i=1}^k a_{i, j_1}^{\alpha_{i,1}} \cdot \prod_{i=1}^k a_{i, j_2}^{\alpha_{i,2}} \cdots \prod_{i=1}^k a_{i, j_d}^{\alpha_{i,d}})_{j_1, j_2, \dots, j_d \in S} \\ &= \bigotimes_{j=1}^d \left( \prod_{i=1}^k a_{i,j}^{\alpha_{i,j}} \right)_{s \in S} \in \text{Spec } B_d(S), \end{aligned}$$

that is,  $\varphi^*(a)$  is the projection of a rank  $\leq 1$  tensor. Thus  $\text{Im}(\varphi^*)$  is contained in the projection of rank  $\leq 1$  tensors.

Now, if  $y_{1,2,\dots,d} \in B_d([d])$  is mapped under  $\varphi$  to  $f = \sum_{c=1}^l M_c$  which is a sum of, say  $l$ , monomials in the variables  $x_{i,j}$  where  $1 \leq i \leq k$  and  $1 \leq j \leq d$ , then the subadditivity of tensor rank implies that  $\text{Im}(\varphi^*)$  lies in the projection  $i^*$  of tensors having rank at most  $l$ . Since flattening does not increase rank, this implies  $\text{Im}(\varphi^*) \subset Z_{d,l}$ . Hence  $\overline{\text{Im}(\varphi^*)} \subset Z_{d,l}$ .  $\square$

**Remark 3.4.2.6.** Let  $\varphi : B[k_0, k_1, \dots, k_d] \rightarrow B_1^{\otimes k}$  be an **FI**-algebra homomorphism. By universality of  $B[k_0, k_1, \dots, k_d]$ ,  $\varphi = \bigotimes_{e=0}^d (\bigotimes_{i=1}^{k_e} \varphi_{e,i})$  where,  $\varphi_{e,i} : B_e \rightarrow B_1^{\otimes k}$  for all  $e$  and for all  $i$ , and  $\text{Im}(\varphi_{e,i}^*) \subset Z_{e,l_{e,i}}$  for some  $l_{e,i}$ . Let  $l$  be the maximum of the set  $\{l_{e,i} : 0 \leq e \leq d \text{ and } 0 \leq i \leq k_e\}$ . Then  $\text{Im}(\varphi^*) \subset \prod_{e=0}^d Z_{e,l}^{k_e}$ .  $\diamond$

For a commutative ring  $R$  with 1, and a prime ideal  $p \in \text{Spec } R$ ,  $\kappa(p)$  denotes the fraction field of the integral domain  $R/p$ .

**Lemma 3.4.2.7.** *Let  $Y$  be a subset of  $\text{Spec } C_d$ . Suppose that there exists  $N \in \mathbb{Z}_{\geq 0}$  such that for all  $S$ , for all  $p \in Y(S)$ , the tensor rank of  $p$  over the field  $\kappa(p)$  is at most  $N$ . Then there exist an **FI**-algebra  $A$  that is finitely generated in width at most 1 and a map  $\varphi_N^* : \text{Spec } A \rightarrow \text{Spec } C_d$  such that  $Y \subset \text{Im}(\varphi_N^*)$ .*

In this lemma,  $Y(S)$  is an arbitrary subset of  $\text{Spec}(C(S))$ , in such a manner that for any  $\pi \in \text{Hom}_{\mathbf{FI}}(S, T)$ , the map  $C(\pi)^*$  maps  $Y(T)$  into  $Y(S)$ .

*Proof.* Define a homomorphism of **FI**-algebras  $\varphi_N : C_d \rightarrow A = C_1^{\otimes [N] \times [d]}$  by sending  $y_{i_1, i_2, \dots, i_d}$  to  $\sum_{j=1}^N x_{j,1,i_1} x_{j,2,i_2} \cdots x_{j,d,i_d}$ . By assumption, for each  $S$  and for every

$$p = (p_{i_1, i_2, \dots, i_d})_{i_1, i_2, \dots, i_d \in S} \in Y(S),$$

there exists  $(q_{j,l,i_l})_{(j,l,i_l) \in [N] \times [d] \times S} \in \text{Spec } C_1^{\otimes [N] \times [d]}(S)(\kappa(p))$  such that

$$p_{i_1, i_2, \dots, i_d} = \sum_{j=1}^N q_{j,1,i_1} q_{j,2,i_2} \cdots q_{j,d,i_d}.$$

Hence  $Y \subset \text{Im}(\varphi_N^*)$ .  $\square$

**Lemma 3.4.2.8.** *Let  $Z$  be a subset of  $\text{Spec } B_d$ . Suppose that there exists  $N \in \mathbb{Z}_{\geq 0}$  such that for all  $S \in \mathbf{FI}$  and for all  $p \in Z(S)$  there exists  $\tilde{p} \in \text{Spec } C_d(S)$  such that the tensor rank of  $\tilde{p}$  over the field  $\kappa(\tilde{p})$  is at most  $N$  and  $\iota^*(\tilde{p}) = p$  where  $\iota : B_d \rightarrow C_d$  is the inclusion map. Then there exists a map  $\psi : B_d \rightarrow A$  such that  $Z \subset \text{Im}(\psi^*)$ , where  $A$  is an **FI**-algebra that is finitely generated in width at most 1.*

**Remark 3.4.2.9.** The condition on the off-diagonal tensor  $p$  is that it can be completed to a (full) tensor  $\tilde{p}$  of rank  $\leq N$ , where  $N$  does not depend on  $p$  or  $S$ .  $\diamond$

*Proof.* We define

$$Y := \{\tilde{p} \in (i^*)^{-1}(Z) : \text{tensor rank of } \tilde{p} \text{ over } \kappa(\tilde{p}) \leq N\}.$$

By Lemma 3.4.2.7,  $Y \subset \text{Im}(\varphi_N^*)$  and by construction,  $Z = i^*(Y)$ . Then  $\psi = \varphi_N \circ i$  is the required map such that  $Z \subset \text{Im}(\psi^*)$ .  $\square$

**Proposition 3.4.2.10.** *Let  $Z_{d,l}$  be the subset of  $\text{Spec } B_d$  defined above, by the vanishing of off-diagonal  $(l+1) \times (l+1)$ -subdeterminants of flattenings. Then there exists  $N \in \mathbb{Z}_{\geq 0}$  such that for all  $S \in \mathbf{FI}$  and for all  $p \in Z_{d,l}(S)$  there exists  $\tilde{p} \in \text{Spec } C_d(S)$  such that the tensor rank of  $\tilde{p}$  over the field  $\kappa(\tilde{p})$  is at most  $N$  and  $\iota^*(\tilde{p}) = p$  where  $\iota : B_d \rightarrow C_d$  is the inclusion map.*

*Proof.* We will prove this by a double induction: an outer induction on  $d$  and an inner induction on  $l$ . For  $d = 0$  and  $d = 1$ , we have  $B_d = C_d$ , and every element of  $\text{Spec } C_d(S)$  has tensor rank at most 1, regardless of  $l$ . Assume the proposition is true for  $\leq d-1$ . When  $l = 0$  note that  $Z_{d,0}$  contains only a zero tensor. Take  $N = 0$  and we are done. Fix  $l > 0$  and assume the proposition is true for all smaller values. Write  $Z_{d,l}(S) = Z_{d,l-1}(S) \cup (Z_{d,l}(S) \setminus Z_{d,l-1}(S))$ . By the inner induction hypothesis, there exists  $N_0$  such that for every  $p \in Z_{d,l-1}(S)$  there exists  $\tilde{p} \in \text{Spec } C_d(S)$  such that the tensor rank of  $\tilde{p}$  over the field  $\kappa(\tilde{p})$  is at most  $N_0$  and  $i^*(\tilde{p}) = p$ .

Now take  $p \in Z_{d,l}([n]) \setminus Z_{d,l-1}([n])$ . This implies that there exists  $i_0 \in [d]$  such that there is an  $l \times l$ -sub-determinant  $h$  of  $b_{i_0}(p)$  which is non-zero. The sub-determinant  $h$  involves at most  $\leq l(d-1) + l = l \cdot d$  indices. Split  $[n]$  into  $S \sqcup T$  such that  $S$  consists of indices that appear in  $h$  and  $T = [n] \setminus S$ ; i.e.,  $T$  is the set of indices that do not appear in  $h$ . For each fixed subset  $I \subset [d]$  and fixed distinct value tuple  $\alpha \in DS^{[d] \setminus I}$ , consider the off-diagonal tensor  $p_\alpha = (p_{\alpha,\beta})_{\beta \in DT^l}$ . We regard  $p_\alpha$  as an element of  $\text{Spec } B_{|I|}(T)$ .

Case 1: If  $I \neq [d]$ , then the outer induction hypothesis applies to  $p_\alpha$ . Indeed, flattening  $b_i(p_\alpha)$  for  $i \in I$  yields a sub-matrix of  $b_i(p)$  and by the induction hypothesis,  $p_\alpha = i^*(q_\alpha)$ , where  $q_\alpha \in \text{Spec } C_{|I|}(T)$  has rank  $\leq N_\alpha$ .

Case 2: If  $I = [d]$ , then we want to find  $q_\emptyset \in \text{Spec } C_d(T)$  such that  $p_\emptyset = i^*(q_\emptyset)$  and the tensor rank of  $q_\emptyset$  over the field  $\kappa(p)$  is at most some  $N_\emptyset$ . Then we will have

$$p = i^*\left(\sum_{I,\alpha} (q_\alpha \text{ padded with zeros})\right)$$

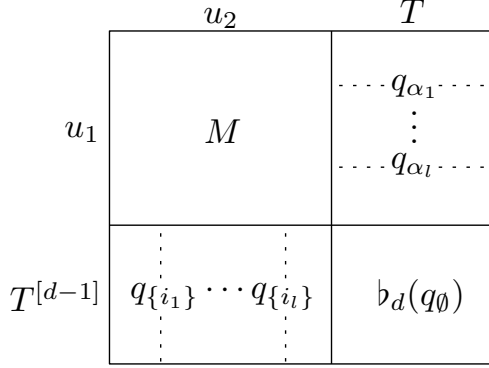


Figure 3.1: Flattening

such that the tensor rank of  $p$  over the field  $\kappa(p)$  is at most  $N_1 = \sum_{\alpha} N_{\alpha}$ . Without loss of generality, assume that  $i_0 = d$ . We have an off-diagonal  $l \times l$ -sub-matrix  $M$  of the matrix  $b_d(p)$  such that  $\det(M) = h \neq 0$ . Define  $q_\emptyset$  as follows:

$$\begin{aligned}
 q_\emptyset &= \sum_{j \in u_2 \text{ and } \alpha \in u_1} (M^{-1})_{j,\alpha} q_{\{j\}} \otimes q_\alpha \\
 &= \sum_{j \in u_2} q_{\{j\}} \otimes \left( \sum_{\alpha \in u_1} (M^{-1})_{j,\alpha} \cdot q_\alpha \right)
 \end{aligned}$$

where  $rk(q_{\{j\}}) \leq N_{\{j\}}$  and  $rk(\sum_{\alpha \in u_1} (M^{-1})_{j,\alpha} \cdot q_\alpha) \leq 1$ . Hence  $rk(q_\emptyset) \leq N_\emptyset := \sum_{j \in u_2} N_{\{j\}}$ .

Notations used in Figure 3.1: Rows and columns of the matrix are labeled by tuples in  $(S \sqcup T)^{d-1}$  and elements in  $S \sqcup T$  respectively.  $M$  is a  $u_1 \times u_2$ -submatrix where  $u_1 \subset DS^{d-1}$ ,  $u_2 \subset S$  such that  $u_1 \times u_2 \subset DS^d$  and  $|u_1| = |u_2| = l$ .

We want to show that  $i^*(q_\emptyset) = p_\emptyset$ . Straightforward computations (from linear algebra) show that the matrix in Figure 3.1 has rank  $= l$ . In particular, all off diagonal  $(l+1) \times (l+1)$ -sub-determinants in positions  $(u_1 \cup \{\alpha\}) \times (u_2 \cup \{i\})$  are zero where  $\alpha \in DT^{d-1}$  and  $i \in T$ . The same holds, by assumption for  $p$ . Hence, since  $M$  is invertible and for all  $j \in u_2$ ,  $i^*(q_{\{j\}}) = p_{\{j\}}$  and for all  $\alpha \in u_1$ ,  $i^*(q_\alpha) = p_\alpha$ , we find that  $(p_\emptyset)_{\alpha,i} = (q_\emptyset)_{\alpha,i}$ , for all  $\alpha \in DT^{d-1}$  and for all  $i \in T$  such that  $(\alpha, i) \in DT^d$ . Hence  $i^*(q_\emptyset) = p_\emptyset$ .  $N = \max\{N_0, N_1\}$  is the required  $N$ . This completes the proof.  $\square$

Recall from Remark 3.2.0.2 that for any nonnegative integers  $e, k_0, k_1, \dots, k_e$ ,  $B[k_0, k_1, \dots, k_e]$  denotes the following **FI**-algebra  $\bigotimes_{i=0}^e B_i^{\otimes k_i}$ .

**Lemma 3.4.2.11.** *For any morphism  $\varphi : B = B[k_0, k_1, \dots, k_e] \rightarrow A = B_1^{\otimes k}$  there exist a closed subset  $Z \subset \operatorname{Spec} B$  defined by finitely many equations in  $B$ , an **FI**-algebra  $A'$  that is finitely generated in width at most 1, and a morphism  $\psi : B \rightarrow A'$  such that  $\overline{\operatorname{Im}(\varphi^*)} \subset Z \subset \operatorname{Im}(\psi^*)$ .*

*Proof.* We will prove this by induction on the number of **FI**-algebras appearing in the tensor product  $B = B[k_0, k_1, \dots, k_e]$ .

For the base case, suppose that  $B = B_d$  for some nonnegative integer  $d$ . Then by Lemma 3.4.2.5, there exists a positive integer  $l$  such that  $\overline{\operatorname{Im}(\varphi^*)}$  is a closed subset of  $Z_{d,l}$ . By Lemma 3.4.2.4,  $Z_{d,l}$  is defined by finitely many equations in  $B_d$ . We take  $Z = Z_{d,l}$ . By Proposition 3.4.2.10 and Lemma 3.4.2.8, there exists a homomorphism  $\psi : B_d \rightarrow A'$  of **FI**-algebras such that  $Z = Z_{d,l}$  is contained in the image  $\psi^*(\operatorname{Spec} A')$ , where  $A'$  is an **FI**-algebra that is finitely generated in width at most 1.

Now suppose that  $B = B_{d_1} \otimes B_{d_2} \otimes \dots \otimes B_{d_n}$ . Here  $d_i$ 's are not necessarily distinct. We write  $B = C \otimes B_{d_n}$  where  $C = B_{d_1} \otimes B_{d_2} \otimes \dots \otimes B_{d_{n-1}}$ . We think of  $C$  and  $B_{d_n}$  as subalgebras of  $B$  via the natural inclusions. Denote by  $\varphi_1$  and  $\varphi_2$  the restrictions of the map  $\varphi : B \rightarrow A$  to  $C$  and  $B_{d_n}$  respectively. Then by the induction hypothesis, there exist closed subsets  $Z_1 \subset \operatorname{spec} C$  and  $Z_2 \subset \operatorname{Spec} B_{d_n}$  defined by finitely many equations in  $C$  and in  $B_{d_n}$  respectively, **FI**-algebras  $A_1$  and  $A_2$  both are finitely generated in width at most 1, and homomorphisms  $\psi_1 : C \rightarrow A_1$ ,  $\psi_2 : B_{d_n} \rightarrow A_2$  such that

$$\overline{\operatorname{Im}(\varphi_1^*)} \subset Z_1 \subset \operatorname{Im}(\psi_1^*) \text{ and } \overline{\operatorname{Im}(\varphi_2^*)} \subset Z_2 \subset \operatorname{Im}(\psi_2^*).$$

We now have  $\overline{\operatorname{Im}(\varphi^*)} \subseteq \overline{\operatorname{Im}(\varphi_1^*)} \times \overline{\operatorname{Im}(\varphi_2^*)}$  and  $\operatorname{Im}(\psi_1^*) \times \operatorname{Im}(\psi_2^*) = \operatorname{Im}((\psi_1 \otimes \psi_2)^*)$ , where  $\psi_1 \otimes \psi_2$  is the **FI**-algebra homomorphism  $C \otimes B_{d_n} \rightarrow A_1 \otimes A_2$  obtained as the tensor product of  $\psi_1$  and  $\psi_2$ .

Note that the **FI**-algebra  $A_1 \otimes A_2$ , being the tensor product of **FI**-algebras that are finitely generated in width at most 1, is itself also finitely generated in width at most 1. Further, the product  $Z := Z_1 \times Z_2$  is defined by the union of the finite sets of equations defining  $Z_1$  and  $Z_2$ . Thus this  $Z$ , and  $A' := A_1 \otimes A_2$ , and  $\psi := \psi_1 \otimes \psi_2$  have the required property  $\overline{\operatorname{Im}(\varphi^*)} \subset Z \subset \operatorname{Im}(\psi^*)$ .  $\square$

*Proof of the Main Theorem.* Recall from Section 3.3 that without loss of generality we can assume that both **FI**-algebras  $B$  and  $A$  are free, that is,  $B = B[k_0, k_1, \dots, k_e]$  and  $A = B_1^{\otimes k}$  for some nonnegative integers  $k, k_0, k_1, \dots, k_e$ .

From Lemma 3.4.2.11,  $\overline{\operatorname{Im}(\varphi^*)}$  is contained in a closed subset  $Z \subset \operatorname{Spec} B$  which is defined by finitely many equations in  $B$ , moreover,  $Z$  is contained

in the image  $\text{Im}(\psi^*)$  of a width-one **FI<sup>op</sup>**-scheme of finite type. Recall from Corollary 1.4.6.4 that width-one **FI<sup>op</sup>**-schemes of finite type are topologically Noetherian and from Lemma 1.4.6.7 that the image of a topologically Noetherian space is topologically Noetherian. By Lemma 1.4.6.7, the subset  $Z$ , being a subset of the topologically Noetherian space  $\text{Im}(\psi^*)$ , is topologically Noetherian. Thus  $\overline{\text{Im}(\varphi^*)}$  is topologically Noetherian being a closed subset of  $Z$  and defined by finitely many equations in the coordinate ring of  $Z$ . Since  $Z$  is defined by finitely many equations in  $B$ , this implies that  $\overline{\text{Im}(\varphi^*)}$  is defined by finitely many equations in  $B$ . This completes the proof.  $\square$

# Chapter 4

## Sym-Noetherianity for powers of GL-varieties

This chapter is based on the paper [Chi+22] with Christopher Chui, Alessandro Danelon, Jan Draisma, and Rob Eggermont.

### 4.1 Introduction

A significant amount of contemporary literature delves into investigating the finiteness properties of algebraic varieties with infinite dimensions, that have an accompanying action from either the infinite symmetric group or the infinite general linear group. In this chapter, we explore a shared extension where both groups act on spaces with infinite dimensions. Our result demonstrates that these spaces are topologically Noetherian under this joint action.

#### 4.1.1 Sym-Noetherianity and GL-Noetherianity

It has been well-established since the 1980s that if  $Z$  is finite-dimensional variety, then the topological space  $Z^{\mathbb{N}}$ , equipped with the inverse-limit topology of the Zariski topologies, has the property that if

$$X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$$

is a descending chain of closed subsets, each stable under the infinite symmetric group  $\text{Sym}$  permuting the copies of  $Z$ , then  $X_n = X_{n+1}$  for all  $n \gg 0$ . We say

that  $Z^{\mathbb{N}}$  is *Sym-Noetherian*; see [Coh67; Coh87; AH07; HS12] for the relevant literature.

On the other hand, in [Dra19] Draisma proved that if  $Z$  is a *GL-variety*: a (typically infinite-dimensional) affine variety equipped with a suitable action of the infinite general linear group  $\mathrm{GL}$ —see below for precise definitions—then  $Z$  is topologically  $\mathrm{GL}$ -Noetherian. See [BDES22] for the structure theory of  $\mathrm{GL}$ -varieties.

### 4.1.2 Our result: $\mathrm{Sym} \times \mathrm{GL}$ -Noetherianity

Given a  $\mathrm{GL}$ -variety  $Z$ , the group  $\mathrm{Sym} \times \mathrm{GL}$  acts naturally on  $Z^{\mathbb{N}}$ , and our main goal in this chapter is to prove the following theorem.

**Theorem 4.1.2.1** (Main Theorem). *Let  $Z$  be a  $\mathrm{GL}$ -variety over a field of characteristic zero. Then  $Z^{\mathbb{N}}$  is topologically  $\mathrm{Sym} \times \mathrm{GL}$ -Noetherian. In other words, every descending chain*

$$X_1 \supseteq X_2 \supseteq \dots$$

*of  $\mathrm{Sym} \times \mathrm{GL}$ -stable closed subsets of  $Z^{\mathbb{N}}$  stabilises eventually. Equivalently, any  $\mathrm{Sym} \times \mathrm{GL}$ -stable closed subset of  $Z^{\mathbb{N}}$  is defined by finitely many  $\mathrm{Sym} \times \mathrm{GL}$ -orbits of polynomial equations.*

### 4.1.3 A generalisation: $\mathrm{Sym}^k \times \mathrm{GL}$ -Noetherianity

We prove the Main Theorem by establishing first the following more general result.

**Theorem 4.1.3.1.** *Let  $Z_1, \dots, Z_k$  be  $\mathrm{GL}$ -varieties over a field of characteristic zero. Then the variety  $Z_1^{\mathbb{N}} \times \dots \times Z_k^{\mathbb{N}}$  is topologically Noetherian up to the natural action of  $\mathrm{Sym}^k \times \mathrm{GL}$ .*

Here there is one copy of  $\mathrm{GL}$  that acts diagonally, and there are  $k$  copies of  $\mathrm{Sym}$  that act on separate copies of  $\mathbb{N}$ . We believe it is impossible to prove the Main Theorem without considering multiple copies of  $\mathrm{Sym}$ . Indeed, the need for this generalization comes from the fact that, in order to cover a proper closed  $\mathrm{Sym} \times \mathrm{GL}$ -stable subset of  $Z^{\mathbb{N}}$ , we often need to partition  $\mathbb{N}$  into finitely many parts, such that for the indices  $i$  in one of these parts, the points in  $Z$  labeled by those indices behave in a similar fashion. The following example illustrates this point.

**Example 4.1.3.2.** Let  $Z$  be the space of  $\mathbb{N} \times \mathbb{N}$ -matrices over a field of characteristic zero, equipped with the GL-action given by  $(g, A) \mapsto gAg^T$ . Let  $X$  be the closed  $\text{Sym} \times \text{GL}$ -stable subvariety of  $Z^{\mathbb{N}}$  consisting of all infinite matrix tuples  $(A_1, A_2, \dots)$  such that each  $A_i$  is either symmetric or skew-symmetric. It is easy to see that  $X$  is defined by the  $\text{Sym} \times \text{GL}$ -orbit of the equation  $(x_{112} + x_{121})(x_{112} - x_{121})$ , where  $x_{ijk}$  is the  $(j, k)$ -entry of the  $i$ th matrix. We will see that the  $\text{Sym} \times \text{GL}$ -Noetherianity of  $X$  follows from the  $\text{Sym}^2 \times \text{GL}$ -Noetherianity of the “smaller” variety  $Z_1^{\mathbb{N}} \times Z_2^{\mathbb{N}}$ , where  $Z_1 \subseteq Z$  is the GL-subvariety of symmetric matrices, and  $Z_2 \subseteq Z$  is the GL-subvariety of skew-symmetric matrices. Here the term “smaller” refers to the fact that both  $Z_1$  and  $Z_2$  are quotients of  $Z$ . The exact meaning of smaller varieties is given in Section 4.2.8.  $\diamond$

## 4.1.4 Relation to existing literature

The Main Theorem generalizes the results mentioned in Section 4.1.1: taking for  $Z$  a finite-dimensional affine variety with trivial GL-action, one recovers the Sym-Noetherianity of  $Z^{\mathbb{N}}$ ; and on the other hand, if  $Z$  is a GL-variety, then considering chains  $X_1 \supseteq X_2 \supseteq \dots$  in which each  $X_i$  is of the form  $Z_i^{\mathbb{N}}$  with  $Z_i \subseteq Z$  a GL-subvariety, one recovers the GL-Noetherianity of  $Z$ .

The proof of the Main Theorem will reflect these two special cases. We will use the proof method from [Dra19] for the GL-Noetherianity of  $Z$ , and similarly, we will use methods for Sym-varieties from [DEF22]. In fact, we do not explicitly use Higman’s lemma in our proofs as is classically done [AH07; HS12; Dra14], and *en passant* we give a new proof of the Sym-Noetherianity of  $Z^{\mathbb{N}}$  for a finite-dimensional variety  $Z$ . However, our proof only yields a *set-theoretic* Noetherianity result, while in the pure Sym-setting (much) stronger results are known: the increasing chains of Sym-stable ideals stabilize [Coh67; Coh87; AH07; HS12], and even finitely generated modules over such rings with a compatible Sym-action are Noetherian [NR19]. In the pure GL-setting, however, such stronger Noetherianity results are known only for very few classes of GL-varieties [NSS16; SS22].

Partitions of  $\mathbb{N}$  into finitely many subsets also feature in the classification of symmetric subvarieties of infinite affine space  $(\mathbb{A}^1)^{\mathbb{N}}$  [NS21b], and while our proofs do not logically depend on this classification, that paper did serve as an inspiration.

### 4.1.5 Organisation of this chapter

This chapter is organized as follows. In Section 4.2 we reformulate the Main Theorem and Theorem 4.1.3.1 in the more convenient language of affine varieties over the categories  $\mathbf{FI}^{\text{op}} \times \mathbf{Vec}$  and  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ , respectively. We introduce a category  $\mathbf{PM}$  with morphisms between such varieties, in which, for the reasons explained in Example 4.1.3.2 and above it,  $k$  varies. Then in Section 4.3 we formulate and prove the Parameterisation Theorem, the core technical result of this chapter. The statement roughly says that if  $X$  is a proper closed  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -subvariety of a variety  $Z$  of *product type*

$$Z : (V; S_1, \dots, S_k) \mapsto \prod_{i=1}^k Z_i(V)^{S_i},$$

where the  $Z_i$  are  $\mathbf{Vec}$ -varieties, then  $X$  is covered by finitely many morphisms in  $\mathbf{PM}$  from  $(\mathbf{FI}^{\text{op}})^l \times \mathbf{Vec}$ -varieties of product form that are, in a suitable (and very subtle!) manner, smaller than  $Z$ . In Section 2.5 we use this to prove that all  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -varieties of product type are Noetherian, and obtain Theorem 4.1.3.1 and the Main Theorem as corollaries.

### 4.1.6 Notation and conventions

- Let  $S$  be a finite set. We denote by  $|S|$  the cardinality of  $S$ .
- Throughout this chapter, we work over a field  $K$  of characteristic zero.
- $\text{Sym}$  denotes the infinite symmetric group  $\text{Sym}(\infty)$ .
- $\text{GL}$  denotes the infinite general linear group  $\text{GL}_{\infty}$ .
- The category of schemes over  $K$  is denoted by  $\mathbf{Sch}_K$ . A product  $X \times Y$  of two schemes will always mean a product in this category.
- A *variety*  $X$  here is a reduced affine scheme of finite type over  $K$ . By  $K[X]$  we denote its coordinate ring, so  $X = \text{Spec } K[X]$ . If  $Y$  is a subvariety of  $X$ , then we write  $\mathcal{I}(Y) \subseteq K[X]$  for the (radical) ideal of functions on  $X$  vanishing on  $Y$ .
- If  $f \in K[X]$  then we write  $X[1/f] := \text{Spec}(K[X]_f)$ .
- Let  $\varphi : X \rightarrow Y$  be a morphism of varieties. We denote by  $\varphi^{\#} : K[Y] \rightarrow K[X]$  the induced morphism on coordinate rings.

- By a point  $x$  of a variety  $X$  we always mean a closed point of  $X$ , i.e. an element of  $X(\bar{K})$ . Recall that, for example,  $(X \times Y)(\bar{K}) \cong X(\bar{K}) \times Y(\bar{K})$  and that a morphism of varieties  $X \rightarrow Y$  is surjective if and only if the map  $X(\bar{K}) \rightarrow Y(\bar{K})$  is.

## 4.2 The category of $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -varieties

### 4.2.1 Polynomial functors

Let  $K$  be a field of characteristic zero, and let  $\mathbf{Vec}$  be the category of finite-dimensional vector spaces over  $K$  with  $K$ -linear morphisms.

**Definition 4.2.1.1.** A *polynomial functor* is a functor  $P : \mathbf{Vec} \rightarrow \mathbf{Vec}$  such that for each  $U, V \in \mathbf{Vec}$  the map  $P : \text{Hom}_{\mathbf{Vec}}(U, V) \rightarrow \text{Hom}_{\mathbf{Vec}}(P(U), P(V))$  is polynomial, and such that the degree of this polynomial map is upper-bounded independently of  $U, V$ . The minimal such bound is called the *degree* of  $P$ .  $\diamond$

**Example 4.2.1.2.** The functor  $\mathbf{Vec} \rightarrow \mathbf{Vec}$  that maps every vector space to a fixed finite-dimensional vector space  $V_0$  and every linear map to the identity map  $id_{V_0}$  is a polynomial functor of degree zero. We call it a constant polynomial functor.  $\diamond$

**Example 4.2.1.3.** The functor  $\mathbf{Vec} \rightarrow \mathbf{Vec}$  that maps all vector spaces and linear maps to themselves is a polynomial functor of degree one.  $\diamond$

**Example 4.2.1.4.** Let  $i \in \mathbb{N}$  be a positive integer. The functor  $T^i : \mathbf{Vec} \rightarrow \mathbf{Vec}$  that maps a vector space  $V$  to the  $i$ -th tensor power  $V^{\otimes i}$  of  $V$  and a linear map  $f : V \rightarrow W$  to the map  $f^{\otimes i} : V^{\otimes i} \rightarrow W^{\otimes i}$  sending  $v^{\otimes i}$  to  $f(v)^{\otimes i}$  is a polynomial functor of degree  $i$ .  $\diamond$

**Definition 4.2.1.5.** Let  $P$  and  $Q$  be two polynomial functors. We define the *direct sum* of  $P$  and  $Q$ , denoted by  $P \oplus Q$ , as the polynomial functor given by  $(P \oplus Q)(V) = P(V) \oplus Q(V)$ , where  $\oplus$  denotes the direct sum of vector spaces and the map  $(P \oplus Q)(f) : P(V) \oplus Q(V) \rightarrow P(W) \oplus Q(W)$  for a linear map  $f : V \rightarrow W$  is defined by  $(P \oplus Q)(f)(v_1, v_2) = (P(f)(v_1), Q(f)(v_2))$ . The degree of  $P \oplus Q$  is the maximum of the degrees of  $P$  and  $Q$ .  $\diamond$

**Remark 4.2.1.6.** The direct sum of a finite collection of polynomial functors is defined in a natural manner.  $\diamond$

We will also regard a polynomial functor  $P$  as a functor  $\mathbf{Vec} \rightarrow \mathbf{Sch}_K$  by composing with the embedding  $\mathbf{Vec} \rightarrow \mathbf{Sch}_K$  given by  $V \mapsto \text{Spec Sym}_K(V^*)$ , the spectrum of the symmetric algebra on  $V^*$ . Every polynomial functor  $P$  equals  $P_0 \oplus \dots \oplus P_d$ , where  $d$  is the degree of  $P$  and  $P_i$  is defined as

$$P_i(V) := \{v \in P(V) \mid \forall t \in K : P(\text{id}_V)v = t^i v\}.$$

Considering  $P$  as a functor  $\mathbf{Vec} \rightarrow \mathbf{Sch}_K$  we have  $P(V) = P_0(V) \times \dots \times P_d(V)$ . We note that  $P_0$  is a constant polynomial functor, which assigns a fixed vector space  $P(0) \in \mathbf{Vec}$  to all  $V \in \mathbf{Vec}$  and the identity map to each linear map. We call  $P$  *pure* if  $P_0 = \{0\}$ .

### 4.2.2 Vec-varieties

This subsection presents the concept of **Vec**-varieties, which serve as the finite-dimensional, functorial counterpart to GL-varieties. See Remark 4.2.2.6 for the connection with GL-varieties.

**Definition 4.2.2.1.** Let  $X, Y : \mathbf{Vec} \rightarrow \mathbf{Sch}_K$  be functors. A *closed immersion*  $\iota : X \rightarrow Y$  is a natural transformation such that  $\iota(V) : X(V) \rightarrow Y(V)$  is a closed immersion for all  $V \in \mathbf{Vec}$ . In particular,  $X$  is then a subfunctor of  $Y$ .  $\diamond$

**Definition 4.2.2.2.** An *affine Vec-scheme* is a functor  $X : \mathbf{Vec} \rightarrow \mathbf{Sch}_K$  that admits a closed immersion  $X \rightarrow P$  with  $P : \mathbf{Vec} \rightarrow \mathbf{Sch}_K$  a polynomial functor. A *Vec-variety* is an affine **Vec**-scheme  $X$  such that  $X(V)$  is reduced for all  $V \in \mathbf{Vec}$ . The *category of affine Vec-schemes* is the full subcategory of the functor category  $\mathbf{Sch}_K^{\mathbf{Vec}}$  whose objects are affine **Vec**-schemes.  $\diamond$

Spelled out explicitly, a **Vec**-variety  $X$  can be described by the data of a polynomial functor  $P$  and a subvariety  $X(V) \subseteq P(V)$  for each  $V \in \mathbf{Vec}$  such that, for each  $\varphi \in \text{Hom}_{\mathbf{Vec}}(U, V)$ , the linear map  $P(\varphi)$  maps  $X(U)$  into  $X(V)$ . A morphism of **Vec**-varieties  $\tau : X \rightarrow Y$  consists of a morphism of varieties  $\tau(V) : X(V) \rightarrow Y(V)$  for each  $V \in \mathbf{Vec}$  such that, for each  $\varphi \in \text{Hom}_{\mathbf{Vec}}(U, V)$ , we have  $\tau(V) \circ X(\varphi) = Y(\varphi) \circ \tau(U)$ .

**Remark 4.2.2.3.** The subcategory of **Vec**-varieties is closed under taking closed immersions and finite products. To see the latter, note that the product of  $X, Y : \mathbf{Vec} \rightarrow \mathbf{Sch}_K$  in  $\mathbf{Sch}_K^{\mathbf{Vec}}$  is given by  $V \mapsto X(V) \times Y(V)$ ; and furthermore, given closed immersions  $X \hookrightarrow P, Y \hookrightarrow Q$  the assignment

$$X(V) \times Y(V) \rightarrow P(V) \times Q(V)$$

defines a closed immersion of the product  $X \times Y$  into the polynomial functor  $P \oplus Q$ .  $\diamond$

**Lemma 4.2.2.4.** *The category of affine **Vec**-schemes admits fiber products.*

*Proof.* First note that for morphisms of affine **Vec**-schemes  $X \rightarrow Y$ ,  $Z \rightarrow Y$  the fibre product  $X \times_Y Z$  of  $X$  and  $Z$  over  $Y$  exists in the functor category  $\mathbf{Sch}_K^{\mathbf{Vec}}$  and is given by

$$(X \times_Y Z)(V) := X(V) \times_{Y(V)} Z(V).$$

Moreover, since  $Y(V)$  is affine (or more generally since  $Y(V)$  is separated, see [Sta20, Tag 01KR]) the natural morphism

$$X(V) \times_{Y(V)} Z(V) \rightarrow X(V) \times Y(V)$$

is a closed immersion. The statement follows by Remark 4.2.2.3.  $\square$

The main result of [Dra19] says that **Vec**-varieties are topologically Noetherian.

**Theorem 4.2.2.5** ([Dra19, Theorem 1]). *Let  $X$  be a **Vec**-variety. Then every descending chain of **Vec**-subvarieties*

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$$

*stabilizes, that is, there exists  $N \geq 0$  such that for each  $n \geq N$  we have  $X_n = X_{n+1}$ .*

**Remark 4.2.2.6.** If  $X$  is a **Vec**-variety, then the inverse limit  $X_\infty := \lim_{\leftarrow n} X(K^n)$  is a GL-variety in the sense of [BDES22]. This yields an equivalence of categories between **Vec**-varieties and GL-varieties. Most of our reasoning will be in the former terminology but could be rephrased in the latter.  $\diamond$

### 4.2.3 $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -varieties

Recall that **FI** denotes the category of finite sets with injections.

**Definition 4.2.3.1.** Let  $k \in \mathbb{Z}_{\geq 0}$ . An  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -variety is a covariant functor  $X$  from  $(\mathbf{FI}^{\mathbf{op}})^k$  to the category of **Vec**-varieties.  $\diamond$

Explicitly, an  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -variety is given by the following data: for any  $k$ -tuple  $(S_1, \dots, S_k)$  we have a **Vec**-variety  $X(S_1, \dots, S_k)$ , and for any  $k$ -tuple of injective maps

$$\iota = (\iota_1 : S_1 \rightarrow T_1, \dots, \iota_k : S_k \rightarrow T_k),$$

we have a corresponding morphism  $X(\iota) : X(T_1, \dots, T_k) \rightarrow X(S_1, \dots, S_k)$  of **Vec**-varieties and the usual requirements that  $X(\tau \circ \iota) = X(\iota) \circ X(\tau)$  and  $X(\text{id}_{S_1}, \dots, \text{id}_{S_k}) = \text{id}_{X(S_1, \dots, S_k)}$ .

Again, there are natural notions of morphism and closed immersion of  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -varieties, and we call an  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -variety Noetherian if every descending chain of closed  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -subvarieties stabilizes.

**Remark 4.2.3.2.** In particular, any contravariant functor from **FI** to finite-dimensional affine varieties, i.e., an  $\mathbf{FI}^{\text{op}}$ -variety, is trivially an  $\mathbf{FI}^{\text{op}} \times \mathbf{Vec}$ -variety. In this generality,  $\mathbf{FI}^{\text{op}}$ -varieties are certainly not Noetherian: see [HS12, Example 3.8].

However, we will be largely concerned with  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -varieties defined as follows. Let  $Z_1, \dots, Z_k$  be **Vec**-varieties, define

$$X(S_1, \dots, S_k) := Z_1^{S_1} \times \dots \times Z_k^{S_k} \quad (4.1)$$

and for  $\iota = (\iota_1, \dots, \iota_k) : (S_1, \dots, S_k) \rightarrow (T_1, \dots, T_k)$  define  $X(\iota)$  as the product of the natural projections  $Z^{T_i} \rightarrow Z^{S_i}$  associated to  $\iota_i$ . We will prove that  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -varieties of this form are, indeed, Noetherian.  $\diamond$

Note that we may also regard a  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -variety as a functor  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec} \rightarrow \mathbf{Sch}_K$ . For fixed  $k$ , the  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -varieties thus form a category by considering it as the full subcategory in the corresponding functor category.

**Remark 4.2.3.3.** If  $X$  is an  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -variety, then the group  $\text{Sym}^k \times \text{GL}$  acts on the inverse limit

$$\varprojlim_{n_1, \dots, n_k, n} X([n_1], \dots, [n_k])(K^n).$$

This gives a functor from  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -varieties to (infinite-dimensional) schemes equipped with a  $\text{Sym}^k \times \text{GL}$ -action. Unlike in Remark 4.2.2.6, this is not quite an equivalence of categories (even under reasonable restrictions on the  $\text{Sym}^k \times \text{GL}$ -action). For example,  $X([n_1], \dots, [n_k])$  could be empty for large  $n_i$  and a fixed nontrivial  $\text{GL}$ -variety for smaller  $n_i$ . We will consider an explicit example of this type later in Example 4.2.8.7. In that case, the inverse limit is empty but the  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -variety is not trivial. Our theorems will be formulated in the richer category of  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -varieties.  $\diamond$

#### 4.2.4 Partition morphisms and the category PM

Suppose that we are given a point  $p$  in some  $X(S_1, \dots, S_k)(V)$ , where  $X$  is as in (4.1). Then the components of  $p$  labelled by one of the finite sets  $S_i$  may

exhibit different behaviours, which prompts us to further partition  $S_i$  into subsets labelling components where the behaviour is similar. In that case,  $p$  will be in the image of some partition morphism, a notion that we define now.

**Definition 4.2.4.1.** Let  $X$  be an  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -variety and let  $Y$  be an  $(\mathbf{FI}^{\text{op}})^l \times \mathbf{Vec}$ -variety. A *partition morphism*  $Y \rightarrow X$  consists of the following data:

1. a map  $\pi : [l] \rightarrow [k]$ ; and
2. for each  $l$ -tuple of finite sets  $(T_1, \dots, T_l)$  a morphism

$$\varphi(T_1, \dots, T_l) : Y(T_1, \dots, T_l) \rightarrow X \left( \bigsqcup_{j \in \pi^{-1}(1)} T_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} T_j \right)$$

of  $\mathbf{Vec}$ -varieties in such a manner that for any  $l$ -tuple  $\iota_j \in \text{Hom}_{\mathbf{FI}}(S_j, T_j)$  the following equality holds:

$$\begin{aligned} X \left( \bigsqcup_{j \in \pi^{-1}(1)} \iota_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} \iota_j \right) \circ \varphi(T_1, \dots, T_l) \\ = \varphi(S_1, \dots, S_l) \circ Y(\iota_1, \dots, \iota_l). \end{aligned}$$

◇

**Remark 4.2.4.2.** Note that if we take  $k = l$  and  $\pi = \text{id}_{[k]}$ , then a partition morphism is just a morphism of  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -varieties. ◇

There is a natural way to compose partition morphisms: if  $(\pi, \varphi)$  is a partition morphism  $Y \rightarrow X$  as above and  $(\rho, \psi)$  is a partition morphism  $Z \rightarrow Y$ , where  $Z$  is an  $(\mathbf{FI}^{\text{op}})^m \times \mathbf{Vec}$ -variety, then  $(\pi, \varphi) \circ (\rho, \psi)$  is the partition morphism given by the data  $\pi \circ \rho : [m] \rightarrow [k]$  and the morphisms

$$\begin{aligned} \varphi \left( \bigsqcup_{n \in \rho^{-1}(1)} R_n, \dots, \bigsqcup_{n \in \rho^{-1}(l)} R_n \right) \circ \psi(R_1, \dots, R_m) : \\ Z(R_1, \dots, R_m) \rightarrow X \left( \bigsqcup_{n \in (\pi \circ \rho)^{-1}(1)} R_n, \dots, \bigsqcup_{n \in (\pi \circ \rho)^{-1}(k)} R_n \right). \end{aligned}$$

A tedious but straightforward computation shows that partition morphisms turn the class of  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -varieties, with varying  $k$ , into a category. We call this category **PM**.

**Definition 4.2.4.3.** Let  $X$  be an  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -variety,  $Y$  an  $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$ -variety, and  $(\pi, \varphi) : Y \rightarrow X$  a partition morphism. Let  $S_1, \dots, S_k \in \mathbf{FI}$  and  $V \in \mathbf{Vec}$ . The (set-theoretic) *image* of  $(\pi, \varphi)$  in  $X(S_1, \dots, S_k)(V)$  is defined as the set of all points of the form  $(X(\iota_1, \dots, \iota_k)(V) \circ \varphi(T_1, \dots, T_l)(V))(q)$  where  $T_1, \dots, T_l$  are finite sets,  $q$  is a point in  $Y(T_1, \dots, T_l)(V)$ , and each  $\iota_i$  is a bijection from  $S_i$  to  $\bigsqcup_{j \in \pi^{-1}(i)} T_j$ . The partition morphism  $(\pi, \varphi)$  is called *surjective* if its image in  $X(S_1, \dots, S_k)(V)$  equals  $X(S_1, \dots, S_k)(V)$  for all choices of  $S_1, \dots, S_k$  and  $V$ .  $\diamond$

**Remark 4.2.4.4.** In the previous definition, each bijection  $\iota_i$  induces a partition of the set  $S_i$ . Furthermore, if a partition morphism  $(\pi, \varphi)$  is surjective and for every  $i$  the  $\mathbf{Vec}$ -variety

$$X(\emptyset, \dots, \emptyset, \{*\}, \emptyset, \dots, \emptyset),$$

where  $\{*\}$  is a singleton in the  $i$ -th position, is nonempty, then the map  $\pi$  is automatically surjective, so that  $\pi$  induces a partition of  $[l]$  into  $k$  labeled, nonempty parts. This is our reason for calling the morphisms in  $\mathbf{PM}$  partition morphisms.  $\diamond$

The following example rephrases Example 4.1.3.2 in the current terminology.

**Example 4.2.4.5.** Let  $Z$  be the  $\mathbf{Vec}$ -variety that maps  $V$  to  $V \otimes V$ , and let  $Z_1, Z_2$  be the closed  $\mathbf{Vec}$ -subvarieties consisting of symmetric and skew-symmetric tensors, respectively. Consider the  $\mathbf{FI}^{\mathbf{op}} \times \mathbf{Vec}$ -variety defined by  $S \mapsto Z^S$ , and for every finite set  $S$  let  $X(S)$  be the closed  $\mathbf{Vec}$ -subvariety given by the points  $x = (x_s)_{s \in S} \in Z(V)^S$  such that each component  $x_s$  is either symmetric or skew-symmetric. Note that  $X$  is a closed  $\mathbf{FI}^{\mathbf{op}} \times \mathbf{Vec}$ -subvariety. Let  $Y$  be the  $(\mathbf{FI}^{\mathbf{op}})^2 \times \mathbf{Vec}$ -variety defined by

$$Y(S_1, S_2) = Z_1^{S_1} \times Z_2^{S_2}.$$

We now construct a partition morphism  $\varphi : Y \rightarrow X$  as follows. The map  $\pi : [2] \rightarrow [1]$  is the only possible, and for every  $V \in \mathbf{Vec}$  and  $(S_1, S_2) \in \mathbf{FI}^{\mathbf{op}^2}$  the map

$$\varphi(S_1, S_2)(V) : Y(S_1, S_2)(V) = Z_1(V)^{S_1} \times Z_2(V)^{S_2} \rightarrow X(S_1 \sqcup S_2)(V)$$

is defined by:

$$((x_{s_1})_{s_1 \in S_1}, (x_{s_2})_{s_2 \in S_2}) \mapsto (x_s)_{s \in S_1 \sqcup S_2}.$$

Note that the partition morphism  $\varphi$  is surjective. In particular, we say that  $X$  is covered by  $Y$ , and, as we have already hinted in Example 4.1.3.2,  $Y$  is in some sense smaller than the assignment  $S \mapsto Z^S$ . The fact that we can do this, in general, is the content of the *Parameterisation Theorem 4.3.1.1*.  $\diamond$

The following lemma is immediate.

**Lemma 4.2.4.6.** *Let  $X$  be an  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -variety,  $X'$  a closed  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -subvariety of  $X$ , and let  $(\pi, \varphi)$  be a partition morphism from an  $(\mathbf{FI}^{\text{op}})^l \times \mathbf{Vec}$ -variety  $Y$  to  $X$ . Then  $Y' := (\pi, \varphi)^{-1}(X')$  defined by*

$$Y'(T_1, \dots, T_l) := \varphi(T_1, \dots, T_l)^{-1} \left( X' \left( \bigsqcup_{j \in \pi^{-1}(1)} T_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} T_j \right) \right)$$

*is a closed  $(\mathbf{FI}^{\text{op}})^l \times \mathbf{Vec}$ -subvariety of  $Y$ , and the data of  $\pi$  together with the restrictions of the morphisms  $\varphi(T_1, \dots, T_l)$  gives a partition morphism from  $Y'$  to  $X$ . Moreover, if  $(\pi, \varphi)$  is surjective, then so is its restriction to  $Y' \rightarrow X'$ .*

The following easy proposition is crucial in our approach to the main theorem.

**Proposition 4.2.4.7.** *If  $(\pi, \varphi)$  is a surjective partition morphism from  $Y$  to  $X$ , and  $Y$  is a Noetherian  $(\mathbf{FI}^{\text{op}})^l \times \mathbf{Vec}$ -variety, then  $X$  is a Noetherian  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -variety.*

*Proof.* Let  $X_1 \supseteq X_2 \supseteq \dots$  be a descending chain of closed  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -subvarieties. By Lemma 4.2.4.6, the preimages  $Y_i := (\pi, \varphi)^{-1}(X_i)$  are closed  $(\mathbf{FI}^{\text{op}})^l \times \mathbf{Vec}$ -subvarieties of  $Y$ . Hence the chain  $Y_1 \supseteq Y_2 \supseteq \dots$  stabilises by assumption. The surjectivity of  $(\pi, \varphi)$  implies the surjectivity of its restriction to  $Y_i \rightarrow X_i$ . This implies that  $X_i$  is uniquely determined by  $Y_i$ , and hence the chain  $X_1 \supseteq X_2 \supseteq \dots$  stabilizes at the same point.  $\square$

## 4.2.5 Product type

We now introduce the  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$  varieties of product type. Essentially, these are the varieties from Remark 4.2.3.2, but for our proofs, we will need finer control over these products. Therefore, we will work over a general base  $\mathbf{Vec}$ -variety  $Y$ , and keep track of the “constant parts”  $B_i$  of the  $\mathbf{Vec}$ -varieties whose products we consider.

**Definition 4.2.5.1.** Let  $Y$  be a  $\mathbf{Vec}$ -variety and  $k, n_1, \dots, n_k \in \mathbb{Z}_{\geq 0}$ . For each  $i \in [k]$ , let  $B_i$  be a  $\mathbf{Vec}$ -subvariety of  $Y \times \mathbb{A}^{n_i}$ , and  $Q_i$  be a pure polynomial

functor. By construction each **Vec**-variety  $B_i \times Q_i$  has a morphism to  $Y$  induced by the projection  $Y \times \mathbb{A}^{n_i} \rightarrow Y$ . We define the  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -variety  $Z = [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$  via

$$Z(S_1, \dots, S_k) := (B_1 \times Q_1) \times_Y \dots \times_Y (B_1 \times Q_1) \times_Y (B_2 \times Q_2) \\ \times_Y \dots \times_Y (B_k \times Q_k),$$

where for every index  $i \in [k]$  the fibre product over  $Y$  of  $B_i \times Q_i$  with itself is taken  $|S_i|$  times, and these copies are labelled by the elements of  $S_i$ ; and where the morphism  $Z(T_1, \dots, T_k) \rightarrow Z(S_1, \dots, S_k)$  corresponding to  $\iota : S \rightarrow T$  is the projection as in Remark 4.2.3.2. We also write the above product in a more compact notation as

$$(B_1 \times Q_1)_Y^{S_1} \times_Y \dots \times_Y (B_k \times Q_k)_Y^{S_k}.$$

We say that  $Z$  is an  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -variety of *product type* (over  $Y$ ).  $\diamond$

Note that  $Z(S_1, \dots, S_k)$  is naturally a closed **Vec**-subvariety of

$$Y \times \prod_{i=1}^k (\mathbb{A}^{n_i} \times Q_i)^{S_i}.$$

Moreover, if  $k = 0$ , then by definition  $Z = Y$ .

When we talk of  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -varieties of product type, we will always specify each  $B_i$  together with its closed embedding in  $Y \times \mathbb{A}^{n_i}$ ; the reason being that, in the proof of the Main Theorem, we aim to argue by induction on both  $Y$  and  $n_i$ .

**Remark 4.2.5.2.** The settings of Theorem 4.1.2.1 and Theorem 4.1.3.1 can be rephrased in our current terminology as follows. Consider **Vec**-varieties  $Z_1, \dots, Z_k$ . Then for every  $i \in [k]$  there exist  $n_i \in \mathbb{Z}_{\geq 0}$ , a finite-dimensional affine variety  $A_i \subseteq \mathbb{A}^{n_i}$ , and a pure polynomial functor  $Q_i$  such that  $Z_i \subseteq A_i \times Q_i$ . Define  $Y$  to be a point, and  $B_i := Y \times A_i$ . Then the variety  $Z_1^{\mathbb{N}} \times \dots \times Z_k^{\mathbb{N}}$  of Theorem 4.1.3.1 is a subvariety of the product-type  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -variety

$$[Y; B_1 \times Q_1, \dots, B_k \times Q_k],$$

with  $k = 1$  being the special case addressed in Theorem 4.1.2.1.  $\diamond$

**Remark 4.2.5.3.** In [DEF22], for  $\mathbf{FI}^{\mathbf{op}}$ -varieties (no dependence on **Vec**), the notion of product type is more restrictive. Essentially, there the last three

authors considered a single finite-dimensional affine variety  $Z$  with a morphism to a finite-dimensional, irreducible, affine variety  $Y$ , with the additional requirement that  $K[Z]$  is a free  $K[Y]$ -module. This then ensures that each irreducible component of  $Z^S$  maps dominantly to  $Y$ . In [DEF22] this is used to count the orbits of  $\text{Sym}(S)$  on these irreducible components.  $\diamond$

The following example describes the partition morphisms between product-type varieties. It is particularly relevant as the partition morphisms we will be dealing with in our proof of the Parameterisation Theorem 4.3.1.1 are of this shape.

**Example 4.2.5.4.** Let  $Z' := [Y'; B'_1 \times Q'_1, \dots, B'_l \times Q'_l]$  and  $Z := [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$  be an  $(\mathbf{FI}^{\text{op}})^l \times \mathbf{Vec}$ -variety and an  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -variety of product type over  $Y'$  and  $Y$ , respectively. We want to construct a partition morphism  $(\pi, \varphi) : Z' \rightarrow Z$ . Consider the following data:

- let  $\pi : [l] \rightarrow [k]$  be any map;
- let  $\alpha : Y' \rightarrow Y$  be a morphism of  $\mathbf{Vec}$ -varieties;
- and for each  $j \in [l]$  let  $\beta_j : B'_j \times Q'_j \rightarrow B_{\pi(j)} \times Q_{\pi(j)}$  be a morphism of  $\mathbf{Vec}$ -varieties such that the following diagram commutes:

$$\begin{array}{ccc} B'_j \times Q'_j & \xrightarrow{\beta_j} & B_{\pi(j)} \times Q_{\pi(j)} \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{\alpha} & Y. \end{array} \quad (4.2)$$

For each  $(T_1, \dots, T_l) \in \mathbf{FI}^l$  we define the morphism of  $\mathbf{Vec}$ -varieties

$$\varphi(T_1, \dots, T_l) : Z'(T_1, \dots, T_l) \rightarrow Z \left( \bigsqcup_{j \in \pi^{-1}(1)} T_j, \dots, \bigsqcup_{j \in \pi^{-1}(k)} T_j \right)$$

as follows. Let  $S_i := \bigsqcup_{j \in \pi^{-1}(i)} T_j$ , then for any  $V \in \mathbf{Vec}$  the element

$$((b'_{j,t}, q'_{j,t})_{t \in T_j})_{j \in [l]} \in (B'_1 \times Q'_1)_{Y'}^{T_1}(V) \times_{Y'} \cdots \times_{Y'} (B'_l \times Q'_l)_{Y'}^{T_l}(V)$$

is mapped to the element

$$(((\beta_j(V)(b'_{j,t}, q'_{j,t}))_{t \in T_j})_{j \in \pi^{-1}(i)})_{i \in [k]} \in (B_1 \times Q_1)_Y^{S_1}(V) \times_Y \cdots \times_Y (B_k \times Q_k)_Y^{S_k}(V).$$

By construction, the pair  $(\pi, \varphi)$  is a partition morphism  $Z' \rightarrow Z$ . Conversely, every partition morphism  $Z' \rightarrow Z$  is of this form. Indeed, from a general partition morphism  $Z' \rightarrow Z$ ,  $\alpha$  is recovered by taking all  $T_j$  empty and  $\beta_j$  is recovered by taking  $T_j$  a singleton and all  $T_{j'}$  with  $j' \neq j$  empty. That (4.2) commutes then follows by applying the commuting diagram from the definition of a partition morphism to the morphism  $(\emptyset, \dots, \emptyset, \dots, \emptyset) \rightarrow (\emptyset, \dots, \{*\}, \dots, \emptyset)$  in  $\mathbf{FI}^l$ .  $\diamond$

## 4.2.6 The leading monomial ideal

We introduce a size measure for a closed subvariety  $B \subseteq Y \times \mathbb{A}^n$ .

**Definition 4.2.6.1.** Let  $Y$  be a **Vec**-variety,  $n \in \mathbb{Z}_{\geq 0}$  and  $B$  a closed **Vec**-subvariety of  $Y \times \mathbb{A}^n$ . For  $V \in \mathbf{Vec}$  consider the ideal  $\mathcal{I}(B(V))$  of

$$K[Y(V)][x_1, \dots, x_n]$$

defining  $B(V)$ . We fix the lexicographic order on monomials in  $x_1, \dots, x_n$ , and denote by  $\text{LM}(B)$  the set of those monomials that appear as leading monomials of *monic* polynomials in  $\mathcal{I}(B(V))$ , i.e., those with leading coefficient 1 in  $K[Y(V)]$ .  $\diamond$

The following lemma shows that  $\text{LM}(B)$  is well-defined.

**Lemma 4.2.6.2.** *The set  $\text{LM}(B)$  does not depend on the choice of  $V$ .*

*Proof.* Let  $V \in \mathbf{Vec}$  and consider the linear maps  $\iota : 0 \rightarrow V$  and  $\pi : V \rightarrow 0$ . If  $f \in \mathcal{I}(B(V))$  is monic with leading monomial  $x^u$ , then applying  $Y(\iota)^\#$  to all coefficients of  $f$  yields a polynomial in  $\mathcal{I}(B(0))$  which is monic with leading monomial  $x^u$ . This shows that the leading monomials of monic polynomials in  $\mathcal{I}(B(V))$  remain leading monomials of monic elements in  $\mathcal{I}(B(0))$ . One obtains the converse inclusion by applying  $Y(\pi)^\#$ .  $\square$

The following lemma monitors the size of  $\text{LM}$  of the constant parts after a base change in product-type varieties. See Proposition 4.2.7.4.

**Lemma 4.2.6.3.** *Let  $Y' \rightarrow Y$  be a morphism of **Vec**-varieties, let  $B$  be a closed **Vec**-subvariety of  $Y \times \mathbb{A}^n$ , and define  $B' := Y' \times_Y B \subseteq Y' \times \mathbb{A}^n$ . Then  $\text{LM}(B') \supseteq \text{LM}(B)$ .*

*Proof.* Pulling back a monic equation for  $B(V)$  along  $Y'(V) \times \mathbb{A}^n \rightarrow Y(V) \times \mathbb{A}^n$  yields a monic equation for  $B'(V)$  with the same leading monomial.  $\square$

## 4.2.7 Shifting over tuples of finite sets

We now introduce an operation that has had much fortune to prove results in the infinite-dimensional setting via the functorial language. The third author first used it in [Dra19] to prove what became “The Embedding Theorem” for GL-varieties in [BDES22, Theorems 4.1, 4.2]. Afterward, the last three authors performed this operation in the **FI**-world [DEF22]. Here we describe this operation in the context of  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -varieties.

**Definition 4.2.7.1.** Let  $X$  be an  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -variety and let  $S = (S_1, \dots, S_k) \in \mathbf{FI}^k$ . Then the *shift*  $\mathrm{Sh}_S X$  of  $X$  over  $S$  is the  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -variety defined by

$$(\mathrm{Sh}_S X)(T_1, \dots, T_k) := X(S_1 \sqcup T_1, \dots, S_k \sqcup T_k)$$

and, for injections  $\iota_i : T_i \rightarrow T'_i$ ,

$$(\mathrm{Sh}_S X)(\iota_1, \dots, \iota_k) := X(\mathrm{id}_{S_1} \sqcup \iota_1, \dots, \mathrm{id}_{S_k} \sqcup \iota_k). \quad \diamond$$

**Remark 4.2.7.2.** Consider an tuple  $S = (S_1, \dots, S_k)$  in  $(\mathbf{FI}^{\mathbf{op}})^k$  and define the covariant functor  $\mathrm{Sh}_S : (\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec} \rightarrow (\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$  by assigning to each tuple  $(T_1, \dots, T_k)$  the tuple  $(S_1 \sqcup T_1, \dots, S_k \sqcup T_k)$  and to each morphism

$$\iota : (\iota_1, \dots, \iota_k) : (T_1, \dots, T_k) \rightarrow (T'_1, \dots, T'_k)$$

the morphism  $\iota \sqcup \mathrm{id}_S$ . In particular  $\mathrm{Sh}_S X$  is the composition  $X \circ \mathrm{Sh}_S$ .  $\diamond$

**Remark 4.2.7.3.** Let  $V$  be a finite-dimensional vector space. While, as sets,  $\mathrm{Sh}_S X(T_1, \dots, T_k)(V)$  and  $X(S_1 \sqcup T_1, \dots, S_k \sqcup T_k)(V)$  coincide, the action of the  $k$  copies of the symmetric group on them is different. Indeed, the groups  $\mathrm{Sym}(S_1 \sqcup T_1) \times \dots \times \mathrm{Sym}(S_k \sqcup T_k)$  and  $\mathrm{Sym}(T_1) \times \dots \times \mathrm{Sym}(T_k)$  act by functoriality on the latter and on the former, respectively.  $\diamond$

With the following proposition, we describe what happens when the shift operation is performed on product-type varieties.

**Proposition 4.2.7.4.** *The shift  $\mathrm{Sh}_S Z$  over  $S = (S_1, \dots, S_k)$  of an  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -variety  $Z := [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$  of product type is itself isomorphic to a variety of product type:*

$$\mathrm{Sh}_S Z \cong [Y'; B'_1 \times Q_1, \dots, B'_k \times Q_k]$$

with  $Y' := (B_1 \times Q_1)_{Y'}^{S_1} \times_Y \dots \times_Y (B_k \times Q_k)_{Y'}^{S_k}$ , and  $B'_i := Y' \times_Y B_i$ .

Furthermore, each  $B'_i$  is naturally a **Vec**-subvariety of  $Y' \times \mathbb{A}^{n_i}$ , and we have  $\mathrm{LM}(B'_i) \supseteq \mathrm{LM}(B_i)$ .

*Proof.* Straightforward; for the last statement use Lemma 4.2.6.3.  $\square$

In analogy with [Dra19, Lemma 14] and [DEF22, Section 3.3], the shift operation doesn't increase the “complexity” of product-type varieties. Indeed, we will have  $\text{Sh}_S Z \preceq Z$  according to the order in Section 4.2.8.

### 4.2.8 Well-founded orders

By a pre-order  $\preceq$  on a class, we will mean a reflexive and transitive relation. We also write  $B \succeq A$  for  $A \preceq B$ . Furthermore, write  $A \prec B$  or  $B \succ A$  to mean that  $A \preceq B$  but not  $B \preceq A$ . The pre-order is well-founded if it admits no infinite strictly decreasing chains  $A_1 \succ A_2 \succ \dots$

In this section, we first recall a well-founded pre-order on polynomial functors. Building on it, we define well-founded pre-orders

- on varieties appearing in the definition of  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -varieties of product type,
- on product-type varieties, and
- on closed subvarieties of a fixed product-type variety.

#### Order on polynomial functors

**Definition 4.2.8.1.** For polynomial functors  $P, Q$ , we write  $P \preceq Q$  if  $P \cong Q$  or else, for the largest  $e$  with  $P_e \not\cong Q_e$ ,  $P_e$  is a quotient of  $Q_e$ .  $\diamond$

This is a well-founded partial order on polynomial functors, see [Dra19, Lemma 12].

#### Order on $\mathbf{Vec}$ -varieties of type $B \times Q$

Consider  $\mathbf{Vec}$ -varieties  $Y, Y'$ , integers  $n, n'$ , pure polynomial functors  $Q, Q'$ , and  $\mathbf{Vec}$ -subvarieties  $B \subset Y \times \mathbb{A}^n$ ,  $B' \subset Y' \times \mathbb{A}^{n'}$ . We say that  $B' \times Q' \preceq B \times Q$  if:

1.  $Q' \prec Q$  in the order of Definition 4.2.8.1; or
2.  $Q' \cong Q$ ,  $n' = n$  and  $\text{LM}(B') \supseteq \text{LM}(B)$ .

This is a pre-order on  $\mathbf{Vec}$ -varieties of this type.

**Remark 4.2.8.2.** We remark that  $\preceq$  is defined on **Vec**-varieties *with a specified product decomposition*  $B \times Q$  where  $B$  is a **Vec**-variety *with a specified closed embedding into a specified product*  $Y \times \mathbb{A}^n$  of a **Vec**-variety  $Y$  and some  $n$ . It is not a pre-order on **Vec**-varieties without further data.  $\diamond$

**Lemma 4.2.8.3.** *The pre-order on **Vec**-varieties defined above is well-founded.*

*Proof.* Suppose we had an infinite strictly decreasing chain

$$B_1 \times Q_1 \succ B_2 \times Q_2 \succ \dots$$

with  $B_i \subseteq Y_i \times \mathbb{A}^{n_i}$ . Then we have  $Q_1 \succeq Q_2 \succeq \dots$ . By the well-foundedness of  $\succeq$  on polynomial functors, there exists a  $j \geq 1$  such that both  $Q_i$  and  $n_i$  are constant for  $i \geq j$ . But then  $\text{LM}(B_i) \subsetneq \text{LM}(B_{i+1}) \subsetneq \dots$ , which contradicts Dickson's lemma.  $\square$

### Order on product-type varieties

Consider an  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -variety  $Z := [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$ , and an  $(\mathbf{FI}^{\text{op}})^l \times \mathbf{Vec}$ -variety  $Z' := [Y'; B'_1 \times Q'_1, \dots, B'_l \times Q'_l]$ . We say that  $Z' \preceq Z$  if there exists a map  $\pi : [l] \rightarrow [k]$  with the following properties:

1.  $B'_j \times Q'_j \preceq B_{\pi(j)} \times Q_{\pi(j)}$  holds for all  $j \in [l]$ , and
2. for all  $j$  whose  $\pi$ -fibre  $\pi^{-1}(\pi(j))$  has cardinality at least 2 we have  $B'_j \times Q'_j \prec B_{\pi(j)} \times Q_{\pi(j)}$ .
3. If  $\pi$  is a bijection, then either at least one of the inequalities in (1) is strict, or else  $Y'$  is a closed **Vec**-subvariety of  $Y$ .

**Lemma 4.2.8.4.** *Suppose  $Z' \preceq Z$  is witnessed by  $\pi : [l] \rightarrow [k]$  and suppose that at least one of the following holds:*

- $l \neq k$ , or
- *at least one of the inequalities in (1) is strict.*

*Then we have  $Z' \prec Z$ .*

*Proof.* Assume, on the contrary, that  $\sigma : [k] \rightarrow [l]$  witnesses  $Z \preceq Z'$ . Construct a directed graph  $\Gamma$  with vertex set  $[l] \sqcup [k]$  and an arrow from each  $j \in [l]$  to  $\pi(j)$  and an arrow from each  $i \in [k]$  to  $\sigma(i)$ . Like any digraph in which each vertex has out-degree 1,  $\Gamma$  is a union of disjoint directed cycles (here of even length) plus a number of trees rooted at vertices in those cycles and directed

towards those roots. Moreover, those cycles have the same number of vertices in  $[l]$  as in  $[k]$ .

The assumptions imply that at least one of the vertices of  $\Gamma$  does not lie on a directed cycle. Without loss of generality, there exists an  $i \in [k]$  not in any cycle such that  $j := \sigma(i)$  lies on a cycle. Let  $n$  be half the length of that cycle, so that  $(\sigma\pi)^n(j) = j$ . Then we have

$$\begin{aligned} B'_j \times Q'_j &\preceq B_{\pi(j)} \times Q_{\pi(j)} \preceq \dots \preceq B_{\pi(\sigma\pi)^{n-1}(j)} \times Q_{\pi(\sigma\pi)^{n-1}(j)} \\ &\prec B'_{(\sigma\pi)^n(j)} \times Q'_{(\sigma\pi)^n(j)} = B'_j \times Q'_j \end{aligned}$$

where the strict inequality holds because  $\sigma^{-1}(j)$  has at least two elements:  $i$  and  $\pi(\sigma\pi)^{n-1}(j)$ . By transitivity of the pre-order from Section 4.2.8, we find  $B'_j \times Q'_j \prec B'_j \times Q'_j$ , which however contradicts the reflexivity of that pre-order.  $\square$

**Lemma 4.2.8.5.** *The relation  $\preceq$  is a well-founded pre-order on varieties in PM of product type.*

*Proof.* For reflexivity, we may take  $\pi$  equal to the identity. For transitivity, if  $\pi : [l] \rightarrow [k]$  witnesses  $Z' \preceq Z$  and  $\sigma : [k] \rightarrow [m]$  witnesses  $Z \preceq Z''$ , then  $\tau := \sigma \circ \pi$  witnesses  $Z' \preceq Z''$ —here we note that if  $|\tau^{-1}(\tau(j))| > 1$  for some  $j \in [l]$ , then either  $|\pi^{-1}(\pi(j))| > 1$  or else  $|\sigma^{-1}(\sigma(\pi(j)))| > 1$ ; in both cases we find that  $B'_j \times Q'_j \prec B''_{\tau(j)} \times Q''_{\tau(j)}$ .

For well-foundedness, suppose that we had a sequence  $Z_1 \succ Z_2 \succ Z_3 \succ \dots$ , where

$$Z_i = [Y_i; B_{i,1} \times Q_{i,1}, \dots, B_{i,k_i} \times Q_{i,k_i}],$$

and where  $\pi_i : [k_{i+1}] \rightarrow [k_i]$  is a witness to  $Z_i \succ Z_{i+1}$ . We note that  $k_i > 0$  for all  $i$ . Otherwise  $0 = k_i = k_{i+1} = \dots$  and then  $Z_i = Y_i \succ Z_{i+1} = Y_{i+1} \succ \dots$  implies that  $Y_i \supsetneq Y_{i+1} \supsetneq \dots$ , which contradicts the Noetherianity of the **Vec**-variety  $Y_i$ , see Theorem 4.2.2.5.

From the chain, we construct an infinite rooted forest with vertex set  $[k_1] \sqcup [k_2] \sqcup \dots$  as follows:  $[k_1]$  is the set of roots, and we attach each  $j \in [k_{i+1}]$  via an edge with  $\pi_i(j)$ ; the latter is called the *parent* of the former. We further label each vertex  $j \in [k_i]$  with the product  $B_{i,j} \times Q_{i,j}$ .

We claim that  $\pi_i$  is an injection for all  $i \gg 0$ , i.e., that there are only finitely many vertices with more than one child. Indeed, if not, then by König's lemma the forest would have an infinite path starting at a root in  $[k_1]$  and passing through infinitely many vertices with at least two children. By construction, the labels  $B \times Q$  decrease weakly along such a path and strictly whenever

going from a vertex to one of its more than one child, a contradiction to Lemma 4.2.8.3.

For even larger  $i$ , the  $k_i$  are constant, say equal to  $k$ , and hence the  $\pi_i$  are bijections. After reordering, we may assume that the  $\pi_i$  all equal the identity on  $[k]$ . Moreover, for all such  $i$  we still have  $B_{i,j} \times Q_{i,j} \succeq B_{i+1,j} \times Q_{i+1,j} \succeq \dots$  for all  $j \in [k]$ , and all these chains stabilise. When they do, we have  $Y_i \supsetneq Y_{i+1} \supsetneq \dots$ , which is a strictly decreasing chain of **Vec**-varieties—but this again contradicts the Noetherianity of **Vec**-varieties.  $\square$

### Order on closed subvarieties of product-type varieties in PM

Consider the  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -variety  $Z = [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$  and let  $X$  be a closed  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -subvariety of  $Z$ ;  $X$  is not required to be of product type. We define

$$\delta_X := \min_{(S_1, \dots, S_k) \in \mathbf{FI}^k} \left\{ \sum_{i=1}^k |S_i| : X(S_1, \dots, S_k) \neq Z(S_1, \dots, S_k) \right\}$$

Let  $X$  and  $X'$  be closed  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -subvarieties of  $Z$ , then we say  $X' \preceq X$  if  $\delta_{X'} \leq \delta_X$ . This is a well-founded pre-order on the  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -subvarieties of  $Z$ .

**Remark 4.2.8.6.** If  $f$  is a nonzero equation for  $X(S_1, \dots, S_k)(V)$  with  $\sum_i |S_i| = \delta_X$ , then  $f$  may still “come from smaller sets”. More specifically, there might exist a  $k$ -tuple  $(S'_1, \dots, S'_k)$  with  $|S'_i| \leq |S_i|$  for all  $i \in [k]$  and with strict inequality for at least one  $i$ , an  $\mathbf{FI}^k$ -morphism  $\iota := (\iota_1, \dots, \iota_k) : (S'_1, \dots, S'_k) \rightarrow (S_1, \dots, S_k)$ , and an element  $f' \in K[Z(S'_1, \dots, S'_k)(V)]$  such that  $Z(\iota)(V)^\#(f') = f$ . This is related to Remark 4.2.3.3. The following example demonstrates this phenomenon.  $\diamond$

**Example 4.2.8.7.** Consider the  $\mathbf{FI}^{\mathbf{op}} \times \mathbf{Vec}$ -variety  $Z := [\{0\}; K]$ . The coordinate ring  $K[Z(S)]$  is isomorphic to the polynomial ring over  $K$  in  $|S|$  variables. Let  $n \in \mathbb{Z}_{>0}$  and define the proper closed variety  $X$  of  $Z$  by

$$X(S) := \begin{cases} Z(S) & \text{for } |S| < n; \\ \emptyset & \text{otherwise.} \end{cases}$$

Then  $\delta_X$  is equal to  $n$  and computed by the element  $1 \in K[Z([n])]$ , which is the image of  $1 \in K[Z(\emptyset)]$  under the natural map  $K[Z(\emptyset)] \rightarrow K[Z([n])]$ .  $\diamond$

## 4.3 Covering $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -varieties by smaller ones

### 4.3.1 The Parameterisation Theorem

The goal of this section is to prove the following core result, which says that any proper closed subset of an  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -variety of product type is covered by finitely many smaller such varieties.

**Theorem 4.3.1.1** (Parameterisation Theorem). *Consider an  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -variety  $Z$  of product type and let  $X \subsetneq Z$  be a proper closed  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -subvariety. Then there exist a finite number of quadruples consisting of:*

- an  $l \in \mathbb{Z}_{\geq 0}$ ;
- an  $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$ -variety  $Z'$  of product type with  $Z' \prec Z$ ;
- a  $k$ -tuple  $S = (S_1, \dots, S_k) \in \mathbf{FI}^k$ ; and
- a partition morphism  $(\pi, \varphi) : Z' \rightarrow \mathrm{Sh}_S Z$ ;

such that for any  $T_1, \dots, T_k \in \mathbf{FI}^k$ , any  $V \in \mathbf{Vec}$ , and any  $p \in X(T_1, \dots, T_k)(V)$  there exist: one of these quadruples; finite sets  $U_1, \dots, U_k$ ; and bijections  $\sigma_i : T_i \rightarrow S_i \sqcup U_i$ ; such that  $p$  lies in the image under  $Z(\sigma_1, \dots, \sigma_k)(V)$  of the image of  $(\pi, \varphi)$  in  $\mathrm{Sh}_S(Z)(U_1, \dots, U_k)(V) = Z(S_1 \sqcup U_1, \dots, S_k \sqcup U_k)(V)$ .

**Remark 4.3.1.2.** Recall Definition 4.2.4.3 of the image of a partition morphism. Explicitly, the conclusion above means that there exist finite sets  $U'_1, \dots, U'_l$  and, for each  $i \in [k]$ , a bijection  $\iota_i : U_i \rightarrow \bigsqcup_{j \in \pi^{-1}(i)} U'_j$ , and a point  $q \in Z'(U'_1, \dots, U'_l)(V)$  such that

$$(Z(\sigma_1, \dots, \sigma_k)(V) \circ (\mathrm{Sh}_S Z)(\iota_1, \dots, \iota_l)(V) \circ \varphi(U'_1, \dots, U'_l)(V))(q) = p.$$

Informally, we will say that all points in  $X$  are *hit* by finitely many partition morphisms from varieties  $Z'$  in  $\mathbf{PM}$  of product type with  $Z' \prec Z$ .  $\diamond$

### 4.3.2 A key proposition

The proof of Theorem 4.3.1.1 uses a key proposition that we establish first. The reader may prefer to read only the statement of this proposition and postpone its proof until after reading the proof of Theorem 4.3.1.1 in Section 4.3.5.

**Proposition 4.3.2.1.** *Let  $Y$  be a **Vec**-variety;  $n \in \mathbb{Z}_{\geq 0}$ ;  $B$  a closed **Vec**-subvariety of  $Y \times \mathbb{A}^n$ ;  $Q$  a pure polynomial functor; and  $X$  a proper closed **Vec**-subvariety of  $B \times Q \subseteq Y \times \mathbb{A}^n \times Q$ .*

*Then there exist a proper closed **Vec**-subvariety  $Y_0$  of  $Y$ , a **Vec**-variety  $Y'$  together with a morphism  $\alpha : Y' \rightarrow Y$ ;  $k \in \mathbb{Z}_{>0}$  and, for each  $l = 0, \dots, k$ , integers  $n_l \in \mathbb{Z}_{\geq 0}$ ; closed **Vec**-subvarieties  $B_l \subseteq Y' \times \mathbb{A}^{n_l}$ ; pure polynomial functors  $Q_l$ ; and morphisms  $\beta_l : B_l \times Q_l \rightarrow B \times Q$  such that the following properties hold:*

1. *For each  $l = 0, \dots, k$ ,  $B_l \times Q_l \prec B \times Q$  in the preorder from Section 4.2.8, and the following diagram commutes:*

$$\begin{array}{ccc} B_l \times Q_l & \xrightarrow{\beta_l} & B \times Q \\ \downarrow & & \downarrow \\ Y' & \xrightarrow{\alpha} & Y. \end{array}$$

2. *Let  $m \in \mathbb{Z}_{\geq 0}$ ,  $V \in \mathbf{Vec}$ , and points  $p_1, \dots, p_m \in X(V) \subseteq Y(V) \times \mathbb{A}^n \times Q(V)$  whose images in  $Y(V)$  are all equal to the same point  $y \in Y(V) \setminus Y_0(V)$ . Then there exist indices  $l_j \in \{0, \dots, k\}$  for  $j \in [m]$  and points  $p'_j \in B_{l_j}(V) \times Q_{l_j}(V)$  whose images in  $Y'(V)$  are all equal to the same point  $y'$  and such that  $\beta_{l_j}(V)(p'_j) = p_j$  for all  $j \in [m]$ .*

**Remark 4.3.2.2.** The condition  $\beta_{l_j}(V)(p'_j) = p_j$ , together with the commuting diagram in (1), implies  $\alpha(y') = y$ .  $\diamond$

To apply Proposition 4.3.2.1 in the proof of Theorem 4.3.1.1 we will do a shift over an appropriate  $k$ -tuple of finite sets. After this shift, we deal with the points of  $X$  lying over  $Y_0$  by induction, while we cover those in the complement by a partition morphism constructed with the morphisms  $\alpha$  and  $\beta_j$ 's, and whose domain is a product-type variety strictly smaller than  $Z$ . Before proving Proposition 4.3.2.1 in Section 4.3.4, we demonstrate its statement in two special cases.

**Example 4.3.2.3.** Consider the case where  $Y = \text{Spec } K$  and  $n = 0$ ; then  $B \subseteq Y \times \mathbb{A}^n$  is also isomorphic to  $\text{Spec } K$ . Let  $Q$  be an arbitrary polynomial functor. In this case,  $X$  is a proper closed **Vec**-subvariety of  $Q$  and by [BDES22] there exist  $k \in \mathbb{Z}_{>0}$ , (finite-dimensional) varieties  $B_1, \dots, B_k$ , pure polynomial functors  $Q_1, \dots, Q_k \prec Q$  and morphisms  $\beta_i : B_i \times Q_i \rightarrow Q$  such that  $X$  is the union of the images of the  $\beta_j$ . This is an instance of Proposition 4.3.2.1 with  $Y_0 = \emptyset$ ,  $Y' = Y$ , and  $\alpha = \text{id}_Y$ . Note that then  $B_j \times Q_j \prec Q$  since  $Q_j \prec Q$ , so the specific choice of embedding  $B_j \subseteq \mathbb{A}^{n_j}$  is not relevant.  $\diamond$

**Example 4.3.2.4.** Consider the case where  $Y$  is constant, that is, just given by a (finite-dimensional) variety, and  $Q = 0$ . Since  $X$  is a proper closed subvariety of  $B \subseteq Y \times \mathbb{A}^n$ , there exists a  $V \in \mathbf{Vec}$  and a nonzero function  $f \in K[B(V)]$  that vanishes identically on  $X(V)$ .

Then  $f$  is represented by a polynomial in  $K[Y(V)][x_1, \dots, x_n]$ , also denoted by  $f$ . We may reduce  $f$  modulo  $\mathcal{I}(B(V))$  in such a manner that its leading term  $c \cdot x^u$  has the property that  $c \in K[Y(V)]$  is nonzero and  $x^u \notin \mathrm{LM}(B)$ . Then we take for  $Y_0$  the closed subvariety of  $Y$  defined by the vanishing of  $c$  and for  $Y'$  the complement  $Y \setminus Y_0$ , with  $\alpha : Y' \rightarrow Y$  being the inclusion. Furthermore, we take  $k = 1$ , and  $B_1$  to be the intersection of  $B$  with  $Y' \times \mathbb{A}^n$  and with the vanishing locus of  $f$  in  $Y \times \mathbb{A}^n$ . Then  $\mathrm{LM}(B_1) \supseteq \mathrm{LM}(B)$  and since  $c$  is invertible on  $Y'$  and  $f$  vanishes on  $B_1$ ,  $x^u \in \mathrm{LM}(B_1) \setminus \mathrm{LM}(B)$ . To verify (2) of Proposition 4.3.2.1, we observe that the  $p_i$  all map to the same point in  $Y' = Y \setminus Y_0$ , i.e.,  $p_i$  lies in the set  $B_1 \subseteq B$ , and we can just take  $p'_j := p_j$  for all  $j$ .  $\diamond$

### 4.3.3 Iterated partial derivatives

The main idea for proving Proposition 4.3.2.1 comes from Lemma 4.3.3.1 below that is an extension (or better an iteration) of [Dra19, Lemma 18]. Indeed, the proof of topological Noetherianity for polynomial functors in [Dra19] states that we can find an equation  $f$  for  $X \subsetneq P$ , an irreducible component  $R$  of  $P$ , and a direction  $r_0$  such that  $\partial f / \partial r_0$  is not identically zero on  $X$ . The lemma below shows that we can iterate this process.

**Lemma 4.3.3.1.** *Let  $B$  be a  $\mathbf{Vec}$ -variety and  $Q$  a pure polynomial functor. Decompose*

$$Q = R_1 \oplus \dots \oplus R_t,$$

*where the  $R_i$  are irreducible objects in the abelian category of polynomial functors, arranged in weakly increasing degrees. Denote with  $R_{\leq s}$  the functor  $\bigoplus_{i=1}^s R_i$ , so that  $R_{\leq 0} = 0$ . Let  $X$  be a proper closed  $\mathbf{Vec}$ -subvariety of  $B \times Q$  and choose  $U_0 \in \mathbf{Vec}$  such that  $X(U_0) \subsetneq B(U_0) \times Q(U_0)$ . Then there exist*

- $a \in \mathbb{Z}_{\geq 0}$ ;
- vector spaces  $U_1, \dots, U_k$  with partial sums  $U_{\leq s} := \bigoplus_{i=0}^s U_i$  for  $s \geq 0$  (note that  $U_0$  is included in each of these);
- indices  $0 = s_0 < s_1 \leq \dots \leq s_k$ ;

- for each  $l \in \{0, \dots, k\}$  a nonzero function  $h_l \in K[B(U_{\leq l}) \times R_{\leq s_l}(U_{\leq l})]$ ; and
- for each  $l \in \{1, \dots, k\}$ , a nonzero coordinate  $x_l \in R(U_{s_l})^*$  and a function  $r_l$  in  $K[P'(U_{\leq l}) \times R_{\leq s_l}(U_{\leq l})/R_{s_l}(U_l)]$  such that

$$h_l = x_l \cdot h_{l-1} + r_l;$$

and such that, moreover, the function  $h_k$  vanishes on  $X(U_{\leq k})$ .

*Sketch.* Let  $f \in K[B(U_0) \times Q(U_0)]$  be a nonzero function that vanishes identically on  $X(U_0)$ . Choose  $s_k$  as the maximal index in  $[t]$  such that  $f$  involves coordinates in  $R_{s_k}(U_0)^*$ ; if no such index exists, then  $k$  is set to zero, and we may take  $h_0 := f \in K[B(U_0)]$ . For a subspace  $U_k$  of sufficiently high dimension (at least  $d_k := \deg(R_{s_k})$  suffices), act on  $f$  with “upper triangular” elements of the Lie algebra  $\mathfrak{gl}(U_0 \oplus U_k)$  that transform coordinates on  $R_{s_k}(U_0)^*$  to coordinates on  $R_{s_k}(U_k)^*$ . This yields a new polynomial that also vanishes on  $X(U_0 \oplus U_k)$  but is now of the form  $\tilde{f} = x_k \cdot \tilde{h} + \tilde{r}$ , where  $\tilde{h} \in K[B(U_0) \times Q(U_0)]$  is (nonzero and) of lower degree than  $f$ , and where  $\tilde{r}$  does not contain coordinates in  $R_{s_k}(U_k)^*$ . Now let  $s_{k-1}$  be the maximal index in such that  $\tilde{h}$  involves coordinates in  $R_{s_{k-1}}(U_0)^*$ . We will allow  $s_{k-1} = s_k$ , which will be the case if  $f$  was not linear in the coordinates in  $R_{s_k}(U_0)^*$ . Again choose a vector space  $U_{k-1}$  of sufficiently high dimension, and act on  $\tilde{f}$  with upper triangular elements of  $\mathfrak{gl}(U_0 \oplus U_{k-1})$  to obtain

$$\hat{f} = x_k \cdot (x_{k-1} \cdot \hat{h} + \hat{r}) + \bar{r}$$

where  $x_{k-1}$  is a coordinate in  $R_{s_{k-1}}(U_{k-1})^*$ ,  $\hat{r}$  does not involve coordinates in  $R_{s_{k-1}}(U_{k-1})^*$ ,  $\bar{r}$  may be different from  $\tilde{r}$ , but still does not involve coordinates in  $R_{s_k}(U_k)^*$ , and  $\hat{h} \in K[B(U_0) \times Q(U_0)]$  has smaller degree than  $\tilde{h}$ . Continuing in this fashion, we eventually find a polynomial

$$h_k = x_k(x_{k-1}(\dots(x_2(x_1 h_0 + r_1) + r_2)\dots) + r_{k-1}) + r_k \quad (4.3)$$

where  $h_0 \in K[B(U_0)]$ . Now it is clear how to define the intermediate  $h_l$ .  $\square$

#### 4.3.4 Proof of Proposition 4.3.2.1

This section contains the proof of the Proposition 4.3.2.1, and, for clarity’s sake, we spell it out in a concrete example at the end.

**Remark 4.3.4.1.** We recall that, for any **Vec**-variety  $Z$  and any  $U \in \mathbf{Vec}$ , the shift  $\mathrm{Sh}_U Z$  of  $Z$  over  $U$  is the **Vec**-variety defined by  $(\mathrm{Sh}_U Z)(V) = Z(U \oplus V)$ . There is a *natural morphism*  $\mathrm{Sh}_U Z \rightarrow Z$  of **Vec**-varieties: for  $V \in \mathbf{Vec}$ , this morphism  $(\mathrm{Sh}_U Z)(V) = Z(U \oplus V) \rightarrow Z(V)$  is just  $Z(\pi_V)$ , where  $\pi_V$  is the projection  $U \oplus V \rightarrow V$ .  $\diamond$

**Lemma 4.3.4.2.** *Let  $Y$  be a **Vec**-variety,  $n \in \mathbb{Z}_{\geq 0}$ , and  $B$  a closed **Vec**-subvariety of  $Y \times \mathbb{A}^n$ . Then for any  $U \in \mathbf{Vec}$ ,  $\mathrm{Sh}_U B$  is a closed **Vec**-subvariety of  $(\mathrm{Sh}_U Y) \times \mathbb{A}^n$ , and  $\mathrm{LM}(B) = \mathrm{LM}(\mathrm{Sh}_U(B))$ .*

*Proof.* This follows from Lemma 4.2.6.3.  $\square$

**Remark 4.3.4.3.** Let  $X$  be a **Vec**-variety,  $U \in \mathbf{Vec}$  and  $f \in K[X(U)]$ . We define  $(\mathrm{Sh}_U X)[1/f]$  to be the **Vec**-variety given by  $V \mapsto X(U \oplus V)[1/f]$ , where we identify  $f$  with its image under the natural map  $K[X(U)] \rightarrow K[X(U \oplus V)]$ . Note that the action of the group  $\mathrm{GL}$  on the coordinate ring of  $\mathrm{Sh}_U X$  is the identity on the element  $f$ . In particular, for every  $V \in \mathbf{Vec}$ ,  $(\mathrm{Sh}_U X[1/f])(V) \subseteq \mathrm{Sh}_U X(V)$  is the distinguished open set of points not vanishing on the single  $f$ .  $\diamond$

*Proof of Proposition 4.3.2.1.* Since  $X$  is a proper closed subvariety of  $B \times Q$ , there exist a  $U_0 \in \mathbf{Vec}$  such that  $X(U_0) \subsetneq B(U_0) \times Q(U_0)$ . As a first step, we apply the machinery of Lemma 4.3.3.1.

Decompose  $Q$  as  $R_1 \oplus \cdots \oplus R_t$ , where the  $R_s$  are irreducible polynomial functors and  $\deg(R_s) \leq \deg(R_{s+1})$  for all  $1 \leq s \leq t-1$ . Write  $R_{\leq s} := R_1 \oplus \cdots \oplus R_s$  and  $R_{>s} := R_{s+1} \oplus \cdots \oplus R_t$ , so that  $R_{\leq 0} = \{0\}$  and  $R_{>t} = \{0\}$ .

By Lemma 4.3.3.1, we can construct a sequence of vector spaces  $U_1, \dots, U_k$  with partial sums  $U_{\leq l} := \bigoplus_{i=0}^l U_i$  (note that  $U_{\leq 0} = U_0$ ), indices  $0 = s_0 < s_1 \leq \cdots \leq s_k \leq t$ , nonzero coordinates  $x_l \in R_{s_l}(U_l)^*$  for  $l \in [k]$ , nonzero functions  $h_l \in K[B(U_{\leq l}) \times R_{\leq s_l}(U_{\leq l})]$  for  $l = 0, \dots, k$  and functions  $r_l \in K[B(U_{\leq l}) \times (R_{\leq s_l}(U_{\leq l})/R_{s_l}(U_l))]$  for  $l \in [k]$  such that

$$h_l = x_l \cdot h_{l-1} + r_l \quad (\text{A})$$

for each  $l = 1, \dots, k$  and such that  $h_k$  vanishes on  $X(U_{\leq k})$ .

Now  $h_0 \in K[B(U_0)]$  is represented by a polynomial in

$$K[Y(U_0)][x_1, \dots, x_n],$$

and after reducing modulo  $\mathcal{I}(B(U_0))$ , we may assume that its leading term equals  $c \cdot x^u$  where  $c \in K[Y(U_0)]$  is nonzero and  $x^u \notin \mathrm{LM}(B)$ .

Now set  $U := U_{\leq k} = U_0 \oplus \cdots \oplus U_k$ . Then we construct the relevant data as follows.

1. Define  $Y_0$  as the closed **Vec**-subvariety of  $Y$  defined by the vanishing of  $c$ , so that

$$Y_0(V) := \{y \in Y(V) \mid \forall \varphi \in \text{Hom}_{\mathbf{Vec}}(V, U_0) : c(Y(\varphi)y) = 0\}.$$

2. Set  $Y' := (\text{Sh}_U Y)[1/c]$  with  $\alpha : Y' \rightarrow Y$  the restriction to  $Y'$  of the natural morphism  $\text{Sh}_U Y \rightarrow Y$ .
3. Let  $B_0$  be the closed **Vec**-subvariety of  $(\text{Sh}_U B)[1/c]$  defined by the vanishing of the single equation  $h_0$ . Note that  $B_0$  is a closed **Vec**-subvariety of  $Y' \times \mathbb{A}^{n_0}$  with  $n_0 := n$ . Define  $Q_0 := Q$  and  $\beta_0 : B_0 \times Q_0 \rightarrow B \times Q$  as the identity on  $Q$  and equal to the restriction to  $B_0$  of the natural morphism  $\text{Sh}_U B \rightarrow B$  on  $B_0$ . Note that  $\text{LM}(B_0) \supseteq \text{LM}(B)$  by virtue of Lemma 4.3.4.2, and since  $h_0 \in \mathcal{I}(B_0(U_0))$  has leading term  $c \cdot x^u$  and  $c$  is invertible on  $Y'$ , we have  $x^u \in \text{LM}(B_0) \setminus \text{LM}(B)$ . Thus  $B_0 \times Q_0 \prec B \times Q$ .
4. For  $l \in [k]$ , set

$$Q_l := ((\text{Sh}_U R_{\leq s_l}) / (R_{\leq s_l}(U) \oplus R_{s_l})) \oplus R_{> s_l}.$$

Here we recall that, for any pure polynomial functor  $R$ , the top-degree part of  $\text{Sh}_U R$  is naturally isomorphic to that of  $R$ , and its constant part is isomorphic to  $R(U)$  (see [Dra19, Lemma 14] for the first statement; the second is proved in a similar fashion). So, since we ordered the irreducible factors  $R_s$  by ascending degrees,  $R_{s_l}$  is naturally a sub-object of the top-degree part of  $\text{Sh}_U R_{\leq s_l}$ ; and the constant polynomial functor  $R_{\leq s_l}(U)$  is the constant part of  $\text{Sh}_U R_{\leq s_l}$ . Both are modded out, and we have  $Q_l \prec Q$ .

5. For  $l \in [k]$ , we define  $B_l$  as

$$\begin{aligned} B_l &:= (\text{Sh}_U B)[1/c] \times R_{\leq s_l}(U) \times \mathbb{A}^1 \\ &\subseteq Y' \times \mathbb{A}^n \times R_{\leq s_l}(U) \times \mathbb{A}^1 \cong Y' \times \mathbb{A}^{n_l}. \end{aligned}$$

where  $n_l := n + \dim(R_{\leq s_l}(U)) + 1$ . Note that the factor  $R_{\leq s_l}(U)$  is precisely the constant term modded out in the definition of  $Q_l$ ; the role of the factor  $\mathbb{A}^1$  will become clear below.

6. To construct  $\beta_l : B_l \times Q_l \rightarrow B \times Q$  we proceed as follows. Let  $X_l$  be the closed **Vec**-subvariety of  $B \times R_{\leq s_l}$  defined by the vanishing of  $h_l$ . Then (A) shows that, on the distinguished open subset  $(\text{Sh}_{U_{\leq l-1}} X_l)[1/h_{l-1}]$ , the coordinate  $x_l$  can be expressed as a function on

$$\text{Sh}_{U_{\leq l-1}} B \times ((\text{Sh}_{U_{\leq l-1}} R_{\leq s_l}) / R_{s_l})$$

evaluated at  $U_l$ . Since  $R_{s_l}$  is irreducible, *each* coordinate on it can be thus expressed; this is a crucial point in the proof of [Dra19, Lemma 25]. This implies that the projection

$$\mathrm{Sh}_{U_{\leq l-1}} B \times \mathrm{Sh}_{U_{\leq l-1}} R_{\leq s_l} \rightarrow (\mathrm{Sh}_{U_{\leq l-1}} B) \times (\mathrm{Sh}_{U_{\leq l-1}} R_{\leq s_l})/R_{s_l}$$

restricts to a closed immersion of  $(\mathrm{Sh}_{U_{\leq l-1}} X_l)[1/h_{l-1}]$  into the open subset of the right-hand side where  $h_{l-1}$  is nonzero. This statement remains true when we replace  $U_{\leq l-1}$  everywhere by the larger space  $U$ . After also inverting  $c$ , we find a closed immersion

$$(\mathrm{Sh}_U X_l)[1/h_{l-1}][1/c] \rightarrow (\mathrm{Sh}_U B)[1/c] \times (\mathrm{Sh}_U R_{\leq s_l})/R_{s_l} \times \mathbb{A}^1,$$

where the map to the last factor is given by  $1/h_{l-1}$ . By [Bik20, Proposition 1.3.22] the inverse morphism from the image of this closed immersion lifts to a morphism of ambient **Vec**-varieties

$$\begin{aligned} \iota : B_l \times (\mathrm{Sh}_U R_{\leq s_l})/(R_{\leq s_l}(U) \oplus R_{s_l}) \\ \cong (\mathrm{Sh}_U B)[1/c] \times (\mathrm{Sh}_U R_{\leq s_l})/R_{s_l} \times \mathbb{A}^1 \\ \rightarrow \mathrm{Sh}_U(B \times R_{\leq s_l}) \end{aligned}$$

that hits all the points in  $(\mathrm{Sh}_U X_l)[1/h_{l-1}][1/c]$ . Finally, we define  $\beta_l := \beta'_l \times \mathrm{id}_{R_{>s_l}}$  where  $\beta'_l$  is the composition of  $\iota$  and the natural morphism  $\mathrm{Sh}_U(B \times R_{\leq s_l}) \rightarrow B \times R_{\leq s_l}$ .

Property (1) in the proposition holds by construction. We now verify property (2). Thus let  $V \in \mathbf{Vec}$ ,  $m \in \mathbb{Z}_{\geq 0}$ , and let  $p_1, \dots, p_m \in X(V) \subseteq Y(V) \times \mathbb{A}^n \times Q(V)$ . Assume that the images of  $p_1, \dots, p_m$  in  $Y(V)$  are all equal to  $y$ , and that  $y \notin Y_0(V)$ . By definition of  $Y_0$ , this means that there exists a  $\varphi \in \mathrm{Hom}_{\mathbf{Vec}}(V, U)$  such that  $c(Y(\varphi)(y)) \neq 0$ .

On the other hand, we have  $h_k(X(\psi)(p_j)) = 0$  for all  $j$  and all  $\psi : V \rightarrow U$ , because  $h_k$  vanishes identically on  $X$ . For  $j \in [k]$  define

$$l_j := \min\{l \mid \forall \psi \in \mathrm{Hom}_{\mathbf{Vec}}(V, U) : h_l(X(\psi)(p_j)) = 0\}.$$

Put differently,  $l_j$  is the smallest index  $l$  such that the projection of  $p_j$  in  $B \times R_{\leq s_l}$  lies in  $X_l \subseteq B \times R_{\leq s_l}$ . Note that, if  $l_j > 0$ , then there exists a linear map  $\psi : V \rightarrow U$  such that  $h_{l_j-1}(X(\psi)(p_j)) \neq 0$ .

Since  $\mathrm{Hom}_{\mathbf{Vec}}(V, U)$  is irreducible, there exists a linear map  $\varphi : V \rightarrow U$  such that both,  $c(Y(\varphi)(y))$  and  $h_{l_j-1}(X(\varphi)(p_j))$  are not equal to zero for all  $j$  with  $l_j > 0$ .

We now define the  $p'_j$  as follows. First, we decompose  $p_j = (p_{j,1}, p_{j,2})$  where  $p_{j,1} \in B(V) \times R_{\leq s_{l_j}}(V)$  and  $p_{j,2} \in R_{> s_{l_j}}(V)$ . Similarly, we decompose the point  $p'_j = (p'_{j,1}, p'_{j,2})$  to be constructed.

1. Set  $p'_{j,2} := p_{j,2}$  for all  $j$ . Recall that we had defined  $s_0 := 0$ , so that this implies that if  $l_j = 0$ , then the component  $p'_{j,2}$  of  $p'_j$  in  $Q$  equals the component  $p_{j,2}$  of  $p_j$  in  $Q$ .
2. If  $l_j = 0$ , then  $p_{j,1}$  is a point in  $B(V)$ , and  $p'_{j,1} \in B_0(V) \subseteq (\text{Sh}_U B)[1/c](V)$  is defined as  $B(\varphi \oplus \text{id}_V)(p_{j,1})$ . Note that  $p'_{j,1}$  does indeed lie in  $B_0(V)$ ; this follows from the fact  $l_j = 0$ , so that  $h_0(B(\psi)(p_{j,1})) = 0$  for all  $\psi : V \rightarrow U_0$ , and hence also for all  $\psi$  that decompose as  $\psi' \circ (\varphi \oplus \text{id}_V)$ .

Furthermore, note that  $\beta_0(V)(p'_j) = p_j$ ; this follows from the equality  $\pi_V \circ (\varphi \oplus \text{id}_V) = \text{id}_V$ . Also, the image of  $p'_j$  in  $Y'(V)$  equals  $Y(\varphi \oplus \text{id}_V)(y) =: y'$ .

3. If  $l := l_j > 0$ , then  $p_{j,1} \in B(V) \times R_{\leq s_l}(V)$  with  $s_l \geq 1$ , and  $p'_{j,1}$  is constructed as follows. First apply  $(B \times R_{\leq s_l})(\varphi \oplus \text{id}_V)$  to  $p_{j,1}$  and then forget the component in  $R_{s_l}(V)$ . The morphism  $\beta'_l$  was constructed in such a manner that  $\beta'_l(V)(p'_{j,1}) = p_{j,1}$  and therefore  $\beta_l(V)(p'_j) = p_j$ . Note that also the image of  $p'_j$  in  $Y'(V)$  equals  $y'$ . This concludes the proof.  $\square$

**Example 4.3.4.4.** Let  $Y$  be the polynomial functor defined by  $Y(V) := V \oplus V$  and take  $B \subseteq Y \times \mathbb{A}^1$  the closed subvariety

$$B(V) := \{(v, tv, t) \mid v \in V, t \in K\}$$

defined by the equation  $y_1 - sx_1$ , where we write  $x_i, y_i$  for the coordinates on  $Y(K^n) = K^n \oplus K^n$  and  $s$  for the coordinate on  $\mathbb{A}^1$ . Note that  $\text{LM}(B) = \emptyset$ . Consider  $Q(V) := S^2V$ , the symmetric tensors of  $V \otimes V$ , and let  $X \subseteq B(V) \times Q(V)$  be defined by the  $2 \times 2$ -minors expressing that, in a quadruple  $(v, w, t, q)$ , the pair  $w^2, q$  is linearly dependent. Take for the original equation  $f$  one of these  $2 \times 2$ -minors, and compute all the data as in the proof.  $\diamond$

**Example 4.3.4.5.** Write  $Y$  for the polynomial functor  $V \rightarrow V \oplus V$  and write  $K[x_i, y_i \mid i \in [n]]$  for the coordinate ring of  $Y(K^n)$ . Consider the **Vec**-subvariety  $B$  of  $Y \times \mathbb{A}^1$  defined by  $y_1 - t \cdot x_1$ , where  $t$  is the coordinate of  $\mathbb{A}^1$ . Then  $\text{LM}(B) = \emptyset$  and  $B(V)$  is the set of triples  $(v, \lambda v, \lambda)$  with  $v \in V$  and  $\lambda \in K$ . Set  $Q(V) := S^2V$ , and choose coordinates  $z_{ij}, i \leq j$  on  $Q(K^n)$  by

writing an arbitrary element of  $Q(K^n)$  as

$$\sum_{i=1}^n z_{ii} e_i^2 + \sum_{1 \leq i < j \leq n} 2z_{ij} e_i e_j.$$

Note that  $Q$  is an irreducible polynomial functor, so, in the notation of Proposition 4.3.2.1, we have  $R = R_1 = Q$ . Define the **Vec**-subvariety  $X \subset B \times Q \subset Y \times \mathbb{A}^1 \times Q$  by

$$X(V) := \{(v, w, \lambda, q) \mid (v, w, \lambda) \in B(V) \text{ and } w^2, q \text{ are linearly dependent}\}.$$

An equation for  $X(K^2)$  is the determinant

$$f := z_{12}y_1^2 - z_{11}y_1y_2 = t^2(z_{12}x_1^2 - z_{11}x_1x_2) \in K[B(U_0) \times Q(U_0)]$$

with  $U_0 := K^2$ . Define  $U_1 := \langle e_3, e_4 \rangle \cong K^2$ , so that  $U_0 \oplus U_1 = K^4$ . Acting on  $f$  equation with the (upper triangular) elements  $E_{1,3}$  and  $E_{2,4}$  of the Lie algebra  $\mathfrak{gl}(U_0 \oplus U_1)$  gives the equation:

$$h_1 := z_{34}(x_1^2 t^2) + (2z_{14}x_1x_3 - 2z_{13}x_1x_4 - z_{11}x_3x_4)t^2$$

that, by construction, vanishes on  $X(U_0 \oplus U_1)$ . Note that  $z_{34} \in Q(U_1)^*$ ,  $h_0 := x_1^2 t^2 \in K[B(U_0)]$  (and we let  $c$  be the leading coefficient:  $c := x_1^2$ ), and the rest belongs to  $K[B(U_0 \oplus U_1) \times Q(U_0 \oplus U_1)/Q(U_1)]$ .

By acting with permutations  $(3, i)$  and  $(4, j)$  with  $i < j$  on  $h_1$  we find that, where  $h_0$  is nonzero, on  $X$  we have

$$z_{ij} = -\frac{1}{h_0} \cdot (2z_{1j}x_1x_i - 2z_{1i}x_1x_j - z_{11}x_ix_j)t^2. \quad (4.4)$$

A similar expression can be found for  $z_{ii}$ , with the same denominator  $h_0$ .

In this case,  $Y_0$  from the proposition is the **Vec**-subvariety of  $Y$  defined by  $c = x_1^2$ . This consists of all pairs  $(0, w) \in V \oplus V$ . The preimage in  $X$  consists of all quadruples  $(0, 0, \lambda, q)$  with  $q$  arbitrary.

Set  $U := U_0 \oplus U_1$ ,  $Y' := \text{Sh}_U Y[1/c]$ , and let  $B_0$  be the vanishing locus of  $h_0$  in  $\text{Sh}_U B[1/c] \subset Y' \times \mathbb{A}^1$ . Note that we have  $t^2 \in \text{LM}(B_0)$ —indeed,  $t$  even vanishes identically on  $B_0$ . With  $Q_0 := Q$  we find  $B_0 \times Q_0 \prec B \times Q$ , and we define the map:

$$\beta_0 : B_0 \times Q_0 \rightarrow B \times Q$$

as  $B(\pi_V)|_{B_0} \times \text{id}_{Q(V)}$  for every  $V \in \mathbf{Vec}$ . This covers all the points in  $X(V)$  of the form  $(v, 0, 0, q)$  with  $v, q$  arbitrary.

Finally, consider the map:

$$\begin{aligned} \mathrm{Sh}_U(B \times Q)[1/h_0][1/c] &\rightarrow \mathrm{Sh}_U(B \times Q)/Q \times \mathbb{A}^1 \\ &\cong (\mathrm{Sh}_U B \times Q(U) \times \mathbb{A}^1) \times (\mathrm{Sh}_U Q/(Q(U) \oplus Q)) \\ &=: B_1 \times Q_1 \end{aligned}$$

where the coordinate on  $\mathbb{A}^1$  is given by  $1/h_0$ . This is a closed immersion because where  $h_0$  is nonzero, coordinates on  $Q(V)$  can be recovered from the coordinates on the right-hand side via (4.4). We use this to construct the map

$$\beta_1 : B_1 \times Q_1 = \mathrm{Sh}_U(B \times Q)/Q \times \mathbb{A}^1 \rightarrow \mathrm{Sh}_U(B \times Q) \rightarrow B \times Q.$$

The first arrow is given by the identity on the coordinates not in  $Q(V)$ , while the coordinates on  $Q(V)$  are computed via (4.4). The second arrow projects into  $B(V) \times Q(V)$ . This map hits points in  $X(V)$  of the form  $(v, \lambda v, \lambda, \mu(\lambda v)^2)$  with  $v, \lambda$  nonzero.  $\diamond$

### 4.3.5 Proof of Theorem 4.3.1.1

*Proof of Theorem 4.3.1.1.* The  $(\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec}$ -variety  $Z$  is of product type, hence by Definition 4.2.5.1 it can be written as

$$Z = [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$$

for some  $\mathbf{Vec}$ -subvarieties  $B_i$  of  $Y \times \mathbb{A}^{n_i}$  and pure polynomial functors  $Q_i$ . Furthermore,  $X$  is a proper closed  $(\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec}$ -subvariety of  $Z$ .

We prove, by induction on the quantity  $\delta_X$ , that all points in  $X$  can be hit by partition morphisms from finitely many  $(\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec}$ -varieties  $Z'$  of product type with  $Z' \prec Z$ . So in the proof we may assume that this is true for all proper closed  $(\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec}$ -subvarieties  $X' \subsetneq Z$  with  $\delta_{X'} < \delta_X$ .

Let  $(S_1, \dots, S_k) \in \mathbf{FI}^k$  be such that  $\sum_i |S_i| = \delta_X$  and  $X(S_1, \dots, S_k) \neq Z(S_1, \dots, S_k)$ . If all  $S_i$  are empty, then set  $Y' := X(\emptyset, \dots, \emptyset)$ , a proper closed  $\mathbf{Vec}$ -subvariety of  $Y$ ,  $B'_i := Y' \times_Y B_i$ , and  $Z := [Y'; B'_1 \times Q_1, \dots, B'_k \times Q_k]$ . The partition morphism  $(\mathrm{id}_{[k]}, \varphi)$  with  $\varphi(T_1, \dots, T_k)$  the inclusion  $\prod_i (B'_i \times Q_i)^{T_i} \rightarrow \prod_i (B_i \times Q_i)^{T_i}$  has  $X$  in its image, and we have  $Z' \prec Z$  because the  $Q_i$  remain the same,  $\mathrm{LM}(B'_i) \supseteq \mathrm{LM}(B_i)$  by Lemma 4.2.6.3, and  $Y'$  is a proper closed  $\mathbf{Vec}$ -subvariety of  $Y$ . In this case, no shift of  $Z$  is necessary.

Next assume that not all  $S_i$  are empty. First, we argue that the points of  $X(T_1, \dots, T_k)$  where, for some  $i$ ,  $|T_i|$  is strictly smaller than  $|S_i|$ , are hit by partition morphisms from finitely many  $Z' \prec Z$ . We give the argument for

$i = k$ . Define the  $k$ -tuple  $S$  to be shifted over as  $S := (\emptyset, \dots, \emptyset, T_k) \in \mathbf{FI}^k$ , and define the  $(\mathbf{FI}^{\mathbf{op}})^{k-1} \times \mathbf{Vec}$ -variety  $Z'$  of product type

$$Z' := [(B_k \times Q_k)^{T_k}; B'_1 \times Q_1, \dots, B'_{k-1} \times Q_{k-1}]$$

with  $B'_i = (B_k \times Q_k)^{T_k} \times_Y B_i$ . Consider the partition morphism  $(\pi, \varphi) : Z' \rightarrow \mathrm{Sh}_S Z$  where  $\pi : [k-1] \rightarrow [k]$  is the inclusion and  $\varphi(T_1, \dots, T_{k-1})$  is the natural isomorphism of  $\mathbf{Vec}$ -varieties

$$Z'(T_1, \dots, T_{k-1}) \rightarrow (\mathrm{Sh}_S Z)(T_1, \dots, T_{k-1}, \emptyset) = Z(T_1, \dots, T_{k-1}, T_k).$$

Note that  $\pi$  witnesses  $Z' \preceq Z$  since the  $Q_i$  with  $i \leq k-1$  remain the same and  $\mathrm{LM}(B'_i) \supseteq \mathrm{LM}(B_i)$  by Lemma 4.2.6.3. Furthermore, since  $k-1 < k$ , we have  $Z' \prec Z$  by Lemma 4.2.8.4. All points in  $X$  where the last index set has cardinality  $|T_k|$  are hit by this partition morphism. Since there are only finitely many values of  $|T_k|$  that are strictly smaller than  $|S_k|$ , we are done.

So it remains to hit points in  $X(T_1, \dots, T_k)$  where  $|T_i| \geq |S_i|$  for all  $i$ . In this phase, we will apply Proposition 4.3.2.1.

As by assumption not all  $S_i$  are empty, after a permutation of  $[k]$  we may assume that  $S_k \neq \emptyset$ . Let  $*$  be an element of  $S_k$  and define  $\widetilde{S}_k := S_k \setminus \{*\}$ . Consider the  $\mathbf{Vec}$ -varieties

$$\begin{aligned} & Z(S_1, \dots, S_k) \\ &= (B_1 \times Q_1)_{Y^1}^{S_1} \times_Y \cdots \times_Y (B_k \times Q_k)_{Y^k}^{\widetilde{S}_k} \times_Y (B_k \times Q_k)^{\{*\}} \text{ and} \\ & \widetilde{Y} := Z(S_1, \dots, S_{k-1}, \widetilde{S}_k) = (B_1 \times Q_1)_{Y^1}^{S_1} \times_Y \cdots \times_Y (B_k \times Q_k)_{Y^k}^{\widetilde{S}_k}. \end{aligned}$$

Set  $\widetilde{B}_k := \widetilde{Y} \times_Y B_k \subseteq \widetilde{Y} \times \mathbb{A}^{n_k}$ , and note that  $X(S_1, \dots, S_k)$  is a proper closed  $\mathbf{Vec}$ -subvariety of  $\widetilde{B}_k \times Q_k$ . We may therefore apply Proposition 4.3.2.1 to  $\widetilde{Y}, n_k, \widetilde{B}_k, Q_k$  and  $X(S_1, \dots, S_k)$ .

First consider the proper closed  $\mathbf{Vec}$ -subvariety  $Y_0$  of  $\widetilde{Y}$  promised by Proposition 4.3.2.1, and let  $X'$  be the largest closed  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -subvariety of  $Z$  that intersects  $Z(S_1, \dots, S_{k-1}, \widetilde{S}_k)$  in  $Y_0$ . Then  $X'(S_1, \dots, \widetilde{S}_k) \neq Z(S_1, \dots, \widetilde{S}_k)$ , and therefore  $\delta_{X'} \leq \delta_X - 1 < \delta_X$ . Hence, by the induction hypothesis, all points in  $X'(T_1, \dots, T_k)$  can be hit by finitely many partition morphisms from varieties  $Z' \prec Z$  of product type.

Next, we consider the remaining pieces of data from Proposition 4.3.2.1. First, we have the  $\mathbf{Vec}$ -variety  $Y'$  with a morphism  $\alpha : Y' \rightarrow \widetilde{Y}$ . Further, we have an integer  $s \in \mathbb{Z}_{\geq 0}$  and for each  $i = 0, \dots, s$  we have integers  $n'_{k+i}$ ;  $\mathbf{Vec}$ -varieties  $B'_{k+i} \subseteq Y' \times \mathbb{A}^{n'_{k+i}}$ ; pure polynomial functors  $Q'_{k+i}$ ; and morphisms  $\beta_{k+i} : B'_{k+i} \times Q'_{k+i} \rightarrow \widetilde{B}_k \times Q_k$  satisfying the conditions (1) and (2).

Define  $B'_i := Y' \times_Y B_i$  for  $i = 1, \dots, k-1$  and the  $(\mathbf{FI}^{\mathbf{op}})^{k+s} \times \mathbf{Vec}$ -variety

$$Z' := [Y'; B'_1 \times Q_1, \dots, B'_{k-1} \times Q_{k-1}, B'_k \times Q'_k, \dots, B'_{k+s} \times Q'_{k+s}].$$

Now the map  $\pi : [k+s] \rightarrow [k]$  that is the identity on  $[k-1]$  and maps  $[k+s] \setminus [k-1]$  to  $\{k\}$  witnesses that  $Z' \preceq Z$ ; here we use that  $B'_{k+j} \times Q'_{k+j} \prec B_k \times Q_k$  for  $j \in \{0, \dots, s\}$  by the conclusion of Proposition 4.3.2.1, and also Lemma 4.2.6.3 to show that  $B'_i \times Q_i \preceq B_i \times Q_i$  for  $i \in [k-1]$ . In fact, we have  $Z' \prec Z$  by Lemma 4.2.8.4.

Now the base variety  $Y'$  of  $Z'$  comes with a morphism  $\alpha$  to the base variety  $\tilde{Y}$  of  $\mathrm{Sh}_S Z$ ; we have morphisms  $\beta_i : B'_i \times Q_i \rightarrow \widetilde{B_i \times Q_i}$  for  $i = 1, \dots, k-1$  (the natural map  $B'_i \rightarrow \widetilde{B_i}$  times the identity on  $Q_i$ ) and the morphisms  $\beta_{k+j} : B'_{k+j} \times Q'_{k+j} \rightarrow \widetilde{B_k \times Q_k}$  defined earlier. By Example 4.2.5.4, these data yield a partition morphism  $(\pi, \varphi) : Z' \rightarrow \mathrm{Sh}_S Z$ . We have to show that this partition morphism hits all points in  $X$  that are not in  $X'$ .

First we show, for a finite-dimensional vector space  $V$ , that a point  $p \in \mathrm{Sh}_S X(\widetilde{T_1}, \dots, \widetilde{T_k})(V)$  whose projection to  $\tilde{Y}(V)$  is not in  $Y_0(V)$  lies in the image of  $\varphi(\widetilde{T_1}, \dots, \widetilde{T_k})(V)$ . To this end, we write

$$p = ((p_{i,t})_{t \in \widetilde{T_i}})_{i \in [k]}$$

with

$$\begin{aligned} p_{i,t} &\in \mathrm{Sh}_S X(\emptyset, \dots, \emptyset, \{t\}, \emptyset, \dots, \emptyset)(V) \\ &= \tilde{Y}(V) \times_{Y(V)} B_i(V) \times Q_i(V) \subset \tilde{Y}(V) \times \mathbb{A}^{n_i} \times Q_i(V) \end{aligned}$$

where the singleton  $\{t\}$  is in the  $i$ -th position. We write  $p_{i,t} = (\tilde{y}, a_{i,t}, b_{i,t})$  with  $\tilde{y} \in \tilde{Y}(V)$ ,  $a_{i,t} \in \mathbb{A}^{n_i}$ , and  $b_{i,t} \in Q_i(V)$ .

By definition of a fibre product, the  $p_{i,t}$  all have the same projection  $\tilde{y}$  in  $\tilde{Y}(V) \setminus Y_0(V)$ , and hence we can apply (2) of Proposition 4.3.2.1 to the points  $p_{k,t}$  with  $t \in \widetilde{T_k}$ . This yields integers  $l_t \in \{0, \dots, s\}$  and points  $p'_{k,t} \in B'_{k+l_t}(V) \times Q'_{k+l_t}(V)$  for  $t \in \widetilde{T_k}$  whose images in  $Y'(V)$  are all equal, say to  $y' \in Y'(V)$ , and which satisfy  $\beta_{k+l_t}(V)(p'_{k,t}) = p_{k,t}$  for all  $t$ . This implies that  $\alpha(y') = \tilde{y}$ .

Define

$$T'_{k+j} := \{t \in \widetilde{T_k} \mid l_t = j\}$$

$j = 0, \dots, s$ , and set  $T'_i := \widetilde{T_i}$  for  $i = 1, \dots, k-1$ . In  $Z'(T'_1, \dots, T'_{k+s})$  we define the point  $q = ((q_{i,t})_{t \in T'_i})_{i \in [k+s]}$  as follows. We set  $q_{i,t}$  to be  $(y', a_{i,t}, b_{i,t})$  for

$i = 1, \dots, k-1$  and  $t \in T'_i$ , and  $q_{i,t} = p'_{k,t}$  for  $i = k, \dots, k+s$  and  $t \in T'_i$ . Then

$$\varphi(T'_1, \dots, T'_{k+s})(q) = p,$$

as desired.

Now, more generally, consider a point  $p$  in  $X(T_1, \dots, T_k)(V) \setminus X'(T_1, \dots, T_k)(V)$ , where the cardinalities satisfy  $|T_i| \geq |S_i|$ . Then there exists an  $\mathbf{FI}^k$ -morphism  $\iota = (\iota_1, \dots, \iota_k) : S \rightarrow (T_1, \dots, T_k)$  such that  $X(\iota)(p) \notin Y_0(V)$ . Define  $\widetilde{T}_i := T_i \setminus \text{Im}(\iota_i)$  and extend  $\iota$  to an isomorphism  $\iota^e : S \sqcup (\widetilde{T}_1, \dots, \widetilde{T}_k) \rightarrow (T_1, \dots, T_k)$  by defining  $\iota_i$  on  $\widetilde{T}_i$  to be the inclusion. Consider  $X(\iota^e)(p) \in X(S \sqcup (\widetilde{T}_1, \dots, \widetilde{T}_k))(V)$ . This is also a point in  $\text{Sh}_S X(\widetilde{T}_1, \dots, \widetilde{T}_k)(V)$  whose projection to  $\widetilde{Y}(V)$  does not lie in  $Y_0(V)$ . We can therefore find a point  $q$  as described above showing that  $X(\iota^e)(p)$  is in the image of  $(\pi, \varphi) : Z' \rightarrow \text{Sh}_S Z$ ; by Definition 4.2.4.3, then so is  $p$ .  $\square$

## 4.4 Proof of the main theorem

The most general version of our Noetherianity result is the following.

**Theorem 4.4.0.1.** *Any  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -variety of product type is Noetherian.*

*Proof.* We proceed by induction along the well-founded order on objects of product type in  $\mathbf{PM}$  from Section 4.2.8.

Let  $Z$  be an  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -variety of product type and let  $X_1 \supseteq X_2 \supseteq \dots$  be a descending chain of closed  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -subvarieties. Then either all  $X_i$  are equal to  $Z$ , or there exists an  $i_0$  such that  $X := X_{i_0}$  is a proper closed  $(\mathbf{FI}^{\text{op}})^k \times \mathbf{Vec}$ -subvariety of  $Z$ . In the latter case, by Theorem 4.3.1.1, there exist a finite number of objects  $Z_1, \dots, Z_N$  in  $\mathbf{PM}$  of product type, along with  $k$ -tuples  $S_1, \dots, S_N \in \mathbf{FI}^k$  and partition morphisms  $(\pi_j, \varphi_j) : Z_j \rightarrow \text{Sh}_{S_j} Z$  such that every point of  $X$  is hit by one of these. By the induction hypothesis, all  $Z_j$ s are Noetherian. For each  $j$ , by Lemma 4.2.4.6, the preimage in  $Z_j$  of the chain  $(\text{Sh}_{S_j} X_i)_{i \geq i_0}$  is a chain of closed subvarieties, which therefore stabilizes. As soon as these  $N$  chains have all stabilized, then so has the chain  $(X_i)_i$ —here we have used a version of Proposition 4.2.4.7.  $\square$

To deduce from this Theorems 4.1.2.1 and 4.1.3.1, we consider GL-varieties  $Z_1, \dots, Z_k$  as well as the product  $Z := Z_1^{\mathbb{N}} \times \dots \times Z_k^{\mathbb{N}}$ . Recall Remark 4.2.3.3.

*Proof of Theorem 4.1.3.1.* We need to prove that any descending chain  $Z \supseteq X_1 \supseteq \dots$  of  $\text{Sym}^k \times \text{GL}$ -stable subsets of  $Z$  stabilizes.

To each  $Z_i$  is associated a **Vec**-variety, which by abuse of notation we also denote  $Z_i$ ; see Remark 4.2.2.6. Furthermore,  $Z_i$  is a closed subvariety of  $B_i \times Q_i$  for some finite-dimensional variety  $B_i$  and some pure polynomial functor  $Q_i$ , and hence  $Z$  is a closed subvariety of

$$(B_1 \times Q_1)^{\mathbb{N}} \times \cdots \times (B_k \times Q_k)^{\mathbb{N}}.$$

Now each  $X_i$  defines a closed  $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$ -subvariety  $\widetilde{X}_i$  of

$$\widetilde{Z} := [Y; B_1 \times Q_1, \dots, B_k \times Q_k],$$

where  $Y$  is a point. By Theorem 4.4.0.1, the  $\widetilde{X}_i$  stabilize. As soon as they do, so do the  $X_i$ . □

*Proof of the Main Theorem.* Apply Theorem 4.1.3.1 with  $k = 1$ . □



# Bibliography

- [AH07] Matthias Aschenbrenner and Christopher J. Hillar. “Finite generation of symmetric ideals”. In: *Trans. Amer. Math. Soc.* 359.11 (2007), pp. 5171–5192.
- [AKS23] Yulia Alexandr, Joe Kileel, and Bernd Sturmfels. *Moment Varieties for Mixtures of Products*. 2023. arXiv: 2301.09068 [math.AG].
- [BD11] Andries E. Brouwer and Jan Draisma. “Equivariant Gröbner bases and the Gaussian two-factor model”. In: *Math. Comput.* (2011), pp. 1123–1133.
- [BDE19] Arthur Bik, Jan Draisma, and Rob H. Eggermont. “Polynomials and tensors of bounded strength”. English. In: *Commun. Contemp. Math.* 21.7 (2019). Id/No 1850062, p. 24.
- [BDES22] Arthur Bik, Jan Draisma, Rob H. Eggermont, and Andrew Snowden. “The Geometry of Polynomial Representations”. In: *International Mathematics Research Notices* (2022).
- [Bie+20] Jennifer Biermann, Hernán de Alba, Federico Galetto, Satoshi Murai, Uwe Nagel, Augustine O’Keefe, Tim Römer, and Alexandra Seceleanu. “Betti numbers of symmetric shifted ideals”. In: *Journal of Algebra* 560 (2020), pp. 312–342.
- [Bik20] Arthur Bik. “Strength and noetherianity for infinite tensors”. Ph.D. thesis. Universität Bern, June 2020.
- [CEF15] Thomas Church, Jordan S. Ellenberg, and Benson Farb. “FI-modules and stability for representations of symmetric groups”. In: *Duke Mathematical Journal* 164.9 (2015).
- [Chi+22] Christopher Chiu, Alessandro Danelon, Jan Draisma, Rob H. Eggermont, and Azhar Farooq. *Sym Noetherianity for powers of GL-varieties*. 2022. arXiv: 2212.05790.

- [Coh67] D. E. Cohen. “On the laws of a metabelian variety”. In: *J. Algebra* 5 (1967), pp. 267–273.
- [Coh87] Daniel E. Cohen. “Closure relations. Buchberger’s algorithm, and polynomials in infinitely many variables”. In: *Computation theory and logic*. Vol. 270. Lecture Notes in Comput. Sci. Springer, Berlin, 1987, pp. 78–87.
- [DEF22] Jan Draisma, Rob Eggermont, and Azhar Farooq. “Components of symmetric wide-matrix varieties”. In: *J. Reine Angew. Math.* 793 (2022), pp. 143–184.
- [DEFM22] Jan Draisma, Rob H. Eggermont, Azhar Farooq, and Leandro Meier. *Image closure of symmetric wide-matrix varieties*. 2022. arXiv: 2212.12458.
- [DEKL16] Jan Draisma, Rob H. Eggermont, Robert Krone, and Anton Leykin. “Noetherianity for infinite dimensional toric varieties”. In: *Algebra & Number Theory* 9.8 (2016), pp. 1857–1880.
- [DES17] Harm Derksen, Rob Eggermont, and Andrew Snowden. “Topological noetherianity for cubic polynomials”. English. In: *Algebra Number Theory* 11.9 (2017), pp. 2197–2212.
- [DK14] Jan Draisma and Jochen Kuttler. “Bounded-rank tensors are defined in bounded degree”. In: *Duke Mathematical Journal* 163.1 (2014).
- [DKK] Jan Draisma, Alexei Krasilnikov, and Robert Krone. **FI-Algebras, OI-Algebras,  $S_\infty$ -ALGEBRAS: examples and counterexamples**. Manuscript in preparation.
- [DLL19] Jan Draisma, Michał Lasoń, and Anton Leykin. “Stillman’s conjecture via generic initial ideals”. English. In: *Commun. Algebra* 47.6 (2019), pp. 2384–2395.
- [Dra10] Jan Draisma. “Finiteness for the  $k$ -factor model and chirality varieties”. In: *Adv. Math.* 223.1 (2010), pp. 243–256.
- [Dra14] Jan Draisma. “Noetherianity up to symmetry”. In: *Combinatorial algebraic geometry*. Vol. 2108. Lecture Notes in Math. Springer, Cham, 2014, pp. 33–61.
- [Dra19] Jan Draisma. “Topological noetherianity of polynomial functors”. In: *J. Am. Math. Soc.* 32.3 (2019), pp. 691–707.
- [DSS06] Mathias Drton, Bernd Sturmfels, and Seth Sullivant. “Algebraic factor analysis: tetrads, pentads and beyond”. In: *Probability Theory and Related Fields* 138.3-4 (2006), pp. 463–493.

- [Eis95] David Eisenbud. *Commutative algebra. With a view toward algebraic geometry*. Vol. 150. Berlin: Springer-Verlag, 1995, pp. xvi + 785.
- [ES21] Rob H. Eggermont and Andrew Snowden. “Topological noetherianity for algebraic representations of infinite rank classical groups”. In: *Transformation Groups* 27 (2021), pp. 1251–1260.
- [ESS19a] Daniel Erman, Steven V Sam, and Andrew Snowden. “Generalizations of Stillman’s Conjecture via Twisted Commutative Algebra”. In: *International Math. Research Notices* 2021.16 (2019), pp. 12281–12304.
- [ESS19b] Daniel Erman, Steven V. Sam, and Andrew Snowden. “Big polynomial rings and Stillman’s conjecture”. In: *Invent. Math.* 218.2 (2019), pp. 413–439.
- [GN18] Sema Güntürkün and Uwe Nagel. “Equivariant Hilbert series of monomial orbits”. English. In: *Proc. Am. Math. Soc.* 146.6 (2018), pp. 2381–2393.
- [HS12] Christopher J. Hillar and Seth Sullivant. “Finite Gröbner bases in infinite dimensional polynomial rings and applications”. In: *Adv. Math.* 229.1 (2012), pp. 1–25.
- [JLR20] Martina Juhnke-Kubitzke, Dinh Van Le, and Tim Römer. “Asymptotic Behavior of Symmetric Ideals: A Brief Survey”. In: *Combinatorial Structures in Algebra and Geometry*. Cham: Springer International Publishing, 2020, pp. 73–94.
- [Joy81] Andre Joyal. “Une théorie combinatoire des séries formelles”. In: *Adv. Math.* 42 (1981), pp. 1–82.
- [KLS17] Robert Krone, Anton Leykin, and Andrew Snowden. “Hilbert series of symmetric ideals in infinite polynomial rings via formal languages”. In: *J. Algebra* 485 (2017), pp. 353–362.
- [KR22] Mario Kummer and Cordian Riener. *Equivariant algebraic and semi-algebraic geometry of infinite affine space*. 2022.
- [KVR22] Thomas Kahle, Dinh Van Le, and Tim Römer. “Invariant chains in algebra and discrete geometry”. In: *SIAM J. Discrete Math.* 36.2 (2022), pp. 975–999.
- [LN22] Dinh Van Le and Hop D. Nguyen. *On regularity and projective dimension up to symmetry*. 2022. arXiv: 2206.15141 [math.AC].

- [LNNR19] Dinh Van Le, Uwe Nagel, Hop D. Nguyen, and Tim Römer. “Codi-  
mension and projective dimension up to symmetry”. In: *Mathe-  
matische Nachrichten* 293.2 (2019), pp. 346–362.
- [NR17] Uwe Nagel and Tim Römer. “Equivariant Hilbert series in non-  
noetherian polynomial rings”. In: *Journal of Algebra* 486 (2017),  
pp. 204–245.
- [NR19] Uwe Nagel and Tim Römer. “FI- and OI-modules with varying  
coefficients”. In: *Journal of Algebra* 535 (2019), pp. 286–322.
- [NS21a] Rohit Nagpal and Andrew Snowden. “Symmetric ideals of the infi-  
nite polynomial ring”. In: (2021). arXiv: 2107.13027 [math.AC].
- [NS21b] Rohit Nagpal and Andrew Snowden. *Symmetric subvarieties of  
infinite affine space*. 2021. arXiv: 2011.09009 [math.AG].
- [NSS16] Rohit Nagpal, Steven V. Sam, and Andrew Snowden. “Noetheri-  
anity of some degree two twisted commutative algebras”. In: *Sel.  
Math., New Ser.* 22.2 (2016), pp. 913–937.
- [SS22] Steven V. Sam and Andrew Snowden. “Sp-equivariant modules  
over polynomial rings in infinitely many variables”. In: *Trans.  
Amer. Math. Soc.* 375.3 (2022), pp. 1671–1701.
- [Sta20] The Stacks project authors. *The Stacks project*. 2020.
- [Sta82] Richard P. Stanley. “Linear Diophantine equations and local co-  
homology”. In: *Invent. Math.* 68 (1982), pp. 175–193.
- [Sta97] Richard P. Stanley. *Enumerative combinatorics. Vol. 1*. Vol. 49.  
Cambridge: Cambridge University Press, 1997, pp. xi + 325.

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# Curriculum Vitae

Azhar Farooq was born on the fourteenth of September in 1990 in Layyah Punjab, Pakistan. After finishing his higher secondary education in 2009 at Punjab Group of Colleges in Multan, he studied BS Mathematics at Bahauddin Zakariya University Multan Pakistan. After completing his BS in 2015 he went to Abdus Salam School of Mathematical Sciences, Government College, and University Lahore where he studied MPhil in Mathematics. He wrote his MPhil thesis in 2017 titled Symmetries and Conserved Quantities under the supervision of Dr. Amer Iqbal. In 2019 he started a Ph.D. project at Eindhoven University of Technology in Netherlands of which the results are presented in this dissertation.