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Citation for published version (APA):
Meijer, T. J., Holicki, T., van den Eijnden, S. J. A. M., Scherer, C. W., & Heemels, W. P. M. H. (2023). The Non-Strict Projection Lemma. *arXiv*, *2023*, 1-7. Article 2305.08735. https://doi.org/10.48550/arXiv.2305.08735

DOI:
10.48550/arXiv.2305.08735

Document status and date:
Published: 01/01/2023

Document Version:
Publisher's PDF, also known as Version of Record (includes final page, issue and volume numbers)

The Non-Strict Projection Lemma

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***Abstract*—The projection lemma (often also referred to as the elimination lemma) is one of the most powerful and useful tools in the context of linear matrix inequalities for system analysis and control. In its traditional formulation, the projection lemma only applies to strict inequalities, however, in many applications we naturally encounter non-strict inequalities. As such, we present, in this note, a non-strict projection lemma that generalizes both its original strict formulation as well as an earlier non-strict version. We demonstrate several applications of our result in robust linear-matrix-inequality-based marginal stability analysis and stabilization, a matrix S-lemma, which is useful in (direct) data-driven control applications, and matrix dilation.**

***Index Terms*—Linear matrix inequalities (LMIs), parameter elimination, data-driven control, semi-definite programming, marginal stability**

I. Introduction

LINEAR matrix inequalities (LMIs) have found their way into a wide variety of control applications [1]. In parallel to this adoption, an incredible collection of tools has been developed that enables us to formulate LMIs for different and increasingly complicated applications. The projection lemma (PL), see, e.g., [2], [3], is a crucial part of this LMI toolkit, which has been an enabler for developing powerful results, such as, e.g., H_∞ controller synthesis [2], robust control design [1], [4], [5], and gain-scheduled control design [6], [7] to name but a few. In fact, [5, Chapter 9] presents a unified approach based on the projection lemma to solve 17 seemingly different control problems, including the characterization of all stabilizing controllers for a linear time-invariant (LTI) plant, covariance control, H_∞ control, L_∞ control, LQG control, and H_2 control of LTI systems. It is also used to introduce slack variables for reducing the conservatism in certain robust control designs, see, e.g., [8], [9]. In other words, the PL has had–without a doubt–a significant impact in the field of system and control theory. All of the above developments are based on the strict version of the projection lemma (strict in the sense of strictness of the involved matrix inequalities). Given the impact of this strict projection lemma (SPL) and the emergence of various control problems that call for non-strict versions of the PL (see the discussion in Sections II and IV below), we will formulate a non-strict generalization of this powerful result.

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Funded by the European Union (ERC, Proacthis, no. 101055384). Views and opinions expressed are however those of the author(s) only and do not necessarily reflect those of the European Union or the European Research Council. Neither the European Union nor the granting authority can be held responsible for them.

Funded by Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy -EXC 2075 -390740016. We acknowledge the support by the Stuttgart Center for Simulation Science (SimTech).

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The classical projection lemma [1], [2], stated next, is formulated in terms of strict inequalities.

Lemma 1. *Consider the complex matrices $U \in \mathbb{C}^{m\times p}$ and $V \in \mathbb{C}^{n\times p}$ and the Hermitian matrix $Q \in \mathbb{H}^p$, there exists a matrix $X \in \mathbb{C}^{m\times n}$ such that*

$$Q + U^{\mathsf{H}}XV + V^{\mathsf{H}}X^{\mathsf{H}}U \succ 0, \tag{1}$$

if and only if

$$U_\perp^{\mathsf{H}}QU_\perp \succ 0 \text{ and } V_\perp^{\mathsf{H}}QV_\perp \succ 0, \tag{2}$$

where, for any matrix, $(\cdot)^{\mathsf{H}}$ denotes its conjugate transpose and $(\cdot)_\perp$ denotes an arbitrary matrix with columns that form a basis for its kernel (null space).

When replacing the *strict* inequalities ($\succ$) in Lemma 1 by *non-strict* inequalities ($\succcurlyeq$), the implication (1) $\Rightarrow$ (2) still holds, however, the converse is no longer true in general. An earlier non-strict version of the PL with additional conditions on U and V under which the PL extends to non-strict inequalities, see Lemma 2 below, was presented in [10, Lemma 6.3].

Lemma 2. *Suppose U and V are such that*

$$\operatorname{im} U^{\mathsf{H}} \cap \operatorname{im} V^{\mathsf{H}} = \{0\}. \tag{3}$$

Then, there exists X such that

$$Q + U^{\mathsf{H}}XV + U^{\mathsf{H}}X^{\mathsf{H}}V \succcurlyeq 0,$$

if and only if

$$U_\perp^{\mathsf{H}}QU_\perp \succcurlyeq 0 \text{ and } V_\perp^{\mathsf{H}}QV_\perp \succcurlyeq 0.$$

The result above, in which $\operatorname{im}(\cdot)$ denotes the image of a matrix, is shown to be useful in the context of, e.g., robust control using μ-synthesis [10]. However, Lemma 2 only applies to matrices Q, U and V that satisfy the additional condition (3), which can be restrictive in many applications, such as marginal/Lyapunov stability, as we will illustrate in Section II. Another noteworthy observation regarding Lemma 2 is that it does *not* include the SPL as a special case and, thus, Lemma 2 is not a true generalization of the SPL. Although there are prior applications of the non-strict projection lemma, see, e.g., [3], [11], the precise result and a rigorous proof are, to the best of the authors' knowledge, not found in the literature.

The main contribution of this note is to present a non-strict projection lemma (NSPL) that (a) is general in the sense that it applies to all matrices Q, U and V, and (b) generalizes both the SPL and the non-strict formulation in Lemma 2. The usefulness of our result is demonstrated by applying it to derive LMI-based marginal stability conditions, which could not be achieved using the SPL or Lemma 2. We also apply the NSPL to a matrix dilation problem and to derive a useful interpolation result with weaker assumptions than existing results. Interestingly, as we will see, a matrix S-lemma, similar to those developed in [12] for (direct) data-driven control, follows naturally from the aforementioned interpolation result.

The remainder of this note is organized as follows. After introducing notational conventions in Section I-A, we discuss a motivating example, in Section II, for which the SPL and Lemma 2 fall short. In Section III, we present the NSPL, which forms our main contribution. Finally, we present several applications of this NSPL in Section IV and conclusions in Section V. All proofs are found in the Appendix.

A. Notation

The sets of real, complex and non-negative natural numbers are denoted, respectively, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N} = \{0, 1, 2, \ldots\}$. We denote $\mathbb{R}_{\geqslant 0} = [0, \infty)$ and $\mathbb{R}_{>0} = (0, \infty)$. The sets of n-dimensional real and complex vectors are, respectively, $\mathbb{R}^n$ and $\mathbb{C}^n$. Let (u, v) denote $[u^{\mathsf{H}} \quad v^{\mathsf{H}}\ ]^{\mathsf{H}}$ for any $u, v \in \mathbb{C}^n$. The sets of n-by-n Hermitian and symmetric matrices are denoted by, respectively, $\mathbb{H}^n = \{A \in \mathbb{C}^{n\times n} \mid A = A^{\mathsf{H}}\}$ and $\mathbb{S}^n = \{A \in \mathbb{R}^{n\times n} \mid A = A^\top\}$. We use the symbol $\star$ to complete a Hermitian matrix, e.g., $\left[\begin{smallmatrix} A & B \\ \star & C \end{smallmatrix}\right] = \left[\begin{smallmatrix} A & B \\ B^{\mathsf{H}} & C \end{smallmatrix}\right]$, and I is an identity matrix of appropriate dimensions. For a Hermitian matrix $H \in \mathbb{H}^n$, $H \succ 0$, $H \succcurlyeq 0$ and $H \prec 0$ mean, respectively, that H is positive definite, i.e., $x^{\mathsf{H}} H x > 0$ for all $x \in \mathbb{C}^n \setminus \{0\}$, positive semi-definite, i.e., $x^{\mathsf{H}} H x \geqslant 0$ for all $x \in \mathbb{C}^n$, and negative definite, i.e., $-H \succ 0$. We denote, respectively, the sets of such matrices of size n-by-n as $\mathbb{H}^n_{\succ 0}$ and $\mathbb{H}^n_{\succcurlyeq 0}$ and their real-valued counterparts as $\mathbb{S}^n_{\succ 0}$, $\mathbb{S}^n_{\succcurlyeq 0}$ and $\mathbb{S}^n_{\prec 0}$. For a complex matrix $A \in \mathbb{C}^{n\times m}$, $\operatorname{im} A = \{x \in \mathbb{C}^n \mid x = Ay \text{ for some } y \in \mathbb{C}^m\}$ denotes its image, $\ker A = \{x \in \mathbb{C}^m \mid Ax = 0\}$ its kernel and A^+ its (Moore-Penrose) pseudoinverse. Finally, $\operatorname{diag}\{A_1, A_2, \ldots, A_n\}$ denotes a block-diagonal matrix with diagonal blocks A_i, $i \in \{1, 2, \ldots, n\}$.

II. Motivating example

Consider the discrete-time LTI system

$$x_{k+1} = Ax_k, \tag{4}$$

where $x_k \in \mathbb{R}^{n_x}$ denotes the state at time $k \in \mathbb{N}$. We are interested in marginal stability of the system (4), i.e., whether, for all x_0, the solution x_k is uniformly bounded in the sense that there exists $c \in \mathbb{R}_{\geqslant 0}$ such that $\|x_k\| \leqslant c\|x_0\|$ for all $k \in \mathbb{N}$. The system (4) is marginally stable, if and only if there exists a symmetric matrix $P \in \mathbb{S}^{n_x}$ such that [13, p. 211]

$$P \succ 0 \text{ and } P - A^\top P A \succcurlyeq 0, \tag{5}$$

or, equivalently, if there exists $S \in \mathbb{S}^{n_x}$ such that

$$S \succ 0 \text{ and } S - ASA^\top \succcurlyeq 0, \tag{6}$$

in which case $V(x) = x^\top P x$ (with $P = S^{-1}$) is a weak Lyapunov function, i.e., V is positive definite, radially unbounded and non-increasing along solutions to (4) [14], [15].

While (6) can be used to guarantee marginal stability of (4), it cannot easily be extended to more complicated applications such as synthesis or even robust control, e.g., when A is uncertain, by applying (6) to the relevant closed-loop dynamics. In the context of *asymptotic* stability, many different LMI-based conditions have been proposed that not only guarantee that (4) is asymptotically stable, but also accommodate such more complicated applications, see, e.g., [16]–[19]. Many of these results, see, e.g., [16]–[18] (or [20] for continuous time), follow by application of the SPL. For illustrative purposes, we consider the following condition: The system (4) is asymptotically stable if and only if there exist a symmetric matrix $S \in \mathbb{S}^{n_x}$ and a matrix $X \in \mathbb{R}^{n_x\times n_x}$ such that [8]

$$\begin{bmatrix} S & AX \\ \star & X + X^\top - S \end{bmatrix} \succ 0, \tag{7}$$

which follows from the conditions $S \succ 0$ and $S - ASA^\top \succ 0$, which are necessary and sufficient for asymptotic stability, by applying Lemma 1. To see this, note that (7) is (1) with

$$Q = \begin{bmatrix} S & 0 \\ \star & -S \end{bmatrix}, \; U^\top = \begin{bmatrix} 0 \\ I \end{bmatrix} \text{ and } V^\top = \begin{bmatrix} A \\ I \end{bmatrix}.$$

This condition is useful for stabilizing controller synthesis by replacing $A \leftarrow A + BK$ and applying the linearizing change of variables $X = KY$, where K is the to-be-designed controller gain. What makes the condition in (7) even more powerful is the absence of products between A and S, which enables a natural extension to robust controller synthesis as presented in, e.g., [16], [18], in which both A and S depend on some uncertain parameter. Clearly, we cannot use the SPL to obtain a non-strict counterpart to (7) that is also equivalent to (6). Also the condition in (3) is not always satisfied and, hence, we cannot apply Lemma 2 either. To see this, observe that, if there exists $x \in \mathbb{R}^{n_x} \setminus \{0\}$ with $Ax = 0$, then $(0, x)$ is contained in both $\operatorname{im} U^\top$ and $\operatorname{im} V^\top$ and, hence, their intersection is non-trivial. In the next section, we will present a generalization of Lemma 2 after which we will revisit this example in Section IV-A and derive a non-strict version of (7).

III. Main result

The main contribution of this note, i.e., a non-strict generalization of the well-known SPL, is stated below.

Theorem 1. *There exists X such that*

$$Q + U^{\mathsf{H}} X V + V^{\mathsf{H}} X^{\mathsf{H}} U \succcurlyeq 0, \tag{8}$$

if and only if

$$U_\perp^{\mathsf{H}} Q U_\perp \succcurlyeq 0 \text{ and } V_\perp^{\mathsf{H}} Q V_\perp \succcurlyeq 0, \tag{9}$$

and

$$\ker U \cap \ker V \cap \{\xi \in \mathbb{C}^p \mid \xi^{\mathsf{H}} Q \xi = 0\} \subset \ker Q. \tag{10}$$

It is worth mentioning that the proof of Theorem 1, which is constructive, shows that, if U, V and Q are real-valued, then also X satisfying (8) is real-valued. Interestingly, Theorem 1 generalizes the SPL (Lemma 1) because (2) implies that

$$\ker U \cap \ker V \cap \{\xi \in \mathbb{C}^p \mid \xi^{\mathsf{H}} Q \xi = 0\} = \{0\} \subset \ker Q. \tag{11}$$

Observe also that, in contrast with Lemma 2, the equivalence relation in Theorem 1 holds for any matrices Q, U and V and that, thereby, Theorem 1 generalizes Lemma 2, which holds only for matrices that satisfy (3). In fact, it follows, by combining Theorem 1 and Lemma 2, that (3) and (9) together must imply (10). To make this more insightful, we will now show this fact directly (without using Theorem 1 or Lemma 2). Suppose that (3) and (9) hold and let $x \in \ker U \cap \ker V$ be such that $x^{\mathsf{H}}Qx = 0$. Then, there exist ξ and η such that $x = U_\perp \xi$ and $x = V_\perp \eta$. It follows that $0 = x^{\mathsf{H}}Qx = \xi^{\mathsf{H}}U_\perp^{\mathsf{H}}QU_\perp\xi = \eta^{\mathsf{H}}V_\perp^{\mathsf{H}}QV_\perp\eta$, which, using (9), implies that $U_\perp^{\mathsf{H}}Qx = 0$ and $V_\perp^{\mathsf{H}}Qx = 0$. Hence, it holds that $Qx \in \ker U_\perp^{\mathsf{H}} = (\operatorname{im} U_\perp)^\perp = (\ker U)^\perp = \operatorname{im} U^{\mathsf{H}}$ and, similarly, $Qx \in \operatorname{im} V^{\mathsf{H}}$, i.e.,

$$Qx \in \operatorname{im} U^{\mathsf{H}} \cap \operatorname{im} V^{\mathsf{H}} \overset{(3)}{=} \{0\}.$$

Thereby, (10) holds.

To make the conservatism introduced by (3) more concrete, we provide an algebraic example below.

Example. *Consider the matrices*

$$Q = \begin{bmatrix} 3 & 1 & -2 \\ \star & 1 & -1 \\ \star & \star & 1 \end{bmatrix},\ U^\top = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } V^\top = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

The relevant annihilators are given by

$$U_\perp = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^\top \text{ and } V_\perp = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}^\top$$

and it is straightforward to verify that

$$U_\perp^\top Q U_\perp = 1 \succcurlyeq 0 \text{ and } V_\perp^\top Q V_\perp = \begin{bmatrix} 1 & -1 \\ \star & 1 \end{bmatrix} \succcurlyeq 0.$$

Here, $\operatorname{im} U^\top \cap \operatorname{im} V^\top = \operatorname{im} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^\top \neq \{0\}$ *such that* (3) *does not hold. However, we can still use Theorem 1 to conclude that there exists a matrix X satisfying* (8)*, because* (11) *holds and, thereby,* (10) *is satisfied. To see this, note that* $\ker U \cap \ker V = \ker U$ *and* $U_\perp^\top Q U_\perp = 1 \succ 0$*, such that there does not exist a nonzero* $x \in \ker U \cap \ker V$ *with* $x^\top Q x = 0$.

Although a rigorous proof has not been published before, the NSPL has already proved useful in formulating LMI relaxations in robust control [11]. In the next section, we revisit the motivating example from Section II, for which we showed that the SPL and Lemma 2 did not apply, and utilize Theorem 1 to find a solution. We will also demonstrate applications of Theorem 1 to matrix dilation theory, interpolation and a matrix S-lemma, which is used in modern data-driven techniques.

IV. Applications

In this section, we present several relevant applications of Theorem 1. For each of the applications below, the crucial steps in their respective proofs are achieved through the NSPL.

A. Marginal stability and stabilizability revisited

First, we revisit the motivating example discussed in Section II, for which we demonstrated that neither the strict projection lemma nor Lemma 2 could be applied. Using Theorem 1 we derive a non-strict version of (7) that can be used to test marginal stability of the system (4). The resulting condition is presented in (P1.3), below, along with its counterpart that can be used for observer synthesis in (P1.2).

Proposition 1. *The following statements are equivalent:*

(P1.1) The system (4) *is marginally stable.*

(P1.2) There exist a symmetric positive-definite matrix $P \in \mathbb{S}^{n_x}_{\succ 0}$ *and a matrix* $X \in \mathbb{R}^{n_x \times n_x}$ *such that*

$$\begin{bmatrix} P & A^\top X^\top \\ \star & X + X^\top - P \end{bmatrix} \succcurlyeq 0. \tag{12}$$

(P1.3) There exist a symmetric positive-definite matrix $S \in \mathbb{S}^{n_x}_{\succ 0}$ *and a matrix* $X \in \mathbb{R}^{n_x \times n_x}$ *such that*

$$\begin{bmatrix} S & AX \\ \star & X + X^\top - S \end{bmatrix} \succcurlyeq 0. \tag{13}$$

Moreover, any matrix X satisfying (12) *or* (13) *is non-singular.*

Although this is not the main focus of this note, let us elaborate somewhat on the conditions obtained in Proposition 1. All statements in Proposition 1 are equivalent, however, each is useful in different applications. For instance, (P1.1) can be used to verify marginal stability of a given system (4), whereas (P1.2) is useful for observer synthesis by replacing $A \leftarrow A + LC$ and applying the linearizing change of variables $Z_L = XL$. Similarly, (P1.3) can be used for controller synthesis by substituting $A \leftarrow A + BK$ and performing the linearizing change of variables $KX = Z_K$. Due to the fact that no products between A and P or S appear, both (P1.2) and (P1.3) can be used, following the same development as in [16], to synthesize polytopic/switched observers/controllers for polytopic or switched linear systems, while guaranteeing marginal stability of the closed-loop (or estimation error) system using a polytopic/switched weak Lyapunov function.

B. Interpolation and the matrix S-procedure

We can also apply the NSPL to derive the interpolation result below, which is a version of [11, Lemma A.2] with slightly weaker assumptions.

Lemma 3. *Let* $R \in \mathbb{S}^m_{\prec 0}$ *and*

$$P = \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \in \mathbb{S}^{n+m}$$

with $Q - SR^{-1}S^\top \succcurlyeq 0$*. Then, for any* $z \in \mathbb{R}^n$ *and* $w \in \mathbb{R}^m$*, there exists some matrix* $\Delta \in \mathbb{R}^{m \times n}$ *such that*

$$w = \Delta z \text{ and } \begin{bmatrix} I \\ \Delta \end{bmatrix}^\top P \begin{bmatrix} I \\ \Delta \end{bmatrix} \succcurlyeq 0, \tag{14}$$

if and only if

$$\begin{bmatrix} z \\ w \end{bmatrix}^\top P \begin{bmatrix} z \\ w \end{bmatrix} \geqslant 0. \tag{15}$$

Interpolation results, such as the one in Lemma 3, have been used to formute LMI relaxations in robust control [11]. Interestingly, Lemma 3 can also be used to derive the matrix S-lemma below.

Lemma 4. *Let $M \in \mathbb{S}^{n+m}$ and*

$$N = \begin{bmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{bmatrix} \in \mathbb{S}^{n+m},$$

with $N_{22} \prec 0$ and $N_{11} - N_{12}N_{22}^{-1}N_{12}^\top \succcurlyeq 0$. Then, the following statements are equivalent:

(L4.1) $\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} \succ 0$ *for all Z with* $\begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} \succcurlyeq 0$.

(L4.2) There exists $\alpha \geqslant 0$ such that $M - \alpha N \succ 0$.

Lemma 4 is essentially [12, Corollary 12] without the requirement that N is non-singular. Similar non-strict results, see, e.g., [12], [21], have proved instrumental in recent (direct) data-driven control applications, see, e.g., [12], [21]–[24], which, in turn, demonstrates the relevance of the NSPL, also for data-driven control applications.

C. Matrix dilations

Finally, we indicate that Theorem 1 also has applications in matrix (or operator) dilation theory [25], such as in the result stated below.

Lemma 5. *Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times p}$ and $C \in \mathbb{R}^{q\times n}$. Then, there exists $D \in \mathbb{R}^{q\times p}$ with*

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leqslant 1, \tag{16}$$

if and only if

$$\left\| \begin{bmatrix} A & B \end{bmatrix} \right\| \leqslant 1 \text{ and } \left\| \begin{bmatrix} A \\ C \end{bmatrix} \right\| \leqslant 1. \tag{17}$$

A proof based on Theorem 1 can be found in the Appendix.

V. Conclusions

In this technical note, we presented a non-strict generalization of the projection lemma. This non-strict projection lemma was shown to include both the strict projection lemma and an earlier non-strict version of the projection lemma as special cases, thereby showing that our contribution generalizes these existing results. In addition, we showed several applications of this novel non-strict projection lemma, for which existing results could not be applied. One such application is analyzing marginal stability (or performing marginal stabilization) of discrete-time LTI systems, for which we derived several LMI-based conditions. The resulting stability conditions are such that they can be used for controller and observer synthesis. They may be further extended to accommodate synthesis for polytopic/switched linear systems using a polytopic/switched weak Lyapunov function to guarantee marginal stability for the corresponding closed-loop system. We also demonstrate applications of our results to dilation theory as well as interpolation, where, for the latter, we show that the matrix S-lemma, which proves instrumental in the context of (direct) data-driven control, naturally follows.

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Appendix

A. Preliminaries

Lemma 6 (Schur complement [1, p. 8, 28])**.** *Let $Q \in \mathbb{H}^m$, $R \in \mathbb{H}^n$ and $S \in \mathbb{C}^{m\times n}$. It holds that*

$$\begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \succcurlyeq 0, \tag{18}$$

if and only if $R \succcurlyeq 0$, $Q - SR^+S^\mathsf{H} \succcurlyeq 0$ and $S(I - RR^+) = 0$. If R is non-singular, (18) *holds if and only if $R \succcurlyeq 0$ and $Q - SR^{-1}S^\mathsf{H} \succcurlyeq 0$.*

Lemma 7 (S-lemma [1, p. 24])**.** *Let $M, N \in \mathbb{S}^n$ and suppose that there exists some $\bar{x} \in \mathbb{R}^n$ such that $\bar{x}^\top N \bar{x} > 0$. Then, the following statements are equivalent:*
(L7.1) $x^\top M x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ such that $x^\top N x \geqslant 0$.
(L7.2) There exists $\alpha \geqslant 0$ such that $M - \alpha N \succ 0$.

Lemma 8 (Finsler's lemma [26, Theorem 2.2], [27])**.** *Let $M, N \in \mathbb{S}^n$. Then, the following statements are equivalent:*
(L8.1) $x^\top M x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ such that $x^\top N x = 0$.
(L8.2) There exists $\alpha \in \mathbb{R}$ such that $M - \alpha N \succ 0$.

B. Proof of Theorem 1

Necessity: Suppose there exists $X \in \mathbb{C}^{m\times n}$ such that (8) holds. It follows, by projection, that

$$U_\perp^\mathsf{H}\left(Q + U^\mathsf{H}XV + V^\mathsf{H}X^\mathsf{H}U\right)U_\perp = U_\perp^\mathsf{H} Q U_\perp \succcurlyeq 0,$$

and

$$V_\perp^\mathsf{H}\left(Q + U^\mathsf{H}XV + V^\mathsf{H}X^\mathsf{H}U\right)V_\perp = V_\perp^\mathsf{H} Q V_\perp \succcurlyeq 0.$$

Thus, we conclude that (9) is satisfied. It remains to show that (10) holds. Let $x \in \ker U \cap \ker V \cap \{\xi \in \mathbb{R}^p \mid \xi^\mathsf{H} Q \xi = 0\}$ and let $S \in \mathbb{H}^n_{\succcurlyeq 0}$ be any square matrix with $S^\mathsf{H}S = Q + U^\mathsf{H}XV + V^\mathsf{H}X^\mathsf{H}U$. Then,

$$\|Sx\|^2 = x^\mathsf{H}(Q + U^\mathsf{H}XV + V^\mathsf{H}X^\mathsf{H}U)x = x^\mathsf{H}Qx = 0,$$

from which we obtain

$$0 = Sx = SSx = (Q + U^\mathsf{H}XV + V^\mathsf{H}X^\mathsf{H}U)x = Qx,$$

or, equivalently, $x \in \ker Q$, such that (10) holds.

Sufficiency: Suppose that (9) and (10) hold. Let $T \in \mathbb{C}^{p\times p}$ be a non-singular matrix, partitioned as $T = \begin{bmatrix} T_1 & T_2 & T_3 & T_4 & T_5 \end{bmatrix}$ with

$$\operatorname{im}\begin{bmatrix} T_1 & T_3 & T_4 \end{bmatrix} = \ker U, \tag{19}$$
$$\operatorname{im}\begin{bmatrix} T_2 & T_3 & T_4 \end{bmatrix} = \ker V, \tag{20}$$
$$\operatorname{im}\begin{bmatrix} T_3 & T_4 \end{bmatrix} = \ker U \cap \ker V, \tag{21}$$
$$\operatorname{im} T_4 = \ker U \cap \ker V \cap \ker Q. \tag{22}$$

By congruence transformation with T, (8) holds if and only if

$$Y := T^\mathsf{H}QT + (UT)^\mathsf{H}XVT + (VT)^\mathsf{H}X^\mathsf{H}UT \succcurlyeq 0. \tag{23}$$

Using (22), $W := T^\mathsf{H}QT$, partitioned according to T, reads

$$W = \left[\begin{array}{ccc:c:c} W_{11} & W_{12} & W_{13} & 0 & W_{15} \\ \star & W_{22} & W_{23} & 0 & W_{25} \\ \star & \star & W_{33} & 0 & W_{35} \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ \hdashline \star & \star & \star & 0 & W_{55} \end{array}\right]. \tag{24}$$

Similarly, using (19), (20) and (22), the term $(UT)^\mathsf{H}XVT$ in (23) reads as

$$(UT)^\mathsf{H}XVT = \begin{bmatrix} 0 \\ (UT_2)^\mathsf{H} \\ 0 \\ 0 \\ (UT_5)^\mathsf{H} \end{bmatrix} X \begin{bmatrix} (VT_1)^\mathsf{H} \\ 0 \\ 0 \\ 0 \\ (VT_5)^\mathsf{H} \end{bmatrix}^\mathsf{H}. \tag{25}$$

It follows from (19) and (20), respectively, that $\begin{bmatrix} UT_2 & UT_5 \end{bmatrix}$ and $\begin{bmatrix} VT_1 & VT_5 \end{bmatrix}$ are full column rank. Using (24) and (25), Y in (23) reads as

$$\left[\begin{array}{c:c:c} Y_1 & 0 & Y_2 \\ \hdashline \star & 0 & 0 \\ \hdashline \star & \star & Y_3 \end{array}\right] = \tag{26}$$
$$\left[\begin{array}{ccc:c:c} W_{11} & W_{12} + K^\mathsf{H} & W_{13} & 0 & W_{15} + M^\mathsf{H} \\ \star & W_{22} & W_{23} & 0 & W_{25} + L \\ \star & \star & W_{33} & 0 & W_{35} \\ \hdashline 0 & 0 & 0 & 0 & 0 \\ \hdashline \star & \star & \star & 0 & W_{55} + N + N^\mathsf{H} \end{array}\right] \succcurlyeq 0,$$

where

$$\begin{bmatrix} K & L \\ M & N \end{bmatrix} = \begin{bmatrix} (UT_2)^\mathsf{H} \\ (UT_5)^\mathsf{H} \end{bmatrix} X \begin{bmatrix} VT_1 & VT_5 \end{bmatrix}.$$

Observe that, since $\begin{bmatrix} UT_2 & UT_5 \end{bmatrix}$ and $\begin{bmatrix} VT_1 & VT_5 \end{bmatrix}$ are full column rank,

$$X = \begin{bmatrix} (UT_2)^\mathsf{H} \\ (UT_5)^\mathsf{H} \end{bmatrix}^+ \begin{bmatrix} K & L \\ M & N \end{bmatrix} \begin{bmatrix} VT_1 & VT_5 \end{bmatrix}^+, \tag{27}$$

satisfies (8) for any K, L, M and N that satisfy (26). In the remainder of this proof, we construct such K, L, M and N.

First, we construct K that renders Y_1 in (26) positive semi-definite. To this end, note that, due to (9), (19) and (20),

$$\begin{bmatrix} W_{11} & W_{13} \\ \star & W_{33} \end{bmatrix} \succcurlyeq 0 \text{ and } \begin{bmatrix} W_{22} & W_{23} \\ \star & W_{33} \end{bmatrix} \succcurlyeq 0. \tag{28}$$

It also follows from (10) that

$$x^\top Q x \neq 0, \text{ for all } x \in \ker U \cap \ker V \text{ with } x \notin \ker Q,$$

and, thus, by construction of T_3 and using (28), it holds that

$$W_{33} = T_3^\mathsf{H} Q T_3 \succ 0.$$

Hence, we can apply Lemma 6 to (28) to obtain

$$W_{11} - W_{13}W_{33}^{-1}W_{13}^\mathsf{H} \succcurlyeq 0 \text{ and } W_{22} - W_{23}W_{33}^{-1}W_{23}^\mathsf{H} \succcurlyeq 0. \tag{29}$$

Since $W_{33} \succ 0$, Lemma 6 reveals that $Y_1 \succcurlyeq 0$ if and only if

$$\begin{bmatrix} W_{11} & W_{12} + K^\mathsf{H} \\ \star & W_{22} \end{bmatrix} - \begin{bmatrix} W_{13} \\ W_{23} \end{bmatrix} W_{33}^{-1} \begin{bmatrix} W_{13} \\ W_{23} \end{bmatrix}^\mathsf{H} \succcurlyeq 0.$$

By (29), $K = -W_{12}^\mathsf{H} + W_{23}W_{33}^{-1}W_{13}^\mathsf{H}$ renders the latter inequality valid and, hence, $Y_1 \succcurlyeq 0$.

Next, we apply Lemma 6 to see that (26) is equivalent to

$$Y_1 \succcurlyeq 0,\ Y_3 - Y_2^\mathsf{H}Y_1^+Y_2 \succcurlyeq 0 \text{ and } Y_2^\mathsf{H}(I - Y_1Y_1^+) = 0.$$

We have already constructed a matrix K such that $Y_1 \succcurlyeq 0$. Let us now construct L and M such that $Y_2^\mathsf{H}(I - Y_1Y_1^+) = 0$, which, due to the symmetry of $Y_1Y_1^+$, is equivalent to $(I -$

$Y_1Y_1^+)Y_2$. Hence, it suffices here to construct L and M such that we can write Y_2 as $Y_2 = Y_1\tilde{P}$ for some $\tilde{P}$, i.e.,

$$\begin{bmatrix} W_{15}+M^{\mathsf{H}} \\ W_{25}+L \\ W_{35} \end{bmatrix} = \begin{bmatrix} W_{11} & W_{13}W_{33}^{-1}W_{23}^{\mathsf{H}} & W_{13} \\ \star & W_{22} & W_{23} \\ \star & \star & W_{33} \end{bmatrix} \tilde{P}, \quad (30)$$

where we have substituted the K that we constructed earlier. A particular choice of $\tilde{P}$, L and M that satisfies (30) is

$$\tilde{P} = \begin{bmatrix} 0 \\ 0 \\ W_{33}^{-1}W_{35} \end{bmatrix}, \quad \begin{bmatrix} M^{\mathsf{H}} \\ \hdashline L \end{bmatrix} = \begin{bmatrix} -W_{15}+W_{13}W_{33}^{-1}W_{35} \\ \hdashline -W_{25}+W_{23}W_{33}^{-1}W_{35} \end{bmatrix}.$$

It remains to construct N such that

$$0 \preccurlyeq Y_3 - Y_2^{\mathsf{H}}Y_1^+Y_2 = W_{55}+N+N^{\mathsf{H}} - Y_2^{\mathsf{H}}Y_1^+Y_2, \quad (31)$$

which we achieve by choosing $N = \alpha I$ with $\alpha > 0$ sufficiently large to ensure that (31) holds. Since we have found K, L, M and N for which (26) holds, X as in (27) satisfies (8), which completes the proof.

C. Proof of Proposition 1

(P1.2): It is well-known, see, e.g., [13, p. 211], that (P1.1) is equivalent to the existence of $P \in \mathbb{S}^{n_x}_{\succ 0}$ satisfying (5), which is equivalent to the existence of $P \in \mathbb{S}^{n_x}_{\succ 0}$ such that (9) holds with

$$Q = \begin{bmatrix} P & 0 \\ \star & -P \end{bmatrix}, \; U_\perp = \begin{bmatrix} I \\ 0 \end{bmatrix} \text{ and } V_\perp = \begin{bmatrix} -I \\ A \end{bmatrix}.$$

To complete the key step in this proof, we aim to apply Theorem 1. We can take $U = \begin{bmatrix} 0 & I \end{bmatrix}$ and $V = \begin{bmatrix} A & I \end{bmatrix}$. Next, we will show that if (9) holds, then also (10) holds. To this end, it suffices to show that (11) holds. Let $x \in \ker U \cap \ker V$ with $x^\top Qx = 0$. Note that $x \in \ker U$ implies that $x = (y, 0)$ for some $y \in \mathbb{R}^{n_x}$. Moreover, $(y,0)^\top Q(y,0) = y^\top Py = 0$ implies that $y = 0$, since $P \succ 0$ and, thus, (11) holds. By Theorem 1, we find that $P \in \mathbb{S}^{n_x}_{\succ 0}$ satisfies (5) if and only if there exists $X \in \mathbb{R}^{n_x \times n_x}$ such that

$$\begin{bmatrix} P & 0 \\ \star & -P \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} X \begin{bmatrix} A & I \end{bmatrix} + \begin{bmatrix} A^\top \\ I \end{bmatrix} X^\top \begin{bmatrix} 0 & I \end{bmatrix} \succcurlyeq 0. \quad (32)$$

Thus, (P1.1) and (P1.2) are equivalent. To see that X satisfying (12) is non-singular, we note that (12) implies that $X + X^\top \succcurlyeq P \succ 0$, which holds only if X is non-singular.

(P1.3): The proof is completed using (6) and following the same steps as for (P1.2) with $P \leftarrow S$ and $A \leftarrow A^\top$.

D. Proof of Lemma 3

Necessity: Suppose there exists $\Delta \in \mathbb{R}^{m\times n}$ for which $w = \Delta z$. It immediately follows that (15) holds, since

$$\begin{bmatrix} z \\ w \end{bmatrix}^\top P \begin{bmatrix} z \\ w \end{bmatrix} = z^\top \begin{bmatrix} I \\ \Delta \end{bmatrix}^\top P \begin{bmatrix} I \\ \Delta \end{bmatrix} z \overset{(14)}{\geqslant} 0.$$

Sufficiency: Suppose that (15) holds. If $z = 0$, this inequality reads as $w^\top Rw \geqslant 0$, which, due to $R \prec 0$, implies that $w = 0$. Then, by choosing $\Delta = -R^{-1}S^\top$, we infer $w = \Delta z$ and, by assumption,

$$\begin{bmatrix} I \\ \Delta \end{bmatrix}^\top P \begin{bmatrix} I \\ \Delta \end{bmatrix} = Q - SR^{-1}S^\top \succcurlyeq 0.$$

Next, suppose $z \neq 0$. Using Lemma 6 (Schur complement) and $R \prec 0$, the desired inequality in (14) can be expressed as

$$\begin{bmatrix} Q + S\Delta + \Delta^\top S^\top & \Delta^\top \\ \star & -R^{-1} \end{bmatrix} \succcurlyeq 0. \quad (33)$$

To guarantee $w = \Delta z$, we must have, for some $H \in \mathbb{R}^{m\times n}$,

$$\Delta = wz^+ + H(I - zz^+). \quad (34)$$

It follows, by substituting (34) into (33), that there exists Δ satisfying (14), if and only if there exists H such that

$$\Psi + U^\top HV + V^\top H^\top U \succcurlyeq 0, \quad (35)$$

with $U = \begin{bmatrix} S^\top & I \end{bmatrix}$, $V = \begin{bmatrix} I - zz^+ & 0 \end{bmatrix}$ and

$$\Psi = \begin{bmatrix} Q + Swz^+ + (Swz^+)^\top & (wz^+)^\top \\ \star & -R^{-1} \end{bmatrix}.$$

The key step of this proof is done by applying Theorem 1 to find that a matrix H satisfying (35) exists, if and only if (9) and (10) hold (with $Q \leftarrow \Psi$). We introduce the annihilators

$$U_\perp = \begin{bmatrix} I \\ -S^\top \end{bmatrix} \text{ and } V_\perp = \begin{bmatrix} z & 0 \\ Rw & I \end{bmatrix}.$$

We have $U_\perp^\top \Psi U_\perp = Q - SR^{-1}S^\top \succcurlyeq 0$, by assumption, and

$$V_\perp^\top \Psi V_\perp = \begin{bmatrix} z^\top Qz + z^\top Sw + (z^\top Sw)^\top + w^\top Rw & 0 \\ \star & -R^{-1} \end{bmatrix} \overset{(15)}{\succcurlyeq} 0.$$

Next, we show that (10) holds. Since $z \neq 0$, we have

$$\ker U \cap \ker V = \operatorname{im} \begin{bmatrix} Iz \\ -S^\top z \end{bmatrix} = \operatorname{im} \begin{bmatrix} I \\ -S^\top \end{bmatrix} z. \quad (36)$$

It follows from (36) that there exists $x \in \ker U \cap \ker V \setminus \{0\}$ such that $x^\top \Psi x = 0$ if and only if

$$z^\top \begin{bmatrix} I \\ -S^\top \end{bmatrix}^\top \Psi \begin{bmatrix} I \\ -S^\top \end{bmatrix} z = \|(Q - SR^{-1}S^\top)^{\frac{1}{2}} z\|^2 = 0, \quad (37)$$

where we used the fact that, by assumption, $Q - SR^{-1}S^\top \succcurlyeq 0$. It follows that $x^\top \Psi x = 0$ for any $x \in \ker U \cap \ker V$ and, hence, (10) holds if $\ker U \cap \ker V \subset \ker \Psi$. We have

$$\Psi \begin{bmatrix} U \\ V \end{bmatrix}_\perp = \begin{bmatrix} Q + Swz^+ \\ wz^+ + R^{-1}S^\top \end{bmatrix} z = \begin{bmatrix} Qz + Sw \\ w + R^{-1}S^\top z \end{bmatrix}. \quad (38)$$

From (15) and the fact that $R \prec 0$, we have

$$\begin{aligned} 0 \leqslant \begin{bmatrix} z \\ w \end{bmatrix}^\top P \begin{bmatrix} z \\ w \end{bmatrix} &= z^\top Qz + z^\top Sw + w^\top S^\top z + w^\top Rw, \\ &= (w + R^{-1}S^\top z)^\top R (w + R^{-1}S^\top z) \leqslant 0, \end{aligned}$$

and, hence, $w = -R^{-1}S^\top z$. Substitution in (38) yields

$$\Psi \begin{bmatrix} U \\ V \end{bmatrix}_\perp = \begin{bmatrix} (Q - SR^{-1}S^\top)z \\ 0 \end{bmatrix} \overset{(37)}{=} 0,$$

such that (10) holds. We apply Theorem 1 to conclude that there exists H satisfying (35) and, thus, Δ in (34) satisfies (14).

E. Proof of Lemma 4

$(L4.2) \Rightarrow (L4.1)$**:** Suppose that (L4.2) holds. Then, for all $Z \in \mathbb{R}^{m\times n}$ such that $\begin{bmatrix} I & Z^\top \end{bmatrix} N \begin{bmatrix} I & Z^\top \end{bmatrix}^\top \succcurlyeq 0$, we have

$$\begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} \succ \alpha \begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} \succcurlyeq 0.$$

$(L4.1) \Rightarrow (L4.2)$**:** Suppose that (L4.1) holds. By Lemma 3, for any $x = (w, z)$ such that $x^\top N x \geqslant 0$, there exists $Z \in \mathbb{R}^{m\times n}$ such that

$$w = Zz \text{ and } \begin{bmatrix} I \\ Z \end{bmatrix}^\top N \begin{bmatrix} I \\ Z \end{bmatrix} \succcurlyeq 0. \tag{39}$$

Firstly, we consider the case where N has at least one positive eigenvalue. Take any $x = (w, z) \neq 0$ such that $x^\top N x \geqslant 0$ and let Z be as in (39). It follows, by (L4.1), that $\begin{bmatrix} I & Z^\top \end{bmatrix} M \begin{bmatrix} I & Z^\top \end{bmatrix}^\top \succ 0$ and, hence,

$$0 < z^\top \begin{bmatrix} I \\ Z \end{bmatrix}^\top M \begin{bmatrix} I \\ Z \end{bmatrix} z \overset{(39)}{=} x^\top M x. \tag{40}$$

Thus, (L7.1) holds and we can apply Lemma 7 to conclude that (L4.2) holds. Secondly, we consider the case where N has no positive eigenvalues, i.e., $N \preccurlyeq 0$. Take any $x = (w, z) \neq 0$ such that $x^\top N x = 0$ and let Z be as in (39). Again, it follows, by (L4.1), that $\begin{bmatrix} I & Z^\top \end{bmatrix} M \begin{bmatrix} I & Z^\top \end{bmatrix}^\top \succ 0$. Thus, (40) holds and we can apply Lemma 8 to conclude that there exists some $\alpha \in \mathbb{R}$ such that $M - \alpha N \succ 0$. If $\alpha \in \mathbb{R}_{\geqslant 0}$, we are done. Otherwise, since $N \preccurlyeq 0$, we obtain $\alpha N \succcurlyeq 0$ and, thus,

$$M - 0 \cdot N = M \succ \alpha N \succcurlyeq 0, \tag{41}$$

which completes our proof as well.

F. Proof of Lemma 5

The norm inequality (16) can be expressed as an LMI, which, using Lemma 6, can be, equivalently, expressed as

$$0 \preccurlyeq \left[\begin{array}{cc:cc} I & 0 & A & B \\ \star & I & C & D \\ \hdashline \star & \star & I & 0 \\ \star & \star & \star & I \end{array}\right] = Q + U^\top D V + V^\top D^\top U, \tag{42}$$

with

$$Q = \left[\begin{array}{cc:cc} I & 0 & A & B \\ \star & I & C & 0 \\ \hdashline \star & \star & I & 0 \\ \star & \star & \star & I \end{array}\right], \; U = \left[\begin{array}{c} 0 \\ I \\ \hdashline 0 \\ 0 \end{array}\right]^\top \text{ and } V = \left[\begin{array}{c} 0 \\ 0 \\ \hdashline 0 \\ I \end{array}\right]^\top.$$

We introduce the relevant annihilators

$$U_\perp = \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & 0 & 0 \\ \hdashline 0 & I & 0 \\ 0 & 0 & I \end{array}\right] \text{ and } V_\perp = \left[\begin{array}{ccc} I & 0 & 0 \\ 0 & I & 0 \\ \hdashline 0 & 0 & I \\ 0 & 0 & 0 \end{array}\right]. \tag{43}$$

Necessity: By assumption, (42) holds. It follows that

$$U_\perp^\top Q U_\perp = \begin{bmatrix} I & A & B \\ \star & I & 0 \\ \star & \star & I \end{bmatrix} \succcurlyeq 0 \text{ and } V_\perp^\top Q V_\perp = \begin{bmatrix} I & 0 & A \\ \star & I & C \\ \star & \star & I \end{bmatrix} \succcurlyeq 0, \tag{44}$$

which implies the norm inequalities in (17).

Sufficiency: Suppose (17) holds, which is equivalent to (44). It remains to show that (10) holds, such that we can apply the NSPL which finishes the proof. To this end, note that (44) implies, using Lemma 6, that

$$\begin{bmatrix} I & A \\ \star & I \end{bmatrix} \succcurlyeq 0, \begin{bmatrix} I - BB^\top & A \\ \star & I \end{bmatrix} \succcurlyeq 0 \text{ and } \begin{bmatrix} I - C^\top C & A \\ \star & I \end{bmatrix} \succcurlyeq 0. \tag{45}$$

Let $x = (x_1, x_2, x_3, x_4) \in \ker U \cap \ker V$ with $x^\top Q x = 0$. Due to (43), we have $x_2 = 0$ and $x_4 = 0$. It follows that

$$0 = x^\top Q x = \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}^\top \begin{bmatrix} I & A \\ \star & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \overset{(45)}{=} \left\| \begin{bmatrix} I & A \\ \star & I \end{bmatrix}^{\frac{1}{2}} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \right\|^2,$$

which implies that

$$\begin{bmatrix} I & A \\ \star & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = 0. \tag{46}$$

Next, we observe that

$$0 \overset{(45)}{\preccurlyeq} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix}^\top \begin{bmatrix} I - BB^\top & A \\ \star & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} \overset{(46)}{=} -\|B^\top x_1\|^2, \tag{47}$$

from which we conclude that $B^\top x_1 = 0$. Similarly, we infer from (45) that $Cx_3 = 0$. Combining (46), $B^\top x_1 = 0$ and $Cx_3 = 0$, we obtain

$$Qx = \left[\begin{array}{cc:cc} I & 0 & A & B \\ \star & I & C & 0 \\ \hdashline \star & \star & I & 0 \\ \star & \star & \star & I \end{array}\right] \left[\begin{array}{c} x_1 \\ 0 \\ \hdashline x_3 \\ 0 \end{array}\right] = \left[\begin{array}{c} x_1 + Ax_3 \\ Cx_3 \\ \hdashline A^\top x_1 + x_3 \\ B^\top x_1 \end{array}\right] = 0. \tag{48}$$

Hence, we can apply the NSPL to conclude that there exists $D \in \mathbb{C}^{q\times p}$ satisfying (16).