Stabilisation for varieties in polynomial functors

Stabilisation for varieties in polynomial functors

PROEFSCHRIFT

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A mia mamma Anna,

per avermi insegnato a fruire della bellezza.

Contents

Part I

Overture

Notation

Set of numbers:

- Let n be a nonnegative integer, we denote with $[n]$ the set $\{1, \dots, n\}$. In particular, if $n = 0$ then $[0] = \emptyset$.
- $\mathbf{N}$ denotes the set of natural numbers $\{0, 1, 2, \dots\}$.
- $\mathbf{Z}$ denotes the set of integer numbers $\{\dots, -2, -1, 0, 1, 2, \dots\}$.
- $\mathbf{R}$ denotes the field of real numbers.
- $\mathbf{C}$ denotes the field of complex numbers.

(Linear) algebra:

- R denotes a commutative ring with unit.
- A denotes a commutative R-algebra.
- K denotes a field.
- V, U, W denote finite dimensional vector spaces over a field K.
- $\mathrm{Hom}(U, V)$ denotes the K-module of K-linear maps from U to V.
- $\mathrm{End}(V)$ denotes $\mathrm{Hom}(V, V)$.
- $\dim(V)$ denotes the dimension of V as a K-vector space.
- Fix a vector space U. The map $\iota_V : V \to U \oplus V$ is defined by $\iota_V(v) = (0, v)$: it is the inclusion map.
- Fix a vector space U. The map $\pi_V : U \oplus V \to V$ is defined by $\pi_V(u, v) = v$: it is the projection map.
- For $m \leq n$ the maps $\pi_{n,m} : K^n \to K^m$ and $\iota_{m,n} : K^m \to K^n$ denote, respectively, the projection on the first m coordinates, and the inclusion map.
- V_∞ denotes the projective limit of K^n with maps $\pi_{n,n-1}$. It is an uncoutable dimensional vector space.
- V^* denotes the dual of a vector space V.
- $\mathrm{Sym}_K^\bullet(V^*)$ denotes the symmetric K-algebra over the vector space V^*. If the characteristic of the field is zero, its elements are the polynomial functions on V.

- $\mathrm{Mat}(n)$ denotes the space of $n \times n$-matrices with coefficients in the field K.
- Let M be a $n \times m$-matrix. Then M^t denotes its transpose.
- $\mathrm{Inc}(\mathbf{N})$ denotes the monoid of order-preserving maps from $\mathbf{N}$ to $\mathbf{N}$.
- An infinite-by-infinite matrix refers to a matrix with entries labelled by $(i, j) \in \mathbf{N} \times \mathbf{N}$.

Geometry:

- $\mathrm{Spec}(R)$ denotes the spectrum (set of prime ideals) of a ring R.
- Let V be a vector space. The *coordinates* of V are the elements of a dual basis.
- With *variety* we mean a reduced affine scheme of finite-type over a field K.
- $K[X]$ denotes the coordinate ring of a variety X.
- $\mathcal{V}(I)$ denotes the vanishing locus of I in the variety X for an ideal $I \subset K[X]$.
- Let $\varphi : X \to Y$ be a morphism of varieties, then the corresponding map on rings is denoted by $\varphi^{\#} : K[Y] \to K[X]$.
- $\dim(X)$ denotes the dimension of the variety X.

Groups:

- G denotes a group. We use the multiplicative notation.
- 1_G denotes the unit element of the group G.
- $\mathrm{Sym}(n)$ denotes the symmetric group on the set $[n]$.
- GL_n is the general linear group on K^n.
- $\mathrm{GL}(V)$ is the general linear group of the vector space V.
- Sym denotes the *infinite symmetric group*. It is defined as the direct limit over $\mathrm{Sym}(n)$ with the obvious inclusion maps.
- GL is the *infinite general linear group*. It is defined as the direct limit of GL_n with inclusion maps $\mathrm{GL}_n \to \mathrm{GL}_{n+1}$ mapping g to $\left(\begin{array}{c|c} g & 0 \\ \hline 0 & 1 \end{array}\right)$.

Categories:

- **Alg** denotes the category of K-algebras of finite-type.

- **FI** denotes the category of finite sets with injections, and $\mathbf{FI}^{\mathbf{op}}$ the corresponding opposite category.
- **PF** denotes the Abelian category of polynomial functors over an infinite field K with natural transformations, and
- $\mathbf{PF}_d$, $\mathbf{PF}_{\leq d}$ denote the full Abelian subcategories of, respectively, homogeneous polynomial functors of degree d and polynomial functors of degree at most d with natural transformations.
- $\mathbf{PF}^{\mathrm{pol}}$ denotes the non-Abelian category of polynomial functors over an infinite field K with polynomial transformations, and
- $\mathbf{PF}^{\mathrm{pol}}_d$, $\mathbf{PF}^{\mathrm{pol}}_{\leq d}$ denote the subcategories of, respectively, homogeneous polynomial functors of degree d and polynomial functors of degree at most d with polynomial transformations.
- **Sch** denotes the category of schemes over K,
- **Top** denotes the category of topological spaces.
- **Var** denotes the category of varieties over K.
- **Vec** denotes the category of finite dimensional vector spaces over K.
- Let C, D be objects in some category $\mathcal{C}$. Then $\mathrm{Hom}_{\mathcal{C}}(C, D)$ denotes the collection of arrows from C to D.

Chapter 1

Equivariant algebraic geometry

In general, one can study quantities attached to mathematical objects once these quantities are finite. Notions like the height of an ideal or the dimension of a variety are finite while working in a finite dimensional frame. In the context of infinite dimensional algebraic geometry we can look at quantities and qualities up to the action of a symmetry group. In this thesis we take this point of view: the study of infinite dimensional varieties is always meant *"up to the symmetry"* of the infinite general linear group, or the infinite symmetric group, or both together.

1.1 Stillman's conjecture

I think that the moving question for this area of mathematics is the following.

Which classical properties can be carried over to the infinite dimensional world if we take into account the action of a group, and which groups allow this?

However, in my experience, Stillman's conjecture—which is no longer a conjecture— is best cited for motivating this area of mathematics and, following this line, I recall it below. Mike Stillman in [PS09, Problem 3.14] asked if, given natural numbers $d_1, \ldots, d_k$, the projective dimension of an ideal generated by k homogeneous polynomials of degrees $d_1, \ldots, d_k$ can be bounded independently of the number of variables involved in these polynomials. This question was proven in by Tigran Ananyan and Mel Hochster in [AH20a] using the notion of strength of polynomials (and without passing to the infinite dimensional setting). In [ESS19] there are two other proofs of the conjecture: the first extends the approach of [AH20a] and establishes general results in the infinite dimensional setting that imply the results of [AH20a]; the second approach uses topological Noetherianity of polynomial functors of [Dra19], a powerful tool in infinite dimensional GL-equivariant algebraic geometry. What matters for us is that the infinite dimensional setting is the natural environment for formalising the techniques used in the proofs of Stillman's conjecture.

1.2 Up to symmetry

In this section we explain what *"up to the symmetry"* of a group means. So let G be a group, let X be a set, and recall we use the multiplicative notation for G.

Definition 1.2.1. A *left action* of the group G on the set X is a binary operation

$$\varphi : G \times X \to X$$

such that

- for any $g_1, g_2 \in G$ and any $x \in X$ we have $\varphi(g_2, \varphi(g_1, x)) = \varphi(g_2 g_1, x)$, and
- $\varphi(1_G, x) = x$ for any $x \in X$.

We denote by $g \cdot x$ the left action of $g \in G$ on $x \in X$. ♪

Analogously, one defines the *right action* of a group G on a set X. As any right action can be turned into a left action, we will only be using the left action notation.

Definition 1.2.2. If a group G has an action (either left or right) on a set X, we say that X is a *G-set*. ♪

In the case a G-set X lies in some category $\mathcal{C}$, and the group G acts on X via morphisms in $\mathcal{C}$ we say that X is a *G-object*. For example, if X is a ring and the group G acts via ring-homomorphisms on X, we say that X is a *G-ring*. Also, if Y is a subset of a G-set X we say that Y is a *G-stable subset* if the restriction to Y of the action of G on X turns Y into a G-set.

Example 1.2.3. In this example we work over $\mathbf{C}$, the field of complex numbers. Consider the space of $n \times n$-matrices $\mathrm{Mat}(n)$ with entries in $\mathbf{C}$ and let GL_n be the general linear group on $\mathbf{C}^n$. The action $g \cdot M := gMg^t$ for any $g \in \mathrm{GL}_n$ and $M \in \mathrm{Mat}(n)$ turns $\mathrm{Mat}(n)$ into a GL_n-set. As $\mathrm{Mat}(n)$ is also an algebraic variety and the above action of GL_n is given by automorphisms of $\mathrm{Mat}(n)$, then $\mathrm{Mat}(n)$ is also a GL_n-variety. Consider the subset $\mathcal{M}_{\leq r}(n)$ of $n \times n$-matrices of rank at most r for some positive integer r. If $r \geq n$, then $\mathcal{M}_{\leq r}(n) = \mathrm{Mat}(n)$; otherwise $\mathcal{M}_{\leq r}(n)$ is a proper subset of $\mathrm{Mat}(n)$. The multiplication of a matrix by any invertible matrix doesn't change its rank and hence $\mathcal{M}_{\leq r}(n)$ is a GL_n-stable subset of $\mathrm{Mat}(n)$. For $r < n$ the GL_n-set $\mathcal{M}_{\leq r}(n)$ is also a proper Zariski-closed subset of $\mathrm{Mat}(n)$: it is defined by the vanishing locus of the $(r+1) \times (r+1)$-minors. In particular, it is a closed GL_n-stable subset of $\mathrm{Mat}(n)$, or a GL_n-subvariety of $\mathrm{Mat}(n)$, or just a GL_n-variety. ♫

Definition 1.2.4. Let X be a G-set and $x \in X$ be an element. We call the set $G \cdot x = \{y \in X \mid y = g \cdot x \text{ for some } g \in G\}$ the *orbit of x under G* or the *G-orbit of x*. ♪

Example 1.2.5. Consider a nonzero complex symmetric matrix $M \in \mathrm{Mat}(n)$ of rank $r < n$. Then its GL_n-orbit is a GL_n-subset that is not closed. Indeed, it consists of all symmetric matrices in $\mathrm{Mat}(n)$ of exactly rank r. However, the Zariski-closure of the GL_n-orbit of M is the space of symmetric matrices of rank at most r that is a GL_n-subvariety of $\mathrm{Mat}(n)$. ♫

Example 1.2.6. Let I be an ideal of a G-ring R, and suppose I is a G-stable subset. Then I is a G-ideal of R. ♫

When we define notions in the equivariant setting we take into account the action of G. E.g. in the definition of *G-Noetherianity* of a G-ring R we look at ascending chains of G-ideals.

Example 1.2.7. This is the dual of Example 1.2.3. We look at $\mathrm{Mat}(n)$ as an algebraic variety. Its coordinate ring $\mathbf{C}[\mathrm{Mat}(n)]$ is $\mathbf{C}[x_{i,j} \mid i,j \in [n]]$. The action of GL_n induces an action on $\mathbf{C}[\mathrm{Mat}(n)]$ (by ring automorphisms) and the defining ideal of $\mathcal{M}_{\leq r}(n)$ (generated by the $(r+1) \times (r+1)$-minors) is stable with respect to this action. In particular it is a GL_n-ideal. ♫

Definition 1.2.8. Let X and Y be G-sets, and let $\varphi : X \to Y$ be a morphism. We say that φ is *G-equivariant* if for every $x \in X$ we have:

$$g \cdot \varphi(x) = \varphi(g \cdot x).$$

♪

Proposition 1.2.9. *Consider a G-equivariant morphism $\varphi : X \to Y$ and let Y' be a G-stable subset of Y. Then its preimage $\varphi^{-1}(Y')$ along φ is a G-stable subset of X.*

Proof. Let $g \in G$ act on $x \in \varphi^{-1}(y)$ for some $y \in Y'$. Then $\varphi(g \cdot x) = g \cdot \varphi(x)$ by the G-equivariance of φ, and $g \cdot y \in Y'$ because Y' is G-stable. Hence, $g \cdot x \in \varphi^{-1}(Y')$. □

1.2.1 Noetherianity

Noetherianity is one of the conditions for stabilisation we are interested in. Recall that a ring R is Noetherian if every ascending chain of ideals stabilises. Equivalently, a ring is Noetherian if and only if every ideal is generated by a finite number of elements. We now extend this definition in our G-equivariant setting.

Definition 1.2.10. Let R be a G-ring, and let Σ be the collection of G-ideals of R ordered by inclusion. We say that R is *G-Noetherian* if every ascending chain of elements in Σ stabilises. ♪

Example 1.2.11 (Hilbert's Basis Theorem)**.** Let A be a finite-type algebra over a Noetherian ring R and let the group $G = \{1_G\}$ act. Then A is Noetherian in the classical sense. We say A is $\{1_G\}$-Noetherian. ♫

Proposition 1.2.12. *Let H be a subgroup of G and consider a G-ring R. If R is H-Noetherian, then R is also G-Noetherian.*

Proof. Every ascending chain of G-ideals is an ascending chain of H-ideals that therefore stabilises. □

Proposition 1.2.13. *Let R' and R be G-rings, and let $\varphi : R \to R'$ be a surjective G-equivariant morphism of rings. If R is G-Noetherian then so is R'.*

Proof. The pull-back in R of a chain of G-ideals in R' is a chain of G-ideals by Proposition 1.2.9 and this chain stabilises by the G-Noetherianity of R. Hence, the initial chain stabilises too by the surjectivity of φ. □

Below we propose an example of an infinite dimensional ring that *is not* Noetherian with respect to the action of the trivial group but that *is* Noetherian when the infinite symmetric group Sym acts.

Example 1.2.14. Let R be a Noetherian ring, and consider the polynomial ring $R[x_1, x_2, \dots]$ in infinitely many variables, and let $G = \{1_G\}$ act. Then the chain of ideals

$$(x_1) \subsetneq (x_1, x_2) \subsetneq \cdots$$

doesn't stabilise. Hence this polynomial ring is not $\{1_G\}$-Noetherian. ♫

Theorem 1.2.15 ([Coh87][1])**.** *Let R be a Noetherian ring and k be a positive integer. Consider the action of the group* Sym *on $R[x_{i,j} \mid i \in [k], j \in \mathbf{N}]$ defined by $\sigma \cdot x_{i,j} := x_{i,\sigma(j)}$ for any $\sigma \in \mathrm{Sym}$. Then $R[x_{i,j} \mid i \in [k], j \in \mathbf{N}]$ is* Sym*-Noetherian.*

Theorem 1.2.15 with $k = 1$ states that $R[x_1, x_2, \dots]$ is Sym-Noetherian.

Proposition 1.2.16. *A G-ring R is G-Noetherian if and only if every G-ideal is generated by the G-orbits of finitely many elements.*

Proof. Suppose R is G-Noetherian and suppose that a G-ideal I is not generated by the G-orbits of finitely many elements. Let $r_1, r_2, \dots$ be elements of I such that:

$$(G \cdot r_1) \subsetneq (G \cdot \{r_1, r_2\}) \subsetneq \cdots$$

[1] Theorem 1.2.15 was first proven with the action of the monoid $\mathrm{Inc}(\mathbf{N})$ for $k = 1$ and R a Noetherian domain by Daniel H. Cohen; see [Coh67, Proposition 2]. Its strengthening to any k and to any Noetherian ring R (and again under the action of $\mathrm{Inc}(\mathbf{N})$) is [Coh87, Theorem 7] by the same author. Similar results were then independently rediscovered for the action of Sym and $k = 1$ [AH07, Theorem 1.1] and also for any k but over fields [HS12, Theorem 1.1].

where $(G \cdot \{r_1, \ldots, r_k\})$ is the G-ideal generated by the set $G \cdot \{r_1, \ldots, r_k\}$. These elements exist by the assumption on I and the ascending chain of G-ideals above doesn't stabilise. This is in contradiction with the G-Noetherianity of R.

Suppose now that every G-ideal of R is generated by the G-orbits of finitely many elements. Consider a chain of G-ideals in R:

$$I_1 \subset I_2 \subset \cdots$$

Then $\bigcup_k I_k$ is a G-ideal of R and hence it is generated by the orbits of $r_1, \ldots, r_t$. In particular there exists a k_0 such that $r_i \in I_{k_0}$ for every $i = 1, \ldots, t$ and therefore the chain stabilises at k_0. □

1.2.2 Topological Noetherianity

The *ring-theoretic* G-Noetherianity for the varieties we are interested in "is like the Arab Phoenix: everybody swears it exists, but no one knows where it is"[2]—to be fair, we know the Phoenix exists for some cases ([SS16; SS19; NSS16; SS22]). We will therefore be looking at a weaker condition for stabilisation and seek for topological G-Noetherianity. The price we pay is that our results are *set-theoretic*.

So, let now X be a topological space and consider the poset Σ of closed G-stable subsets ordered by inclusion.

Definition 1.2.17. We say that X is *topologically G-Noetherian* if every descending chain of elements in Σ stabilises. ♪

Example 1.2.18. Recall the definitions of the infinite general linear group GL, and of the infinite dimensional $\mathbf{C}$-vector space V_∞ from Notation at page 15. Consider two copies of V_∞. The group GL acts on each of them: an element $g \in \mathrm{GL}$ gives a linear automorphism of V_∞ sending a vector $v \in V_\infty$ to a vector $g \cdot v$. Hence a point $(v, w) \in V_\infty \oplus V_\infty$ is mapped via g to $(g \cdot v, g \cdot w)$. By Theorem 1.2.15 together with Proposition 1.2.12, this space is ring-theoretically GL-Noetherian as Sym is a subgroup of GL. However, we give a direct proof of topological Noetherianity. Let $\mathbf{C}[x_i, y_i \mid i \in \mathbf{N}]$ be the coordinate ring of $V_\infty \oplus V_\infty$ where the x_i's are coordinates for the first copy of V_∞ and the y_i's for the second one. We want to describe all the closed GL-stable subsets of $V_\infty \oplus V_\infty$. Clearly, the following are closed subsets:

1. the empty set,
2. the whole space,
3. the point $(0, 0)$, where 0 is the zero vector,

[2] From "Cosí fan tutte" by W.A. Mozart. The libretto, by L. Da Ponte, reads: "è come l'araba fenice: che ci sia ognun lo dice, dove sia nessun lo sa.".

4. the set $C_{\lambda,\mu} := \{(\lambda v, \mu v) \mid v \in \mathrm{V}_\infty\}$ for a fixed $(\lambda : \mu) \in \mathbb{P}^1$, with defining ideal $(\mu x_j - \lambda y_j \mid j \in \mathbf{N})$, and

5. finite unions of the latter.

We prove that the above list is complete. Let C be a closed subset. If C is empty, we are in case 1. Suppose C is nonempty. Let (v, w) be a point in C. If $v = w = 0$, and there are no other points in C we are in case 3. If the vectors v, w are linearly independent, then the closure of the GL-orbit of (v, w) fills the whole space— case 2. Suppose $(\lambda v, \mu v) \in C$ for a nonzero vector $v \in \mathrm{V}_\infty$ and $(\lambda : \mu) \in \mathbb{P}^1$. Then its GL-orbit gives the closed subset $C_{\lambda,\mu}$, and, as C is stable under GL, we have $C_{\lambda,\mu} \subset C$. If $C = C_{\lambda,\mu}$ or it is a finite union of these closed subsets, we are in either case 4 or 5. Suppose it is not, and that C is not the whole space. Let $p_1, p_2, \dots$ be nonzero points in C with distinct GL-orbits C_{λ_i,μ_i} for some $(\lambda_i : \mu_i) \in \mathbb{P}^1$ and $i \in \mathbf{N}$. Let $f \in \mathbf{C}[x_i, y_i \mid i \in \mathbf{N}]$ be a nonzero element of the defining ideal of C. Then, for each $i \in \mathbf{N}$ the polynomial f must have a factor from the ideal $(\mu_i x_j - \lambda_i y_j \mid j \in \mathbf{N})$, but $\mathbf{C}[x_i, y_i \mid i \in \mathbf{N}]$ is a UFD, $\mu_i x_j - \lambda_i y_j$ are irreducible for every $i, j \in \mathbf{N}$, and the degree of f is bounded, hence this is impossible. In particular, the list above is complete and the space $\mathrm{V}_\infty \oplus \mathrm{V}_\infty$ is topologically GL-Noetherian. ♫

Proposition 1.2.19. *Let X and Y be topological G-spaces and let $\varphi : X \to Y$ be a continuous G-equivariant morphism. Suppose that φ is surjective and X is topologically G-Noetherian then Y is topologically G-Noetherian too.*

Proof. The pull-back along φ of any descending chain of closed G-stable subsets of Y is a descending chain of closed G-stable subsets of X by Proposition 1.2.9 and the continuity of φ. The pulled-back chain therefore stabilises because X is topologically G-Noetherian. By surjectivity, the initial chain stabilises too. □

Proposition 1.2.20. *Let A be a G-ring and I be a G-ideal. Define $Y := \operatorname{Spec}(A)$ and let X be the G-subscheme $\mathcal{V}(I)$. Suppose Y is topologically G-Noetherian. Then there exist $a_1, \dots, a_k \in I$ such that*

$$X = \mathcal{V}((G \cdot \{a_1, \dots, a_k\})).$$

Proof. Suppose not. Then there exist elements $a_1, a_2, \dots$ in I such that

$$\mathcal{V}((G \cdot a_1)) \supsetneq \mathcal{V}((G \cdot \{a_1, a_2\})) \supsetneq \cdots$$

The above is an infinite descending chain of closed G-stable subsets of Y. This is against the topological G-Noetherianity of Y. □

Remark 1.2.21. In this thesis, either when the action of a group is clear, or it is clear we are looking up to an action, we might drop the notation referring to the group. For example, if X is a topological G-space we might just say "X is a topological space", and in the case X is topologically G-Noetherian, we might just write "X is topologically Noetherian". However, in this chapter we try to mention the group involved. ♪♫

1.3 Summary of the thesis

This Ph.D. thesis comprises three papers and a half: the papers [BDD22] and [CDDEF22]—treating topological Noetherianity; the paper [BDDE22]—about properties of tensors with high strength; and the first half of [CDD]— regarding the singular locus of GL-varieties. The papers [BDS22] and [BDFK22] do not appear in this thesis but have been written during my Ph.D. time.

This thesis is divided into four parts. You are reading Part I, The Overture. This first chapter aims to be a gentle and informal introduction to the subject (I hope you agree!), and now it describes the content of the thesis. Part II, The Players, presents the objects and the notions we will be working with for proving our results: Chapter 2 introduces polynomial functors over infinite fields and their subvarieties, while the subsequent Chapter 3 presents the theory of polynomial functors defined over rings. These chapters are based on [Bik20; BDD22; BDDE22; BDES22; Dra19; FS97; Tou14]. The successive Parts III and IV contain all the results. Part III, composed of Chapters 4,5, and 6, studies stabilisation of varieties when the group GL acts. In Chapter 4, based on [BDD22], we prove topological GL-Noetherianity for polynomial functors defined over rings with Noetherian spectrum. Chapter 5, based on the forthcoming paper [CDD], deals with stabilisation of the singular locus of varieties in polynomial functors, while Chapter 6, based on [BDDE22], investigates strength and universality of tensors.

Our study of geometric objects with a combined action of the infinite symmetric group and the infinite general linear group is in Part IV consisting of Chapters 7 and 8. In Chapter 7 we first formalise our point of view on varieties with action of Sym, and then of $\mathrm{Sym} \times \mathrm{GL}$. Eventually, we take Chapter 8 to prove that k infinite products of polynomial functors are topologically $\mathrm{Sym}^k \times \mathrm{GL}$-Noetherian. These last chapters are based on, respectively, [CEF15; CEFN14; DEF22] and [CDDEF22].

1.3.1 The threads

The possibly most striking result in this area is that infinite dimensional algebraic varieties acted upon by the infinite general linear group GL are topologically GL-Noetherian. This means that any descending chain of closed subsets stable under GL stabilises or, equivalently, these varieties are cut out by finitely many equations up to the action of GL. The result has first been shown for quadrics and cubics, respectively in [Egg15] and [DES17]. It was then re-formulated and proven with the language of polynomial functors defined over infinite fields in [Dra19]. After that, the geometry of polynomial functors has taken more directions and, under the assumption of working over a field of characteristic zero, it has been formalised with the language of GL-varieties in [BDES22]. In this paper we find the starting point of the first thread of this thesis: "The Embedding Theorem" [BDES22, Theorems 4.1, 4.2]. This theorem was already in [Dra19] but more like a silent actor rather than one of the main characters. In this thesis, The Embedding Theorem is Theorem 2.4.1.

It allows the use of induction on "the size" of varieties in polynomial functors as it describes a distinguished open subset of the variety in terms of "smaller" varieties. The thread for Chapters 4,5, 6 and 8 relates to the above use of The Embedding Theorem. As mentioned before, in Chapter 4 we prove that polynomial functors defined over rings with Noetherian spectrum are topological GL-Noetherian. Methods come from commutative algebra and representation theory while the strategy relies on the line of the original proof of [Dra19]. Indeed, we make use of a (very intricate) triple induction after establishing the corresponding Embedding Theorem for this setting. The results of Chapter 5 make use of The Embedding Theorem of [BDES22] *tout-court*. In Chapter 6 we prove Theorem 6.1.6, the "Parameterisation Theorem for GL-subsets": under the assumption of working in characteristic zero we prove that a proper GL-subset (not necessarily closed!) of a **Vec**-variety can be covered by finitely many "smaller" **Vec**-varieties. In Chapter 8, restricting again to characteristic zero, we prove Theorem 8.4.1: it is a parameterisation theorem for a specific type of varieties (hence closed) enjoying the action of the group $\mathrm{Sym}^k \times \mathrm{GL}$.

The second thread of this Ph.D. thesis regards the notions of strength and universality and it is contained in Chapter 6. In the setting of infinite dimensional GL-varieties, there are elements that can specialise to any element "defined in finite dimension" (in a suitable sense). We name such elements *universal* and we find a link with (a generalisation of) the notion of *strength*, a founding and fundamental concept in this area.

Part II

The players

Chapter 2

Polynomial functors over fields

In this chapter we describe polynomial functors over an *infinite* field K. These objects firstly raised in the setting of representation theory [FLS94; FS97] but our interest lies in their geometry. An introduction to polynomial functors is [FS97; Tou14], while a comprehensive geometric study (in their incarnation as spectra of finitely generated GL-algebras) is in [BDES22]. This chapter is based on these works together with [Bik20; Dra19] and it is intended to make this thesis self-contained.

2.1 Polynomial representations

Let V be a K-vector space and G be a group. Recall that

- a *representation* (V, ρ) of G is a morphism of groups:

$$\rho : G \to \mathrm{GL}(V),$$

- and an *algebraic group* is an algebraic variety G that admits a group structure whose multiplication map $m : G \times G \to G$ (mapping $(g_1, g_2) \mapsto g_1 g_2$), and inversion map $\iota : G \to G$ (mapping $g \mapsto g^{-1}$) are regular morphisms.

Example 2.1.1. For every $n \in \mathbf{N}$ the group GL_n is an algebraic group. ♫

Let V be a finite dimensional K-vector space and let (V, ρ) be a representation of GL_n. We are interested in representations coming from geometry, namely, those for which the map

$$\rho : \mathrm{GL}_n \to \mathrm{GL}(V)$$

is a morphism of algebraic groups. These representations are called *"rational"*. Indeed, after picking a basis for V, we have

$$\rho((x_{i,j})_{i,j=1}^n) = \left(\frac{p_{k,l}(x_{i,j} \mid i,j \in [n])}{\det((x_{i,j})_{i,j=1}^n)^e}\right)_{k,l=1}^{\dim(V)}$$

where $p_{k,l}$ is a polynomial in the variables $x_{i,j}$ for $i,j \in [n]$, $\det((x_{i,j})_{i,j=1}^n)$ is the determinant of the matrix $(x_{i,j})_{i,j=1}^n$, and e is a natural number. In particular, the entries of the right-hand-side are rational functions whose denominator is a power of the determinant of $(x_{i,j})_{i,j=1}^n$. We are interested when $e = 0$, and in this case we speak about *"polynomial representations"*.

Example 2.1.2. Let V be a vector space of dimension n together with basis $\{v_1, \ldots, v_n\}$. Every matrix $g = (x_{i,j})_{i,j=1}^n$ in GL_n gives an isomorphism $g : V \to V$ defined by the usual matrix-vector multiplication: $g(v_j) = \sum_{i=1}^n x_{i,j} v_j$. This is a polynomial representation with $p_{k,l}(x_{i,j}) = x_{k,l}$ for all $k, l \in [n]$. ♫

Example 2.1.3. Let V be as above, and consider the tensor product $V \otimes V$. Then every $g = (x_{i,j})_{i,j=1}^n \in \mathrm{GL}_n$ gives the automorphism $g \otimes g \in \mathrm{GL}(V \otimes V)$ defined by $v \otimes w \mapsto g \cdot v \otimes g \cdot w$ and extended by linearity. This is a polynomial representation: consider the ordered basis $\{v_i \otimes v_j \mid i,j \in [n]\}$ where $v_i \otimes v_j$ precedes $v_k \otimes v_l$ if $(i,j) < (k,l)$ in the lexicographic order. Then the $((k,l),(i,j))$-th entry of the matrix $g \otimes g$ is $x_{k,i}x_{l,j}$. ♫

2.1.1 Polynomial functors

Recall that **Vec** is the category of finite dimensional K-vector spaces with K-linear maps. Let V and W be vector spaces in **Vec**. A *polynomial map* from V to W is an element of $\mathrm{Sym}_K^\bullet(V^*) \otimes W$, where $\mathrm{Sym}_K^\bullet(V^*)$ denotes the symmetric K-algebra over the dual space of V.

Example 2.1.4. Let $\{x_1, \ldots, x_n\}$ be the dual of a basis $\{v_1, \ldots, v_n\}$ of V, and let $\{w_1, \ldots, w_m\}$ be a basis for W. An element f of $\mathrm{Sym}_K^\bullet(V^*) \otimes W$ is of the form:

$$f := \sum_{i=1}^m p_i(x_1, \ldots, x_n) \otimes w_i,$$

where p_i's are polynomials in $K[x_1, \ldots, x_n]$. Then f defines the (polynomial) map $f : V \to W$ mapping the vector $v = \sum_{j=1}^n \lambda_j v_j$ with $\lambda_i \in K$ to the vector $\sum_{i=1}^m p_i(\lambda_1, \ldots, \lambda_n) w_i$. The *degree* of f is the highest degree of the p_i's, and f is said *homogeneous of degree d* if all the p_i's are homogeneous of degree d. As we are working over an infinite field, the assignment $\mathrm{Sym}_K^\bullet(V^*) \otimes W \to \mathrm{Map}(V, W)$—the maps from V to W— is injective. ♫

Example 2.1.5. Let $V \cong K^2$ be a vector space with basis $\{v_1, v_2\}$ and coordinates x, y, and let $W \cong K^3$ be a vector space with basis w_1, w_2, w_3. Let $f \in \mathrm{Sym}^\bullet_K(V^*) \otimes W$ be the element $x \otimes w_1 + xy \otimes w_2 + y^3 \otimes w_3$. Then, with respect to the above bases, f maps a vector $(u, v) \in V$, with $u, v \in K$, to the vector $(u, uv, v^3) \in W$. ♫

Definition 2.1.6. A *polynomial functor* is a functor P from the category **Vec** to itself such that for all vector spaces $U, V \in$ **Vec** the assignment

$$P_{U,V} : \mathrm{Hom}(U, V) \to \mathrm{Hom}(P(U), P(V))$$

is a polynomial map. If the degree of the map $P_{U,V}$ has a fixed bound independent from U and V we say that the polynomial functor P has *finite (or bounded) degree*. If for every U, V the map $P_{U,V}$ is homogeneous of a fixed degree d, we say that the polynomial functor P is *homogeneous of degree d*. To simplify notation we write $P(\varphi)$ instead of $P_{U,V}(\varphi)$ when the domain U and the codomain V of a map φ is clear from the context. ♪

Remark 2.1.7. Note that after choosing bases, a map $\varphi \in \mathrm{Hom}(U, V)$ corresponds to a matrix with coefficients in K. The fact that $P_{U,V}$ is a polynomial map means that $P_{U,V}(\varphi)$ is a matrix whose entries are polynomials in the entries of φ. ♪♫

Example 2.1.8. Consider the functor $S^1 \oplus S^1 \colon$ **Vec** $\to$ **Vec**. It assigns to each $V \in$ **Vec** the space $V \oplus V$ and to each $\varphi \in \mathrm{Hom}(U, V)$ the map $\varphi \oplus \varphi$ that sends (v, w) to $(g \cdot v, g \cdot w)$. Then, $S^1 \oplus S^1$ is a homogeneous polynomial functor of degree 1. ♫

Example 2.1.9. Consider the functor $T^2(V) := V \otimes V$ assigning to a map $\varphi : U \to V$ the map $\varphi \otimes \varphi : U \otimes U \to V \otimes V$ that sends $u \otimes \overline{u}$ to $\varphi(u) \otimes \varphi(\overline{u})$ for $u, \overline{u} \in U$, and is extended by linearity. Let $\{u_1, \dots, u_m\}$ be a basis for U, $\{v_1, \dots, v_n\}$ be a basis for V, and let $(x_{i,j})_{i \in n, j \in [m]}$ be corresponding matrix for φ. After ordering lexicographically the bases $\{u_i \otimes u_j\}_{i,j=1}^m$ and $\{v_i \otimes v_j\}_{i,j=1}^n$, the map $P(\varphi) = \varphi \otimes \varphi$ corresponds to the $n^2 \times m^2$ matrix where the $((k, l), (i, j))$-th entry is $x_{k,i} x_{l,j}$. So, the *second tensor power* T^2 is a homogeneous polynomial functor of degree two. ♫

Example 2.1.10. Consider the functor $T^2 \oplus S^1$ assigning to each $V \in$ **Vec** the space $(V \otimes V) \oplus V$, and to each $\varphi \in \mathrm{Hom}(U, V)$ the map $\varphi \otimes \varphi \oplus \varphi$. This is a non-homogeneous polynomial functor of degree 2. ♫

Other examples of polynomial functors are: the d-th symmetric power S^d, the d-th alternating power $\bigwedge^d$.

Example 2.1.11. Consider the d-the symmetric power $V \mapsto S^d(V)$. It is a homogeneous polynomial functor of degree d. If V has basis $x_1, \dots, x_n$, then $S^d(V)$ is isomorphic to $K[x_1, \dots, x_n]_d$, the K-module of homogeneous degree-d polynomials (together with 0). In this setting, linear maps $S^d(\varphi)$ for $\varphi \colon V \to W$ are induced by substitutions of the variables $x_1, \dots, x_n$ with linear forms in the variables $y_1, \dots, y_m$, with $K[y_1, \dots, y_m]_d \cong S^d(W)$. ♫

Remark 2.1.12. Polynomial functors show another connection with polynomiality: they behave like ordinary univariate polynomials. Indeed, they can be added via direct sums, multiplied via taking the tensor products, subtracted via taking quotients (in the case one is included in the other), and composed. Moreover, they are direct sums of unique homogeneous polynomial functors of non-negative degree (see Remark 2.1.13), we can shift them by a constant as explained in Section 2.1.4, and we can define their derivative as in Section 6.2.1. ♪♫

Remark 2.1.13. A polynomial functor P is the direct sum of its homogeneous components. For every $V \in \mathbf{Vec}$ the i-th homogeneous component is defined by

$$P_i(V) = \{ p \in P(V) : P(\lambda\, \mathrm{id}_V)(p) = \lambda^i p \text{ for every } \lambda \in K \},$$

and each P_i is a homogeneous polynomial functor of degree i. By the semi-simplicity of GL we have $P = \bigoplus_{i \geq 0} P_i$ ([Tou14, Section 4.1.3]). If P is of bounded degree, then the above is a finite direct sum. We note that P_0 is a constant polynomial functor, which assigns a fixed vector space $P(0) \in \mathbf{Vec}$ to all $V \in \mathbf{Vec}$ and the identity map to each linear map. We call P *pure* if $P_0 = \{0\}$. ♪♫

Remark 2.1.14. **PF** denotes the category of polynomial functors with natural transformations. In particular, a natural transformation $\alpha : P \to Q$ is given by *linear* maps $\alpha(V) : P(V) \to Q(V)$ for every $V \in \mathbf{Vec}$ plus the usual commutativity condition. A check shows that **PF** is an Abelian category. ♪♫

Remark 2.1.15. For reasons that will become clear later, our proofs rely on the fact that the polynomial functors we are working with have bounded degree. Hence we will be working with the full subcategories $\mathbf{PF}_d$, and $\mathbf{PF}_{\leq d}$ that denote, respectively, homogeneous polynomial functors of degree d, and polynomial functors of degree at most d. By [Gre07, Corollary 2.6e] these categories are semisimple when $\mathrm{Char}(K) = 0$ or $\mathrm{Char}(K) > d$. ♪♫

Remark 2.1.16. Let $V \in \mathbf{Vec}$. Note that $\mathrm{GL}(V)$ sits inside $\mathrm{End}(V)$. In particular, $P(V)$ is a polynomial $\mathrm{GL}(V)$-representation. ♪♫

Definition 2.1.17. We say that a polynomial functor P is *irreducible* if the only nonzero subobject of P in **PF** is P itself. ♪

Remark 2.1.18. A polynomial functor P is irreducible if and only if for every $V \in \mathbf{Vec}$ we have that $P(V)$ is an irreducible $\mathrm{GL}(V)$-representation. Indeed, this is true by the Friedlander and Suslin's lemma of the following section. ♪♫

Example 2.1.19. Consider the polynomial functor T^2. For every $V \in \mathbf{Vec}$ we have $T^2(V) = S^2(V) \oplus \bigwedge^2(V)$. Then T^2 is not irreducible but S^2 and $\bigwedge^2$ are. ♫

Remark 2.1.20. When the characteristic of the field is zero, an irreducible polynomial functor corresponds to a Schur functor. Therefore, a generic polynomial functor is isomorphic to direct sums of Schur functors. As Schur functors are quotients of tensors powers, we may refer to the elements of polynomial functors as *"tensors"*. We point at [FH91] for the theory of Schur functors. ♪♫

2.1.2 Friedlander and Suslin's lemma

Consider a polynomial functor P. Recall that by Remark 2.1.16 for every vector space $V \in$ **Vec** the assignment $P \mapsto P(V)$ gives a $\mathrm{GL}(V)$-representation $P(V)$. Denote by

- $\mathbf{Rep}^f_{\mathrm{GL}(V)}$ the category of finite-dimensional polynomial representations of $\mathrm{GL}(V)$,
- $\mathbf{Rep}^f_{\mathrm{GL}(V),\leq d}$ the subcategory of finite-dimensional polynomial $\mathrm{GL}(V)$-representations of degree at most d,
- $\mathbf{Rep}^f_{\mathrm{GL}(V),d}$ the subcategory of homogeneous degree-d $\mathrm{GL}(V)$-representations.

Then we have the following lemma by Erik Friedlander and Andrei Suslin.

Lemma 2.1.21 ([FS97, Lemma 3.4])**.** *Let $n \geq d$ and V be a vector space of dimension n. Then the functor*

$$\mathbf{PF}_d \to \mathbf{Rep}^f_{\mathrm{GL}(V),d}$$

assigning to a polynomial functor P the representation $P(V)$ is an equivalence of categories.

In particular, when the dimension of V is bigger than d the functor above extends to an equivalence of categories

$$\mathbf{PF}_{\leq d} \to \mathbf{Rep}^f_{\mathrm{GL}_n,\leq d}.$$

Example 2.1.22. The condition $n \geq d$ is necessary. Consider the polynomial functor $\bigwedge^d$, the d-th alternating tensor power. It is a homogeneous polynomial functor of degree d. The $\bigwedge^d(V)$ is trivial for any V of dimension strictly smaller than d. ♫

2.1.3 Well-founded order

By a *pre-order* $\preceq$ on a class we will mean a reflexive and transitive relation. We also write $B \succeq A$ for $A \preceq B$. Furthermore, write $A \prec B$ or $B \succ A$ to mean that $A \preceq B$ but not $B \preceq A$. The pre-order is *well-founded* if it admits no infinite strictly decreasing chains $A_1 \succ A_2 \succ \ldots$.

(Isomorphism classes of) polynomial functors are partially ordered by the relation $\prec$ defined by $Q \prec P$ if $Q \ncong P$ and for the largest e with $Q_e \ncong P_e$ the former is a quotient of the latter. This partial order is well-founded by Lemma 2.1.21. See [Dra19, Lemma 12] for a proof.

2.1.4 Shift operation

Definition 2.1.23. Given a finite dimensional vector space U, define the *shift functor* $\mathrm{Sh}_U : \mathbf{Vec} \to \mathbf{Vec}$ by assigning to each $V \in \mathbf{Vec}$ the vector space $U \oplus V$, and to each map $\varphi \in \mathrm{Hom}(V, W)$ the map $\mathrm{id}_U \oplus \varphi$. ♪

Let P be a polynomial functor. We denote with $\mathrm{Sh}_U P$ the composition $P \circ \mathrm{Sh}_U$. Then $\mathrm{Sh}_U P$ is a polynomial functor assigning to each $V \in \mathbf{Vec}$ the vector space $P(U \oplus V)$, and to a morphism $\varphi \in \mathrm{Hom}(V, W)$ the morphism $P(\mathrm{id}_U \oplus \varphi)$.

Example 2.1.24. Consider $P := T^2$, then

$$\mathrm{Sh}_U T^2(V) = T^2(U \oplus V) = (U \otimes U) \oplus (U \otimes V) \oplus (V \otimes U) \oplus (V \otimes V).$$

Note in particular that the right-hand-side is different than $T^2(U) \oplus T^2(V)$ ♫

Remark 2.1.25. The group $\mathrm{GL}(U \oplus V)$ acts on $P(U \oplus V)$. The space $\mathrm{Sh}_U P(V)$ coincides with $P(U \oplus V)$ but by functoriality only the stabiliser $\mathrm{GL}(V)$ of U acts on it. ♪♫

Remark 2.1.26. Let P be a polynomial functor of degree d. After choosing bases we have

$$P(\mu\,\mathrm{id}_U \oplus \lambda\,\mathrm{id}_V) = \sum_{\substack{i,j \geq 0 \\ i+j \leq d}} \lambda^i \mu^j M_{i,j}$$

for some matrices $M_{i,j} \in K^{\dim(P(U \oplus V))^2}$. Consider $p \in (\mathrm{Sh}_U P)_d(V)$. By definition, $P(\mathrm{id}_U \oplus \lambda\,\mathrm{id}_V)(p) = \lambda^d p$ for every λ, and, since the field is infinite, $M_{i,j}(p) = 0$ for every $(i,j) \neq (0,d)$. In particular, $P(\mu\,\mathrm{id}_U \oplus \lambda\,\mathrm{id}_V)(p) = \lambda^d p$ for every μ. ♪♫

Remark 2.1.27. Let P be a polynomial functor of degree d. We show below that

$$\mathrm{Sh}_U P \cong P \oplus P'$$

with P' a strictly smaller polynomial functor than P (note that $P' \cong \mathrm{Sh}_U P/P$). The inclusion map $\iota_V : V \to U \oplus V$ and the projection map $\pi_V : U \oplus V \to V$ give linear morphisms $P(\iota_V) : P(V) \to P(U \oplus V)$ and $P(\pi_V) : P(U \oplus V) \to P(V)$. As $\pi_V \circ \iota_V = \mathrm{id}_V$, we have that $P(\iota_V)$ is injective and $P(\pi_V)$ is surjective, and a check shows that both of them respect the grading (they map elements of the e-degree part to elements of degree e). Hence, $\mathrm{Sh}_U P \cong P \oplus P'$ for some polynomial functor P'. Consider $p \in (\mathrm{Sh}_U P)_d(V)$, and note $0 \cdot \mathrm{id}_U \oplus \mathrm{id}_V = \iota_V \circ \pi_V$. By Remark 2.1.26 $P(0 \cdot \mathrm{id}_U \oplus \mathrm{id}_V)(p) = p$, so that $P(\iota_V)(P(\pi_V)(p)) = p$ and therefore $p \in P_d(V)$. Then the two polynomial functors have isomorphic homogeneous degree-d components

$$P_d \cong (\mathrm{Sh}_U P)_d,$$

and P' has strictly smaller degree than P. ♪♫

Example 2.1.28. In the setting of Example 2.1.24, $P'(V) = U \otimes U \oplus U \otimes V \oplus V \otimes U$, and it has degree one. ♫

2.1.5 Dimension

In characteristic zero, when polynomial functors are direct sums of Schur functors, their dimension is a polynomial in $\dim(V)$ with rational coefficients. The same holds for general polynomial functors of bounded degree:

Proposition 2.1.29. *Let P be a polynomial functor of bounded degree. Then there exists a univariate polynomial f_P in $\dim(V)$ of degree at most $\deg P$ with rational coefficients such that for every $V \in \mathbf{Vec}$:*

$$\dim(P(V)) = f_P(\dim(V)).$$

We call the polynomial f_P the dimension function *of P.*

Proof. We do induction on the order $\prec$ of Section 2.1.3. If the degree of P is zero, then $P(V)$ is a constant vector space, hence its dimension function is a constant polynomial and the statement is true. Suppose now that $\deg P > 0$ and that the statement holds for every polynomial functor of degree strictly smaller than the degree of P. In particular, we can assume P homogeneous of degree d. Let U be a one-dimensional vector space and consider $\mathrm{Sh}_U P \cong P \oplus P'$ with $P' := \mathrm{Sh}_U P/P$. By Remark 2.1.27 $\deg P' < \deg P$. Therefore we have:

$$f_P(n+1) = f_{P \oplus P'}(n) = f_P(n) + f_{P'}(n),$$

where, by induction, $f_{P'}$ is a polynomial of degree at most $d-1$. There exists only one function p satisfying both $p(n+1) - p(n) = f_{P'}(n)$ and $p(0) = \dim(P(\{0\}))$. The function p is a polynomial of degree at most d[1], so we set $f_P = p$ and get the statement. □

Example 2.1.30. For every $V \in \mathbf{Vec}$ we have

- $f_{T^2}(\dim(V)) = \dim(T^2(V)) = \dim(V)^2$,
- $f_{S^2}(\dim(V)) = \dim(S^2(V)) = \binom{\dim(V)+1}{2} = \frac{1}{2}(\dim(V)^2 + \dim(V))$,
- and $f_{\bigwedge^2}(\dim(V)) = \dim(\bigwedge^2(V)) = \frac{1}{2}(\dim(V)^2 - \dim(V))$.

♫

[1] Actually, p has degree one more than the degree of $f_{P'}$.

2.2 Geometry

In this section, we equip polynomial functors with their natural *Zariski topology*. We describe the regular maps, the closed subsets (the **Vec**-varieties), and the connection with infinite dimensional GL-varieties.

2.2.1 Polynomial functors as affine spaces

For every $V \in \mathbf{Vec}$ the vector space $P(V)$ is naturally an affine space. Indeed, choose a basis for $P(V)$ and consider the polynomial functions on $P(V)$ with respect to that basis. These are polynomials on the duals of the basis vectors. The Zariski-closed subsets are given by the vanishing loci of these polynomials. In other words, the coordinate ring $K[P(V)]$ of $P(V)$ is given by the symmetric K-algebra $\mathrm{Sym}_K^\bullet(P(V)^*)$ on $P(V)^*$.

Example 2.2.1. Let $V \in \mathbf{Vec}$, then the space $S^1(V) = V$ is an affine space with coordinate ring $K[x_1, \dots, x_{\dim(V)}]$ where x_i are the dual elements of a basis of V. ♫

This topology behaves well with respect to functoriality: a polynomial functor P is also a functor into the category of affine varieties. Indeed, for any linear morphism $\varphi : V \to W$ the corresponding map $P(\varphi)$ is a morphism of affine varieties as $P(\varphi)$ is a linear map.

Moreover, one gets a contravariant functor to K-algebras of finite-type by considering the symmetric K-algebra on the dual. Summing up, one can look at a polynomial functor P as a functor to

- finite dimensional vector spaces: $V \mapsto P(V) \in \mathbf{Vec}$;
- topological spaces: $V \mapsto P(V) \in \mathbf{Top}$;
- affine K-varieties: $V \mapsto P(V) \in \mathbf{Var}$;
- K-algebras of finite-type: $V \mapsto \mathrm{Sym}_K^\bullet(P(V)^*) \in \mathbf{Alg}$;
- affine K-schemes of finite-type: $V \mapsto \mathrm{Spec}\left(\mathrm{Sym}_K^\bullet(P(V)^*)\right) \in \mathbf{Sch}$.

When we take one of the geometric points of view on P, we change the notation accordingly. For example, let P and Q be polynomial functors, then:

$$(P \oplus Q)(V) = P(V) \times Q(V)$$

where the right-hand-side is the fibre product over $\mathrm{Spec}\, K$. In particular, according to the splitting into homogeneous components, we have $P(V) = P_0(V) \times \ldots \times P_d(V)$.

Remark 2.2.2. The coordinate ring $K[P(V)]$ is a polynomial $\mathrm{GL}(V)$-representation over K. The (right) action of $g \in \mathrm{GL}(V)$ on $K[P(V)]$ is given by $P(g)^\#$. ♪♫

The following proposition is nothing but a very well-known fact in geometry. We single it out as we will be using this property a lot.

Proposition 2.2.3. *Let P and Q be polynomial functors. Then*

$$K[(P \oplus Q)(V)] = K[P(V)] \otimes K[Q(V)].$$

In particular, we will be looking at equations in $K[(P \oplus Q)(V)]$ as polynomials with variables in $(Q(V))^*$ and coefficients in $K[P(V)]$. Another consequence of the above proposition is the definition of the GL-*grading*. Suppose that the polynomial functor P has degree d and consider its decomposition into homogeneous components, namely $P = \bigoplus_{i=0}^{d} P_i$. The above proposition implies that

$$K[P(V)] = K[P_0 V] \otimes K[P_1 V] \otimes \cdots \otimes K[P_d V].$$

Let x be a coordinate of $P_i(V)^*$ and consider $g = \lambda \operatorname{id}_V \in \mathrm{GL}(V)$. Then the action of g on x gives $\lambda^i x$. Hence, besides the standard grading on $K[P(V)]$ that assigns degree one to each element of $P(V)^*$, we have the GL-grading: the coordinates of $P_i(V)$ have GL-*degree* i.

2.2.2 Morphisms

In this section we introduce the morphisms between polynomial functors. They play the role of regular maps in classical algebraic geometry.

Definition 2.2.4. Let P and Q be polynomial functors. A *polynomial transformation* $\alpha : P \to Q$ of polynomial functors is given by a polynomial map $\alpha(V) : P(V) \to Q(V)$ for every $V \in \mathbf{Vec}$ such that for every $\varphi \in \operatorname{Hom}(U, V)$ the diagram

$$\begin{tikzcd} P(U) \arrow[r, "\alpha(U)"] \arrow[d, "P_{U,V}(\varphi)"'] & QU \arrow[d, "Q_{U,V}(\varphi)"] \\ P(V) \arrow[r, "\alpha(V)"] & Q(V) \end{tikzcd}$$

commutes. ♪

Example 2.2.5. In this example we think about elements of $S^1(V)$ and $S^2(V)$ as, respectively, linear and quadratic forms. Consider $\alpha : S^1 \oplus S^1 \to S^2$ defined as follows. For every $V \in \mathbf{Vec}$ the maps $\alpha(V) : S^1(V) \oplus S^1(V) \to S^2(V)$ is given by $\alpha(V)(l_1, l_2) = l_1 l_2$, the multiplication of the two linear forms. Note that for every $\varphi \in \operatorname{Hom}(U, V)$ the diagram

$$\begin{tikzcd} S^1(U) \oplus S^1(U) \arrow[r, "\alpha(U)"] \arrow[d, "\varphi\oplus\varphi"'] & S^2(U) \arrow[d, "S^2(\varphi)"] \\ S^1(V) \oplus S^1(V) \arrow[r, "\alpha(V)"] & S^2(V), \end{tikzcd} \quad \text{given by} \quad \begin{tikzcd} (l_1, l_2) \arrow[r, "\alpha(U)"] \arrow[d, "\varphi\oplus\varphi"'] & l_1 l_2 \arrow[d, "S^2(\varphi)"] \\ (\varphi(l_1), \varphi(l_2)) \arrow[r, "\alpha(V)"] & \varphi(l_1)\varphi(l_2) \end{tikzcd}$$

commutes. Then, α is a polynomial transformation. ♫

Remark 2.2.6. Homogeneous polynomial transformations of degree one are the natural transformations we used in Section 2.1.2. With respect to these latter maps, polynomial functors form the Abelian category $\mathbf{PF}$. On the other hand, the category $\mathbf{PF}^{\mathrm{pol}}$ of polynomial functors equipped with polynomial transformations is not Abelian: the image might no longer be a polynomial functor. ♪♫

Remark 2.2.7. For us a *subfunctor* Q of a polynomial functor P is a subobject of P in the category $\mathbf{PF}$. In particular, Q is a polynomial functor, and $P = Q \oplus P'$ where P' is a polynomial functor. ♪♫

2.2.3 Subvarieties of polynomial functors

After we established how a polynomial functor P is an affine space we now want to define its "varieties". Look at a polynomial functor P as a functor from $\mathbf{Vec}$ to $\mathbf{Sch}$.

Definition 2.2.8. Let $X, Y : \mathbf{Vec} \to \mathbf{Sch}$ be functors. Let $\alpha : X \to Y$ be a natural transformation. We say that α is a *closed embedding* if $\alpha(V) : X(V) \to Y(V)$ is a closed embedding for every $V \in \mathbf{Vec}$. We say that a functor $X : \mathbf{Vec} \to \mathbf{Sch}$ admitting a closed embedding $\alpha : X \to P$ in some polynomial functor P is an *affine* **Vec**-*scheme*. The category of affine **Vec**-schemes is the full subcategory (whose objects are affine **Vec**-schemes) in the functor category $\mathbf{Sch}^{\mathbf{Vec}}$. ♪

The following is our main object of study: it is our notion of variety in polynomial functors.

Definition 2.2.9. Let X be an affine **Vec**-scheme. If $X(V)$ is a reduced affine scheme for every $V \in \mathbf{Vec}$, we say that X is a **Vec**-*variety*. ♪

We can think of a **Vec**-variety X of P as a functor $X : \mathbf{Vec} \to \mathbf{Sch}$ such that $X(V) \subset P(V)$ is a subvariety and $X(\varphi) = P(\varphi)_{|X(U)}$ for every $\varphi \in \mathrm{Hom}(U, V)$.

Example 2.2.10. Let $\mathcal{M}_{\leq r}$ be the functor assigning to every $V \in \mathbf{Vec}$, the subset $\mathcal{M}_{\leq r}(V)$ of tensors of rank at most r in $T^2(V)$, and to every $\varphi \in \mathrm{Hom}(U, V)$ the restriction to $\mathcal{M}_{\leq r}(U)$ of $\varphi \otimes \varphi$. Note that $\mathcal{M}_{\leq r}(V)$ is a Zariski-closed subset of $V \otimes V$ for every $V \in \mathbf{Vec}$, and $\varphi \otimes \varphi(\mathcal{M}_{\leq r}(U)) \subset \mathcal{M}_{\leq r}(V)$ for every $\varphi \in \mathrm{Hom}(U, V)$. Then, together with the reduced structure, $\mathcal{M}_{\leq r}$ is a **Vec**-variety. ♫

Let X and Y be **Vec**-varieties. A *morphism* of **Vec**-varieties is a natural transformation $\alpha : X \to Y$ such that for every $\varphi \in \mathrm{Hom}(U, V)$ we have $Y(\varphi) \circ \alpha(U) = \alpha(V) \circ X(\varphi)$. The category of **Vec**-varieties is the full subcategory whose objects are **Vec**-varieties in the category of affine **Vec**-schemes.

Remark 2.2.11. Clearly, the category of affine **Vec**-schemes is closed under taking closed embeddings. It is also closed under taking finite products. Indeed, the product of $X, Y : \mathbf{Vec} \to \mathbf{Sch}$ in $\mathbf{Sch}^{\mathbf{Vec}}$ is given by $V \mapsto X(V) \times Y(V)$; and furthermore, given closed embeddings $X \to P$ and $Y \to Q$, the assignment

$$X(V) \times Y(V) \to P(V) \times Q(V)$$

defines a closed embedding of the product $X \times Y$ into the polynomial functor $P \oplus Q$. ♪♫

Lemma 2.2.12. *The category of affine* **Vec***-schemes admits fibre products.*

Proof. First note that for morphisms of affine **Vec**-schemes $X \to Y, Z \to Y$ the fibre product $X \times_Y Z$ of X and Z over Y exists in the functor category $\mathbf{Sch}^{\mathbf{Vec}}$ and is given by

$$(X \times_Y Z)(V) := X(V) \times_{Y(V)} Z(V).$$

Moreover, since $Y(V)$ is affine (or more generally since $Y(V)$ is separated, see [Stacks, Tag 01KR]) the natural morphism $X(V) \times_{Y(V)} Z(V) \to X(V) \times Z(V)$ is a closed embedding. The statement then follows by Remark 2.2.11. □

Definition 2.2.13. Let $\alpha : P \to Q$ be a polynomial transformation. We define the *image* of α to be the functor $\mathrm{Im}(\alpha)$ defined by assigning $\alpha(V)(P(V))$ to every $V \in \mathbf{Vec}$ and the maps $Q(\varphi)|_{\alpha(U)(P(U))}$ to every $\varphi \in \mathrm{Hom}(U, V)$. By taking the closure of $\alpha(V)(P(V))$ inside $Q(V)$, we define the *closure of the image* of α and denote it by $\overline{\mathrm{Im}\,\alpha}$. Note that $\overline{\mathrm{Im}\,\alpha}$ is a **Vec**-variety of Q. ♪

The following proposition is immediate.

Proposition 2.2.14. *Let X be a* **Vec***-variety of a polynomial functor P, and let $\varphi \in \mathrm{Hom}(V, W)$. Then:*

1. *if φ is injective, then $P(\varphi)$ restricts to a closed embedding $X(V) \to X(W)$;*
2. *if φ is surjective, then $P(\varphi)$ restricts to a surjective morphism $X(V) \to X(W)$; and*
3. *if $V = W$ and φ is a linear isomorphism, then $P(\varphi)$ restricts to an automorphism $X(V) \to X(V)$—and indeed, the map $\mathrm{GL}(V) \times X(V) \to X(V)$, $(\varphi, p) \mapsto P(\varphi)p$ is an algebraic group action.*

Proof. We prove the first item; the rest is proved in a similar fashion. If φ is injective, then let $\psi \in \mathrm{Hom}(W, V)$ be such that $\psi \circ \varphi = \mathrm{id}_V$. Then $P(\psi) \circ P(\varphi) = P(\psi \circ \varphi) = P(\mathrm{id}_V) = \mathrm{id}_{P(V)}$ by functoriality. It follows that $P(\varphi)$ is an injective linear map, hence defines a closed embedding $P(V) \to P(W)$, and this restricts to a closed embedding $X(V) \to X(W)$. □

Definition 2.2.15. Let X be a nonempty **Vec**-variety such that if $X = Z_1 \cup Z_2$ for Z_1, Z_2 two **Vec**-subvarieties of X, then either $X = Z_1$ or $X = Z_2$. In this case we say that X is *irreducible*. ♪

Remark 2.2.16. Let X be a **Vec**-variety of a finite degree polynomial functor P. Then X is *irreducible* if and only if $X(V)$ is an irreducible variety for every $V \in \mathbf{Vec}$. The "only if" part is a consequence of Theorem 2.4.3. ♪♫

Remark 2.2.17. Polynomial functors are always irreducible **Vec**-varieties, but they might not be irreducible objects in **PF**. ♪♫

2.2.4 Infinite dimensional varieties

We now show how the language of polynomial functors fits in the frame of infinite dimensional GL-equivariant algebraic geometry.

For $n \in \mathbf{N}_{\geq 1}$ define the map $\pi_n : K^n \longrightarrow K^{n-1}$ to be the projection onto the first $n-1$ components. The collection $P(K^n)$ with the morphisms $P(\pi_n)$ forms an inverse system. Denote with P_∞ its inverse limit. Explicitly, up to isomorphism one has:

$$P_\infty = \left\{ (p_n)_{n\in\mathbf{N}} \in \prod_{n\in\mathbf{N}} P(K^n) \mid P(\pi_j)(p_j) = p_{j-1} \text{ for all } j \in \mathbf{N}_{\geq 1} \right\},$$

and the natural projections $\pi_{\infty,n} : P_\infty \longrightarrow P(K^n)$ are given by mapping $(p_i)_{i\in\mathbf{N}}$ to p_n. It is a topological space with respect to the inverse limit topology.

Remark 2.2.18. The inverse limit topology coincides with the Zariski topology induced by the ring $\varinjlim_n K[P(K^n)]$: the direct limit of $K[P(K^n)]$ with maps $P(\pi_n)^\#$. Up to isomorphism we have

$$\varinjlim_n K[P(K^n)] = \bigcup_n K[P(K^n)]$$

and the direct limit maps are the natural inclusions of $K[P(K^n)]$ into the direct limit. Let I be an ideal of $\bigcup_n K[P(K^n)]$, and let I_n be its preimages in $K[P(K^n)]$ along the natural inclusions. Then one has:

$$\mathcal{V}(I) = \bigcap_n \pi_{\infty,n}^{-1}(\mathcal{V}(I_n)),$$

showing that the Zariski topology sits inside the inverse limit topology. As the inverse limit topology is the coarsest topology making the maps $\pi_{\infty,n}$ continuous and these are continuous with respect to the Zariski topology, one gets the opposite inclusion too. In particular, P_∞ coincides with the K-points of the scheme:

$$\operatorname{Spec}\left(\varinjlim_n K[P(K^n)]\right).$$

♪♫

Remark 2.2.19. We now describe the action of GL on the affine space P_∞. Consider an element $g \in \mathrm{GL}$. In particular g belongs to GL_n for a fixed n. For every i denote by g_i the matrix

$$g_i \coloneqq \left(\begin{array}{c|c} g & 0 \\ \hline 0 & \mathbb{I}_i \end{array}\right),$$

where $\mathbb{I}_i$ denotes an identity matrix of size $i \times i$. Define $g \cdot (p_{n+i})_{i \geq 0}$ as $(P(g_i)(p_{n+i}))_{i \geq 0}$. This action is well defined by the commutativity of the diagram:

$$\begin{tikzcd} P(K^{n+i-1}) \arrow[d, "P(g_{i-1})"'] & P(K^{n+i}) \arrow[l, "P(\pi_{n+i})"'] \arrow[d, "P(g_i)"] \\ P(K^{n+i-1}) & P(K^{n+i}). \arrow[l, "P(\pi_{n+i})"] \end{tikzcd}$$

♪♫

Let X be a **Vec**-variety of P. Let X_∞ be the inverse limit of $X(K^n)$ with $P(\pi_n)$. Let I_n be the defining ideal of $X(K^n)$ inside $K[P(K^n)]$. Then the inverse limit X_∞ is a subvariety of P_∞ given by $\mathcal{V}\left(\bigcup_n I_n\right)$ and it is stable under GL, in particular it is a GL-variety. Viceversa, given a polynomial functor P with inverse limit P_∞, and given a GL-variety Y in P_∞, we can construct a **Vec**-variety X of P such that its inverse limit X_∞ satisfies $X_\infty = Y$. Details follow. Consider the (radical) ideal of Y in $K[P_\infty]$, and let $I_i \subset K[P(K^i)]$ be the preimages along the inclusions. Let $X(K^i)$ be the vanishing locus of I_i in $P(K^i)$ and note that for every i the inclusion $I_i \subset I_{i+1}$ holds. For every vector space V of dimension i consider an isomorphism $\varphi : K^i \to V$, and define $X(V) = P(\varphi)(X(K^i))$. One can check that these data give a **Vec**-variety X of P, and, by construction, its projective limit satisfies $X_\infty = Y$.

Example 2.2.20. T^2_∞ is the space of infinite-by-infinite matrices, while $\mathcal{M}_{\leq r\infty}$ is the subspace of infinite-by-infinite matrices of rank at most r. ♫

Remark 2.2.21. The above shows that there is a one-to-one correspondence between **Vec**-varieties of P and GL-varieties in P_∞. ♪♫

Remark 2.2.22. A polynomial transformation $\alpha \colon P \to Q$ naturally yields a continuous map $P_\infty \to Q_\infty$ also denoted by α. ♪♫

2.2.5 Linear endomorphisms

Elements of GL are $\mathbf{N} \times \mathbf{N}$ matrices of the block form

$$\begin{pmatrix} g & 0 \\ 0 & \mathbb{I}_\infty \end{pmatrix}$$

where $g \in \mathrm{GL}_n$ for some n and $\mathbb{I}_\infty$ is the infinite identity matrix.

Definition 2.2.23. Let $E \supset \mathrm{GL}$ be the monoid of $\mathbf{N} \times \mathbf{N}$ matrices with the property that each *row* contains only finitely many nonzero entries. ♪

Example 2.2.24. For every integer $i \geq 1$, let $\varphi_i \in K^{n_i \times m_i}$ be a matrix. Then the block matrix

$$\begin{pmatrix} \varphi_1 & & \\ & \varphi_2 & \\ & & \ddots \end{pmatrix}$$

is an element of E. ♫

We define an action of E on P_∞ as follows. Let $p = (p_0, p_1, \dots) \in P_\infty$ and $\varphi \in E$. For each integer $i \geq 0$, to compute q_i in

$$q = (q_0, q_1, \dots) = P(\varphi)p$$

we choose $n_i \geq 0$ such that all the nonzero entries of the first i rows of φ are in the first n_i columns. Now, we let $\psi_i \in K^{i \times n_i}$ be the $i \times n_i$ block in the upperleft corner of φ, so that

$$\varphi = \begin{pmatrix} \psi_i & 0 \\ * & * \end{pmatrix},$$

and we set $q_i := P(\psi_i)p_{n_i}$. Note that if we replace n_i by a larger number $\tilde{n}_i$, then the resulting matrix $\tilde{\psi}_i$ satisfies $\tilde{\psi}_i = \psi_i \circ \pi$, where $\pi : K^{\tilde{n}_i} \to K^{n_i}$ is the projection. Consequently, we then have

$$P(\tilde{\psi}_i)p_{\tilde{n}_i} = P(\psi_i)P(\pi)p_{\tilde{n}_i} = P(\psi_i)p_{n_i},$$

so that q_i is, indeed, well-defined. A straightforward computation shows that, for $\varphi, \psi \in E$, we have $P(\psi) \circ P(\varphi) = P(\psi \circ \varphi)$, so that E does indeed act on P_∞. For infinite degree-d forms, the action of $\varphi \in E$ is by linear variable substitutions $x_j \mapsto \sum_{i=1}^{\infty} \varphi_{ij} x_i$. Note that, since each x_i appears in the image of only finitely many x_j, this substitution does indeed make sense on infinite degree-d series. Since $\mathrm{GL} \subseteq E$, an E-stable subset of P_∞ is also GL-stable. The converse does not hold, since for instance E also contains the zero matrix, and $P(0)f = 0 \neq P(g)f$ for all nonzero $f \in P_\infty$ and $g \in \mathrm{GL}$ when the polynomial functor P is pure. However, it is easy to see that closed GL-stable subsets of P_∞ are also E-stable. In particular, we have $\overline{\mathrm{GL} \cdot f} = \overline{P(E)f}$.

2.3 Strength

We now introduce a classical measure for homogeneous polynomials and we then extend it to elements in any polynomial functor. The strength of polynomials plays a key role in the resolution of Stillman's conjecture by Tigran Ananyan and Mel Hochster [AH20a;

AH20b], the subsequent work by Daniel Erman, Steven Sam and Andrew Snowden [ESS19; ESS21d; ESS21a], and in David Kazhdan and Tamar Ziegler's work [KZ18b; KZ20]. Also see [BBOV22; BBOV21; BV21; BDE19; BO21; DES17] for other recent papers studying strength.

2.3.1 Definitions and examples

Definition 2.3.1. Let $n \geq 1$ be an integer and let $f \in K[x_1, \ldots, x_n]_d$ be a homogeneous polynomial of degree $d \geq 2$. Then the *strength* of f, denoted $\mathrm{str}(f)$, is the minimal integer $k \geq 0$ such that there exists an expression

$$f = g_1 h_1 + \ldots + g_k h_k$$

where $g_i \in K[x_1, \ldots, x_n]_{d_i}$ and $h_i \in K[x_1, \ldots, x_n]_{d-d_i}$ for some integer $0 < d_i < d$ for each $i \in [k]$. ♪

Example 2.3.2. Fix integers $d \geq 2$ and $k \geq 0$. The elements in $S^d(V)$ of strength $\leq k$ form a subset of S^d. Over an algebraically closed field of characteristic zero this set is closed for $d = 2, 3$ but not for $d = 4$; see [BBOV22]. ♫

Example 2.3.3. In the context of Definition 2.3.1, we set $P := \bigoplus_{i=1}^k (S^{d_i} \oplus S^{d-d_i})$ and $Q := S^d$ and define α by

$$\alpha(g_1, h_1, \ldots, g_k, h_k) := g_1 h_1 + \ldots + g_k h_k.$$

This is a polynomial transformation $P \to Q$. ♫

Example 2.3.4. Let Q, R be polynomial functors and $\alpha \colon Q \otimes R \to P$ a linear morphism. Then $(q, r) \mapsto \alpha(q \otimes r)$ defines a *bilinear* polynomial transformation $Q \oplus R \to P$. ♫

Inspired by these examples, we propose the following definition of strength for elements of homogeneous polynomial functors. We are not sure that this is the best definition in arbitrary characteristic, so we restrict ourselves to characteristic zero.

Definition 2.3.5. Assume that $\mathrm{char}\, K = 0$. Let P be a homogeneous polynomial functor of degree $d \geq 2$ and let $V \in \mathbf{Vec}$. The *strength* of $p \in P(V)$ is the minimal integer $k \geq 0$ such that

$$p = \alpha_1(q_1, r_1) + \ldots + \alpha_k(q_k, r_k)$$

where, for each $i \in [k]$, Q_i, R_i are irreducible polynomial functors with positive degrees adding up to d, $\alpha_i \colon Q_i \oplus R_i \to P$ is a bilinear polynomial transformation and $q_i \in Q_i(V)$ and $r_i \in R_i(V)$ are tensors. ♪

Remark 2.3.6. Positive degrees of two polynomial functors cannot add up to 1. So nonzero tensors $p \in P(V)$ of homogeneous polynomial functors P of degree 1 cannot have finite strength. We say that such tensors p have *infinite strength*. Note that the strength of $0 \in P(V)$ always equals 0. ♪♫

Proposition 2.3.7. *Assume that* $\operatorname{char} K = 0$. *For each integer* $d \geq 2$, *the strength of a polynomial* $f \in S^d(V)$ *according to Definition 2.3.1 equals that according to Definition 2.3.5.*

Proof. The inequality $\geq$ follows from the fact that $\alpha_i : S^{d_i} \oplus S^{d-d_i} \to S^d$, $(g, h) \mapsto g \cdot h$ is a bilinear polynomial transformation. For the inequality $\leq$, suppose that $\alpha : Q \oplus R \to S^d$ is a nonzero bilinear polynomial transformation, where Q and R are irreducible of degrees $e < d$ and $d-e < d$. So Q and R are Schur functors corresponding to Young diagrams with e and $d-e$ boxes, respectively, and $Q \otimes R$ admits a nonzero linear morphism to S^d, whose Young diagram is a row of d boxes. The Littlewood-Richardson rule then implies that the Young diagrams of Q and R must be a single row as well, so that $Q = S^e$ and $R = S^{d-e}$, and also that there is (up to scaling) a unique morphism $Q \otimes R = S^e \otimes S^{d-e} \to S^d$, namely, the one corresponding to the polynomial transformation $(g, h) \mapsto g \cdot h$. □

2.3.2 Some properties

The strength of a tensor in P quickly becomes very difficult when P is not irreducible as the following examples show.

Example 2.3.8. Take $P = (S^d)^{\oplus e}$ for some integer $e \geq 1$. Then the strength of a tuple $(f_1, \dots, f_e) \in P(V)$ is the minimum number $k \geq 0$ such that

$$f_1, \dots, f_e \in \operatorname{span}\{g_1, \dots, g_k\}$$

where $g_1, \dots, g_k \in S^d(V)$ are reducible polynomials. ♫

Example 2.3.9. Consider $P = S^2 \oplus \bigwedge^2$, so that $P(V) = V \otimes V$, and assume that K is algebraically closed. The only possibilities for Q and R are $Q(V) = R(V) = V$. The bilinear polynomial transformations $\alpha : Q \oplus R \to P$ are of the form

$$\alpha(u, v) = au \otimes v + bv \otimes u = c(u \otimes v + v \otimes u) + d(u \otimes v - v \otimes u)$$

for certain $a, b, c, d \in K$. We note that $\operatorname{str}(A) = \lceil \operatorname{rk}(A)/2 \rceil$ when $A \in S^2(V)$ and $\operatorname{str}(A) = \operatorname{rk}(A)/2$ when $A \in \bigwedge^2(V)$. In general, we have

$$\operatorname{rk}(A)/2, \operatorname{rk}(A+A^\top)/2, \operatorname{rk}(A-A^\top)/2 \leq \operatorname{str}(A) \leq \operatorname{rk}(A), \operatorname{rk}(A+A^\top)/2 + \operatorname{rk}(A-A^\top)/2$$

for all $A \in V \otimes V$, where each bound can hold with equality. For example, for the matrix

$$A = \begin{pmatrix} 0 & 1 & & & \\ 0 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & 0 & 0 \end{pmatrix}$$

we have $\mathrm{rk}(A + A^\top)/2 = \mathrm{rk}(A - A^\top)/2 = \mathrm{str}(A) = \mathrm{rk}(A)$. ♫

Example 2.3.10. Again take $P = S^2 \oplus \bigwedge^2$ and consider $P(K^2) = K^{2\times 2}$. Assume K is algebraically closed. The matrix

$$A = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$$

clearly has strength ≤ 2. We will show that A has strength 2 whenever $x = \pm 2$ and strength 1 otherwise. In particular, this shows that the subset of $P(K^2)$ of matrices of strength ≤ 1 is not closed. Suppose A has strength 1. Then we can write A as $au \otimes v + bv \otimes u$ with $a, b \in K$ and $v, u \in K^2$. Let e_1, e_2 be the standard basis of K^2. Without loss of generality, we may assume that $u = e_1 + \lambda e_2$ and $v = e_1 + \mu e_2$ for some $\lambda, \mu \in K$. We get

$$\begin{aligned} a + b &= 1, & a\mu + b\lambda &= x, \\ a\lambda + b\mu &= 0, & \lambda\mu &= 1. \end{aligned}$$

Using $\lambda = \mu^{-1}$ and $b = 1 - a$, we are left with $a\mu^2 + (1 - a) = x\mu$ and $a + (1 - a)\mu^2 = 0$. The latter gives us $\mu \neq \pm 1$ and $a = \mu^2/(\mu^2 - 1)$. We get $\mu^2 + 1 = x\mu$. Now, if $x \neq \pm 2$, then such a $\mu \neq \pm 1$ exists. So in this case A indeed has strength 1. If $x = \pm 2$, the only solution is $\mu = \pm 1$. Hence A has strength 2 in this case. ♫

2.3.3 Strength in infinite dimension

Let P be a pure polynomial functor. If $P = S^d$, then the elements of P_∞ can be thought of as homogeneous series of degree d in infinitely many variables $x_1, x_2, \ldots$. Here, closed subsets of P_∞ are defined by polynomial equations in the coefficients of these series. On P_∞ acts the group GL and in the case of degree-d series, an element $g \in \mathrm{GL}_n \subset \mathrm{GL}$ maps each of the first n variables x_i to an invertible linear combination of $x_1, \ldots, x_n$ and the remaining variables to themselves.

Example 2.3.11. On degree-d forms, GL has dense orbits, such as that of

$$f = x_1 x_2 \cdots x_d + x_{d+1} x_{d+2} \cdots x_{2d} + \ldots$$

The reason is that this series can be specialised to any degree-d form *in finitely many variables* by linear variable substitutions. This implies that the image of $\mathrm{GL}\cdot f$ in each $S^d(K^n)$ is dense. Hence $\mathrm{GL}\cdot f$ is dense in S^d_∞. ♫

For every pure polynomial functor P, the group GL has dense orbits on P_∞—in fact, uncountably many of them: see [Bik20, Section 4.5.1]. We now extend the notion of strength to elements in the infinite dimensional setting.

Definition 2.3.12. Assume that char $K = 0$. Let P be a homogeneous polynomial functor. The strength of a tensor $p \in P_\infty$ is the minimal integer $k \geq 0$ such that

$$p = \alpha_1(q_1, r_1) + \ldots + \alpha_k(q_k, r_k)$$

for some irreducible polynomial functors Q_i, R_i whose positive degrees sum up to d, bilinear polynomial transformations $\alpha_i \colon Q_i \oplus R_i \to P$ and elements $q_i \in Q_{i,\infty}$ and $r_i \in R_{i,\infty}$. If no such k exists, we say that p has infinite strength. ♪

2.3.4 A quasi-order on infinite tensors

Definition 2.3.13. For infinite tensors $p, q \in P_\infty$ we write $p \preceq q$ if $p \in \overline{P(E)q}$. In this case, we say that q *specialises* to p. ♪

Remark 2.3.14. The symbol $\prec$ or $\preceq$ was used for the order on polynomial functors in Section 2.1.3. This cannot lead to confusion: the type of objects involved is completely different. ♪♫

From the fact that E is a unital monoid that acts on P_∞, we find that $\preceq$ is transitive and reflexive. Hence it induces an equivalence relation $\simeq$ on P_∞ by

$$p \simeq q \Leftrightarrow p \preceq q \text{ and } q \preceq p,$$

as well as a partial order on the equivalence classes of $\simeq$.

Example 2.3.15. Fix an integer $k \geq 1$ and consider the polynomial functor $P = (S^1)^{\oplus k}$. A tuple $q = (q_1, \ldots, q_k) \in P_\infty$ has a dense GL-orbit if and only if $q_1, \ldots, q_k \in S^1_\infty$ are linearly independent. Suppose that q has a dense GL-orbit and let A be the $\mathbf{N} \times k$ matrix corresponding to q. Then A has full rank. By acting with an element of $\mathrm{GL} \subseteq E$, we may assume that

$$A = \begin{pmatrix} \mathbb{I}_k \\ B \end{pmatrix}$$

where B is again an $\mathbf{N} \times k$ matrix. Now, take

$$\varphi_C := \begin{pmatrix} \mathbb{I}_k & \\ C & \mathbb{I}_\infty \end{pmatrix} \in E$$

and note that $\varphi_{-B} A = (\mathbb{I}_k \ 0)^\top$, so that $P(\varphi_{-B})q = (x_1, \ldots, x_k)$. So any two tuples in P_∞ with a dense GL-orbit are in the same equivalence class. Moreover, the element of E specializing one tuple to the other can be chosen to be invertible in E as $\varphi_C \varphi_{-C} = \mathbb{I}_\infty$. ♫

There is an obvious relation between $\preceq$ and orbit closures, namely: if $p \preceq q$, then $p \in \overline{\mathrm{GL}\cdot q}$. The converse, however, is not true.

Example 2.3.16. Let $p = x_1(x_1^2 + x_2^2 + \dots)$, $q = x_1^3 + x_2^3 + \dots \in S^3_\infty$. As every cubic polynomial is sum of powers of linear forms, the GL-orbit of g is dense in S^3_∞, and hence $p \in S^3_\infty = \overline{\mathrm{GL}\cdot q}$. However, we have $p \not\preceq q$: suppose that

$$f \coloneqq x_1 g(x_1, x_2, \dots) + h(x_2, x_3, \dots) \in S^3(E)q$$

for some $g \in S^2_\infty$ and $h \in S^3_\infty$. As only finitely many variables x_i are substituted by linear forms containing x_1 when specialising q to f, we see that

$$x_1 g(x_1, x_2, \dots) + \tilde{h}(x_2, x_3, \dots) \in S^3(E)(x_1^3 + x_2^3 + \dots + x_n^3)$$

for some integer $n \geq 1$ and $\tilde{h} \in S^3_\infty$. From this, it is easy to see that g has finite strength. Hence $f \neq p$ as $x_1^2 + x_2^2 + \dots$ has infinite strength. So indeed $p \not\preceq q$.

♫

In order to have a tensor $p \in P_\infty$ with a dense GL-orbit, the polynomial functor P must be pure. For some time, we believed that when this is the case all elements $p \in P_\infty$ with a dense GL-orbit might form a single $\simeq$-equivalence class. When P has degree ≤ 2, this is in fact true; see Example 6.4.4. However, it doesn't hold for cubics.

Example 2.3.17. Let $p, q \in S^3_\infty$ be as before. Now also consider $r = p(x_1, x_3, \dots) + q(x_2, x_4, \dots)$. We have $q = r(0, x_1, 0, x_2, \dots) \preceq r$ and so $S^3_\infty = \overline{\mathrm{GL}\cdot q} \subseteq \overline{\mathrm{GL}\cdot r}$. Hence both q and r have dense GL-orbits. And, we have $r \not\preceq q$: indeed, otherwise $p = r(x_1, 0, x_2, 0, \dots) \preceq r \preceq q$, but $p \not\preceq q$. ♫

Indeed, the poset of equivalence classes for points in S^3_∞ of infinite strength is isomorphic to $\mathbf{N}$; see [BDS22].

2.4 The Embedding Theorem

In this section we work with polynomial functors of bounded degree defined over a field K of characteristic zero.

Theorem 2.4.1 (The Embedding Theorem)**.** *Let P be a polynomial functor and let X be a proper* **Vec***-variety of P. Let R be an irreducible subfunctor of P and let $\pi : P \to P/R$ be the projection transformation. Let X' be the closure of the projection of X along π. Then one of the following holds:*

1. $X = \pi^{-1}(X') = X' \times R$,

2. *there exists a finite dimensional vector space U and an equation $h \in K[P(U)]$ not entirely vanishing on X such that the dashed arrow below coming from the compositions of the diagram is a closed embedding:*

$$\begin{tikzcd} \mathrm{Sh}_U X \arrow[r] & \mathrm{Sh}_U P \arrow[d] \\ \mathrm{Sh}_U X[1/h] \arrow[u] \arrow[r, dashed] & \mathrm{Sh}_U P/R[1/h]. \end{tikzcd}$$

Remark 2.4.2. The natural map mentioned in case 2 is the restriction of the projection map $\mathrm{Sh}_U P \to \mathrm{Sh}_U P/R$ to the open subset of $\mathrm{Sh}_U X$ where h doesn't vanish. Moreover, we can rephrase case 2 by saying that $\mathrm{Sh}_U X[1/h]$ is isomorphic to a **Vec**-subvariety Z of $\mathrm{Sh}_U P/R[1/h]$. ♪♫

2.4.1 Applications of the Embedding Theorem

The Embedding Theorem first appeared in the proof of topological Noetherianity of polynomial functors.

Theorem 2.4.3 ([Dra19, Theorem 1])**.** *Let X be a* **Vec***-variety. Then every descending chain of* **Vec***-subvarieties*

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$$

stabilises, that is, there exists $N \geq 0$ such that for each $n \geq N$ we have $X_n = X_{n+1}$.

Another corollary of The Embedding Theorem is the following proposition. It is also known as "The Shift Theorem" in [BDES22, Theorem 5.1].

Proposition 2.4.4. *Let X be a* **Vec***-variety of a polynomial functor P, then there exist a finite-dimensional vector space U, an equation $h \in K[P(U)]$ not identically vanishing on $X(U)$, a polynomial functor Q, and a finite-dimensional affine variety B such that*

$$\mathrm{Sh}_U X[1/h] \cong B \times Q\,.$$

Proof. In the case X is the polynomial functor P the statement of the theorem is true for $Q = P$ and B being a point. Assume then X to be proper and proceed by induction on the ordering of polynomial functors. The base case is when P has degree zero and X is a proper closed subvariety B of the (finite-dimensional) affine space. In this case, $X = B$, so the base case of the induction is settled. Suppose P is not constant, X is proper, and assume the statement true for every **Vec**-variety of a polynomial functors P' with $P' \prec P$. We apply Theorem 2.4.1 to X and P choosing R to be an irreducible subfunctor in the highest degree part of P. The theorem gives two possibilities. In the case 1, we have $X = \pi^{-1}(\overline{\pi(X)})$. Define $P' := P/R$ and note that

$$X \cong \overline{\pi(X)} \times R \subset P.$$

By construction $P' \prec P$, hence we can apply the induction hypothesis to $\overline{\pi(X)}$. Therefore there exist a finite-dimensional vector space U, a polynomial functor Q, an equation h, and a finite-dimensional affine variety B such that $\mathrm{Sh}_U\, \overline{\pi(X)}[1/h] \cong B \times Q$. Hence

$$\mathrm{Sh}_U X[1/h] \cong \mathrm{Sh}_U \left(\overline{\pi(X)} \times R\right)[1/h] = \mathrm{Sh}_U\, \overline{\pi(X)}[1/h] \times \mathrm{Sh}_U R \cong B \times Q \times \mathrm{Sh}_U R.$$

Since $Q \oplus \mathrm{Sh}_U R$ is a polynomial functor, we proved the theorem for this case.
In the possibility 2 of The Embedding Theorem, there exists a **Vec**-subvariety $Z \subset \mathrm{Sh}_U P/R[1/h]$ such that

$$\mathrm{Sh}_U X[1/h] \cong Z. \tag{2.1}$$

As $\mathrm{Sh}_U P/R \prec P$ (because the highest degree parts of P and of any of its shifts are isomorphic and R was taken in the top-degree part of P), the induction hypothesis applies to Z, and hence there are a polynomial functor Q, a finite-dimensional vector space U', an equation h', and a finite-dimensional affine variety B such that

$$\mathrm{Sh}_{U'} Z[1/h'] \cong B \times Q.$$

As the shift of an isomorphism is still an isomorphism and the restriction of an isomorphism is an isomorphism too, the map in (2.1) induces the isomorphism

$$\mathrm{Sh}_{U \oplus U'} X[1/h \cdot h'] \cong \mathrm{Sh}_{U'} Z[1/h'] \cong B \times Q,$$

where h' denotes both the equation in $K[(\mathrm{Sh}_U P/R)(U')]$ and its pull-back along the projection map $\mathrm{Sh}_U P \to \mathrm{Sh}_U P/R$. This proves the theorem. □

Remark 2.4.5. In the case X is proper and there is an R in the top-degree part of P for which case 2 applies, we have that the polynomial functor Q of Theorem 2.4.4 satisfies $Q \prec P$ in the order of polynomial functors given in Subsection 2.1.3. This is true because in the proof Q is (isomorphic to) the subfunctor of $\mathrm{Sh}_{U \oplus U'} P/\mathrm{Sh}_{U'} R$ that is smaller than P. ♪♫

2.4.2 Dimension functions

Definition 2.4.6. Let X be a **Vec**-variety of a polynomial functor P. We define the *dimension function* f_X of X to be:

$$f_X(n) := \dim(X(V)),$$

where V is a vector space of dimension n and $\dim(X(V))$ is the dimension of $X(V)$ as a variety. ♪

Recall Proposition 2.1.29. Below we prove that also the dimension functions of **Vec**-varieties are eventually polynomials with rational coefficients:

Proposition 2.4.7. *If X is a proper* **Vec**-*variety of a polynomial functor P, then its dimension function is eventually a polynomial with rational coefficients whose degree is at most the degree of P. In other words, there exists a polynomial p_X in n with rational coefficients such that for any n big enough we have $f_X(n) = p_X(n)$.*

Proof. Assume X irreducible, so that $\dim(X(U \oplus V)) = \dim(X(U \oplus V)[1/h])$ for any equation h not vanishing identically on X. By Proposition 2.4.4 there are a finite-dimensional vector space U, an equation h, a polynomial functor Q, and a finite-dimensional affine variety B such that $\mathrm{Sh}_U X[1/h] \cong B \times Q$. Say that the dimensions of U, V, and B are, respectively, k, n, and b then

$$f_X(k+n) = f_{\mathrm{Sh}_U X[1/h]}(n) = f_Q(n) + b.$$

In particular for any n bigger than k we have

$$f_X(n) = f_Q(n-k) + b.$$

By Proposition 2.1.29 the function f_Q is a polynomial with rational coefficients, and hence for $n > k$ the function $f_X(n)$ is the polynomial $f_Q(n-k) + b$ thought as a polynomial in n. Since b is an integer, $f_X(n)$ has rational coefficients too.
In the case X is not irreducible, we can consider its irreducible components. The number of irreducible components is finite because P is Noetherian by Theorem 2.4.3. Let $X_1, \dots, X_t$ be the irreducible components of X. The dimension function of each of them is eventually a polynomial with rational coefficients by the above reasoning. Hence, the polynomial with the biggest leading term among those is eventually the dimension function of X and this proves the theorem. □

We want to prove the following

Proposition 2.4.8. *Let X be an irreducible* **Vec**-*variety of a polynomial functor P of degree d. Suppose that case 1 of Theorem 2.4.1 never applies for any irreducible subfunctor R of P of highest degree. Then the dimension function f_X of X is eventually a polynomial of degree strictly less than d.*

Proof. There exists an U, a h, a Q, and a B as in Proposition 2.4.4 such that

$$\mathrm{Sh}_U X[1/h] \cong B \times Q, \tag{2.2}$$

and $Q \prec P$ by Remark 2.4.5. In particular, the degree of Q is smaller than or equal to the degree of P. Say that the dimension of U is k and the dimension of B is b. By the irreducibility of X we have that

$$f_X(n) = f_{\mathrm{Sh}_U X[1/h]}(n-k) = f_Q(n-k) + b$$

for $n > k$. In particular, the dimension function f_X is eventually a polynomial of degree at most the degree of Q by Proposition 2.1.29. Suppose that the degree of Q is d and let S be an irreducible subfunctor of Q_d. Define $Q' := Q/S$, so that $Q = Q' \oplus S$, and consider the morphism $B \times Q' \times S \to \mathrm{Sh}_U P$ induced by the isomophism (2.2). Fix a point $p = (b, q') \in B \times Q'$, and consider the restriction of the above map to $\{p\} \times S$. Its image gives an irreducible component of degree d in $\mathrm{Sh}_U P$. With an abuse of notation, name it S too. In particular, $\mathrm{Sh}_U X[1/h] = S \times X''$ for some **Vec**-variety $X'' \subset \mathrm{Sh}_U P/S$. The morphism $\mathrm{Sh}_U P \to P$ induced by the map $P_{U\oplus V \to V}$ restricts to an isomorphism on the top degree part, namely, on part of degree d. As this morphism is also surjective, we deduce that $X = S \times X'$ with X' being a **Vec**-variety of P/R. This is against the assumption that 1 of Theorem 2.4.1 never applies to X.
We deduce that the degree of Q is strictly smaller than d and therefore the dimension function of X is eventually a polynomial of degree strictly smaller than d. □

Chapter 3

Polynomial functors over rings

In this chapter we introduce polynomial functors defined over rings and their associated topological spaces. In Section 3.1 we set the notation up and recall some classic results. Polynomial laws and the coordinate ring of a module are the topics of Section 3.2, while Section 3.3 describes the topological space $\mathbb{A}_M$ associated to a module M. Finally, we introduce polynomial functors defined over rings and develop the geometry needed for the proof of topological Noetherianity in Chapter 4. This chapter is fully based on parts of [BDD22]. The polynomial functors that we study are often referred to as "strict polynomial functors" in the literature, e.g. in [FS97; Tou14]; we will drop the adjective "strict". We do not know whether the polynomial functors over finite fields studied in [Pir02] admit a similar theory. We will use work of Roby on polynomial laws [Rob63] and work of Touzé on polynomial functors [Tou14] but indeed only more elementary parts of their work such as the generalisation of Friedlander-Suslin's [FS97, Theorem 3.2] to general base rings R; see [Tou14, Théorème 7.2].

3.1 Notation and classical results

For this chapter and Chapter 4, since we will be working over a ring R, we modify our notation from page 15 as follows. Recall that R denotes a commutative ring with unit. Now, the letters U, V denote R-modules, $\mathrm{Hom}(U, V)$ denotes the R-module of R-module homomorphisms from U to V. In particular, $\mathrm{End}(U) := \mathrm{Hom}(U, U)$, and $U^* := \mathrm{Hom}(U, R)$. When we pick an R-algebra A —namely, when we write something like:"Let A be an R-algebra"— we assume it is commutative. However, not all R-algebras we deal with are commutative. When we speak about a morphism of rings, we assume it is unital. If A is an R-algebra and M, N are A-modules, then $\mathrm{Hom}_A(M, N)$ denotes the A-module of A-module homomorphisms from M to N. If F is a field and M is an F-module, $\dim_F(M)$

denotes the dimension of M as an F-vector space. If D is an R-algebra that is a domain, $\mathrm{GL}_n(D)$ denotes the invertible $n \times n$-matrices with coefficients in D. $\mathbf{Alg}_R$ denotes the category of R-algebras.

3.1.1 Additional notation

Let $\mathfrak{p}$ be a prime ideal of R, then we write $K_\mathfrak{p}$ for the fraction field of the domain $R/\mathfrak{p}$. If R is a domain, then we write $K := K_{(0)}$ for the fraction field of R. Except where specified otherwise, tensor products are over R, and we use the terms R-domain and R-field for R-algebras that, as rings, are domains and fields, respectively.

3.1.2 From finitely generated to free modules.

The following lemma, which we will later generalise to polynomial functors, is well-known; we give a proof for completeness.

Lemma 3.1.1. *Let R be a domain, let M be a finitely generated R-module, and let N be a submodule of M. Then there exists a nonzero $r \in R$ and elements $v_1, \dots, v_n \in N$ such that $R[1/r] \otimes N$ is a finitely generated free submodule of $R[1/r] \otimes M$ with basis $1 \otimes v_1, \dots, 1 \otimes v_n$, and such that $R[1/r] \otimes M$ is the direct sum of $R[1/r] \otimes N$ and another free $R[1/r]$-module.*

Note that tensoring with K yields that $n = \dim_K(K \otimes N)$.

Proof. The vector space $K \otimes N$ is contained in the finite-dimensional vector space $K \otimes M$. Hence there exist $v_1, \dots, v_n \in N$ such that $1 \otimes v_1, \dots, 1 \otimes v_n$ is a basis of $K \otimes N$, and $v_{n+1}, \dots, v_m \in M$ such that $1 \otimes v_{n+1}, \dots, 1 \otimes v_m$ is a basis of a complement of $K \otimes N$ in $K \otimes M$. We claim that both statements hold with K replaced by $R[1/r]$ for some nonzero r.

To see this, extend $v_1, \dots, v_m$ with $v_{m+1}, \dots, v_l$ to a generating set of the R-module M. Then for each $j = m+1, \dots, l$ we have, in $K \otimes M$,

$$1 \otimes v_j = \sum_{i=1}^{m} c_{ij} \otimes v_i$$

for certain coefficients $c_{ij} \in K$. This identity means that there exists a non-zero element $r \in R$ and suitable coefficients c'_{ij} in R such that

$$1 \otimes v_j = \sum_{i=1}^{m} (c'_{ij}/r) \otimes v_i$$

holds in $R[1/r] \otimes M$. Hence $R[1/r] \otimes M$ is generated by $1 \otimes v_1, \dots, 1 \otimes v_m$, and these elements do not have any nontrivial linear relation over $R[1/r]$ since their images in $K \otimes$

M do not satisfy any such relation over K. It follows that $R[1/r] \otimes M$ is free with basis $1 \otimes v_1, \dots, 1 \otimes v_m$. Furthermore, $R[1/r] \otimes N$ contains the $R[1/r]$-module spanned by $1 \otimes v_1, \dots, 1 \otimes v_n$; and conversely, if $v \in R[1/r] \otimes M$ is an element of $R[1/r] \otimes N$, then it cannot have a nonzero coefficient on any of the last $m - n$ basis elements, because in $K \otimes M$ the image of v is a linear combination of the first m basis elements and the basis elements do not satisfy any linear relation there. Hence $R[1/r] \otimes N \subseteq R[1/r] \otimes M$ is free with basis $1 \otimes v_1, \dots, 1 \otimes v_n$. □

3.2 Polynomial laws and the coordinate ring of a module

3.2.1 Polynomial laws

We follow [Rob63, Chapter 1]. Let M, N be R-modules. Recall that $\mathbf{Alg}_R$ is the category of R-algebras.

Definition 3.2.1. A *polynomial law* $\varphi\colon M \to N$ is a collection of maps

$$(\varphi_A : A \otimes M \to A \otimes N)_{A \in \mathbf{Alg}_R}$$

such that for every R-algebra homomorphism $\alpha\colon A \to B$ the following diagram commutes:

$$\begin{tikzcd} A \otimes M \arrow[r, "\varphi_A"] \arrow[d, "\alpha \otimes \mathrm{id}_M"'] & A \otimes N \arrow[d, "\alpha \otimes \mathrm{id}_N"] \\ B \otimes M \arrow[r, "\varphi_B"] & B \otimes N. \end{tikzcd}$$

♪

Example 3.2.2. Suppose that M and N are the free modules R^2 and R, respectively, so that $A \otimes M$ and $A \otimes N$ are canonically identified with A^2 and A. Then the collection $(\varphi_A)_A$ defined by $\varphi_A(x, y) = xy + y^2$ for $x, y \in A$ is a polynomial law $M \to N$, and indeed one that is *homogeneous* of degree 2 in the sense of Definition 3.2.5 below. ♫

More generally, the name polynomial law derives from the following fact.

Lemma 3.2.3. *Consider two R-modules M and N. Suppose that M is finitely generated and let $\{v_1, \dots, v_n\}$ be a set of generators. Let $\varphi\colon M \to N$ be a polynomial law. Then φ is completely determined by the element:*

$$\iota(\varphi) := \varphi_{R[x_1,\dots,x_n]}(x_1 \otimes v_1 + \cdots + x_n \otimes v_n) \in R[x_1, \dots, x_n] \otimes N.$$

This gives an injective map ι from the collection of polynomial laws from M to N to the module $R[x_1, \dots, x_n] \otimes N$. In the case where M is free with basis $v_1, \dots, v_n$, this injective map is a bijection.

Proof. Let A be an R-algebra, let $a_1, \ldots, a_n \in A$ be elements and let $\alpha\colon R[x_1, \ldots, x_n] \to A$ be the R-algebra homomorphism sending $x_i \mapsto a_i$. Then the diagram associated to α shows that $\varphi_A(a_1 \otimes v_1 + \cdots + a_n \otimes v_n) = (\alpha \otimes \mathrm{id}_N)\iota(\varphi)$ and hence ι is injective. If M is free with basis $v_1, \ldots, v_n$, then $\varphi_A(a_1 \otimes v_1 + \cdots + a_n \otimes v_n) = \sum_j f_j(a_1, \ldots, a_n) \otimes w_j$ defines a polynomial law $\varphi\colon M \to N$ for every $\sum_j f_j \otimes w_j \in R[x_1, \ldots, x_n] \otimes N$. □

Example 3.2.4. If R is an infinite field, then a polynomial law φ from $M = R^n$ to $N = R^m$ is in fact uniquely determined by φ_R, which is required to be a polynomial map, i.e., a map all of whose m coordinate functions are polynomials in the n coordinates on M. So then the set of polynomial laws from M to N is precisely the set of polynomial maps from the vector space M to the vector space N.

For a general ring R, we denote by $\mathbb{A}^n_R$ the affine scheme $\mathrm{Spec}(R[x_1, \cdots, x_n])$. The set of polynomial laws from R^n to R^m is the set of morphisms $\mathbb{A}^n_R \to \mathbb{A}^m_R$ defined over R. Of course, such a morphism need not be determined by its map $\varphi_R\colon R^n \to R^m$, but it *is* determined by the maps $\varphi_A\colon A^n \to A^m$ for all R-algebras A. This motivates the definition of polynomial laws. ♫

Definition 3.2.5. A polynomial law $\varphi\colon M \to N$ is *homogeneous of degree d* if for each R-algebra A and all $a \in A$, $m \in A \otimes M$, we have $\varphi_A(am) = a^d \varphi_A(m)$. ♪

Writing $R[x_1, \ldots, x_n]_d$ for the set of homogeneous polynomials of degree d, we see that the injection from Lemma 3.2.3 maps a homogeneous polynomial law $M \to N$ of degree d to an element of $R[x_1, \ldots, x_n]_d \otimes N$.

Proposition 3.2.6. *Let $M_1, \ldots, M_d, N$ be R-modules and let $\varphi\colon M_1 \times \cdots \times M_d \to N$ be a multilinear map. Then φ extends to a homogeneous polynomial law of degree d (also denoted φ). After identifying $A \otimes (M_1 \times \cdots \times M_d) \cong A \otimes M_1 \times \cdots \times A \otimes M_d$, we have*

$$\varphi_A\left(\sum_{i_1} a_{i_1} \otimes m_{i_1}, \ldots, \sum_{i_d} a_{i_d} \otimes m_{i_d}\right) = \sum_{i_1, \ldots, i_d} a_{i_1} \cdots a_{i_d} \otimes \varphi(m_{i_1}, \ldots, m_{i_d})$$

for all R-algebras A, $a_{i_1}, \ldots, a_{i_d} \in A$ and $m_{i_1} \in M_1, \ldots, m_{i_d} \in M_d$.

Proof. The maps φ_A are well-defined as the maps $A^d \times M_1 \times \cdots \times M_d \to A \otimes N$ sending $(a_1, \ldots, a_d, m_1, \ldots, m_d) \mapsto a_1 \cdots a_d \varphi(m_1 \cdots m_d)$ are multilinear. The collection $(\varphi_A)_A$ is a homogeneous polynomial law of degree d and $\varphi_R = \varphi$. □

Remark 3.2.7. Composition of R-module homomorphisms is a bilinear map. By the proposition, we can thus view this operation as a polynomial law. ♪♫

A homogeneous polynomial law $\varphi\colon M \to N$ of degree 0 is the same thing as an element of N (namely, the element $\varphi_R(0)$, which equals $\varphi_A(m)$ for any R-algebra A and any element $m \in A \otimes M$); we call these polynomial laws *constant*. A homogeneous polynomial law $M \to N$ of degree 1 is the extension of an R-module homomorphism $M \to N$ as in the proposition above (namely, the map $\varphi_R : M \to N$, which in this case is R-linear and uniquely determines φ_A for all $A \in \mathbf{Alg}_R$); we call these polynomial laws *linear*.

The following proposition says that, in many ways, polynomial laws behave like ordinary polynomial maps between vector spaces. For proofs we refer to [Rob63].

Proposition 3.2.8. *Let $\varphi, \psi : M \to N$, $\gamma : N \to O$ be polynomial laws between R-modules.*

1. *The collection $\varphi + \psi := (\varphi_A + \psi_A)_A$ is a polynomial law $M \to N$, homogeneous of degree d if φ, ψ are.*
2. *We have $\varphi = \sum_{d=0}^{\infty} \varphi_d$ for unique polynomial laws $\varphi_d : M \to N$ of degree d, where for each R-algebra A and each $m \in A \otimes M$ we have $\varphi_{d,A}(m) = 0$ for all but finitely many d's (φ_d is called the* homogeneous component *of φ of degree d); moreover, if M is finitely generated, then only finitely many of the φ_d are nonzero.*
3. *The collection $\gamma \circ \varphi := (\gamma_A \circ \varphi_A)_A$ is a polynomial law $M \to O$, homogeneous of degree de if φ, ψ are homogeneous of degrees d, e, respectively.*
4. *If $N = R$, then $\varphi \cdot \psi := (m \mapsto \varphi_A(m)\psi_A(m))_A$ is a polynomial law $M \to R$, homogeneous of degree $d + e$ if φ, ψ are homogeneous of degrees d, e, respectively.*

Proposition 3.2.9. *Let $\varphi : M \oplus M' \to N$ be a polynomial law between R-modules. Then φ has a unique decomposition $\varphi = \sum_{i,j=0}^{\infty} \varphi_{(i,j)}$ such that $\varphi_{(i,j)} : M \oplus M' \to N$ is a bihomogeneous polynomial law of degree (i, j), i.e., after identifying $A \otimes (M \oplus M') \cong A \otimes M \oplus A \otimes M'$, we have $\varphi_{(i,j),A}(am, bm') = a^i b^j \varphi_{(i,j),A}(m, m')$ for all R-algebras A, $a, b \in A$, $m \in A \otimes M$ and $m' \in A \otimes M'$. Moreover, if φ is homogeneous of degree d, then $\varphi_{(i,j)} = 0$ for all $i+j \neq d$.*

Proof. Suppose that such a decomposition exists and let A be an R-algebra. Then we have

$$\varphi_{A[s,t]}(sm, tm') = \sum_{i,j} \varphi_{(i,j),A[s,t]}(sm, tm') = \sum_{i,j} s^i t^j \varphi_{(i,j),A}(m, m') \in \bigoplus_{i,j=0}^{\infty} s^i t^j A \otimes N$$

for all $m \in A \otimes M$ and $m' \in A \otimes M'$. This shows that the $\varphi_{(i,j)}$ are unique. If φ is homogeneous of degree d, setting $s = t$, we see that $\varphi = \sum_{i+j=d} \varphi_{(i,j)}$ and hence $\varphi_{(i,j)} = 0$ for $i + j \neq d$. What remains to show the existence of the decomposition. In fact, defining $\varphi_{(i,j),A}(m, m')$ to be the coefficient of $s^i t^j$ in $\varphi_{A[s,t]}(sm, tm')$, it is easy to show that the $\varphi_{(i,j)}$ are bihomogeneous polynomial laws of degree (i, j) adding up to φ. □

The class of R-modules, in addition to its structure of Abelian category with R-module homomorphisms as morphisms, has the structure of a (non-Abelian) category with polynomial laws as morphisms. Both structures will be important to us, but we reserve the notation $\mathbf{Mod}_R$ for the category in which the morphisms are R-module homomorphisms (i.e., homogeneous polynomial laws of degree 1).

Definition 3.2.10 (Base change)**.** If B is an R-algebra, then the tensor product functor $\mathbf{Mod}_R \to \mathbf{Mod}_B$, which sends *linear* polynomial laws over R to linear polynomial laws over B, can be extended to a functor from the category of R-modules with polynomial laws over R to the category of B-modules with polynomial laws over B: on objects, the functor is just $M \mapsto B \otimes M$, and a polynomial law $(\varphi_A)_{A \in \mathbf{Alg}_R} : M \to N$ is mapped to $(\varphi_A)_{A \in \mathbf{Alg}_B}$ where, for a B-algebra A, the map φ_A is interpreted as a map $A \otimes_B (B \otimes_R M) \cong A \otimes_R M \to A \otimes_R N \cong A \otimes_B (B \otimes_R N)$. ♪

3.2.2 The coordinate ring of a module

Let M be a finitely generated R-module.

Definition 3.2.11. We write $R[M]$ for the set of polynomial laws $M \to R$ and $R[M]_d \subseteq R[M]$ for the subset of homogeneous polynomial laws of degree d. The addition and multiplication from Proposition 3.2.8, the grading from Definition 3.2.5 and the identification $R[M]_0 = R$ give $R[M] = \bigoplus_{d=0}^{\infty} R[M]_d$ the structure of a $\mathbf{Z}_{\geq 0}$-graded commutative R-algebra. We call this R-algebra the *coordinate ring* of M. ♪

Remark 3.2.12. In [Rob63, Chapitre III], various algebras associated to an R-module M are introduced, but they are different from our R-algebra $R[M]$. One important difference is that for us, the elements of M play the role of geometric objects, whereas there, the algebras consist of elements in divided or symmetric powers of M. ♪♫

As usual with coordinate rings, the association $M \mapsto R[M]$ is a contravariant functor from the category of R-modules with polynomial laws to the category of R-algebras: a polynomial law $\varphi \colon M \to N$ has a pull-back map $\varphi^{\#} : R[N] \to R[M]$ sending $f \mapsto f \circ \varphi$. If φ is linear, then $\varphi^{\#}$ is a graded homomorphism.

If M is generated by $v_1, \dots, v_n$, then the injection $\iota \colon R[M] \to R[x_1, \dots, x_n]$ of Lemma 3.2.3 is a graded ring homomorphism. The following lemma says precisely which subalgebra its image is.

Lemma 3.2.13. *Let $\psi \colon N \to M$ be a surjective R-module homomorphism. Then the map $\psi^{\#}$ is a graded isomorphism from $R[M]$ to the graded R-subalgebra of $R[N]$ whose degree-d part equals*

$$\{f \in R[N]_d \mid \forall u \in \ker(\psi) : f \circ t_u = f\}$$

where $t_u : N \to N$ (called translation by *u) is the affine-linear polynomial law $v \mapsto v + u$.*

Proof. Let $g \in R[M]_d$ and write $f = \psi^{\#}(g) = g \circ \psi$. To see that $\psi^{\#}$ is injective, note that $f_A = g_A \circ (\mathrm{id}_A \otimes \psi)$ for all R-algebras A. So if $f_A = 0$, then $g_A = 0$ as $\mathrm{id}_A \otimes \psi$ is surjective. To see that the image is contained in the subalgebra, it is enough to note that $\psi_A = \mathrm{id}_A \otimes \psi$ and $t_{u,A}(m) = m + 1 \otimes u$ and so $\psi \circ t_u = \psi$ as polynomial laws. Now, let $f \in R[N]_d$ be a polynomial law such that $f \circ t_u = f$ for all $u \in \ker(\psi)$. It remains to show that $f = g \circ \psi$ for some $g \in R[M]_d$. As $\mathrm{id}_A \otimes \psi$ is surjective, we set $g_A(m) := f_A(n)$ for any $n \in A \otimes N$ mapping to m. To do this, we need to show that $f_A(n) = f_A(n')$ whenever $n - n' \in \ker(\mathrm{id}_A \otimes \psi)$. Since the functor $A \otimes -$ from R-modules to A-modules is right-exact, we have $\ker(\mathrm{id}_A \otimes \psi) = A \otimes \ker(\psi)$. Take $h = f \circ ((n, n') \mapsto n + n')$. Then we see that

$$h_A(n, 1 \otimes u) = f_A(n + 1 \otimes u) = (f \circ t_u)_A(n) = f_A(n) = h_A(n, 0)$$

for all R-algebras A, $n \in A \otimes N$ and $u \in \ker(\psi)$. It follows that $h_{(i,j),A}(n, 1 \otimes u) = 0$ whenever $j > 0$. And, we have $h_{(d,0),A}(n, n') = f_A(n)$. So

$$\begin{aligned} f_A(n + a \otimes u) &= h_A(n, a \otimes u) \\ &= h_{(d,0),A}(n, a \otimes u) + \textstyle\sum_{i=1}^d h_{(d-i,i),A}(b, a \otimes u) \\ &= f_A(n) + \textstyle\sum_{i=1}^d a^i h_{(d-i,i),A}(b, 1 \otimes u) \\ &= f_A(n) \end{aligned}$$

for all $n \in A \otimes N$, $a \in A$ and $u \in \ker(\psi)$. So if $n - n' \in \ker(\mathrm{id}_A \otimes \psi)$, then $f_A(n) = f_A(n')$. This shows g_A is well-defined. It is straightforward to check that $g = (g_A)_A$ is a homogeneous polynomial law of degree d. □

Example 3.2.14. When R is an infinite field and both M and N are finite-dimensional vector spaces over R, $R[M]$ is just the subring of $R[N]$ consisting of all polynomials that are constant on fibres of the projection $N \to M$. ♫

The following example shows that, even when R is Noetherian and M is finitely generated, $R[M]$ need not be Noetherian.

Example 3.2.15. Let $R := K[t]/(t^2)$ where K is a field of characteristic zero, and let $M := K[t]/(t)$. Then $M = R/(t)$ is an R-module generated by a single element $v := 1 + (t)$ and $R[M]$ is the subring of $R[x]$ spanned by all homogeneous polynomials $f = cx^d$ such that $f(x + at) = f(x)$ for all $a \in K$. Now $c(x + at)^d = cx^d + cdatx^{d-1}$ and hence we need that $c \in (t)$ whenever $d \geq 1$. Hence $R[M]$ is the vector space over K spanned by $1, t, tx, tx^2, \dots$ with the multiplication $(t^i x)(t^i x) = 0$. Observe that $R[M]$ is not Noetherian, since the ideal $\mathrm{span}\{t, tx, tx^2, \dots\}$ is not finitely generated. On the other hand, the quotient $R[M]^{\mathrm{red}}$ of $R[M]$ by its ideal of nilpotent elements is K. ♫

However, we will see later that if $\mathrm{Spec}(R)$ is Noetherian and M is finitely generated, then a certain topological space $\mathbb{A}_M$ defined using $R[M]$ is also Noetherian. In Example 3.2.15, this is a consequence of the fact that $\mathrm{Spec}(R[M]) = \mathrm{Spec}(K)$ is Noetherian. See also Remark 3.3.17.

Example 3.2.16. Now consider a field K of characteristic 2 and set $R := K[t]/(t^2)$. The same computation as above shows that cx^i with *odd* i can only be in $R[M] \subseteq R[x]$ if c

is in (t). But for *even* i, cx^i is in $R[M]$ regardless of $c \in R$. Hence $R[M]$ is the K-vector space with basis

$$1, t, tx, x^2, tx^2, tx^3, x^4, tx^4, \ldots$$

and $R[M]^{\mathrm{red}} \cong K[x^2]$ as a graded algebra. ♫

If B is an R-algebra, then the base change functor from Definition 3.2.10 sends polynomial laws $M \to R$ to polynomial laws $B \otimes M \to B$. This yields an R-algebra homomorphism $R[M] \to B[B \otimes M]$ and hence a B-algebra homomorphism $B \otimes R[M] \to B[B \otimes M]$. The following example shows that this needs not be an isomorphism.

Example 3.2.17. Let $R = \mathbf{Z}$ and $M = \mathbf{Z}/2\mathbf{Z}$, generated by a single element $v = 1+2\mathbf{Z}$. Then by Lemma 3.2.13, $R[M]$ is the subring of $R[x]$ spanned by all homogeneous univariate polynomials f such that $f(x+2a) = f(x)$ for all $a \in \mathbf{Z}$. Only the constant polynomials have that property, so $R[M] = R$. Now take the $\mathbf{Z}$-algebra $B = \mathbf{Z}/2\mathbf{Z} =: \mathbb{F}_2$, which is a field, and $B \otimes M$ is the one-dimensional vector space over that field, so $B[B \otimes M] \cong \mathbb{F}_2[x]$. ♫

However, when B is a localisation of a domain R, then the map *is* an isomorphism:

Proposition 3.2.18. *Suppose that R is a domain. Let M be a finitely generated R-module and let S be a multiplicative subset of R not containing* 0. *Set $R' := S^{-1}R$. Then*

$$R' \otimes R[M] \cong S^{-1}R[M] \cong R'[R' \otimes M] \cong R'[S^{-1}M].$$

Proof. The first and last isomorphisms are standard. For the middle isomorphism, we choose generators $m_1, \ldots, m_n$ of M and embed $R[M]$ as a graded R-subalgebra A of $R[x_1, \ldots, x_n]$. Since localisation is exact, $S^{-1}R[M]$ is then isomorphic to the R'-algebra $S^{-1}A \subseteq R'[x_1, \ldots, x_n]$. On the other hand, using the generators $1 \otimes m_1, \ldots, 1 \otimes m_n$, the R'-algebra $R'[R' \otimes M]$ also embeds as a graded R'-subalgebra B of $R'[x_1, \ldots, x_n]$. The canonical map $R' \otimes R[M] \to R'[R' \otimes M]$ translates into an inclusion $S^{-1}A \subseteq B$, so it remains to show that $B \subseteq S^{-1}A$. For this, let O be the kernel of the R-module homomorphism $R^n \to M$ given by the generators $m_1, \ldots, m_n$. Again since localisation is exact, $S^{-1}O \cong R' \otimes O$ is the kernel of the corresponding R'-module homomorphism $(R')^n \to R' \otimes M$. Let $f \in B$ and let $s \in S$ be such that $g := sf \in R[x_1, \ldots, x_n]$. Then, since $f \in B$, one has that $f \circ t_u = f$ for all $u \in S^{-1}O \subseteq (R')^n$, by Lemma 3.2.13 applied to the R'-module $R' \otimes M$. In particular, the multiplication by s gives $g \circ t_u = g$ over R' for all $u \in O \subseteq R^n$. Since R is a domain, the same holds over R and hence $g \in A$, again by Lemma 3.2.13 but now applied to the R-module M. Hence $f = s^{-1}g \in S^{-1}A$, as desired. □

Like in ordinary algebraic geometry, the coordinate ring of a direct sum is the tensor product of the coordinate rings.

Proposition 3.2.19. *Let M, N be finitely generated R-modules. Then*

$$R[M \oplus N] \cong R[M] \otimes R[N].$$

Proof. Elements of $R[M]$ and $R[N]$ induce elements of $R[M \oplus N]$ via composition with the projections $M \oplus N \to M$ and $M \oplus N \to N$, respectively. The product of such induced polynomial laws $M \oplus N \to R$ gives a bilinear map $R[M] \times R[N] \to R[M \oplus N]$. This induces an R-linear map $R[M] \otimes R[N] \to R[M \oplus N]$, which is in fact a homomorphism of R-algebras. Denote by $R[M \oplus N]_{(d,e)}$ the R-submodule of $R[M \oplus N]$ consisting of all bihomogeneous polynomial laws of degree (d, e). It suffices to show that $R[M \oplus N]_{(d,e)} \cong R[M]_d \otimes R[N]_e$. To see this, first suppose that M, N are free. In this case, we get $R[x_1, \ldots, x_n, y_1, \ldots, y_m]_{(d,e)} \cong R[x_1, \ldots, x_n]_d \otimes R[y_1, \ldots, y_m]_e$ when x_i, y_j have degrees $(1, 0)$, $(0, 1)$, respectively. In general, let $\varphi\colon M' \to M$ and $\psi\colon N' \to N$ be surjective R-linear maps from finitely generated free R-modules. Then we see that

$$\{f \in R[M' \oplus N']_{(d,e)} \mid \forall u_1 \in \ker(\varphi)\forall u_1 \in \ker(\psi) : f \circ t_{(u_1,u_2)} = f\} \cong$$

$$\{f \in R[M']_d \mid \forall u_1 \in \ker(\varphi) : f \circ t_{u_1} = f\} \otimes \{g \in R[N']_e \mid \forall u_1 \in \ker(\psi) : g \circ t_{u_2} = g\}$$

and hence $R[M \oplus N]_{(d,e)} \cong R[M]_d \otimes R[N]_e$. □

Example 3.2.15 shows that the coordinate ring of a module is quite a subtle notion. However, we will see that in the proof of our Theorem 4.1.1, by a localisation we can always pass to a case where the module M is free. In that case, by Lemma 3.2.13, $R[M]$ is just a polynomial ring over R.

3.3 The topological space $\mathbb{A}_M$

We first make precise what we mean with a topological space over a category. Denote by $\mathrm{Forget}_{\mathcal{D}}$ the forgetful functor from the category $\mathcal{D}$ to **Set** (when it exists). For convenience, if P is a functor from $\mathcal{C}$ to $\mathcal{D}$, we denote by P the composition $\mathrm{Forget}_{\mathcal{D}} \circ P$.

Definition 3.3.1. Let $F : \mathcal{C} \to \mathbf{Set}$ be a functor. A *subset* X of F is a functor from $\mathcal{C} \to \mathbf{Set}$ such that $X(C) \subset F(C)$ for every $C \in \mathcal{C}$ and $X_{C,D}(\varphi) = F_{C,D}(\varphi)_{|X(C)}$ for every $C, D \in \mathcal{C}$ and for every $\varphi \in \mathrm{Hom}_{\mathcal{C}}(C, D)$. ♪

Remark 3.3.2. Let X and Y be subsets of a functor $P : \mathcal{C} \to \mathbf{Set}$. The *union* $X \cup Y$ of X and Y is the subset $(X \cup Y)(A) = X(A) \cup Y(A)$. Analogously, one defines the *intersection* $X \cap Y$ of X and Y, infinite unions, and infinite intersections. Moreover, we denote by $\emptyset$ the subset of P such that $\emptyset(A) = \emptyset \subset P(A)$ for every $A \in \mathcal{C}$. ♪♫

Definition 3.3.3. Consider a functor $P : \mathcal{C} \to \mathbf{Set}$ and let I be an index set. A *topology* $\mathcal{T}$ on P is given by a collection of subsets $\{X_i\}_{i \in I}$ of P containing P and $\emptyset$, and that is closed under taking arbitrary intersections and finite unions. We say that the couple $(P, \mathcal{T})$ is a *topological space* over $\mathcal{C}$. ♪

Remark 3.3.4. A functor P with a topology $\mathcal{T}$ of closed subsets $\{X_i\}_{i\in I}$ gives rise to a functor $P : \mathcal{C} \to \mathbf{Top}$, assigning to an object C the topological space $P(C)$ with $\{X_i(C)\}_{i\in I}$ as collection of closed subsets. ♪♫

3.3.1 The space $\mathbb{A}_M$

We now construct the topological space $\mathbb{A}_M$ for M a finitely generated R-module. To be precise, $\mathbb{A}_M$ is a topological space over the category $\mathbf{Dom}_R$ of R-domains with R-algebra monomorphisms, in the sense of Definition 3.3.3. In what follows, we use the term *"injections"* to refer to R-algebra monomorphisms.

Definition 3.3.5. Define $\mathbb{A}_M$ to be the rule assigning to each $D \in \mathbf{Dom}_R$ the set $D \otimes M$. A subset of $\mathbb{A}_M$ is a rule X that assigns to each $D \in \mathbf{Dom}_R$ a subset $X(D)$ of $D \otimes M$ in such a manner that $\iota \otimes \mathrm{id}_M$ maps $X(D)$ into $X(E)$ for all injections $\iota\colon D \to E$. For every subset $S \subseteq R[M]$, the rule $\mathcal{V}(S)$ assigning

$$D \mapsto \mathcal{V}(S)(D) := \{m \in D \otimes M \mid \forall f \in S : f_D(m) = 0\}$$

is a subset of $\mathbb{A}_M$. We say that $X \subseteq \mathbb{A}_M$ is *closed* if $X = \mathcal{V}(S)$ for some $S \subseteq R[M]$. This collection of closed turns $\mathbb{A}_M$ into a topological space over $\mathbf{Dom}_R$ in the sense of Definition 3.3.3. We call this topology the *Zariski topology* on $\mathbb{A}_M$. ♪

Remark 3.3.6. If D is an R-domain, then we can make $D \otimes M$ into an topological space by defining the closed subsets to be $\mathcal{V}(S)(D)$ for $S \subseteq R[M]$; we will call this the Zariski topology (over R) on $D \otimes M$. To see that these sets are preserved under finite unions, one uses $\mathcal{V}(S)(D) \cup \mathcal{V}(T)(D) = \mathcal{V}(S \cdot T)(D)$, which holds since D is a domain. For any R-algebra homomorphism $D \to E$ between R-domains (not necessarily injective), the induced map $D \otimes M \to E \otimes M$ sends $\mathcal{V}(S)(D)$ into $\mathcal{V}(S)(E)$. Furthermore, if $D \to E$ is injective, then that induced map is continuous with respect to the topologies on $D \otimes M$ and $E \otimes M$. So $\mathbb{A}_M$ induces a functor from $\mathbf{Dom}_R$ to $\mathbf{Top}$ and the $\mathcal{V}(S)$ are closed subsets. We will not consider closed subsets of $D \otimes M$ on their own. ♪♫

Remark 3.3.7. We think of $\mathbb{A}_M$ as the "affine space" corresponding to M. Note that in the definition of closed subsets of $\mathbb{A}_M$ we require S to be independent of D, i.e., not every rule assigning to $D \in \mathbf{Dom}_R$ a subset of the form $\mathcal{V}(S)(D)$ is a closed subset of $\mathbb{A}_M$. To see that this is desirable, consider $R = \mathbf{Z}$, $M = R$ and let X_n be the rule such that $X_n(D) = \{0\} = \mathcal{V}(\{x\})(D)$ when $0 < \operatorname{char} D \leq n$ and $X_n(D) = D = \mathcal{V}(\emptyset)(D)$ otherwise. Then $X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$ is a descending chain of rules and $X_{p-1}(\mathbb{F}_p) = \mathbb{F}_p \neq \{0\} = X_p(\mathbb{F}_p)$ for every prime number $p > 0$. ♪♫

Definition 3.3.8 (Base change)**.** If B is an R-algebra, and D is a B-domain, then $D \otimes M \cong D \otimes_B (B \otimes M)$ also carries a Zariski topology over B, coming from closed sets defined by subsets of $B[B \otimes M]$. This refines the Zariski topology on $D \otimes M$ over R. If X is a closed subset of $\mathbb{A}_M$, then we write X_B for the closed subset of $\mathbb{A}_{B\otimes M}$ that maps a B-domain D to $X(D)$. ♪

Let X be a subset of $\mathbb{A}_M$. Then we define the *ideal* of X to be

$$\mathcal{I}_X := \{f \in R[M] \mid \forall D \in \mathbf{Dom}_R \forall x \in X(D) : f_D(x) = 0\}$$

As f_D maps elements into a domain, we see that $\mathcal{I}_X$ is a radical ideal of $R[M]$. We define the *closure* of X in $\mathbb{A}_M$ to be the closed subset $\overline{X} := \mathcal{V}(\mathcal{I}_X)$ of $\mathbb{A}_M$.

Let $\varphi\colon M \to N$ be a polynomial law between finitely generated R-modules. Then the maps $(\varphi_D)_{D\in\mathbf{Dom}_R}$ define a continuous map $\mathbb{A}_M \to \mathbb{A}_N$, i.e., for every injection $\iota\colon D \to E$, the diagram

$$\begin{tikzcd} \mathbb{A}_M(D) \arrow[r, "\varphi_D"] \arrow[d, "\iota\otimes\mathrm{id}_M"'] & \mathbb{A}_N(D) \arrow[d, "\iota\otimes\mathrm{id}_N"] \\ \mathbb{A}_M(E) \arrow[r, "\varphi_E"] & \mathbb{A}_N(E) \end{tikzcd}$$

commutes, so $\varphi(X) = (D \mapsto \varphi_D(X(D)))$ is a subset of $\mathbb{A}_N$ for each subset X of $\mathbb{A}_M$, and for every subset $S \subseteq R[N]$, the subset

$$\varphi^{-1}(\mathcal{V}(S)) = (D \mapsto \varphi_D^{-1}(\mathcal{V}(S)(D)))_D$$

of $\mathbb{A}_M$ is closed (as $\varphi_D^{-1}(\mathcal{V}(S)(D)) = \mathcal{V}(\varphi^{\#}S)(D)$ holds). As usual, we have

$$\varphi(\overline{X}) \subseteq \overline{\varphi(X)}$$

for all subsets X of $\mathbb{A}_M$.

When M is free and finitely generated, we have the usual correspondence between closed subsets and radical ideals.

Proposition 3.3.9. *Let M be a finitely generated free R-module of rank n. Then the rule sending an element $x \in D \otimes M$ of $\mathbb{A}_M$ to $\mathfrak{q}_x := \{f \in R[M] \mid f_D(x) = 0\} \in \mathbb{A}_R^n :=$ $\mathrm{Spec}(R[M])$ is surjective and maps closed subsets of $\mathbb{A}_M$ to closed subsets of $\mathbb{A}_R^n$. Moreover, that map from closed subsets of $\mathbb{A}_M$ to closed subsets of $\mathbb{A}_R^n$ is a bijection. In particular, we have $\mathcal{I}_{\mathcal{V}(S)} = \mathrm{rad}(S)$ for any subset $S \subseteq R[M]$.*

Proof. Note that for every R-domain D and element $x \in D \otimes M$, the set $\mathfrak{q}_x \subseteq R[M]$ is a prime ideal. Let $\mathfrak{q} \subseteq R[M] = R[x_1, \ldots, x_n]$ be a prime ideal. Then we have $\mathfrak{q} = \mathfrak{q}_x$ for $x = (x_1 + \mathfrak{q}, \ldots, x_n + \mathfrak{q}) \in (R[M]/\mathfrak{q}) \otimes M$. Next, let $S \subseteq R[M]$ be a set. Then we see that $\{\mathfrak{q}_x \mid x \in \mathcal{V}(S)(D), D \in \mathbf{Dom}_R\} = \{\mathfrak{q} \in \mathrm{Spec}(R[M]) \mid \mathfrak{q} \supseteq S\}$. So closed subsets of $\mathbb{A}_M$ are mapped to closed subsets of $\mathbb{A}_R^n$. Clearly, every closed subset arises from a closed subset of $\mathbb{A}_M$. To see that this map is injective, we note that

$$\mathcal{I}_{\mathcal{V}(S)} = \bigcap_{\substack{x\in\mathcal{V}(S)(D)\\ D\in\mathbf{Dom}_R}} \mathfrak{q}_x = \bigcap_{\substack{\mathfrak{q}\in\mathrm{Spec}(R[M])\\ \mathfrak{q}\supseteq S}} \mathfrak{q} = \mathrm{rad}(S) \text{ and } \mathcal{V}(S) = \mathcal{V}(\mathrm{rad}(S)).$$

Hence $\mathcal{V}(S)$ is uniquely determined by its associated subset of $\mathbb{A}^n_R$. $\square$

While we have defined closed subsets of $\mathbb{A}_M$ by looking at all R-domains D, it actually suffices to look at algebraic closures $\overline{K_\mathfrak{p}}$ where $\mathfrak{p} \in \operatorname{Spec}(R)$. For $\mathfrak{p} \in \operatorname{Spec}(R)$, we write $K_\mathfrak{p} := \operatorname{Frac}(R/\mathfrak{p})$ for the fraction field of $R/\mathfrak{p}$.

Proposition 3.3.10. *Let X be a subset of $\mathbb{A}_M$. Then*

$$\mathcal{I}_X = \bigcap_{\mathfrak{p} \in \operatorname{Spec}(R)} \left\{ f \in R[M] \mid f_{\overline{K_\mathfrak{p}}} \in \mathcal{I}_{X(\overline{K_\mathfrak{p}})} \right\}.$$

Proof. Clearly, the inclusion $\subseteq$ holds. Let $f \in R[M]$ be such that $f_{\overline{K_\mathfrak{p}}} \in \mathcal{I}_{X(\overline{K_\mathfrak{p}})}$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. Let D be an R-domain and let $\mathfrak{p}$ be the kernel of the homomorphism $R \to D$. Then there exists a field L containing $\operatorname{Frac}(D)$ and $\overline{K_\mathfrak{p}}$. By the Nullstellensatz, the fact that $f_{\overline{K_\mathfrak{p}}} \in \mathcal{I}_{X(\overline{K_\mathfrak{p}})}$ implies that $f_L \in \mathcal{I}_{X(L)}$. It follows that f_D vanishes on $X(D)$. $\square$

Corollary 3.3.11. *A closed subset X of $\mathbb{A}_M$ is uniquely determined by its values $X(\overline{K_\mathfrak{p}})$ where $\mathfrak{p}$ runs over* $\operatorname{Spec}(R)$.

Proof. This follows from the previous proposition since $X = \mathcal{V}(\mathcal{I}_X)$. $\square$

The proof of Theorem 4.1.1 follows a divide-and-conquer strategy in which the following two lemmas and their generalisations to closed subsets of polynomial functors (Lemmas 3.4.20 and 3.4.21), play a crucial role.

Lemma 3.3.12. *Let R be a ring with Noetherian spectrum and r an element of R. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the minimal primes of $R/(r)$. Then two closed subsets $X, Y \subseteq \mathbb{A}_M$ are equal if and only if $X_{R[1/r]} = Y_{R[1/r]}$ and $X_{R/\mathfrak{p}_i} = Y_{R/\mathfrak{p}_i}$ for all $i = 1, \dots, k$.*

Proof. Suppose that $X_{R[1/r]} = Y_{R[1/r]}$ and $X_{R/\mathfrak{p}_i} = Y_{R/\mathfrak{p}_i}$ for all $i = 1, \dots, k$. Let K be an R-field and let $R \to K$ be the corresponding homomorphism. If the image of r in K is zero, then $R \to K$ factors via $R/\mathfrak{p}_i$ for some $i = 1, \dots, k$ and hence K is a $(R/\mathfrak{p}_i)$-domain. In this case, we have $X(K) = X_{R/\mathfrak{p}_i}(K) = Y_{R/\mathfrak{p}_i}(K) = Y(K)$. If the image of r in K is nonzero, then K naturally is an $R[1/r]$-field. In this case, we have $X(K) = X_{R[1/r]}(K) = Y_{R[1/r]}(K) = Y(K)$. So $X = Y$ by Corollary 3.3.11. $\square$

Lemma 3.3.13. *Let $R \subseteq R'$ be a finite extension of domains and let $X, Y \subseteq \mathbb{A}_M$ be closed subsets. Then $X = Y$ if and only if $X_{R'} = Y_{R'}$.*

Proof. The extension $R \subseteq R'$ satisfies lying over, i.e., for every prime $\mathfrak{p} \in \operatorname{Spec}(R)$ there is a prime $\mathfrak{q} \in \operatorname{Spec}(R')$ with $\mathfrak{p} = \mathfrak{q} \cap R$. The lemma follows by Corollary 3.3.11. $\square$

3.3.2 Noetherianity of $\mathbb{A}_M$

We now prove that the topological space $\mathbb{A}_M$ is Noetherian.

Proposition 3.3.14. *If R has a Noetherian spectrum and M is a finitely generated R-module, then the topological space $\mathbb{A}_M$ over $\mathbf{Dom}_R$ is Noetherian.*

Thus let R be a ring.

Lemma 3.3.15. *If* $\mathrm{Spec}(R)$ *is Noetherian, then so is* $\mathrm{Spec}(R[x])$.

Proof. This is an application of [ESS20, Theorem 1.1] with trivial group. □

Lemma 3.3.16. *Assume that* $\mathrm{Spec}(R)$ *is Noetherian and set $N := R^n$. Then $\mathbb{A}_N$ is Noetherian, i.e., any chain $X_1 \supseteq X_2 \supseteq \cdots$ of closed subsets of $\mathbb{A}_N$ stabilises eventually.*

Proof. Consider the chain $\mathcal{I}_{X_1} \subseteq \mathcal{I}_{X_2} \subseteq \cdots$ of radical ideals in $R[N] \cong R[x_1, \ldots, x_n]$. Since the latter ring has a topological spectrum, this chain stabilises. Since $X_i = \mathcal{V}(\mathcal{I}_{X_i})$, so does the chain $X_1 \subseteq X_2 \subseteq \cdots$. □

Proof of Proposition 3.3.14. Let R be a ring with Noetherian spectrum, let M be a finitely generated R-module, and let $X_1 \supseteq X_2 \supseteq \cdots$ be a chain of closed subsets of $\mathbb{A}_M$. Since M is finitely generated, there exists a surjective R-module homomorphism $\varphi\colon N := R^n \to M$ for some n. This defines a (linear) polynomial law $N \to M$ and so a continuous map $\mathbb{A}_N \to \mathbb{A}_M$. Set $Y_i := \varphi^{-1}(X_i)$, which is the closed subset of $\mathbb{A}_N$ such that $Y_i(D) = (1 \otimes \varphi)^{-1}(X_i(D))$ for all R-domains D. By Lemma 3.3.16, the chain $Y_1 \supseteq Y_2 \supseteq \cdots$ stabilises, i.e., $Y_n = Y_{n+1}$ for all $n \gg 0$. So, since $1 \otimes \varphi\colon D \otimes N \to D \otimes M$ is surjective for every R-domain D, we have $X_i(D) = (1 \otimes \varphi)(Y_i(D))$ for every i and D, and therefore $X_n = X_{n+1}$ for all $n \gg 0$. □

Remark 3.3.17. If two ideals I and J in $R[M]$ define the same closed subset in $\mathrm{Spec}(R[M])$, then they have the same radical and hence define the same closed subset in $\mathbb{A}_M$. But it could possibly happen that two ideals that define the same closed subset in $\mathbb{A}_M$ do *not* define the same closed subset in $\mathrm{Spec}(R[M])$. In particular, the proof above does not show that $\mathrm{Spec}(R[M])$ is a Noetherian topological space. Indeed, we don't know whether this is the case. ♪♫

Question 3.3.18. *Suppose that* $\mathrm{Spec}(R)$ *is Noetherian and let M be a finitely generated R-module. Is* $\mathrm{Spec}(R[M])$ *Noetherian? Is the map from radical ideals of $R[M]$ to closed subsets of $\mathbb{A}_M$ a bijection?*

3.3.3 Dimension

Proposition 3.3.19. *Let R be a domain, let M be a finitely generated R-module and let X be a closed subset of $\mathbb{A}_M$. Then the function*

$$\begin{aligned}\operatorname{Spec}(R) &\to \mathbf{Z}_{\geq -1}\\ \mathfrak{p} &\mapsto \dim_{\overline{K_\mathfrak{p}}}(X(\overline{K_\mathfrak{p}}))\end{aligned}$$

is constant in some open dense subset $\operatorname{Spec}(R[1/r])$ *of* $\operatorname{Spec}(R)$.

Proof. By Lemma 3.1.1, there exists a nonzero $r \in R$ such that $R[1/r] \otimes M$ is free. It suffices to prove the statement for the domain $R[1/r]$, the $R[1/r]$-module $R[1/r] \otimes M$ and the closed subset $X_{R[1/r]}$ of $\mathbb{A}_{R[1/r]\otimes M}$. So we may assume that M is free, say of rank m, and so X is a closed subset of $\mathbb{A}^m_R$; let $I \subseteq R[x_1, \ldots, x_m]$ be its vanishing ideal. Choose an arbitrary monomial order on monomials in $x_1, \ldots, x_m$. For each nonzero $r \in R$, let M_r be the set of leading monomials of *monic* polynomials in $R[1/r] \otimes I$; this is an upper ideal in the monoid of monomials. By Dickson's lemma applied to the m-tuples in $\mathbf{N}^m$ of the exponents of monomials, there exists an r such that M_r is inclusion-wise maximal. Choose monic polynomials $f_1, \ldots, f_k \in R[1/r][x_1, \ldots, x_n]$ whose leading monomials generate the upper ideal M_r. Then $f_1, \ldots, f_k$ generate the ideal $R[1/r] \otimes I$—indeed, otherwise there would be some element f in the latter ideal whose leading monomial is not divisible by any of the leading monomials of the f_i; and letting r' be the leading coefficient of f we would find that $M_{rr'}$ strictly contains M_r, a contradiction. Moreover, again by maximality of M_r, the f_i satisfy Buchberger's criterion: every S-polynomial of them reduces to zero modulo $f_1, \ldots, f_k$ when working over $R[1/r][x_1, \ldots, x_m]$. Then for each $\mathfrak{p} \in \operatorname{Spec}(R[1/r])$, the images of the f_i generate the ideal $K_\mathfrak{p} \otimes I = K_\mathfrak{p} \otimes_{R[1/r]} (R[1/r] \otimes I)$; and still satisfy Buchberger's criterion. Hence these images form a Gröbner basis, and since the dimension of $X(\overline{K_\mathfrak{p}})$ can be read of from the set of leading monomials, that dimension is constant for $\mathfrak{p} \in \operatorname{Spec}(R[1/r])$. □

Proposition 3.3.20. *Let R be a domain, M a finitely generated R-module, and X a closed subset of $\mathbb{A}_M$. Then there exists a nonzero $r \in R$ such that the following holds: for any $f \in R[M]$, if f vanishes identically on $X(\overline{K})$, then f vanishes identically on $X(\overline{K_\mathfrak{p}})$ for all* $\mathfrak{p} \in \operatorname{Spec}(R[1/r])$.

Proof. As in the previous proof, it suffices to prove the statement in the case that M is free of rank m. Let $I \subseteq R[x_1, \ldots, x_m]$ be the vanishing ideal of X. This time, for each nonzero $r \in R$, let M_r be the set of leading monomials of *monic* polynomials in $R[1/r][x_1, \ldots, x_m]$ *some power of which* lies in $R[1/r] \otimes I$. Choose r such that M_r is maximal, and $f_1, \ldots, f_k \in R[1/r][x_1, \ldots, x_m]$ monic, whose powers lie in $R[1/r] \otimes I$, and whose leading monomials generate the upper ideal M_r. Then the images of $f_1, \ldots, f_k$ form a Gröbner basis of the radical ideal of $K \otimes I$. Now assume that $f \in R[M]$ vanishes identically on $X(\overline{K})$, and let g be the image of f in $R[1/r][x_1, \dots, x_m]$. Then by the Nullstellensatz, some power of g reduces to zero modulo $f_1, \dots, f_k$. But then that reduction holds modulo $\mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Spec}(R[1/r])$, so g vanishes identically on $X(\overline{K_{\mathfrak{p}}})$ for all such $\mathfrak{p}$. □

3.4 Polynomial functors and their properties

3.4.1 Polynomial functors over a ring

For reasons that will become clear later, we will only be interested in polynomial functors from the category $\mathbf{fgfMod}_R$ of finitely generated free R-modules into either $\mathbf{Mod}_R$, the category of modules over R, or $\mathbf{fgMod}_R$, the category of finitely generated modules over R. In all these categories morphisms are R-module homomorphism.

Definition 3.4.1. A *polynomial functor* $P\colon \mathbf{fgfMod}_R \to \mathbf{Mod}_R$ consists of an object $P(U) \in \mathbf{Mod}_R$ for each object $U \in \mathbf{fgfMod}_R$ and a polynomial law

$$P_{U,V}\colon \mathrm{Hom}(U, V) \to \mathrm{Hom}(P(U), P(V))$$

for each $U, V \in \mathbf{fgfMod}_R$ such that the diagram

$$\begin{tikzcd}
\mathrm{Hom}(V, W) \oplus \mathrm{Hom}(U, V) \arrow[r, "- \circ -"] \arrow[d, "P_{V,W} \oplus P_{U,V}"] & \mathrm{Hom}(U, W) \arrow[d, "P_{U,W}"] \\
\mathrm{Hom}(P(V), P(W)) \oplus \mathrm{Hom}(P(U), P(V)) \arrow[r, "- \circ -"] & \mathrm{Hom}(P(U), P(W))
\end{tikzcd}$$

commutes for every $U, V, W \in \mathbf{fgfMod}_R$. Here the bilinear horizontal polynomial laws are given as in Remark 3.2.7. Moreover, for every $U \in \mathbf{fgfMod}_R$, we require that $P_{U,U}(\mathrm{id}_U) = \mathrm{id}_{P(U)}$ and we require that P has *finite (or bounded) degree*, i.e., there is a uniform bound $d \in \mathbf{Z}_{\geq 0}$ such that for all U, V the polynomial law $P_{U,V}$ has degree at most d. ♪

Polynomial functors $\mathbf{fgfMod}_R \to \mathbf{Mod}_R$ form an Abelian category $\mathbf{PF}_R$ in which a morphism $\alpha\colon Q \to P$ is given by an R-linear map $\alpha_U\colon Q(U) \to P(U)$ for each $U \in \mathbf{fgfMod}_R$ such that the diagram of polynomial laws

$$\begin{tikzcd}
\mathrm{Hom}(U, V) \arrow[r, "Q_{U,V}"] \arrow[d, "P_{U,V}"] & \mathrm{Hom}(Q(U), Q(V)) \arrow[d, "\alpha_V \circ -"] \\
\mathrm{Hom}(P(U), P(V)) \arrow[r, "- \circ \alpha_U"] & \mathrm{Hom}(Q(U), P(V))
\end{tikzcd}$$

commutes for all U, V. Note that post-composing with α_V and pre-composing with α_U are R-linear maps and hence, indeed, (linear) polynomial laws.

For every R-algebra A and R-modules U, V, W, let $- \circ_A -$ be the A-bilinear extension of the R-bilinear composition maps $-\circ-\colon \operatorname{Hom}(V, W)\times\operatorname{Hom}(U, V) \to \operatorname{Hom}(U, W)$. So $(- \circ_A -)_A$ is the polynomial law extending $- \circ -$. Then the diagram above says that

$$P_{U,V,A}(\varphi) \circ_A (1 \otimes \alpha_U) = (1 \otimes \alpha_V) \circ_A Q_{U,V,A}(\varphi) \tag{3.1}$$

for all R-algebras A and $\varphi \in A \otimes \operatorname{Hom}(U, V)$. Note that to check that the diagram commutes, it suffices to check that this equality holds for $A = R[x_1, \dots, x_n]$ and $\varphi = x_1 \otimes \varphi_1 + \cdots + x_n \otimes \varphi_n$ where $\varphi_1, \dots, \varphi_n$ is a basis of $\operatorname{Hom}(U, V)$.

Recall that for all R-modules U, V, there is a natural A-linear map

$$A \otimes \operatorname{Hom}(U, V) \to \operatorname{Hom}_A(A \otimes U, A \otimes V).$$

For $U, V \in \mathbf{fgfMod}_R$, this map is an isomorphism. Thus an element φ of $A\otimes\operatorname{Hom}(U, V)$ can be thought of as an "element of $\operatorname{Hom}(U, V)$ with coordinates in A". Viewing $Q_{U,V,A}(\varphi), P_{U,V,A}(\varphi)$ as maps, (3.1) implies that the diagram

$$\begin{tikzcd}
A \otimes Q(U) \arrow[r, "\alpha_{U,A}"] \arrow[d, "Q_{U,V,A}(\varphi)"] & A \otimes P(U) \arrow[d, "P_{U,V,A}(\varphi)"] \\
A \otimes Q(V) \arrow[r, "\alpha_{V,A}"] & A \otimes P(V)
\end{tikzcd}$$

commutes; here $\alpha_{U,A}$ is the A-linear extension of α_U. When A is a polynomial ring over R, the map

$$A \otimes \operatorname{Hom}(Q(U), P(V)) \to \operatorname{Hom}_A(A \otimes Q(U), A \otimes P(V))$$

is injective and so the reverse implication also holds. So the family $(\alpha_U)_U$ is a morphism of polynomial functors if and only if the last diagram above commutes for all A, U, V, φ. This is closer to the definition of polynomial functors over infinite fields, and generalises as follows.

Definition 3.4.2. Let P, Q be polynomial functors. We define a *polynomial transformation* $\alpha\colon Q \to P$ be a rule assigning to every $U \in \mathbf{fgfMod}_R$ a polynomial law $\alpha_U\colon Q(U) \to P(U)$ such that the last diagram above commutes for all R-algebras A and $\varphi \in A \otimes \operatorname{Hom}(U, V)$. ♪

Just like polynomial laws generalise R-module homomorphisms, and the latter are precisely the linear polynomial laws, polynomial transformations generalise morphisms of polynomial laws, and the latter are precisely the linear polynomial transformations.

Remark 3.4.3. If R is an infinite field, then a polynomial functor

$$P\colon \mathbf{fgfMod}_R \to \mathbf{fgMod}_R = \mathbf{fgfMod}_R$$

is the same thing as a functor from the category of finite-dimensional R-vector spaces to itself such that for all $U, V \in \mathbf{fgfMod}_R$ the map

$$P_{U,V}\colon \operatorname{Hom}(U, V) \to \operatorname{Hom}(P(U), P(V))$$

is a polynomial map. This is the set-up of Chapter 2. If R is a field but not necessarily infinite, then a polynomial functor $\mathbf{fgfMod}_R \to \mathbf{fgfMod}_R$ is a strict polynomial functor in the sense of Friedlander-Suslin [FS97]. ♪♫

Many of our proofs will involve passing to the case of (infinite) fields and invoking arguments from [Dra19]. This is facilitated by the following construction.

Definition 3.4.4 (Base change)**.** Let B be an R-algebra and let $P\colon \mathbf{fgfMod}_R \to \mathbf{Mod}_R$ be a polynomial functor. Then P induces a polynomial functor P_B from $\mathbf{fgfMod}_B$ to $\mathbf{Mod}_B$ as follows: first, for each finitely generated free B-module U fix a B-module isomorphism $\psi_U\colon U \to B \otimes U_R$, where U_R is a free R-module of the same R-rank as the B-rank of U. Then, set $P_B(U) := B \otimes P(U_R)$. Next, for each B-algebra A, we need to assign to every $\varphi \in A \otimes_B \operatorname{Hom}_B(U, V)$ an image in $A \otimes \operatorname{Hom}_B(P_B(U), P_B(V))$. For this, note that

$$\begin{aligned} A \otimes_B \operatorname{Hom}_B(U, V) &\cong A \otimes_B \operatorname{Hom}_B(B \otimes U_R, B \otimes V_R) \\ &\cong A \otimes_B (B \otimes \operatorname{Hom}(U_R, V_R)) \\ &\cong A \otimes \operatorname{Hom}(U_R, V_R), \end{aligned}$$

where the isomorphism in the first step is $1_A \otimes_B (\psi_V \circ - \circ \psi_U^{-1})$ and the second isomorphism follows from the freeness of U_R and V_R. Via these isomorphisms, φ is mapped to an element of $A \otimes \operatorname{Hom}(U_R, V_R)$. Applying $P_{U_R,V_R,A}$ to this element yields an element of $A \otimes \operatorname{Hom}(P(U_R), P(V_R)) \cong A \otimes_B (B \otimes \operatorname{Hom}(P(U_R), P(V_R)))$, and applying the natural map $B \otimes \operatorname{Hom}(P(U_R), P(V_R)) \to \operatorname{Hom}_B(B \otimes P(U_R), B \otimes P(V_R))$ in the second factor (which may not be an isomorphism since $P(U_R), P(V_R)$ need not be free) yields an element of $A \otimes_B \operatorname{Hom}_B(P_B(U), P_B(V))$. It is straightforward to check that P_B thus defined is a polynomial functor from $\mathbf{fgfMod}_B$ to $\mathbf{Mod}_B$. A different choice of isomorphisms ψ_U yields a different but isomorphic polynomial functor P_B. ♪

Remark 3.4.5. In this construction we have made use of the fact that P is a polynomial functor from finitely generated *free* R-modules to R-modules. The choice of ψ_U's could have been avoided as follows: instead of working with $\mathbf{fgfMod}_R$, we could have worked with the category whose objects are finite sets and whose morphisms $J \to I$ are given by $I \times J$ matrices with entries in R. Then $P_{J,I}$ would have been a polynomial law from the module of $I \times J$ matrices to $\operatorname{Hom}(P(J), P(I))$. However, the set-up we chose stresses better that we are interested in phenomena that do *not* depend on a choice of basis in our free modules. ♪♫

Definition 3.4.6. A polynomial functor $P\colon \mathbf{fgfMod}_R \to \mathbf{Mod}_R$ is called *homogeneous of degree d* if the polynomial law $P_{U,V}$ is homogeneous of degree d for each $U, V \in \mathbf{fgfMod}_R$. ♪

Every polynomial functor $P\colon \mathbf{fgfMod}_R \to \mathbf{Mod}_R$ is a direct sum $P_0 \oplus \cdots \oplus P_d$, where $P_i\colon \mathbf{fgfMod}_R \to \mathbf{Mod}_R$ is the homogeneous polynomial functor of degree i given on objects by $P_i(V) := \{v \in P(V) \mid P_{V,V,R[t]}(t \otimes \mathrm{id}_V)(v) = t^i \otimes v\}$; and $P_{i,U,V}$ is the restriction of the degree-i component of the polynomial law $P_{U,V}$ to $P_i(U)$. Here we identify $R[t] \otimes \mathrm{Hom}(P(V), P(V))$ with $\mathrm{Hom}(P(V), R[t] \otimes P(V))$. We might denote with $P_{e,B}$ the base change $(P_e)_B$ of P_e, the e-th component of P, with the R-algebra B.

3.4.2 Duality

Definition 3.4.7. Let $P\colon \mathbf{fgfMod}_R \to \mathbf{Mod}_R$ be a polynomial functor over R. Then we obtain another polynomial functor $P^*\colon \mathbf{fgfMod}_R \to \mathbf{Mod}_R$ by setting, for each $V \in \mathbf{fgfMod}_R$, $P^*(V) := P(V^*)^* = \mathrm{Hom}(P(V^*), R)$ and for each $\varphi \in A \otimes \mathrm{Hom}(U, V)$,

$$P^*_{U,V,A}(\varphi) := P_{V^*,U^*,A}(\varphi^\#)^\#,$$

where $\varphi^\#$ is the image of φ under the natural isomorphism

$$A \otimes \mathrm{Hom}(U, V) \cong A \otimes \mathrm{Hom}(V^*, U^*)$$

(here we use that U, V are free) and the outermost $*$ again represents a dual. ♪

The *dual functor* P^* of P has the same degree as P and will play a role in Section 4.2.10. To avoid having too many stars, we will there think of it as the functor that sends V^* to $P(V)^*$. If P takes values in $\mathbf{fgfMod}_R$, then $(P^*)^*$ is canonically isomorphic to P.

3.4.3 Shifting

Let U be a finitely generated free R-module.

Definition 3.4.8. We define the *shift functor* $\mathrm{Sh}_U\colon \mathbf{fgfMod}_R \to \mathbf{fgfMod}_R$ that sends $V \mapsto U \oplus V$ and $\varphi \mapsto \mathrm{id}_U \oplus \varphi$. For a polynomial functor $P\colon \mathbf{fgfMod}_R \to \mathbf{fgMod}_R$ we set $\mathrm{Sh}_U(P) := P \circ \mathrm{Sh}_U$, called the *shift of P by U*. ♪

Lemma 3.4.9. *The composition* $\mathrm{Sh}_U(P)$ *is again a polynomial functor* $\mathbf{fgfMod}_R \to \mathbf{fgMod}_R$*, the projection $U \oplus V \to V$ yields a surjection of polynomial functors* $\mathrm{Sh}_U(P) \to P$ *and inclusion the $V \to U \oplus V$ yields a section $P \to$* $\mathrm{Sh}_U(P)$ *to that surjection. In particular,* $\mathrm{Sh}_U(P) \cong P \oplus (\mathrm{Sh}_U(P)/P)$. *Furthermore,* $\mathrm{Sh}_U(P)/P$ *has degree strictly smaller than the degree of P.*

Proof. The proof in [Dra19, Lemma 14] (in the case where R is an infinite field) carries over to the current more general setting. □

3.4.4 Dimension functions of polynomial functors

Let $P\colon \mathbf{fgfMod}_R \to \mathbf{fgMod}_R$ be a polynomial functor. For $\mathfrak{p} \in \mathrm{Spec}(R)$, set $f_{\mathfrak{p}}(n) := \dim_{K_{\mathfrak{p}}}(K_{\mathfrak{p}} \otimes P(R^n))$. It turns out that these functions are polynomials in n, and depend semicontinuously on $\mathfrak{p}$. To formalise this semicontinuity, we order polynomials in $\mathbf{Z}[x]$ by $f \geq g$ if $f(n) \geq g(n)$ for all $n \gg 0$; this is the lexicographic order on coefficients.

Proposition 3.4.10. *For each $\mathfrak{p} \in \mathrm{Spec}(R)$ the function $f_{\mathfrak{p}}\colon \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ is a polynomial with integral coefficients of degree at most the degree of P. Furthermore, the map $\mathfrak{p} \mapsto f_{\mathfrak{p}}$ is upper semicontinuous on $\mathrm{Spec}(R)$ in a strong sense: both the sets $\{\mathfrak{p} \mid f_{\mathfrak{p}} \geq f\}$ and $\{\mathfrak{p} \mid f_{\mathfrak{p}} > f\}$ are closed for all $f \in \mathbf{Z}[x]$.*

Proof. We proceed by induction on the degree of P. If P has degree 0, then $P(R^n)$ is a fixed R-module U, and $f_{\mathfrak{p}}$ is the constant polynomial that maps n to $\dim_{K_{\mathfrak{p}}}(K_{\mathfrak{p}} \otimes U)$. In this case, if $f \in \mathbf{Z}[x]$ has positive degree, then $f_{\mathfrak{p}} > f$ and $f_{\mathfrak{p}} \geq f$ are either both trivially true for all $\mathfrak{p}$ or both trivially false for $\mathfrak{p}$ (depending on the sign of the leading coefficient of f), so we need only look at constant f.

In this case, the result is known; we recall the argument. Let $R^n \to U$ be a surjective R-module homomorphism, and let N be its kernel. Since tensoring with $K_{\mathfrak{p}}$ is right-exact, $1 \otimes N$ spans the kernel of the surjection $K_{\mathfrak{p}}^n \to K_{\mathfrak{p}} \otimes U$ for each $\mathfrak{p}$.

The statement that $\dim_{K_{\mathfrak{p}}}(K_{\mathfrak{p}} \otimes U)$ is upper semicontinuous is therefore equivalent to the statement that dimension of the span of $1 \otimes N$ in $K_{\mathfrak{p}}^n$ is lower semicontinuous. And indeed, the locus where this dimension is less than k is defined by the vanishing of all $k \times k$ subdeterminants of all $k \times n$ matrices (with entries in R) whose rows are k elements of N.

For the induction step, assume that the proposition is true for all polynomial functors of degree $< d$ and assume that P has degree $d \geq 1$. Then consider the functor $\mathrm{Sh}_R(P)$, which by Lemma 3.4.9 is isomorphic to $P \oplus Q$ for $Q := \mathrm{Sh}_R(P)/P$ of degree $< d$.

By the induction hypothesis, the proposition holds for Q: the function $g_{\mathfrak{p}}(n) := \dim_{K_{\mathfrak{p}}}(K_{\mathfrak{p}} \otimes Q(R^n))$ equals a polynomial with integral coefficients for all $n \geq 0$, and $\mathfrak{p} \mapsto g_{\mathfrak{p}}$ is semicontinuous. Now we have

$$\begin{aligned} f_{\mathfrak{p}}(n+1) &= \dim_{K_{\mathfrak{p}}}(K_{\mathfrak{p}} \otimes P(R^1 \oplus R^n)) \\ &= \dim_{K_{\mathfrak{p}}}(K_{\mathfrak{p}} \otimes P(R^n)) + \dim_{K_{\mathfrak{p}}}(K_{\mathfrak{p}} \otimes Q(R^n)) \quad = f_{\mathfrak{p}}(n) + g_{\mathfrak{p}}(n). \end{aligned}$$

This means that $f_{\mathfrak{p}}(n)$ is the unique polynomial with $(\Delta f_{\mathfrak{p}})(n) := f_{\mathfrak{p}}(n+1) - f_{\mathfrak{p}}(n) = g_{\mathfrak{p}}(n)$ for $n \geq 0$ and $f_{\mathfrak{p}}(0) = \dim_{K_{\mathfrak{p}}}(K_{\mathfrak{p}} \otimes P(0))$; this $f_{\mathfrak{p}}$ has integral coefficients and degree at most d.

For the semi-continuity statement, note that $f_{\mathfrak{p}} \geq f$ is equivalent to either $g_{\mathfrak{p}} = \Delta f_{\mathfrak{p}} > \Delta f$, or else $g_{\mathfrak{p}} \geq \Delta f$ and moreover $f_{\mathfrak{p}}(0) \geq f(0)$. Both possibilities are closed conditions on $\mathfrak{p}$. Similarly, $f_{\mathfrak{p}} > f$ is equivalent to either $g_{\mathfrak{p}} > \Delta f$ or else $g_{\mathfrak{p}} \geq \Delta f$ and $f_{\mathfrak{p}}(0) > f(0)$, which, again, are closed conditions. □

3.4.5 Local freeness

We now generalise Lemma 3.1.1 to polynomial functors.

Proposition 3.4.11. *Let R be a domain, $P\colon \mathbf{fgfMod}_R \to \mathbf{fgMod}_R$ a polynomial functor and S a subobject of P in the larger category of polynomial functors $\mathbf{fgfMod}_R \to \mathbf{Mod}_R$. Then there exists a nonzero $r \in R$ such that $R[1/r] \otimes S(U)$ and $R[1/r] \otimes P(U)$ are finitely generated free $R[1/r]$-modules for all $U \in \mathbf{fgfMod}_R$, and the latter is a direct sum of the former and another free $R[1/r]$-module.*

Note that we do not claim that the complement is itself the evaluation of another subobject; i.e., $S_{R[1/r]}$ needs not be a summand of $P_{R[1/r]}$ in the category of polynomial functors over $R[1/r]$.

Proof. Again, we proceed by induction on the degree of P. If P has degree 0, then so does S and then the statement is just Lemma 3.1.1. Suppose that the degree of P is $d > 0$ and that the proposition holds for all polynomial functors of degree less than d.

By Lemma 3.4.9, for each n we have

$$P(R^{n+1}) = P(R^n) \oplus Q(R^n)$$

where $Q = \mathrm{Sh}_R(P)/P$ has degree $< d$. Similarly, we have

$$S(R^{n+1}) = S(R^n) \oplus N(R^n)$$

where $N = \mathrm{Sh}_R(S)/S \subseteq Q$. It follows that

$$\begin{aligned} P(R^n) &= P(0) \oplus Q(0) \oplus Q(R^1) \oplus \cdots \oplus Q(R^{n-1}) \text{ and} \\ S(R^n) &= S(0) \oplus N(0) \oplus N(R^1) \oplus \cdots \oplus N(R^{n-1}). \end{aligned}$$

Now by Lemma 3.1.1 there exists a nonzero r_0 such that $R[1/r_0] \otimes P(0)$ is the direct sum of a free $R[1/r_0]$-module and $R[1/r_0] \otimes S(0)$, which is also free. And by the induction hypothesis there exists a nonzero $r_1 \in R$ such that for each m, $R[1/r_1] \otimes Q(R^m)$ is a direct sum of two free $R[1/r_1]$-modules, one of which is $R[1/r_1] \otimes N(R^m)$. Then $r := r_0 r_1$ does the trick for the pair P, S. □

3.4.6 The Friedlander-Suslin lemma

The Friedlander-Suslin lemma relates polynomial functors of bounded degree to representations of certain associative algebras called *Schur Algebras*. To introduce these, let $U \in \mathbf{fgfMod}_R$ and let $d \geq 1$ be an integer. The bilinear polynomial law

$$- \circ -\colon\ \mathrm{End}(U) \times \mathrm{End}(U) \to \mathrm{End}(U)$$

given by composition yields an algebra homomorphism

$$R[\mathrm{End}(U)] \to R[\mathrm{End}(U) \times \mathrm{End}(U)] \cong R[\mathrm{End}(U)] \otimes R[\mathrm{End}(U)]$$

which maps the part $R[\mathrm{End}(U)]_{\leq d}$ of degree $\leq d$ into

$$\sum_{\substack{a,b \geq 0 \\ a+b \leq d}} R[\mathrm{End}(U)]_a \otimes R[\mathrm{End}(U)]_b \subseteq R[\mathrm{End}(U)]_{\leq d} \otimes R[\mathrm{End}(U)]_{\leq d}.$$

Taking the dual R-modules, we obtain a map

$$R[\mathrm{End}(U)]^*_{\leq d} \otimes R[\mathrm{End}(U)]^*_{\leq d} \to (R[\mathrm{End}(U)]_{\leq d} \otimes R[\mathrm{End}(U)]_{\leq d})^* \to R[\mathrm{End}(U)]^*_{\leq d}.$$

We set $S_{\leq d}(U) := R[\mathrm{End}(U)]^*_{\leq d}$. The first map is, in fact, an isomorphism due to the fact that $S_{\leq d}(U)$ is finitely generated and free as an R-module. Indeed, if U is free with basis $u_1, \ldots, u_n$, then $\mathrm{End}(U)$ is free with basis $(E_{ij})_{i,j=1}^n$, where $E_{ij} u_k = \delta_{jk} u_i$, and $R[\mathrm{End}(U)]_{\leq d}$ is free with basis the monomials x^α of degree $\leq d$ in the coordinates x_{ij} dual to the E_{ij}, and hence $R[\mathrm{End}(U)]^*_{\leq d}$ is free with the dual basis $(s_\alpha)_\alpha$, where α runs over all multi-indices in $\mathbf{Z}_{\geq 0}^{n \times n}$ such that $|\alpha| := \sum_{i,j} \alpha_{i,j} \leq d$. We let $- * -\colon S_{\leq d}(U) \times S_{\leq d}(U) \to S_{\leq d}(U)$ be the bilinear map associated to the map above.

Definition 3.4.12. The R-module $S_{\leq d}(U)$ with the bilinear map $- * -$ is called the *Schur algebra of degree $\leq d$ on U*, and (given a basis of U), the basis $(s_\alpha)_\alpha$ is called its *distinguished basis*. ♪

The Schur algebra is associative (but not commutative unless $n = 1$); this follows from the associativity of composition in $\mathrm{End}(U)$. Explicitly, the coefficient of s_γ in the product $s_\alpha * s_\beta$ is computed as follows: First, expand the composition $(\sum_{ij} x_{ij} E_{ij}) \circ (\sum_{kl} y_{kl} E_{kl})$, where the x_{ij} and y_{kl} are variables, as $\sum_{i,l} (\sum_j x_{ij} y_{jl}) E_{il} =: \sum_{il} z_{il} E_{il}$. Then expand z^γ as a polynomial in the x_{ij} and the y_{kl}, and take the coefficient of the monomial $x^\alpha y^\beta$.

The map $\mathrm{End}(U) \to S_{\leq d}(U)$ that sends φ to the R-linear evaluation map

$$R[\mathrm{End}(U)]_{\leq d} \to R, \quad f \mapsto f_R(\varphi)$$

is an injective homomorphism of associative R-algebras, so $S_{\leq d}(U)$-modules M are, in particular, representations of the R-algebra $\mathrm{End}(U)$. In fact, they are precisely the *polynomial* $\mathrm{End}(U)$-representations of degree $\leq d$, i.e., those for which the map $\mathrm{End}(U) \to \mathrm{End}(M)$ is not just a homomorphism of (noncommutative) R-algebras but also a polynomial law making certain diagrams commute. Since we will not need this interpretation, we skip the details.

Now suppose that P is a polynomial functor $\mathbf{fgfMod}_R \to \mathbf{Mod}_R$ of degree $\leq d$. Then $P(U)$ naturally carries the structure of an $S_{\leq d}(U)$-module as follows: the polynomial law

$$P_{U,U}\colon \mathrm{End}(U) \to \mathrm{End}(P(U))$$

has degree $\le d$ and therefore we have

$$P_{U,U,R[x_{11},x_{12},\dots,x_{nn}]}\left(\sum_{i,j=1}^{n} x_{ij}\otimes E_{ij}\right)=\sum_{|\alpha|\le d} x^{\alpha}\otimes\varphi_{\alpha}$$

for certain endomorphisms $\varphi_\alpha \in \mathrm{End}(P(U))$. Now the basis element s_α of $S_{\le d}(U)$ acts on $P(U)$ via φ_α; it can be shown that this construction is independent of the choice of basis of U.

Theorem 3.4.13 (Friedlander-Suslin lemma, [Tou14, Théorème 7.2] and [FS97, Theorem 3.2])**.** *Let $U \in$ **fgfMod**$_R$ have rank $\ge d$. Then the association $P \mapsto P(U)$ is an equivalence of Abelian categories from the full subcategory of* $\mathbf{PF}_R$ *consisting of polynomial functors* **fgfMod**$_R$ $\to$ **Mod**$_R$ *of degree $\le d$ to the category of $S_{\le d}(U)$-modules.*

To conclude this section, we observe that Schur algebras behave well under base change: if A is an R-algebra, then we have a commuting diagram (up to natural isomorphisms):

$$\begin{tikzcd}
(\mathbf{PF}_R)_{\le d} \arrow[r] \arrow[d, "P\mapsto P_A"'] & \{S_{\le d}(U)\text{-modules}\} \arrow[d, "M\mapsto A\otimes M"] \\
(\mathbf{PF}_A)_{\le d} \arrow[r] & \{(A\otimes S_{\le d}(U))\text{-modules}\}
\end{tikzcd}$$

where the lower horizontal map is evaluation at $A \otimes U$ and the A-algebra $A \otimes S_{\le d}(U)$ is canonically isomorphic to the Schur algebra $S_{\le d}(A\otimes U)$ on the free A-module $A \otimes U$.

3.4.7 Irreducibility in an open subset of Spec(R)

Let R be a domain and let $P\colon$ **fgfMod**$_R$ $\to$ **fgMod**$_R$ be a polynomial functor. As before, for each prime $\mathfrak{p} \in \mathrm{Spec}(R)$ we set $K_{\mathfrak{p}} := \mathrm{Frac}(R/\mathfrak{p})$; in particular, $K := K_{(0)}$ is the fraction field of R. Recall that the base change functor yields a polynomial functor $P_{K_{\mathfrak{p}}}$ over the field $K_{\mathfrak{p}}$ for each $\mathfrak{p} \in \mathrm{Spec}(R)$, and also a polynomial functor $P_{\overline{K_{\mathfrak{p}}}}$ over the algebraic closure $\overline{K_{\mathfrak{p}}}$ of $K_{\mathfrak{p}}$. The goal of this section is to transfer certain properties of P_K to $P_{K_{\mathfrak{p}}}$ for $\mathfrak{p}$ in an open dense subset of Spec(R).

Proposition 3.4.14. *Let $\overline{Q}$ be an irreducible subobject of P_K in the Abelian category of polynomial functors over K and assume that $\overline{Q}_{\overline{K}}$ is still irreducible. Then there exists a subobject Q of P in the category of polynomial functors* **fgfMod**$_R$ $\to$ **Mod**$_R$ *such that $Q_K = \overline{Q}$ and $Q_{\overline{K_{\mathfrak{p}}}}$ is an irreducible subobject of $P_{\overline{K_{\mathfrak{p}}}}$ in the Abelian category of polynomial functors over $\overline{K_{\mathfrak{p}}}$ for all primes $\mathfrak{p}$ in a dense open subset* $\mathrm{Spec}(R[1/r]) \subseteq \mathrm{Spec}(R)$.

Remark 3.4.15. Note that we don't require that Q is a functor into **fgMod**$_R$; we may not be able to guarantee this if R is not a Noetherian ring. ♪♫

In order to prove this proposition, we use the following lemma.

Lemma 3.4.16. *Let A be a (not necessarily commutative) associative R-algebra and N an A-module that is, as an R-module, finitely generated and free. Suppose that $\overline{K} \otimes N$ is an irreducible $(\overline{K} \otimes A)$-module. Then there exists a dense open subset* $\operatorname{Spec}(R[1/r]) \subseteq \operatorname{Spec}(R)$ *such that $\overline{K_\mathfrak{p}} \otimes N$ is an irreducible $(\overline{K_\mathfrak{p}} \otimes A)$-module for all* $\mathfrak{p} \in \operatorname{Spec}(R[1/r])$.

Proof. Let $v_1, \ldots, v_n$ be an R-basis of N. For each $j \in [n]$ and each $a \in A$ let $c_{a,i,j} \in R$ be the *structure constants* determined by

$$av_j = \sum_i c_{a,i,j} v_i.$$

For each $k = 1, \ldots, n-1$, we will construct a constructible subset Z_k of the Grassmannian $\operatorname{Gr}_R(k, n)$ over R whose set of $\overline{K_\mathfrak{p}}$-points, for $\mathfrak{p} \in \operatorname{Spec}(R)$, is the set of k-dimensional $(\overline{K_\mathfrak{p}} \otimes A)$-submodules of $\overline{K_\mathfrak{p}} \otimes N$. The construction is as follows: for each $J \subseteq [n]$ of size k consider the $k \times n$ matrix X_J whose entries on the columns labelled by J are a $k \times k$ identity matrix over R and whose other entries are variables x_{ij}, $i \in [k], j \in [n] \setminus J$. Recall that $\operatorname{Gr}_R(k, n)$ has an open cover of affine spaces $\mathbb{A}^{k\times(n-k)}_{R,J}$ over R on which the coordinates are precisely these x_{ij} with $j \notin J$. For $j \in J$ we write $x_{ij} \in \{0, 1\}$ for the corresponding entry of X_J. Note that, for each $m = 1, \ldots, k$ and each $a \in A$, we have

$$(1 \otimes a)\left(\sum_{j=1}^{n} x_{mj} \otimes v_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{a,i,j} x_{mj} \otimes v_i \in R\left[x_{ij} \,\middle|\, i \in [k], j \in [n] \setminus J\right] \otimes N$$

and we define the row vector of coefficients

$$y_{a,m} := \left(\sum_{j=1}^{n} c_{a,i,j} x_{mj}\right)_{i=1}^{n}$$

with entries in the coordinate ring $R[x_{ij} \mid i \in [k], j \in [n] \setminus J]$ of $\mathbb{A}^{k\times(n-k)}_{R,J}$.

Let C_J be the closed subset of $\mathbb{A}^{k\times(n-k)}_{R,J}$ defined by the vanishing of all $(k+1)\times(k+1)$-subdeterminants of the matrices

$$\begin{bmatrix} y_{a,m} \\ X_J \end{bmatrix}$$

for all choices of $a \in A$ and $m = 1, \ldots, k$. For each prime $\mathfrak{p} \in \operatorname{Spec}(R)$, the subset $C_J(\overline{K_\mathfrak{p}}) \subseteq \operatorname{Gr}_R(k, n)(\overline{K_\mathfrak{p}})$ parameterises the k-dimensional $(\overline{K_\mathfrak{p}} \otimes A)$-submodules of $\overline{K_\mathfrak{p}} \otimes N \cong \overline{K_\mathfrak{p}}^{[n]}$ that map surjectively to $\overline{K_\mathfrak{p}}^{J}$. In particular, by the assumption that $\overline{K} \otimes N$ is still irreducible, the image of C_J in $\operatorname{Spec}(R)$ does not contain the prime 0, for any k and any k-set $J \subseteq [n]$. In other words, the morphism $C_J \to \mathrm{Spec}(R)$ is not dominant. Set $Z_k := \bigcup_{J\subseteq[n],|J|=k} \overline{C_J}$, a finite union of locally closed subsets of the Grassmannian. Then $Z_k \to \mathrm{Spec}(R)$ is still not dominant, and neither is $\left(\bigcup_{k=1}^{n-1} Z_k\right) \to \mathrm{Spec}(R)$. Hence there exists a nonzero $r \in R$ that lies in the vanishing ideal of the image; the open dense subset $\mathrm{Spec}(R[1/r]) \subseteq \mathrm{Spec}\, R$ then has the desired property. □

Proof of Proposition 3.4.14. By the Friedlander-Suslin Lemma (Theorem 3.4.13) and the fact that the Schur algebra behaves well under base change, it suffices to prove the corresponding statement for all $d \in \mathbf{Z}_{\geq 0}$, $U := R^d$, and all $S_{\leq d}(U)$-modules that are finitely generated over R (which, of course, is equivalent to being finitely generated as an $S_{\leq d}(U)$-module).

So let M be a finitely generated $S_{\leq d}(U)$-module and let $\overline{N}$ be an irreducible $(K \otimes S_{\leq d}(U))$-submodule of $K \otimes M$ that remains irreducible when tensoring with $\overline{K}$. Define

$$N := \{v \in M \mid 1 \otimes v \in \overline{N}\}.$$

A straightforward computation shows that N is a (not necessarily finitely generated) $S_{\leq d}(U)$-submodule of M.

By Lemma 3.1.1 there exist a nonzero $r \in R$ and elements $v_1, \ldots, v_n \in N$ such that $R[1/r] \otimes N$ is a free $R[1/r]$-module with basis $1 \otimes v_1, \ldots, 1 \otimes v_n$. Then Lemma 3.4.16 applied with R equal to $R[1/r]$ and A equal to $R[1/r] \otimes S_{\leq d}(U)$ shows that $\overline{K_\mathfrak{p}} \otimes N$ is an irreducible $(\overline{K_\mathfrak{p}} \otimes S_{\leq d}(U))$-submodule of $\overline{K_\mathfrak{p}} \otimes M$ for all $\mathfrak{p}$ in some nonempty open subset $\mathrm{Spec}\, R[1/(rs)] \subseteq \mathrm{Spec}(R[1/r]) \subseteq \mathrm{Spec}(R)$. □

3.4.8 Closed subsets of polynomial functors

Closed subsets of a polynomial functors play the role of affine varieties in finite-dimensional algebraic geometry. In this subsection, P is a fixed polynomial functor $\mathbf{fgfMod}_R \to \mathbf{fgMod}_R$ of finite degree.

For any $U, V \in \mathbf{fgfMod}_R$ we have a sequence of polynomial laws

$$\mathrm{Hom}(U, V) \times P(U) \xrightarrow{P_{U,V} \times \mathrm{id}} \mathrm{Hom}(P(U), P(V)) \times P(U) \xrightarrow{(\varphi,p)\mapsto\varphi(p)} P(V),$$

whose composition we denote by $\Phi_{U,V}$. We also let $\Pi_{U,V} \colon \mathrm{Hom}(U, V) \times P(U) \to P(U)$ be the linear polynomial law given by projection. Recall that $\Phi_{U,V}$ and $\Pi_{U,V}$ both yield continuous maps from $\mathbb{A}_{\mathrm{Hom}(U,V)\times P(U)} \to \mathbb{A}_{P(V)}$.

Definition 3.4.17. We define $\mathbb{A}_P$ to be P. A *subset* of $\mathbb{A}_P$ is a rule X that assigns to each $U \in \mathbf{fgfMod}_R$ a subset $X(U)$ of $\mathbb{A}_{P(U)}$ (see Definition 3.3.5) in such a manner that

$$\Phi_{U,V}(\Pi_{U,V}^{-1}(X(U))) \subseteq X(V)$$

for all $U, V \in \mathbf{fgfMod}_R$. The subset $X \subseteq \mathbb{A}_P$ is *closed* if $X(U)$ is a closed subset of $\mathbb{A}_{P(U)}$ for all $U \in \mathbf{fgfMod}_R$. The *closure* of X is the closed subset $\overline{X}$ of $\mathbb{A}_P$ assigning $\overline{X(U)}$ to U for all $U \in \mathbf{fgfMod}_R$. ♪

It is worth spelling out what this means. Let U, V be finitely generated free R-modules, let D be an R-domain and let $\varphi \in D \otimes \mathrm{Hom}(U, V)$. Then the condition is that $P_{U,V,D}(\varphi) \in D \otimes \mathrm{Hom}(P(U), P(V))$ maps $X(U)(D) \subseteq D \otimes P(U)$ into $X(V)(D)$. In the particular case where $V = U$, this condition can be informally thought of as the condition that $X(U)$ is preserved under the polynomial action of $\mathrm{End}(U)$. Let $\alpha\colon Q \to P$ be a polynomial transformation and let X be a subset of Q. Then $\alpha(X) = (U \mapsto \alpha_U(X(U)))$ is a subset of P.

Definition 3.4.18. For $X \subseteq \mathbb{A}_P$, we define the ideal $\mathcal{I}_X$ of X to be the rule assigning $\mathcal{I}_{X(U)} \subseteq R[P(U)]$ to U for all $U \in \mathbf{fgfMod}_R$. The rule $\mathcal{I}_X$ is an ideal in the R-algebra over the category $\mathbf{fgfMod}_R$ defined by $U \mapsto R[P(U)]$, i.e., for all $\varphi \in \mathrm{Hom}(U, V)$ we have $\mathcal{I}_X(V) \circ P_{U,V,R}(\varphi) \subseteq \mathcal{I}_X(U)$. ♪

Definition 3.4.19 (Base change)**.** If $X \subseteq \mathbb{A}_P$ is a closed subset and B is an R-algebra, then we obtain a closed subset X_B of $\mathbb{A}_{P_B}$ by letting, for a $U \in \mathbf{fgfMod}_B$, $X_B(U)$ be the closed subset $X(U_R)_B$ of $\mathbb{A}_{P_B(U)} = \mathbb{A}_{B\otimes P(U_R)}$, where U_R is the free R-module such that $U \cong B \otimes U_R$ from the definition of P_B. ♪

We will use the following lemmas very frequently in our proof of Theorem 4.1.1.

Lemma 3.4.20. *Let R be a ring with Noetherian spectrum and r an element of R. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_k$ be the minimal primes of $R/(r)$. Then two closed subsets $X, Y \subseteq \mathbb{A}_P$ are equal if and only if $X_{R[1/r]} = Y_{R[1/r]}$ and $X_{R/\mathfrak{p}_i} = Y_{R/\mathfrak{p}_i}$ for all $i = 1, \dots, k$.*

Proof. This follows from Lemma 3.3.12 with $X(U), Y(U)$ for every $U \in \mathbf{fgfMod}_R$. □

Lemma 3.4.21. *Let $R \subseteq R'$ be a finite extension of domains and let $X, Y \subseteq \mathbb{A}_P$ be closed subsets. Then $X = Y$ if and only if $X_{R'} = Y_{R'}$.*

Proof. This follows from Lemma 3.3.13 with $X(U), Y(U)$ for every $U \in \mathbf{fgfMod}_R$. □

Lemma 3.4.22. *Let $U \in \mathbf{fgfMod}_R$ and $g \in R[P(U)]$. Then*

$$Y(V)(D) = \{p \in D \otimes P(V) \mid \forall \varphi \in D \otimes \mathrm{Hom}(V, U) : g_D(P_{V,U,D}(\varphi)(p)) = 0\}$$

for all $V \in \mathbf{fgfMod}_R$ and R-domains D defines a closed subset $Y \subseteq \mathbb{A}_P$. The subset Y is the biggest closed subset of $\mathbb{A}_P$ such that g is in the ideal of $Y(U)$.

Proof. It is easy to check that $Y(V)$ is a subset of $\mathbb{A}_{P(V)}$ for all $V \in \mathbf{fgfMod}_R$ and that Y is a subset of $\mathbb{A}_P$. We need to check that Y is a closed subset of $\mathbb{A}_P$, i.e., that $Y(V)$ is a closed subset of $\mathbb{A}_{P(V)}$ for every $V \in \mathbf{fgfMod}_R$.

Let $\varphi_1, \ldots, \varphi_n$ be a basis of $\mathrm{Hom}(V, U)$. For every R-algebra A, consider the map

$$g_{A[x_1,\ldots,x_n]} \circ P_{V,U,A[x_1,\ldots,x_n]}(x_1 \otimes \varphi_1 + \cdots + x_n \otimes \varphi_n) \colon A[x_1, \ldots, x_n] \otimes P(V) \to A[x_1, \ldots, x_n].$$

We have

$$g_{A[x_1,\ldots,x_n]} \circ P_{V,U,A[x_1,\ldots,x_n]}(x_1 \otimes \varphi_1 + \cdots + x_n \otimes \varphi_n)|_{A\otimes P(V)} = \sum_{\alpha \in \mathbf{Z}^n_{\geq 0}} x^\alpha g_{\alpha,A}$$

where $g_{\alpha,A} \colon A \otimes P(V) \to A$. We get polynomial laws $g_\alpha = (g_{\alpha,A})_A \in R[P(V)]$. Set $S_V = \{g_\alpha \mid \alpha \in \mathbf{Z}^n_{\geq 0}\}$. We claim that $Y(V) = \mathcal{V}(S_V)$. Let D be an R-domain and take $p \in Y(V)(D)$. Then, viewing p as an element of $Y(V)(D[x_1, \ldots, x_n])$, we see that

$$g_{D[x_1,\ldots,x_n]}(P_{V,U,D[x_1,\ldots,x_n]}(\varphi)(p)) = 0$$

for all $\varphi \in D[x_1, \ldots, x_n] \otimes \mathrm{Hom}(V, U)$. Using $\varphi = x_1 \otimes \varphi_1 + \cdots + x_n \otimes \varphi_n$, we get $p \in \mathcal{V}(S_V)(D)$. Conversely, suppose that $p \in \mathcal{V}(S_V)(D)$. Then

$$g_{D[x_1,\ldots,x_n]}(P_{V,U,D[x_1,\ldots,x_n]}(x_1 \otimes \varphi_1 + \cdots + x_n \otimes \varphi_n)(p)) = 0$$

Specializing the x_i to elements of D, we find that

$$g_D(P_{V,U,D}(a_1 \otimes \varphi_1 + \cdots + a_n \otimes \varphi_n)(p)) = 0$$

for all $a_1, \ldots, a_n \in D$. So $p \in Y(V)(D)$. So $Y(V) = \mathcal{V}(S_V)$ is indeed closed. □

Remark 3.4.23. It is not true in general that

$$Y(V)(D) = \{p \in X(V)(D) \mid \forall \varphi \in \mathrm{Hom}(V, U) : h_D(P(\varphi)_D(p)) = 0\}.$$

For an example, take $R = \mathbb{F}_p$, $P(V) = V$ and $h = x^p - x \in R[x] = R[P(R)]$. Then the right hand side above consists of all $p \in D \otimes V \cong D^n$ such that $x^p = x$ for every coordinate of p while the left hand side also has the requirement that $(\alpha x)^p = \alpha x$ for all $\alpha \in E$ for every D-domain E. So $Y(V)(D) = 0$. ♪♫

3.4.9 Gradings

Let $P \colon \mathbf{fgfMod}_R \to \mathbf{fgMod}_R$ be a polynomial functor. For each $U \in \mathbf{fgfMod}_R$, the R-algebra $R[P(U)]$ has two natural gradings: first, the *ordinary* grading that each coordinate ring $R[M]$ of a module M has (see Definition 3.2.11); and second, a grading that takes into account the degrees of the homogeneous components P, as follows. Write $P = P_0 \oplus P_1 \oplus \cdots \oplus P_d$, so that $R[P(U)]$ is the tensor product of the $R[P_i(U)]$ by Proposition 3.2.19. Then multiply the ordinary grading on $R[P_i(U)]$ by i and use these to define a grading on $R[P(U)]$, called the *standard* grading. The standard grading has an alternative characterisation, as follows: $f \in R[P(U)]$ is homogeneous of degree j if $f_A(P_{U,U,A}(a \otimes \mathrm{id}_U)(v)) = a^j f_A(v)$ for all $A \in \mathbf{Alg}_R$ and all $v \in A \otimes P(U)$. We have

$$f_{A[t]}(v_0 + tv_1 + \cdots + t^d v_d) = \sum_{j=0}^{\infty} t^j f_{j,A}(v_0 + v_1 + \cdots + v_d)$$

for all $A \in \mathbf{Alg}_R$ and $v_i \in A \otimes P_i(U)$ where f_j is the part of f of standard degree j.

Lemma 3.4.24. *For any closed subset $X \subseteq \mathbb{A}_P$ and any $U \in \mathbf{fgfMod}_R$, the ideal $\mathcal{I}_X(U)$ is homogeneous with respect to the standard grading.*

Proof. Take $f \in \mathcal{I}_X(U)$ and let D be an R-domain. Then

$$0 = f_{D[t]}(P_{U,U,D[t]}(t \otimes \mathrm{id}_U)(v_0 + v_1 + \cdots + v_d)) = f_{D[t]}(v_0 + tv_1 + \cdots + t^d v_d)$$

for all $v_i \in D \otimes P_i(U)$ such that $v_0 + v_1 + \cdots + v_d \in X(U)(D)$. Hence the homogeneous parts of f are also contained in $\mathcal{I}_X(U)$. $\square$

Part III

Stabilisation under GL

Chapter 4

Topological GL-Noetherianity over rings

This chapter contains the proof of topological Noetherianity of polynomial functors defined over rings with Noetherian spectrum. It is fully based on the paper [BDD22] and it relies on the notions of Chapter 3 from which we also import the notation of Section 3.1. In Section 4.1 we review the main objects and ideas and explain the motivation together with the relations with the literature. In the subsequent Section 4.2 we present the proof. In the final Section 4.3 we present some applications.

4.1 Introduction

Theorem 2.4.3 on topological Noetherianity of polynomial functors defined over infinite fields was used in work by Daniel Erman, Steven Sam, and Andrew Snowden [ESS19; ESS21b; ESS21c] and by Jan Draisma, Michał Lasoń, and Anton Leykin [DLL19] in new proofs of Stillman's conjecture. In this context, Erman-Sam-Snowden asked whether the Noetherianity of polynomial functors also holds over $\mathbf{Z}$; this would show that their proof of Stillman's conjecture yields bounds that are independent of the characteristic, just like another proof by Erman-Sam-Snowden [ESS19] and the original proof by Tigran Ananyan and Mel Hochster [AH20a].

In Section 4.2 we settle Erman-Sam-Snowden's question in the affirmative. Indeed, rather than working over $\mathbf{Z}$, we use the language of Chapter 3 and work over a ring R whose spectrum is Noetherian—this turns out to be precisely the setting where topological Noetherianity also holds for polynomial functors.

4.1.1 The frame and the result

So let R be a ring. In Section 3.2 we reviewed the notion of *polynomial laws* from an R-module M to an R-module N. In the special case where $N = R$, these polynomial laws form a graded ring $R[M]$ (see Section 3.2.2). This ring, playing the role of "the coordinate ring of a module", is used in Section 3.3 to define a topological space $\mathbb{A}_M$, in such a manner that any polynomial law $\varphi\colon M \to N$ yields a continuous map, also denoted φ, from $\mathbb{A}_M \to \mathbb{A}_N$. Recall that $\mathbb{A}_M$ is a topological space over the category $\mathbf{Dom}_R$ of R-domains with R-algebra monomorphisms in the sense of Definition 3.3.3.

If M is freely generated by n elements, then $R[M]$ is the polynomial ring $R[x_1, \ldots, x_n]$ and the poset of closed sets in $\mathbb{A}_M$ is the same as that in the spectrum of $R[M]$. In general, however, we do not completely understand the relation between $\mathbb{A}_M$ and the spectrum of $R[M]$ (see Remark 3.3.17), and we work with the former rather than the latter. The following result is a topological version of Hilbert's basis theorem in this setting.

Proposition (Proposition 3.3.14)**.** *If R has a Noetherian spectrum and M is a finitely generated R-module, then the topological space $\mathbb{A}_M$ over $\mathbf{Dom}_R$ is Noetherian.*

Interestingly, it is not true that if R is Noetherian and M is finitely generated, then $R[M]$ is Noetherian (see Example 3.2.15), so "topologically Noetherian" is the most natural setting here. A special case of the theorem (taking M free of rank 1) is that if R has a Noetherian spectrum, then so does the polynomial ring $R[x]$. This special case, a topological version of Hilbert's basis theorem, is easy and well-known; e.g., it also follows from [ESS20, Theorem 1.1] with a trivial group G.

Following [Rob63], in Section 3.4 we recalled the notion of polynomial functors from the category $\mathbf{fgfMod}_R$ of finitely generated free R-modules to the category $\mathbf{Mod}_R$ of R-modules. These polynomial functors form an Abelian category. The subcategory of polynomial functors from $\mathbf{fgfMod}_R$ to the category $\mathbf{fgMod}_R$ of finitely generated, but not necessarily free, R-modules is not an Abelian subcategory when R is not Noetherian, but it is closed under taking quotients, and this will suffice for our purposes.

Given a polynomial functor $P\colon \mathbf{fgfMod}_R \to \mathbf{fgMod}_R$, a closed subset of $\mathbb{A}_P$ is a rule X that assigns to each finitely generated free R-module U a closed subset $X(U)$ of $\mathbb{A}_{P(U)}$ such that the continuous map corresponding to the polynomial law

$$\mathrm{Hom}(U, V) \times P(U) \to P(V), \quad (\varphi, p) \mapsto P_{U,V}(\varphi)(p)$$

maps the pre-image of $X(U)$ under the projection on $P(U)$ in $\mathbb{A}_{\mathrm{Hom}(U,V)\times P(U)}$ into $X(V)$ (see Section 3.4.8 for details). If Y is a second such rule, then we say that X is a subset of Y if $X(U)$ is a subset of $Y(U)$ for each $U \in \mathbf{fgfMod}_R$. Our main result, then, is the following.

Theorem 4.1.1. *Let R be a commutative ring whose spectrum is a Noetherian topological space and let P be a finite-degree polynomial functor $\mathbf{fgfMod}_R \to \mathbf{fgMod}_R$. Then every descending chain $X_1 \supseteq X_2 \supseteq \ldots$ of closed subsets of $\mathbb{A}_P$ stabilises: for all sufficiently large n we have $X_n = X_{n+1}$.*

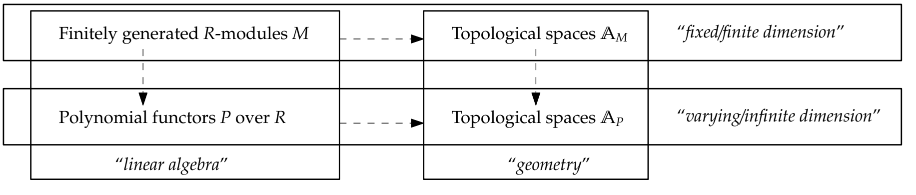
Proposition 3.3.14 is the special case of Theorem 4.1.1 where the polynomial functor has degree 0, i.e., sends each U to a fixed module M and each morphism to the identity id_M. It is the base case in our inductive proof of Theorem 4.1.1.

We recall two fundamental properties of polynomial functors over rings that are fundamental in our proof. The first is the Friedlander-Suslin's lemma stating equivalences of Abelian categories between polynomial functors $\mathbf{fgfMod}_R \to \mathbf{fgMod}_R$ of degree $\leq d$ and finitely generated modules for the non-commutative R-algebra $R[\mathrm{End}(U)]^*_{\leq d}$ (called the Schur algebra) for any $U \in \mathbf{fgfMod}_R$ of rank $\geq d$; see Section 3.4.6.

The second fundamental fact, discussed in Section 3.4.7, is that if R is a domain and P a polynomial functor from $\mathbf{fgfMod}_R$ to $\mathbf{Mod}_R$ such that $\mathrm{Frac}(R) \otimes P$ is irreducible, then $\mathrm{Frac}(R/\mathfrak{p}) \otimes P$ is irreducible for all primes $\mathfrak{p}$ in some open dense subset of $\mathrm{Spec}(R)$. This is an incarnation of the philosophy in representation theory that irreducibility is a generic condition.

The global proof strategy is as follows: we show that the induction steps in [Dra19], where Theorem 4.1.1 is proved when R is an infinite field, can be made global in the sense that they hold for $\mathrm{Frac}(R/\mathfrak{p})$ for all $\mathfrak{p}$ in some open dense subset of $\mathrm{Spec}(R)$; and then we use Noetherian induction on $\mathrm{Spec}(R)$ to deal with the remaining primes $\mathfrak{p}$. The details of this approach are quite subtle and beautiful.

The big picture is depicted in the following diagram:



Building on the notion of finitely generated R-modules, on the left we pass to polynomial functors over R. Here many results carry over, such as the fact that the rank is a semicontinuous function on $\mathrm{Spec}(R)$; see Proposition 3.4.10. We regard this as "linear algebra in varying dimensions". In the other direction, we construct the topological space $\mathbb{A}_M$ and enter the realm of algebraic geometry; the closed subsets generalise affine algebraic varieties. Finally, both constructs come together in the construction of the topological space associated to a polynomial functor P. Here we use both results from the "linear algebra" of polynomial functors, such as Friedlander-Suslin's lemma, and results about the topological spaces $\mathbb{A}_M$, to prove that $\mathbb{A}_P$ is Noetherian. Furthermore, we establish the fundamental result that the dimension function of a closed subset of $\mathbb{A}_P$ depends on primes in $\mathrm{Spec}(R)$ in a constructible manner; see Proposition 4.3.6.

4.1.2 Further relations to the literature

The paper [ESS21b] establishes finiteness results for (cone-stable and weakly upper semi-continuous) ideal invariants in polynomial rings over a fixed field. As Daniel Erman pointed out to us, at least part of their results carry over to arbitrary base rings with Noetherian spectrum. In particular, [ESS21b] establishes the Noetherianity of a space $Y_{\mathbf{d}}$ that parameterises homogeneous ideals generated in degrees $\mathbf{d} = (d_1, \ldots, d_r)$. While they work with certain limit spaces, the "functor analogue" of their $Y_{\mathbf{d}}$ in our setting would be a functor from $\mathbf{fgfMod}_R$ to the category of functors from $\mathbf{Dom}_R$ to sets that sends a finitely generated free R-module $U = R^n$ to the functor that maps an R-domain D to the set of $\mathrm{GL}_n(D)$-orbits of ideals in $R[x_1, \ldots, x_n]$ generated by homogeneous polynomials of degrees $d_1, \ldots, d_r$. Then $Y_{\mathbf{d}}$ admits a surjective map from the space $\mathbb{A}_{S^{d_1} \oplus \cdots \oplus S^{d_r}}$—a functor from $\mathbf{fgfMod}_R$ to functors from $\mathbf{Dom}_R$ to topological spaces, and one can give $Y_{\mathbf{d}}$ the quotient topology. Theorem 4.1.1 implies that $Y_{\mathbf{d}}$ is then Noetherian, provided that $\mathrm{Spec}(R)$ is Noetherian.

Our work does not say much about Noetherianity of the coordinate rings $R[\mathbb{A}_P]$, let alone about Noetherianity of finitely generated modules over them. Currently, these much stronger results are known only when R is a field of characteristic zero and P is either a direct sum of copies of S^1 [SS16; SS19] or $P = S^2$ or $P = \bigwedge^2$ [NSS16] or $P = S^1 \oplus S^2$ or $P = S^1 \oplus \bigwedge^2$ [SS22].

Like [AH20a], recent work [KZ18a; KZ18b] implies that polynomials of high strength, and high-strength sequences of polynomials, behave very much like generic polynomials or sequences. Like Corollary 4.3.1, their results are uniform in the characteristic of the field. But the route that David Kazhdan and Tamar Ziegler take is entirely different: first a theorem is proved over finite fields by algebraic-combinatorial means, with uniform constants that do not depend on the finite field, and then model theory is used to transfer the result to arbitrary algebraically closed fields.

In [BDE19] it is shown that in any closed subset of the polynomial functor S^d defined over $\mathbf{Z}$, the strength of polynomials over a ground field of characteristic 0 or characteristic $> d$ is uniformly bounded from above. While of a similar flavour as Corollary 4.3.1, that result—in which the restriction on the characteristic cannot be removed—does not follow from our current work. Far-reaching generalisations of [BDE19], but only over fields of characteristic zero, are topics of [BDDE22; BDES22].

4.2 Proof of GL-Noetherianity over rings

In this section we prove Theorem 4.1.1. Let R be a ring whose spectrum is Noetherian and let $P\colon \mathbf{fgfMod}_R \to \mathbf{fgMod}_R$ a polynomial functor of finite degree. We will prove that any chain $\mathbb{A}_P \supseteq X_1 \supseteq X_2 \supseteq \cdots$ of closed subsets eventually stabilises.

4.2.1 Reduction to the case of a domain

Since $\mathrm{Spec}(R)$ is Noetherian, the ring R has finitely many minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$. By Lemma 3.4.20 with $r = 1$, the sequence $\mathbb{A}_P \supseteq X_1 \supseteq X_2 \supseteq \cdots$ stabilises if and only if the sequence $\mathbb{A}_{P_{R/\mathfrak{p}_i}} \supseteq X_{1,R/\mathfrak{p}_i} \supseteq X_{2,R/\mathfrak{p}_i} \supseteq \cdots$ stabilises for each $i \in [k]$. So from now on we assume that R is a domain, we write $K_\mathfrak{p} := \mathrm{Frac}(R/\mathfrak{p})$ for $\mathfrak{p} \in \mathrm{Spec}(R)$, $K := K_{(0)} = \mathrm{Frac}(R)$, and we let $\overline{K}, \overline{K_\mathfrak{p}}$ be algebraic closures of $K, K_\mathfrak{p}$, respectively.

4.2.2 A stronger statement

We will prove the following stronger statement which clearly implies Theorem 4.1.1.

Theorem 4.2.1. *Let (R, P, X) be a triple consisting of a domain R with Noetherian spectrum, a polynomial functor $P\colon \mathbf{fgfMod}_R \to \mathbf{fgMod}_R$ of finite degree and a closed subset $X \subseteq \mathbb{A}_P$. Then (R, P, X) satisfies the following conditions:*

1. *Every descending chain $X = X_1 \supseteq X_2 \supseteq \cdots$ of closed subsets of X eventually stabilises.*
2. *There exists a nonzero $r \in R$ such that the following holds for all $U \in \mathbf{fgfMod}_R$: if $f \in R[P(U)]$ vanishes identically on $X(U)(\overline{K})$, then f vanishes identically on $X(U)(\overline{K_\mathfrak{p}})$ for all primes $\mathfrak{p} \in \mathrm{Spec}(R[1/r])$.*

Remark 4.2.2. Condition (2) of the theorem means that $\mathcal{I}_{X_{R[1/r]}}$ is determined by $\mathcal{I}_{X_{\overline{K}}}$. More precisely, setting $R' = R[1/r]$, for every $U \in \mathbf{fgfMod}_{R'}$, the ideal

$$\mathcal{I}_{X_{R'}}(U) = \mathcal{I}_{X_{R'}(U)} \subseteq R'[P_{R'}(U)]$$

is the pull-back of the ideal in $\overline{K}[P_{R'}(\overline{K} \otimes U)]$ of the affine variety $X_{R'}(\overline{K} \otimes U)$. ♪♫

The proof of Theorem 4.2.1 is a somewhat intricate induction, combining induction on P, Noetherian induction on $\mathrm{Spec}(R)$ and induction on minimal degrees of functions in the ideal of X—for details, see below.

Notation 4.2.3. For any fixed triple (R, P, X), we denote conditions (1) and (2) of Theorem 4.2.1 by $\Sigma(R, P, X)$. ♪

4.2.3 The induction base

If P has degree zero, then X is just a closed subset of $\mathbb{A}_{P(0)}$. Here, the Noetherianity statement is Proposition 3.3.14 and the statement about vanishing functions is Proposition 3.3.20.

4.2.4 The outer induction

To prove the theorem for P of positive degree, we will show that $\Sigma(R, P, X)$ is implied by $\Sigma(R', P', X')$ where X' is a closed subset of $\mathbb{A}_{P'}$ and (R', P') ranges over pairs that have one of the following forms:

(i) $(R', P') = (R/\mathfrak{p}, P_{R/\mathfrak{p}})$ for some nonzero prime $\mathfrak{p}$ of R; or

(ii) (R', P') where R' is a domain that is a finite extension of a localisation $R[1/r]$ of R, $\deg P' \le \deg P =: d$, for $K' := \mathrm{Frac}(R')$ we have $P'_{K'} \not\cong P_{K'}$ and for the largest e such that the homogeneous parts $P'_{e,K'}$ and $P_{e,K'}$ are not isomorphic, the former is a quotient of the latter.

In both cases, we write $(R, P) \to (R', P')$. We consider the class Π of all the pairs (R, P). The reflexive and transitive closure of the relation $\to$ is a partial order on Π.

Lemma 4.2.4. *The partial order on Π is well-founded.*

Proof. Suppose that we had an infinite sequence

$$(R_0, P_0) \to (R_1, P_1) \to (R_2, P_2) \to \cdots$$

of such steps. By the Friedlander-Suslin's lemma, any sequence of steps of type (ii) only must terminate (see also [Dra19, Lemma 12]). So our sequence contains infinitely many steps of type (i).

Each step $(R, P) \to (R', P')$ induces a morphism $\alpha\colon \mathrm{Spec}(R') \to \mathrm{Spec}(R)$. This morphism α has the property that for irreducible closed subsets $C \subsetneq D \subseteq \mathrm{Spec}(R')$, we have $\overline{\alpha(C)} \subsetneq \overline{\alpha(D)}$. This holds trivially for steps of type (i), where the morphism $\alpha\colon \mathrm{Spec}(R/\mathfrak{p}) \to \mathrm{Spec}(R)$ is a closed embedding, and also for steps of type (ii) by elementary properties of localisation and of integral extensions of rings (see, e.g., [Eis95, Corollary 4.18 (Incomparibility)]).

Let $\alpha_i\colon \mathrm{Spec}(R_i) \to \mathrm{Spec}(R_{i-1})$ be the morphism induced by $(R_{i-1}, P_{i-1}) \to (R_i, P_i)$ and take $\beta_i = \alpha_1 \circ \cdots \circ \alpha_i\colon \mathrm{Spec}(R_i) \to \mathrm{Spec}(R_0)$. Then the maps β_i have the same incomparability property as the α_i. Hence, whenever the step $(R_{i-1}, P_{i-1}) \to (R_i, P_i)$ is of type (i), there is the inclusion of irreducible closed sets $\overline{\mathrm{im}\,\alpha_i} \subsetneq \mathrm{Spec}(R_{i-1})$ and therefore $\overline{\mathrm{im}\,\beta_i} \subsetneq \overline{\mathrm{im}\,\beta_{i-1}}$ is a strict inclusion. This contradicts the Noetherianity of $\mathrm{Spec}(R_0)$. □

By Lemma 4.2.4 we can proceed by induction on Π, namely, in proving that $\Sigma(R, P, X)$ holds, we may assume $\Sigma(R', P', X')$ whenever $(R', P') \leftarrow (R, P)$.

Lemma 4.2.5. *Let $r \in R$ be a nonzero element and let $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ be the minimal primes of $R/(r)$. Assume that $\Sigma(R[1/r], P_{R[1/r]}, X_{R[1/r]})$ and $\Sigma(R/\mathfrak{p}_i, P_{R/\mathfrak{p}_i}, X_{R/\mathfrak{p}_i})$ for each $i \in [k]$ hold. Then $\Sigma(R, P, X)$ holds as well.*

Proof. By Lemma 3.4.20, we see that condition (1) for (R, P, X) follows from condition (1) for $(R[1/r], P_{R[1/r]}, X_{R[1/r]})$ together with $\Sigma(R/\mathfrak{p}_i, P_{R/\mathfrak{p}_i}, X_{R/\mathfrak{p}_i})$ for each $i \in [k]$. Condition (2) for (R, P, X) follows from condition (2) for $(R[1/r], P_{R[1/r]}, X_{R[1/r]})$. □

Combining this lemma with our induction hypothesis, we see that in order to prove $\Sigma(R, P, X)$ it suffices to prove $\Sigma(R[1/r], P_{R[1/r]}, X_{R[1/r]})$ for some $r \in R$. So we may replace (R, P, X) by $(R[1/r], P_{R[1/r]}, X_{R[1/r]})$ whenever this is convenient.

4.2.5 Finding an irreducible factor

Now let $P\colon \mathbf{fgfMod}_R \to \mathbf{fgMod}_R$ be a fixed polynomial functor of degree $d > 0$ over a domain R with Noetherian spectrum. Recall that K is the fraction field of R.

Suppose first that the base change P_K has degree $< d$. Then $K \otimes P_d(U) = 0$ for all $U \in \mathbf{fgfMod}_R$. In particular, this holds for $U = R^d$. So since $P_d(U)$ is a finitely generated R-module, there exists a nonzero $r \in R$ such that $R[1/r] \otimes P_d(U) = 0$. By the Friedlander-Suslin's lemma (Theorem 3.4.13), we then find $(P_d)_{R[1/r]} = 0$. In this case, we replace (R, P, X) by $(R[1/r], P_{R[1/r]}, X_{R[1/r]})$. By repeating this at most d times, we may assume that the base change P_K has the same degree as P.

We want a polynomial subfunctor M of the top-degree part P_d of P whose base change with $\overline{K}$ is an irreducible polynomial subfunctor of $(P_d)_{\overline{K}}$. In the next lemma, we show that such an M exists after passing from R to a suitable finite extension of one of its localisations.

Proposition 4.2.6. *There exist a finite extension R' of a localisation $R[1/r]$ of R and a polynomial subfunctor M of the top-degree part of the polynomial functor $P_{R'}$ such that the base change $M_{\overline{K}}$ is an irreducible polynomial subfunctor of $P_{d,\overline{K}}$.*

Proof. The $S_d(\overline{K}^d)$-module $P_{d,\overline{K}}(\overline{K}^d) = \overline{K} \otimes P_d(R^d)$ is finite-dimensional and hence has an irreducible submodule N'. It is finitely generated, say of dimension $n > 0$. Let $\sum_j \alpha_{ij} \otimes m_{ij}$ for $i = 1, \dots, n$ be a $\overline{K}$-basis. By the Friedlander-Suslin's lemma, the irreducible submodule N' corresponds to an irreducible polynomial subfunctor N of $P_{d,\overline{K}}$. The elements α_i are algebraic over the fraction field K of R. Let $r \in R$ be the product of all the denominators appearing in their minimal polynomials. Then $R' = R[1/r][\alpha_1, \dots, \alpha_n]$ is a finite extension of the localisation $R[1/r]$ of R since the α_i are integral over $R[1/r]$. Consider the submodule M' of the $S_d(R'^d)$-module $P_{d,R'}((R')^d)$ generated by the elements $\sum_j \alpha_{ij} \otimes m_{ij}$. By the Friedlander-Suslin's lemma, M' corresponds to a polynomial subfunctor M of $P_{d,R'}$ whose base change $M_{\overline{K}} = N$ is an irreducible polynomial subfunctor of $P_{d,\overline{K}}$. □

Let $r \in R$ and R' be as in the previous proposition. We would like to reduce to the case where $R' = R$. As before, we can replace (R, P, X) by $(R[1/r], P_{R[1/r]}, X_{R[1/r]})$, so that R' is a finite extension of R. We now prove a version of Lemma 4.2.5 for such extensions.

Lemma 4.2.7. *Assume that* $\Sigma(R', P_{R'}, X_{R'})$ *holds. Then* $\Sigma(R, P, X)$ *holds as well.*

Proof. By Lemma 3.4.21, condition (1) for $(R', P_{R'}, X_{R'})$ implies condition (1) for (R, P, X). Let $r' \in R'$ be a nonzero element as in condition (2) for $(R', P_{R'}, X_{R'})$, i.e., for every $U \in \mathbf{fgfMod}_{R'}$, every $f \in R'[P_{R'}(U)]$ vanishing identically on $X_{R'}(U)(\overline{K})$ also vanishes identically on $X_{R'}(U)(\overline{K_{\mathfrak{p}}})$ for every prime ideal $\mathfrak{p} \in \mathrm{Spec}(R'[1/r'])$. Now $(r') \cap R$ is not the zero ideal, since r' is nonzero and integral over R. Pick any nonzero $r \in (r') \cap R$. We claim that condition (2) holds for (R, P, X) with this particular r.

Indeed, let $U_R \in \mathbf{fgfMod}_R$ and take $U := R' \otimes U_R$. Let f be an element of $R[P(U_R)]$ vanishing identically on $X(U_R)(\overline{K})$. Then f is naturally induces an element of $R'[P_{R'}(U)]$ vanishing identically on $X_{R'}(U)(\overline{K}) = X(U_R)(\overline{K})$. So we see that f vanishes on $X_{R'}(U)(\overline{K_{\mathfrak{q}}})$ for each $\mathfrak{q} \in \mathrm{Spec}(R'[1/r'])$. Since R' is integral over R, for any $\mathfrak{p} \in \mathrm{Spec}(R)$ there exists an $\mathfrak{q} \in \mathrm{Spec}(R')$ with $\mathfrak{q} \cap R = \mathfrak{p}$; and if, moverover, the prime ideal $\mathfrak{p}$ does not contain r, then the prime ideal $\mathfrak{q}$ does not contain r'. Hence f vanishes identically on $\overline{K_{\mathfrak{p}}}$, as desired. □

We replace (R, P, X) by $(R', P_{R'}, X_{R'})$, so that there exists a polynomial subfunctor M of the top-degree part P_d of P such that the base change $M_{\overline{K}}$ is an irreducible polynomial subfunctor of $P_{d,\overline{K}}$.

4.2.6 Splitting off M

Proposition 3.4.11 guarantees that after passing to a further localisation (and using Noetherian induction for the complement), we may assume that for each $U \in \mathbf{fgfMod}_R$, the R-module $P(U)$ is the direct sum of a finitely generated free R-module and the (also finitely generated free) R-module $M(U)$. In particular, both P and $P' := P/M$ are polynomial functors $\mathbf{fgfMod}_R \to \mathbf{fgfMod}_R$.

Let $\pi\colon P \to P'$ be the projection morphism. For a closed subset $X \subseteq \mathbb{A}_P$, we define the closed subset $X' \subseteq \mathbb{A}_{P'}$ as the closure of $\pi(X)$. Note that $(R, P) \to (R, P')$ and hence $\Sigma(R, P', X')$ holds. In particular, we may and will replace R by a further localisation $R[1/r]$ which ensures that, if $f \in R[P'(U)]$ vanishes identically on $X'(U)(\overline{K})$, then it vanishes identically on $X'(U)(\overline{K_{\mathfrak{p}}})$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

4.2.7 The inner induction

We perform the same inner induction as in [Dra19, Section 2.9]. Let $\delta_X \in \{0, 1, \ldots, \infty\}$ denote the smallest degree, in the standard grading, of a homogeneous element of $R[P(U)] \cong R[M(U)] \otimes R[P'(U)]$ (here we use that $P(U)$ is the direct sum of the R-modules $M(U)$ and $P'(U)$), over all $U \in \mathbf{fgfMod}_R$, that lies in the vanishing ideal of $X(U)$ but does not lie in the vanishing ideal of the pre-image in $\mathbb{A}_{P(U)}$ of $X'(U) \subseteq \mathbb{A}_{P'(U)}$. Note that $\delta_X = 0$ is, in fact, impossible, since the coordinates on $R[M(U)]$ have positive degree, so that a degree-0 homogeneous element of $R[P(U)]$ that lies in the ideal of $X(U)$ is an element of $R[P'(U)]$ that lies in the ideal of $X'(U)$. At the other extreme, $\delta_X = \infty$ means that $X(U)$ is the Cartesian product of $X'(U)$ with $\mathbb{A}_{M(U)}$ for all U. We order closed subsets of $\mathbb{A}_P$ by $Y < X$ if either $Y' \subsetneq X'$ or else $Y' = X'$ but $\delta_Y < \delta_X$. Note that, by the outer induction hypothesis for $\Sigma(R, P', X')$ and since $\{0, 1, \ldots, \infty\}$ is well-ordered, this order is well-founded. Hence when proving $\Sigma(P, R, X)$, we may assume that $\Sigma(P, R, Y)$ holds for all $Y < X$.

First suppose that $\delta_X = \infty$. Then, for all proper closed subsets Y of X, we have $Y < X$ and so $\Sigma(R, P, Y)$ holds by the inner induction hypothesis. It follows that condition (1) holds for (R, P, X). Condition (2) for (R, P, X) follows from condition (2) for (R, P', X'), with the same $r \in R$ to be inverted. Indeed, if $f \in R[P(U)] \cong R[M(U)] \otimes R[P'(U)]$ vanishes identically $X(U)(\overline{K}) \cong \mathbb{A}_{M(U)}(\overline{K}) \times X'(U)(\overline{K})$, then, regarding f as a polynomial in the coordinates on $M(U)$ with coefficients in $R[P'(U)]$, those coefficients must all vanish identically on $X'(U)(\overline{K})$, hence on $X'(U)(\overline{K_\mathfrak{p}})$ for all $\mathfrak{p} \in \mathrm{Spec}(R[1/r])$.

4.2.8 A directional derivative

Next, suppose that $1 \leq \delta_X < \infty$. Let $f \in R[P(U)] \cong R[M(U)] \otimes R[P'(U)]$ be a homogeneous polynomial of degree δ_X in the standard grading, which lies in the ideal of $X(U)$ but not on the preimage in $\mathbb{A}_{P(U)}$ of $X'(U)$. Expanding f as a polynomial in the coordinates on $R[M(U)]$ with coefficients in $R[P'(U)]$, one of those coefficients does not lie in the ideal of $X'(U)$. Our assumptions together with Corollary 3.3.11 guarantee that, in fact, that coefficient does not vanish identically on $X'(U)(\overline{K})$, so that f does not vanish identically on the pre-image of $X'(U)(\overline{K})$ in $\mathbb{A}_{P(U)}(\overline{K})$. We then proceed as in [Dra19, Lemma 18]. Let $v_1, \ldots, v_m$ be an R-basis of $M(U)$ and extend this with $v_{m+1}, \ldots, v_n$ to an R-basis of $P(U)$, inducing an isomorphism $R[P(U)] \cong R[x_1, \ldots, x_n]$. The expression

$$f_{R[x_1,\ldots,x_n,y_1,\ldots,y_m,t]}\left(\sum_{i=1}^{n} x_i \otimes v_i + \sum_{j=1}^{m} ty_j \otimes v_j\right) \in R[x_1, \ldots, x_n, y_1, \ldots, y_m, t]$$

explicitly reads as

$$f(x_1 + ty_1, x_2 + ty_2, \ldots, x_m + ty_m, x_{m+1}, \ldots, x_n).$$

Take $p = 1$ if $\mathrm{char}\, R = 0$ and $p = \mathrm{char}\, R$ otherwise. A Taylor expansion in t turns this expression into

$$f(x_1, \ldots, x_n) + t^{p^e} \cdot \left(h_1(x_1, \ldots, x_n)y_1^{p^e} + \cdots + h_m(x_1, \ldots, x_n)y_m^{p^e}\right) + t^{p^e+1} \cdot g$$

for some integer $e \geq 0$, polynomial $g \in R[x_1, \ldots, x_n, y_1, \ldots, y_m, t]$ and homogeneous polynomials $h_i \in R[P(U)]$ of (standard) degree $\delta_X - p^e d$ not all vanishing identically on

$X(U)(\overline{K})$. Specialising the variables y_i to values $a_i \in \{0,1\}$, we get that

$$h(x_1,\ldots,x_n) := \sum_{i=1}^{m} a_i^{p^e} h_i(x_1,\ldots,x_n) \in R[P(U)]$$

does not vanish identically on $X(U)(\overline{K})$.

Let $p \in \overline{K} \otimes P(U)$ be a point in $X(U)(\overline{K})$ such that $h_{\overline{K}}(p) \neq 0$. Relative to the chosen basis of $P(U)$, we may write $p = (\alpha_1,\ldots,\alpha_n)$. Reasoning as before, let $r \in R$ be the product of all the denominators appearing in the minimal polynomials of the α_i over K so that $R' = R[1/r][\alpha_1,\cdots,\alpha_k]$ is a finite extension of $R[1/r]$ containing all α_i. Replacing R by R' and using Lemma 4.2.7, we can therefore assume that $p \in X(U)(R)$ satisfies $h_R(p) \neq 0$. Further replacing R by $R[1/h_R(p)]$, we find that $h_D(p) \neq 0$ for all R-domains D. Define Y to be the biggest closed subset of X where h does vanish.

Lemma 4.2.8. *We have*

$$Y(V)(D) = \{p \in X(V)(D) \mid \forall \varphi \in D \otimes \mathrm{Hom}(V,U) : h_D(P_{V,U,D}(\varphi)(p)) = 0\}$$

for all $V \in \mathbf{fgfMod}_R$ and R-domains D.

Proof. The closed subset Y is the intersection of X with the biggest closed subset of $\mathbb{A}_P$ where h vanishes. So the lemma follows from Lemma 3.4.22. □

Let $X = X_1 \supseteq X_2 \supseteq \cdots$ be a sequence of closed subsets of X. Since $Y < X$, the statement $\Sigma(R,P,Y)$ holds by the inner induction. In particular, the intersections of the X_i with Y stabilise. This settles part of condition (1) of $\Sigma(R,P,X)$. We now develop the theory to deal with the complement of Y. This will afterwards be used to settle both condition (2) for $\Sigma(R,P,X)$ in Section 4.2.10 and complete the proof of condition (1) in Section 4.2.11.

4.2.9 Dealing with the localised shift

In [Dra19, Lemma 25], it is proved that for all $\mathfrak{p} \in \mathrm{Spec}(R)$ and $V \in \mathbf{fgfMod}_R$, the projection $\mathrm{Sh}_U(P) \to \mathrm{Sh}_U(P)/M$ induces a homeomorphism of $\mathrm{Sh}_U(X)[1/h](V)(\overline{K_\mathfrak{p}})$ with a closed subset of the basic open $(\mathrm{Sh}_U(P)/M)[1/h](V)(\overline{K_\mathfrak{p}})$. This proof uses that $M_{\overline{K_\mathfrak{p}}}$ is irreducible, which is why we have localised so as to make this true. The proof shows that, indeed, for each linear function $x \in (\overline{K_\mathfrak{p}} \otimes M(V))^*$, the p^e-th power x^{p^e} lies in the sum of the ideal of $\mathrm{Sh}_U(X)[1/h](V)(\overline{K_\mathfrak{p}})$ in $\overline{K_\mathfrak{p}}[\overline{K_\mathfrak{p}} \otimes P(U \oplus V)][1/h]$ and the subring $\overline{K_\mathfrak{p}}[\overline{K_\mathfrak{p}} \otimes (P(U \oplus V)/M(V))]$. We globalise this result as follows: for all $V \in \mathbf{fgfMod}_R$, define

$$N(V) := \left\{x \in M(V)^* \,\middle|\, x^{p^e} \in \mathcal{I}_{\mathrm{Sh}_U(X)[1/h]}(V) + R[P(U \oplus V)/M(V)][1/h]\right\}.$$

There is a slight abuse of notation here: $M(V)$ is a submodule of $P(U \oplus V)$, so $M(V)^*$ is naturally a quotient of $P(U \oplus V)^*$ rather than a submodule. But the projection $P(U \oplus V) \to P(U \oplus V)/M(V)$ admits a section (indeed, we have arranged things such that $P(U \oplus V)$ is isomorphic to the direct sum of the free R-modules $M(V)$ and $P(U \oplus V)/M(V)$), and any section yields a section $M(V)^* \to P(U \oplus V)^*$. Two such sections differ by adding elements from $(P(U \oplus V)/M(V))^*$, which is contained in the second term above, so $N(V)$ does not depend on the choice of section.

Recall from Section 3.4.2 that $V^* \mapsto M(V)^*$ is a polynomial functor M^* of degree d.

Lemma 4.2.9. *The association $V^* \mapsto N(V)$ is a polynomial subfunctor of M^*.*

Proof. Let A be an R-algebra and take $V, W \in \mathbf{fgfMod}_R$. Take $y' \in A \otimes N(V)$ and $\varphi^* \in A \otimes \mathrm{Hom}(V^*, W^*)$ corresponding to $\varphi \in A \otimes \mathrm{Hom}(W, V)$. Then

$$\begin{aligned} A \otimes \mathrm{Hom}(M(W), M(V)) &\cong A \otimes \mathrm{Hom}(M(V)^*, M(W)^*) \\ &\cong \mathrm{Hom}_A(A \otimes M(V)^*, A \otimes M(W)^*). \end{aligned}$$

Denote the image of $M^*_{V^*,W^*,A}(\varphi^*) = M_{W,V,A}(\varphi)$ in $\mathrm{Hom}_A(A \otimes M(V)^*, A \otimes M(W)^*)$ by $M_{W,V,A}(\varphi)^*$. We need to show that $M_{W,V,A}(\varphi)^*(y') \in A \otimes N(W)$. This condition is A-linear in y', so we may assume that $y' = 1 \otimes y$ with $y \in N(V)$.

Choose $A = R[x_1, \dots, x_n]$ and $\varphi = \sum_i x_i \otimes \varphi_i$ where the φ_i form a basis of $\mathrm{Hom}(W, V)$. Then in particular we need that

$$M_{W,V,R[x_1,\dots,x_n]}(\textstyle\sum_i x_i \otimes \varphi_i)^*(1 \otimes y) \in R[x_1, \dots, x_n] \otimes N(W).$$

Conversely, by specializing the x_i to $a_i \in A$ for any R-algebra A, this in fact suffices. As M is a subfunctor of P, we may here replace M by P.

Since $P(V)$ is free, the R-linear map

$$P_{W,V,R[x_1,\dots,x_n]}(\textstyle\sum_i x_i \otimes \varphi_i)^*|_{P(V)^*} \colon P(V)^* \to R[x_1, \dots, x_n] \otimes P(W)^*$$

induces a homomorphism $\Phi \colon R[P(V)] \to R[x_1, \dots, x_n] \otimes R[P(W)]$ of R-algebras. As taking the p^e-th power is additive, an element z is contained in $R[x_1, \dots, x_n] \otimes N(W)$ if and only if z^{p^e} is contained in

$$R[x_1, \dots, x_n] \otimes (\mathcal{I}_{\mathrm{Sh}_U(X)[1/h]}(W) + R[P(U \oplus W)/M(W)][1/h]).$$

So we now need to show that $\Phi(y)^{p^e} = \Phi(y^{p^e})$ is contained in this latter set. Since $y \in N(V)$, we have $y^{p^e} = g_1 + g_2$ for some $g_1 \in \mathcal{I}_{\mathrm{Sh}_U(X)[1/h]}(V)$ and $g_2 \in R[P(U \oplus V)/M(V)][1/h]$. Now we note that $\Phi(g_1) \in R[x_1, \dots, x_n] \otimes \mathcal{I}_{\mathrm{Sh}_U(X)[1/h]}(W)$ as in the proof of Lemma 3.4.22 and $\Phi(g_2) \in R[x_1, \dots, x_n] \otimes R[P(U \oplus W)/M(W)][1/h]$. So indeed

$$M_{W,V,R[x_1,\dots,x_n]}(\textstyle\sum_i x_i \otimes \varphi_i)^*(1 \otimes y) \in R[x_1, \dots, x_n] \otimes N(W)$$

holds. □

Lemma 4.2.10. *For every $V \in$ **fgfMod**$_R$, every element of $M(V)^*$ has a nonzero R-multiple in $N(V)$.*

Proof. By [Dra19, Lemma 25], any element x of $M(V)^*$ has $1 \otimes x^{p^e} \in K \otimes N(V) \subseteq K \otimes M(V)^*$; in the symbol $\subseteq$ we use that $M(V)$, and hence $M(V)^*$, are free. Clearing denominators, we find that $rx^{p^e} \in M(V)^*$ for some nonzero $r \in R$. □

Lemma 4.2.11. *There exists a nonzero $r \in R$ such that $R[1/r] \otimes N(V) = R[1/r] \otimes M(V)^*$ holds for all $V \in$ **fgfMod**$_R$.*

Proof. Recall that the degree of the polynomial functor M is d and consider $V = R^d$. By Lemma 4.2.10 and the fact that $M(V)$ is finitely generated, there exists a nonzero $r \in R$ such that $R[1/r] \otimes N(V) = R[1/r] \otimes M(V)^*$. The Friedlander-Suslin's lemma, for polynomial functors over $R[1/r]$, gives that then $R[1/r] \otimes N(V) = R[1/r] \otimes M(V)^*$ for every V. □

We now replace R by the localisation $R[1/r]$ and may henceforth assume that $N(V) = M(V)^*$.

4.2.10 Proof of condition (2)

To establish condition (2) for (P, R, X), we will first prove an analoguous statement for the localised shift.

Lemma 4.2.12. *There exists a nonzero $r \in R$ such that the following holds for all $V \in$ **fgfMod**$_R$: if $g \in R[P(U \oplus V)]$ vanishes identically on $\mathrm{Sh}_U(X)[1/h](V)(\overline{K})$, then g vanishes identically on $\mathrm{Sh}_U(X)[1/h](V)(\overline{K_\mathfrak{p}})$ for all primes $\mathfrak{p} \in \mathrm{Spec}(R[1/r])$.*

Proof. Assume that $g \in R[P(U \oplus V)]$ vanishes identically on $\mathrm{Sh}_U(X)[1/h](V)(\overline{K})$. View g as a polynomial in the coordinates x_i of $M(V)^*$ corresponding to a basis of $M(V)$ with coefficients in $R[P(U \oplus V)/M(V)]$. By the conclusion of Section 4.2.9, we have $N(V) = M(V)^*$, which means that each $x_i^{p_e}$ is a sum of an element in $R[P(U \oplus V)/M(V)][1/h]$ and an element in the ideal of $\mathrm{Sh}_U(X)[1/h](V)$. We then find that also $g^{p^e} = g_1 + g_2$ with $g_1 \in R[P(U \oplus V)/M(V)][1/h]$ and $g_2 \in \mathcal{I}_{\mathrm{Sh}_U(X)[1/h](V)}$. Let Z be the closure of the projection of $\mathrm{Sh}_U(X)[1/h]$ to $(\mathrm{Sh}_U(P)/M)[1/h]$. Since both g and g_2 vanish identically on $\mathrm{Sh}_U(X)[1/h](V)(\overline{K})$, g_1 vanishes identically on $Z(V)(\overline{K})$. By the outer induction hypothesis, after a localisation that doesn't depend on g_1 or on V, one concludes that g_1 vanishes identically on $Z(V)(\overline{K_\mathfrak{p}})$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. But then g^{p^e}, and hence g itself, vanish identically on $\mathrm{Sh}_U(X)[1/h](V)(\overline{K_\mathfrak{p}})$. □

Now we can establish condition (2) of $\Sigma(R, P, X)$:

Proposition 4.2.13. *There exists a nonzero $r \in R$ such that the following holds for all $V \in$ $\mathbf{fgfMod}_R$: if $g \in R[P(V)]$ vanishes identically on $X(V)(\overline{K})$, then g vanishes identically on $X(V)(\overline{K_{\mathfrak{p}}})$ for all primes $\mathfrak{p} \in \mathrm{Spec}(R[1/r])$.*

Remark 4.2.14. For each fixed V, such an r exists by Proposition 3.3.20. Taking the product of such r's, the same applies to a finite number of V's, so we may restrict our attention to all V of sufficiently large rank; we will do this in the proof. ♪♫

Proof of Proposition 4.2.13. By the inner induction hypothesis, after replacing R by a localisation $R[1/r]$, we know that if $g \in R[P(V)]$ vanishes identically on $Y(V)(\overline{K})$, then it vanishes identically on $Y(V)(\overline{K_{\mathfrak{p}}})$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

For any $V \in \mathbf{fgfMod}_R$ and $\mathfrak{p} \in \mathrm{Spec}(R)$, define $Z(V)(\overline{K_{\mathfrak{p}}}) := X(V)(\overline{K_{\mathfrak{p}}}) \setminus Y(V)(\overline{K_{\mathfrak{p}}})$. It suffices to show that with a further localisation we achieve that for any $V \in \mathbf{fgfMod}_R$, if $g \in R[P(V)]$ vanishes identically on all points of $Z(V)(\overline{K})$, then it vanishes identically on all points of $Z(V)(\overline{K_{\mathfrak{p}}})$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. In proving this, by Remark 4.2.14 above, we may assume that V has rank at least that of U. Hence we may replace V by $U \oplus V$.

Such a g that vanishes identically on $Z(U \oplus V)(\overline{K})$ vanishes, in particular, identically on $\mathrm{Sh}_U(X)[1/h](V)(\overline{K})$. Lemma 4.2.12 says that (after replacing R by a localisation that does not depend on g or V), g also vanishes identically on $\mathrm{Sh}_U(X)[1/h](V)(\overline{K_{\mathfrak{p}}})$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$. This basic open is actually dense in $Z(U \oplus V)(\overline{K_{\mathfrak{p}}})$, as one sees as follows: $Z(U \oplus V)(\overline{K_{\mathfrak{p}}})$ is the image of the action

$$\mathrm{GL}(\overline{K_{\mathfrak{p}}} \otimes (U \oplus V)) \times \mathrm{Sh}_U(X)[1/h](V)(\overline{K_{\mathfrak{p}}}) \to X(U \oplus V)(\overline{K_{\mathfrak{p}}}).$$

If the basic open were contained in the union of a proper subset of the irreducible components of $Z(U \oplus V)(\overline{K_{\mathfrak{p}}})$, then, by irreducibility of $\mathrm{GL}(\overline{K_{\mathfrak{p}}} \otimes (U \oplus V))$, so would the image of that action, a contradiction. Hence g then vanishes identically on $Z(V)(\overline{K_{\mathfrak{p}}})$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. □

Remark 4.2.15. Note that, unlike Y, the Z defined in the proof is not a subset of X in the sense of Definition 3.4.17. ♪♫

4.2.11 Proof of the Noetherianity of X

Finally, we prove condition (1) of $\Sigma(R, P, X)$. Let $X = X_1 \supseteq X_2 \supseteq \cdots$ be a sequence of closed subsets of X. Recall from Section 4.2.8 that the intersections of the X_i with Y stabilise. Now, consider again the projection $\mathrm{Sh}_U(P)[1/h] \to (\mathrm{Sh}_U(P)/M)[1/h]$. We let Z_i' be the closure of the image of $\mathrm{Sh}_U(X_i)[1/h]$ in $(\mathrm{Sh}_U(P)/M)[1/h]$. Since the polynomial functor $(\mathrm{Sh}_U(P)/M)$ is smaller then P, we have Noetherianity for $(\mathrm{Sh}_U(P)/M)[1/h]$

and therefore the sequence $Z'_1 \supseteq Z'_2 \supseteq \cdots$ stabilises. We now conclude from this that the sequence of $\mathrm{Sh}_U(X_i)[1/h]$'s also stabilises.

Lemma 4.2.16. *Let $X'' \subseteq X' \subseteq X$ be closed subsets, assume $\mathrm{Sh}_U(X'')[1/h] \subsetneq \mathrm{Sh}_U(X')[1/h]$ and let $Z'' \subseteq Z'$ be the closures of their images in $(\mathrm{Sh}_U(P)/M)[1/h]$. Then $Z'' \subsetneq Z'$.*

Proof. Since $\mathrm{Sh}_U(X'')[1/h] \subsetneq \mathrm{Sh}_U(X')[1/h]$, we have

$$\mathrm{Sh}_U(X'')[1/h](V) \subsetneq \mathrm{Sh}_U(X')[1/h](V)$$

for some $V \in \mathbf{fgfMod}_R$. This means that $\mathcal{I}_{\mathrm{Sh}_U(X'')[1/h]}(V) \supsetneq \mathcal{I}_{\mathrm{Sh}_U(X')[1/h]}(V)$. Let $g \in R[P(U \oplus V)][1/h]$ be an element of the former ideal that is not contained in the latter. Then the same holds for g^{p^e}. By the conclusion of Section 4.2.9, g^{p^e} is a sum of an element g_1 in $R[P(U \oplus V)/M(V)][1/h]$ and an element g_2 of $\mathcal{I}_{\mathrm{Sh}_U(X)[1/h]}(V) \subseteq \mathcal{I}_{\mathrm{Sh}_U(X')[1/h]}(V)$. This means that g_1 is also an element of $\mathcal{I}_{\mathrm{Sh}_U(X'')[1/h]}(V)$ not contained in $\mathcal{I}_{\mathrm{Sh}_U(X')[1/h]}(V)$. Hence

$$\mathcal{I}_{\mathrm{Sh}_U(X'')[1/h]}(V) \cap R[P(U\oplus V)/M(V)][1/h] \supsetneq \mathcal{I}_{\mathrm{Sh}_U(X')[1/h]}(V) \cap R[P(U\oplus V)/M(V)][1/h]$$

holds. The former ideal of $R[P(U\oplus V)/M(V)][1/h]$ equals $\mathcal{I}_{Z''}(V)$ and the latter equals $\mathcal{I}_{Z'}(V)$. So $Z''(V) \subsetneq Z'(V)$ and hence $Z'' \subsetneq Z'$. □

By the lemma, the fact that the sequence of Z'_i stabilises implies that the sequence of $\mathrm{Sh}_U(X_i)[1/h]$'s also stabilises. Now again, we write

$$Z_i(V)(\overline{K_\mathfrak{p}}) = X_i(V)(\overline{K_\mathfrak{p}}) \setminus Y(V)(\overline{K_\mathfrak{p}})$$

for all $V \in \mathbf{fgfMod}_R$ and $\mathfrak{p} \in \mathrm{Spec}(R)$. We consider the descending sequence of Z_i's. What is left to prove for the Noetherianity of X is the following result.

Lemma 4.2.17. *The sequence $Z_1 \supseteq Z_2 \supseteq \cdots$ stabilises.*

Proof. Let m be the rank of U. As in equation $(*)$ in [Dra19, Section 2.9], for every $\mathfrak{p} \in \mathrm{Spec}(R)$, we have:

$$Z_i(U\oplus V)(\overline{K_\mathfrak{p}}) = \{p \in X_i(U\oplus V)(\overline{K_\mathfrak{p}}) : h(g\cdot p) \neq 0 \text{ for some } g \in \mathrm{GL}(\overline{K_\mathfrak{p}}\otimes(U\oplus V))\},$$

and the right-hand-side can be written as:

$$\bigcup_{g\in\mathrm{GL}(\overline{K_\mathfrak{p}}\otimes(U\oplus V))} g\cdot \mathrm{Sh}_U(X_i)[1/h](V)(\overline{K_\mathfrak{p}}).$$

So the sequence of Z_i's restricted to $V \in \mathbf{fgfMod}_R$ of rank $\geq m$ stabilizes. As the sequence of $X_i(R^k)$'s stabilizes for each $k \in \{0, \dots, m-1\}$ by Proposition 3.3.14, the unrestricted sequence of Z_i's also stabilizes. □

Since both the sequence of $X_i \cap Y$'s and Z_i's stabilize, using Corollary 3.3.11, the sequence of X_i's also stabilizes. So the closed subset X is Noetherian. This concludes the proof of condition (1) for (R, P, X) and hence the proof of Theorem 4.1.1.

4.3 Applications

In this section we present some applications. The first part concerns the existence of upper bounds for the degrees of defining equations for closed subsets of polynomial functors. We will not find explicit upper bounds but we show how these are independent on the field of definition and on the dimension. Note that for "defining equations" we mean the set-theoretic defining equations. The second part concerns how dimensions of such closed subsets behave "for n tending at infinity".

4.3.1 Upper bounds for degrees of defining equations

Our original motivation is the following: let P, Q be (finite-degree) polynomial functors from the category of finitely generated free $\mathbf{Z}$-modules to itself and let $\alpha\colon Q \to P$ be a polynomial transformation. Define the closed subset X of $\mathbb{A}_P$ as the closure of the image of α. Specifically, for a natural number n, the pull-back along $\alpha_{\mathbf{Z}^n}$ defines a ring homomorphism $\mathbf{Z}[P(\mathbf{Z}^n)] \to \mathbf{Z}[Q(\mathbf{Z}^n)]$, and $X(\mathbf{Z}^n)$ is the closed subset of $\operatorname{Spec} \mathbf{Z}[P(\mathbf{Z}^n)]$ defined by the kernel of that ring homomorphism. Theorem 4.1.1 implies the following.

Corollary 4.3.1. *There exists a uniform bound d such that for all $n \in \mathbf{Z}_{\geq 0}$ and all fields K, $X(K^n) \subseteq K \otimes P(\mathbf{Z}^n)$ is defined by polynomials of degree $\leq d$.*

This corollary has many applications; here is one. If V is a finite-dimensional vector space over a field K and $T \in V \otimes V \otimes V$ is a tensor, then T is said to have *slice rank* $\leq r$ if T can be written as the sum of r terms of the form $\sigma(v \otimes A)$, where $v \in V$ and $A \in V \otimes V$, and σ is a cyclic permutation of 1, 2, 3 permuting the tensor factors. If K is algebraically closed, then being of slice rank $\leq r$ is a Zariski-closed condition [ST].

Corollary 4.3.2. *Fix a natural number r. There exists a uniform bound d such that for all algebraically closed fields K and for all $n \in \mathbf{Z}_{\geq 0}$, the variety of slice-rank-$\leq r$ tensors in $K^n \otimes K^n \otimes K^n$ is defined by polynomials of degree $\leq d$.*

The same holds when the number of tensor factors is increased to any fixed number, possibly at the expense of increasing d, and similar results hold for the set of cubic forms of bounded q-rank [DES17] or for the closure of the set of degree-e forms of bounded strength in the sense of [AH20a]. We stress again that "defined by" is intended in a purely set-theoretic sense. We do not know whether the vanishing ideals of these varieties are generated in bounded degree, even if the field K were fixed beforehand.

Proof of Corollary 4.3.2. Consider the polynomial functor P that sends a free $\mathbf{Z}$-module $\mathbf{Z}^n$ to $\mathbf{Z}^n \otimes \mathbf{Z}^n \otimes \mathbf{Z}^n$, and the polynomial functor Q that sends $\mathbf{Z}^n$ to $\mathbf{Z}^n \oplus (\mathbf{Z}^n \otimes \mathbf{Z}^n)$. For any r-tuple $(\sigma_1, \ldots, \sigma_r)$ of cyclic permutations of $1, 2, 3$ we have a polynomial transformation

$$Q^r \to P, ((v_1, A_1), \ldots, (v_r, A_r)) \mapsto \sum_{i=1}^{r} \sigma_i(v_i \otimes A_i),$$

whose image closure is defined in uniformly bounded degree e by Corollary 4.3.1. The variety of slice-rank-$\leq r$ tensors is the union of these image closures over all r-tuples of cyclic permutations, hence defined in degree at most $e \cdot 3^r$, independently of the algebraically closed field and independently of n. □

Remark 4.3.3. Over a field K of characteristic zero, the irreducible polynomial functors P are precisely the Schur functors, and any polynomial functor is isomorphic to a direct sum of Schur functors. These always admit a $\mathbf{Z}$-form, i.e., a polynomial functor $P_{\mathbf{Z}}$ over $\mathbf{Z}$ such that $K \otimes P_{\mathbf{Z}} \cong P$, which moreover has the property that it maps free $\mathbf{Z}$-modules to free $\mathbf{Z}$-modules [ABW82]. The $\mathbf{Z}$-form need not be unique; e.g., the Schur functor over K that maps V to its d-th symmetric power $S^d(V)$, comes both from the functor from free $\mathbf{Z}$-modules to free $\mathbf{Z}$-modules that sends U to $S^d(U)$ and from the functor that sends U to the sub-$\mathbf{Z}$-module of $U^{\otimes d}$ consisting of symmetric tensors. These two functors are non-isomorphic $\mathbf{Z}$-forms. In applications such as the above, where one looks for field-independent bounds, it is important to choose the $\mathbf{Z}$-form that captures the problem of interest. ♪♫

Example 4.3.4. Again over $R = \mathbf{Z}$, consider the polynomial transformation $\alpha\colon (S^2)^4 \to S^4$ that maps a quadruple $(q_1, \ldots, q_4)$ of quadratic forms to $q_1^2 + \cdots + q_4^2$. Let X be the image closure as above. If K is algebraically closed of characteristic zero, then $X_K(K^4)$ is a hypersurface in $S^4(K)$ of degree 38475 [BHORS12], so the degree bound from Corollary 4.3.1 must be at least that large. On the other hand, if K is algebraically closed of characteristic 2, then the image of α is just the linear space spanned by all degree-four monomials that are squares, and hence only linear equations are needed to cut out this image. ♫

Remark 4.3.5. Over algebraically closed fields of positive characteristic, irreducible polynomial functors are still parameterised by partitions, but polynomial functors are no longer semisimple, and the $\mathbf{Z}$-forms from Remark 4.3.3 do not always remain irreducible; standard references are [CL74; Gre07]. The typical example is that, in characteristic p, the functor S^p contains a subfunctor that maps V to the linear space of p-th powers of elements of V. ♪♫

4.3.2 Dimension functions of closed subsets of polynomial functors

To illustrate that the proof method for Theorem 4.1.1 can be used to obtain further results on closed subsets of polynomial functors, we establish a natural common variant of

Propositions 3.3.19 and 3.4.10. For each $\mathfrak{p} \in \mathrm{Spec}(R)$ define the function $f_{\mathfrak{p}} : \mathbf{Z}_{\geq 0} \to \mathbf{Z}_{\geq 0}$ as $f_{\mathfrak{p}}(n) := \dim(X(R^n)(\overline{K_{\mathfrak{p}}}))$.

Proposition 4.3.6. *For each $\mathfrak{p} \in \mathrm{Spec}(R)$, $f_{\mathfrak{p}}(n)$ is a polynomial in n with integral coefficients for all $n \gg 0$. Furthermore, the map that sends $\mathfrak{p}$ to this polynomial is constructible.*

Proof sketch. Both statements follow by inductions identical to the one for Theorem 4.1.1, using that, in the most interesting induction step, for $n \geq m := \mathrm{rk}(U)$ the dimension of $X_{\overline{K_{\mathfrak{p}}}}(\overline{K_{\mathfrak{p}}}^n)$ is the maximum of the dimensions of $Y_{\overline{K_{\mathfrak{p}}}}(\overline{K_{\mathfrak{p}}}^n)$ and

$$(\mathrm{Sh}_U(X)[1/h])_{\overline{K_{\mathfrak{p}}}}(\overline{K_{\mathfrak{p}}}^{n-m}).$$

Furthermore, for the case where $X_{\overline{K_{\mathfrak{p}}}}$ is the pre-image of $X'_{\overline{K_{\mathfrak{p}}}}$, we use Proposition 3.4.10, and for the base case in the induction proof for the constructibility statement we use Proposition 3.3.19. □

Example 4.3.7. Take $R = \mathbf{Z}$, take $P = S^3$, and let X be the closed subset defined as the image closure of the polynomial transformation $(S^1)^2 \to S^3$, $(v, w) \mapsto v^3 + w^3$; see Section 4.3 for similar polynomial transformations. Then $X_{\overline{K_{\mathfrak{p}}}}(\overline{K_{\mathfrak{p}}}^n)$ has dimension $2n$ for $\mathfrak{p} \neq (3)$ and dimension n for $\mathfrak{p} = (3)$, since in the latter case the set of cubes of linear forms is a linear subspace of the space of cubics. This is an instance of Proposition 4.3.6. ♫

Chapter 5

Intermezzo: a glance at singularities

In this chapter we consider polynomial functors in bounded degree defined over a field K of characteristic zero and we use the notation of page 15.

5.1 Stabilisation of the singular locus

There are meaningful notions of singularity for the infinite dimensional setting, for example using *embedding codimension*; see [CFD22]. However, we are interested in the behaviour of the singular locus of a **Vec**-variety X in the following sense: for every $V \in$ **Vec** we consider $\mathrm{Sing}(X(V))$, the singular locus of $X(V)$, and we look at its behaviour for $\dim(V)$ tending to infinity. In particular, our notion of singularity remains the classical one of finite dimensional algebraic geometry. The following example was the initial inspiration for our result.

Example 5.1.1. It is well known that the singular locus of the variety of matrices of rank at most r is the variety of matrices of rank at most $r-1$. This statement is independent of the size of the matrices, provided that both sizes are strictly greater than r. In this chapter we establish a far-reaching generalisation of this phenomenon: we will show that the singular locus of a **Vec**-variety is itself a **Vec**-variety. ♫

5.1.1 Description of the result

Our main result is as follows.

Theorem 5.1.2 (Stabilisation of Singularities Theorem)**.** *For any finite-degree polynomial functor P over a field K of characteristic zero, and for any* **Vec***-variety X of P, there exists a unique* **Vec***-subvariety Y of X such that for all $V \in$* **Vec** *of sufficiently large dimension we have $Y(V) = \mathrm{Sing}(X(V))$. Furthermore, for* all *V we then have $\mathrm{Sing}(X(V)) \subseteq Y(V)$.*

We will denote this **Vec**-variety Y by $\mathrm{Sing}(X)$, but warn that, for low-dimensional V, it will in general not be true that $\mathrm{Sing}(X)(V) = \mathrm{Sing}(X(V))$; see the following example, which formalises Example 5.1.1.

Example 5.1.3. Consider the second tensor power T^2 and the **Vec**-variety $\mathcal{M}_{\leq r}$ defined by the vanishing of all $(r+1) \times (r+1)$-subdeterminants. For $\dim(V) \leq r$, the variety $\mathcal{M}_{\leq r}(V)$ coincides with the ambient space, with empty singular locus. On the other hand, for $\dim(V) > r$, the singular locus of $\mathcal{M}_{\leq r}(V)$ is precisely $\mathcal{M}_{\leq r-1}(V)$. Hence we have $\mathrm{Sing}(\mathcal{M}_{\leq r}) = \mathcal{M}_{\leq r-1}$. Indeed, $\mathrm{Sing}(\mathcal{M}_{\leq r})(V) = \mathrm{Sing}(\mathcal{M}_{\leq r}(V))$ for $\dim(V) > r$, while $\mathrm{Sing}(\mathcal{M}_{\leq r}(V)) \subsetneq \mathcal{M}_{\leq r-1}(V)$ for $\dim(V) \leq r$. ♫

5.2 Proof of stabilisation of the singular locus

5.2.1 Reduction to the irreducible case

Lemma 5.2.1. *Suppose that Theorem 5.1.2 holds for irreducible* **Vec***-varieties of P. Then it holds for all* **Vec***-varieties of P.*

Proof. By Noetherianity guaranteed by Theorem 2.4.3, X admits a unique decomposition $X_1 \cup \cdots \cup X_s$ where each X_i is an irreducible **Vec**-variety of P, and not contained in the union $\bigcup_{j \neq i} X_j$; this means that $X_i(V) \not\subseteq \bigcup_{j \neq i} X_j(V)$ for all V of sufficiently large dimension. Then, for each V with $\dim(V) \gg 0$, we have

$$\mathrm{Sing}(X(V)) = \left(\bigcup_i \mathrm{Sing}(X_i(V)) \right) \cup \left(\bigcup_{i \neq j} (X_i(V) \cap X_j(V)) \right)$$

By assumption, $\mathrm{Sing}(X_i(V))$ agrees with $Y_i(V)$ for some closed subvariety Y of X_i and $\dim(V) \gg 0$, and each of the intersections $X_i \cap X_j$ define **Vec**-subvarieties of X, hence the theorem follows from the fact that the class of **Vec**-subvarieties of X is preserved under finite unions. □

5.2.2 The irreducible case

Proof of Theorem 5.1.2. The uniqueness of the **Vec**-variety $Y \subseteq X$ such that $Y(V) = \mathrm{Sing}(X)(V)$ for all V of sufficiently high dimension follows immediately from the fact that $Y(V)$ for $\dim(V) \gg 0$ determines $Y(V)$ for small V by Proposition 2.2.14.

To prove the existence of Y, by Lemma 5.2.1 we may assume that X is irreducible. We proceed by induction on the polynomial functor P, using the well-founded order from Section 2.1.3. For $\deg(P) = 0$, $X(V)$ is just a fixed closed subvariety, independent of V, of the fixed finite-dimensional affine space $P(V)$, and we may set $Y(V) := \mathrm{Sing}(X(V))$ for all V.

Suppose next that $d := \deg(P) > 0$, suppose that the theorem holds for all polynomial functors $Q \prec P$, and let R be an irreducible subfunctor of the top-degree part of P. Since $\mathrm{char}\, K = 0$, we may write $P = P' \oplus R$ where $P' \prec P$. Let $\pi : P \to P'$ be the projection along R and define $X'(V) := \overline{\pi(X(V))}$. Then X' is a **Vec**-variety of P'.

By Theorem 2.4.1, there are two possibilities. In the first case, we have $X(V) = X'(V) \times R(V)$ for all V, and then clearly $\mathrm{Sing}(X(V)) = \mathrm{Sing}(X'(V)) \times R(V)$ for all V. By the induction assumption, there exists a unique closed subvariey Y' of X' such that the latter equals $Y'(V) \times R(V)$ for all V of sufficiently high dimension. Then with $Y := Y' \times R$, we have $Y(V) = \mathrm{Sing}(X(V))$ for all V of sufficiently high dimension.

The previous paragraph applies as long as there exists an irreducible subfunctor R to which the first case applies. We may therefore assume that no such subfunctor exists. Then, by Proposition 2.4.8, the dimension polynomial of X has degree strictly less than d.

So now, fixing any irreducible subfunctor R of P_d, we find U, f, r, h, P'', Z as in the second part of Theorem 2.4.1. Then $\mathrm{Sh}_U(X)[1/h] \cong Z \subseteq P''[1/h]$ where $P'' := \mathrm{Sh}_U(P)/R \prec P$, and by the induction assumption there exists a **Vec**-variety Y' of $P''[1/h]$ such that, for $\dim(V)$ at least some n_1, we have $\mathrm{Sing}(Z(V)) = Y'(V)$.

Now, for any $V \in$ **Vec**, $\mathrm{Sh}_U(X)[1/h](V) \cong Z(V)$, and we use this isomorphism to identify $Y'(V)$ with a locally closed subset of $X(U \oplus V)$. On the latter variety acts $\mathrm{GL}(U \oplus V)$ by automorphisms by Proposition 2.2.14, hence preserving the singular locus. Take $\dim(V) \geq n_1$, so that $Y'(V)$ is the locus of singular points in $X(U \oplus V)$ where h is nonzero. Then for any $p \in Y'(V)$ the map $\mathrm{GL}(U \oplus V) \to X(V), g \mapsto g \cdot p$ maps $\mathrm{GL}(U \oplus V)$ into $\mathrm{Sing}(X(U \oplus V))$, and in fact an open dense subset of $\mathrm{GL}(U \oplus V)$ into $Y'(V)$. This implies that the closure $\overline{Y'(V)}$ of $Y'(V)$ in $X(U \oplus V)$ is a $\mathrm{GL}(U \oplus V)$-stable closed subset of $\mathrm{Sing}(X(U \oplus V))$.

For $W \in$ **Vec**, we define $Y_1(W) \subseteq X(W)$ as follows. Choose any vector space V with $\dim(V) \geq n_1$ and $\dim(U \oplus V) \geq \dim(W)$ and any surjective linear map $\varphi : U \oplus V \to W$ and set

$$Y_1(W) := \overline{P(\varphi) Y'(V)}.$$

Lemma 5.2.2. *The rule Y_1 defines a* **Vec**-*variety of X with $Y_1(W) \subseteq \mathrm{Sing}(X(W))$ for all W with $\dim(W) \geq n_1 + \dim(U)$. More precisely, for such large W, $Y_1(W)$ is the closure in $X(W)$ of the set*

$$\{p \in \mathrm{Sing}(X(W)) \mid \exists \psi \in \mathrm{Hom}(W, U) : h(P(\psi)p) \neq 0\}.$$

Proof. We first show that $Y_1(W)$ does not depend on the choice of (sufficiently large) V and of φ. Let $\varphi' : U \oplus V' \to W$ be another surjective linear map. Without loss of generality, assume that $\dim(V') \geq \dim(V)$. Write $\ker \varphi' = A_1 \oplus A_2$ where $\dim(A_2) = \dim(V') - \dim(V)$. Then φ' factors as

$$\varphi' = \varphi \circ g_1 \circ (\mathrm{id}_U \times \psi) \circ g_2$$

where $g_2 \in \mathrm{GL}(U \oplus V')$ maps A_2 into a subspace B_1 of V', $\psi : V' \to V$ is a linear map with kernel B_1, and $g_1 \in \mathrm{GL}(U \oplus V)$ maps the image of A_1 under $(\mathrm{id}_U \times \psi) \circ g_2$, which has dimension $\dim(U \oplus V) - \dim(W)$, into $\ker \varphi$. Now, by the above, g_2 preserves $\overline{Y'(V')}$, g_1 preserves $\overline{Y'(V)}$, and by definition $\mathrm{id}_U \times \psi$ maps the former onto the latter. We conclude that $\varphi'(\overline{Y'(V')}) = \varphi(\overline{Y(V)})$, and then the same holds when we take the closure on both sides.

Furthermore, that Y_1 is (covariantly) functorial in W is immediate from the definition, and so is the fact that $Y_1(W)$ is closed in $X(W)$ for every $W \in \mathbf{Vec}$. That $Y_1(W)$ is contained in $\mathrm{Sing}(X(W))$ for $\dim(W) \geq n_1 + \dim(U)$ follows from the discussion before the lemma. Finally, for W of this dimension, we may fix any isomorphism $\varphi : U \oplus V \to W$, and then $Y_1(W) = \overline{P(\varphi)Y'(V)}$ by the independence of $Y_1(W)$ of the choice of φ. Then take $\psi = \pi_U \circ \varphi^{-1}$ where $\pi_U : U \oplus V \to U$ is the projection. Then for any point of the form $q := P(\varphi)p$ with $p \in Y'(V)$ we have $h(P(\psi)q) = h(P(\pi_U)p) \neq 0$. □

Next we vary R, U, f, r, and hence h. Let $Z \subseteq X$ be the $\mathbf{Vec}$-subvariety defined by the vanishing of *all* partial derivatives $\tilde{h} := \partial\tilde{f}/(\partial\tilde{r})$ for $\tilde{f}$ an element in the ideal of $X(\tilde{U})$, for some $\tilde{U}$, and $\tilde{r}$ an element in $\tilde{R}(\tilde{U})$ where $\tilde{R}$ is an arbitrary irreducible subfactor of P_d.

Lemma 5.2.3. *For all V of sufficiently large dimension, $Z(V) \subseteq \mathrm{Sing}(X(V))$.*

Proof. First, since polynomial functors in characteristic zero form a semisimple category, P_d is the sum of its irreducible subfunctors $\tilde{R}$. Therefore, the directional derivatives $\partial\tilde{f}/(\partial\tilde{r})$ vanish on $Z(V)$ for all $\tilde{r} \in P_d(V)$. We construct the Jacobi-matrix for $X(V)$ as follows: the rows correspond to a generating set of the ideal, the first $n_{<d} = n_{<d}(V)$ columns correspond to coordinates on $P_{<d}(V)$, and the last $n_d = n_d(V)$ columns correspond to coordinates on $P_d(V)$. By Proposition 2.1.29, $n_{<d}$ grows as a polynomial of degree $< d$ in $\dim(V)$ and n_d grows as a polynomial of degree d in $\dim(V)$. Furthermore, the dimension polynomial of $X(V)$ has degree $< d$ by Proposition 2.4.8. This implies that the codimension $c = c(V)$ of $X(V)$ in $P(V)$ grows as a polynomial of degree d in V and, for $\dim(V) \gg 0$, any $c \times c$-submatrix of the Jacobi matrix intersects the last n_d columns. On $Z(V)$, the last n_d columns are identically zero. Hence $Z(V) \subseteq \mathrm{Sing}(X(V))$, as desired. □

Now, by Noetherianity of Theorem 2.4.3, the closed subvariety Z of X is defined by finitely many $h_i := \partial f_i/\partial r_i$ for $i = 1, \ldots, k$, defined with respect to the vector space U_i.

For each $i = 1, \dots, k$, we construct, as above, a closed subvariety Y_i of X such that, for $\dim(V) \gg 0$, we have

$$Y_i(V) = \overline{\{p \in \mathrm{Sing}(X(V)) \mid \exists \psi : V \to U_i : h_i(P(\psi)(p)) \neq 0\}}.$$

Now, for $\dim(V) \gg 0$, we have

$$Y_1(V) \cup \dots \cup Y_k(V) \cup Z(V) = \mathrm{Sing}(X(V))$$

so that $Y := Y_1 \cup \dots \cup Y_k \cup Z$ is a **Vec**-subvariety of X as desired. $\square$

Chapter 6

Strength and stabilisation

In this chapter we make use of the notation at page 15, and we work with polynomial functors defined over an infinite field K that, for the relevant results, will typically be asked to be algebraically closed and of characteristic zero. In particular, we work with polynomial functors of Chapter 2. Moreover, from Section 2.3 we use the notion of strength, the quasi-order $\preceq$ on infinite tensors, and the equivalence relation $\simeq$ (see Section 2.3.4 for the latter two). In Section 6.1 we present our two main theorems: Theorem 6.1.6, called "The Parameterisation Theorem for GL-subsets"[1], and Theorem 6.1.14, called "The Extreme Elements Theorem"[2]. In this same section we cite the relevant literature. Section 6.2 contains our proof of The Parameterisation Theorem while Section 6.3 spells out the details for proving The Extreme Elements Theorem by constructing the extreme elements. Finally, in Section 6.4 we present some more examples.

6.1 Introduction

Definition 6.1.1. Let P be a polynomial functor. A *subset* X of P is a functor from **Vec** to **Vec** such that $X(V) \subset P(V)$ for every $V \in$ **Vec** and $X_{U,V}(\varphi) = F_{U,V}(\varphi)_{|X(U)}$ for every $U, V \in$ **Vec** and for every $\varphi \in \mathrm{Hom}(U, V)$. ♪

Recall our convention on subfunctors of a polynomial functor of Remark 2.2.7, and note that a subset X of a polynomial functor P such that $X(V)$ is Zariski-closed in $P(V)$ for every $V \in$ **Vec** is a **Vec**-variety of P.

[1] This is a local name appearing only in this thesis.

[2] See above.

6.1.1 Kazhdan-Ziegler's theorem: universality of strength

Theorem 6.1.2 (Kazhdan-Ziegler [KZ20, Theorem 1.9])**.** *Let $d \geq 2$ be an integer. Assume that K is algebraically closed and of characteristic 0 or $> d$. Let X be a subset of S^d. Then either $X = S^d$ or else there exists an integer $k \geq 0$ such that each polynomial in each $X(U)$ has strength $\leq k$.*

This theorem is a strengthening of [BDE19, Theorem 4], where the additional assumption is that X is closed. The condition that K be algebraically closed cannot be dropped. Indeed, consider the following example.

Example 6.1.3. Take $K = \mathbf{R}$ and let $X(V)$ be the set of positive semidefinite elements in $S^2(V)$, i.e., those that are sums of squares of elements of V. Then X is a subset of S^2. ♫

Note now that there is no uniform upper bound on the strength of positive definite quadratic forms. The condition on the characteristic can also not be dropped, but see Remark 6.1.15.

Corollary 6.1.4 (Kazhdan-Ziegler, universality of strength)**.** *With the same assumptions on K, for every fixed number of variables $m \geq 1$ and degree $d \geq 2$ there exists an $r \geq 0$ such that for any number of variables $n \geq 1$, any polynomial $f \in K[x_1, \ldots, x_n]_d$ of strength $\geq r$ and any polynomial $g \in K[y_1, \ldots, y_m]_d$ there exists a linear variable substitution $x_j \mapsto \sum_i c_{ij} y_i$ under which f specialises to g.*

Proof. For each $U \in \mathbf{Vec}$, define $X(U) \subseteq S^d(U)$ as the set of all f such that the map

$$\begin{aligned} \mathrm{Hom}(U, K^m) &\to S^d(K^m) \\ \varphi &\mapsto S^d(\varphi)f \end{aligned}$$

is *not* surjective. A straightforward computation shows that this is a subset of S^d. It is not all of S^d, because if we take U to be of dimension $d(\dim(S^d(K^m)))$, then in $S^d(U)$ we can construct a sum f of $\dim(S^d(K^m))$ squarefree monomials in distinct variables and specialise each of these monomials to a prescribed multiple of a basis monomial in $S^d(K^m)$. Hence $f \notin X(U)$. By Theorem 6.1.2, it follows that the strength of elements of $X(U)$ is uniformly bounded. □

6.1.2 Our generalisation: universality for polynomial functors

Let P, Q be polynomial functors. Recall that Q is smaller than P, denoted $Q \prec P$, when P and Q are not (linearly) isomorphic and Q_d is a quotient of P_d for the highest degree d where P_d and Q_d are not isomorphic. We say that a polynomial functor P is pure when $P(\{0\}) = \{0\}$.

Remark 6.1.5. Let $Q \prec P$ be polynomial functors and suppose that P is homogeneous of degree $d > 0$. Then Q_d must be a quotient of P_d. So we see that $Q \oplus R \prec P$ for any polynomial functor R of degree $< d$. ♪♫

The following is our first main result.

Theorem 6.1.6 (Parameterisation Theorem for GL-subsets)**.** *Assume that K is algebraically closed of characteristic zero. Let X be a subset of a pure polynomial functor P over K. Then either $X(U) = P(U)$ for all $U \in$ **Vec** or else there exist finitely many polynomial functors $Q_1, \dots, Q_k \prec P$ and polynomial transformations $\alpha_i\colon Q_i \to P$ with $X(U) \subseteq \bigcup_{i=1}^k \operatorname{im}(\alpha_{i,U})$ for all $U \in$ **Vec**. In the latter case, X is contained in a proper closed subset of P.*

*If we assume furthermore that P is irreducible, then in the second case there exists a integer $k \geq 0$ such that for all $U \in$ **Vec** and all $p \in X(U)$ the strength of p is at most k.*

This is a strengthening of a theorem from [BDES22] (also in [Bik20, Theorem 4.2.5]), where the additional assumption is that X be closed.

Remark 6.1.7. When P is irreducible of degree 1, then $P(U) = U$. In this case, the subsets of P are P and $\{0\}$. So indeed, the elements of a proper subset of P have bounded strength, namely 0. ♪♫

Again, the condition that K be algebraically closed cannot be dropped, and neither can the condition on the characteristic; however, see Remark 6.1.15. Theorem 6.1.6 has the same corollary as Theorem 6.1.2.

Corollary 6.1.8. *With the same assumptions as in Theorem 6.1.6, let $U \in$ **Vec** be a fixed vector space. Then there exist finitely many polynomial functors $Q_1, \dots, Q_k \prec P$ and polynomial transformations $\alpha_i\colon Q_i \to P$ such that for every $V \in$ **Vec** and every $f \in P(V)$ that is* not *in $\bigcup_{i=1}^k \operatorname{im}(\alpha_{i,V})$ the map $\operatorname{Hom}(V, U) \to P(U)$, $\varphi \mapsto P(\varphi)f$ is surjective.*

If P is irreducible, then the condition that $f \notin \bigcup_{i=1}^k \operatorname{im}(\alpha_{i,V})$ can be replaced by the condition that f has strength greater than some function of $\dim(U)$ *only.*

6.1.3 Application to strength

Recall Section 2.2.4 where P_∞ is introduced. We now look at point $p \in P_\infty$ whose orbit under GL is dense in P_∞. They have interesting properties.

Corollary 6.1.9. *Suppose that* $\mathrm{GL}{\cdot}p$ *is dense in P_∞. Then for each integer $n \geq 1$, the image of* $\mathrm{GL}{\cdot}p$ *in $P(K^n)$ is all of $P(K^n)$.*

Proof. For $V \in$ **Vec**, define

$$X(V) := \left\{P(\varphi)P(\pi_n)p \mid n \geq 1, \varphi \in \operatorname{Hom}(K^n, V)\right\} \subseteq P(V),$$

which is exactly the image of $\mathrm{GL}\cdot p$ under the projection $P_\infty \to P(K^m)$ followed by an isomorphism $P(\varphi)$, where $\varphi\colon K^m \to V$ is a linear isomorphism. We see that X is a subset of P. For each $V \in \mathbf{Vec}$, the subset $X(V)$ is dense in $P(V)$ since $\mathrm{GL}\cdot p$ is dense in P_∞. So $X = P$ by Theorem 6.1.6. □

Corollary 6.1.10. *Assume that* $\operatorname{char} K = 0$ *and that* P *is irreducible of degree* ≥ 2*. Then an element of* P_∞ *has infinite strength if and only if its* GL*-orbit is dense.*

Proof. If $p \in P_\infty$ has finite strength, then let $\alpha_i\colon Q_i \times R_i \to P$ be as in the definition above and let

$$\alpha \coloneqq \alpha_1 + \ldots + \alpha_k \colon Q \coloneqq \bigoplus_{i=1}^{k} (Q_i \otimes R_i) \to P$$

be their sum, so that $p \in \operatorname{im}(\alpha)$. Consider the closed subset $X = \overline{\operatorname{im}(\alpha)}$, i.e., the closed subset defined by $X(V) = \overline{\operatorname{im}(\alpha_V)}$ for all $V \in \mathbf{Vec}$. As $\dim(Q(K^n))$ is a polynomial in n of degree $< d$, while $\dim(P(K^n))$ is a polynomial in n of degree d, we see that $X(K^n)$ is a proper subset of $P(K^n)$ for all $n \gg 0$. Since $p \in X_\infty$, it follows that $\mathrm{GL}\cdot p$ is not dense.

Suppose, conversely, that $\mathrm{GL}\cdot p$ is not dense. Then it is contained in X_∞ for some proper closed subset X of P. Hence p has finite strength by Theorem 6.1.6. □

Example 6.1.11. Let P, Q be homogeneous functors of the same degree $d \geq 2$ and let $p \in P_\infty$ be an element of infinite strength. Then $(p, 0) \in P_\infty \oplus Q_\infty$ also has infinite strength, but the orbit $\mathrm{GL}\cdot(p, 0)$ is not dense. ♫

Remark 6.1.12. In Section 6.3 we will use a generalisation of notation introduced here: for an integer $m \geq 0$ we will write $P_{\infty-m}$ for the limit $\varprojlim_n P(K^{[n]-[m]})$ over all integers $n \geq m$. This space is isomorphic to P_∞, but the indices have been shifted by m. On $P_{\infty-m}$ acts the group $\mathrm{GL}_{\infty-m} \cong \mathrm{GL}$, which is defined as the union of $\mathrm{GL}(K^{[n]-[m]})$ over all $n \geq m$. We denote the image of an element $p \in P_{\infty-m}$ in $P(K^{[n]-[m]})$ by $p_{[n]-[m]}$. The inclusions $\iota_n\colon K^{[n]-[m]} \to K^n$ sending $v \mapsto (0, v)$ allow us to view $P_{\infty-m}$ as a subset of P_∞. ♪♫

Corollary 6.1.13. *Let* P *be a homogeneous polynomial functor of degree* $d \geq 2$ *and* $m \geq 0$ *an integer. Let* $p \in P_{\infty-m}$ *be a tensor whose* $\mathrm{GL}_{\infty-m}$*-orbit is not dense and let* $q \in P_\infty$ *be an element with finite strength. Then the* GL*-orbit of* $p + q \in P_\infty$ *is also not dense.*

Proof. Note that p is contained in the image of $\alpha\colon Q_{\infty-m} \to P_{\infty-m}$ for some polynomial transformation $\alpha\colon Q \to P$ with $Q \prec P$ [Bik20, Theorem 4.2.5] and q is contained in the image of $\beta\colon R_\infty \to P_\infty$ for some polynomial transformation $\beta\colon R \to P$ with $\deg(R) < d$. So since $Q \oplus R \prec P$ by Remark 6.1.5, we see that $p+q$ is contained in a proper closed subset of P. Hence its GL-orbit is not dense. □

6.1.4 Minimal classes of elements with dense orbits

With respect to the quasi-order defined in Section 2.3.4, there always exists *minimal* element f with dense orbits. Recall that this minimality relates to a monoid E of linear endomorphisms extending GL, described in Section 2.2.5. This result is a consequence of the following theorem. Note that the theorem states also the existence of a maximal element.

Theorem 6.1.14 (Extreme Elements Theorem)**.** *Suppose that K is algebraically closed of characteristic zero. Let P be a pure homogeneous polynomial functor over K. Then there exist tensors $p, r \in P_\infty$ whose* GL*-orbits are dense such that $p \preceq q \preceq r$ for all $q \in P_\infty$ whose* GL*-orbit is dense.*

The elements p that have this property form a single $\simeq$-class which lies below the $\simeq$-classes of all other $q \in P_\infty$ whose GL-orbit is dense. For the construction of such a tensor $p \in P_\infty$, see Section 6.3.1. For the construction of the tensor $r \in P_\infty$, see Section 6.3.4.

Remark 6.1.15. In both our Main Theorems, we require that the characteristic be zero. This is because the results in [Bik20] and [BDES22] require this. However, the proof of topological Noetherianity for polynomial functors in [Dra19] does not require characteristic zero, and shows that after a shift and a localisation, a closed subset of a polynomial functor admits a homeomorphism into an open subset of a smaller polynomial functor. In characteristic zero, this is in fact a closed embedding, so that it can be inverted and yields a parameterisation of (part of) the closed subset. In positive characteristic, it is not a closed embedding, but the map still becomes invertible if one formally inverts the Frobenius morphism; this is touched upon in [BDES22]. This might imply variants of our Theorems 6.1.6 and 6.1.14 in arbitrary characteristic, but we have not yet pursued this direction in detail. ♪♫

6.2 Proof of the Parameterisation Theorem for GL-subsets

6.2.1 The linear approximation of a polynomial functor

Let P be a polynomial functor over an infinite field and let $U, V \in \mathbf{Vec}$. Then $P(U \oplus V) = \bigoplus_{d,e=0}^{\infty} Q_{d,e}(U, V)$ where

$$Q_{d,e}(U, V) := \{v \in P(U \oplus V) \mid \forall s, t \in K : P(s\,\mathrm{id}_U \oplus t\,\mathrm{id}_V)v = s^d t^e v\} \subseteq P_{d+e}(U \oplus V).$$

The terms with $e = 0$ add up to $P(U)$, and the terms with $e = 1$ add up to a polynomial bifunctor evaluated at (U, V) that is linear in V. This is necessarily of the form $P'(U) \otimes V$, where P' is a polynomial functor. In other words, we have

$$P(U \oplus V) = P(U) \oplus (P'(U) \otimes V) \oplus \text{ higher-degree terms in } V.$$

We informally think of the first two terms as the linear approximation of P around U. Now suppose that we have a short exact sequence

$$0 \to P \to Q \to R \to 0$$

of polynomial functors. This implies that for all U, V we have a short exact sequence

$$\{0\} \to P(U \oplus V) \to Q(U \oplus V) \to R(U \oplus V) \to \{0\}$$

and inspecting the degree-1 parts in V we find a short exact sequence

$$0 \to P' \to Q' \to R' \to 0.$$

This, and further straightforward computations, shows that $P \mapsto P'$ is an exact functor from the category of polynomial functors to itself.

Remark 6.2.1. For $U \in \mathbf{Vec}$ fixed we defined $\mathrm{Sh}_U P$ in Section 2.1.4. Then we have

$$\mathrm{Sh}_U(P)_e(V) = \{v \in P(U \oplus V) \mid \forall t \in K : P(\mathrm{id}_U \oplus t\,\mathrm{id}_V)v = t^e v\}$$

and from this we see that $Q_{d,e}(U, V) = \mathrm{Sh}_U(P)_e(V) \cap P_{d+e}(U \oplus V)$. In particular, when P is homogeneous of degree d, we see that $P(U \oplus V) = \bigoplus_{e=0}^{d} Q_{d-e,e}(U, V)$ where $Q_{d-e,e}(U, V) = \mathrm{Sh}_U(P)_e(V)$. Also note that, in this case, $\mathrm{Sh}_U(P)_0(V) = P(U)$ and $\mathrm{Sh}_U(P)_d(V) = P(V)$ via the inclusions of U, V into $U \oplus V$. ♪♫

Example 6.2.2. If $P = S^d$, then the formula

$$S^d(U \oplus V) \cong \bigoplus_{e=0}^{d} S^{d-e}(U) \otimes S^e(V) = S^d(U) \oplus (S^{d-1}(U) \otimes V) \oplus \cdots$$

identifies P' with S^{d-1}. ♫

Example 6.2.3. Let K be an algebraically closed field of characteristic p. Then S^p contains the subfunctor $P(V) := \{v^p \mid v \in V\}$. We have $P(U \oplus V) = P(U) \oplus P(V)$, and hence $P' = 0$. ♫

6.2.2 Proof of the Parameterisation Theorem

In this subsection we prove Theorem 6.1.6. We start with a result of independent interest.

Theorem 6.2.4. *Let P be a pure polynomial functor over an algebraically closed field K of characteristic* 0 *or* $>$ $\deg(P)$ *and let X be a subset of P such that $X(V)$ is dense in $P(V)$ for all $V \in$* **Vec**. *Then, in fact, $X(V)$ is* equal *to $P(V)$ for all $V \in$* **Vec**.

Example 6.1.3 shows that the condition that K be algebraically closed cannot be dropped. We do not know if the condition on the characteristic of K can be dropped, but the proof will use that the polynomial functor P' introduced in Section 6.2.1 is sufficiently large, which, by Example 6.2.3, need not be the case when char K is too small.

Proof. Let $q \in P(K^n)$. For each $k \geq n$, we consider the incidence variety

$$Z_k := \{(\varphi, r) \in \operatorname{Hom}(K^k, K^n) \times P(K^k) \mid \operatorname{rk}(\varphi) = n \text{ and } P(\varphi)r = q\}.$$

We write $e_k := \dim_K P(K^k)$. Since for every $\varphi \in \operatorname{Hom}(K^k, K^n)$ of rank n the linear map $P(\varphi)$ is surjective, Z_k is a vector bundle of rank $e_k - e_n$ over the rank-n locus in $\operatorname{Hom}(K^k, K^n)$. Hence Z_k is an irreducible variety with $\dim(Z_k) = kn + e_k - e_n$. We therefore expect the projection $\Pi\colon Z_k \to P(K^k)$ to be dominant for $k \gg n$. To prove that this is indeed the case, we need to show that for $z \in Z_k$ sufficiently general, the local dimension at z of the fibre $\Pi^{-1}(\Pi(z))$ is (at most) $\dim(Z_k) - e_k = kn - e_n$. By the upper semicontinuity of the fibre dimension [Har92, Theorem 11.12], it suffices to exhibit a single point z with this property, and indeed, it suffices to show that the tangent space to the fibre at z has dimension (at most) $kn - e_n$.

To find such a point z, set $U := K^n$ and $V := K^{k-n}$ and consider

$$z := (\pi_U, P(\iota_U)q + r) \in Z_k,$$

where $\pi_U\colon U \oplus V \to U$ is the projection and $\iota_U\colon U \to U \oplus V$ is the inclusion and where we will choose $r \in P'(U) \otimes V \subseteq P(U \oplus V)$. Note that then

$$P(\iota_U)q + r \in P(U) \oplus (P'(U) \otimes V) \subseteq P(U \oplus V)$$

and that $P(\pi_U)r = 0$ so that z does, indeed, lie in Z_k.

The tangent space $T_z\Pi^{-1}(\Pi(z))$ (projected into $\operatorname{Hom}(K^k, K^n)$) is contained in the solution space of the linear system of equations

$$P(\pi_U + \varepsilon\psi)(P(\iota_U)q + r) = q \mod \varepsilon^2$$

for ψ. By the rank theorem, the dimension of this solution space equals $kn = \dim(\operatorname{Hom}(K^k, K^n))$ minus the rank of the linear map

$$\operatorname{Hom}(U \oplus V, U) \to P(U), \psi \mapsto \text{the coefficient of } \varepsilon \text{ in } P(\pi_U + \varepsilon\psi)(P(\iota_U)q + r).$$

So it suffices to prove that for all $k \gg n$ there is a suitable r such that this linear map is surjective. In fact, we will restrict the domain to those $\psi \in \operatorname{Hom}(U \oplus V, U)$ of the form $\omega \circ \pi_V$ where $\pi_V\colon U \oplus V \to V$ is the projection and $\omega \in \operatorname{Hom}(V, U)$. Then

$$P(\pi_U + \varepsilon\psi)(P(\iota_U)q) = P((\pi_U + \varepsilon\omega \circ \pi_V) \circ \iota_U)q = P(\mathrm{id}_U)q = q$$

So $P(\iota_U)q$ does not contribute to the coefficient of ε and this coefficient equals

$$P(\mathrm{id}_U + \mathrm{id}_U)(\mathrm{id}_{P'(U)} \otimes \omega)r$$

where $\mathrm{id}_U + \mathrm{id}_U \colon U \oplus U \to U$ is the map sending (u_1, u_2) to $u_1 + u_2$. Note that the codomain of $\mathrm{id}_{P'(U)} \otimes \omega$ equals $P'(U) \otimes U \subseteq P(U \oplus U)$, so that the composition above makes sense. Below we will show that for $k - n = \dim(V) \gg n$ and suitable $r \in P'(U) \otimes V$ the linear map

$$\begin{aligned} \Omega_{P,V,r} \colon \mathrm{Hom}(V, U) &\to P(U) \\ \omega &\mapsto P(\mathrm{id}_U + \mathrm{id}_U)(\mathrm{id}_{P'(U)} \otimes \omega)r \end{aligned}$$

is surjective.

Hence there exists a k such that $Z_k \to P(K^k)$ is dominant. By Chevalley's theorem, the image contains a dense open subset of $P(K^k)$, and this dense open subset intersects the dense set $X(K^k)$. Hence there exists an element $p \in X(K^k)$ and a $\varphi \in \mathrm{Hom}(K^k, K^n)$ such that $P(\varphi)p = q$. Finally, since X is a subset of P, also q is a point in $X(K^n)$. Hence $X(K^n) = P(K^n)$ for each n, as desired. □

Lemma 6.2.5. *Let P be a polynomial functor over an infinite field K with* $\mathrm{char}(K) = 0$ *or* $\mathrm{char}(K) > \deg(P)$ *and let $U \in$* **Vec**. *Then for $V \in$* **Vec** *with* $\dim(V) \gg \dim(U)$, *there exists an $r \in P'(U) \otimes V$ such that*

$$\begin{aligned} \Omega_{P,V,r} \colon \mathrm{Hom}(V, U) &\to P(U) \\ \omega &\mapsto P(\mathrm{id}_U + \mathrm{id}_U)(\mathrm{id}_{P'(U)} \otimes \omega)r \end{aligned}$$

is surjective.

Proof. When $\mathrm{char}(K) = 0$, the Abelian category of polynomial functors is semisimple, with the Schur functors as a basis. When $\mathrm{char}(K) = p > 0$, the situation is more complicated. The irreducible polynomial functors still correspond to partitions [Gre07, Theorem 3.5]. A degree-d irreducible polynomial functor is a submodule of the functor $T(V) = V^{\otimes d}$ if and only if the corresponding partition is column p-regular [Jam80, Theorem 3.2]. Luckily, this is always the case when $d < p$. And, the Abelian category of polynomial functors of degree $< p$ is semisimple [Gre07, Corollary 2.6e]. Now, if P, Q are such polynomial functors and $r_1 \in P'(U) \otimes V$ and $r_2 \in Q'(U) \otimes W$ have the required property for P, Q, respectively, then

$$\begin{aligned} r := (r_1, r_2) \in (P'(U) \otimes V) \oplus (Q'(U) \otimes W) &\subseteq (P'(U) \oplus Q'(U)) \otimes (V \oplus W) \\ &= (P \oplus Q)'(U) \otimes (V \oplus W) \end{aligned}$$

has the required property for $P \oplus Q$. Hence it suffices to prove the lemma in the case where P is an irreducible polynomial functor of degree d. We then have $T = P \oplus Q$, where

$T(V) = V^{\otimes d}$ and Q is another polynomial functor. By a similar argument as above, if $r \in T'(U) \otimes V$ has the required property for T, then its image in $P'(U) \otimes V$ has the required property for P. Hence it suffices to prove the lemma for T.

Now we have

$$\begin{aligned} T(U \oplus V) = T(U) &\oplus (V \otimes U \otimes U \otimes \cdots \otimes U) \oplus (U \otimes V \otimes U \otimes \cdots \otimes U) \\ &\oplus \cdots \oplus (U \otimes U \otimes U \otimes \cdots \otimes V) \oplus \text{ terms of higher degree in } V, \end{aligned}$$

so that T' is a direct sum of d copies of $U \mapsto U^{\otimes d-1}$. We take r in the first of these copies, as follows. Let $e_1, \ldots, e_n$ be a basis of U and set

$$r := \sum_{\alpha \in [n]^{d-1}} v_\alpha \otimes e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_{d-1}}$$

where the v_α are a basis of a space V of dimension n^{d-1}. For every $\beta \in [n]^{d-1}$ and $i \in [n]$, the linear map ω that maps v_β to e_i and all other v_α to zero is a witness to the fact that $e_i \otimes e_{\beta_1} \otimes \cdots \otimes e_{\beta_{d-1}}$ is in the image of $\Omega_{T,V,r}$. Hence this linear map is surjective. □

Lemma 6.2.6. *Assume that K is algebraically closed of characteristic zero. Let P, Q be polynomial functors. Assume that P is irreducible of degree d, Q has degree $< d$ and let $\alpha\colon Q \to P$ be a polynomial transformation, then there is a uniform bound on the strength of elements of* $\mathrm{im}(\alpha_V)$ *that is independent of V.*

Proof. Let R be the sum of the components of Q of strictly positive degree. Any element in $\mathrm{im}(\alpha_V)$ is also in $\mathrm{im}(\beta_V)$ for a polynomial transformation $\beta_V\colon R \to P$ obtained from α by a suitable specialisation. Write $R = R^{(1)} \oplus \cdots \oplus R^{(k)}$, where the $R^{(i)}$ are Schur functors of degrees $0 < d_i < d$. The polynomial transformation β factors uniquely as the polynomial transformation

$$\begin{aligned} \delta\colon R^{(1)} \oplus \cdots \oplus R^{(k)} &\to F := \bigoplus_{\substack{e_1,\ldots,e_k \geq 0 \\ \sum_i e_i d_i = d}} \bigotimes_{i=1}^{k} S^{e_i} R^{(i)} \\ (r_1, \ldots, r_k) &\mapsto (r_1^{\otimes e_1} \otimes \cdots \otimes r_k^{\otimes e_k})_{e_1,\ldots,e_k} \end{aligned}$$

and a *linear* polynomial transformation $\gamma\colon F \to P$. As γ is linear, we see that $\mathrm{str}(\gamma_V(v)) \leq \mathrm{str}(v)$ for all $V \in \mathbf{Vec}$ and $v \in F(V)$. So it suffices to prove that the elements of the subset $\mathrm{im}(\delta)$, which depends only on Q and d, have bounded strength. We have

$$\mathrm{str}(r_1^{\otimes e_1} \otimes \cdots \otimes r_k^{\otimes e_k})_{e_1,\ldots,e_k} \leq \sum_{\substack{e_1,\ldots,e_k \geq 0 \\ \sum_i e_i d_i = d}} \mathrm{str}(r_1^{\otimes e_1} \otimes \cdots \otimes r_k^{\otimes e_k}) \leq \sum_{\substack{e_1,\ldots,e_k \geq 0 \\ \sum_i e_i d_i = d}} 1$$

as $\sum_i e_i \geq 2$ whenever $\sum_i e_i d_i = d$. So this is indeed the case. □

Proof of Theorem 6.1.6 (Parameterisation Theorem for GL*-subset).* Let X be a subset of a pure polynomial functor P over an algebraically closed field K of characteristic zero. For each $V \in \mathbf{Vec}$ define $Y(V) := \overline{X(V)}$. If Y is a proper closed subset of P, then by [Bik20, Theorem 4.2.5] there exist finitely many polynomial transformations $\alpha_i : Q_i \to P$ with $Q_i \prec P$ and $Y(V) \subseteq \bigcup_i \operatorname{im}(\alpha_{i,V})$ for all $V \in \mathbf{Vec}$. Since $X \subseteq Y$, we are done. Otherwise, if $Y(V) = P(V)$ for all V, then Theorem 6.2.4 implies that also $X(V) = P(V)$ for all V. The last statement follows from the previous lemma. □

Proof of Corollary 6.1.8. Let X be the subset of P consisting of all elements $f \in P(V)$ such that

$$\begin{aligned} \operatorname{Hom}(V, U) &\to P(U) \\ \varphi &\mapsto P(\varphi)f \end{aligned}$$

is not surjective. By Theorem 6.1.6, it suffices to prove that $X \neq P$. As before, we claim that in fact $X(V) \neq P(V)$ already when $\dim(V) \geq (\deg(P))(\dim(P(U)))$.

First suppose that P is irreducible. Then P is a Schur functor. Take $V_0 = K^d$ and $\ell = \dim P(U)$. Then it is known that $\operatorname{Hom}(V_0, U) \cdot P(V_0)$ spans $P(U)$. Let $P(\varphi_1)p_1, \dots, P(\varphi_\ell)p_\ell$ be a basis of $P(U)$, let $\iota_i : V_0 \to V_0^{\oplus \ell}$ and $\pi_i : V_0^\ell \to V_0$ be the inclusion and projection maps and take

$$p = P(\iota_i)p_1 + \ldots + P(\iota_\ell)p_\ell \in P(V_0^{\oplus \ell}).$$

Then $P(\varphi_i \circ \pi_i)(p) = P(\varphi_i)p_i$. Hence

$$\begin{aligned} \operatorname{Hom}(V_0^{\oplus \ell}, U) &\to P(U) \\ \varphi &\mapsto P(\varphi)p \end{aligned}$$

is surjective.

Next, suppose that $P = Q \oplus R$ and that there exist $f \in Q(V)$ and $g \in R(W)$ such that

$$\begin{aligned} \operatorname{Hom}(V, U) &\to Q(U) \\ \varphi &\mapsto Q(\varphi)f \end{aligned} \qquad \text{and} \qquad \begin{aligned} \operatorname{Hom}(W, U) &\to R(U) \\ \varphi &\mapsto R(\varphi)g \end{aligned}$$

are surjective. By induction, we can assume such f, g exist when $\dim(V) \geq (\deg(P))(\dim(Q(U)))$ and $\dim(W) \geq (\deg(P))(\dim(R(U)))$. Now, we see that

$$\begin{aligned} \operatorname{Hom}(V \oplus W, U) &\to P(U) \\ \varphi &\mapsto P(\varphi)(P(\iota_1)(f) + P(\iota_2)(g)) \end{aligned}$$

is surjective. This proves the first part of the corollary. For the second statement, we note that when P is irreducible the elements of $\operatorname{im}(\alpha_i)$ have bounded strength. As the bound depends only on X and X only depends on $\dim(U)$, we see that $f \notin \bigcup_{i=1}^k \operatorname{im}(\alpha_i)$ for all f with strength greater than some function of $\dim(U)$ only. □

6.3 Proof of The Extreme Elements Theorem

6.3.1 Construction of the minimal class

Let P be a homogeneous polynomial functor of degree $d > 0$ over an algebraically closed field K of characteristic zero. Decompose

$$P = P^{(1)} \oplus \cdots \oplus P^{(\ell)}$$

into Schur functors. For each $U \in \mathbf{Vec}$ of dimension $\geq d$ the $\mathrm{GL}(U)$-module $P^{(i)}(U)$ is irreducible (and in particular nonzero). Let $V \in \mathbf{Vec}$ be a vector space of dimension d. Let $V^{(1,i)}$ be a copy of V for each $i = 1, \dots, \ell$ and choose any nonzero $q^{(1,i)} \in P^{(i)}(V^{(1,i)})$. We write

$$q^{(1)} := q^{(1,1)} + \ldots + q^{(1,\ell)} \in P^{(1)}(V^{(1,1)}) \oplus \cdots \oplus P^{(\ell)}(V^{(1,\ell)}) \subseteq P(W^{(1)})$$

where $W^{(1)} = V^{(1,1)} \oplus \cdots \oplus V^{(1,\ell)}$. We take independent copies $W^{(j)} = V^{(j,1)} \oplus \cdots \oplus V^{(j,\ell)}$ of $W^{(1)}$ and copies $q^{(j)} = q^{(j,1)} + \ldots + q^{(j,\ell)} \in P(W^{(j)})$ of q_1 and set

$$q := q^{(1)} + q^{(2)} + \ldots \in P_\infty$$

where we concatenate copies of a basis in the ℓd-dimensional space $W^{(1)}$ to identify $W^{(1)} \oplus \cdots \oplus W^{(k)}$ with $K^{k\ell d}$.

Example 6.3.1. Let $P = S^d \oplus \bigwedge^d$, so that we may take $V = K^d$. We may take $q^{(1,1)} := x_1^d \in S^d(V^{(1,1)})$ and $q^{(1,2)} := x_{d+1} \wedge \cdots \wedge x_{2d} \in \bigwedge^d(V^{(1,2)})$, where $x_1, \dots, x_d$ and $x_{d+1}, \dots, x_{2d}$ are bases of $V^{(1,1)}$ and $V^{(1,2)}$, respectively. We then have

$$q = (x_1^d + x_{d+1} \wedge \cdots \wedge x_{2d}) + (x_{2d+1}^d + x_{3d+1} \wedge \cdots \wedge x_{4d}) + \ldots \qquad ♫$$

We will prove, first, that any q constructed in this manner has a dense GL-orbit in P_∞, and second, that $q \preceq p$ for all $p \in P_\infty$ with a dense GL-orbit.

6.3.2 Density of the orbit of q

Proposition 6.3.2. *The* GL*-orbit of q is dense in P_∞.*

Proof. It suffices to prove that for each $U \in \mathbf{Vec}$ and each $p \in P(U)$ there exists a $k \geq 1$ and a linear map $\varphi\colon W^{(1)} \oplus \cdots \oplus W^{(k)} \to U$ such that $P(\varphi)(q^{(1)} + \ldots + q^{(k)}) = p$. Furthermore, we may assume that U has dimension at least d. Fix a linear injection $\iota\colon V \to U$. Now $\tilde{q}^{(i)} := P(\iota)(q^{(j,i)})$ is a nonzero vector in the $\mathrm{GL}(U)$-module $P^{(i)}(U)$, which is irreducible. Hence the component $p^{(i)}$ of p in $P^{(i)}(U)$ can be written as

$$p^{(i)} = P(g^{(1,i)})\tilde{q}^{(i)} + \ldots + P(g^{(k_i,i)})\tilde{q}^{(i)}$$

for suitable elements $g^{(1,i)}, \dots, g^{(k_i,i)} \in \mathrm{End}(U)$. Do this for all $i = 1, \dots, \ell$. By taking the maximum of the numbers k_i (and setting the irrelevant $g^{(j,i)}$ equal to zero) we may assume that the k_i are all equal to a fixed number k; this is the k that we needed. Now we may define φ by declaring its restriction on $V^{(j,i)}$ to be equal to $g^{(j,i)} \circ \iota$. We then have

$$P(\varphi)(q_1 + \dots + q_k) = \sum_{j=1}^{k} \sum_{i=1}^{\ell} P(g^{(j,i)})\tilde{q}^{(i)} = \sum_{i=1}^{\ell} p^{(i)} = p,$$

as desired. □

6.3.3 Minimality of the class of q

Proposition 6.3.3. *We have $q \preceq p$ for every $p \in P_\infty$ with a dense* GL*-orbit.*

Proof. Let $p \in P_\infty$ be a tensor with a dense GL-orbit and write $p = (p_0, p_1, p_2, \dots)$ with $p_i \in P(K^i)$. Take $m_0 = n_0 = 0$. There exists a linear map $\varphi_0 \colon K^{m_0} \to K^{n_0}$ such that $P(\varphi_0)p_{m_0} = q_{n_0} = 0$, namely the zero map. Write $n_i = n_0 + i\ell d$. Our goal is to contruct, for each integer $i \geq 1$, an integer $m_i \geq m_{i-1}$ and a linear map $\psi_i \colon K^{[m_i]-[m_{i-1}]} \to W^{(i)}$ such that the linear map $\varphi_i \colon K^{m_i} \to K^{n_i}$ making the diagram

$$\begin{tikzcd}
K^{m_i} = K^{m_{i-1}} \oplus K^{[m_i]-[m_{i-1}]} \arrow[rr, "\varphi_i"] \arrow[dr, "\mathrm{id}_{m_{i-1}} \oplus \psi_i"'] & & K^{n_{i-1}} \oplus W^{(i)} = K^{n_i} \\
& K^{m_{i-1}} \oplus W^{(i)} \arrow[ur, "\varphi_{i-1} \oplus \mathrm{id}_{W^{(i)}}"'] &
\end{tikzcd}$$

commute satisfies $P(\varphi_i)p_{m_i} = q_{n_i} = q^{(1)} + \dots + q^{(i)}$.

Let $i \geq 1$ be an integer. As observed in Section 6.2.1, we can write

$$P(K^{m_{i-1}} \oplus V) = P(K^{m_{i-1}}) \oplus R_1(V) \oplus \cdots \oplus R_{d-1}(V) \oplus P(V)$$

where $R_j = \mathrm{Sh}_{K^{m_{i-1}}}(P)_j$ is a homogeneous polynomial functor of degree j. Writing $K^{\mathbf{N}}$ as $K^{m_{i-1}} \oplus K^{\mathbf{N}-[m_{i-1}]}$, we obtain a corresponding decomposition

$$p = p_{m_{i-1}} + r_1 + \dots + r_{d-1} + p'$$

where $r_j \in R_{j,\infty-m_{i-1}}$ and $p' \in P_{\infty-m_{i-1}}$ and we claim that p' has a dense $\mathrm{GL}_{\infty-m_{i-1}}$-orbit; here we use the notation from Remark 6.1.12.

The polynomial bifunctor $(U, V) \mapsto P(U \oplus V)$ is a direct sum of bifunctors of the form $(U, V) \mapsto Q(U) \otimes R(V)$ where Q, R are Schur functors. It follows that $R_j(V)$ is the direct sum of spaces $Q(K^{m_{i-1}}) \otimes R(V)$ where Q, R are Schur functors of degrees $d - j, j$, respectively. Hence the elements $r_1, \dots, r_{d-1}$ have finite strength. Also note that

$p_{m_{i-1}} \in P(K^{m_{i-1}})$ has finite strength. So by Corollary 6.1.13, we see that the $\mathrm{GL}_{\infty - m_{i-1}}$-orbit of p' must be dense.

The tuple $(r_1, \ldots, r_{d-1}) \in \bigoplus_{j=1}^{d-1} R_{j,\infty-m_{i-1}}$ may not have a dense $\mathrm{GL}_{\infty-m_{i-1}}$-orbit. However, there exists a polynomial functor R less than or equal to $R_1 \oplus \cdots \oplus R_{d-1}$ with $R(\{0\}) = \{0\}$, an $r \in R_{\infty-m_{i-1}}$ and a polynomial transformation

$$\alpha = (\alpha_1, \ldots, \alpha_{d-1}) \colon R \to R_1 \oplus \cdots \oplus R_{d-1}$$

such that r has a dense $\mathrm{GL}_{\infty-m_{i-1}}$-orbit and $\alpha(r) = (r_1, \ldots, r_{d-1})$. Since P is homogeneous of degree $d > \deg(R)$, the pair (r, p') has a dense orbit in $R_{\infty-m_{i-1}} \oplus P_{\infty-m_{i-1}}$ by [Bik20, Lemma 4.5.3]. Hence, by Corollary 6.1.9, there exists an $m_i \geq m_{i-1} + \ell d$ and a linear map $\psi_i \colon K^{[m_i]-[m_{i-1}]} \to W^{(i)}$ such that $R(\psi_i) r_{[m_i]-[m_{i-1}]} = 0$ and $P(\psi_i) p'_{[m_i]-[m_{i-1}]} = q^{(i)}$.

Since polynomial transformations between polynomial functors with zero constant term map zero to zero, the first equality implies that, for all $j = 1, \ldots, d-1$,

$$R_j(\psi_i) r_{j,[m_i]-[m_{i-1}]} = R_j(\psi_i) \alpha_j(r_{[m_i]-[m_{i-1}]}) = \alpha_j(R(\psi_i) r_{[m_i]-[m_{i-1}]}) = \alpha_j(0) = 0.$$

Thus, informally, applying the map ψ_i makes p' specialise to the required $q^{(i)}$, while the terms $r_1, \ldots, r_{d-1}$ are specialised to zero.

We define φ_i as above and we have

$$\begin{aligned} P(\varphi_i) p_{m_i} &= P(\varphi_{i-1} \oplus \mathrm{id}_{W^{(i)}}) P(\mathrm{id}_{m_{i-1}} \oplus \psi_i) \left(p_{m_{i-1}} + \sum_{j=1}^{d-1} r_{j,[m_i]-[m_{i-1}]} + p'_{[m_i]-[m_{i-1}]} \right) \\ &= P(\varphi_{i-1} \oplus \mathrm{id}_{W^{(i)}}) \left(p_{m_{i-1}} + \sum_{j=1}^{d-1} R_j(\psi_i) r_{j,[m_i]-[m_{i-1}]} + P(\varphi_i) p'_{[m_i]-[m_{i-1}]} \right) \\ &= P(\varphi_{i-1} \oplus \mathrm{id}_{W^{(i)}})(p_{m_{i-1}} + q^{(i)}) = q_{n_{i-1}} + q^{(i)} = q^{(1)} + \ldots + q^{(i)}. \end{aligned}$$

Iterating this argument, we find that the infinite matrix

$$\begin{pmatrix} \varphi_0 & & & & \\ & \psi_1 & & & \\ & & \psi_2 & & \\ & & & \psi_3 & \\ & & & & \ddots \end{pmatrix} =: e$$

has the property that $P(e)p = q^{(1)} + q^{(2)} + \ldots = q$, as desired. □

Remark 6.3.4. Note that the element $e \in E$ constructed above has only finitely many nonzero entries in each row *and* in each column! ♪♫

Remark 6.3.5. Fix an integer $k \geq 0$. Then we have the following strengthening of the previous theorem: we have $(x_1, \ldots, x_k, q) \preceq (\ell_1, \ldots, \ell_k, p)$ for every $(\ell_1, \ldots, \ell_k, p) \in (S^1_\infty)^{\oplus k} \oplus P_\infty$ with a dense GL-orbit. Here q is defined as before in variables distinct from $x_1, \ldots, x_k$. To see this, note that a tensor in $(S^1_\infty)^{\oplus k} \oplus P_\infty$ with a dense GL-orbit is of the form $(\ell_1, \ldots, \ell_k, p)$ where $\ell_1, \ldots, \ell_k \in S^1_\infty$ are linearly independent and $p \in P_\infty$ has a dense GL-orbit. By acting with an invertible element of E as in Example 2.3.15, we may assume that $\ell_i = x_i$. Take $n_0 = k$. Similar to induction step in the proof of the previous theorem, there exists an integer $m_0 \geq k$ and a linear map $\psi\colon K^{[m_0]-[k]} \to K^{n_0}$ such that the linear map $\varphi_0 = \mathrm{id}_k + \psi\colon K^k \oplus K^{[m_0]-[k]} \to K^{n_0}$ satisfies $P(\varphi_0)p_{m_0} = q_{n_0} = 0$. We now proceed as in the proof of the theorem with these m_0, n_0, φ_0 to find the result. ♪♫

Proof of Theorem 6.1.14, existence of p. The existence of a minimal p among all elements with a dense GL-orbit follows directly from Propositions 6.3.2 and 6.3.3. $\square$

6.3.4 Maximal tensors

Next, we construct maximal elements with respect to $\preceq$ of P_∞ for any pure polynomial functor P. We start with n-way tensors, then do Schur functors and finally general polynomial functors. Let $d \geq 1$ be an integer and let T^d be the polynomial functor sending $V \mapsto V^{\otimes d}$.

Lemma 6.3.6. *There exists a tensor $r_d \in T^d_\infty$ such that $p \preceq r_d$ for all $p \in T^d_\infty$.*

Proof. For $d = 1$, we know that the element $r_1 := x_1 \in T^1_\infty$ satisfies $p \preceq r_1$ for all $p \in T^1_\infty$. Now suppose that $d \geq 2$ and that $r_{d-1} = r_{d-1}(x_1, x_2, \ldots) \in T^{d-1}_\infty$ satisfies $p \preceq r_{d-1}$ for all $p \in T^{d-1}_\infty$. We define a $r_d \in T^d_\infty$ satisfying $p \preceq r_d$ for all $p \in T^d_\infty$.

For $j \in \{1, \ldots, d\}$, we define the map $- \otimes_j -\colon T^1_\infty \times T^{d-1}_\infty \to T^d_\infty$ as the inverse limit of the bilinear maps $- \otimes_j -\colon V \times V^{\otimes d-1} \to V^{\otimes d}$ such that $v_j \otimes_j (v_1 \otimes \cdots \otimes v_{j-1} \otimes v_{j+1} \otimes \cdots \otimes v_d) = v_1 \otimes \cdots \otimes v_d$ for all finite-dimensional vector space V and all vectors $v_1, \ldots, v_d \in V$. Now, we take

$$r_d := \sum_{i=1}^{\infty} \sum_{j=1}^{d} x_{\iota(i,j,1)} \otimes_j r_{d-1}(x_{\iota(i,j,2)}, x_{\iota(i,j,3)}, \ldots)$$

where $\iota\colon \mathbf{N} \times \{1, \ldots, d\} \times \mathbf{N} \to \mathbf{N}$ is any injective map. We claim that $p \preceq r_d$ for all $p \in T^d_\infty$. Indeed, any such p can we written as

$$p = \sum_{i=1}^{\infty} \sum_{j=1}^{d} x_i \otimes_j p_i(x_i, x_{i+1}, \ldots)$$

with $p_1, p_2, \ldots \in T^{d-1}_\infty$ and by assumption we can specialize r_{d-1} to p_i using an element of E for all i. Combined, this yields a specialization of r_d to p. Note here that $x_{\iota(i,j,1)} \mapsto x_i$

and $x_{\iota(i,j,k)} \mapsto \ell_{i,j,k}$ for $k > 1$ in such a way that x_ℓ occurs, when ranging over k, in only finitely many $\ell_{i,j,k}$ when $i \le \ell$ and x_ℓ does not occur in $\ell_{i,j,k}$ when $i > \ell$. This means that the specialization of r_d to p indeed goes via an element of E. So for all $d \ge 1$, the space T^d_∞ has a maximal element with respect to $\preceq$. ◻

Lemma 6.3.7. *Let P be a Schur functor of degree $d \ge 1$. Then there exists a tensor $r \in P_\infty$ such that $p \preceq r$ for all $p \in P_\infty$.*

Proof. The space P_∞ is a direct summand of T^d_∞. Let r be the component in P_∞ of r_d from the previous lemma. Then $p \preceq r$ for all $p \in P_\infty$. ◻

Proof of Theorem 6.1.14, the existence of r. Let P be a polynomial functor and write

$$P = P^{(1)} \oplus \cdots \oplus P^{(k)}$$

as a direct sum of Schur functors. For each $i \in \{1, \ldots, k\}$, let $r_i = r_i(x_1, x_2, \ldots) \in P^{(i)}_\infty$ be a tensor such that $p_i \preceq r_i$ for all $p_i \in P^{(i)}_\infty$ and take $r = (r_1(x_1, x_{k+1}, \ldots), \ldots, r_k(x_k, x_{2k}, \ldots)) \in P_\infty$. Then $p \preceq r$ for all $p \in P_\infty$. ◻

6.4 Further examples

In this section we give more examples: we prove that tensors in P_∞ with a dense GL-orbit for a single equivalence class when P has degree ≤ 2, we compare candidates for minimal tensors in a direct sum of S^d's of distinct degrees and we construct maximal elements in P_∞ for all P with $P(\{0\}) = \{0\}$.

6.4.1 Polynomial functors of degree ≤ 2

Example 6.4.1. Take $P = S^1 \oplus S^1$. Then a pair $(v, w) \in S^1_\infty \oplus S^1_\infty$ has one of the following forms:

1. the pair (v, w) with $v, w \in S^1_\infty$ linearly independent vectors;
2. the pair $(\lambda u, \mu u)$ with $u \in S^1_\infty$ nonzero and $[\lambda : \mu] \in \mathbb{P}^1$; or
3. the pair $(0, 0)$.

In the first case, the pair (v, w) has a dense GL-orbit and is equivalent to (x_1, x_2). When $\mu v - \lambda w = 0$ for some $\lambda, \mu \in K$, then this also holds for all specialisations of (v, w). So the poset of equivalence classes is given by:

$$\begin{array}{c} (x_1, x_2) \\ | \\ \mathbb{P}^1 \\ | \\ (0, 0) \end{array}$$

where a point $[\lambda : \mu] \in \mathbb{P}^1$ corresponds to the class of $(\lambda u, \mu u)$ with $u \in S^1_\infty$ nonzero and all points in $\mathbb{P}^1$ are incomparable. ♫

Example 6.4.2. Take $P = S^2$. By Proposition 6.3.3 each infinite quadric

$$p = \sum_{1 \leq i \leq j} a_{ij} x_i x_j$$

of infinite rank specialises to the quadric $q = x_1 x_2 + x_3 x_4 + \dots$ via a suitable linear change of coordinates. Here each variable is only allowed to occur in only finitely many of the linear forms that $x_1, x_2, \dots$ are substituted by. Conversely, it is not difficult to see that q specialises to p as well by applying the following element of E:

$$\begin{pmatrix} 1 & a_{11} & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_{12} & 1 & a_{22} & 0 & 0 & \cdots \\ 0 & a_{13} & 0 & a_{23} & 1 & a_{33} & \cdots \\ 0 & a_{14} & 0 & a_{24} & 0 & a_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

We conclude that the infinite-rank quadrics form a single equivalence class under $\simeq$ and that the rank function is an isomorphism from the poset of equivalence classes to the well-ordered set $\{0, 1, 2, \dots, \infty\}$. ♫

Example 6.4.3. Take $P = \bigwedge^2$. By Proposition 6.3.3 each infinite alternating tensor

$$p = \sum_{1 \leq i < j} a_{ij} x_i \wedge x_j$$

of infinite rank specialises to $q = x_1 \wedge x_2 + x_3 \wedge x_4 + \dots$. And, q specialises to p as well by applying the following element of E:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_{12} & 1 & 0 & 0 & 0 & \cdots \\ 0 & a_{13} & 0 & a_{23} & 1 & 0 & \cdots \\ 0 & a_{14} & 0 & a_{24} & 0 & a_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}.$$

As before, we conclude that the infinite-rank alternating tensors form a single $\simeq$-equivalence class and that the rank function is an isomorphism from the poset of equivalence classes to the well-ordered set $\{0, 1, 2, \dots, \infty\}$. ♫

Example 6.4.4. Take $P = (S^1)^{\oplus a} \oplus (S^2)^{\oplus b} \oplus (\bigwedge^2)^{\oplus c}$ for integers $a, b, c \geq 0$. By Remark 6.3.5, any tuple in P_∞ with a dense GL-orbit specialises to the tuple

$$(x_1, \dots, x_a, y_1y_2 + y_{2b+1}y_{2b+2} + \dots, \dots, y_{2b-1}y_{2b} + y_{4b-1}y_{4b} + \dots,$$
$$z_1 \wedge z_2 + z_{2c+1} \wedge z_{2c+2} + \dots, \dots, z_{2c-1} \wedge z_{2c} + z_{4c-1} \wedge z_{4c} + \dots)$$

where $y_{2ib+j} = x_{a+2ib+2ic+j}$ for $i \geq 0$ and $1 \leq j < 2b$ and $z_{2ic+j} = x_{a+2(i+1)b+2ic+j}$ for $i \geq 0$ and $1 \leq j < 2c$. By the previous examples, each of the entries in this latter tuple independently specialises to any tensor in the same space. So the entire tuple also specialises to any other tuple in P_∞. So the tuple with a dense GL-orbit again form a single $\simeq$-equivalence class. ♫

6.4.2 Non-homogeneous polynomial functors

The proof of Proposition 6.3.3 relies on the fact that P is homogeneous. Apart from the slight generalisation from Remark 6.3.5, we don't know if such a result holds in a more general setting.

Question 6.4.5. *Take $P = S^2 \oplus S^3$. Does there exist a tensor $q \in P_\infty$ with a dense* GL*-orbit such that $q \preceq p$ for all $p \in P_\infty$ with a dense* GL*-orbit?*

The next example compares different candidates for such a minimal element.

Example 6.4.6. Take $P = S^{d_1} \oplus S^{d_2} \oplus \dots \oplus S^{d_k}$ with $1 < d_1 < d_2 < \dots < d_k$. By [Bik20, Lemma 4.5.3], an element $(f_1, \cdots, f_k) \in P_\infty$ has dense GL-orbit if and only if $f_i \in S^{d_i}_\infty$ has dense GL-orbit for all $i = 1, \dots, k$. In particular, the elements

$$q = (q^{(1)}, \dots, q^{(k)}) = (x_1^{d_1} + x_2^{d_1} + \dots, \dots, x_1^{d_k} + x_2^{d_k} + \dots)$$

and

$$p = (p^{(1)}, \dots, p^{(k)}) = (x_1^{d_1} + x_{k+1}^{d_1} + \dots, \cdots, x_k^{d_k} + x_{2k}^{d_k} + \dots)$$

have dense GL-orbits. Clearly $q \preceq p$. By Corollary 6.1.9, there exists an $n \geq 1$ and linear forms $\ell_1, \dots, \ell_n$ in $x_1, \dots, x_k$ such that $q_n^{(j)}(\ell_1, \dots, \ell_n) = x_j^{d_j}$ for $j = 1, \dots, k$. Take

$$\ell_{hn+i} = \ell_i(x_{hn+1}, \dots, x_{hn+n})$$

for $h \geq 1$ and $i \in \{1, \dots, k\}$. Then we see that $q_n^{(j)}(\ell_{hn+1}, \dots, \ell_{hn+n}) = x_{hn+j}^{d_j}$ for $j = 1, \dots, k$. So since

$$q^{(j)} = q_n^{(j)} + q_n^{(j)}(x_{n+1}, \dots, x_{2n}) + \dots$$

we see that $q^{(j)}(\ell_1, \ell_2, \ldots) = p^{(j)}$. Let A be the $k \times n$ matrix corresponding to $\ell_1, \ldots, \ell_n$ and take

$$e := \begin{pmatrix} A & & \\ & A & \\ & & \ddots \end{pmatrix} \in E$$

Then $P(e)q^{(j)} = q^{(j)}(\ell_1, \ell_2, \ldots)$. So $p \preceq q$. Hence $p \simeq q$. ♫

Part IV

Stabilisation under $\mathrm{Sym}^k \times \mathrm{GL}$

Chapter 7

Vec-varieties over FI

In this chapter and in the following Chapter 8 we look at varieties with the actions of Sym and GL combined. In Section 7.1 we introduce varieties with Sym-action and their functorial counterpart, that makes use of the the category **FI** of finite sets with injections. In Section 7.2 we describe varieties enjoying both the symmetries of Sym and of GL via the functorial language of $\mathbf{FI^{op}} \times \mathbf{Vec}$-varieties, and, for reasons that become clear in Chapter 8, we will study the even larger category of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties for varying k. On these varieties the group $\mathrm{Sym}^k \times \mathrm{GL}$ acts, and we will see that in this generality topological Noetherianity doesn't hold; see Remark 7.2.2. In Chapter 8 we restrict our attention to $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties of *product-type*, and we will prove they are topologically Noetherian. From now to the end of the thesis K denotes a field of characteristic zero.

7.1 Varieties over FI

The literature on **FI**-modules is [CEF15; CEFN14; NR19]. However, [DEF22] is the main reference for this section.

We denote by **FI** the category where the objects are finite sets, and the morphisms are injections. In particular, given a finite set S, the group $\mathrm{Sym}(S)$ of the permutations of the elements of S are morphisms in this category. Any functor $F : \mathbf{FI} \to \mathcal{C}$ encodes objects having symmetries of a symmetric group. Indeed, for every finite set S the group $\mathrm{Sym}(S)$ acts on $F(S)$ via $F(\sigma)$ for each $\sigma \in \mathrm{Sym}(S)$. The following example hints at the connection between functors over **FI** and infinite dimensional varieties with an action of Sym.

Example 7.1.1. The *infinite affine space* is defined as the spectrum of $K[x_1, x_2, \dots]$ the polynomial ring in infinitely many variables. The group Sym acts on this algebra (and hence on the variety) by the K-algebra automorphism $\sigma \cdot x_i = x_{\sigma(i)}$ for $\sigma \in \mathrm{Sym}$. ♫

The ring of Example 7.1.1 can be seen as a direct limit of a certain **FI**-algebra that we define below.

Definition 7.1.2. An **FI**-*algebra* is a covariant functor F from **FI** to the category **Alg** of K-algebras. ♪

Let F be an **FI**-algebra, then F assigns to any finite set S a K-algebra $F(S)$ and to each injective map $\iota : S \to T$ a morphisms of K-algebras $F(\iota) : F(S) \to F(T)$ such that $F(\mathrm{id}_S) = \mathrm{id}_{F(S)}$ and $F(\iota \circ \tau) = F(\iota) \circ F(\tau)$. Let F be an **FI**-algebra, and consider the direct limit F_∞ over $n \in \mathbf{N}$ of the $F(\iota_n) : F([n]) \to F([n+1])$ where $\iota_n : [n] \to [n+1]$ are the inclusions. Then F_∞ is a K-algebra stable under the action of Sym.

Example 7.1.3. Consider the functor $F : \mathbf{FI} \to \mathbf{Alg}$ that assigns to each finite set S the K-algebra $K[x_i \mid i \in S]$. Given an injection $\iota : S \to T$, define the corresponding morphism $F(\iota) : F(S) \to F(T)$ mapping x_i to $x_{\iota(i)}$. Then F is an **FI**-algebra over K and the direct limit F_∞ is the Sym-ring $K[x_1, x_2, \dots]$ of Example 7.1.1. ♫

Recall Theorem 1.2.15: let R be a Noetherian ring and let Sym act on the algebra $R[x_{i,j} \mid i \in [k], j \in \mathbf{N}]$ by $\sigma \cdot x_{i,j} = x_{i,\sigma(j)}$. Then $R[x_{i,j} \mid i \in [k], j \in \mathbf{N}]$ is Sym-Noetherian (see footnote 1 at page 22 for the relevant literature). This theorem implies that if Z is a finite-dimensional variety, then the topological space $Z^{\mathbf{N}}$, equipped with the inverse-limit topology of the Zariski topologies, is topologically Sym-Noetherian: if

$$X_1 \supseteq X_2 \supseteq X_3 \supseteq \dots$$

is a descending chain of closed subsets, each stable under the infinite symmetric group Sym permuting the copies of Z, then $X_n = X_{n+1}$ for all $n \gg 0$. The condition of k being a fixed positive integer is necessary as the following example shows.

Example 7.1.4. Consider the ring $K[x_{i,j} \mid i,j \in \mathbf{N}]$ with the action of $\mathrm{Sym} \times \mathrm{Sym}$ given by $(\sigma, \tau) \cdot x_{i,j} = x_{\sigma(i),\tau(j)}$. This ring is *not* $\mathrm{Sym} \times \mathrm{Sym}$-Noetherian by [HS12, Example 3.8], and with their same argument one can also show that the spectrum is not topologically $\mathrm{Sym} \times \mathrm{Sym}$-Noetherian. We sketch the argument below.

Recall by Proposition 1.2.20 that every closed subscheme stable under G of a G-scheme is cut out by the G-orbits of a finite number of elements of its defining ideal. Consider the ideal

$$I := (x_{1,1}x_{1,2}x_{2,2}x_{2,1}, x_{1,1}x_{1,2}x_{2,2}x_{2,3}x_{3,3}x_{3,1}, \dots),$$

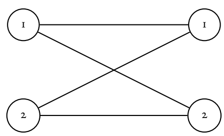
and let X be its corresponding vanishing locus in $\mathrm{Spec}(K[x_{i,j} \mid i,j \in \mathbf{N}])$. Let

$$f_k := \left(\prod_{i=1}^{k} x_{i,i}x_{i,i+1}\right) x_{k+1,k+1}x_{k+1,1}$$

be the k-th generator of I. Suppose that the orbits of $f_{i_1}, \cdots, f_{i_t}$ for $i_1 < \cdots < i_t$ cut out set-theoretically X. Let $E_{i,j}$ be the infinite zero matrix with a one in the (i, j)-th position. Then the point

$$p = \sum_{i=1}^{i_t+1} (E_{i,i} + E_{i,i+1}) + E_{i_t+2,i_t+2} + E_{i_t+2,1}$$

is in the vanishing locus of (the orbits of) $f_{i_1}, \cdots, f_{i_t}$ but it is not a point of X because $f_{i_t+1}(p) \neq 0$. The proof uses that to each polynomial one can associate a bipartite graph, e.g. the monomial f_1 corresponds to the bipartite graph:



♫

Denote by $\mathbf{FI}^{\mathbf{op}}$ the opposite category of **FI**.

Definition 7.1.5. An $\mathbf{FI}^{\mathbf{op}}$*-scheme* is a covariant functor from $\mathbf{FI}^{\mathbf{op}}$ to the category of schemes. ♪

In particular the composition of an **FI**-algebra with the functor Spec : **Alg** $\to$ **Sch** gives an $\mathbf{FI}^{\mathbf{op}}$-scheme.

7.2 The categories of $(\mathbf{FI}^{\mathbf{op}})^k \times$ Vec-varieties

In this section we look at varieties where both the group Sym and the group GL act together. Recall Definition 2.2.3 of **Vec**-variety from Chapter 2.

7.2.1 $(\mathbf{FI}^{\mathbf{op}})^k \times$ Vec-varieties

Definition 7.2.1. Let $k \in \mathbf{Z}_{\geq 0}$. An $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$*-variety* is a covariant functor X from $(\mathbf{FI}^{\mathbf{op}})^k$ to the category of **Vec**-varieties. ♪

Explicitly, an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety is given by the following data: for any k-tuple $(S_1, \dots, S_k)$ we have a **Vec**-variety $X(S_1, \dots, S_k)$, and for any k-tuple of injective maps $\iota = (\iota_1 : S_1 \to T_1, \dots, \iota_k : S_k \to T_k)$, we have a corresponding morphism $X(\iota) : X(T_1, \dots, T_k) \to X(S_1, \dots, S_k)$ of **Vec**-varieties and the usual requirements that $X(\tau \circ \iota) = X(\iota) \circ X(\tau)$ and $X(\mathrm{id}_{S_1}, \dots, \mathrm{id}_{S_k}) = \mathrm{id}_{X(S_1,\dots,S_k)}$.

As for **Vec**-varieties, there are natural notions of morphism and closed immersion of $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties, and for fixed k, the $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties form a category: the full subcategory in the corresponding functor category. We call an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety *Noetherian* if every descending chain of closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvarieties stabilises.

Remark 7.2.2. In particular, any contravariant functor from **FI** to finite-dimensional affine varieties, i.e., an $\mathbf{FI^{op}}$-variety, is trivially an $\mathbf{FI^{op}} \times \mathbf{Vec}$-variety. In this generality, $\mathbf{FI^{op}}$-varieties are certainly not Noetherian: see the already mentioned [HS12, Example 3.8] in Section 7.1.4.

However, we will be largely concerned with $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties defined as follows. Let $Z_1, \dots, Z_k$ be **Vec**-varieties, define

$$X(S_1, \dots, S_k) := Z_1^{S_1} \times \cdots \times Z_k^{S_k} \tag{7.1}$$

and for $\iota = (\iota_1, \dots, \iota_k) : (S_1, \dots, S_k) \to (T_1, \dots, T_k)$ define $X(\iota)$ as the product of the natural projections $Z^{T_i} \to Z^{S_i}$ associated to ι_i. We will prove that $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties of this form are, indeed, Noetherian. ♪♫

Remark 7.2.3. If X is an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety, then the group $\mathrm{Sym}^k \times \mathrm{GL}$ acts on the inverse limit

$$\varprojlim_{n_1,\dots,n_k,n} X([n_1], \dots, [n_k])(K^n).$$

This gives a functor from $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties to (infinite dimensional) schemes equipped with a $\mathrm{Sym}^k \times \mathrm{GL}$-action. Unlike the correspondence between polynomial functors and their inverse limits, this is not quite an equivalence of categories (even under reasonable restrictions on the $\mathrm{Sym}^k \times \mathrm{GL}$-action). For example, $X([n_1], \dots, [n_k])$ could be empty for large n_i and a fixed nontrivial GL-variety for smaller n_i. We will consider an explicit example of this type later in Example 8.3.18. In that case, the inverse limit is empty but the $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety is not trivial. Our theorems will be formulated in the richer category of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties. ♪♫

7.2.2 Partition morphisms and the category PM

Suppose that we are given a point p in some $X(S_1, \dots, S_k)(V)$, where X is as in (7.1). Then the components of p labelled by one of the finite sets S_i may exhibit different behaviours, which prompts us to further partition S_i into subsets labelling components where the behaviour is similar. For instance we can think about the following example.

Example 7.2.4. Let Z be the space of $\mathbf{N} \times \mathbf{N}$-matrices over the field K, equipped with the GL-action given by $g \cdot A := gAg^T$. Let X be the closed $\mathrm{Sym} \times \mathrm{GL}$-stable subvariety of $Z^{\mathbf{N}}$ consisting of all infinite matrix tuples $(A_1, A_2, \dots)$ such that each A_i is either symmetric or skew-symmetric. It is easy to see that X is defined by the $\mathrm{Sym} \times \mathrm{GL}$-orbit of the equation $(x_{112}+x_{121})(x_{112}-x_{121})$, where x_{ijk} is the (j,k)-entry of the ith matrix. In this case we would like to sort the components of a point into the ones consisting of a symmetric matrix and the ones consisting of a skew-symmetric matrix. ♫

Inspired by the above example, we want to define a type of morphisms capturing this feature. Indeed, we will say that p lies in the *image* of some *partition morphism* defined below.

Definition 7.2.5. Let X be an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety and let Y be an $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-variety. A *partition morphism* $Y \to X$ consists of the following data:

1. a map $\pi : [l] \to [k]$; and
2. for each l-tuple of finite sets $(T_1, \dots, T_l)$ a morphism

$$\varphi(T_1, \dots, T_l) : Y(T_1, \dots, T_l) \to X\left(\bigsqcup_{j\in\pi^{-1}(1)} T_j, \dots, \bigsqcup_{j\in\pi^{-1}(k)} T_j\right)$$

of $\mathbf{Vec}$-varieties in such a manner that for any l-tuple $\iota_j \in \mathrm{Hom}_{\mathbf{FI}}(S_j, T_j)$ the following diagram of $\mathbf{Vec}$-variety morphisms commutes:

$$\begin{tikzcd}
Y(T_1, \dots, T_l) \arrow[r, "\varphi(T_1,\dots,T_l)"] \arrow[d, "Y(\iota_1,\dots,\iota_l)"'] & X\left(\bigsqcup_{j\in\pi^{-1}(1)} T_j, \dots, \bigsqcup_{j\in\pi^{-1}(k)} T_j\right) \arrow[d, "X\left(\bigsqcup_{j\in\pi^{-1}(1)} \iota_j,\dots,\bigsqcup_{j\in\pi^{-1}(k)} \iota_j\right)"] \\
Y(S_1, \dots, S_l) \arrow[r, "\varphi(S_1,\dots,S_l)"'] & X\left(\bigsqcup_{j\in\pi^{-1}(1)} S_j, \dots, \bigsqcup_{j\in\pi^{-1}(k)} S_j\right).
\end{tikzcd}$$

♪

Remark 7.2.6. Note that if we take $k = l$ and $\pi = \mathrm{id}_{[k]}$, then a partition morphism is just a morphism of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties. ♪♫

There is a natural way to compose partition morphisms: if (π, φ) is a partition morphism $Y \to X$ as above and (ρ, ψ) is a partition morphism $Z \to Y$, where Z is an $(\mathbf{FI^{op}})^m \times \mathbf{Vec}$-variety, then $(\pi, \varphi) \circ (\rho, \psi)$ is the partition morphism given by the data $\pi \circ \rho : [m] \to [k]$ and the morphisms

$$\varphi\left(\bigsqcup_{n\in\rho^{-1}(1)} R_n, \dots, \bigsqcup_{n\in\rho^{-1}(l)} R_n\right) \circ \psi(R_1, \dots, R_m) :$$
$$Z(R_1, \dots, R_m) \to X\left(\bigsqcup_{n\in(\pi\circ\rho)^{-1}(1)} R_n, \dots, \bigsqcup_{n\in(\pi\circ\rho)^{-1}(k)} R_n\right).$$

A tedious but straightforward computation shows that partition morphisms turn the class of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties, with varying k, into a category. We call this category $\mathbf{PM}$.

Remark 7.2.7. In [NS20] partitions of $\mathbf{N}$ into finitely many subsets feature in the classification of symmetric subvarieties of infinite affine space $(\mathbb{A}^1)^{\mathbf{N}}$, and while our proofs in Chapter 8 do not logically depend on this classification, that paper did serve as an inspiration. ♪♫

Definition 7.2.8. Let X be an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety, Y an $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-variety, and $(\pi, \varphi) : Y \to X$ a partition morphism. Let $S_1, \ldots, S_k \in \mathbf{FI}$ and $V \in \mathbf{Vec}$. The (set-theoretic) *image* of (π, φ) in $X(S_1, \ldots, S_k)(V)$ is defined as the set of all points of the form $(X(\iota_1, \ldots, \iota_k)(V) \circ \varphi(T_1, \ldots, T_l)(V))(q)$ where $T_1, \ldots, T_l$ are finite sets, q is a point in $Y(T_1, \ldots, T_l)(V)$, and each ι_i is a bijection from S_i to $\bigsqcup_{j \in \pi^{-1}(i)} T_j$. The partition morphism (π, φ) is called *surjective* if its image in $X(S_1, \ldots, S_k)(V)$ equals $X(S_1, \ldots, S_k)(V)$ for all choices of $S_1, \ldots, S_k$ and V. ♪

Remark 7.2.9. In the previous definition, each bijection ι_i induces a partition of the set S_i. Furthermore, if a partition morphism (π, φ) is surjective and for every i the $\mathbf{Vec}$-variety

$$X(\emptyset, \ldots, \emptyset, \{*\}, \emptyset, \ldots, \emptyset),$$

where $\{*\}$ is a singleton in the i-th position, is nonempty, then the map π is automatically surjective, so that π induces a partition of $[l]$ into k labelled, nonempty parts. This is our reason for calling the morphisms in $\mathbf{PM}$ partition morphisms. ♪♫

The following example rephrases Example 7.2.4 in the current terminology.

Example 7.2.10. Let Z be the $\mathbf{Vec}$-variety that maps V to $V \otimes V$, and let Z_1, Z_2 be the closed $\mathbf{Vec}$-subvarieties consisting of symmetric and skew-symmetric tensors, respectively. Consider the $\mathbf{FI^{op}} \times \mathbf{Vec}$-variety defined by $S \mapsto Z^S$, and for every finite set S let $X(S)$ be the closed $\mathbf{Vec}$-subvariety given by the points $x = (x_s)_{s \in S} \in Z(V)^S$ such that each component x_s is either symmetric or skew-symmetric. Note that X is a closed $\mathbf{FI^{op}} \times \mathbf{Vec}$-subvariety. Let Y be the $(\mathbf{FI^{op}})^2 \times \mathbf{Vec}$-variety defined by

$$Y(S_1, S_2) := Z_1^{S_1} \times Z_2^{S_2}.$$

We now construct a partition morphism $\varphi : Y \to X$ as follows. The map $\pi : [2] \to [1]$ is the only possible, and for every $V \in \mathbf{Vec}$ and $(S_1, S_2) \in (\mathbf{FI^{op}})^2$ the map

$$\varphi(S_1, S_2)(V) : Y(S_1, S_2)(V) = Z_1(V)^{S_1} \times Z_2(V)^{S_2} \to X(S_1 \sqcup S_2)(V)$$

is defined by:

$$((x_{s_1})_{s_1 \in S_1}, (x_{s_2})_{s_2 \in S_2}) \mapsto (x_s)_{s \in S_1 \sqcup S_2}.$$

Note that the partition morphism φ is surjective. ♫

Chapter 8

$\mathrm{Sym}^k \times$GL-Noetherianity for products of polynomial functors

In this chapter we prove topological $\mathrm{Sym}^k \times \mathrm{GL}$-Noetherianity for k infinite products of polynomial functors. In Section 8.1 we describe our initial question that served as motivation for the development of the theory in Section 7.2. Indeed, Theorem 8.1.1 and Theorem 8.2.1 can be reformulated in the language of $\mathbf{FI^{op}} \times \mathbf{Vec}$-varieties and $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties, respectively; see Section 8.3. Recall that in Section 7.2.2 we introduced the category **PM** with morphisms between such varieties, in which, for the reasons explained in Example 7.2.4 and above it, k varies. In Section 8.4 we formulate and prove the Parameterisation Theorem for $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvarieties of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties of *product-type*, the core technical result of this chapter. The statement says that if X is a proper closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvariety of a variety Z of product-type

$$Z : (S_1, \ldots, S_k; V) \mapsto \prod_{i=1}^{k} Z_i(V)^{S_i},$$

where the Z_i are **Vec**-varieties, then X is covered by finitely many morphisms in **PM** from $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-varieties of product-type that are, in a suitable (and very subtle!) manner, smaller than Z. The details of the order on product-type varieties are in Section 8.3.3. In Section 8.5 we use the Parameterisation Theorem to prove that all $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties of product-type are Noetherian, and obtain Theorem 8.2.1 and Theorem 8.1.1 as corollaries.

8.1 Description of the result

Given a GL-variety Z, the group $\mathrm{Sym} \times \mathrm{GL}$ acts naturally on $Z^{\mathbf{N}}$, and our main goal in this chapter is to prove the following theorem.

Theorem 8.1.1. *Let Z be a* GL*-variety over a field of characteristic zero. Then $Z^{\mathbf{N}}$ is topologically* $\mathrm{Sym} \times \mathrm{GL}$*-Noetherian. In other words, every descending chain*

$$X_1 \supseteq X_2 \supseteq \dots$$

of closed $\mathrm{Sym} \times \mathrm{GL}$*-stable subsets of $Z^{\mathbf{N}}$ eventually stabilises. Equivalently, any* $\mathrm{Sym} \times \mathrm{GL}$*-stable closed subset of $Z^{\mathbf{N}}$ is defined by finitely many* $\mathrm{Sym} \times \mathrm{GL}$*-orbits of polynomial equations.*

Theorem 8.1.1 generalises the results mentioned in Section 7.1: taking for Z a finite-dimensional affine variety with trivial GL-action, one recovers the Sym-Noetherianity of $Z^{\mathbf{N}}$; and on the other hand, if Z is a GL-variety, then considering chains $X_1 \supseteq X_2 \supseteq \dots$ in which each X_i is of the form $Z_i^{\mathbf{N}}$ with $Z_i \subseteq Z$ a GL-subvariety, one recovers the GL-Noetherianity of Z.

The proof of Theorem 8.1.1 will reflect these two special cases. We will use the proof method from [Dra19] for the GL-Noetherianity of Z, and similarly, we will use methods for Sym-varieties from [DEF22]. In fact, we do not explicitly use Higman's lemma in our proofs as is classically done [AH07; HS12; Dra14], and *en passant* we give a new proof of the Sym-Noetherianity of $Z^{\mathbf{N}}$ for a finite-dimensional variety Z. However, our proof only yields a *set-theoretic* Noetherianity result, while in the pure Sym-setting (much) stronger results are known (recall Theorem 1.2.15), and even finitely generated modules over such rings with a compatible Sym-action are Noetherian [NR19]. In the pure GL-setting, however, such stronger Noetherianity results are known only for very few classes of GL-varieties; see the second paragraph of Section 4.1.2.

8.2 Setting up the proof

8.2.1 A generalisation

Interestingly, we prove Theorem 8.1.1 by establishing first the following more general result.

Theorem 8.2.1. *Let $Z_1, \dots, Z_k$ be* GL*-varieties over a field of characteristic zero. Then the variety $Z_1^{\mathbf{N}} \times \dots \times Z_k^{\mathbf{N}}$ is* $\mathrm{Sym}^k \times \mathrm{GL}$*-Noetherian.*

Here $\mathrm{Sym}^k \times \mathrm{GL}$ is to be read as $(\mathrm{Sym}^k) \times \mathrm{GL}$, i.e., there is one copy of GL that acts diagonally, and there are k copies of Sym that act on separate copies of $\mathbf{N}$. We believe it is impossible to prove Theorem 8.1.1 without considering multiple copies of Sym. Indeed, the need for this generalisation comes from the fact that, in order to cover a proper closed $\mathrm{Sym} \times \mathrm{GL}$-stable subset of $Z^{\mathbf{N}}$, we often need to partition $\mathbf{N}$ into finitely many parts, such that for the indices i in one of these parts, the points in Z labelled by those indices behave in a similar fashion. Example 7.2.10 illustrates this point.

8.2.2 Surjective partition morphisms and Noetherianity

We want to prove Theorem 8.2.1 exploiting the theory of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties. This section is dedicated to explain how we derive Noetherianity from surjective partition morphisms. The reader should wait Section 8.3 for the definition of product-type varieties and the meaning of "smaller"— that comes from a well-quasi-order $\preceq$ on product-type varieties. Also the fact that we can cover a proper closed subvariety X in a product-type variety with some smaller product-type varieties comes later: it is indeed the content of the *Parameterisation Theorem 8.4.1* in Section 8.4.

In the setting of Theorem 8.2.1 we consider the GL-variety $Z_1^{\mathbf{N}} \times \cdots \times Z_k^{\mathbf{N}}$. As a first step, we translate it into a closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvariety Z of an appropriate $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety Z' of *product-type*. For product-type varieties we define a well-quasi-ordering, so we can apply induction and assume that Noetherianity holds for all $\mathbf{FI^{op}}^l \times \mathbf{Vec}$-varieties of product-type smaller than Z'. A closed $\mathrm{Sym}^k \times \mathrm{GL}$-stable subset of $Z_1^{\mathbf{N}} \times \cdots \times Z_k^{\mathbf{N}}$ gives a closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvariety X in Z. Clearly, one of the inclusions $X \subset Z \subset Z'$ can be assumed to be strict. We then construct partition morphisms into Z' such that their domains are $\mathbf{FI^{op}}^l \times \mathbf{Vec}$-varieties of product-type strictly smaller than Z', and such that the union of their images contains X. The following two results—that are immediate—guarantee that X is Noetherian too.

Lemma 8.2.2. *Let X be an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety, X' a closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvariety of X, and let (π, φ) be a partition morphism from an $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-variety Y to X. Then $Y' := (\pi, \varphi)^{-1}(X')$ defined by*

$$Y'(T_1, \ldots, T_l) := \varphi(T_1, \ldots, T_l)^{-1}\left(X'\left(\bigsqcup_{j \in \pi^{-1}(1)} T_j, \ldots, \bigsqcup_{j \in \pi^{-1}(k)} T_j\right)\right)$$

is a closed $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-subvariety of Y, and the data of π together with the restrictions of the morphisms $\varphi(T_1, \ldots, T_l)$ gives a partition morphism from Y' to X. Moreover, if (π, φ) is surjective, then so is its restriction to $Y' \to X'$.

Proposition 8.2.3. *If (π, φ) is a surjective partition morphism from Y to X, and Y is a Noetherian $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-variety, then X is a Noetherian $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety.*

Proof. Let $X_1 \supseteq X_2 \supseteq \ldots$ be a descending chain of closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvarieties. By Lemma 8.2.2, the preimages $Y_i := (\pi, \varphi)^{-1}(X_i)$ are closed $(\mathbf{FI^{op}})^l \times \mathbf{Vec}$-subvarieties of Y.

Hence the chain $Y_1 \supseteq Y_2 \supseteq \dots$ stabilises by assumption. The surjectivity of (π, φ) implies the surjectivity of its restriction to $Y_i \to X_i$. This implies that X_i is uniquely determined by Y_i, and hence the chain $X_1 \supseteq X_2 \supseteq \dots$ stabilises at the same point. □

Finally, as explained in Section 8.5, we go back to the original setting of Theorem 8.2.1 via inverse limits of these $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties.

Remark 8.2.4. Note that in the Example 7.2.10 the partition morphism φ is surjective, and the $(\mathbf{FI^{op}})^2 \times \mathbf{Vec}$-variety Y (of product type) is *"smaller"* than Z in the sense that both Z_1 and Z_2 are quotients of Z. ♪♫

8.3 Product-type varieties

We now introduce the $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$ varieties of product type. Essentially, these are the varieties from Remark 7.2.2, but for our proofs we will need a finer control over these products. Therefore, we will work over a general base $\mathbf{Vec}$-variety Y, and keep track of the "constant parts" B_i of the $\mathbf{Vec}$-varieties whose products we consider.

Definition 8.3.1. Let Y be a $\mathbf{Vec}$-variety and $k, n_1, \dots, n_k \in \mathbf{Z}_{\geq 0}$. For each $i \in [k]$, let B_i be a $\mathbf{Vec}$-subvariety of $Y \times \mathbb{A}^{n_i}$, and Q_i be a pure polynomial functor. By construction each $\mathbf{Vec}$-variety $B_i \times Q_i$ has a morphism to Y induced by the projection $Y \times \mathbb{A}^{n_i} \to Y$. We define the $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety $Z = [Y; B_1 \times Q_1, \dots, B_k \times Q_k]$ via

$$Z(S_1, \dots, S_k) := (B_1 \times Q_1) \times_Y \dots \times_Y (B_1 \times Q_1) \times_Y (B_2 \times Q_2) \times_Y \dots \times_Y (B_k \times Q_k),$$

where for every index $i \in [k]$ the fibre product over Y of $B_i \times Q_i$ with itself is taken $|S_i|$ times, and these copies are labelled by the elements of S_i. The morphism $Z(T_1, \dots, T_k) \to Z(S_1, \dots, S_k)$ corresponding to $\iota : S \to T$ is the projection as in Remark 7.2.2. We also write the above product in a more compact notation as

$$(B_1 \times Q_1)_Y^{S_1} \times_Y \cdots \times_Y (B_k \times Q_k)_Y^{S_k}.$$

We say that Z is an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety of *product-type* (over Y). ♪

Note that $Z(S_1, \dots, S_k)$ is naturally a closed $\mathbf{Vec}$-subvariety of

$$Y \times \prod_{i=1}^{k} (\mathbb{A}^{n_i} \times Q_i)^{S_i},$$

where each product is over $\mathrm{Spec}(K)$. Clearly, if $k = 0$, then by definition $Z = Y$.

When we talk of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties of product-type, we will always specify each B_i together with its closed embedding in $Y \times \mathbb{A}^{n_i}$; the reason being that, in the proof of Theorem 8.1.1, we aim to argue by induction on both Y and n_i.

Remark 8.3.2. The settings of Theorem 8.1.1 and Theorem 8.2.1 can be rephrased in our current terminology as follows. Consider **Vec**-varieties $Z_1, \ldots, Z_k$. Then for every $i \in [k]$ there exist $n_i \in \mathbf{Z}_{\geq 0}$, a finite dimensional affine variety $A_i \subseteq \mathbb{A}^{n_i}$, and a pure polynomial functor Q_i such that $Z_i \subseteq A_i \times Q_i$. Define Y to be a point, and $B_i := Y \times A_i$. Then the variety $Z_1^{\mathbf{N}} \times \cdots \times Z_k^{\mathbf{N}}$ of Theorem 8.2.1 is a subvariety of the product-type $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety

$$[Y; B_1 \times Q_1, \ldots, B_k \times Q_k],$$

with $k = 1$ being the special case addressed in Theorem 8.1.1. ♪♫

Remark 8.3.3. In [DEF22], for $\mathbf{FI}^{\mathbf{op}}$-varieties (no dependence on **Vec**), the notion of product-type is more restrictive. Essentially, there the last three authors considered a single finite-dimensional affine variety Z with a morphism to a finite-dimensional, irreducible, affine variety Y, with the additional requirement that $K[Z]$ is a free $K[Y]$-module. This then ensures that each irreducible component of Z^S maps dominantly to Y. In [DEF22] this is used to count the orbits of $\mathrm{Sym}(S)$ on these irreducible components. ♪♫

The following example describes the partition morphisms between product-type varieties. It is particularly relevant as the partition morphisms we will be dealing with in our proof of the Parameterisation Theorem 8.4.1 are of this shape.

Example 8.3.4. Let $Z' := [Y'; B_1' \times Q_1', \ldots, B_l' \times Q_l']$ and $Z := [Y; B_1 \times Q_1, \ldots, B_k \times Q_k]$ be an $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$-variety and an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety of product-type over Y' and Y, respectively. We want to construct a partition morphism $(\pi, \varphi) : Z' \to Z$. Consider the following data:

- let $\pi : [l] \to [k]$ be any map;
- let $\alpha : Y' \to Y$ be a morphism of **Vec**-varieties;
- and for each $j \in [l]$ let $\beta_j : B_j' \times Q_j' \to B_{\pi(j)} \times Q_{\pi(j)}$ be a morphism of **Vec**-varieties such that the following diagram commutes:

$$\begin{tikzcd} B_j' \times Q_j' \arrow[r, "\beta_j"] \arrow[d] & B_{\pi(j)} \times Q_{\pi(j)} \arrow[d] \\ Y' \arrow[r, "\alpha"'] & Y. \end{tikzcd} \tag{8.1}$$

For each $(T_1, \ldots, T_l) \in \mathbf{FI}^l$ we define the morphism of **Vec**-varieties

$$\varphi(T_1, \ldots, T_l) \colon Z'(T_1, \ldots, T_l) \to Z\left(\bigsqcup_{j \in \pi^{-1}(1)} T_j, \ldots, \bigsqcup_{j \in \pi^{-1}(k)} T_j\right)$$

as follows. Let $S_i := \bigsqcup_{j\in\pi^{-1}(i)} T_j$, then for any $V \in \mathbf{Vec}$ the element

$$((b'_{j,t}, q'_{j,t})_{t\in T_j})_{j\in[l]} \in (B'_1 \times Q'_1)^{T_1}_{Y'}(V) \times_{Y'} \cdots \times_{Y'} (B'_l \times Q'_l)^{T_l}_{Y'}(V)$$

is mapped to the element

$$(((\beta_j(V)(b'_{j,t}, q'_{j,t}))_{t\in T_j})_{j\in\pi^{-1}(i)})_{i\in[k]} \in (B_1 \times Q_1)^{S_1}_{Y}(V) \times_Y \cdots \times_Y (B_k \times Q_k)^{S_k}_{Y}(V).$$

By construction, the pair (π, φ) is a partition morphism $Z' \to Z$. Conversely, every partition morphism $Z' \to Z$ is of this form. Indeed, from a general partition morphism $Z' \to Z$, α is recovered by taking all T_j empty and β_j is recovered by taking T_j a singleton and all $T_{j'}$ with $j' \neq j$ empty. That (8.1) commutes then follows by applying the commuting diagram from the definition of a partition morphism to the morphism $(\emptyset, \dots, \emptyset, \dots, \emptyset) \to (\emptyset, \dots, \{*\}, \dots, \emptyset)$ in $\mathbf{FI}^l$. ♫

8.3.1 The leading monomial ideal

We introduce a size measure for a closed subvariety $B \subseteq Y \times \mathbb{A}^n$.

Definition 8.3.5. Let Y be a **Vec**-variety, $n \in \mathbf{Z}_{\geq 0}$ and B a closed **Vec**-subvariety of $Y \times \mathbb{A}^n$. For $V \in \mathbf{Vec}$ consider the ideal $\mathcal{I}(B(V))$ of $K[Y(V)][x_1, \dots, x_n]$ defining $B(V)$. We fix the lexicographic order on monomials in $x_1, \dots, x_n$, and denote by $\mathrm{LM}(B)$ the set of those monomials that appear as leading monomials of *monic* polynomials in $\mathcal{I}(B(V))$, i.e., those with leading coefficient $1 \in K[Y(V)]$. ♪

The following lemma shows that $LM(B)$ is well-defined.

Lemma 8.3.6. *The set* $\mathrm{LM}(B)$ *does not depend on the choice of* V.

Proof. Let $V \in \mathbf{Vec}$ and consider the linear maps $\iota : 0 \to V$ and $\pi : V \to 0$. If $f \in \mathcal{I}(B(V))$ is monic with leading monomial x^u, then applying $Y(\iota)^\#$ to all coefficients of f yields a polynomial in $\mathcal{I}(B(0))$ which is monic with leading monomial x^u. This shows that the leading monomials of monic polynomials in $\mathcal{I}(B(V))$ remain leading monomials of monic elements in $\mathcal{I}(B(0))$. One obtains the converse inclusion by applying $Y(\pi)^\#$. □

The following lemma monitors the size of LM of the constant parts after a base change in product-type varieties. See Proposition 8.3.11.

Lemma 8.3.7. *Let* $Y' \to Y$ *be a morphism of* **Vec**-*varieties, let* B *be a closed* **Vec**-*subvariety of* $Y \times \mathbb{A}^n$*, and define* $B' := Y' \times_Y B \subseteq Y' \times \mathbb{A}^n$*. Then* $\mathrm{LM}(B') \supseteq \mathrm{LM}(B)$.

Proof. Pulling back a monic equation for $B(V)$ along $Y'(V) \times \mathbb{A}^n \to Y(V) \times \mathbb{A}^n$ yields a monic equation for $B'(V)$ with the same leading monomial. □

8.3.2 Shifting over tuples of finite sets

We now describe the shift operation in the context of $(\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec}$-varieties.

Definition 8.3.8. Let X be an $(\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec}$-variety and let $S = (S_1, \ldots, S_k) \in \mathbf{FI}^k$. Then the *shift* $\mathrm{Sh}_S X$ *of* X *over* S is the $(\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec}$-variety defined by

$$(\mathrm{Sh}_S X)(T_1, \ldots, T_k) := X(S_1 \sqcup T_1, \ldots, S_k \sqcup T_k)$$

and, for injections $\iota_i : T_i \to T_i'$,

$$(\mathrm{Sh}_S X)(\iota_1, \ldots, \iota_k) := X(\mathrm{id}_{S_1} \sqcup \iota_1, \ldots, \mathrm{id}_{S_k} \sqcup \iota_k).$$ ♪

Remark 8.3.9. Consider an tuple $S = (S_1, \ldots, S_k)$ in $(\mathbf{FI}^{\mathrm{op}})^k$ and define the covariant functor $\mathrm{Sh}_S : (\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec} \to (\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec}$ by assigning to each tuple $(T_1, \ldots, T_k)$ the tuple $(S_1 \sqcup T_1, \ldots, S_k \sqcup T_k)$ and to each morphism $\iota : (\iota_1, \ldots, \iota_k) : (T_1, \ldots, T_k) \to (T_1', \ldots, T_k')$ the morphism $\iota \sqcup \mathrm{id}_S$. In particular $\mathrm{Sh}_S X$ is the composition $X \circ \mathrm{Sh}_S$. ♪♫

Remark 8.3.10. Let $V \in \mathbf{Vec}$. While, as sets, $\mathrm{Sh}_S X(T_1, \ldots, T_k)(V)$ and $X(S_1 \sqcup T_1, \ldots, S_k \sqcup T_k)(V)$ coincide, the action of the k copies of the symmetric group on them is different. Indeed, the groups $\mathrm{Sym}(S_1 \sqcup T_1) \times \cdots \times \mathrm{Sym}(S_k \sqcup T_k)$ and $\mathrm{Sym}(T_1) \times \cdots \times \mathrm{Sym}(T_k)$ act by functoriality on the latter and on the former, respectively. ♪♫

With the following proposition we describe what happens when the shift operation is performed on product-type varieties.

Proposition 8.3.11. *The shift* $\mathrm{Sh}_S Z$ *over* $S = (S_1, \ldots, S_k)$ *of an* $(\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec}$*-variety* $Z := [Y; B_1 \times Q_1, \ldots, B_k \times Q_k]$ *of product type is itself isomorphic to a variety of product-type:*

$$\mathrm{Sh}_S Z \cong [Y'; B_1' \times Q_1, \ldots, B_k' \times Q_k]$$

with

$$\begin{aligned} Y' &:= (B_1 \times Q_1)_Y^{S_1} \times_Y \ldots \times_Y (B_k \times Q_k)_Y^{S_k}, \text{ and} \\ B_i' &:= Y' \times_Y B_i. \end{aligned}$$

Furthermore, each B_i' *is naturally a* $\mathbf{Vec}$*-subvariety of* $Y' \times \mathbb{A}^{n_i}$*, and we have* $\mathrm{LM}(B_i') \supseteq \mathrm{LM}(B_i)$.

Proof. Straightforward; for the last statement we use Lemma 8.3.7. □

In analogy with [Dra19, Lemma 14] and [DEF22, Section 3.3], the shift operation doesn't increase the "complexity" of product-type varieties. Indeed, we have $\mathrm{Sh}_S Z \leq Z$ according to the order in Section 8.3.3.

8.3.3 Well-founded orders

In this section we recall the well-founded pre-order on polynomial functors of Section 2.1.3. Building on it, we define well-founded pre-orders

- on varieties appearing in the definition of $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-varieties of product-type,
- on product-type varieties, and
- on closed subvarieties of a fixed product-type variety.

Order on polynomial functors

Definition 8.3.12. For polynomial functors P, Q, we write $P \preceq Q$ if $P \cong Q$ or else, for the largest e with $P_e \not\cong Q_e$, P_e is a quotient of Q_e. ♪

Order on Vec-varieties of type $B \times Q$

Consider **Vec**-varieties Y, Y', integers n, n', pure polynomial functors Q, Q', and **Vec**-subvarieties $B \subset Y \times \mathbb{A}^n, B' \subset Y' \times \mathbb{A}^{n'}$. We say that $B' \times Q' \preceq B \times Q$ if:

1. $Q' \prec Q$ in the order of Definition 8.3.12; or
2. $Q' \cong Q, n' = n$ and $\mathrm{LM}(B') \supseteq \mathrm{LM}(B)$.

This is a pre-order on **Vec**-varieties of this type.

Remark 8.3.13. We remark that $\preceq$ is defined on **Vec**-varieties *with a specified product decomposition* $B \times Q$ where B is a **Vec**-variety *with a specified closed embedding into a specified product* $Y \times \mathbb{A}^n$ of a **Vec**-variety Y and some n. It is not a pre-order on **Vec**-varieties without further data. ♪♫

Lemma 8.3.14. *The pre-order on* **Vec***-varieties defined as above is well-founded.*

Proof. Suppose we had an infinite strictly decreasing chain

$$B_1 \times Q_1 \succ B_2 \times Q_2 \succ \dots$$

with $B_i \subseteq Y_i \times \mathbb{A}^{n_i}$. Then we have $Q_1 \succeq Q_2 \succeq \dots$. By the well-foundedness of $\succeq$ on polynomial functors, there exists a $j \gg 0$ such that both Q_i and n_i are constant for $i \geq j$. But then $\mathrm{LM}(B_i) \subsetneq \mathrm{LM}(B_{i+1}) \subsetneq \dots$, which contradicts Dickson's lemma. □

Order on product-type varieties

Consider an $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety $Z := [Y; B_1 \times Q_1, \ldots, B_k \times Q_k]$, and an $(\mathbf{FI}^{\mathbf{op}})^l \times \mathbf{Vec}$-variety $Z' := [Y'; B_1' \times Q_1', \ldots, B_l' \times Q_l']$. We say that $Z' \leq Z$ if there exists a map $\pi : [l] \to [k]$ with the following properties:

1. $B_j' \times Q_j' \leq B_{\pi(j)} \times Q_{\pi(j)}$ holds for all $j \in [l]$, and

2. for all j whose π-fibre $\pi^{-1}(\pi(j))$ has cardinality at least 2 we have $B_j' \times Q_j' \prec B_{\pi(j)} \times Q_{\pi(j)}$.

3. If π is a bijection, then either at least one of the inequalities in (1) is strict, or else Y' is a closed $\mathbf{Vec}$-subvariety of Y.

Lemma 8.3.15. *Suppose $Z' \leq Z$ is witnessed by $\pi : [l] \to [k]$ and suppose that at least one of the following holds:*

- *$l \neq k$, or*

- *at least one of the inequalities in (1) is strict.*

Then we have $Z' \prec Z$.

Proof. Assume, on the contrary, that $\sigma : [k] \to [l]$ witnesses $Z \leq Z'$. Construct a directed graph Γ with vertex set $[l] \sqcup [k]$ and an arrow from each $j \in [l]$ to $\pi(j)$ and an arrow from each $i \in [k]$ to $\sigma(i)$. Like any digraph in which each vertex has out-degree 1, Γ is a union of disjoint directed cycles (here of even length) plus a number of trees rooted at vertices in those cycles and directed towards those roots. Moreover, those cycles have the same number of vertices in $[l]$ as in $[k]$.

The assumptions imply that at least one of the vertices of Γ does not lie on a directed cycle. Without loss of generality, there exists an $i \in [k]$ not in any cycle such that $j := \sigma(i)$ lies on a cycle. Let n be half the length of that cycle, so that $(\sigma\pi)^n(j) = j$. Then we have

$$B_j' \times Q_j' \leq B_{\pi(j)} \times Q_{\pi(j)} \leq \ldots \leq B_{\pi(\sigma\pi)^{n-1}(j)} \times Q_{\pi(\sigma\pi)^{n-1}(j)} \prec B'_{(\sigma\pi)^n(j)} \times Q'_{(\sigma\pi)^n(j)} = B_j' \times Q_j'$$

where the strict inequality holds because $\sigma^{-1}(j)$ has at least two elements: i and $\pi(\sigma\pi)^{n-1}(j)$. By transitivity of the pre-order from Section 8.3.3, we find $B_j' \times Q_j' \prec B_j' \times Q_j'$, which however contradicts the reflexivity of that pre-order. □

Lemma 8.3.16. *The relation $\leq$ is a well-founded pre-order on varieties in* **PM** *of product-type.*

Proof. For reflexivity we may take π equal to the identity. For transitivity, if $\pi : [l] \to [k]$ witnesses $Z' \leq Z$ and $\sigma : [k] \to [m]$ witnesses $Z \leq Z''$, then $\tau := \sigma \circ \pi$ witnesses $Z' \leq Z''$—here we note that if $|\tau^{-1}(\tau(j))| > 1$ for some $j \in [l]$, then either $|\pi^{-1}(\pi(j))| > 1$ or else $|\sigma^{-1}(\sigma(\pi(j)))| > 1$; in both cases we find that $B'_j \times Q'_j < B''_{\tau(j)} \times Q''_{\tau(j)}$.

For well-foundedness, suppose that we had a sequence $Z_1 > Z_2 > Z_3 > \ldots$, where

$$Z_i = [Y_i; B_{i,1} \times Q_{i,1}, \ldots, B_{i,k_i} \times Q_{i,k_i}],$$

and where $\pi_i : [k_{i+1}] \to [k_i]$ is a witness to $Z_i > Z_{i+1}$. We note that $k_i > 0$ for all i. Otherwise $0 = k_i = k_{i+1} = \ldots$ and then $Z_i = Y_i > Z_{i+1} = Y_{i+1} > \ldots$ implies that $Y_i \supsetneq Y_{i+1} \supsetneq \ldots$, which contradicts the Noetherianity of the **Vec**-variety Y_i, see Theorem 2.4.3.

From the chain, we construct an infinite rooted forest with vertex set $[k_1] \sqcup [k_2] \sqcup \ldots$ as follows: $[k_1]$ is the set of roots, and we attach each $j \in [k_{i+1}]$ via an edge with $\pi_i(j)$; the latter is called the *parent* of the former. We further label each vertex $j \in [k_i]$ with the product $B_{i,j} \times Q_{i,j}$.

We claim that π_i is an injection for all $i \gg 0$, i.e., that there are only finitely many vertices with more than one child. Indeed, if not, then by König's lemma the forest would have an infinite path starting at a root in $[k_1]$ and passing through infinitely many vertices with at least two children. By construction, the labels $B \times Q$ decrease weakly along such a path and strictly whenever going from a vertex to one of its more than one children, a contradiction to Lemma 8.3.14.

For even larger i, the k_i are constant, say equal to k, and hence the π_i are bijections. After reordering, we may assume that the π_i all equal the identity on $[k]$. Moreover, for all such i we still have $B_{i,j} \times Q_{i,j} \geq B_{i+1,j} \times Q_{i+1,j} \geq \ldots$ for all $j \in [k]$, and all these chains stabilise. When they do, we have $Y_i \supsetneq Y_{i+1} \supsetneq \ldots$, which is a strictly decreasing chain of **Vec**-varieties—but this again contradicts the Noetherianity of **Vec**-varieties. □

Order on closed subvarieties of product-type varieties in PM

Consider the $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety $Z := [Y; B_1 \times Q_1, \ldots, B_k \times Q_k]$ and let X be a closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvariety of Z; X is not required to be of product-type. We define

$$\delta_X := \min_{(S_1,\ldots,S_k) \in \mathbf{FI}^k} \left\{ \sum_{i=1}^{k} |S_i| \, : \, X(S_1, \ldots, S_k) \neq Z(S_1, \ldots, S_k) \right\}$$

Let X and X' be closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvarieties of Z, then we say $X' \leq X$ if $\delta_{X'} \leq \delta_X$. This is a well-founded pre-order on the $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvarieties of Z.

Remark 8.3.17. If f is a nonzero equation for $X(S_1, \ldots, S_k)(V)$ with $\sum_i |S_i| = \delta_X$, then f may still "come from smaller sets". More specifically, there might exist a k-tuple $(S'_1, \ldots, S'_k)$ with $|S'_i| \leq |S_i|$ for all $i \in [k]$ and with strict inequality for at least one i, an $\mathbf{FI}^k$-morphism

$\iota := (\iota_1, \dots, \iota_k) : (S'_1, \dots, S'_k) \to (S_1, \dots, S_k)$, and an element $f' \in K[Z(S'_1, \dots, S'_k)(V)]$ such that $Z(\iota)(V)^\#(f') = f$. This is related to Remark 7.2.3. The following example demonstrates this phenomenon. ♪♫

Example 8.3.18. Consider the $\mathbf{FI}^{\mathrm{op}} \times \mathbf{Vec}$-variety $Z := [\{0\}; K]$. The coordinate ring $K[Z(S)]$ is isomorphic to the polynomial ring over K in $|S|$ variables. Let $n \in \mathbf{Z}_{>0}$ and define the proper closed variety X of Z by

$$X(S) := \begin{cases} Z(S) & \text{for } |S| < n; \\ \emptyset & \text{otherwise.} \end{cases}$$

Then δ_X is equal to n and computed by the element $1 \in K[Z([n])]$, which is the image of $1 \in K[Z(\emptyset)]$ under the natural map $K[Z(\emptyset)] \to K[Z([n])]$. ♫

8.4 The Parameterisation Theorem in product-type varieties

The goal of this section is to prove the following core result, which says that any proper closed subset of an $(\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec}$-variety of product-type is covered by finitely many smaller such varieties.

Theorem 8.4.1 (Parameterisation Theorem)**.** *Consider an $(\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec}$-variety Z of product-type and let $X \subsetneq Z$ be a proper closed $(\mathbf{FI}^{\mathrm{op}})^k \times \mathbf{Vec}$-subvariety. Then there exist a finite number of quadruples consisting of:*

- *an $l \in \mathbf{Z}_{\geq 0}$;*
- *an $(\mathbf{FI}^{\mathrm{op}})^l \times \mathbf{Vec}$-variety Z' of product type with $Z' \prec Z$;*
- *a k-tuple $S = (S_1, \dots, S_k) \in \mathbf{FI}^k$; and*
- *a partition morphism $(\pi, \varphi) : Z' \to \mathrm{Sh}_S Z$;*

such that for any $T_1, \dots, T_k \in \mathbf{FI}^k$, any $V \in \mathbf{Vec}$, and any $p \in X(T_1, \dots, T_k)(V)$ there exist: one of these quadruples; finite sets $U_1, \dots, U_k$; and bijections $\sigma_i : T_i \to S_i \sqcup U_i$; such that p lies in the image under $Z(\sigma_1, \dots, \sigma_k)(V)$ of the image of (π, φ) in $\mathrm{Sh}_S(Z)(U_1, \dots, U_k)(V) = Z(S_1 \sqcup U_1, \dots, S_k \sqcup U_k)(V)$.

Remark 8.4.2. Recall Definition 7.2.8 of the image of a partition morphism. Explicitly, the conclusion above means that there exist finite sets $U'_1, \dots, U'_l$ and, for each $i \in [k]$, a bijection $\iota_i : U_i \to \bigsqcup_{j \in \pi^{-1}(i)} U'_j$, and a point $q \in Z'(U'_1, \dots, U'_l)(V)$ such that

$$(Z(\sigma_1, \dots, \sigma_k)(V) \circ (\mathrm{Sh}_S Z)(\iota_1, \dots, \iota_l)(V) \circ \varphi(U'_1, \dots, U'_l)(V))(q) = p.$$

Informally, we will say that all points in X are *hit* by finitely many partition morphisms from varieties Z' in $\mathbf{PM}$ of product-type with $Z' \prec Z$. ♪♫

8.4.1 A key proposition

The proof of Theorem 8.4.1 uses a key proposition that we establish first. The reader may prefer to read only the statement of this proposition and postpone its proof until after reading the proof of Theorem 8.4.1 in Section 8.4.4.

Proposition 8.4.3 (The Key Proposition)**.** *Let Y be a* **Vec**-*variety; $n \in \mathbf{Z}_{\geq 0}$; B a closed* **Vec**-*subvariety of $Y \times \mathbb{A}^n$; Q a pure polynomial functor; and X a proper closed* **Vec**-*subvariety of $B \times Q \subseteq Y \times \mathbb{A}^n \times Q$.*

Then there exist a proper closed **Vec**-*subvariety Y_0 of Y, a* **Vec**-*variety Y' together with a morphism $\alpha : Y' \to Y$; $k \in \mathbf{Z}_{>0}$ and, for each $l = 0, \ldots, k$, integers $n_l \in \mathbf{Z}_{\geq 0}$; closed* **Vec**-*subvarieties $B_l \subseteq Y' \times \mathbb{A}^{n_l}$; pure polynomial functors Q_l; and morphisms $\beta_l : B_l \times Q_l \to B \times Q$ such that the following properties hold:*

1. *For each $l = 0, \ldots, k$, $B_l \times Q_l \prec B \times Q$ in the preorder from Section 8.3.3, and the following diagram commutes:*

$$\begin{array}{ccc} B_l \times Q_l & \xrightarrow{\beta_l} & B \times Q \\ \downarrow & & \downarrow \\ Y' & \xrightarrow[\alpha]{} & Y. \end{array}$$

2. *Let $m \in \mathbf{Z}_{\geq 0}$, $V \in$ **Vec**, and points $p_1, \ldots, p_m \in X(V) \subseteq Y(V) \times \mathbb{A}^n \times Q(V)$ whose images in $Y(V)$ are are all equal to the same point $y \in Y(V) \setminus Y_0(V)$. Then there exist indices $l_j \in \{0, \ldots, k\}$ for $j \in [m]$ and points $p'_j \in B_{l_j}(V) \times Q_{l_j}(V)$ whose images in $Y'(V)$ are all equal to the same point y' and such that $\beta_{l_j}(V)(p'_j) = p_j$ for all $j \in [m]$.*

Remark 8.4.4. The condition $\beta_{l_j}(V)(p'_j) = p_j$, together with the commuting diagram in (1), implies $\alpha(y') = y$. ♪♫

To apply Proposition 8.4.3 in the proof of Theorem 8.4.1 we will do a shift over an appropriate k-tuple of finite sets. After this shift, we deal with the points of X lying over Y_0 by induction, while we cover those in the complement by a partition morphism constructed with the morphisms α and β_j's, and whose domain is a product-type variety strictly smaller than Z. Before proving Proposition 8.4.3 in Section 8.4.3, we demonstrate its statement in two special cases.

Example 8.4.5. Consider the case where $Y = \operatorname{Spec} K$ and $n = 0$; then $B \subseteq Y \times \mathbb{A}^n$ is also isomorphic to $\operatorname{Spec} K$. Let Q be an arbitrary polynomial functor. In this case, X is a proper closed **Vec**-subvariety of Q and by [BDES22] there exist $k \in \mathbf{Z}_{\geq 0}$, (finite-dimensional) varieties $B_1, \ldots, B_k$, pure polynomial functors $Q_1, \ldots, Q_k \prec Q$ and morphisms $\beta_i : B_i \times Q_i \to Q$ such that X is the union of the images of the β_j. This is an instance of Proposition 8.4.3 with $Y_0 = \emptyset$, $Y' = Y$, and $\alpha = \mathrm{id}_Y$. Note that then $B_j \times Q_j \prec Q$ since $Q_j \prec Q$, so the specific choice of embedding $B_j \subseteq \mathbb{A}^{n_j}$ is not relevant. ♫

Example 8.4.6. Consider the case where Y is constant, that is, just given by a (finite-dimensional) variety, and $Q = 0$. Since X is a proper closed subvariety of $B \subseteq Y \times \mathbb{A}^n$, there exist a $V \in \mathbf{Vec}$ and a nonzero function $f \in K[BV]$ that vanishes identically on XV.

Then f is represented by a polynomial in $K[YV][x_1, \dots, x_n]$, also denoted by f. We may reduce f modulo $\mathcal{I}(BV)$ in such a manner that its leading term $c \cdot x^u$ has the property that $c \in K[YV]$ is nonzero and $x^u \notin \mathrm{LM}(B)$. Then we take for Y_0 the closed subvariety of Y defined by the vanishing of c and for Y' the complement $Y \setminus Y_0$, with $\alpha : Y' \to Y$ being the inclusion. Furthermore, we take $k = 1$, and B_1 to be the intersection of B with $Y' \times \mathbb{A}^n$ and with the vanishing locus of f in $Y \times \mathbb{A}^n$. Then $\mathrm{LM}(B_1) \supseteq \mathrm{LM}(B)$ and since c is invertible on Y' and f vanishes on B_1, $x^u \in \mathrm{LM}(B_1) \setminus \mathrm{LM}(B)$. To verify (2) of Proposition 8.4.3, we observe that the p_i all map to the same point in $Y' = Y \setminus Y_0$, i.e., p_i lies in the set $B_1 \subseteq B$, and we can just take $p_j' := p_j$ for all j. ♫

8.4.2 Iterated partial derivatives

The main idea for proving Proposition 8.4.3 comes from Lemma 8.4.7 below, that is an extension (or better an iteration) of [Dra19, Lemma 18].

Lemma 8.4.7. *Let P be a polynomial functor and let $R_1, \dots, R_t$ be irreducible polynomial functors of positive degree such that*

$$P = P' \oplus R_1 \oplus \cdots \oplus R_t.$$

Denote with $R_{\leq s}$ the functor $\bigoplus_{i=1}^s R_i$. Let B be a $\mathbf{Vec}$-subvariety of P' and let X be a proper closed $\mathbf{Vec}$-subvariety of $B \times R_{\leq s}$ such that X is not isomorphic to $X' \times R_1 \times \cdots \times R_t$ with X' a closed $\mathbf{Vec}$-subvariety of B. Let f be a non-zero defining equation of $X(U_0)$ in $K[B(U_0) \times R_{\leq t}(U_0)]$ for some $U_0 \in \mathbf{Vec}$. Then there exist

- *vector spaces $U_1, \dots, U_k$ with partial sums $U_{\leq s} := \bigoplus_{i=0}^s U_i$,*
- *indices $0 = s_0 < s_1 \leq \cdots \leq s_k$, and,*
- *for each $l \in [k]$, nonzero coordinates $x_l \in R(U_{s_l})^*$, nonzero functions $h_l \in K[P'(U_{\leq l}) \times R_{\leq s_l}(U_{\leq l})]$, and functions r_l in $K[P'(U_{\leq l}) \times R_{\leq s_l}(U_{\leq l})/R_{s_l}(U_l)]$ such that*

$$h_l = x_l \cdot h_{l-1} + r_l,$$

and moreover the function h_k vanishes on $X(U_{\leq k})$.

Sketch. First, choose s_k as the maximal index in $[t]$ such that f involves coordinates in $R_{s_k}(U_0)^*$; if no such index exists, then k is set to zero, and we may take $h_0 := f \in K[B(U_0)]$. For a subspace U_k of sufficiently high dimension (at least $d_k := \deg(R_{s_k})$ suffices), act on f with "upper triangular" elements of the Lie algebra $\mathfrak{gl}(U_0 \oplus U_k)$ that transform coordinates on $R_{s_k}(U_0)^*$ to coordinates on $R_{s_k}(U_k)^*$. This yields a new polynomial that vanishes on $X(U_0 \oplus U_k)$ but is now of the form $\widetilde{f} = x_k \cdot \widetilde{h} + \widetilde{r}$, where $\widetilde{h} \in K[B(U_0) \times Q(U_0)]$ is (nonzero and) of lower degree than f, and where $\widetilde{r}$ does not contain coordinates in $R_{s_k}(U_k)^*$. Now let s_{k-1} be the maximal index in such that $\widetilde{h}$ involves coordinates in $R_{s_{k-1}}(U_0)^*$. We will allow $s_{k-1} = s_k$, which will be the case if f was not linear in the coordinates in $R_{s_k}(U_0)^*$. Again choose a vector space U_{k-1} of sufficiently high dimension, and act on $\widetilde{f}$ with upper triangular elements of $\mathfrak{gl}(U_0 \oplus U_{k-1})$ to obtain

$$\widehat{f} = x_k \cdot (x_{k-1} \cdot \widehat{h} + \widehat{r}) + \bar{r}$$

where x_{k-1} is a coordinate in $R_{s_{k-1}}(U_{k-1})^*$, $\widehat{r}$ does not involve coordinates in $R_{s_{k-1}}(U_{k-1})^*$, $\bar{r}$ may be different from $\widetilde{r}$, but still does not involve coordinates in $R_{s_k}(U_k)^*$, and $\widehat{h} \in K[B(U_0) \times Q(U_0)]$ has smaller degree than $\widetilde{h}$. Continuing in this fashion, we eventually find a polynomial

$$h_k = x_k(x_{k-1}(\dots(x_2(x_1 h_0 + r_1) + r_2)\dots) + r_{k-1}) + r_k \tag{8.2}$$

where $h_0 \in K[B(U_0)]$. Now it is clear how to define the intermediate h_l. □

8.4.3 Proof of The Key Proposition

This section contains the proof of the Proposition 8.4.3, and, for clarity's sake, we spell it out in a concrete example at the end.

Remark 8.4.8. We recall that, for any **Vec**-variety Z and any $U \in$ **Vec**, the shift $\mathrm{Sh}_U Z$ of Z over U is the **Vec**-variety defined by $\mathrm{Sh}_U Z(V) = Z(U \oplus V)$. There is a *natural morphism* $\mathrm{Sh}_U Z \to Z$ of **Vec**-varieties: for $V \in$ **Vec**, this morphism $(\mathrm{Sh}_U Z)(V) = Z(U \oplus V) \to Z(V)$ is just $Z(\pi_V)$, where π_V is the projection $U \oplus V \to V$. ♪♫

Lemma 8.4.9. *Let Y be a* **Vec***-variety, $n \in \mathbf{Z}_{\geq 0}$, and B a closed* **Vec***-subvariety of $Y \times \mathbb{A}^n$. Then for any $U \in$* **Vec**, *$\mathrm{Sh}_U B$ is a closed* **Vec***-subvariety of $(\mathrm{Sh}_U Y) \times \mathbb{A}^n$, and* $\mathrm{LM}(B) = \mathrm{LM}(\mathrm{Sh}_U(B))$.

Proof. This follows from Lemma 8.3.7. □

Remark 8.4.10. Let X be a **Vec**-variety, $U \in$ **Vec** and $f \in K[X(U)]$. We define $(\mathrm{Sh}_U X)[1/f]$ to be the **Vec**-variety given by $V \mapsto X(U \oplus V)[1/f]$, where we identify f with its image under the natural map $K[X(U)] \to K[X(U \oplus V)]$. Note that the action of the group GL on the coordinate ring of $\mathrm{Sh}_U X$ is the identity on the element f. In particular, for every $V \in \mathbf{Vec}$, $(\mathrm{Sh}_U X[1/f])(V) \subseteq \mathrm{Sh}_U X(V)$ is the distinguished open set of points not vanishing on the single equation f. ♪♫

Proof of Proposition 8.4.3. Since X is a proper closed subvariety of $B \times Q$, there exist a $U_0 \in$ **Vec** and a nonzero $f \in K[B(U_0) \times Q(U_0)]$ that vanishes on X. As a first step, we apply the machinery of Lemma 8.4.7.

Decompose Q as $R_1 \oplus \cdots \oplus R_t$, where the R_s are irreducible polynomial functors and $\deg(R_s) \leq \deg(R_{s+1})$ for all $s = 1, \ldots, t-1$. Write $R_{\leq s} := R_1 \oplus \cdots \oplus R_s$ and $R_{>s} := R_{s+1} \oplus \cdots \oplus R_t$, so that $R_{\leq 0} = \{0\}$ and $R_{>t} = \{0\}$.

By Lemma 8.4.7, from f we can construct a sequence of vector spaces $U_1, \ldots, U_k$ with partial sums $U_{\leq l} := \bigoplus_{i=0}^{l} U_i$ (note that $U_{\leq 0} = U_0$), indices $0 = s_0 < s_1 \leq \cdots \leq s_k \leq t$, nonzero coordinates $x_l \in R_{s_l}(U_l)^*$ for $l \in [k]$, nonzero functions $h_l \in K[B(U_{\leq l}) \times R_{\leq s_l}(U_{\leq l})]$ for $l = 0, \ldots, k$ and functions $r_l \in K[B(U_{\leq l}) \times (R_{\leq s_l}(U_{\leq l})/R_{s_l}(U_l))]$ for $l \in [k]$ such that

$$h_l = x_l \cdot h_{l-1} + r_l \tag{A}$$

for each $l = 1, \ldots, k$ and such that h_k that vanishes on $X(U_{\leq k})$.

Now $h_0 \in K[B(U_0)]$ is represented by a polynomial in $K[Y(U_0)][x_1, \ldots, x_n]$, and after reducing modulo $\mathcal{I}(B(U_0))$, we may assume that its leading term equals $c \cdot x^u$ where $c \in K[Y(U_0)]$ is nonzero and $x^u \notin \mathrm{LM}(B)$.

Now set $U := U_{\leq k} = U_0 \oplus \cdots \oplus U_k$. Then we construct the relevant data as follows.

1. Define Y_0 as the closed **Vec**-subvariety of Y defined by the vanishing of c, so that

$$Y_0(V) := \{y \in Y(V) \mid \forall \varphi \in \mathrm{Hom}(V, U_0) : c(Y(\varphi)y) = 0\}.$$

2. Set $Y' := (\mathrm{Sh}_U Y)[1/c]$ with $\alpha : Y' \to Y$ the restriction to Y' of the natural morphism $\mathrm{Sh}_U Y \to Y$.

3. Let B_0 be the closed **Vec**-subvariety of $(\mathrm{Sh}_U B)[1/c]$ defined by the vanishing of the single equation h_0. Note that B_0 is a closed **Vec**-subvariety of $Y' \times \mathbb{A}^{n_0}$ with $n_0 := n$. Define $Q_0 := Q$ and $\beta_0 : B_0 \times Q_0 \to B \times Q$ as the identity on Q and equal to the restriction to B_0 of the natural morphism $\mathrm{Sh}_U B \to B$ on B_0. Note that $\mathrm{LM}(B_0) \supseteq \mathrm{LM}(B)$ by virtue of Lemma 8.4.9, and since $h_0 \in \mathcal{I}(B_0(U_0))$ has leading term $c \cdot x^u$ and c is invertible on Y', we have $x^u \in \mathrm{LM}(B_0) \setminus \mathrm{LM}(B)$. Thus $B_0 \times Q_0 \prec B \times Q$.

4. For $l \in [k]$, set

$$Q_l := ((\mathrm{Sh}_U R_{\leq s_l})/(R_{\leq s_l}(U) \oplus R_{s_l})) \oplus R_{>s_l}.$$

   Here we recall that, for any pure polynomial functor R, the top-degree part of $\mathrm{Sh}_U R$ is naturally isomorphic to that of R, and its constant part is isomorphic to $R(U)$ (see

[Dra19, Lemma 14] for the first statement; the second is proved in a similar fashion). So, since we ordered the irreducible factors R_s by ascending degrees, R_{s_l} is naturally a sub-object of the top-degree part of $\mathrm{Sh}_U R_{\leq s_l}$; and the constant polynomial functor $R_{\leq s_l}(U)$ is the constant part of $\mathrm{Sh}_U R_{\leq s_l}$. Both are modded out, and we have $Q_l \prec Q$.

5. For $l \in [k]$, we define B_l as

$$\begin{aligned} B_l &:= (\mathrm{Sh}_U B)[1/c] \times R_{\leq s_l}(U) \times \mathbb{A}^1 \\ &\subseteq Y' \times \mathbb{A}^n \times R_{\leq s_l}(U) \times \mathbb{A}^1 \cong Y' \times \mathbb{A}^{n_l}. \end{aligned}$$

   where $n_l := n + \dim(R_{\leq s_l}(U)) + 1$. Note that the factor $R_{\leq s_l}(U)$ is precisely the constant term modded out in the definition of Q_l; the role of the factor $\mathbb{A}^1$ will become clear below.

6. To construct $\beta_l : B_l \times Q_l \to B \times Q$ we proceed as follows. Let X_l be the closed **Vec**-subvariety of $B \times R_{\leq s_l}$ defined by the vanishing of h_l. Then (A) shows that, on the distinguished open subset $(\mathrm{Sh}_{U_{\leq l-1}} X_l)[1/h_{l-1}]$, the cooordinate x_l can be expressed as a function on $\mathrm{Sh}_{U_{\leq l-1}} B \times ((\mathrm{Sh}_{U_{\leq l-1}} R_{\leq s_l})/R_{s_l})$ evaluated at U_l. Since R_{s_l} is irreducible, *each* coordinate on it can be thus expressed; this is a crucial point in the proof of [Dra19, Lemma 25]. This implies that the projection

$$\mathrm{Sh}_{U_{\leq l-1}} B \times \mathrm{Sh}_{U_{\leq l-1}} R_{\leq s_l} \to (\mathrm{Sh}_{U_{\leq l-1}} B) \times (\mathrm{Sh}_{U_{\leq l-1}} R_{\leq s_l})/R_{s_l}$$

   restricts to a closed immersion of $(\mathrm{Sh}_{U_{\leq l-1}} X_l)[1/h_{l-1}]$ into the open subset of the right-hand side where h_{l-1} is nonzero. This statement remains true when we replace $U_{\leq l-1}$ everywhere by the larger space U. After also inverting c, we find a closed immersion

$$(\mathrm{Sh}_U X_l)[1/h_{l-1}][1/c] \to (\mathrm{Sh}_U B)[1/c] \times (\mathrm{Sh}_U R_{\leq s_l})/R_{s_l} \times \mathbb{A}^1,$$

   where the map to the last factor is given by $1/h_{l-1}$. By [Bik20, Proposition 1.3.22] the inverse morphism from the image of this closed immersion lifts to a morphism of ambient **Vec**-varieties

$$\begin{aligned} \iota : &B_l \times (\mathrm{Sh}_U R_{\leq s_l})/(R_{\leq s_l}(U) \oplus R_{s_l}) \\ &\cong (\mathrm{Sh}_U B)[1/c] \times (\mathrm{Sh}_U R_{\leq s_l})/R_{s_l} \times \mathbb{A}^1 \\ &\to \mathrm{Sh}_U(B \times R_{\leq s_l}) \end{aligned}$$

   that hits all the points in $(\mathrm{Sh}_U X_l)[1/h_{l-1}][1/c]$. Finally, we define $\beta_l := \beta_l' \times \mathrm{id}_{R_{>s_l}}$ where β_l' is the composition of ι and the natural morphism $\mathrm{Sh}_U(B \times R_{\leq s_l}) \to B \times R_{\leq s_l}$.

Property (1) in the proposition holds by construction. We now verify property (2). Thus let $V \in \mathbf{Vec}$, $m \in \mathbf{Z}_{\geq 0}$, and let $p_1, \ldots, p_m \in X(V) \subseteq Y(V) \times \mathbb{A}^n \times Q(V)$. Assume that the images of $p_1, \ldots, p_m$ in $Y(V)$ are all equal to y, and that $y \notin Y_0(V)$. By definition of Y_0, this means that there exists a $\varphi \in \mathrm{Hom}(V, U)$ such that $c(Y(\varphi)(y)) \neq 0$.

On the other hand, we have $h_k(X(\psi)(p_j)) = 0$ for all j, because h_k vanishes identically on X. For $j \in [k]$ define

$$l_j := \min\{l \mid \forall \psi \in \mathrm{Hom}(V, U) : h_l(X(\psi)(p_j)) = 0\}.$$

Put differently, l_j is the smallest index l such that the projection of p_j in $B \times R_{\leq s_l}$ lies in $X_l \subseteq B \times R_{\leq s_l}$. Note that, if $l_j > 0$, then there exists a linear map $\psi : V \to U$ such that $h_{l_j - 1}(X(\psi)(p_j)) \neq 0$.

Since $\mathrm{Hom}(V, U)$ is irreducible, there exists a linear map $\varphi \colon V \to U$ such that first, $c(Y(\varphi)(y)) \neq 0$; and second, $h_{l_j-1}(X(\varphi)(p_j)) \neq 0$ for all j with $l_j > 0$.

We now define the p'_j as follows. First, we decompose $p_j = (p_{j,1}, p_{j,2})$ where $p_{j,1} \in B(V) \times R_{\leq s_{l_j}}(V)$ and $p_{j,2} \in R_{> s_{l_j}}(V)$. Similarly, we decompose the point $p'_j = (p'_{j,1}, p'_{j,2})$ to be constructed.

1. Set $p'_{j,2} := p_{j,2}$ for all j. Recall that we had defined $s_0 := 0$, so that this implies that if $l_j = 0$, then the component $p'_{j,2}$ of p'_j in Q equals the component $p_{j,2}$ of p_j in Q.

2. If $l_j = 0$, then $p_{j,1} \in B(V)$, and $p'_{j,1} \in B_0(V) \subseteq (\mathrm{Sh}_U B)[1/c](V)$ is defined as $B(\varphi \oplus \mathrm{id}_V)(p_{j,1})$. Note that $p'_{j,1}$ does indeed lie in $B_0(V)$; this follows from the fact $l_j = 0$, so that $h_0(B(\psi)(p_{j,1})) = 0$ for all $\psi : V \to U_0$, and hence also for all ψ that decompose as $\psi' \circ (\varphi \oplus \mathrm{id}_V)$.

   Furthermore, note that $\beta_0(V)(p'_j) = p_j$; this follows from the equality $\pi_V \circ (\varphi \oplus \mathrm{id}_V) = \mathrm{id}_V$. Also, the image of p'_j in $Y'(V)$ equals $Y(\varphi \oplus \mathrm{id}_V)(y) =: y'$.

3. If $l := l_j > 0$, then $p_{j,1} \in B(V) \times R_{\leq s_l}(V)$ with $s_l \geq 1$, and $p'_{j,1}$ is constructed as follows. First apply $(B \times R_{\leq s_l})(\varphi \oplus \mathrm{id}_V)$ to $p_{j,1}$ and then forget the component in $R_{s_l}(V)$. The morphism β'_l was constructed in such a manner that $\beta'_l(V)(p'_{j,1}) = p_{j,1}$ and therefore $\beta_l(V)(p'_j) = p_j$. Note that also the image of p'_j in $Y'(V)$ equals y'.

□

Example 8.4.11. Let Y be the polynomial functor $V \to V \oplus V$ and let $K[x_i, y_i \mid i \in [\dim(V)]]$ be the coordinate ring of $Y(V)$. We interpret vectors of V as linear polynomial functions on V^*. Consider the **Vec**-subvariety B of $Y \times \mathbb{A}^1$ defined by $y_i - t x_i$ (where t is the coordinate of $\mathbb{A}^1$), consisting of the points $(v, \lambda v, \lambda)$ with $v \in V$ and $\lambda \in K$. Note that $K[B(V)] = K[t, x_i \mid i \in [\dim(V)]]$ and $\mathrm{LM}(B) = \{0\} \subset K[t]$. Let $Q(V)$ be the space of homogeneous degree-2 polynomial functions on V^* and let $K[z_{i,j} \mid i,j \in$

$[\dim(V)]$ and $i \leq j]$ be its coordinate ring. Note that Q is an irreducible polynomial functor so, in the notation of Proposition 8.4.3, we have $R = R_1 = Q$. Define the **Vec**-subvariety

$$X \subset B \times Q \subset Y \times \mathbb{A}^1 \times Q$$

to be given by the points (v, w, λ, q) such that the set $\{w^2, q\}$ is linearly dependent. An equation for X is

$$y_1^2 z_{1,2} - 2y_1 y_2 z_{1,1} = 0.$$

The minor in the left hand side can be written as

$$t^2(x_1^2 z_{1,2} - 2x_1 x_2 z_{1,1}) \in K[B(U_0) \times Q(U_0)]$$

with $U_0 \cong K^2$, the vector space spanned by the elements of the canonical basis e_1, e_2. Say that e_3, e_4 span $U_1 \cong K^2$. Acting on the above element with the (upper triangular) elements $E_{1,3}$ and $E_{2,4}$ of the Lie algebra $\mathfrak{gl}(U_0 \oplus U_1)$ gives:

$$h_1 := z_{3,4}(x_1^2 t^2) + 2(z_{1,4} x_1 x_3 - 2z_{1,3} x_1 x_4 - z_{1,1} x_3 x_4) t^2$$

that, by construction, vanishes on $X(U_0 \oplus U_1)$. Note that $z_{3,4} \in Q(U_1)^*$, $h_0 := x_1^2 t^2 \in K[B(U_0)]$ (hence set $c := x_1^2$), and the rest belongs to $K[B(U_0 \oplus U_1) \times Q(U_0 \oplus U_1)/QU_1]$. Moreover, for any $j > i \geq 4$ the action on h_1 of first $E_{4,j}$ and then $E_{3,i}$ gives:

$$z_{i,j}(x_1^2 t^2) + 2(z_{1,j} x_1 x_i - 2z_{1,i} x_1 x_j - z_{1,1} x_i x_j) t^2.$$

Hence, setting the above equation to zero gives the relation:

$$z_{i,j} = \frac{2(z_{1,j} x_1 x_i - 2z_{1,i} x_1 x_j - z_{1,1} x_i x_j) t^2}{x_1^2 t^2}.$$

Define $r_{i,j}$ to be the numerator of the right hand side above. For $i \in [n]$ and $j \in [m]$ let $p_{0,i}$ and $p_{1,j}$ be points in $X(V)$ such that

- they are all not contained in the vanishing locus of the orbit of c,
- they have all the same projection to $Y(V)$. Moreover,
- the points $p_{0,i}$ are contained in vanishing locus of the orbit of h_0, while
- the points $p_{1,j}$ are not.

Let v be a nonzero linear form and λ a nonzero scalar. Then the points $p_{0,i}$'s are of the form $(v, 0, 0, q_i)$ —where the second entry is the zero vector, the third is the zero scalar, and q_i is any quadratic form; while the points $p_{1,j}$'s are $(v, \lambda v, \lambda, \mu_j v^2)$ where μ_j is a scalar.[1]

[1] In this example either we have all points of the first type or of the second as they need to have the same projection on $Y(V)$.

Let $\varphi : V \to U$ be such that $h_0(X_\varphi(p_{1,j})) \neq 0$ for every $j \in [m]$ and $c(X_\varphi(p_{k,l})) \neq 0$ for $(k,l) \in \{0\} \times [n] \cup \{1\} \times [m]$. We now construct the data of the proposition.

Set $U := U_0 \oplus U_1$, $Y' := \mathrm{Sh}_U\, Y[1/c]$, define B_0 to be the vanishing locus of h_0 in $\mathrm{Sh}_U\, B[1/c] \subset Y' \times \mathbb{A}^1$. In particular, $t^2 \in \mathrm{LM}(B_0)$. Set $Q_0 := Q$, so $B_0 \times Q_0 \prec B \times Q$, and let the map:

$$\beta_0 : B_0 \times Q_0 \to B \times Q$$

be $B(\pi_V)_{|B_0} \times \mathrm{id}_{QV}$ for every $V \in \mathbf{Vec}$. Define $p'_{0,i} := (B_{\varphi\oplus\mathrm{id}_V}((v,0,0)), q)$ and note that by definition of φ these points are in $B_0 \times Q_0$. Clearly they all have the same projection to Y' and $\beta_0(V)(p'_{0,i}) = p_{0,i}$.

Consider now $p_{1,i}$ and note that $X_{\varphi\oplus\mathrm{id}_V}(p_{1,i}) \in \mathrm{Sh}_U(B\times Q)(V)[1/ch_0]$. Consider the map:

$$\mathrm{Sh}_U(B\times Q)[1/ch_0] \to \mathrm{Sh}_U(B\times Q)/Q\times\mathbb{A}^1 \cong \mathrm{Sh}_U\, B\times QU\times\mathbb{A}^1\times\mathrm{Sh}_U\, Q/(QU\oplus Q) =: B_1\times Q_1$$

where the coordinate on $\mathbb{A}^1$ is given by $1/h_0(X_\varphi(p_{1,i}))$, and let $p'_{1,i}$ be the image of $X_{\varphi\oplus\mathrm{id}_V}(p_{1,i})$ along this map. Consider the map

$$\beta_1 : \mathrm{Sh}_U(B\times Q)/Q\times\mathbb{A}^1 \to \mathrm{Sh}_U(B\times Q) \to B\times Q.$$

The first arrow is given by the identity on the coordinates not in QV, while the coordinates $z_{i,j}$ in QV are given by $r_{i,j}t$ where t is the coordinate of $\mathbb{A}^1$. The second arrow instead projects on the part $BV \times QV$. Clearly, the image of $p'_{1,i}$ along the above map is $p_{1,i}$. ♫

8.4.4 Proof of The Parameterisation Theorem

This section contains the proof of The Parameterisation Theorem for $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvarieties of $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-varieties of product-type.

Proof of Theorem 8.4.1. The $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-variety Z is of product-type, hence by Definition 8.3.1 it can be written as

$$Z = [Y; B_1 \times Q_1, \ldots, B_k \times Q_k]$$

for some $\mathbf{Vec}$-subvarieties B_i of $Y \times \mathbb{A}^{n_i}$ and pure polynomial functors Q_i. Furthermore, X is a proper closed $(\mathbf{FI}^{\mathbf{op}})^k \times \mathbf{Vec}$-subvariety of Z.

We prove, by induction on the quantity δ_X, that all points in X can be hit by partition morphisms from finitely many $(\mathbf{FI}^{\mathbf{op}})^k\times\mathbf{Vec}$-varieties Z' of product-type with $Z' \prec Z$. So in the proof we may assume that this is true for all proper closed $(\mathbf{FI}^{\mathbf{op}})^k\times\mathbf{Vec}$-subvarieties $X' \subsetneq Z$ with $\delta_{X'} < \delta_X$.

Let $(S_1,\ldots,S_k) \in \mathbf{FI}^k$ be such that $\sum_i |S_i| = \delta_X$ and $X(S_1,\ldots,S_k) \neq Z(S_1,\ldots,S_k)$. If all S_i are empty, then set $Y' := X(\emptyset,\ldots,\emptyset)$, a proper closed $\mathbf{Vec}$-subvariety of Y, $B'_i := Y' \times_Y B_i$, and $Z := [Y'; B'_1 \times Q_1, \ldots, B'_k \times Q_k]$. The partition morphism $(\mathrm{id}_{[k]}, \varphi)$ with

$\varphi(T_1, \ldots, T_k)$ the inclusion $\prod_i (B'_i \times Q_i)^{T_i} \to \prod_i (B_i \times Q_i)^{T_i}$ has X in its image, and we have $Z' \prec Z$ because the Q_i remain the same, $\mathrm{LM}(B'_i) \supseteq \mathrm{LM}(B_i)$ by Lemma 8.3.7, and Y' is a proper closed **Vec**-subvariety of Y. In this case, no shift of Z is necessary.

Next assume that not all S_i are empty. First we argue that the points of $X(T_1, \ldots, T_k)$ where, for some i, $|T_i|$ is strictly smaller than $|S_i|$, are hit by partition morphisms from finitely many $Z' \prec Z$. We give the argument for $i = k$. Define the k-tuple S to be shifted over as $S := (\emptyset, \ldots, \emptyset, T_k) \in \mathbf{FI}^k$, and define the $(\mathbf{FI^{op}})^{k-1} \times \mathbf{Vec}$- variety Z' of product-type

$$Z' := [(B_k \times Q_k)^{T_k}; B'_1 \times Q_1, \ldots, B'_{k-1} \times Q_{k-1}]$$

with $B'_i = (B_k \times Q_k)^{T_k} \times_Y B_i$. Consider the partition morphism $(\pi, \varphi) : Z' \to \mathrm{Sh}_S Z$ where $\pi : [k-1] \to [k]$ is the inclusion and $\varphi(T_1, \ldots, T_{k-1})$ is the natural isomorphism of **Vec**-varieties

$$Z'(T_1, \ldots, T_{k-1}) \to (\mathrm{Sh}_S Z)(T_1, \ldots, T_{k-1}, \emptyset) = Z(T_1, \ldots, T_{k-1}, T_k).$$

Note that π witnesses $Z' \preceq Z$ since the Q_i with $i \leq k-1$ remain the same and $\mathrm{LM}(B'_i) \supseteq \mathrm{LM}(B_i)$ by Lemma 8.3.7. Furthermore, since $k - 1 < k$, we have $Z' \prec Z$ by Lemma 8.3.15. All points in X where the last index set has cardinality $|T_k|$ are hit by this partition morphism. Since there are only finitely many values of $|T_k|$ that are strictly smaller than $|S_k|$, we are done.

So it remains to hit points in $X(T_1, \ldots, T_k)$ where $|T_i| \geq |S_i|$ for all i. In this phase we will apply Proposition 8.4.3.

As by assumption not all S_i are empty, after a permutation of $[k]$ we may assume that $S_k \neq \emptyset$. Let $*$ be an element of S_k and define $\widetilde{S_k} := S_k \setminus \{*\}$. Consider the **Vec**-varieties

$$\begin{aligned} Z(S_1, \ldots, S_k) &= (B_1 \times Q_1)^{S_1}_Y \times_Y \cdots \times_Y (B_k \times Q_k)^{\widetilde{S_k}}_Y \times_Y (B_k \times Q_k)^{\{*\}} \text{ and} \\ \widetilde{Y} := Z(S_1, \ldots, S_{k-1}, \widetilde{S_k}) &= (B_1 \times Q_1)^{S_1}_Y \times_Y \cdots \times_Y (B_k \times Q_k)^{\widetilde{S_k}}_Y. \end{aligned}$$

Set $\widetilde{B_k} := \widetilde{Y} \times_Y B_k \subseteq \widetilde{Y} \times \mathbb{A}^{n_k}$, and note that $X(S_1, \ldots, S_k)$ is a proper closed **Vec**-subvariety of $\widetilde{B_k} \times Q_k$. We may therefore apply Proposition 8.4.3 to $\widetilde{Y}$, n_k, $\widetilde{B_k}$, Q_k and $X(S_1, \ldots, S_k)$.

First consider the proper closed **Vec**-subvariety Y_0 of $\widetilde{Y}$ promised by Proposition 8.4.3, and let X' be the largest closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvariety of Z that intersects $Z(S_1, \ldots, S_{k-1}, \widetilde{S_k})$ in Y_0. Then $X'(S_1, \ldots, \widetilde{S_k}) \neq Z(S_1, \ldots, \widetilde{S_k})$, and therefore $\delta_{X'} \leq \delta_X - 1 < \delta_X$. Hence, by the induction hypothesis, all points in $X'(T_1, \ldots, T_k)$ can be hit by finitely many partition morphisms from varieties $Z' \prec Z$ of product-type.

Next we consider the remaining pieces of data from Proposition 8.4.3. First, we have the **Vec**-variety Y' with a morphism $\alpha : Y' \to \widetilde{Y}$. Further, we have an integer $s \in \mathbf{Z}_{\geq 0}$ and for each $i = 0, \ldots, s$ we have integers n'_{k+i}; **Vec**-varieties $B'_{k+i} \subseteq Y' \times \mathbb{A}^{n'_{k+i}}$; pure polynomial functors Q'_{k+i}; and morphisms $\beta_{k+i} : B'_{k+i} \times Q'_{k+i} \to \widetilde{B_k} \times Q_k$ satisfying the conditions (1) and (2).

Define $B'_i := Y' \times_Y B_i$ for $i = 1, \dots, k-1$ and the $(\mathbf{FI^{op}})^{k+s} \times \mathbf{Vec}$-variety

$$Z' := [Y'; B'_1 \times Q_1, \dots, B'_{k-1} \times Q_{k-1}, B'_k \times Q'_k, \dots, B'_{k+s} \times Q'_{k+s}].$$

Now the map $\pi : [k+s] \to [k]$ that is the identity on $[k-1]$ and maps $[k+s] \setminus [k-1]$ to $\{k\}$ witnesses that $Z' \preceq Z$; here we use that $B'_{k+j} \times Q'_{k+j} \prec B_k \times Q_k$ for $j \in \{0, \dots, s\}$ by the conclusion of Proposition 8.4.3, and also Lemma 8.3.7 to show that $B'_i \times Q_i \preceq B_i \times Q_i$ for $i \in [k-1]$. In fact, we have $Z' \prec Z$ by Lemma 8.3.15.

Now the base variety Y' of Z' comes with a morphism α to the base variety $\widetilde{Y}$ of $\mathrm{Sh}_S Z$; we have morphisms $\beta_i : B'_i \times Q_i \to \widetilde{B}_i \times Q_i$ for $i = 1, \dots, k-1$ (the natural map $B'_i \to \widetilde{B}_i$ times the identity on Q_i) and the morphisms $\beta_{k+j} : B'_{k+j} \times Q'_{k+j} \to \widetilde{B}_k \times Q_k$ defined earlier. By Example 8.3.4, these data yield a partition morphism $(\pi, \varphi) : Z' \to \mathrm{Sh}_S Z$. We have to show that this partition morphism hits all points in X that are not in X'.

First we show, for a $V \in \mathbf{Vec}$, that a point $p \in \mathrm{Sh}_S X(\widetilde{T_1}, \dots, \widetilde{T_k})(V)$ whose projection to $\widetilde{Y}(V)$ is not in $Y_0(V)$ lies in the image of $\varphi(\widetilde{T_1}, \dots, \widetilde{T_k})(V)$. To this end, we write

$$p = ((p_{i,t})_{t \in \widetilde{T_i}})_{i \in [k]}$$

with

$$p_{i,t} \in \mathrm{Sh}_S X(\emptyset, \dots, \emptyset, \{t\}, \emptyset, \dots, \emptyset)(V) = \widetilde{Y}(V) \times_{Y(V)} B_i(V) \times Q_i(V) \subset \widetilde{Y}(V) \times \mathbb{A}^{n_i} \times Q_i(V)$$

where the singleton $\{t\}$ is in the i-th position. We write $p_{i,t} = (\widetilde{y}, a_{i,t}, b_{i,t})$ with $\widetilde{y} \in \widetilde{Y}(V)$, $a_{i,t} \in \mathbb{A}^{n_i}$, and $b_{i,t} \in Q_i(V)$.

By definition of a fibre product, the $p_{i,t}$ all have the same projection $\widetilde{y}$ in $\widetilde{Y}(V) \setminus Y_0(V)$, and hence we can apply (2) of Proposition 8.4.3 to the points $p_{k,t}$ with $t \in \widetilde{T_k}$. This yields integers $l_t \in \{0, \dots, s\}$ and points $p'_{k,t} \in B'_{k+l_t}(V) \times Q'_{k+l_t}(V)$ for $t \in \widetilde{T_k}$ whose images in $Y'(V)$ are all equal, say to $y' \in Y'(V)$, and which satisfy $\beta_{k+l_t}(V)(p'_{k,t}) = p_{k,t}$ for all t. This implies that $\alpha(y') = \widetilde{y}$.

Define

$$T'_{k+j} := \{ t \in \widetilde{T_k} \mid l_t = j \}$$

$j = 0, \dots, s$, and set $T'_i := \widetilde{T}_i$ for $i = 1, \dots, k-1$. In $Z'(T'_1, \dots, T'_{k+s})$ we define the point $q = ((q_{i,t})_{t \in T'_i})_{i \in [k+s]}$ as follows. We set $q_{i,t}$ to be $(y', a_{i,t}, b_{i,t})$ for $i = 1, \dots, k-1$ and $t \in T'_i$, and $q_{i,t} = p'_{k,t}$ for $i = k, \dots, k+s$ and $t \in T'_i$. Then

$$\varphi(T'_1, \dots, T'_{k+s})(q) = p,$$

as desired.

Now, more generally, consider a point p in $X(T_1, \dots, T_k)(V) \setminus X'(T_1, \dots, T_k)(V)$, where the cardinalities satisfy $|T_i| \geq |S_i|$. Then there exists an $\mathbf{FI}^k$-morphism $\iota =$

$(\iota_1, \ldots, \iota_k) : S \to (T_1, \ldots, T_k)$ such that $X(\iota)(p) \notin Y_0(V)$. Define $\widetilde{T}_i := T_i \setminus \mathrm{Im}(\iota_i)$ and extend ι to an isomorphism $\iota^e : S \sqcup (\widetilde{T}_1, \ldots, \widetilde{T}_k) \to (T_1, \ldots, T_k)$ by defining ι_i on $\widetilde{T}_i$ to be the inclusion. Consider $X(\iota^e)(p) \in X(S \sqcup (\widetilde{T}_1, \ldots, \widetilde{T}_k))(V)$. This is also a point in $\mathrm{Sh}_S X(\widetilde{T}_1, \ldots, \widetilde{T}_k)(V)$ whose projection to $\widetilde{Y}(V)$ does not lie in $Y_0(V)$. We can therefore find a point q as described above showing that $X(\iota^e)(p)$ is in the image of $(\pi, \varphi) : Z' \to \mathrm{Sh}_S Z$; by Definition 7.2.8, then so is p. □

8.5 Proof of Sym×GL-Noetherianity

The most general version of our Noetherianity result is the following.

Theorem 8.5.1. *Any* $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$*-variety of product-type is Noetherian.*

Proof. We proceed by induction along the well-founded order on objects of product-type in **PM** from Section 8.3.3.

Let Z be an $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-variety of product-type and let $X_1 \supseteq X_2 \supseteq \ldots$ be a descending chain of closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-sub-varieties. Then either all X_i are equal to Z, or there exists an i_0 such that $X := X_{i_0}$ is a proper closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvariety of Z. In the latter case, by Theorem 8.4.1, there exist a finite number of objects $Z_1, \ldots, Z_N$ in **PM** of product-type, along with k-tuples $S_1, \ldots, S_N \in \mathbf{FI}^k$ and partition morphisms $(\pi_j, \varphi_j) : Z_j \to \mathrm{Sh}_{S_j} Z$ such that every point of X is hit by one of these. By the induction hypothesis, all Z_js are Noetherian. For each j, by Lemma 8.2.2, the preimage in Z_j of the chain $(\mathrm{Sh}_{S_j} X_i)_{i \geq i_0}$ is a chain of closed subvarieties, which therefore stabilises. As soon as these N chains have all stabilised, then so has the chain $(X_i)_i$—here we have used a version of Proposition 8.2.3. □

To deduce from this Theorems 8.1.1 and 8.2.1, we consider GL-varieties $Z_1, \ldots, Z_k$ as well as the product $Z := Z_1^{\mathbf{N}} \times \cdots \times Z_k^{\mathbf{N}}$. Recall Remark 7.2.3.

Proof of Theorem 8.2.1. We need to prove that any descending chain $Z \supseteq X_1 \supseteq \ldots$ of $\mathrm{Sym}^k \times \mathrm{GL}$-stable subsets of Z stabilises.

To each Z_i is associated a **Vec**-variety, which by abuse of notation we also denote Z_i; see Remark 2.2.21. Furthermore, Z_i is a closed subvariety of $B_i \times Q_i$ for some finite-dimensional variety B_i and some pure polynomial functor Q_i, and hence Z is a closed subvariety of

$$(B_1 \times Q_1)^{\mathbf{N}} \times \cdots \times (B_k \times Q_k)^{\mathbf{N}}.$$

Now each X_i defines a closed $(\mathbf{FI^{op}})^k \times \mathbf{Vec}$-subvariety $\widetilde{X}_i$ of

$$\widetilde{Z} := [Y; B_1 \times Q_1, \ldots, B_k \times Q_k],$$

where Y is a point. By Theorem 8.5.1, the $\widetilde{X}_i$ stabilise. As soon as they do, so do the X_i. □

Proof of Theorem 8.1.1. Apply Theorem 8.2.1 with $k = 1$. □

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Curriculum Vitae

Alessandro was born on the 19th of May 1992 in Bolzano, Italy, and moved to Milano, Italy, when he was six years old.

In Milano, he attended the Liceo-Ginnasio A. Manzoni, and he obtained the high-school diploma in Humanities in 2011. From 2006, during the high-school, he started studying Music at the Conservatory G. Verdi, where he graduated with a master degree in Clarinet performance in 2013 under the guidance of Sergio Del Mastro.

Afterwards, he started a bachelor in Mathematics at Università degli Studi of Milano, and he graduated in 2016. He continued his studies with the double-degree master ALGANT (ALgebra, Geometry, And Number Theory) at the University of Leiden, The Netherlands, and at the University of Duisburg-Essen, Germany. In 2018, he completed the ALGANT master with a thesis on modular forms under the lead of Massimo Bertolini.

In February 2019 he started his Ph.D. in Jan Draisma's project "Stabilisation in Algebra and Geometry" at the University of Technology in Eindhoven. His supervisors are Jan Draisma and Rob Eggermont. Some of the results of these years of research are presented in this dissertation.

From March 2023 he will be employed at the University of Bern as a postdoctoral researcher, and from August 2023 as a postdoctolar assistant professor at the University of Michigan in Ann Arbor.