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Distributed Coverage Control of Constrained Constant-Speed Unicycle Multi-Agent Systems

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***Abstract*—This paper proposes a novel distributed coverage controller for a multi-agent system with constant-speed unicycle robots (CSUR). The work is motivated by the limitation of the conventional method that does not ensure the satisfaction of hard state- and input-dependent constraints and leads to feasibility issues for multi-CSUR systems. In this paper, we solve these problems by designing a novel coverage cost function and a saturated gradient-search-based control law. Invariant set theory and Lyapunov-based techniques are used to prove the state-dependent confinement and the convergence of the system state to the optimal coverage configuration, respectively. The controller is implemented in a distributed manner based on a novel communication standard among the agents. A series of simulation case studies are conducted to validate the effectiveness of the proposed coverage controller in different initial conditions and with control parameters. A comparison study in simulation reveals the advantage of the proposed method in terms of avoiding infeasibility. The experiment study verifies the applicability of the method to real robots with uncertainties. The development procedure of the method from theoretical analysis to experimental validation provides a novel framework for multi-agent system coordinate control with complex agent dynamics.**

***Note to Practitioners*—This paper gives a novel solution for multiple robots to effectively cover a polygonal area. Compared to the conventional approaches, our method allows the robots to cover a target region using circular orbits, which is suitable for constant-speed unicycle robots (CSUR) like fixed-wing unmanned aerial vechicles (fUAV). The main advantage of this method is to ensure that the coverage is always successful by preventing the robots from departing the target region. Also, the method satisfies common control saturation constraints in practice and can be implemented in a reliable decentralized scheme. The method is validated to be effective for wheeled robots in experiment studies, although it can also be applied to fUAVs in theory.**

***Index Terms*—multi-agent systems, coverage control, barrier-Lyapunov function, invariance, input-saturation control.**

I. Introduction

THE effective coverage of a target region using robots is an important task for various practical applications in industry, agriculture, and public services. It is also the prototype for more complicated tasks, such as event monitoring, production measuring, and resource allocation. The essential objective of a coverage problem is to effectively allocate the robots in the target region, such that a certain criterion is optimized. Previously, a coverage task is usually executed by a single robot using its trajectory [1]. Nevertheless, multi-agent systems that consist of multiple networked robots are increasingly used. In this case, each agent only dominates a local partition of the target region, which results in high efficiency and superior reliability due to effective collaboration and coordination among the agents. For multi-agent optimal coverage, the most widely used is the closest-distance criterion, i.e., every spot of the target region is dominated by its closest agent [2]. The corresponding solution is depicted as a Centroidal Voronoi Tessellation (CVT) [3], where each agent is positioned in the geometric center or the *centroid* of a Voronoi partition. Solving a multi-agent coverage problem is equivalent to finding a Voronoi partition scheme subject to the optimized criterion. This inspires the design of an optimal coverage controller that drives the agents to move along the negative gradient direction of the coverage criterion and ultimately reach the optimal coverage configuration [3]. However, most of the conventional coverage controllers are only effective for agents that are formulated as single integrators, or single-integrator robots (SIR), such as quadcopters. Optimal coverage control using agents with complex dynamics is still an open and challenging question.

A typical agent with complex dynamics is a constant-speed unicycle robot (CSUR) which moves at a constant linear speed and is steered by its angular velocity [4]. Different from SIRs, a CSUR does not freeze in a fixed position but always moves until the power is used up. Our focus on CSURs is motivated by the interest to solve optimal coverage using fixed-wing unmanned aerial vehicles (fUAV), a class of vehicles that are maneuvered by two fixed wings [5]. Compared to quadcopters, an fUAV can carry heavier loads and cruise at a higher speed with less power, offering higher efficiency in terms of longer air-borne time and a larger coverage capability [5]. However, the conventional coverage control methods used for SIRs can not be directly applied to CSURs due to the difference between their dynamic properties. A CSUR is typically controlled to orbit around a fixed point [6]. In this sense, optimal coverage can be realized by treating the orbiting center of a CSUR as a conventional agent and regulating each CSUR to orbit around the geometric center of its Voronoi partition [7]. However, this may lead to a feasibility issue of the CVT when the orbiting centers move out of the target region and the number of CSURs is reduced before optimal coverage is reached.

The main reason for the feasibility issue is that the orbiting

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motion of a CSUR renders an under-actuated dynamic model that brings up an additional state-dependent perturbation term. This term may deflect the desired motion direction of a CSUR and drive it toward the outside of the target region. This issue only shows up in a multi-agent system with complex agent models but not in one with simple and fully-actuated agent dynamics like SIRs. To our best knowledge, the feasibility issue of a coverage control problem has not been well defined and studied by existing work, due to the lack of studies on the coverage control of complex agents. Fixing this problem requires an additional switching law which, however, brings discontinuity to the controller [7]. Another solution that has not been explored is to use several hard constraints to forcibly confine the orbiting centers of the CSURs within the target region. These hard constraints can be embedded in the coverage control problem with barrier functions, such as a control barrier function (CBF) [8] or a barrier Lyapunov function (BLF) [9] which are widely used for safe-critic control. Nevertheless, using them to construct feasible coverage controllers for multi-agent systems with complex dynamic models has not been investigated by previous work.

Besides the feasibility issue, distributed realization is also important for coverage control. In practice, it is very common that robots are not fully connected, which brings up challenges to centralized control approaches. A distributed controller that only requires the communication locally performed among adjacent agents is more robust to faults and anomalies than a centralized one since the agents are not affected when their non-adjacent agents are defective. The distributed realization has been an essential requirement for many multi-agent coordinate control problems, such as consensus [10], formation [11], and distributed optimization [12]. The conventional coverage controllers for SIRs can also be implemented in a distributed manner [13]. However, whether a multi-CSUR system admits a distributed coverage controller is still an open problem. In general, designing a distributed coverage controller should not only incorporate the complex dynamic models of the agents but also redefine the communication standard among them.

In this paper, we propose a novel distributed controller for coverage control of a multi-CSUR system with the feasibility issue fixed. The controller is designed based on a novel coverage cost function which serves as a barrier-Lyapunov function (BLF) that encodes the hard state-dependent constraints to the coverage problem. It ensures that the orbiting centers of the CSURs asymptotically approach the optimal configuration while being confined within the target region. Thus, optimal coverage is ultimately achieved without causing infeasibility. The achievement of optimal coverage and the satisfaction of the state-dependent constraints are proved using a Lypapunov-based method and the controlled invariance theory, respectively. Also, the control-saturation constraints are satisfied via a Sigmoid function. The controller is designed in a distributed manner with the communication standard properly redefined. The rest of this paper is organized as follows. In Sec. II, we briefly review the related work on optimal coverage control and address the main challenges of our study. Sec. III introduces the preliminaries and formulates the problem. Sec. IV proposes the theoretical results of our proposed coverage controller. The simulation and experimental studies conducted to validate the proposed method are presented in Sec. V and Sec. VI, respectively. Finally, Section VIII concludes the paper.

Notations: $\mathbb{R}$ is the set of real scalars. $\mathbb{R}_+$ and $\mathbb{R}_{\geq 0}$ denote the positive and non-negative real scalars. $\mathbb{N}$ and $\mathbb{N}^+$ are the sets of non-negative and positive integers. For a real scalar $a \in \mathbb{R}$, $|a| \in \mathbb{R}_{\geq 0}$ is its absolute value. $x \in \mathbb{R}^n$ represents an n dimensional vector and $A \in \mathbb{R}^{n\times m}$ is an n by m matrix. $\|x\|$ is the 2-norm of x and $\|x\|_Q = \sqrt{x^\top Q x}$ is its weighted norm, $Q \in \mathbb{R}^{n\times n}$, $Q > 0$. For a closed compact set $\Omega \in \mathbb{R}^n$, Ω represents the interior of Ω and $\partial\Omega$ is its boundary. For a set $\mathcal{A} \subset \Omega$, $\Omega - \mathcal{A}$ denotes the set difference of Ω and $\mathcal{A}$.

II. Related Work

The optimal coverage problem is originally introduced in [3] based on a facility location problem [14] which also addresses the relation between its solution and a CVT. In [15], optimal coverage is defined as a coordination control problem for multi-agent systems with time-variant network topology and nonsmooth dynamics, based on which a general distributed coverage control law is proposed using nonsmooth gradient flows. The stability of the controlled system is analyzed using nonsmooth Lyapunov functions. This work has formed the theoretical foundation of optimal coverage control problems. Then, a general gradient searching law is designed for a team of SIRs [16]. The gradient-based control framework is then extended to generic multi-agent coordination control problems in [17]. In [18], this control framework is further extended to various coverage cost criteria, where the non-convexity of the coverage problem is clarified. All these efforts have provided us with a strong theoretical foundation to analyze the feasibility and stability of the coverage control solutions.

Recent work attempts to improve the flexibility of the control methods against imperfect environmental knowledge. In [19], a radius basis function (RBF) is used to approximate the unknown distribution function of the coverage criterion, such that the robots can incrementally learn the environment knowledge during the movement. In [20] and [21], an adaptive controller is proposed for a time-variant coverage criterion. Besides, many efforts are devoted to the optimal coverage over nontrivial geometric manifolds like circles [22], [23], spherical surfaces [24], or arbitrary curves designated by certain vector-fields [25]. Complementary results toward complicated coverage tasks are also introduced. In [26], a control scheme is proposed to ensure a smooth transference between coverage and other coordinate tasks. The work in [27] attempts to seek a global optimal coverage solution. In [28], the coverage control problem is investigated for a team of disk-shaped robots with heterogeneous sizes. Also, [29] studies coverage control of robots with adjustable sensor ranges, which leads to Voronoi partitions with soft margins instead of the conventional ones with clear boundaries. A survey on other recent development of multi-agent coverage control can be referred to in [30].

Compared to SIRs, the coverage control of complex agents attracts less attention. In [7], [31], coverage controllers are developed for CSURs, where the ultimate optimal coverage configuration corresponds to the solution where the orbiting

centers of the CSURs coincide with the Voronoi centroids. The feasibility issue is solved using hard switching schemes which have obvious shortcomings. Firstly, they may lead to instability for an oddly shaped region due to the finite discrete-sampling rate. Secondly, they require a large control effort on the boundary of the target region, which is difficult to satisfy considering the practical control limits. Thirdly, the closed-loop system under hard switching is not robust to disturbances. To avoid hard switching in the controller inputs, a feasible solution is to formulate the feasibility requirement as a group of state-dependent constraints and encode them into the coverage controller using barrier functions [32], which may result in a controller subject to the *controlled invariance* property [33]. Although the barrier functions are widely applied to practical control systems due to the advantage of continuous control inputs, they have not been used for coverage control of complex agents. We recognize them as powerful tools to solve the feasibility issue for multi-CUSR systems.

III. Preliminaries and Formulation

This section introduces the preliminaries and the problem formulation. We first recall the classical multi-robot optimal coverage problem and present a conventional distributed coverage controller for SIRs. Then, we introduce the dynamic model of a CSUR. Finally, we formulate the multi-CSUR optimal coverage control problem studied in this paper.

A. The Optimal Coverage Problem with Multiple Agents

Let $\Omega \in \mathbb{R}^2$ be a closed convex polygonal set surrounded by $M \in \mathbb{N}^+$ linear edges, i.e.,

$$\Omega = \left\{ \omega \in \mathbb{R}^2 \,\middle|\, h_j(\omega) \geq 0 \,,\, \forall\, j \in \mathcal{M} \right\}, \tag{1}$$

where $\mathcal{M} = \{1, 2, \cdots, M\}$ and $h_j(\omega)$ is defined as

$$h_j(\omega) = b_j - a_j^\top \omega,\ \omega \in \mathbb{R}^2,\ j \in \mathcal{M}, \tag{2}$$

where $a_j \in \mathbb{R}^2$, $b_j \in \mathbb{R}$, are coefficients to denote the edges. Also, we denote the boundary $\partial\Omega$ and the interior $\operatorname{int}\Omega$ of the region respectively as

$$\begin{aligned} \partial\Omega &= \left\{ \omega \in \mathbb{R}^2 \,\middle|\, h_j(\omega) = 0,\, \exists\, j \in \mathcal{M} \right\}, \\ \operatorname{int}\Omega &= \left\{ \omega \in \mathbb{R}^2 \,\middle|\, h_j(\omega) > 0 \,,\, \forall\, j \in \mathcal{M} \right\}. \end{aligned} \tag{3}$$

Note that $\operatorname{int}\Omega$ is open. To simplify the representation, we assume that the origin O of the coordinate is within Ω or on its boundary, i.e., $O \in \Omega$ without losing generality. Actually, for any other case, we can always apply a coordinate transformation to make it satisfied for the new coordinate frame. Thus, we can regulate $\|a_j\| = 1$ and $b_j > 0$ for all $j \in \mathcal{M}$ to uniquely define the edges. N agents are placed in region Ω for coverage. The position of each agent is denoted as $z_k \in \mathbb{R}^2$, $k \in \mathcal{N}$, where $\mathcal{N} = \{1, 2, \cdots, N\}$. We define $\mathcal{Z} = \{z_1, z_2, \cdots, z_N\}$, $z_i \neq z_j$ for any $i, j \in \mathcal{N}$, $i \neq j$, as a *configuration* which is defined on a joint domain $\Omega^N = \underbrace{\Omega \times \cdots \times \Omega}_{N}$ with $\mathcal{Z} \in \Omega^N$ denoting $z_1 \in \Omega \cap z_2 \in \Omega \cap \cdots \cap z_N \in \Omega$.

The objective of the optimal coverage problem is to properly locate the N agents to minimize the following coverage cost,

$$H_\Omega(\mathcal{Z}) = \int_\Omega f(\omega, \mathcal{Z}) \Phi(\omega) \mathrm{d}\omega,\ \mathcal{Z} \in \Omega^N, \tag{4}$$

where $\omega \in \Omega$ denotes an event in the region Ω, $\Phi : \Omega \to \mathbb{R}^+$ is a function that depicts the distribution of events $\omega \in \Omega$, and $f : \Omega \times \Omega^N \to \mathbb{R}^+$ is a function that assigns a real weight to an event $\omega \in \Omega$. In this paper, the weight function is [2],

$$f(\omega, \mathcal{Z}) = \min_{k \in \mathcal{N}} \frac{1}{2} \|\omega - z_k\|^2 \,,\ \mathcal{Z} \in \Omega^N, \tag{5}$$

which calculates the squared Euclidean distance between an event $\omega \in \Omega$ and its closest agent. This is equivalent to splitting the region Ω into N mutually exclusive Voronoi partitions Ω_1, Ω_2, $\cdots$, Ω_N using the N agents. Each partition is defined as

$$\Omega_k = \left\{ \omega \in \Omega \,\middle|\, \|\omega - z_k\| \leq \|\omega - z_i\|,\, \forall\, i \neq k, i \in \mathcal{N} \right\}. \tag{6}$$

Then, function (5) can be rewritten as

$$f(\omega, \mathcal{Z}) = \frac{1}{2} \|\omega - z_k\|^2 \,,\ \text{if}\, \omega \in \Omega_k \tag{7}$$

which takes off the minimum operator in (5) and converts it to a piece-wise quadratic form. Substituting the weight function (7) to (4), the coverage cost becomes

$$H_\Omega(\mathcal{Z}) = \sum_{k=1}^{N} \frac{1}{2} \int_{\Omega_k} \|\omega - z_k\|^2 \Phi(\omega) \mathrm{d}\omega \tag{8}$$

which transfers the integration over the entire region Ω to the summary of the individual integrals on all Voronoi partitions Ω_k, $k \in \mathcal{N}$. Thus, the optimal coverage problem is solved by placing the agents at the following optimal configuration

$$\mathcal{Z}^* = \arg \min_{\mathcal{Z} \in \Omega^N} H_\Omega(\mathcal{Z}). \tag{9}$$

Note that the coverage cost (8) is a nonconvex function of which a global minimum solution is difficult to find [18]. Similar to the previous work [7], [34], in this paper, we are only concerned with its local optimal solutions which can be solved using gradient-based control laws [3], [35]. Therefore, we refer to the optimal configuration given by (9) as a *local optimal configuration* (LOC). It is worth mentioning that there may exist multiple LOCs in the domain Ω^N. Enumerating all LOCs and discussing which is the best is beyond this paper.

B. Distributed Coverage Controller for A Multi-SIR System

Given the Voronoi partitions defined in (6), we say two partitions are adjacent if they share common boundaries, i.e., $\exists\, \omega \in \Omega$, $\omega \in \Omega_i \cap \Omega_j$. Based on this, we claim that agents $i, j \in \mathcal{N}$, $i \neq j$, are *adjacent* if their Voronoi partitions Ω_i and Ω_j are adjacent. We define an adjacency mapping $\mathscr{A} : \mathcal{N} \to 2^{\mathcal{N}}$ to depict the adjacency relation between the agents. Specifically, $\mathscr{A}_k$, $k \in \mathcal{N}$ is the set of all adjacent agents of agent k. Note that the adjacency relation is bidirectional, i.e., for any $i, j \in \mathcal{N}$, $i \neq k$, $i \in \mathscr{A}_k \Leftrightarrow k \in \mathscr{A}_i$. Also, we define a commonly used set $\overline{\mathscr{A}_k} = \mathscr{A}_k \cup k$, $k \in \mathcal{N}$. The adjacency relation is needed to incorporate a common practical condition that communication can only be effective within a

certain range [15], [36]. For the optimal coverage problem, this range refers to the largest distance between adjacent agents, which renders a common and practical assumption that only adjacent agents can conduct bidirectional communication [34].

Then, we proceed with the discussion on the solution to the optimal coverage problem. It is known that a LOC of the coverage cost (8) can be obtained by solving

$$\nabla H(\mathcal{Z}) = \left[\frac{\partial H(\mathcal{Z})}{\partial z_1} \ \frac{\partial H(\mathcal{Z})}{\partial z_2} \ \cdots \ \frac{\partial H(\mathcal{Z})}{\partial z_N} \right]^\top = 0. \quad (10)$$

According to [35], the k-th element of the gradient $\nabla H(\mathcal{Z})$ is calculated as

$$\begin{aligned} \frac{\partial H(\mathcal{Z})}{\partial z_k} &= \int_{\Omega_k} \frac{1}{2} \frac{\partial \|\omega - z_k\|^2}{\partial z_k} \Phi(\omega) \mathrm{d}\omega \\ &= M(\mathcal{Z}_{\overline{\mathscr{A}_k}}) \Big(z_k - C(\mathcal{Z}_{\overline{\mathscr{A}_k}}) \Big), \end{aligned} \quad (11)$$

where $\mathcal{Z}_{\overline{\mathscr{A}_k}} \in \Omega^{|\overline{\mathscr{A}_k}|}$ is the set of all z_j with $j \in \overline{\mathscr{A}_k}$ where $|\overline{\mathscr{A}_k}|$ is the number of elements in the finite set $\overline{\mathscr{A}_k}$, and $M(\mathcal{Z}_{\overline{\mathscr{A}_k}}) \in \mathbb{R}$ and $C(\mathcal{Z}_{\overline{\mathscr{A}_k}}) \in \mathbb{R}^2$ are the geometric mass and the centroid of the Voronoi partition Ω_k, defined as

$$M(\mathcal{Z}_{\overline{\mathscr{A}_k}}) = \int_{\Omega_k} \Phi(\omega) \mathrm{d}\omega, \ C(\mathcal{Z}_{\overline{\mathscr{A}_k}}) = \int_{\Omega_k} \frac{\omega \Phi(\omega) \mathrm{d}\omega}{M(\mathcal{Z}_{\overline{\mathscr{A}_k}})}, \quad (12)$$

Here, we refer to $\mathcal{Z}_{\overline{\mathscr{A}_k}}$ as a *partial configuration* since it only contains the positions of z_k and its adjacent agents. It is noticed in (11) that the computation of gradient $\frac{\partial H(\mathcal{Z})}{\partial z_k}$ only needs the positions of agent k and its adjacent agents contained in $\mathcal{Z}_{\overline{\mathscr{A}_k}}$, which is an important property for the implementation of a distributed coverage controller to be discussed later.

The relation among the agent positions, the Voronoi partition, and the centroids is illustrated in Fig. 1. Since $M(\mathcal{Z}_{\overline{\mathscr{A}_k}}) > 0$ holds for all $k \in \mathcal{N}$, by solving $\nabla H(\mathcal{Z}) = 0$, we know that a necessary condition for $\mathcal{Z}$ being a LOC is,

$$z_k = C(\mathcal{Z}_{\overline{\mathscr{A}_k}}), \ \forall k \in \mathcal{N}. \quad (13)$$

Therefore, if a configuration is a LOC, the agent positions and the Voronoi centroids must coincide with each other, but the coinciding condition does not necessarily indicate a LOC. Additional conditions should be used for further judgment.

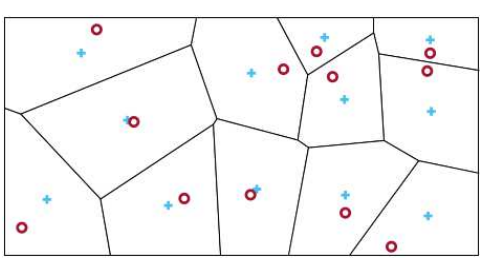


Fig. 1: A coverage example of a rectangular region. The red 'o' marks are the agent positions. The solid lines define the Voronoi partitions. The blue '+' marks are the Voronoi centroids. The fact that the 'o' marks do not coincide with the blue '+' marks indicate that this is not a LOC.

A LOC indicated by condition (13) can be found using the following gradient-based method

$$\dot{z}_k = -\frac{\partial H(\mathcal{Z})}{\partial z_k}, \ k \in \mathcal{N}, \quad (14)$$

which is the main technical point of the conventional methods for the multi-agent coverage problem. For a multi-SIR system with the following single-integrator-based models [15], [36],

$$\dot{z}_k(t) = u_k(t), \ k \in \mathcal{N}, \quad (15)$$

where $u_k(t) \in \mathbb{R}^2$ is the velocity of a SIR as its control input, a trivial optimal coverage controller can be designed as

$$u_k(t) = -\frac{\partial H(\mathcal{Z})}{\partial z_k}, \ k \in \mathcal{N}. \quad (16)$$

This renders a distributed controller since the computation of the gradient $\frac{\partial H(\mathcal{Z})}{\partial z_k}$ only requires the positions of agent z_k and its adjacent agents.

C. The Dynamic Model of A CSUR

The dynamic model of a CSUR is depicted as follows [37],

$$\begin{aligned} \dot{\zeta}(t) &= v_0 r(\theta) \\ \dot{\theta}(t) &= u(t), \end{aligned} \quad (17)$$

where $\zeta(t) \in \mathbb{R}^2$ and $\theta(t) \in \mathbb{R}$ are the position and the orientation of the CSUR at time $t \in \mathbb{R}_{\geq 0}$, respectively, $v_0 \in \mathbb{R}^+$ is the constant linear speed of the robot, $u(t) \in \mathbb{R}$ is the angular velocity input of the robot, and $r(\theta) = [\cos(\theta) \ \sin(\theta)]^\top$ is a transformation vector. It is easy to verify that $r(\theta)$ satisfies

$$\|r(\theta)\| = 1, \text{ and } \frac{\partial^2 r(\theta)}{\partial \theta^2} = -r(\theta), \ \forall \theta \in \mathbb{R}. \quad (18)$$

For the CSUR input $u(t)$ in (17), we regulate that $u(t) < 0$ and $u(t) > 0$ indicate clockwise and anticlockwise orientation directions, respectively. When $u(t) \equiv 0$, the CSUR moves along a straight line. Note that the robot model (17) is underactuated since the three-dimensional state $[\zeta^\top(t) \ \theta(t)]^\top$ is excited by a one-dimensional input signal $u(t)$. Also, it is not possible to let a CSUR freeze in a fixed position like a SIR since it always moves at a constant speed v_0. Following [7], [38], we use the following virtual center of a CSUR, instead of its position $\zeta(t)$, to perform the coverage task,

$$z(t) = \zeta(t) + \frac{v_0}{\omega_0} \frac{\partial r(\theta)}{\partial \theta}, \quad (19)$$

where $\omega_0 \in \mathbb{R}$, $\omega_0 \neq 0$ is a constant parameter that represents the nominal angular velocity of CSUR. Taking the derivative of (19), the dynamic model of the virtual center is

$$\dot{z}(t) = \dot{\zeta}(t) + \frac{v_0}{\omega_0} \frac{\partial^2 r(\theta)}{\partial \theta^2} \dot{\theta}(t) = v_0 r(\theta) - \frac{v_0}{\omega_0} r(\theta) u(t). \quad (20)$$

The meaning of the virtual center $z(t)$ is not straightforward for an arbitrary robot trajectory $\zeta(t)$ but is clear for a special case $u(t) \equiv \omega_0$. Substituting it to (20), we have $\dot{z}(t) = 0$ which denotes that the virtual center $z(t)$ is a static point in this case. Then, equation (19) indicates that the robot is moving around $z(t)$ along a circular orbit with a linear speed

v_0, an angular velocity ω_0, and orbit radius $v_0/|\omega_0|$. Thus, $z(t)$ can be interpreted as the center of the circular orbit of the CSUR when it is a static point, which is why it is referred to as a *virtual* center. The relation between the CSUR position $\zeta(t)$ and its virtual center $z(t)$ is shown in Fig. 2.

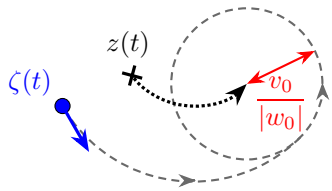


Fig. 2: The trajectories of the CSUR position $\zeta(t)$ and its virtual center $z(t)$, represented as a gray dashed line and a black dotted line, with the arrows pointing out the directions. The black cross and the blue dot are respectively the positions of the CSUR and its virtual center at a certain time $t \in \mathbb{R}$, where the blue arrow indicates the orientation of the robot. The trajectory of the CSUR converges to a circle ultimately, as $z(t)$ reaches a static point. The radius of the circle is $v_0/|\omega_0|$.

Different from the CSUR position $\zeta(t)$ that has to always move at a constant linear speed, the virtual center $z(t)$ can remain static at a certain position when the CSUR is controlled with a constant input $u(t) \equiv \omega_0$. This property is similar to the dynamics of a SIR as shown in (15), which provides the possibility to extend the existing results for SIRs to the virtual centers of CSURs. Therefore, in his paper, we refer to the CSUR virtual centers as *CSUR agents* and use them to conduct the coverage task. Nevertheless, it is noticed that the dynamic model of a CSUR agent in (20) is more complicated than a SIR model (15), which brings up challenges to this extension. Sec. III-D explains the challenges in detail.

D. The Optimal Coverage Control of Multiple CSUR Agents

Derived from (20), the dynamic model of each agent in a multi-CSUR system is depicted by

$$\dot{z}_k(t) = v_0 r(\theta_k) - \frac{v_0}{\omega_0} r(\theta_k) u_k(t), \;\; k \in \mathcal{N}, \tag{21}$$

where $\theta_k(t)$ and $u_k(t)$ are respectively the orientation and the control input of agent k. Here, we assume that all agents have the same speed parameters v_0, ω_0, for the simplification of the problem. The nonlinear projection gain $r(\theta_k)$ and the additive perturbation term $v_0 r(\theta_k)$ in the dynamic model (21) make the coverage control problem more challenging than SIRs. From (17), we know $r(\theta_k)$ has a constant norm 1, which means that these nonlinear terms constantly perturb the agent velocity $\dot{z}(t)$ from the desired gradient-searching direction $-\dfrac{\partial H(\mathcal{Z})}{\partial z_k}$ and prevent the agent position $z_k(t)$ from converging to the optimal configuration $\mathcal{Z}^*$. In some cases, the CUSR agents may even move out of Ω, such that the optimization problem (9) becomes infeasible. Note that SIRs do not have the feasibility issue since the closed-loop dynamic model (14) is not twisted by these nonlinear terms and the control law (16) always guides the SIRs toward the interior of the target region Ω. To ensure feasibility, the CSUR agents must be confined within the target region. Meanwhile, the control inputs should follow certain saturation restrictions for the concern of limited energy or resources. Based on this consideration, we formulate the multi-CSUR optimal coverage control problem as follows.

Problem 1. *Given a convex set $\Omega \subset \mathbb{R}^2$ defined in* (1) *and N CSUR agents depicted by (21), design a distributed control law $u_k(t)$ for all $k \in \mathcal{N}$ subject to the adjacency relation $\mathscr{A}$, such that the following objectives are achieved.*

1) *For all $k \in \mathcal{N}$ and $t \in \mathbb{R}_{\geq 0}$, the control inputs satisfy*

$$|u_k(t)| \leq \overline{U}, \;\; \overline{U} \in \mathbb{R}^+. \tag{22}$$

2) *For all $t \in \mathbb{R}_{\geq 0}$, the agent configuration $\mathcal{Z}(t)$ satisfies*

$$\mathcal{Z}(t) \in \Omega^N, \;\; \forall \mathcal{Z}(0) \in \Omega^N. \tag{23}$$

3) *The agent configuration $\mathcal{Z}(t)$ asymptotically converges to a LOC $\mathcal{Z}^*$ denoted by* (9).

The main difference between Problem 1 and the multi-SIR coverage problem in previous work [15], [36] is indicated by objectives 1) and 2), respectively corresponding to the requirements on the input- and state-dependent constraints. Another difference is that the optimal coverage configuration $\mathcal{Z}^*$ is for the CUSR agents or the virtual centers of the CUSRs, instead of the positions of the CUSRs. When a LOC is achieved, the CSURs are expected to move along their circular orbits around their static virtual centers assigned by the optimal configuration $\mathcal{Z}^*$. Note that Problem 1 is only concerned with a LOC instead of a globally optimal solution. The LOC solutions may be multiple. To which LOC $\mathcal{Z}$ converges mainly depends on the initial robot configuration [39]. Also, the adjacency relation $\mathscr{A}$ may not be constant but changes over time [15]. In this paper, we are only concerned with minimizing the coverage cost (8) without incorporating additional requirements like collision avoidance or time limits. These requirements render more nontrivial challenges that can hardly be fully addressed in this paper. In fact, the parameters v_0, ω_0 can be changed to reduce the radius of the circular orbits of the CSURs to reduce the chance of collisions. Extensions to these challenging problems will be explored in future work.

E. Positively Invariant Set and Tangent Cone

In this subsection, we introduce the *positively invariant set* and the *tanget cone* which are important to the analysis of the satisfaction of the hard state-dependent constraints addressed by objective 2) of Problem 1.

Definition 1. *[33] $\mathcal{S} \subset \mathbb{R}^n$ is a positively invariant set for system $\dot{x}(t) = f(x(t))$ if $\forall\, x(0) \in \mathcal{S}$, $x(t) \subset \mathcal{S}$ for $t \in \mathbb{R}_+$.*

Definition 2. *[33] The tangent cone of a convex set $\mathcal{S} \subset \mathbb{R}^n$ in $x \in \mathbb{R}^n$ is a set*

$$\mathscr{C}_{\mathcal{S}}(x) = \left\{ z \in \mathbb{R}^n \,\middle|\, \lim_{\tau \to 0} \frac{\mathscr{D}(x + \tau z, \mathcal{S})}{\tau} = 0 \right\}, \tag{24}$$

where $\mathscr{D} : \mathbb{R}^n \times 2^{\mathbb{R}^n} \to \mathbb{R}_{\geq 0}$ is a function that specifies the distance between a vector and a set,

$$\mathscr{D}(x, \mathcal{S}) = \inf_{s \in \mathcal{S}} \|x - s\|. \tag{25}$$

The hard state-dependent constraints addressed in (23) can be formulated in a way that the region Ω becomes a *positively invariant set* of the system. The tangent cone of a closed set $\mathcal{S}$ is the set of all feasible directions $\dot{x}$ of the system at state x, such that $\mathcal{S}$ is a positively invariant set. Whether a closed set is positively invariant is determined by the following Lemma.

Lemma 1. *[33] Consider a system $\dot{x}(t) = f(x(t))$ of which each initial condition $x(0) \in \mathcal{X} \subseteq \mathbb{R}^n$ admits a globally unique solution. Then, a closed and convex set $\mathcal{S} \subseteq \mathcal{X}$ is positively invariant for the system if and only if $f(x) \in \mathscr{C}_{\mathcal{S}}(x)$, $\forall\, x \in \partial\mathcal{S}$, where $\partial\mathcal{S}$ is the boundary of $\mathcal{S}$.*

Lemma 1 provides an easy approach to validate whether a designed controller achieves objective 2) of Problem 1 by only investigating the tangent cone on the boundary of the set. Note that Lemma 1 only applies to closed and convex sets.

IV. Main Results

In this section, we present the main theoretical results of the proposed optimal coverage controller. We first introduce an off-LOC cost function and a novel BLF-based coverage cost function. Based on these functions, we propose a novel distributed coverage controller. Then, we validate the objectives in Problem 1 for the proposed controller one by one.

A. The Off-LOC Cost Function

For any agent $k \in \mathcal{N}$ and its adjacent agents $\mathscr{A}_k$, we define the following off-LOC cost function,

$$W(\mathcal{Z}_{\overline{\mathscr{A}_k}}) = \frac{1}{2}\left\| z_k(t) - C(\mathcal{Z}_{\overline{\mathscr{A}_k}}) \right\|_Q^2,\ \mathcal{Z}_{\overline{\mathscr{A}_k}} \in \Omega^{|\overline{\mathscr{A}_k}|}, \tag{26}$$

where $Q \in \mathbb{R}^{2\times 2}$ is a symmetrically positive-definite matrix. It can be verified that $W(\mathcal{Z}_{\overline{\mathscr{A}_k}}) \geq 0$ for all $\mathcal{Z}_{\overline{\mathscr{A}_k}} \in \Omega^{|\overline{\mathscr{A}_k}|}$ and $W(\mathcal{Z}_{\overline{\mathscr{A}_k}}) = 0$ holds if and only if (13) is satisfied, i.e., $\mathcal{Z}_{\overline{\mathscr{A}_k}}$ belong to a LOC. Therefore, this function measures how close a partial configuration $\mathcal{Z}_{\overline{\mathscr{A}_k}}$ to a LOC, which is why we refer to it as the *off-LOC cost*. The following proposition is granted.

Proposition 1. *$W(\mathcal{Z}_{\overline{\mathscr{A}_k}})$ has the following properties for all $\mathcal{Z}_{\overline{\mathscr{A}_k}} \in \Omega^{|\overline{\mathscr{A}_k}|}$ and any $k \in \mathcal{N}$.*
1). There always exists $\overline{W} \in \mathbb{R}_+$, such that $W(\mathcal{Z}_{\overline{\mathscr{A}_k}}) < \overline{W}$.
2). $W(\mathcal{Z}_{\overline{\mathscr{A}_k}}) > 0$ always holds if $z_k \in \partial\Omega$.
3). There always exists $\epsilon \in \mathbb{R}_+$, $\epsilon < \min\limits_{j\in\mathcal{M}} \sup\limits_{\omega\in\Omega} h_j(\omega)$, such that $W(\mathcal{Z}_{\overline{\mathscr{A}_k}}) > 0$ holds for any $x_k \in \Omega - \Omega_\epsilon$, where $\Omega_\epsilon \subset \Omega$ is a closed convex set defined as

$$\Omega_\epsilon = \left\{ \omega \in \mathbb{R}^2 \,\middle|\, h_j(\omega) \geq \epsilon\ ,\ \forall\, j \in \mathcal{M} \right\}. \tag{27}$$

Proof. For property 1), we know that any configuration defined in the region Ω, i.e., $\mathcal{Z} \in \Omega^N$, corresponds to a certain Voronoi partition of Ω, such that $\Omega_k \neq \varnothing$ and $M(\mathcal{Z}_{\overline{\mathscr{A}_k}}) > 0$ hold for all $k \in \mathcal{N}$. As a result, z_k and $C(\mathcal{Z}_{\overline{\mathscr{A}_k}})$ are both bounded, which means that $W(\mathcal{Z}_{\overline{\mathscr{A}_k}})$ always has an upper bound $\overline{W} \in \mathbb{R}_+$, $\forall\, k \in \mathcal{N}$. For property 2), we consider its negative proposition by supposing that there exists $z_k \in \partial\Omega$, $\exists\, k \in \mathcal{N}$, such that $W(\mathcal{Z}_{\overline{\mathscr{A}_k}}) = 0$, which indicates that $z_k = C(\mathcal{Z}_{\overline{\mathscr{A}_k}})$. From the definition (12), however, we know $C(\mathcal{Z}_{\overline{\mathscr{A}_k}}) \notin \partial\Omega$, which breaks this equality. Thus, the negative proposition does not hold and property 2) is satisfied. For 3), we know that $W(\mathcal{Z}_{\overline{\mathscr{A}_k}})$ is a continuous function of z_k since $C(\mathcal{Z}_{\overline{\mathscr{A}_k}})$ is also continuous to z_k, according to (26). Then, since property 2) addresses that $W(\mathcal{Z}_{\overline{\mathscr{A}_k}}) > 0$ holds for any $z_k \in \partial\Omega$, $k \in \mathcal{N}$, we know that there always exists $\epsilon \in \mathbb{R}_+$, $\epsilon < \min\limits_{j\in\mathcal{M}} \sup\limits_{\omega\in\Omega} h_j(\omega)$, such that $\Omega_\epsilon \neq \varnothing$ and there exists $z_k \in \partial\Omega_\epsilon$ that belongs to a LOC, such that $W(\mathcal{Z}_{\overline{\mathscr{A}_k}}) = 0$ and $W(\mathcal{Z}_{\overline{\mathscr{A}_k}}) > 0$ holds for all $x_k \in \Omega - \Omega_\epsilon$. □

Proposition 1 provides several important statements on the off-LOC cost functions $W(\mathcal{Z}_{\overline{\mathscr{A}_k}})$, $k \in \mathcal{N}$. Property 1) addresses its boundedness and Property 2) indicates that LOC does not occur on the boundary $\partial\Omega$. Furthermore, 3) points out that there exists a margin $\Omega - \Omega_\epsilon$ around the convex region Ω where no LOC exists. They both address that all LOCs are distributed in the interior of the region and do not show up in the marginal area close to its boundary. This proposition is important for our theoretical results in Sec. IV-D.

Note that $W(\mathcal{Z}_{\overline{\mathscr{A}_i}})$, $i \in \mathcal{N}$ is also a differentiable function of z_k, $k \in \mathcal{N}$, since $C(\mathcal{Z}_{\overline{\mathscr{A}_i}})$ is differentiable to z_k. According to [40], the partial derivative of $C(\mathcal{Z}_{\overline{\mathscr{A}_i}})$ to z_k reads

$$\frac{\partial C^\top(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{\partial z_k} = \frac{D(\mathcal{Z}_{\overline{\mathscr{A}_i}}, z_k)}{M(\mathcal{Z}_{\overline{\mathscr{A}_i}})} - P(\mathcal{Z}_{\overline{\mathscr{A}_i}}, z_k) C^\top(\mathcal{Z}_{\overline{\mathscr{A}_i}}), \tag{28}$$

where, for $z_i, z_k \in \Omega$, $i, k \in \mathcal{N}$, $i \neq k$, $z_i \neq z_k$,

$$D(\mathcal{Z}_{\overline{\mathscr{A}_i}}, z_k) = \int_{\partial\Omega_k^i} \frac{(\omega - z_k)\omega^\top}{\|z_k - z_i\|} \Phi(\omega) \mathrm{d}\omega, \tag{29a}$$

$$P(\mathcal{Z}_{\overline{\mathscr{A}_i}}, z_k) = \int_{\partial\Omega_k^i} \frac{\omega - z_k}{\|z_k - z_i\|} \Phi(\omega) \mathrm{d}\omega, \tag{29b}$$

where $\partial\Omega_k^i$ is the shared boundary of adjacent partitions Ω_i, Ω_k, $i, k \in \mathcal{N}$. Then, the partial derivative of $W(\mathcal{Z}_{\overline{\mathscr{A}_i}})$ to z_k is

$$\frac{\partial W(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{\partial z_k} = \begin{cases} \left(I - \dfrac{\partial C^\top(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{\partial z_k} \right) Q\Big(z_i - C(\mathcal{Z}_{\overline{\mathscr{A}_i}}) \Big), & i = k, \\ -\dfrac{\partial C^\top(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{\partial z_k} Q\Big(z_i - C(\mathcal{Z}_{\overline{\mathscr{A}_i}}) \Big), & i \neq k. \end{cases} \tag{30}$$

Proposition 2. *For any $i, k \in \mathcal{N}$, $i \neq k$, $\dfrac{\partial C^\top(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{\partial z_k} = 0$ and $\dfrac{\partial W(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{\partial z_k} = 0$ hold, if $i \notin \mathscr{A}_k$ or $k \notin \mathscr{A}_i$.*

Proof. According to (29), for any $i, k \in \mathcal{N}$, $i \neq k$, we have $D(\mathcal{Z}_{\overline{\mathscr{A}_i}}, z_k) = 0$ and $P(\mathcal{Z}_{\overline{\mathscr{A}_i}}, z_k) = 0$ if $i \notin \mathscr{A}_k$ or $k \notin \mathscr{A}_i$. Substituting (28) to (30), we can prove this proposition. □

Proposition 3. $W_1 < \dfrac{\partial W(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{\partial z_k} < W_2$, $\exists\, W_1, W_2 \in \mathbb{R}_+$.

Proof. In (30), it is noticed that $\dfrac{\partial C^\top(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{\partial z_k}$ is continuous and bounded since $M(\mathcal{Z}_{\overline{\mathscr{A}_i}}) > 0$ holds on Ω and $D(\mathcal{Z}_{\overline{\mathscr{A}_i}}, z_k)$, $P(\mathcal{Z}_{\overline{\mathscr{A}_i}}, z_k)$, and $C(\mathcal{Z}_{\overline{\mathscr{A}_i}})$ are all continuous and bounded. Therefore, $\dfrac{\partial W(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{\partial z_k}$ is also continuous and bounded. □

B. A Novel Coverage Cost Function for CSURs

One of the main technical points of this paper is to design a BLF-based coverage cost for a novel coverage controller for multi-CSUR systems. A BLF is a non-negative function that reaches zero at the system equilibria but approaches infinity near the boundary of the confined region [41]. It forces the system states to move towards the interior of the region when they tend to violate the constraints defined by the region boundaries. For a multi-CSUR system with the agent model in (21), we define the following BLF-based coverage cost,

$$V(\mathcal{Z}) = \sum_{i=1}^{N}\sum_{j=1}^{M} \frac{W(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{h_j(z_i)},\ \mathcal{Z} \in \operatorname{int}\Omega^N, \tag{31}$$

where $W(\mathcal{Z}_{\overline{\mathscr{A}_i}})$ is the off-LOC cost function defined in Sec. IV-A and $\operatorname{int}\Omega^N = \underbrace{\operatorname{int}\Omega\times\cdots\times\operatorname{int}\Omega}_{N}$ depicts the product of N open sets. Note that $V(\mathcal{Z})$ is defined on an open domain and has the following properties.

Property 1. *The coverage cost function $V(\mathcal{Z})$ satisfies the following conditions for all $\mathcal{Z} \in \operatorname{int}\Omega^N$.*
1). $V(\mathcal{Z}) = 0$ holds if and only if $\mathcal{Z}$ is a LOC that satisfies the condition in (13)*, otherwise $V(\mathcal{Z}) > 0$.*
2). For any $\overline{V} \in \mathbb{R}_+$, there always exists $\epsilon \in \mathbb{R}_+$, such that for any $h_j(z_i) < \epsilon$, $i \in \mathcal{N}$, $\exists\, j \in \mathcal{M}$, $V(\mathcal{Z}) > \overline{V}$ holds.
3). For any $\epsilon \in \mathbb{R}+$, $\epsilon < \min_{j\in\mathcal{M}} \sup_{\omega\in\Omega} h_j(\omega)$, there always exist $V_* \in \mathbb{R}_+$, such that $V(\mathcal{Z}) < V_*$, for all $\mathcal{Z} \in \operatorname{int}\Omega_\epsilon^N$, where Ω_ϵ is the closed set defined in (27)*.*

The proof for Property 1 is not provided since the properties are straightforward to verify using the definition (31) and the boundedness of the off-LOC cost function $W(\mathcal{Z}_{\overline{\mathscr{A}_k}})$, $k \in \mathcal{N}$, addressed in Proposition 1. Property 1-1) indicates the equivalence between $V(\mathcal{Z}) = 0$ and $\mathcal{Z}$ being a LOC, which is important to the verification of whether a configuration is a LOC. Property 1-2) addresses that the cost function $V(\mathcal{Z})$ becomes unbounded when the configuration $\mathcal{Z}$ approaches any point of the region boundary $\partial\Omega$. Property 1-3) means that $V(\mathcal{Z})$ only becomes unbounded when a position in $\mathcal{Z}$ approaches the region boundary $\partial\Omega$. It is bounded when $\mathcal{Z}$ remains in the interior of region Ω. Therefore, $V(\mathcal{Z})$ is a BLF.

As mentioned in Sec. III-B, solving the necessary condition $\nabla V(\mathcal{Z}) = 0$ is important to obtain a LOC. The k-th element of $\nabla V(\mathcal{Z}) = 0$, $k \in \mathcal{N}$, is calculated as

$$\frac{\partial V(\mathcal{Z})}{\partial z_k} = \sum_{j=1}^{M}\left(\sum_{i=1}^{N}\frac{1}{h_j(z_i)}\frac{\partial W(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{\partial z_k} + a_j\frac{W(\mathcal{Z}_{\overline{\mathscr{A}_k}})}{h_j^2(z_k)}\right), \tag{32}$$

$\mathcal{Z} \in \operatorname{int}\Omega^N$, Substituting (30) to (32), we obtain

$$\begin{aligned}\frac{\partial V(\mathcal{Z})}{\partial z_k} = \sum_{j=1}^{M}\Bigg(&\frac{Q\Big(z_k - C(\mathcal{Z}_{\overline{\mathscr{A}_k}})\Big)}{h_j(z_k)} + \frac{a_j W(\mathcal{Z}_{\overline{\mathscr{A}_k}})}{h_j^2(z_k)}\\ &- \sum_{i=1}^{N}\frac{\partial C^\top(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{\partial z_k}\frac{Q\left(z_i - C(\mathcal{Z}_{\overline{\mathscr{A}_i}})\right)}{h_j(z_i)}\Bigg).\end{aligned} \tag{33}$$

Although $\nabla V(\mathcal{Z})$ shows a complicated form, it can be calculated in a distributed manner. Applying Proposition 2 to (33), we rewrite it as

$$\begin{aligned}\frac{\partial V(\mathcal{Z}_{\overline{\mathscr{A}_k}})}{\partial z_k} = \sum_{j=1}^{M}\Bigg(&\frac{Q\Big(z_k - C(\mathcal{Z}_{\overline{\mathscr{A}_k}})\Big)}{h_j(z_k)} + \frac{a_j W(\mathcal{Z}_{\overline{\mathscr{A}_k}})}{h_j^2(z_k)}\\ &- \sum_{i\in\overline{\mathscr{A}_k}}\frac{\partial C^\top(\mathcal{Z}_{\overline{\mathscr{A}_i}})}{\partial z_k}\frac{Q\Big(z_i - C(\mathcal{Z}_{\overline{\mathscr{A}_i}})\Big)}{h_j(z_i)}\Bigg),\end{aligned} \tag{34}$$

for $\mathcal{Z}_{\overline{\mathscr{A}_i}} \in \operatorname{int}\Omega^{|\overline{\mathscr{A}_i}|}$. This indicates that the gradient $\dfrac{\partial V(\mathcal{Z}_{\overline{\mathscr{A}_k}})}{\partial z_k}$ for each agent $k \in \mathcal{N}$ only needs the information from its own and its adjacent agents, i.e., $i \in \overline{\mathscr{A}_k}$. In Sec. IV-C, we explain how to use this property to implement a distributed coverage controller for multi-CSUR systems based on a proper communication standard.

Note that the partial derivative $\nabla V(\mathcal{Z})$ is continuous, considering the continuity of the linear constraint functions $h_j(z_i)$, the virtual centers $z_i(t)$, and the Voronoi centroids $C(\mathcal{Z}_{\overline{\mathscr{A}_i}})$, $i \in \mathcal{N}$, $j \in \mathcal{M}$. Also, $\nabla V(\mathcal{Z})$ satisfies the following condition.

Proposition 4. *$\nabla V(\mathcal{Z}) = 0$ holds, $\mathcal{Z} \in \operatorname{int}\Omega^N$, if and only if* (13) *holds.*

Proof. The sufficiency of this proposition is straightforward to verify by substituting (13) to (33). For the necessity, we investigate (32). Since all CSURs have identical dynamic models, the number N should not affect the equality of (32). Therefore, according to (33), considering $h_j(z_i) > 0$ for all $z_i \in \Omega$, $i \in \mathcal{N}$ and all $j \in \mathcal{M}$, we can infer that $\nabla V(\mathcal{Z}) = 0$ holds if and only if $W(\mathcal{Z}^+_{\mathscr{A}_i}) = 0$ and $z_k = C(\mathcal{Z}^+_{\mathscr{A}_i})$ hold for all $k \in \mathcal{N}$, which is equivalent to (13). This verifies the necessity of Proposition 4. □

C. The Distributed Coverage Controller with Input Saturation

For the multi-CSUR system (21), We design the following controller for the optimal coverage control problem 1,

$$u_k(t) = \omega_0 + \gamma\omega_0\,\rho\Big(\sigma(\mathcal{Z}_{\overline{\mathscr{A}_k}}, \theta_k)\Big), \tag{35}$$

where $\sigma(\mathcal{Z}_{\overline{\mathscr{A}_k}}, \theta_k) = r^\top(\theta_k)\dfrac{\partial V(\mathcal{Z}_{\overline{\mathscr{A}_k}})}{\partial z_k}$, $\gamma \in \mathbb{R}^+$ is the control gain, and $\rho : \mathbb{R} \to (-1, 1)$ is the following Sigmoid function,

$$\rho(x) = \frac{x}{|x| + \varepsilon},\ x \in \mathbb{R}, \tag{36}$$

where $\varepsilon \in \mathbb{R}^+$ is a constant scalar. Saturation functions like Sigmoid functions are commonly used by previous work to design controllers subject to input constraints [42], [43]. It is straightforward to verify that $\rho(\cdot)$ is continuous on $\mathbb{R}$ and $\big|\rho(x)\big| < 1$ holds for any $x \in \mathbb{R}$. Thus, it is straightforward to propose the following property for the control input $u_k(t)$.

Property 2. *The control input $u_k(t)$, in (35), for all $k \in \mathcal{N}$, is bounded by $|u_k(t) - \omega_0| < \gamma\omega_0$, for all $t \in \mathbb{R}_{\geq 0}$.*

Property 2 indicates that the proposed controller (35) is subject to the input-dependent constraint $|u_k(t) - \omega_0| < \gamma\omega_0$

which leads to $|u_k(t)| < (1+\gamma)\,\omega_0$. To ensure the input saturation constraint (22), we may as well set

$$(1+\gamma)\,\omega_0 \leq \overline{U}, \tag{37}$$

for which we can adjust the control gain γ or the nominal angular velocity ω_0 to achieve target 1) of Problem 1.

Note that (35) is a distributed controller. Substituting (28) to (34) and then to (35), we know that the control input $u_k(t)$ of each agent $k \in \mathcal{N}$ requires the following information.
1). Its own orientation θ_k.
2). The positions z_i and the Voronoi centroids $C(\mathcal{Z}_{\overline{\mathscr{A}_i}})$ of its own and all its adjacent agents $i \in \overline{\mathscr{A}_k}$.
3). The Voronoi mass $M(\mathcal{Z}_{\overline{\mathscr{A}_i}})$ of all adjacent agents $i \in \mathscr{A}_k$.
4). The adjacency relations $\mathscr{A}_i$ of its own all its adjacent agents $i \in \overline{\mathscr{A}_k}$, which are used to determine $\partial\Omega_k^i$ and calculate the Voronoi functions $D(\mathcal{Z}_{\overline{\mathscr{A}_i}}, z_k)$ and $P(\mathcal{Z}_{\overline{\mathscr{A}_i}}, z_k)$.

Based on this, we redefine the communication standard for the multi-CSUR systems to realize a distributed optimal coverage controller. That is, every agent $k \in \mathcal{N}$ should broadcast its position z_k, Voronoi centroid $C(\mathcal{Z}_{\overline{\mathscr{A}_k}})$, Voronoi mass $M(\mathcal{Z}_{\overline{\mathscr{A}_k}})$, and adjacency relations $\mathscr{A}_k$ to all its adjacent agents. In the meantime, agent k also receives the corresponding information from its adjacent agents and uses them to calculate the control input according to (35). This indicates that designing a distributed optimal coverage controller for a multi-CSUR system is feasible and implementable by defining a proper communication standard. Due to the nontrivial dynamic model of the multi-CSUR system, its communication standard is far more complicated than that of a multi-SIR system which only includes the agent positions. Note that the adjacency relations of all agents may change during the motion of the agents. The work [44], [45] provided distributed methods to solve the Voronoi partition of a convex region, which can be used to calculate the adjacency relations in a distributed manner.

D. Invariance to State-Dependent Constraints

Substituting the controller (35) to (21), the closed-loop dynamic model of each CSUR agent is

$$\dot{z}_k(t) = -\gamma\omega_0\, r(\theta_k)\, \rho(\sigma(\mathcal{Z}_{\overline{\mathscr{A}_k}}, \theta_k)),\ \ k \in \mathcal{N}. \tag{38}$$

We use the invariance theory introduced in Sec. III-E to validate whether the closed-loop dynamic model (38) achieves objective 2) in Problem 1. Note that Lemma 1 is only applicable to closed sets. Nevertheless, the closed-loop dynamic model (38) is defined on a open domain $\mathcal{Z}_{\overline{\mathscr{A}_i}} \in \operatorname{int}\Omega^{|\overline{\mathscr{A}_i}|}$, which brings up the challenges of the invariance analysis. In this paper, we perform an indirect manner by investigating the invariance of the closed set Ω_ϵ defined in (27) with a small ϵ, which addresses the following theorem.

Theorem 1. *There always exists $\epsilon_0 \in \mathbb{R}_+$, such that for all $\epsilon < \epsilon_0$, $\Omega_\epsilon \neq \varnothing$ and Ω_ϵ is positively invariant for system* (38).

Proof. The critical point is to solve the tangent cone $\mathscr{C}_{\Omega_\epsilon}(z_k)$ for any $z_k \in \Omega_\epsilon$, $k \in \mathcal{N}$, with given ϵ and validate whether the trajectory admitted by (38) falls in $\mathscr{C}_{\Omega_\epsilon}(z_k)$. Inspired by Lemma 1, we just need to calculate $\mathscr{C}_{\Omega_\epsilon}(z_k)$ for $z_k \in \partial\Omega_\epsilon$ since $\mathscr{C}_{\Omega_\epsilon}(z_k) = \mathbb{R}^2$ for all $z_k \in \operatorname{int}\Omega_\epsilon$. Without losing the generality, we assume that z_k is closest to the boundary $\partial\Omega$ among all agent positions z_r, $r \in \mathcal{N}$, i.e., we always assign ϵ such that $z_k \in \partial\Omega_\epsilon$ while $z_r \in \Omega_\epsilon$, $\forall\, r \in \mathcal{N}$, $r \neq k$.

According to Proposition 1, there always exists a $\epsilon_0 \in \mathbb{R}_+$, such that $W(\mathcal{Z}_{\overline{\mathscr{A}_k}}) > 0$ for all $\epsilon < \epsilon_0$. Also, $\Omega_\epsilon \neq \varnothing$ is ensured if $\epsilon_0 < \min\limits_{j\in\mathcal{M}} \sup\limits_{\omega\in\Omega} h_j(\omega)$. Thus, we define the following function for $z_k \in \partial\Omega_\epsilon$, $\epsilon < \epsilon_0$ with an arbitrary vector $\iota \in \mathbb{R}^2$,

$$\mathscr{V}_\epsilon(z_k, \iota) = \frac{\overline{h}^2(z_k)}{W_k}\iota^\top \frac{\partial V(\mathcal{Z}_{\overline{\mathscr{A}_k}})}{\partial z_k}, \tag{39}$$

where W_k is the brief form of $W(\mathcal{Z}_{\overline{\mathscr{A}_k}})$, $k \in \mathcal{N}$, and $\overline{h}(z_k) = \min\limits_{j\in\mathcal{M}} h_j(z_k)$. Substituting (32) to (39), we have

$$\mathscr{V}_\epsilon(z_k, \iota) = \sum_{j=1}^{M}\left(\sum_{i\in\overline{\mathscr{A}_k}} \frac{\overline{h}^2(z_k)}{h_j(z_i)}\frac{\iota^\top}{W_k}\frac{\partial W_i}{\partial z_k} + \iota^\top a_j \frac{\overline{h}^2(z_k)}{h_j^2(z_k)}\right).$$

According to Proposition 1 and Property 3, we know that both $\dfrac{\partial W_i}{\partial z_k}$, $\forall\, i \in \overline{\mathscr{A}_k}$, and W_k are all bounded. Thus, we know that $\mathscr{V}_\epsilon(z_k, \iota)$ has the following limit as $\epsilon \to 0$,

$$\mathscr{V}(z_k, \iota) = \lim_{\epsilon\to 0} \mathscr{V}_\epsilon(z_k, \iota) = \iota^\top a_r, \tag{40}$$

where $r = \arg\min\limits_{j\in\mathcal{M}} h_j(z_k)$ is the number of the edge to which z_k is the most close. Be reminded that a_r is the normal vector of not only the r-th edge of Ω but also the r-th edge of Ω_ϵ for all $\epsilon < \epsilon_0$. Moreover, the direction of a_r points to the interior of Ω and Ω_ϵ. Then, we inspect the case when $\iota = \dot{z}_k \in \mathbb{R}^2$,

$$\mathscr{V}(z_k, \dot{z}_k) = \dot{z}_k^\top a_r \tag{41}$$

which is the inner product of the system trajectory direction $\dot{z}_k$ and the normal vector a_j. The sign of $\mathscr{V}(z_k, \dot{z}_k)$ indicates whether $\dot{z}_k$ points to the interior of Ω and Ω_ϵ for all $\epsilon < \epsilon_0$. Based on this, we can make a relation between the function $\mathscr{V}(z_k, \dot{z}_k)$ and the tangent cone $\mathscr{C}_{\Omega_\epsilon}(z_k)$. Recalling the definition of the distance function $\mathscr{D}$ in (25), it is not difficult to find, for any $z_k \in \partial\Omega_\epsilon$ with any $\epsilon < \epsilon_0$ and $\dot{z}_k \in \mathbb{R}^2$,

$$\lim_{\tau\to 0} \frac{\mathscr{D}(z_k + \tau\dot{z}_k, \Omega_\epsilon)}{\tau} > 0 \Leftrightarrow \mathscr{V}(z_k, \dot{z}_k) > 0, \tag{42}$$

$$\lim_{\tau\to 0} \frac{\mathscr{D}(z_k + \tau\dot{z}_k, \Omega_\epsilon)}{\tau} = 0 \Leftrightarrow \mathscr{V}(z_k, \dot{z}_k) \leq 0. \tag{43}$$

This indicates that the tangent cone $\mathscr{C}_{\Omega_k}(z_k)$ for any $z_k \in \partial\Omega_\epsilon$ and $\epsilon < \epsilon_0$ is

$$\mathscr{C}_{\Omega_\epsilon}(z_k) = \left\{\dot{z}_k \in \mathbb{R}^2 \,\middle|\, \mathscr{V}(z_k, \dot{z}_k) \leq 0\right\}. \tag{44}$$

Now, let us validate whether the trajectory direction $\dot{z}_k$ admitted by (38) falls in the tangent cone $\mathscr{C}_{\Omega_k}(z_k)$. Substituting the closed-loop dynamics (38) to (39), we have

$$\mathscr{V}_\epsilon(z_k, \dot{z}_k) = -\frac{\gamma\omega_0\overline{h}^2(z_k)|\sigma(\mathcal{Z}_{\overline{\mathscr{A}_k}}, \theta_k)|}{W_k(1 + \varepsilon/|\sigma(\mathcal{Z}_{\overline{\mathscr{A}_k}}, \theta_k)|)}. \tag{45}$$

Note that

$$\lim_{\epsilon\to 0} \frac{\overline{h}^2(z_k)\sigma(\mathcal{Z}_{\overline{\mathscr{A}_k}}, \theta_k)}{W_k} = \lim_{\epsilon\to 0} \mathscr{V}_\epsilon(z_k, r(\theta_k)) = r^\top(\theta_k)a_r \tag{46}$$

and $\lim_{\epsilon\to 0}\frac{1}{|\sigma(\mathcal{Z}_{\overline{\mathscr{A}_k}},\theta_k)|}=0$. Taking the limit of (45), we have

$$\lim_{\epsilon\to 0}\mathscr{V}_\epsilon(z_k,\dot{z}_k)=\mathscr{V}(z_k,\dot{z}_k)=-\gamma\omega_0\left|r^\top(\theta_k)a_r\right|\le 0, \quad (47)$$

which indicates that the dynamic model (38) ensures

$$\dot{z}_k\in\mathscr{C}_{\Omega_\epsilon}(z_k), z_k\in\partial\Omega_\epsilon,\ \forall\,\epsilon<\epsilon_0. \quad (48)$$

According to Lemma 1, the condition (48) means that Ω_ϵ is invariant for z_k, i.e., for any initial condition $z_k(0)\in\Omega_\epsilon$, $z_k(t)\in\Omega_\epsilon$ holds for all $t\in\mathbb{R}_+$. Note that this generally holds for any $k\in\mathcal{N}$ that is closest to the boundary $\partial\Omega$. Thus, we claim that Ω_ϵ is positively invariant for system (38). □

Theorem 1 indicates that there always exists a cluster of closed sets $\Omega_\epsilon\subset\Omega$ that are positively invariant for the closed-loop system (38). This indicates that there always exists ϵ, such that for any initial state $\mathcal{Z}(0)\in\Omega_\epsilon^N$, $\mathcal{Z}(t)\in\Omega_\epsilon^N\subset\Omega^N$ holds for all $t\in\mathbb{R}^+$, which satisfies the state-dependent constraint in (23). Therefore, both objectives 1) and 2) of Problem 1 are achieved by the proposed coverage controller (35). An illustration of Ω_ϵ being a positively invariant set of the system is shown in Fig. 3. The following subsection interprets the convergence of the multi-CSUR system to a LOC.

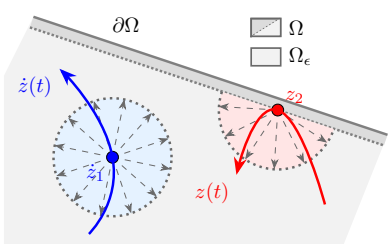


Fig. 3: The illustration of Ω_ϵ being a positive invariant set of the system. For any states $z_1,z_2\in\Omega_\epsilon$, the directions of the system trajectories $\dot{z}(t)$ (the colored arrows) are confined by the corresponding tangent cones (the colored regions). The gray arrows in the tangent cones indicate the feasible trajectory directions. For any interior state $z_1\in\Omega_\epsilon$, the tangent cone at z_1 is $\mathbb{R}^2$ which allows arbitrary trajectory directions $\dot{z}(t)$. The tangent cone of a state on the boundary $z_2\in\partial\Omega_\epsilon$, however, only allows $\dot{z}(t)$ pointing to the interior of Ω_ϵ. Theorem 1 ensures that the ϵ making Ω_ϵ invariant always exists.

E. Convergence of the System Configuration to A LOC

In Sec. III-D, we have introduced the condition of a LOC for the coverage control of multi-agent systems. A LOC $\mathcal{Z}^*$ subject to (13) can be recognized as an equilibrium of the closed-loop system (38). The stability of this equilibrium is addressed by the following theorem.

Theorem 2. *For the dynamics of the CSUR agents in (21) with the control law as in (35), the equilibrium $\mathcal{Z}^*$ subject to (13) is asymptotically stable.*

Proof. We take the time derivative of the energy function $V(\mathcal{Z})$ defined in (31) as follows,

$$\dot{V}(\mathcal{Z})=\sum_{k=1}^{N}\dot{z}_k^\top\frac{\partial V(\mathcal{Z}_{\mathscr{A}_k})}{\partial z_k}. \quad (49)$$

Substituting (38) to (49), we have

$$\begin{aligned}\dot{V}(\mathcal{Z})&=-\sum_{k=1}^{N}Pv_0\left(\sigma(\mathcal{Z}_{\overline{\mathscr{A}_k}},\theta_k)\right)\rho\left(\sigma(\mathcal{Z}_{\overline{\mathscr{A}_k}},\theta_k)\right)\\&=-\sum_{k=1}^{N}\frac{Pv_0\left|\sigma(\mathcal{Z}_{\overline{\mathscr{A}_k}},\theta_k)\right|^2}{\left|\sigma(\mathcal{Z}_{\overline{\mathscr{A}_k}},\theta_k)\right|+\varepsilon}\le 0.\end{aligned} \quad (50)$$

We notice that $\dot{V}(\mathcal{Z})=0$ holds if and only if

$$\sigma(\mathcal{Z}_{\overline{\mathscr{A}_k}},\theta_k)=r^\top(\theta)\frac{\partial V(\mathcal{Z}_{\mathscr{A}_k})}{\partial z_k}=0,\ \forall\,k\in\mathcal{N}, \quad (51)$$

which contains the following conditions, namely $\frac{\partial V(\mathcal{Z}_{\mathscr{A}_k})}{\partial z_k}=0$ or $\frac{\partial V(\mathcal{Z}_{\mathscr{A}_k})}{\partial z_k}\neq 0$ but $r(\theta)$ and $\frac{\partial V(\mathcal{Z}_{\mathscr{A}_k})}{\partial z_k}$ are orthogonal. We use the La Salle invariant principle [46] to verify under which condition the configurations $\mathcal{Z}$ are stable. We take the time derivative of both sides of (51) and obtain

$$\dot{r}^\top(\theta)\frac{\partial V(\mathcal{Z}_{\mathscr{A}_k})}{\partial z_k}+r^\top(\theta)\frac{\partial}{\partial z_k^\top}\left(\frac{\partial V(\mathcal{Z}_{\mathscr{A}_k})}{\partial z_k}\right)\dot{z}_k=0,\ k\in\mathcal{N}. \quad (52)$$

A configuration $\mathcal{Z}$ serving as a stable equilibrium must satisfy both conditions (51) and (52). Note that a necessary condition for a stable equilibrium is $\dot{z}_k=0$ which leads (52) to

$$\dot{r}^\top(\theta)\frac{\partial V(\mathcal{Z}_{\mathscr{A}_k})}{\partial z_k}=0,\ \forall\,k\in\mathcal{N}. \quad (53)$$

Then, the only solution to (51) and (53) is $\frac{\partial V(\mathcal{Z}_{\mathscr{A}_k})}{\partial z_k}=0$ for all $k\in\mathcal{N}$ or $\nabla V(\mathcal{Z})=0$ which is equivalent to condition (13), according to Proposition 4. Thus, any LOC given by (13) is an asymptotically stable equilibrium of the system. □

Theorem 2 indicates that the closed-loop dynamic model of the multi-CSUR system in (38) asymptotically converges to a LOC with any initial conditions. Note that there may exist multiple LOCs in the target region and the stability of each LOC may not be global. Nevertheless, Theorem 2 guarantees that any initial condition must converge to some LOC in the end. Therefore, target 3) of problem 1 is also achieved.

Three control parameters play important roles in our proposed coverage controller. γ is the control gain that adjusts the amplitude of the control input, ϵ is the boundary layer scalar that smooths up the control inputs in zero vicinity, and Q is a gain matrix that tunes the coverage cost function. Increasing γ and Q and decreasing ε can improve the convergence speed of the system to the local optimal coverage solutions. A simulation case study on how these control parameters affect the performance of the system will be conducted in Sec. V-B.

V. Simulation Studies

In this section, we validate the performance of the proposed coverage controller in a series of simulation studies. We first test the effectiveness of the coverage controller for six CSUR agents with different initial conditions and control parameters. Then, we apply the controller to a larger system with more

agents to verify its scalability. Finally, we conduct a comparison study to address the advantage of the conventional method in terms of avoiding infeasibility. All studies are simulated in MATLAB R2021a at a discrete sampling time 0.05 s.

A. Method Test with Initial Conditions

This study tests the performance of the proposed method for a system with six CSUR agents with different initial conditions. The target region Ω is a $4\,\text{m} \times 2.8\,\text{m}$ rectangular region. The boundary functions $h_j(\omega)$, $j = 1, 2, 3, 4$, $\omega \in \Omega$, are parameterized by $a_1 = [-1\ 0]$, $b_1 = 0$, $a_2 = [1\ 0]$, $b_2 = 4$, $a_3 = [0\ 1]$, $b_3 = 2.8$, $a_4 = [0\ -1]$, $b_4 = 0$. The linear speed and the nominal angular velocity of the CSURs are $v_0 = 0.16$ m/s and $\omega_0 = 0.8$ rad/s. Three different initial configurations are randomly generated and assigned to the CSURs, as shown in Tab. I, where $[\,\zeta_x\ \zeta_y\,]^\top$ and θ are the planar coordinate and the orientation of a CSUR. For all cases, the control parameters are selected as $\gamma = 1$, $Q = I$, and $\varepsilon = 2$.

TABLE I: The Initial Configurations of Cases # 1, # 2, and # 3

# Agent		1	2	3	4	5	6
# 1	ζ_x	0.2546	0.1247	1.793	0.3006	1.187	3.144
	ζ_y	1.392	2.629	0.1781	0.4191	0.1445	0.0658
	θ	3.060	3.160	4.610	3.030	4.500	4.680
# 2	ζ_x	0.9549	0.8286	3.148	0.2219	0.1023	3.823
	ζ_y	0.0310	2.702	0.4426	2.705	0.3783	0.7863
	θ	6.130	3.690	2.610	3.370	4.060	0.8600
# 3	ζ_x	0.8690	1.3810	3.610	0.7773	0.3674	0.4060
	ζ_y	0.1436	2.6980	0.2723	2.726	2.610	0.2589
	θ	4.760	4.560	4.390	4.650	1.430	1.340

The simulation results of this study are illustrated in Fig. 4. The trajectories of the robot positions, virtual centers, and Voronoi centroids of the three cases are presented in Fig. 4a, Fig. 4b, and Fig. 4c, respectively. It can be seen that the trajectories of all agents (the virtual centers of the CSURs) are confined within the region for all time, which indicates the achievement of objective 2) of Problem 1. All virtual centers and their corresponding Voronoi centroids, both marked as 'o' but with different colors, coincide with each other ultimately, which verifies the ultimate achievement of optimal coverage. The coinciding points indicate the corresponding LOC. The CSURs ultimately orbit around these points at a radius 50 m which allows a low likelihood of collisions. The achievement of optimal coverage is also reflected in Fig. 4d, Fig. 4e, and Fig. 4f, where the coverage function decays to zero within 100 s for all initial conditions. Besides, the control inputs of all robots shown in Fig. 4g, Fig. 4h, and Fig. 4i are all strictly confined by $|u_k(t) - \omega_0| < \gamma\omega_0 = 0.8$ for all agents and all time, which achieves objective 1) of Problem 1. Note that different initial conditions ultimately lead to different LOCs. They may also affect the convergence speed of the coverage cost. Therefore, we can conclude that the proposed coverage controller (35) achieves all three objectives of Problem 1 with different initial conditions.

B. The Influence of the Control Parameters

This study evaluates the influence of the control parameters, namely the input gain γ, the coverage gain Q, and the boundary layer scalar ε, on the performance of the proposed coverage controller. The size of the target region and the robot parameters v_0, ω_0 are the same as those in Sec. V-A. The initial conditions of the agents are determined as Case # 2 in Tab. I. The simulation results with different control parameters are illustrated in Fig. 5. Note that we also compare the simulation results in Fig. 5 with Case # 2 of Fig. 4 since they have the same initial conditions. Similar to Sec. V-A, Fig. 5 indicates that optimal coverage is achieved for all cases with the trajectories of the virtual centers confined within the target region. All control inputs are restricted by $|u_k(t) - \omega_0| < \gamma\omega_0$, although the bounds are different due to various γ. Therefore, we can conclude that the proposed coverage controller (35) well solves Problem 1 with different control parameters.

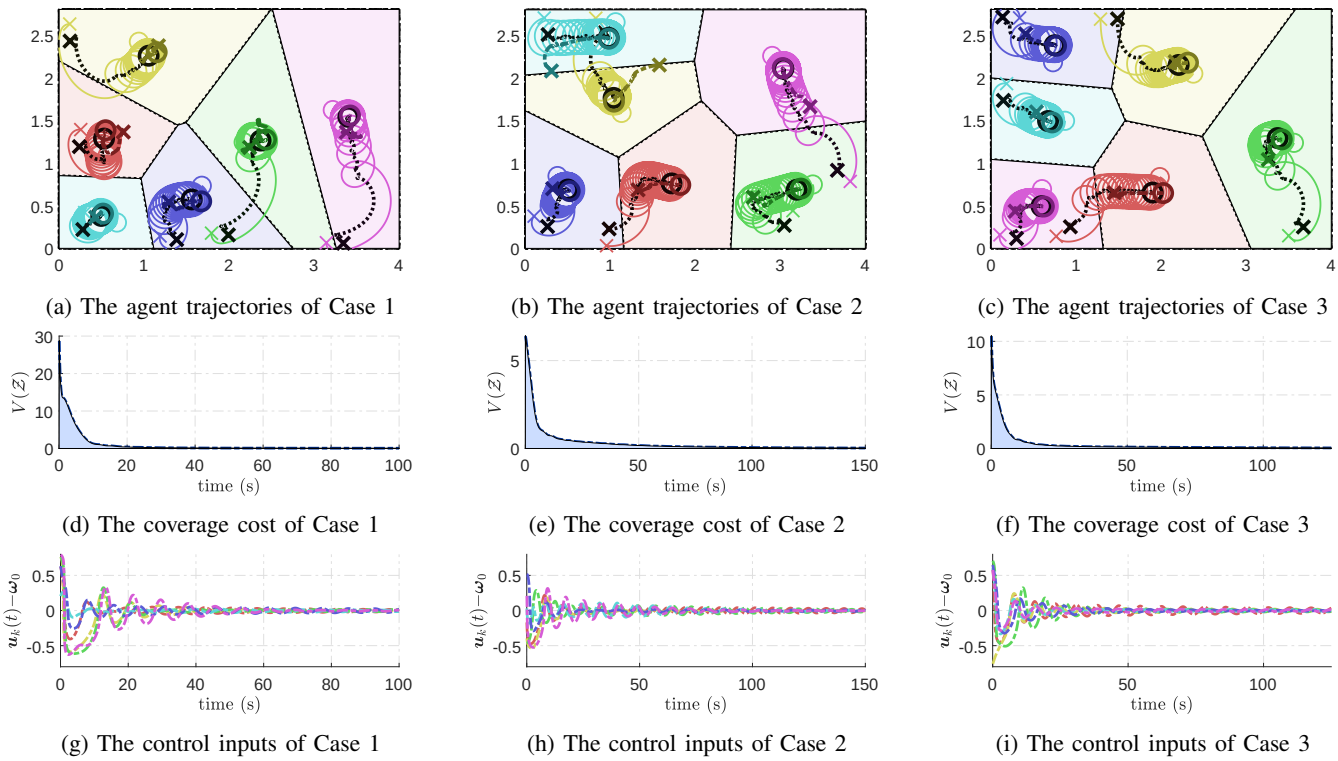
Comparing Fig. 5 with Case 2 in Fig. 4, we notice that these parameters affect the control performance differently. Firstly, a large γ increases the convergence rate of the coverage cost but also causes chattering to the control inputs. This is because the system tends to become unstable as the control gain becomes over-large due to the discrete sampling. Secondly, an over-large ϵ may slow down the convergence to a LOC. Thirdly, a large Q can effectively increase the convergence rate of the coverage cost without causing chatting to the control inputs. Therefore, we suggest only using γ to restrict the control inputs while increasing the value of the coverage gain Q to improve the convergence rate. The boundary layer scalar ε is suggested to be small to maintain a decent convergence rate while ensuring the smoothness of the control inputs.

C. Optimal Coverage of A Larger-Scale System

In this study, we test the proposed coverage controller on a larger-scale multi-agent system that contains 100 CSURs. The coverage is performed on a $800\,\text{m} \times 600\,\text{m}$ rectangular region with the same boundary coefficients as Sec. V-A, except that $b_2 = 800$ and $b_3 = 600$. The linear speed and the nominal angular velocity of the CSURs are $v_0 = 10$ m/s and $\omega_0 = 2$ rad/s which correspond to a small orbit radius 5 m such that the CSURs are not likely to collide with each other. The control parameters are selected as $\gamma = 1$, $Q = 10\,I$, and $\varepsilon = 2$. The initial positions of the robots are randomly sampled from the target region and are not listed here. The simulation results are illustrated in Fig. 6. Fig. 6a shows that the virtual centers of all CSURs ultimately coincide with the Voronoi centroids and Fig. 6b indicates that the coverage cost decays to zero. Thus, optimal coverage is successfully achieved for this 100-agent system. Fig. 6a also shows that all virtual centers are strictly confined within the target region. The control inputs are limited by $|u_k(t) - \omega_0| < \gamma\omega_0 = 2$ according to Fig. 6c. Thus, we can conclude that the proposed coverage controller is also effective for a large-scale multi-CSUR system.

D. A Comparison Study With the Conventional Method

As mentioned in Sec. IV, the main advantage of our proposed coverage controller (35) over the conventional gradient-



(a) The agent trajectories of Case 1 (b) The agent trajectories of Case 2 (c) The agent trajectories of Case 3

(d) The coverage cost of Case 1 (e) The coverage cost of Case 2 (f) The coverage cost of Case 3

(g) The control inputs of Case 1 (h) The control inputs of Case 2 (i) The control inputs of Case 3

Fig. 4: The simulation results of the proposed coverage controller under different initial conditions: (a)-(c) are the trajectories of the CSUR positions $\zeta_i(t)$ (thin solid lines), virtual centers $z_i(t)$ (thick dotted lines), and the corresponding Voronoi centroids $C(\mathcal{Z}_{\overline{\mathscr{A}_i}})$ (thick dashed lines), $i \in \mathcal{N}$, where 'x' and 'o' respectively indicate the starting and ending points of the trajectories. (d)-(f) are the values of the coverage costs as time increases. (g)-(i) are the control inputs of the agents as time changes.

based controller (16) is the additional state-dependent constraints (23) that are critical to solving the feasibility issue for a multi-CSUR system. This subsection conducts a comparison study between these two methods to address the advantage of the proposed coverage controller. The detailed formulation of the conventional coverage controller is provided in [7], which corresponds to the following closed-loop dynamics,

$$\dot{z}_k(t) = -\gamma \frac{\partial H(\mathcal{Z})}{\partial z_k} \tag{54}$$

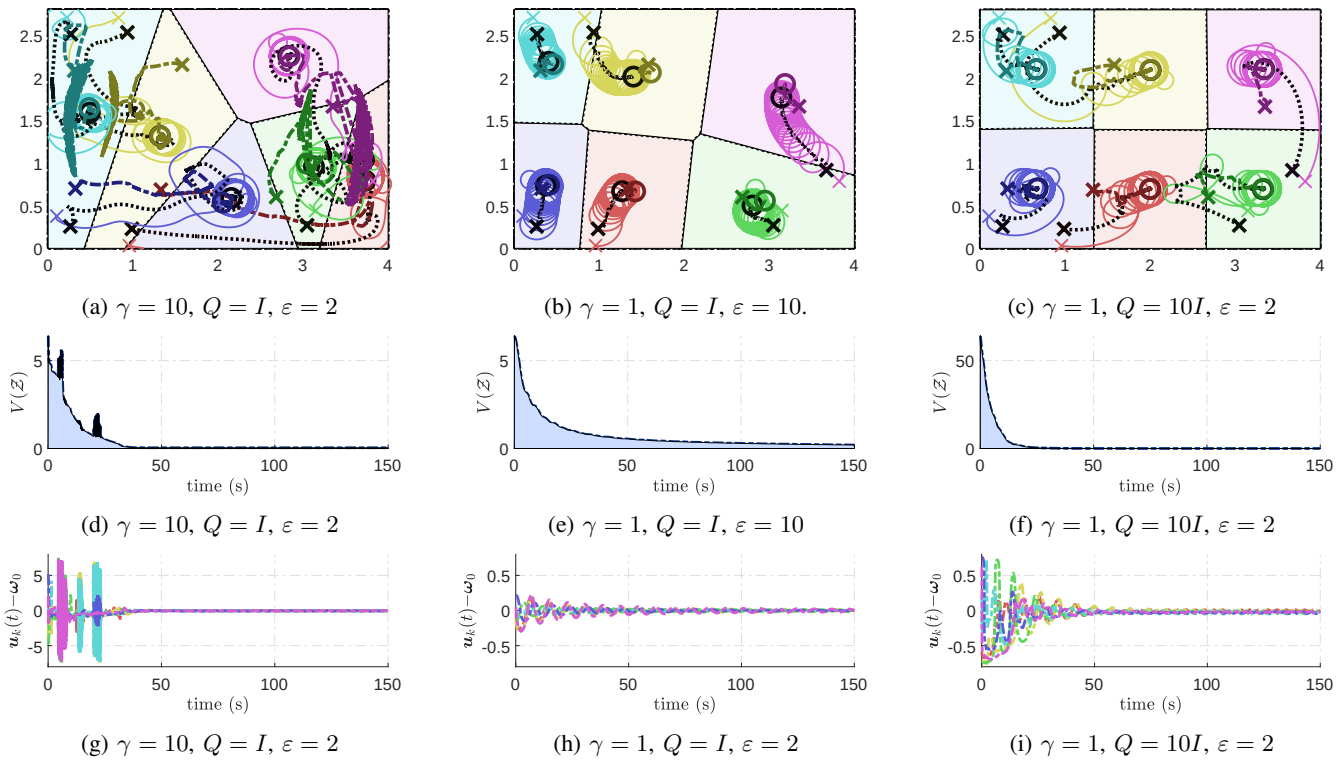
where $\frac{\partial H(\mathcal{Z})}{\partial z_k}$ is calculated using (11). This study is conducted in a 800 m×600 m rectangular region with six CSURs. For both controllers, we set the same velocity constants $v_0 = 40$ m/s and $\omega_0 = 0.8$ rad/s, the same initial positions as shown in Tab. II, and the same control gain $\gamma = 0.1$. For the conventional controller (54), the coverage cost $H(\mathcal{Z})$ is defined as in (8) with $\Phi(\omega) = 1$, $\omega \in \Omega$. For the proposed controller (35), the other control parameters are $Q = I$ and $\varepsilon = 2$. The trajectories of the CSUR positions, virtual centers, and Voronoi centroids are illustrated in Fig. 7. Fig. 7a clearly shows that one virtual center is about to cross the region boundary and move towards outside of the target region while the optimal coverage is not reached yet. The situation after this is not drawn since the Voronoi partition is no more feasible. Nevertheless, the proposed controller guarantees that all virtual centers are confined within the target region and ultimately coincide with the Voronoi centroids, as shown in Fig. 7b. This clearly verifies that the proposed controller ensures the feasibility of the optimal coverage problem even though the conventional one does not under the same conditions.

TABLE II: The Initial Condition of the Comparison Study

# Agent	1	2	3	4	5	6
ζ_x	60.68	624.4	350.6	579.2	782.5	430.3
ζ_y	301.0	43.43	161.5	299.7	408.0	482.4
θ	2.394	0.414	1.810	5.715	1.341	2.841

VI. Experiment Validation

In this section, we conduct an experimental study on real robot platforms to verify the applicability of the proposed method. The target region is a 4 m ×2.8 m indoor area, as shown in Fig. 8a. We use six two-wheel unicycle mobile robots provided by the Arduino Engineering Kit ®, as shown in Fig. 8b, to serve as the CSURs. Each robot is attached with four infra-tracking markers such that its motion can be tracked by a Qualisys ® motion tracking system which captures the motion of the robots at a frequency of 300 Hz with 16 cameras deployed around the target region. A Lenovo Thinkpad laptop with an Intel core I5-6200U CPU and 8GB RAM, running with the Ubuntu 16.04 operating system, is used to receive the robot motion data from the tracking system and send control commands to the robots. Each robot is encapsulated by an independent thread on the laptop within the robotic operating



(a) $\gamma = 10$, $Q = I$, $\varepsilon = 2$ (b) $\gamma = 1$, $Q = I$, $\varepsilon = 10$. (c) $\gamma = 1$, $Q = 10I$, $\varepsilon = 2$

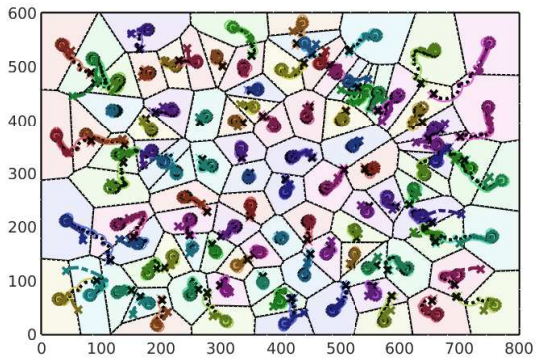
(d) $\gamma = 10$, $Q = I$, $\varepsilon = 2$ (e) $\gamma = 1$, $Q = I$, $\varepsilon = 10$ (f) $\gamma = 1$, $Q = 10I$, $\varepsilon = 2$

(g) $\gamma = 10$, $Q = I$, $\varepsilon = 2$ (h) $\gamma = 1$, $Q = I$, $\varepsilon = 2$ (i) $\gamma = 1$, $Q = 10I$, $\varepsilon = 2$

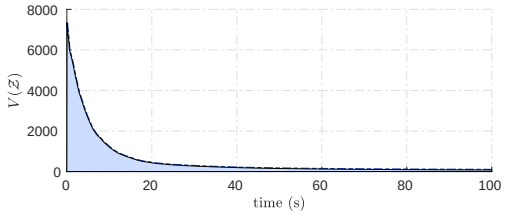
Fig. 5: The simulation results of the proposed coverage controller subject to different control parameters: (a)-(c) are the trajectories of the robot positions $\zeta_i(t)$ (thin solid lines), virtual centers $z_i(t)$ (thick dotted lines), and the corresponding Voronoi centroids $C(\mathcal{Z}_{\overline{\mathcal{A}_i}})$ (thick dashed lines) of the CSURs with different control parameters, $i \in \mathcal{N}$, where 'x' and 'o' are respectively the starting and ending points of the trajectories. (d)-(f) are the values of the coverage cost functions $V(\mathcal{Z}(t))$ as time changes. (g)-(i) are the control inputs $u(t) - \omega_0$ as time changes.

system (ROS) framework with an update frequency of 100 Hz for movement control. The control commands include the constant-speed $v_0 = 0.16$ m/s and the nominal angular velocity 0.8 rad/s which are converted to the motor commands for the robot wheels with a simple PD controller. The motion tracking system, the laptop, and the mobile robots are connected using a common wireless network. The adjacency relation among the robots is computed using the distributed algorithm introduced in in [44], [45]. It is worth mentioning that the ROS network used to coordinate the control and measurement of the robot is not subject to hard real-time and does not ensure constant discrete sampling. Also, there exists communication delay on the network due to its limited bandwidth. Moreover, the linear speed and the nominal angular velocity of the mobile robots are not ideally constant due to the friction forces and the dynamic features of the robot motors. All these factors lead to uncertainties to the experiment. Therefore, the main purpose of this experiment is to investigate the difference between the experiment and simulation results under the same conditions and evaluate how the uncertainties affect the performance of the proposed coverage controller. For a fair comparison, the initial conditions and control parameters of the experiment study are the same as the simulation study in Sec. V-A.

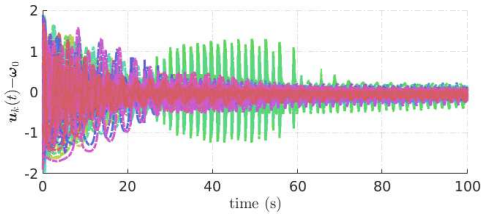
The results of this experiment study are illustrated in Fig. 9. We omit the trajectories of the robots, the virtual centers, and the Voronoi centroids in the transient stage in Fig. 9a, Fig. 9b, and Fig. 9c and only show the ultimate virtual centers, Voronoi centroids, and circular orbits. Besides, The background of these figures is filled with the screenshots of the robot positions from a top-to-down perspective. It is clearly shown that all virtual centers coincide with their Voronoi centroids and all robots orbit around these coinciding positions, which indicates the ultimate achievement of optimal coverage. Fig. 9d, Fig. 9e, and Fig. 9f show that the coverage costs monotonously decay to zero for all three cases. The control inputs shown in Fig. 9g, Fig. 9h, and Fig. 9g are strictly confined by $\|u_k(t) - \omega_0\| < \gamma\omega_0 = 0.8$, which indicates the satisfaction of the input saturation constraints. These observations clearly show that the proposed control method can well solve Problem 1 on real robot platforms. Comparing Fig. 9 with Fig. 4, it is noticed that the simulation and the experiment studies have different ultimate Voronoi partitions and LOCs, even under the same initial conditions and with the same control parameters. This is mainly due to the existence of the uncertainties of the real robots, such as the network delay, the friction forces, and the system noise. Achieving optimal coverage with these uncertainties indicates the robustness of the proposed control method, even though the ultimate LOCs may be different. A video of the experiment can be referred to at https://youtu.be/NAvVDMRWqN8.



(a) The trajectories of the robot positions $\zeta_i(t)$ (thin solid lines), virtual centers $z_i(t)$ (thick dotted lines), and Voronoi centroids $C(\mathcal{Z}_{\overline{\mathscr{A}_i}})$ (thick dashed lines) of the CSURs, $i \in \mathcal{N}$, where 'x' and 'o' are the starting and ending points of the trajectories. The orbit radius is determined as small to avoid collisions.

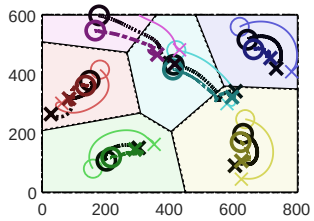


(b) The value of the coverage cost as time changes

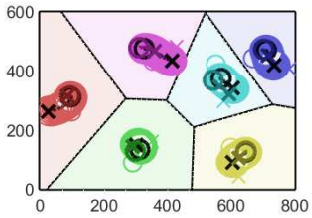


(c) The control inputs as time changes

Fig. 6: The optimal coverage of a 100-agent robot team.

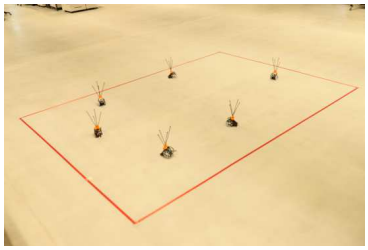


(a) Conventional controller



(b) Proposed controller

Fig. 7: The trajectories of the CSUR positions $\zeta_i(t)$ (thin solid lines), virtual centers $z_i(t)$ (thick dotted lines), and the Voronoi centroids $C(\mathcal{Z}_{\mathscr{A}_i})$ (thick dashed lines) of the multi-CSUR system with the *conventional* and the *proposed* coverage control methods, $i \in \mathcal{N}$, where 'x' and 'o' are respectively the starting and ending points of the trajectories.



(a) The experimental ground region.

(b) Robot

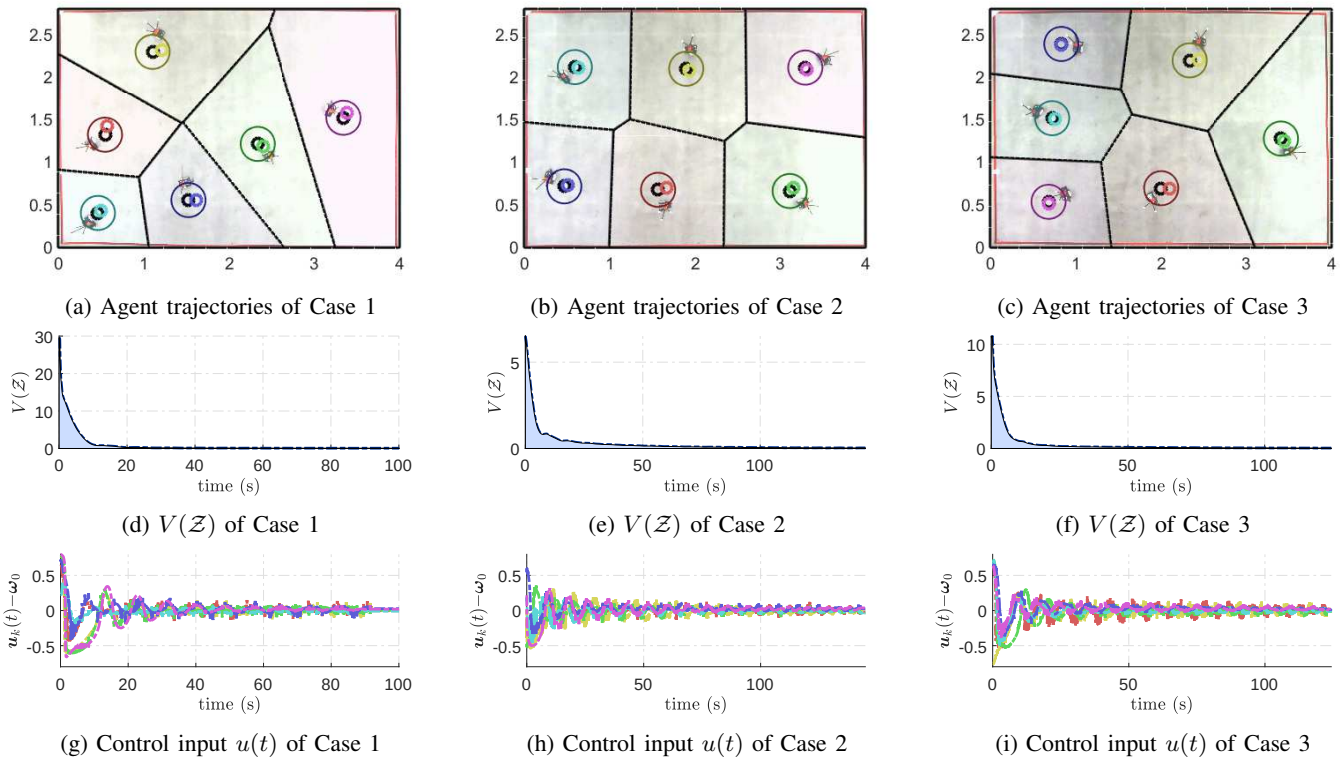
Fig. 8: The experimental setup and the mobile robot.

VII. Discussion

In general, the coordination control of a multi-agent system with complex agent dynamics is a challenging problem. Additional challenges for optimal coverage control of a multi-CSUR system include the non-convex coverage metric function, the state- and input-dependent constraints, and the distributed realization. This paper provides the first feasible solution that solves all these issues. The main technical points of the proposed coverage controller can be summarized as follows. Firstly, to overcome the limitation of the conventional coverage metric that is intended for multi-SIR systems, we propose a novel coverage metric function for multi-CSUR systems. The gradient-based controller derived from this metric function ensures the ultimate achievement of optimal coverage and the satisfaction of state confinement. Secondly, a new communication standard allows the controller to be designed in a distributed manner. Thirdly, a Sigmoid function is used such that the control inputs satisfy the given saturation constraints. Another remark is about the applicability and generalizability of the proposed method to a wider range of practical cases. Elaborating the studies on every possible configuration is not realistic. Instead, a series of simulation and experiment studies in this paper have validated that the proposed method is effective for various initial conditions, control parameters, and number of agents. Also, the advantage and necessity of this method are addressed via a comparison study with the conventional control method. Besides, the experiment study indicates that the proposed method is still effective with the existence of uncertainties except that the ultimate coverage configuration may be changed. Thus, we can confirm the effectiveness and applicability of the proposed coverage controller in a generic sense.

VIII. Conclusion

In this paper, we propose a novel optimal coverage controller for a type of multi-agent system with complex agent dynamics. We solve this non-trivial problem by proposing a novel coverage cost function and comprehensively using several theoretical tools including BLF, Lyapunov asymptotic stability, and the invariance theory. Decentralization of the controller is feasible by redefining the communication standard for the system. The effectiveness of the proposed controller and its advantage over the conventional method are validated via simulation and experiment studies. This work does not only solve a challenging problem but also can inspire the controller design of other coordinate control problems for a multi-agent system with complex agents, which is going to be investigated in future work. Also, our future work will incorporate more flexible collision avoidance into the controller design.



(a) Agent trajectories of Case 1 (b) Agent trajectories of Case 2 (c) Agent trajectories of Case 3

(d) $V(\mathcal{Z})$ of Case 1 (e) $V(\mathcal{Z})$ of Case 2 (f) $V(\mathcal{Z})$ of Case 3

(g) Control input $u(t)$ of Case 1 (h) Control input $u(t)$ of Case 2 (i) Control input $u(t)$ of Case 3

Fig. 9: The results of the experimental study on six unicycle mobile robots: (a)-(c) show the Voronoi partitions of the target region subject to the ultimate LOCs, with the virtual centers (dark small circles), their corresponding Voronoi centroids (shadow small circles), and the orbits of the robots (thin large circles). The background is filled with screenshots of the robot positions taken by a camera from above, (d)-(f) are the coverage cost functions, and (g)-(i) are the control inputs of the robots.

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