Analysis and Control of Nonlinear Systems with Stability and Performance Guarantees

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A Linear Parameter-Varying Approach

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Summary

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Across various domains in engineering, e.g., in the mechatronics, the automotive and the aerospace industry, there is a constant push for higher performance in terms of speed, accuracy, throughput, and power efficiency. These ever-growing performance requirements have led to both increasingly complex engineering systems and increasingly complicated models to describe their behavior with sufficient accuracy. The resulting dynamic system models have become dominantly nonlinear, time-varying, or even exhibiting spatial-varying behavior.

While such complexity of the dynamic models has increased rapidly over the past decades, most tools commonly used in the industry to analyze and control these systems still rely on results from the *Linear Time-Invariant* (LTI) framework, which limits the ability to reach the currently targeted performance requirements. The main advantages of the current LTI framework are that it provides systematic, intuitive, computationally efficient, and relatively easy-to-use tools for analysis and design of controllers to ensure stability and performance guarantees, which contribute to its popularity. Moreover, it also provides effective methods to shape the performance of the resulting closed-loop system. Some attempts have been made to extend the systematic LTI results to nonlinear systems through the use of so-called surrogate models. Surrogate modeling approaches, e.g., the *Linear Parameter-Varying* (LPV) framework, approximate or embed the nonlinear system dynamics in an easier-to-work-with model structure. This simplifies the analysis and controller design in exchange for some conservativeness in the representation. Although many successful applications of surrogate techniques have led to high industrial interest, naive use of these approaches can lead to improper guarantees of global stability and performance for the underlying nonlinear systems. On the other hand, for nonlinear dynamic systems, there also exists a plethora of results on stability analysis and controller synthesis techniques which directly operate on the nonlinear model. Nonetheless, for these methods, there is a lack of available approaches for performance shaping. Furthermore, most of the methods are too complex, requiring significant expertise and complicated design choices from the user compared to the easy-to-use methods available for LTI controller design. This

heavily hampers the adoption of nonlinear analysis and synthesis techniques in the industry.

To this end, a novel framework is proposed for systematic and computationally efficient analysis and control of nonlinear systems to ensure global stability and performance guarantees. In addition, the proposed framework provides intuitive performance shaping just like in the LTI case. To achieve this, crucial questions that are addressed are: (i) how to systematically and computationally efficiently analyze stability and performance of nonlinear systems, (ii) how to use surrogate representations to achieve convex controller synthesis for nonlinear systems under general performance shaping concepts, (iii) how to reduce the used surrogate model complexity for control to ensure computational efficiency for large, complex models.

To achieve true systematic analysis of nonlinear systems with global stability and performance guarantees, we propose the use of stronger, equilibrium independent notions, in particular, universal shifted and incremental stability and performance. Using these concepts, stability and performance are analyzed w.r.t. all (forced) equilibrium points of the system (universal shifted stability) or w.r.t. convergence to all steady state trajectories of the system (incremental stability). This makes these notions advantageous to achieve global stability performance analysis, as they are independent of a particular equilibrium point or trajectory of the system.

One of our main contributions is to provide a systematic and computationally efficient framework for both universal shifted and incremental stability and performance analysis of *Continuous-Time* (CT) and *Discrete-Time* (DT), *Multi-Input-Multi-Output* (MIMO) nonlinear systems. This is achieved by developing analysis results for universal shifted and incremental dissipativity analysis with quadratic supply functions of nonlinear systems. We show that these analysis conditions can be cast as classical dissipativity tests of linearization-like system representations. Furthermore, we show that these linearization-like system representations can naturally be represented through LPV embeddings. This significantly simplifies the analysis problem, and, importantly, allows the problem to be solved computationally efficiently by using various existing LPV techniques.

Based on these analysis results, we also develop both universal shifted and incremental controller synthesis algorithms with a crucial nonlinear realization step of the resulting controller for CT and DT, MIMO nonlinear systems. Similar to the analysis results, we show that through the LPV framework, the proposed controller design procedures can be cast as a standard LPV synthesis problem. To this end, an intuitive and easy-to-use software implementation to realize the proposed framework-based analysis and controller synthesis has been developed in the LPVcore Toolbox for MATLAB.

Finally, we provide a novel, neural network based, scheduling dimension reduction method for LPV models. This allows one to reduce the computational burden for the provided, LPV-based, analysis and controller synthesis results for universal shifted and incremental dissipativity. This is especially of high importance when the developed methods are applied to large-scale, complex systems. Together, these results constitute a systematic and computationally efficient framework for analysis and control of nonlinear systems with global stability and performance guarantees while providing intuitive performance shaping.

The theoretical results are demonstrated on a wide range of academic examples, lab setups, and real-world applications. Through these demonstrations, the benefits of the proposed approaches are shown in terms of improved closed-loop stability and performance properties. Moreover, it is shown that the proposed approaches simultaneously achieve significantly better performance than existing standard LPV control algorithms.

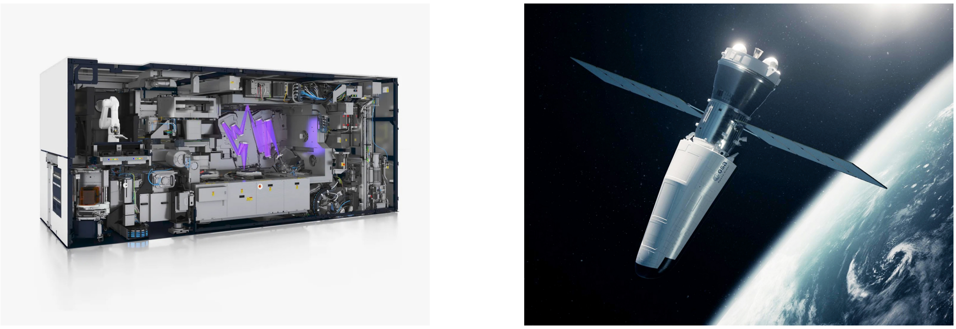
Contents

1 Introduction

In our society, we are constantly improving systems and processes across various domains to satisfy increasingly higher performance requirements, for example in terms of speed, accuracy, throughput, and power efficiency. The engineering solutions to cope with these increasingly higher performance requirements have led to more complex systems and consequently also more complicated dynamic models in order to describe these system sufficiently accurately. This has resulted in dynamic models with dominantly nonlinear, time-varying, or even spatial-varying behavior. Nonetheless, methods for *Linear Time-Invariant* (LTI) systems are still widely used in industry today, as they provide systematic and computationally efficient tools in order to analyze and design controllers which allows to ensure and shape stability and performance guarantees of LTI systems. Over the last decades, various methods have been developed in order to also analyze and control nonlinear dynamic systems. Nonetheless, for such systems, there does not exist the same systematic and computationally efficient framework for analysis and controller design as in the LTI case. Therefore, as the main objective of this thesis, we aim to develop a systematic, computationally efficient framework for the analysis and control of nonlinear systems in order to ensure stability and performance guarantees. In this introductory chapter, we give an overview of tools that are available in order to analyze and design controllers for nonlinear systems and highlight the open problems and challenges. Moreover, we present an outline of the thesis and its main contributions.

1.1 Demand for Increased Performance

In engineering, we are constantly improving systems and processes to satisfy the increasingly demanding performance requirements of society, e.g., in terms of power efficiency, speed, accuracy, etc. These demands for higher performance requirements stem from the challenges that we face today, such as reducing our environmental footprint and increasing energy efficiency, and increasing automation and throughput of production. In order to cope with these demands for increased performance for systems and processes, the engineering solutions for them and the domain(s) in which they operate have become more complex, e.g. in terms of scale or inclusion of multi-physical dynamic phenomenon. For example, for high precision mechatronic systems, like wafer scanners and wire bonders, to achieve the desired production output despite the increasingly smaller scales at which they operate, not only mechanical effects, but also multi-physical effects, such as thermal and electromagnetic dynamics, are needed to be taken into account. Also in aerospace engineering, to extend the operational lifetime and increase the guidance and navigation control accuracy of spacecraft, control algorithms are required that take into account aerodynamical effects and account for wind disturbances in order to safely and reliably land spacecraft back on Earth. (Lamnabhi-Lagarrigue et al. 2017).



(a) ASML Wafer Scanner.

(b) ESA Space Rider.

Figure 1.1: Complex Dynamic Systems.

Therefore, in order to realize the increasing performance demands for engineering systems, more accurate descriptions of dynamics of these systems are required, which take into account the corresponding effects. This has led to the behavior of these systems becoming increasingly dominated by nonlinear, time-varying or sometimes even spatial-varying dynamics, hence, it becomes inevitable to deal with these effects.

While the systems and their dynamical models have grown significantly in complexity, much of the tools used nowadays in industry to analyze and control these systems based on a dynamical model still rely on the *Linear Time-Invariant* (LTI) theory and the corresponding tools that were developed in the 20th century. These tools are still commonly used because they are systematic, intuitive, computationally efficient, and relatively easy-to-use in order to analyze and design controllers to ensure stability and shape the performance of LTI systems. Moreover, for LTI systems, analysis and controller design are widely supported by various popular software packages, such as MATLAB (The MathWorks, Inc. 2022) and Python (Python Control 2022). However, with systems becoming more complex with increasing performance requirements, the ability of the LTI tools to cope with these dynamics and achieve the desired stability and performance requirements is becoming increasingly difficult. While there is a plethora of results available for analysis and control of nonlinear dynamical systems, there unfortunately does not exist a comparable systematic framework for both the analysis and controller design as is available in the LTI case. Many of the tools in the nonlinear domain are often cumbersome to use, specific to their application domain, computationally intensive, and/or require expert knowledge. Furthermore, they also heavily focus on guaranteeing stability of the nonlinear system, much like the early linear tools, without being concerned about what kind of performance they achieve or how to shape this performance to the specific users needs. Finally, there is a lack of reliable software support for analysis and controller synthesis tools for nonlinear systems. The lack of a systematic framework and software support heavily contributes to the slow and limited adoption of the plethora of tools available for nonlinear systems. Hence, with the increase in system complexity, in most if not all domains, there is an increasing need for systematic, computationally efficient, easy-to-use tools to analyze and synthesize controllers for nonlinear dynamical systems in order to ensure stability and shape performance of systems in engineering.

In the next section, we will discuss the various methods that are available to analyze and control nonlinear systems to ensure stability and performance requirements.

1.2 Analysis and Control of Nonlinear Systems

1.2.1 Nonlinear systems

When considering dynamical systems with nonlinear behavior, they are often represented by nonlinear state-space models of the form:

$$\begin{aligned} \xi x(t) &= f(x(t), w(t)); \\ z(t) &= h(x(t), w(t)). \end{aligned} \tag{1.1}$$

For (1.1), $t \in \mathcal{T} \subseteq \mathbb{R}$ is time, where for *Continuous-Time* (CT) systems, $\xi = \frac{d}{dt}$ and $\mathcal{T} = \mathbb{R}$ (or an interval of $\mathbb{R}$), and for *Discrete-Time* (DT) systems, $\xi = q$, i.e., the discrete time-shift operator with $qx(t) := x(t+1)$, and $\mathcal{T} = \mathbb{Z}$ (or an interval of $\mathbb{Z}$). Furthermore, $x(t) \in \mathbb{R}^{n_\mathrm{x}}$ is the vector-valued state variable associated with (1.1) at time t, $w(t) \in \mathbb{R}^{n_\mathrm{w}}$ is the vector-valued input of the system, and $z(t) \in \mathbb{R}^{n_\mathrm{z}}$ is the vector-valued output of the system. Moreover, the functions $f : \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}} \to \mathbb{R}^{n_\mathrm{x}}$ and $h : \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}} \to \mathbb{R}^{n_\mathrm{z}}$ are assumed to be sufficiently smooth such that the solutions of (1.1) exists for all $t \in \mathcal{T}$.

1.2.2 Analysis and Synthesis for Nonlinear Systems

Early Works and Frequency Domain Methods

The analysis of dynamical systems represented by nonlinear differential equations, goes back to the late 19th century with the early works of mathematicians like Poincaré, in the form of analysis of differential equations (Poincaré 1881), and Lyapunov, with stability analysis of dynamical systems (Lyapunov 1892). However, these were not immediately picked up by the engineers at that time, and would only later be rediscovered and reapplied in the systems and control theory.

Early analysis of nonlinear systems from an engineering perspective focussed on analyzing the response of nonlinear systems in terms of periodic and frequency domain behavior, like other LTI methods at that time. For example, Duffing and Van der Pol, among others, focussed on the analysis of the periodic and frequency domain behavior of mechanical and electrical oscillators with nonlinear behavior (Duffing 1918; Van der Pol 1926). This analysis from a frequency domain perspective for nonlinear systems was later formalized in terms of describing functions, which was an early method to analyze the response of a systems for a single sinusoidal input (Mees and Bergen 1975). This also led to the development of other approaches to represent and analyze frequency domain behavior of nonlinear systems, such as nonlinear Bode diagrams (Pavlov, Van de Wouw, et al. 2007), best linear approximations (Schoukens et al. 2009) and generalized frequency response functions (Schetzen 1980), see (Rijlaarsdam et al. 2017) for an overview. However, these approaches are only valid for specific systems types, are restricted to single sinusoidal excitations, and/or are only valid for *Single-Input-Single-Output* (SISO) systems. While many of these analysis methods have been developed, their usage for controller design and synthesis methods has remained limited, as they are not able to capture all the relevant dynamics, and/or the high complexity of the resulting analysis of the system.

Modern Control Theory

Around the 1950s, the development on what we now call modern control methods started. These modern control methods predominantly relied on the state-space representation of dynamical systems, e.g., (1.1), and these methods are still commonly used today. Maybe one of the most influential developments during this era, was the (re)discovery of the work on stability analysis of unforced autonomous dynamical systems by Lyapunov (Lyapunov 1892) in the West, later popularized by Kalman (Kalman and Bertram 1960). The key insight of Lyapunov was that stability of these systems could be analyzed without explicit knowledge of their solutions. By analyzing the rate of change of a function representing the energy of the system, which we nowadays know as a Lyapunov function, see also Figure 1.2, stability could be concluded. Even nowadays, the most widely used method for the analysis of stability of (equilibrium points of) nonlinear systems is through Lyapunov functions. The Lyapunov concept of stability analysis has been extended in many directions, including to systems of the form (1.1) with inputs leading to input-to-state stability (Sontag 1989), to controllability of nonlinear systems, and to control Lyapunov functions (Sontag 1983). This also led to various developments of control methods, based on ensuring stability through construction of Lyapunov functions and controllers such that Lyapunov stability is guaranteed. For example, Lyapunov redesign (Khalil 2015), backstepping (Kokotović and Sussmann 1989), sliding mode control (Young et al. 1999), and many other methods.

However, these methods predominantly focus on guaranteeing stability, where performance mostly only comes in the form of enforcing a decay rate of the Lyapunov function. Moreover, expert knowledge is required to apply these methods, as the design is often done by hand. For example, in terms of constructing a Lyapunov function or feedback control law to ensure stability of a desired equilibrium point. This also makes application of these methods to more complex, larger, systems increasingly difficult or even infeasible.

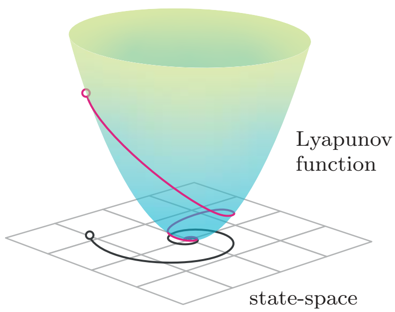


Figure 1.2: Depiction of a Lyapunov function ( ), a trajectory of a system (—), and evaluation of the Lyapunov function along the trajectory (—).

Luré Systems and Linear Matrix Inequalities

Other early modern control developments, focussed on the analysis of so-called Luré systems, named after Anatoliy Luré (Luré and Postnikov 1944). These systems can be modeled as an LTI system G in feeback with a nonlinear, possible time-varying, static nonlinearity ψ (Khalil 2002), see also Figure 1.3. Early developments for these systems focussed on analyzing stability of the origin of the state-space for certain classes of nonlinearities ψ. This led to the famous Circle criterion and Popov Criterion (Luré and Postnikov 1944; Popov 1961), which, for a SISO G, could be used to verify stability through analyzing its Nyquist plot. This significantly simplified the analysis of these types of nonlinear systems, as the well-known LTI tools could be redeployed for this purpose. The Circle and Popov criterion were later also reformulated to *Linear Matrix Inequality* (LMI) conditions using Lyapunov stability theory by Yakubovich (Yakubovich 1964), and led to the well known *Kalman-Yakubovich-Popov* (KYP) Lemma (Rantzer 1996), giving a connection between time and frequency domain inequalities for LTI systems.

These developments have led to analysis and control design tools for Luré type systems (Arcak and Kokotović 2001; Arcak, Larsen, et al. 2003). Furthermore,

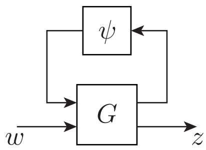


Figure 1.3: Luré System.

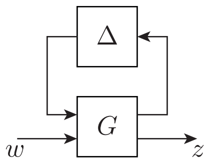


Figure 1.4: LFR System.

analysis of Luré systems also lead to extensions to systems represented by *Linear Fractional Representations* (LFRs), whereby an LTI system G is connected to a time-varying and/or nonlinear function/operator Δ (see Figure 1.4), with analysis through *Integral Quadratic Constraints* (IQCs) (Megretski and Rantzer 1997; Veenman et al. 2016). As the techniques for Luré type systems and LFRs extend LTI methods, they are attractive for analysis and control of nonlinear systems. However, Luré type systems and LFRs remain limited in the types of nonlinear systems they can represent. Moreover, they are also limited by the types of nonlinearities in ψ and Δ they can handle to arrive at feasible analysis and synthesis methods.

Input-Output Analysis

The early results on Luré systems also led to notions of input-output stability of systems (Zames 1966), whereby a system was seen stable if its outputs were bounded and continuous. In this way, performance, could be further generalized to express input-output behavior of systems. This also resulted in various concepts to quantify the performance of systems, such as through passivity and the $\mathcal{L}_2$-gain (for LTI systems connected to positiveness realness and the $\mathcal{H}_\infty$ norm of the transfer function representation), and in the widely used small-gain theorem, connecting to robustness properties of interconnections of nonlinear systems (Khalil 2002; Van der Schaft 2017). This subsequently led to control design methods based on these concepts, for example passivity and $\mathcal{L}_2$ based control (Ortega and Garcia-Canseco 2004; Van der Schaft 2017). Later on, these performance concepts, such as passivity and induced $\mathcal{L}_2$-gain analysis, were unified for LTI systems with Lyapunov based stability analysis through the so-called dissipativity concept by Willems (Willems 1972). These results were also extended to nonlinear system descriptions and general input-output maps (Hill and Moylan 1980), but initially without results for controller design. Besides providing a joined framework for stability and performance analysis, dissipativity theory has also been used in developing various control strategies for nonlinear systems (Van der Schaft 2017; Brogliato et al. 2020). Nonetheless, while these methods allow to provide stability and performance guarantees for nonlinear systems, direct applications of them is often difficult and requires high expertise of the user. In addition, only stability of an a priori chosen equilibrium point or trajectory can be guaranteed, hence, requiring reanalysis and/or redesign of the controller when multiple reference trajectories or disturbances are considered.

Equilibrium Free Analysis

Besides the results based on the standard Lyapunov notion of stability for a single equilibrium point, in the recent decade, increasing attention has been devoted towards equilibrium free notions of stability. We will also refer to these equilibrium free stability concepts, and their related performance notions, as *global stability and performance*, as they consider stability and performance in a global sense, not limited to a particular equilibrium point or trajectory. These notions are especially relevant for nonlinear systems as these can have multiple equilibrium points or even equilibrium trajectories, but also for stability analysis of networks and interconnections of systems, as determining equilibria for these systems is difficult. Equilibrium independent stability/dissipativity is one such notion (Jayawardhana 2006; Hines et al. 2011; Simpson-Porco 2019), which we will refer to as *universal shifted* stability/dissipativity such that it is not confused with other stability/dissipativity notions that are also independent of the equilibrium points of the system. Universal shifted stability/dissipativity considers stability/dissipativity w.r.t. to each (forced) equilibrium point of the system, instead of w.r.t. *one* fixed point, e.g., the origin, as is the case in 'standard' stability analysis, see also Figure 1.5. Some results also exist in ensuring these stability and passivity based performance concepts through controller design (Jayawardhana 2006; Castaños et al. 2009; Kawano et al. 2021).

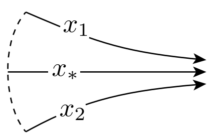


Figure 1.5: Universal shifted stability – stability w.r.t. each (forced) equilibrium point x_*.

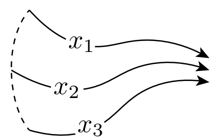


Figure 1.6: Incremental stability – stability between each trajectory.

Another, stronger, equilibrium free concept is that of *incremental stability*, whereby stability is analyzed between all trajectories of the system, see also Figure 1.6. The earliest work considering a similar notion of stability have been developed independently by Opial (Opial 1960) and Demidovich (Demidovich 1961), see also the historical overview in (Jouffroy 2005). The notion of incremental stability as an input-output property was first proposed by Zames (Zames 1966) as a property on the continuity and stability of nonlinear systems and later extended to an induced system gain concept (Romanchuk and James 1996). These notions have later been modernized to various similar notions, such as a Lyapunov framework for incremental stability (Angeli 2002), convergence theory (Pavlov, Pogromsky, et al. 2004), and contraction theory (Lohmiller and Slotine 1998). The relations between these notions have also been studied (Jouffroy and Fossen 2010; Rüffer et al. 2013; Tran et al. 2018). Furthermore, these notions of stability have proven to be useful for other problems, such as synchronization of networked systems and observer design (Pogromsky et al. 2002; Sharma and Kar 2011) and some early results exist for controller design (Pavlov, Van de Wouw, et al. 2006; Scorletti, Fromion, et al. 2015; Manchester and Slotine 2018). Extensions of contraction theory towards passivity (Forni, Sepulchre, and Van der Schaft 2013; Van der Schaft 2013) and dissipativity (Forni and Sepulchre 2013) based formulations have also been made.

While results in this direction are promising, a full framework for analysis and controller synthesis to guarantee these global stability and performance properties has not been developed. The current results remain limited to only $\mathcal{L}_2$-gain and passivity based performance notions and are mainly available only to CT systems. In addition, shaping the closed-loop behavior of systems such certain desired global performance specifications are satisfied has also not been well explored. Moreover, for a large part of the current methods using these notions, the analysis or controller design needs to be done by hand, which requires high expertise of the user and becomes infeasible for more complex systems. Computationally efficient tools for analysis and controller synthesis have also not been well explored.

Geometric Methods and Port-Hamiltonian Systems

Another notable development were the geometric methods in the 1980s (Isidori 1995; Nijmeijer and Van der Schaft 2016). In these methods, techniques from differential and Riemannian geometry were applied to the analysis of nonlinear dynamical systems. These techniques largely shaped our current understanding of properties of nonlinear systems such as controllability and observability. Some notable methods resulting from these geometric methods are feedback linearization (Isidori 1995), widely used for the control of robotic systems, and the development of Port-Hamiltonian methods, including the (energy) shaping of the closed-loop behavior of Port-Hamiltonian systems (Van der Schaft and Jeltsema 2014), which also uses results from passivity and dissipativity theory.

From a mathematical perspective, these methods are elegant. However, this has also resulted in these approaches requiring high expertise of the user. The expert knowledge required to apply these methods to systems has severely limited their applicability in industry.

Online Control Methods

Besides the controller design methods discussed previously for nonlinear systems, whereby a controller is designed/synthesized a priori, there also exist various *online control methods* that compute the control action by solving an online optimization problem. These methods have the advantage that any constraint, on the state, input, and output, can directly be taken into account in the optimization problem. One such method is nonlinear *Model Predictive Control* (MPC) (Mayne and Michalska 1990; Grüne and Pannek 2017), which builds on top of the previously discussed stability and performance principles from optimal control. Several nonlinear MPC algorithms have also been implemented in popular software packages such as MATLAB (The MathWorks, Inc. 2022).

The downside of applying online control methods, such as nonlinear MPC, to nonlinear systems is that the resulting optimization problem is almost exclusively non-convex. The non-convexity of the optimization problem makes it difficult to find a global minimum and it is computationally more expensive to solve compared to convex problems. Moreover, for nonlinear MPC algorithms it is also difficult to ensure the optimization problem is recursively feasible, making it more difficult to ensure safe operation. In some cases the problems can be convexified, but at the cost of increased conservatism in terms of the solution, e.g. through the use of surrogate models (see also Section 1.2.3), such as based on the *Linear Parameter-Varying* (LPV) framework (Hanema et al. 2017), or through an iterative method which uses a convex approximation of the original problem as in *Sequential Quadratic Programming* (SQP) (Diehl et al. 2009).

Learning based Methods

The final branch of nonlinear methods that we will discuss, which have received increasing amounts of attention over the last decade are *learning based methods*. For learning based methods the controller is adapted or learned through interactions with, or based on data of, the system to be controlled. Adaptive control methods, such as *Model Reference Adaptive Control* (MRAC) and *Self-Tuning Control* (STC), were the earliest of such approaches, whereby the controller is adapted to achieve stabilization or asymptotic tracking under plant uncertainties and/or disturbance uncertainties (Tao 2014; Annaswamy and Fradkov 2021). These adaptive control methods build on top of general stability results for nonlinear systems. However, they also carry the downsides of these methods, such as the control designs requiring to be designed by hand by an expert and the lack of general tools for performance shaping. This makes them generally difficult to apply to complex systems and makes it difficult to ensure the desired performance specifications.

Modern learning based methods use recent advances in machine learning, whereby *Gaussian Processes* (GPs) or *Artifical Neural Networks* (ANNs) are used as function approximators to learn (parts of) the system dynamics and/or to train the controller based on data. This has resulted in GP and ANN based feedforward methods (Bolderman et al. 2021; Poot et al. 2022), various supervised and unsupervised, model free and model based reinforcement learning techniques (Sutton and Barto 2018), such as deep Q-learning (Mnih et al. 2013), Proximal Policy Optimization (Schulman et al. 2017), and many others (Arulkumaran et al. 2017). However, while these more recent learning based techniques have achieved incredible results in many complex tasks (OpenAI et al. 2019; Vinyals et al. 2019), they are very costly to train, and require vasts amounts of data and expertise. More importantly, these methods generally do not have formal stability and performance guarantees like the other previously discussed nonlinear control methods. For implementation of these methods in real worlds systems this is crucial, as stability and performance guarantees allow for guarantees w.r.t. the behavior and safety of these systems. Although there have been recent developments which try to establish such results (Gouk et al. 2021; Pauli et al. 2021; Revay et al. 2021).

Discussion

While the various nonlinear methods discussed above have successfully been applied in numerous applications, analyzing nonlinear systems and synthesizing controllers for them still remains a non-systematic and difficult to be applied process compared to LTI framework. Additionally, various software tools exist, such as the Control Systems and Robust Control Toolbox in MATLAB (The MathWorks, Inc. 2022), in order to analyze and design controllers for LTI systems, while off-the-shelf software support for nonlinear systems is lacking.

1.2.3 Surrogate Model Approaches

As mentioned in Section 1.1, most of the tools for direct analysis and controller design for a nonlinear system might be difficult or cumbersome to use. To circumvent this, one option is to construct a surrogate model, i.e., a representation of the system that is easier to work with. For example, by constructing a surrogate model which has a linear relation between inputs and outputs. However, as a consequence of this, the model might be a more conservative representation of the dynamics of the nonlinear system, e.g., by being only valid in a certain operating range or modeling a larger set of behavior. Through a surrogate model, the analysis and control of the original nonlinear system is then simplified, however, the analysis and controller design will also be conservative. Next, we will discuss some commonly used surrogate models for analysis and controller design for nonlinear systems given in terms of a state-space representation (1.1).

LTI Models

Linearization:

While nowadays most systems exhibit some form of nonlinear dynamics, a large part of engineering systems can still be (robustly) approximated with sufficient accuracy using (multiple) LTI models. This allows the analysis and control design for the system to be done through LTI methods. To construct the LTI model, oftentimes a linearization of the nonlinear model is taken. Concretely, a certain operating point $(x_\mathrm{o}, w_\mathrm{o}) \in \mathcal{X} \times \mathcal{W}$ is considered for the linearization. The nonlinear functions f and h are then linearized at this operating point via a Taylor expansion, whereby the 0th and 1st order terms are kept and higher order terms are dropped. This results in the following differential/difference equation

$$\begin{aligned}\xi x(t) &\approx f(x_\mathrm{o}, w_\mathrm{o}) + \left(\frac{\partial f}{\partial x}(x_\mathrm{o}, w_\mathrm{o})\right)(x(t) - x_\mathrm{o}) + \left(\frac{\partial f}{\partial w}(x_\mathrm{o}, w_\mathrm{o})\right)(w(t) - w_\mathrm{o});\\ z(t) &\approx h(x_\mathrm{o}, w_\mathrm{o}) + \left(\frac{\partial h}{\partial x}(x_\mathrm{o}, w_\mathrm{o})\right)(x(t) - x_\mathrm{o}) + \left(\frac{\partial h}{\partial w}(x_\mathrm{o}, w_\mathrm{o})\right)(w(t) - w_\mathrm{o}).\end{aligned} \tag{1.2}$$

In case that $(x_\mathrm{o}, w_\mathrm{o})$ corresponds to a linearization point of the system, i.e., $f(x_\mathrm{o}, w_\mathrm{o}) = 0$ in CT or $f(x_\mathrm{o}, w_\mathrm{o}) = x_\mathrm{o}$ in DT, then, (1.2) can then written as an LTI model in state-space representation of the form

$$\begin{aligned}\xi\tilde{x}(t) &= A\tilde{x}(t) + B\tilde{w}(t);\\ \tilde{z}(t) &= C\tilde{x}(t) + D\tilde{w}(t);\end{aligned} \tag{1.3}$$

with $\tilde{x} := x - x_\mathrm{o}$, $\tilde{w} := w - w_\mathrm{o}$, and $\tilde{z} := z - h(x_\mathrm{o}, w_\mathrm{o})$, where $A := \frac{\partial f}{\partial x}(x_\mathrm{o}, w_\mathrm{o}) \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$, $B := \frac{\partial f}{\partial w}(x_\mathrm{o}, w_\mathrm{o}) \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{w}}$, $C := \frac{\partial h}{\partial x}(x_\mathrm{o}, w_\mathrm{o}) \in \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{x}}$, $D := \frac{\partial h}{\partial w}(x_\mathrm{o}, w_\mathrm{o}) \in \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{w}}$ (Skogestad and Postlethwaite 2001). In case a reliable first principles based model of the system is missing, linear system identification techniques are commonly used to identify the dynamics in a neighborhood around the operating point and obtain an LTI model of the form (1.3) (Ljung 2010).

Classical Methods:

Having obtained an LTI model (1.1) to represent the nonlinear system allows us to use the plethora of analysis and controller design methods for LTI systems. The early methods that were developed are mostly based on frequency domain analysis, and commonly use a transfer-function representation of the LTI system, i.e., $G(s) = C(sI - A)^{-1}B + D$ for (1.3). Nowadays, these methods are often referred to as *classical control methods* (Doyle, Francis, et al. 1990), and are still commonly used today. For stability and performance analysis this includes the Bode and Nyquist diagrams (Nyquist 1932; Bode 1945), which are relatively easy to use as qualitative properties of the system can directly be read from the respective diagrams plotted for the system. One classical controller design method, also still commonly used in industry, is loop shaping (Doyle, Francis, et al. 1990). Using loop shaping, the desirable frequency domain behavior of the closed-loop interconnection of the plant and the controller is shaped directly.

Modern LTI Methods:

The development of most modern control methods for LTI systems, during the second half of the 20th century, led to analysis and controller synthesis for optimal control problems. These optimal control problems were first formulated in terms of the linear quadratic control problems, such as the *Linear-Quadratic Regulator* (LQR) and *Linear-Quadratic-Gaussian* (LQG) controller (Kalman 1960), which later led to $\mathcal{H}_\infty$ and $\mathcal{H}_2$ analysis and controller design (Doyle, Glover, et al. 1989). Importantly, it was also shown how these problems were linked in the time and frequency domain through the KYP lemma (Rantzer 1996). As aforementioned, the dissipativity concept (Willems 1972) unified the concepts of (Lyapunov) stability and performance, such as passivity and $\mathcal{H}_\infty$ analysis, providing a general framework through which both analysis and controller synthesis concepts could be formulated. Through LMIs, these analysis and controller synthesis problems could then be convexified, providing a computationally efficient framework for both analysis and controller synthesis for LTI systems to guarantee stability and performance. Additionally, the shaping of the desired performance was also unified through the generalized plant concept (Doyle 1983). The generalized plant concept enables systematic controller design in terms of different controller structures and performance objectives, e.g., four block mixed sensitivity design, model matching problem, loop shaping design, controllers with a two degree of freedom structure, and observer designs (Zhou et al. 1996; Skogestad and Postlethwaite 2001), see also Figure 1.7.

The analysis and controller synthesis through this framework has paved the way to many extensions such as robust analysis and control of uncertain LTI systems, analysis and control through IQCs and the LPV framework, to name a few.

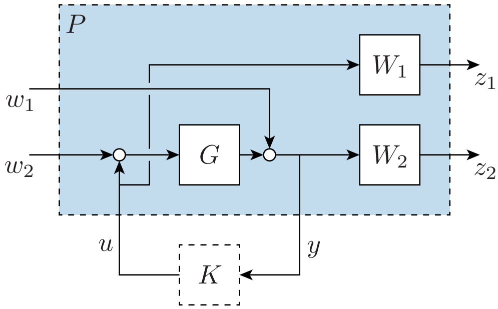


Figure 1.7: Example of a generalized plant P, which includes the plant G, and weighting filters W_*, connected to the (to-be-designed) controller K.

This systematic and computationally efficient framework for LTI systems for analysis and controller design, along with the inclusion of these tools into popular software packages, such as MATLAB (The MathWorks, Inc. 2022) and Python (Python Control 2022), has significantly contributed to their wide spread use in the engineering domain.

However, while stability and dissipativity properties of/around an equilibrium point of the nonlinear system can be inferred from the stability and dissipativity properties of the linearization (1.2) at that point, these properties are only valid in some (small) neighborhood are the considered equilibrium point (Hartman 2002; Khalil 2002; Van der Schaft 2017). The size of this neighborhood is strongly dependent on the dynamics of the original nonlinear system, and for control dependent on the designed controller. In some cases, this can severely limit the applicability of using LTI models (1.3) for analysis and controller design for nonlinear systems (1.1). Robust or multi model LTI concepts can extend this range to which LTI methods can be applied slightly further. Nonetheless, their applicability is still limited as they do not directly take into account the nonlinear dynamics of the system.

LPV Models

Due to the success and the extensive toolset available for LTI systems, but limitations for their application to nonlinear systems, there have been various attempts to extend these results to be used for nonlinear systems. The first attempts that were made, resulted in the 'ad-hoc' methods of gain-scheduling originating form the 1960s for controller design for nonlinear systems (Leith and Leithead 2000). Using gain-scheduling, a collection of LTI controllers would be synthesized for a collection of LTI models based on local linearizations of the nonlinear system at different operating points. This still allowed the existing LTI toolset to be used by doing analysis and controller design at each individual operating point. By interpolating between these LTI controllers, performance could be improved when compared to the use of a single LTI controller, see also Figure 1.8. However, later research showed that using gain-scheduling, formal stability and performance could only be achieved if the operating condition changes 'slowly' (Desoer 1969; Shamma and Athans 1991). Hence, this form of gain-scheduling is limited in applicability.

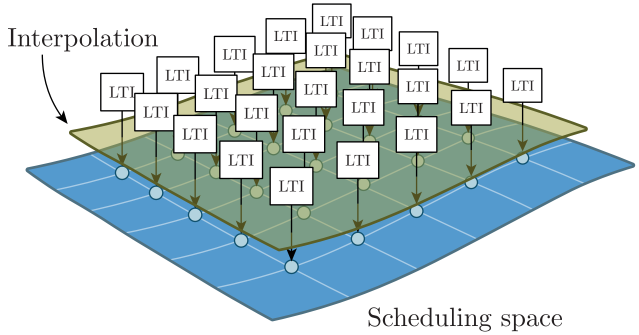


Figure 1.8: Gain-scheduling illustration, interpolation of different local LTI models.

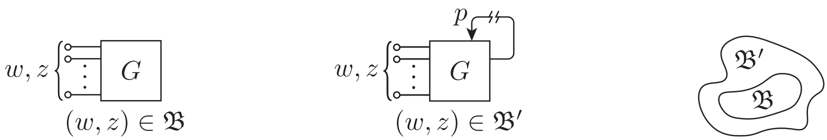
Later on, gain-scheduling was further extended into what would become the *LPV framework* by Shamma (Shamma 1988; Shamma and Athans 1990). In the LPV framework, information about an external variable, the so-called *scheduling-variable*, is used in order to obtain a proxy 'description' of the dynamics of the original nonlinear system, see also Figure 1.9. This corresponding LPV embedding of the system is formulated in order to infer stability, performance and robustness requirements of the original nonlinear system. Concretely, for a nonlinear system (1.1), an LPV model of the form

$$\begin{aligned} \xi x(t) &= A(p(t))x(t) + B(p(t))w(t); \\ z(t) &= C(p(t))x(t) + D(p(t))w(t); \end{aligned} \tag{1.4}$$

is constructed, where $p(t) \in \mathcal{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$ is the scheduling-variable, with $\mathcal{P}$ often taken to be a convex set. Moreover, $A : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$, $B : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{w}}$, $C : \mathcal{P} \to \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{x}}$, $D : \mathcal{P} \to \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{w}}$, where $A, \ldots, D$ are often restricted to a certain function class, such as affine, polynomial, or rational. Furthermore, for this LPV model, there exists some function $\eta : \mathcal{X} \times \mathcal{W} \to \mathcal{P}$, called the scheduling-map, such that $p(t) = \eta(x(t), w(t))$.

Embedding the nonlinear model in an LPV representation was first done using so-called *local methods*, whereby linearizations of the system around operating points, similar to gain-scheduling, or along trajectories are taken, which are then interpolated to construct the LPV model (Tóth 2010). While controller design using local methods often works in practice, they are no strict guarantees as the LPV model only matches the dynamics near the considered operating points. Hence, later extensions to so-called *global methods* were made, where the LPV model is constructed such that it is able to exactly represent the nonlinear system, i.e., constructing $A, \ldots D$ and η such that $f(x, w) = A(\eta(x, w))x + B(\eta(x, w))w$ and $h(x, w) = C(\eta(x, w))x + D(\eta(x, w))w$ for all $(x, w) \in \mathcal{X} \times \mathcal{W}$.

The LPV embedding of a nonlinear system, obtained through either local or global methods, is non-unique, which has resulted in multiple techniques being developed



(a) Nonlinear system. (b) LPV embedding. (c) Relation of resulting behaviors.

Figure 1.9: LPV embedding of a nonlinear system G and the resulting behaviors: $\mathfrak{B}$: solution set of G; and $\mathfrak{B}'$: solution set of the LPV model for $p(t) \in \mathcal{P}$.

to perform this step (Shamma and Cloutier 1993; Leith and Leithead 1998b; Marcos and Balas 2004; Kwiatkowski, Bol, et al. 2006; Donida et al. 2009; Tóth 2010; Hoffmann and Werner 2015b; Abbas, Tóth, Petreczky, Meskin, Mohammadpour Velni, and Koelewijn 2021). Various works have also investigated how to reduce the complexity or conservativeness of the LPV model for analysis or synthesis techniques (Hecker and Varga 2005; Beck 2006; Kwiatkowski and Werner 2008; Hoffmann 2016; Theis et al. 2018; Sadeghzadeh and Tóth 2020). However, these embedding techniques are far from automated and how the embedding step exactly influences the obtainable performance for the corresponding controller design is not straightforward, see e.g. (Kwiatkowski, Bol, et al. 2006). Consequently, the embedding step is often still performed in an ad-hoc fashion or through local methods. Despite lack of guarantees for analysis and controller design using a local LPV embedding of the nonlinear model, this method still sees wide, and successful, application in industry due to its simplicity (Hoffmann and Werner 2015a). Moreover, it also possible to obtain an LPV model directly using various data-driven methods, such as system identification techniques (Tóth 2010; Bachnas et al. 2014; Cox 2018; Verhoek, Beintema, et al. 2022).

Using the LPV framework, the synthesis and analysis results from the LTI framework, such as dissipativity based analysis and synthesis and the generalized plant concept, were extended to be used with LPV models while retaining the computational efficiency of the convex tools that were developed for the LTI framework using LMIs. This allowed for stability and performance guarantees, such as $\mathcal{L}_2$-gain boundedness or passivity, of nonlinear systems by means of the LPV framework. The most common approaches for LPV synthesis are based on polytopic LPV synthesis (Apkarian, Gahinet, and G. Becker 1995; Apkarian and Adams 1998), *Linear Fraction Transformation* (LFT) based LPV synthesis using the S-procedure (Packard 1993; Scorletti and El Ghaoui 1998; Scherer 2001) and gridding-based LPV synthesis (Wu 1995; Wu 2001). In order to make the use of these methods more accessible, some of them have also been implemented in various toolboxes for MATLAB, such as LPVTools (Hjartarson et al. 2015), LCToolbox (Verbandt et al. 2018), IQClab (Veenman 2022), and the LPVcore Toolbox (Boef et al. 2021).

Due to the attractive properties of the LPV synthesis methods, these approaches have successfully been applied in aerospace, automotive, and renewable energy applications, to name a few (Mohammadpour Velni and Scherer 2012; Hoffmann and Werner 2015a). Nonetheless, besides the challenge of (optimally) embedding the nonlinear system in an LPV representation, naive application of the LPV approaches to analyze and design controllers for nonlinear systems can have undesired results. Namely, the LTI approaches, on which the LPV results are based on, generally investigate/ensure stability and performance w.r.t. the origin, which due to the superposition principle, also directly hold w.r.t. other points or trajectories of the system. It was thought that due to the linear dynamical input-output relations of LPV systems, the same principle holds. However, it has been shown through examples that the application of these surrogate model approaches can lead to undesired results (Scorletti, Fromion, et al. 2015; Koelewijn, Sales Mazzoccante, et al. 2020). The aforementioned global stability and performance notions, discussed in Section 1.2.2, have been proposed as a direction to ‘fix’ these problems, however, the notions themselves and also their connection to the LPV framework have not been well explored.

Other Surrogate Model Approaches

Besides LTI and LPV models, other surrogate modeling methods have been introduced such as *Piecewise-Affine* (PWA) models (Sontag 1981; Johansson 1999) and *Takagi-Sugeno* (TS) (or sometimes referred to as *Takagi-Sugeno-Kang* (TSK)) fuzzy models (Babuška and Verbruggen 1996). Both these model classes consider a set of affine models which are switched between. Concretely, these models are of the form

$$\begin{aligned} \xi x(t) &= A_i x(t) + B_i w(t) + b_{\mathrm{x},i}; \\ z(t) &= C_i x(t) + D_i w(t) + b_{\mathrm{z},i}; \end{aligned} \quad \text{for } (x(t), w(t)) \in \Omega_i, \text{ and } i \in [1, \ldots, N], \tag{1.5}$$

where $b_{\mathrm{x},i} \in \mathbb{R}^{n_\mathrm{x}}$ and $b_{\mathrm{z},i} \in \mathbb{R}^{n_\mathrm{z}}$ are bias terms, $N \in \mathbb{N}$ is the number of local models that is considered, and $\Omega_i \subset \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}}$ is a partitioning of the state and input space which determines when which local model, or weighted combination of local models for TS fuzzy models, is active.

Similar to LPV methods, PWA and fuzzy models can be obtained through various conversion and modeling methods (Sontag 1981; Babuška 1998), or are obtained through identification procedures (Takagi and Sugeno 1985; Garulli et al. 2012).

For PWA and fuzzy models, various approaches exist in order to analyze stability and performance of the model or design controllers in order to ensure stability and performance requirements (Sontag 1981; Guerra et al. 2009). Similar to the LTI and LPV methods, these methods exploit the linear/affine properties of their dynamic relationships to reduce the computational complexity of their analysis and synthesis procedures. However, the price for reduced complexity is paid in terms of conservativeness of the system representation, in terms of modeling a larger set of behavior or only being valid in a certain region. This exact trade-off is very complex and has not been analyzed in literature.

Besides the aforementioned high level similarities between these various surrogate model approaches, on the level of the model representation, there are also close connections between PWA, fuzzy models, and the LPV framework. Namely, under certain considerations, the LPV framework allows for the representation of both PWA and fuzzy models (Petreczky and Mercère 2012; Rotondo et al. 2015). Due to the good support for controller design and relative flexibility on how to obtain the model (through either global or local approaches) in the LPV framework, it has become a more popular choice for analysis and controller design compared to PWA and fuzzy based methods.

1.3 Open Problems and Challenges

In Section 1.2, we have given an overview of the various techniques that have been developed and are still used to analyze stability and performance of nonlinear systems and to design controllers to ensure these notions. However, as we have also discussed, the plethora of techniques that are available all have their own advantages and disadvantages, resulting in various open problems and challenges when it comes to analysis and control design for nonlinear systems. Next, we will summarize these open problems and challenges.

We have seen that on the one hand we have the nonlinear methods, which are directly applied to and operate on the nonlinear model, giving guarantees on the nonlinear system based various extensive theoretical results. However, these nonlinear methods still have issues that are yet to be addressed that prevent their application and widespread use in industry:

- **Application and computational complexity**: Most of the nonlinear methods for analyzing nonlinear models and designing controllers for them are generally difficult to apply. For example, for backstepping and sliding mode control, the resulting controller design problem to ensure stability are non-constructive, e.g., in terms of constructing a Lyapunov function and/or feedback control law to ensure stability of the origin of the nonlinear system. This requires expert knowledge and becomes infeasible for large, complex nonlinear systems. Alternatively an optimization problem can be formulated for analysis, e.g., in terms of finding a Lyapunov function (Bobiti and Lazar 2014). However, for nonlinear systems, the resulting optimization problem is almost exclusively non-convex, for which it is generally more difficult to find a global minimum and it is computationally more expensive than convex problems. In some cases the problems can be convexified, but at the cost of increased conservatism in terms of the solution, e.g. through *Sum-of-Squares* (SOS) approaches or via iterative approximation like in SQP. Furthermore, setting up these optimization problems still requires expert knowledge of the user.

- **Performance considerations**: Many of the nonlinear analysis and controller design methods focus solely on stability of the closed-loop interconnection, with much less attention towards what kind of performance they obtain or how to shape the closed-loop performance. While stability is a prerequisite when designing and implementing controllers for real world applications, quantifying and shaping the performance that these controllers obtain is one of the most important design aspects for a control engineer. For LTI systems, there exists a systematic framework for shaping the controller and/or closed-loop behavior of the system. Namely, through the generalized plant concept together with various approaches for designing weighting filters to achieve the desired closed-loop behavior. In the LTI domain, these shaping techniques also nicely connect to shaping the frequency domain behavior through the KYP lemma. On the other hand, for nonlinear systems, there does not exist a systematic approach for shaping their closed-loop performance. While there are some results for nonlinear methods regarding passivity, with respect to energy shaping, and induced $\mathcal{L}_2$-gain, there is no systematic framework on how to achieve the desired performance. Furthermore, while there are methods for frequency domain analysis of nonlinear systems (Rijlaarsdam et al. 2017), they are restrictive and of high complexity, which has limited their applicability for controller synthesis.

- **Equilibrium dependent guarantees**: Most existing analysis and control techniques for nonlinear systems check or ensure stability and performance w.r.t. a single point in the state-space, often the origin of the associated state-space. The systematic approaches in the LTI framework also focus on showing stability and performance w.r.t. the origin, however, due to the superposition principle these properties also hold w.r.t. other enforced equilibria and other trajectories of the system. For nonlinear systems this is not the case, hence, one would need to specifically check stability and performance w.r.t. to each desired (forced) equilibrium or trajectory, meaning for specific reference and disturbance signals. This makes the analysis or controller synthesis excessively complex. Equilibrium independent, or global, stability and performance notions aim to solve this problem by decoupling the stability and performance analysis from a specific equilibrium point. However, these notions have not been well-explored in general with most of them focussing on CT systems only. Additionally, similar to the previous issues pointed out, also for these global notions the existing results are difficult and/or computationally expensive to apply and how to shape the performance of the system, for controller design, is also not well explored. Hence, there is a lack of a systematic overall framework for the analysis and synthesis of these global stability and performance notions.

On the other hand we also have the various surrogate model approaches, which construct a 'simpler' model, often with linear properties, to simplify the analysis or controller design problem. These surrogate model approaches seem like an excellent choice as they build on the systematic framework that already exists for LTI methods. Moreover, they offer computationally efficient results, in terms of convex optimization problems, and offer a systematic framework for performance shaping. Nonetheless, these methods still have issues that are to be addressed:

- **Model conservativeness and complexity**: Due to the use of a surrogate model, the analysis or controller design for a nonlinear system is inherently conservative. Moreover, construction of the surrogate model is often non-unique, which raises the question which one is the 'best', e.g., in terms of a trade-off between complexity and conservativeness. Furthermore, not much research has been performed on quantifying how conservative a particular surrogate model is. Similarly, when the surrogate model is used to enable controller synthesis, the impact of the construction of the surrogate model on the obtainable closed-loop performance is also generally unknown *a priori* and hence it is difficult to compensate for it in advance.

- **Equilibrium dependent guarantees**: Many of the analysis and controller synthesis results for surrogate models are extensions of those from the LTI domain. While this has allowed to formulate powerful results in terms of convex optimization problems, it has been shown that naive application of these methods can lead to undesired results (Scorletti, Fromion, et al. 2015; Koelewijn, Sales Mazzoccante, et al. 2020). Namely, as previously mentioned, the current state-of-the-art LPV methods do not ensure global stability and performance notions. However, the notions themselves and also their connection to surrogate model approaches have not been well explored.

1.4 Problem Statement

As aforementioned, taking into account the nonlinear behavior of systems is becoming more important to push for higher performance in engineering. In the previous sections, we gave an overview of the existing methods and open problems for the analysis and control of nonlinear systems that can be used to tackle this. On the one hand, we have nonlinear methods, which directly take into account the nonlinear behavior to achieve the desired guarantees. Nonetheless, they are cumbersome to use, are often not computationally efficient, and/or lack of a systematic framework for performance analysis and shaping. For engineers in industry to be able to use these methods effectively, systematic and computationally efficient methods are essential. Moreover, a framework to shape the performance of systems is also crucial in order to allow engineers to effectively achieve the desired performance. On the other hand, we have the surrogate model approaches, which construct a 'simpler' model to obtain computationally efficient analysis and controller synthesis results and build on top of the systematic approaches from the LTI framework, but lack in guaranteeing stronger global stability and performance notions. These global stability and performance notions are crucial, as it will ensure that the guaranteed stability and performance of the nonlinear system is independent of any specific equilibrium point or trajectory. This allows to investigate stability of all equilibria at once or convergence of trajectories globally, similar to the stability guarantees for LTI systems. Therefore, is a need for a systematic framework which combines the advantages of surrogate model approaches and nonlinear methods, while addressing their shortcomings, to analyze and shape global stability and performance of nonlinear systems. This constitutes the formulation of the following research objective that will be aimed for in this thesis:

Research Objective

Develop a framework for systematic, computationally efficient, analysis and control of nonlinear systems to ensure and shape global stability and performance guarantees.

As previously mentioned, by 'global' we mean stability and performance notions that are independent of a particular reference/equilibrium point, also called 'equilibrium independent'. We intend to analyze and ensure these notions through (global) extensions of dissipativity to guarantee global stability and performance simultaneously. We will specifically focus on the global notions of *universal shifted* and *incremental* dissipativity. These notions will allow us to guarantee stability and performance w.r.t. all equilibria and/or all feasible trajectories of the systems at once, respectively. The 'performance' characterization that we will consider will be in terms of (extensions of) norm based notions, such as the $\mathcal{L}_2$-gain. We will focus on norm based performance characterizations as they can effectively characterize bounds on the amplification of disturbances, connect to dissipativity, and are also widely used in the LTI framework. Moreover, this also connects to the intention of the to-be-developed framework to be able to 'shape performance'. Namely, like in the LTI framework, through weighting of the norm based performance characterizations, the closed-loop performance can effectively be shaped, e.g., in terms of disturbance amplification. Furthermore, the intended framework should be 'systematic', meaning the procedures to analyze or perform controller synthesis should be the same irrespective of the particular dynamics of a nonlinear system. Moreover, by systematic we also mean that the developed analysis and control design tools are constructive, such that e.g. Lyapunov/storage functions and controllers are constructed automatically based on specifications of the user. The methods in the framework for analysis and controller synthesis should moreover be 'computationally efficient', by which we mean that the computations times should be comparable or a reasonable multiple of the computation time for LTI methods. This will allow engineers to iterate their analysis and controller designs in a comparable time frame as in the LTI framework.

While this research objective seems very ambitious, various assumptions will be taken at different points in the thesis to limit the scope of research. One assumption in particular that we will take is that the dynamics of the nonlinear systems that we will consider can be represented by a nonlinear time-invariant state-space representation, e.g., of the form (1.1). This means we do not explicitly consider nonlinear systems with

- Time-varying or spatial-varying behavior;
- Infinite dimensional state, input or output spaces;

Moreover, we will also assume that the solutions of the considered systems are well-posed in some sense, e.g., they exists for all time and are unique. More concrete assumptions will be taken along the way. However, even with these assumptions, the considered class of nonlinear models is still wide and covers a large generality of phenomena in various domains from electrical and mechanical to chemical and biological systems.

Next, we will formulate various crucial research questions that are to be addressed to achieve the overall specified research objective. These research questions relate to analysis of nonlinear systems, controller synthesis for nonlinear systems, and handling complexity of nonlinear systems for analysis and synthesis.

Remark 1.1. Analyzing or designing a controller for a nonlinear system starts at obtaining a model describing the system. This modeling step plays an important role when analyzing or designing controllers, as the complexity of the model can heavily influence the computational complexity and feasibility of the analysis and controller synthesis methods. However, the modeling step is also heavily dependent on the considered nonlinear system and the analysis and controller design objectives set by the user. As such, we assume that a model of the considered system we want to analyze or do controller design for is available to us, in order to focus on the analysis and controller design steps and not on the modeling.

1.4.1 Analysis

The first key objective in developing the intended framework is analysis. Analyzing the stability and performance of a given nonlinear system is an important aspect in analyzing their qualitative behavior, analyzing they satisfy the desired performance requirements, and analyzing their safety. Moreover, analysis methods are the foundation in a large part of the development of modern control methods, hence, why it also plays a key role in our framework. We discussed in Section 1.3 the shortcomings of existing universal shifted and incremental analysis methods for nonlinear systems. At a high-level, these shortcomings are due to a lack of a systematic and computationally efficient framework, with existing results only being present for specific applications, only focussing on stability or one specific performance notion, or only focussing on CT systems. This leads us to the following research question:

Research Question 1

How to systematically and computationally efficiently analyze universal shifted and incremental dissipativity of CT and DT nonlinear systems?

1.4.2 Synthesis

After answering Research Question 1, the next key part in the development of the framework is controller synthesis. Like for the analysis results, as discussed in

Section 1.3, also for the controller synthesis there is a lack of systematic results for controller synthesis. Not only is it important for the to-be-designed controller synthesis methods to ensure universal shifted or incremental stability of the closed-loop interconnection of plant and controller, they also need to be designed in such a way that desired corresponding global performance criteria, set by the user, are satisfied. Furthermore, like for analysis, also for synthesis we want to ensure that the developed controller synthesis methods are computationally efficient. For controller synthesis this is even more important, as often times multiple controller designs are made, which are iterated on till the desired closed-loop behavior is achieved. Hence, to allow for quick iterations, computational efficiency is also key for controller synthesis. Finally, another important aspect of controller synthesis is how to shape the performance in order to ensure user defined desired behavior in terms of the aforementioned (weighted) norm based bounds, such as (global extensions of) the $\mathcal{L}_2$-gain. As discussed in Section 1.2.3 for LTI and LPV systems we have the generalized plant concept, which, combined with weighting filters, allows for systematic and intuitive shaping of the closed-loop in terms of norm based performance bounds. For LTI systems, these weighting filters also have a clear connection to the resulting frequency domain behavior of the closed-loop system through the KYP lemma. For nonlinear systems these general performance shaping concepts are mostly only considered in conjunction with surrogate models, like through the LPV framework. However, for general controller design of nonlinear systems, and more specifically for controller design ensuring global stability and performance guarantees, performance shaping through these general performance concepts remains largely unexplored. This leads us to the following research question:

Research Question 2

How to systematically and computationally efficiently design and shape controllers for CT and DT nonlinear systems such that universal shifted and incremental performance requirements and stability are ensured?

1.4.3 Complexity

As mentioned in Section 1.1, systems have significantly increased in complexity over the last few decades, both in terms of more dominant nonlinear behavior, and in terms of increase in size, i.e., the order of the dynamics and number of *Input-Output* (IO) variables. Hence, why it is of importance to make sure the developed framework can also cope with these increasingly complex systems. While computational efficiency of the to-be-developed algorithms will play one part in this, even for the most computationally efficient algorithms the 'curse of dimensionality' or specific nonlinearities can result in the algorithms taking a very long time to compute or becoming infeasible to run. Hence, a way to reduce the complexity for analysis and synthesis algorithms of resulting from answering Research Questions 1 and 2, will be an important part in dealing with complex systems. This leads us to our final research question:

Research Question 3

How to reduce the complexity for the to-be-developed analysis and controller synthesis methods to address complex systems?

1.5 Contributions and Outline of the Thesis

In the previous section, we formulated the research objective of developing a systematic, computationally efficient framework for global stability and performance analysis and controller synthesis of nonlinear systems, along with crucial research questions to be answered to achieve this objective. As previously mentioned, for global stability and performance, we will focus on universal shifted and incremental dissipativity. As we will show in the upcoming chapters, these dissipativity notions can be analyzed through linearization-like representations of the system. We will propose the use of LPV systems in order to represent these linearization-like representations. We will show that this will allow us to use methods from the LPV framework in order to formulate systematic and computationally efficient analysis and controller synthesis methods for nonlinear systems that guarantee the aforementioned global stability and performance notions. These developed methods will form the framework in order to achieve our intended research objective.

In Chapter 2, we will introduce preliminaries on the LPV framework, specifically on analysis and synthesis of LPV systems and their current application to nonlinear systems. Furthermore, we will also introduce preliminaries on general dissipativity theory of nonlinear systems.

Next, Chapter 3 will demonstrate the pitfalls of the current use of the LPV framework for analysis of nonlinear systems and controller synthesis for them. This will highlight the need for a different framework for systematic analysis and controller design for nonlinear systems. We will furthermore also go deep into why, and in which cases, the current LPV framework fails to ensure the desired stability and performance guarantees. This chapter is based on the paper (Koelewijn, Sales Mazzoccante, et al. 2020).

In the next chapters, we first focus on analysis and controller synthesis in CT only. In Chapter 4, we introduce the first global dissipativity notion, universal shifted dissipativity. First, as contribution, we present for CT nonlinear systems how universal shifted dissipativity can be analyzed through dissipativity analysis of the time-differentiated dynamics of the system. As a second contribution, we then show that these time-differentiated dynamics can naturally be represented by an LPV system. Using convex tools from the LPV framework, this then allows us to computationally efficiently analyze universal shifted dissipativity of nonlinear systems. Finally, as a contribution, we present how the analysis tools can then be extended to controller synthesis tools, again making use of methods from LPV framework, in order to ensure universal shifted stability and performance of nonlinear systems. These contributions then address Research Questions 1 and 2 w.r.t. universal shifted dissipativity for CT systems. The contents of this chapter are based on the paper (Koelewijn, Tóth, and Weiland 2022a).

Subsequently, in Chapter 5, results on incremental dissipativity are presented. In the chapter, as contribution, a computationally efficient approach will be presented to analyze incremental dissipativity of CT nonlinear systems. We present how the dynamics of the variation of a system around trajectories can be used to analyze incremental dissipativity. We then show these variational dynamics can naturally be represented by an LPV system. Using this, we then show how we computationally efficiently analyze incremental dissipativity of nonlinear systems through the use of the methods from LPV framework. This contribution then addresses Research Question 1 w.r.t. incremental dissipativity for CT systems. The chapter is based on the paper (Verhoek, Koelewijn, et al. 2020).

In Chapter 6, we then present how the analysis results of Chapter 5 can be used to develop a systematic controller synthesis method in order to ensure incremental dissipativity of nonlinear systems. To achieve this, we will present as contribution a convex controller synthesis procedure which first synthesizes a controller for the variational dynamics of the system using the LPV framework. Next, we develop a realization method for this controller that ensures closed-loop incremental dissipativity for the original nonlinear system. This contribution then addresses Research Question 2 w.r.t. incremental dissipativity for CT systems. This chapter is based on the paper (Koelewijn, Tóth, Nijmeijer, et al. 2022).

With the previous chapters answering Research Questions 1 and 2 for CT systems, the next two chapters then address these Research Questions for DT nonlinear systems. In Chapter 7, we first present DT extensions to the incremental dissipativity based results for analysis and control from Chapters 5 and 6. As contributions, we show how results similar to the CT results can be obtained for incremental dissipativity based analysis and controller synthesis of DT systems. These contributions then address Research Questions 1 and 2 w.r.t. incremental dissipativity for DT systems. This chapter is based on the papers (Koelewijn and Tóth 2021b; Koelewijn, Tóth, and Weiland 2021).

Next, in Chapter 8, we present the DT extensions of the universal shifted analysis and controller synthesis results of Chapter 4. As contribution, we first present how convex universal shifted dissipativity analysis of DT system can be achieved using the time *difference* dynamics and methods from LPV framework. As second contribution, we present how the DT incremental controller synthesis results of Chapter 7 can be used and simplified to synthesize controllers to ensure universal shifted stability and performance for DT systems. For this reason, the DT universal shifted extensions are presented after the DT incremental extensions. These contributions then address Research Questions 1 and 2 w.r.t. universal shifted dissipativity for DT systems. Combined with the previous chapters, this then fully addresses Research Questions 1 and 2. The contents of this chapter are based on the paper (Koelewijn, Tóth, and

Weiland 2022b).

Chapter 9 contains results on scheduling dimension reduction of LPV models. As contribution, we propose a new ANN based method for scheduling dimension reduction that achieves a significantly decreased approximation error and has advantageous properties compared to existing methods. Combined with the results in Chapters 4 to 8, which use LPV models as part of their procedures, it allows to reduce the complexity of these procedures, thereby addressing Research Question 3. The content of the chapter is based on (Koelewijn and Tóth 2020).

Together, the results of Chapters 3 to 9, present methods in order to systematically and computationally efficiently analyze and design controllers for (complex) nonlinear systems in order to guarantee universal shifted and incremental dissipativity. With these results, we address the research questions and which together combine to an overall framework for analysis and controller synthesis for nonlinear systems, therefore addressing the overall research objective. In Chapter 10 we demonstrate the capabilities of the developed framework on two realistic applications for controller design. Firstly, an application to a *Control Momement Gyroscope* (CMG) is presented, for which a universal shifted controller is developed to track and reject piecewise constant signals, based on the method discussed in Chapter 4. Secondly, results are presented on the incremental flight control of a *Generic Parafoil Return Vehicle* (GPRV), based on the methods discussed in Chapters 5, 6 and 9. In both applications, the performance improvements of the ensured global stability and performance guarantees are demonstrated. Furthermore, in both applications, the designed controllers achieve significantly better performance compared to current LPV methods.

Finally, in Chapter 11, we give concluding remarks; highlighting the main contributions of the thesis, and providing recommendations for possible future research directions that are of interest.

In each of the chapters where we present technical results, we will briefly (re)introduce the system that is considered and we will briefly refresh the reader on the notation. This is done for the convenience of the reader, as the thesis is of substantial length and a variety of different results are presented.

2

Preliminaries

THIS chapter provides an overview of some existing methods for the analysis and control of *Linear Parameter-Varying* (LPV) and nonlinear systems which will serve as the cornerstone for the results presented in the following chapters. More specifically, we will review the general stability and performance analysis of nonlinear systems and LPV systems together with how they connect to dissipativity theory. Furthermore, we will discuss the implication of the stability and performance of LPV systems on the underlying nonlinear system.

2.1 Introduction

In Chapter 1, we discussed the objective of this thesis to develop a systematic framework for analysis and control of nonlinear systems to ensure global stability and performance. We briefly mentioned in Section 1.5 that to achieve this, two key ingredients will be used, namely global dissipativity notions, which are a generalization of the dissipativity concept introduced by (Willems 1972), and the *Linear Parameter-Varying* (LPV) framework. Hence, in this chapter, we will review some existing results on the stability and performance of nonlinear systems and how they connect to dissipativity. Moreover, we will review existing analysis and controller synthesis results for the LPV framework, which also use the dissipativity concept. We will also discuss how the LPV framework is currently used for analysis and control of nonlinear systems.

The chapter is structured as follows. First, in Section 2.2, we give a more in-depth description on the types of nonlinear systems that we will consider in this thesis. Next, in Section 2.3, we introduce stability and performance concepts for nonlinear systems. In Section 2.4, we will introduce dissipativity of nonlinear systems and how it connects to stability and performance. In Section 2.5, we will give an overview the analysis and controller synthesis results for LPV systems and briefly discuss how they can be used to analyze and design controllers for nonlinear systems. Finally, in Section 2.6, we give a brief summary.

First, some general notation that will be considered in this chapter and throughout this thesis: $\mathbb{R}$ is the set of real numbers, while $\mathbb{R}_0^+$ and $\mathbb{R}^+$ stand for the set of non-negative and positive reals, respectively. $\mathbb{N}$ and $\mathbb{N}_0$ are the set of natural numbers excluding and including zero, respectively. $\mathbb{Z}$ is the set of integers. The set $\{n,\, n+1,\, \ldots,\, m\}$, with $n, m \in \mathbb{Z}$ and $n \leq m$, is denoted by $\mathbb{I}_n^m$. We denote by $\mathbb{S}^n$ the set of real symmetric matrices of size $n \times n$ with $n \in \mathbb{N}$. The projection of $\mathcal{D} := \mathcal{A} \times \mathcal{B}$ with elements (a, b) onto $\mathcal{A}$ is denoted by $\pi_{\mathrm{a}}\,\mathcal{D}$, meaning $a \in \pi_{\mathrm{a}}\,\mathcal{D} = \mathcal{A}$. The set of functions from $\mathcal{X}$ to $\mathcal{Y}$ is denoted by $\mathcal{Y}^{\mathcal{X}}$. For a function $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ for which $V(a, b) \in \mathbb{R}$ with $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$, its gradient w.r.t. a is denoted by $\nabla_a V(a, b) = \begin{bmatrix} \frac{\partial V}{\partial a_1}(a, b) & \cdots & \frac{\partial V}{\partial a_n}(a, b) \end{bmatrix}$, for functions V with one argument the subscript for ∇ is dropped. For a vector $x \in \mathbb{R}^n$, its Euclidian norm is denoted by $\|x\| = \sqrt{x^\top x}$. For a matrix $A \in \mathbb{R}^{n \times m}$, its spectral norm is denoted by $\|A\| = \sqrt{\lambda_{\max}(A^\top A)}$ (where $\lambda_{\max}$ is the largest eigenvalue), corresponding to the largest singular value of A. We use $(\star)$ to denote a symmetric term in a quadratic expression, e.g., $(\star)^\top Q(a - b) = (a - b)^\top Q(a - b)$ for $Q \in \mathbb{S}^n$ and $a, b \in \mathbb{R}^n$. The notation $A \succ 0$ ($A \succeq 0$) indicates that $A \in \mathbb{S}^n$ is positive (semi-) definite, while $A \prec 0$ ($A \preceq 0$) denotes a negative (semi-)definite $A \in \mathbb{S}^n$. The zero-matrix and the identity matrix of appropriate dimensions are denoted as 0 and I, respectively. Furthermore, $\mathrm{col}(x_1, \ldots, x_n)$ denotes the column vector $[x_1^\top \cdots x_n^\top]^\top$. The (block) diagonal concatenation of $a_1, \ldots, a_n$ is denoted by $\mathrm{diag}(a_1, \ldots, a_n)$.

2.2 Nonlinear Systems

In this thesis, when talking about nonlinear systems, we generally consider nonlinear time-invariant systems in state-space representation of the form:

$$\begin{aligned} \xi x(t) &= f(x(t), w(t)); \\ z(t) &= h(x(t), w(t)); \end{aligned} \tag{2.1}$$

where $t \in \mathcal{T}$ is time, $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$ is the state variable associated with (2.1) with initial condition $x(0) = x_0 \in \mathcal{X}$, $w(t) \in \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{w}}$ is the input of the system, and $z(t) \in \mathcal{Z} \subseteq \mathbb{R}^{n_\mathrm{z}}$ is the output of the system. Moreover, $n_\mathrm{x}, n_\mathrm{w}, n_\mathrm{z} \in \mathbb{N}$, i.e., we only consider finite dimensional systems. For *Continuous-Time* (CT) systems, $\xi = \frac{d}{dt}$, such that $\xi x(t) = \frac{d}{dt}x(t) = \dot{x}(t)$, and $\mathcal{T} = \mathbb{R}_0^+$. For *Discrete-Time* (DT) systems, $\xi = q$, i.e., the discrete time-shift operator, such that $\xi x(t) = qx(t) = x(t+1)$, and $\mathcal{T} = \mathbb{N}_0$. We assume that the function $f : \mathcal{X} \times \mathcal{W} \to \mathbb{R}^{n_\mathrm{x}}$ (or in DT $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$) is continuously differentiable, and we assume that f and the function $h : \mathcal{X} \times \mathcal{W} \to \mathcal{Z}$ are such that the solutions of (2.1) are forward complete, i.e., they exists for all $t \in \mathcal{T}$, and are unique. Furthermore, we define the state-transition map $\phi_\mathrm{x} : \mathcal{T} \times \mathcal{T} \times \mathcal{X} \times \mathcal{W}^\mathcal{T} \to \mathcal{X}$, such that

$$x(t) = \phi_\mathrm{x}(t, 0, x(0), w). \tag{2.2}$$

Finally, we denote the corresponding behavior of (2.1), i.e., the set of all solutions, by

$$\mathfrak{B} := \left\{ (x, w, z) \in (\mathcal{X}, \mathcal{W}, \mathcal{Z})^\mathcal{T} \mid (x, w, z) \text{ satisfy } (2.1) \right\}, \tag{2.3}$$

in CT we additionally assume that $x \in \mathcal{C}_1$.

Moreover, we will also consider the following definitions of controllability and observability for nonlinear state-space representations of the form (2.1), adapted from (Nijmeijer and Van der Schaft 2016):

Definition 2.1 (Controllability)**.** *The nonlinear state-space representation (2.1) is called controllable, if for any $x_1, x_2 \in \mathcal{X}$, there exists a finite time T and an input signal $w \in \pi_\mathrm{w} \mathfrak{B}$ such that $x_2 = \phi_\mathrm{x}(T, 0, x_1, w)$.*

Definition 2.2 (Observability)**.** *The nonlinear state-space representation (2.1) is called observable, if for any $x_{0,1}, x_{0,2} \in \mathcal{X}$ and for every $w \in \mathcal{W}^\mathcal{T}$, the two trajectories $z_1, z_2 \in \mathcal{Z}^\mathcal{T}$ with $z_1(t) = h(\phi_\mathrm{x}(t, 0, x_{0,1}, w), w(t))$ and $z_2(t) = h(\phi_\mathrm{x}(t, 0, x_{0,2}, w), w(t))$ being identical, i.e., $z_1(t) = z_2(t)$, $\forall\, t \in \mathcal{T}$, implies that $x_{0,1} = x_{0,2}$.*

2.3 Stability and Performance

2.3.1 Stability

In this section, we will discuss stability analysis of nonlinear systems described by (2.1). When we talk about stability of nonlinear systems in the conventional sense, stability is characterized w.r.t. a particular equilibrium point of the system. The equilibrium points of (2.1) can be described as

$$\text{CT:} \quad \begin{cases} 0 = f(x_*, w_*); \\ z_* = h(x_*, w_*); \end{cases} \qquad \text{DT:} \quad \begin{cases} x_* = f(x_*, w_*); \\ z_* = h(x_*, w_*). \end{cases} \tag{2.4}$$

Let us denote the corresponding set of equilibrium points by

$$\mathscr{E} := \{(x_*, w_*, z_*) \in \mathcal{X} \times \mathcal{W} \times \mathcal{Z} \mid (x_*, w_*, z_*) \text{ satisfy } (2.4)\}, \tag{2.5}$$

and define $\mathscr{X} := \pi_{\mathrm{x}_*} \mathscr{E}$, $\mathscr{W} =: \pi_{\mathrm{w}_*} \mathscr{E}$, and $\mathscr{Z} := \pi_{\mathrm{z}_*} \mathscr{E}$. We assume that w.l.o.g.[1] $(0,0,0) \in \mathscr{E}$. In this thesis, when we talk about stability of a system (2.1) at an equilibrium point $x_* \in \mathscr{X}$, we always consider the behavior of the system for the constant input trajectory corresponding to the equilibrium point x_*, i.e., $w \equiv w_*$ with $(x_*, w_*, z_*) \in \mathscr{E}$. We use the notation $w \equiv w_*$ to denote that $w(t) = w_* \in \mathscr{W}$, $\forall\, t \in \mathcal{T}$. Let us denote the behavior of (2.1) for an input trajectory $\bar{w} \in \mathcal{W}^{\mathcal{T}}$, by

$$\mathfrak{B}_{\mathrm{w}}(\bar{w}) := \{(x, w, z) \in \mathfrak{B} \mid w = \bar{w} \in \mathcal{W}^{\mathcal{T}}\}. \tag{2.6}$$

Furthermore, we define the shorthand notation $\mathfrak{B}_0 := \mathfrak{B}_{\mathrm{w}}(w \equiv 0)$, specifically for the zero input trajectory behavior.

The following definition is adapted from (Khalil 2002; Van der Schaft 2017).

Definition 2.3 (Stability)**.** *The nonlinear system (2.1), at the equilibrium point $x_* \in \mathscr{X}$, and corresponding $w_* \in \mathscr{W}$, is*

- *stable, if for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that*

$$\|x(0) - x_*\| < \delta(\epsilon) \implies \|x(t) - x_*\| < \epsilon, \tag{2.7}$$

  for all $t \in \mathcal{T}$ and $x \in \pi_{\mathrm{x}} \mathfrak{B}_{\mathrm{w}}(w \equiv w_*)$.

- *asymptotically stable, if it is stable and attractive, i.e., there exists a $\delta > 0$ such that for $w \equiv w_*$*

$$\|x(0) - x_*\| < \delta \implies \lim_{t \to \infty} \|\phi_{\mathrm{x}}(t, 0, x(0), w) - x_*\| = 0. \tag{2.8}$$

- *unstable, if it is not stable.*

When referring to stability of a system without mentioning the specific equilibrium point, we always refer to stability of the origin of the state-space, corresponding to $(0,0,0) \in \mathscr{E}$, unless specified otherwise.

A common tool to analyze stability of systems is Lyapunov stability theory. However, before introducing it, we briefly give the following definition:

[1] We can always do a coordinate transformation such that this is the case (Nijmeijer and Van der Schaft 2016).

Definition 2.4 ($\mathcal{C}_n$-space)**.** *A function $f : \mathbb{R}^p \to \mathbb{R}^q$ is in $\mathcal{C}_0$ if it is continuous. For $n \in \mathbb{N}$, the function f is in $\mathcal{C}_n$, it is n-times continuously differentiable, i.e., its first n derivatives exists and are continuous. The function f is in $\mathcal{C}_\infty$ if its derivatives of all orders exist and are continuous.*

Definition 2.5 (Class $\mathcal{K}$ functions)**.** *A function $\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+$, with $\alpha \in \mathcal{C}_0$, belongs to the set $\mathcal{K}$, i.e., it is called a class $\mathcal{K}$ function, if it is strictly increasing and $\alpha(0) = 0$.*

Definition 2.6 (Definite functions (Scherer and Weiland 2015))**.** *Consider a function $V : \mathcal{X} \to \mathbb{R}$ and $x_* \in \mathcal{X}$. The function V is:*

- *positive definite (w.r.t. x_*), if there exists a function $\alpha \in \mathcal{K}$ such that $V(x) \geq \alpha(\|x - x_*\|)$ for all $x \in \mathcal{X}$.*
- *positive semi-definite, if $V(x) \geq 0$ for all $x \in \mathcal{X}$.*
- *decrescent (w.r.t. x_*), if there exists a function $\alpha \in \mathcal{K}$ such that $V(x) \leq \alpha(\|x - x_*\|)$ for all $x \in \mathcal{X}$.*
- *negative definite (w.r.t. x_*) or negative semi-definite, if $-V$ is positive definite or positive semi-definite respectively.*

Definition 2.7 (Class $\mathcal{Q}$ functions)**.** *A function $V : \mathcal{X} \to \mathbb{R}_0^+$ belongs to the set $\mathcal{Q}_{x_*}$, where $x_* \in \mathcal{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$, if there exists functions $\alpha_1, \alpha_2 \in \mathcal{K}$, such that*

$$\alpha_1(\|x - x_*\|) \leq V(x) \leq \alpha_2(\|x - x_*\|), \tag{2.9}$$

for all $x \in \mathcal{X}$. Note that this equivalently means that V is positive definite and decrescent w.r.t. x_* (Scherer and Weiland 2015).

Remark 2.1. Note that quadratic functions of the form $V(x) = (x - x_*)^\top M(x - x_*)$ with $M \in \mathbb{S}^{n_\mathrm{x}}$ and $M \succ 0$ satisfy that $V \in \mathcal{Q}_{x_*}$. Namely, in that case it holds that

$$\lambda_{\min}(M)\,\|x - x_*\|^2 \leq V(x) \leq \lambda_{\max}(M)\,\|x - x_*\|^2\,, \tag{2.10}$$

where $\lambda_{\min}(M)$ and $\lambda_{\max}(M)$ are the smallest and largest eigenvalues of M, respectively (Scherer and Weiland 2015).

Next, we give the following theorem on Lyapunov stability, adopted from (Khalil 2002; Haddad and Chellaboina 2008; Lazar et al. 2009):

Theorem 2.1 (Lyapunov stability)**.** *The nonlinear system given by (2.1) is stable at $x_* \in \mathscr{X}$ with the corresponding $w_* \in \mathscr{W}$, if there exists a Lyapunov function $V : \mathcal{X} \to \mathbb{R}_0^+$ with $V \in \mathcal{C}_1$ and $V \in \mathcal{Q}_{x_*}$ such that*

$$\text{CT:}\quad \frac{\partial}{\partial t} V(x(t)) \leq 0, \tag{2.11a}$$

$$\text{DT:}\quad V(x(t+1)) - V(x(t)) \leq 0, \tag{2.11b}$$

for all $t \in \mathcal{T}$ and $x \in \pi_\mathrm{x}\mathfrak{B}_\mathrm{w}(w \equiv w_*)$. Moreover, the nonlinear system is asymp-

totically stable at $x_* \in \mathscr{X}$, if there exists a Lyapunov function $V : \mathcal{X} \to \mathbb{R}_0^+$ with $V \in \mathcal{C}_1$ and $V \in \mathcal{Q}_{x_*}$ and there exists a function $\alpha \in \mathcal{K}$ such that

$$\text{CT:} \quad \frac{\partial}{\partial t} V(x(t)) \leq -\alpha(\|x(t) - x_*\|), \tag{2.12a}$$

$$\text{DT:} \quad V(x(t+1)) - V(x(t)) \leq -\alpha(\|x(t) - x_*\|), \tag{2.12b}$$

for all $t \in \mathcal{T}$ and $x \in \pi_{\mathrm{x}} \mathfrak{B}_{\mathrm{w}}(w \equiv w_*)$.

As we consider nonlinear systems given by (2.1) with $f \in \mathcal{C}_1$, we can also formulate the following equivalent Lyapunov based conditions for (asymptotic) stability (Khalil 2002; Bof et al. 2018):

Theorem 2.2 (Lyapunov stability, relaxed form)**.** *The nonlinear system given by (2.1) is stable at $x_* \in \mathscr{X}$ with the corresponding $w_* \in \mathscr{W}$, if there exists a Lyapunov function $V : \mathcal{X} \to \mathbb{R}_0^+$ with $V \in \mathcal{C}_1$ and $V \in \mathcal{Q}_{x_*}$ such that*

$$\text{CT:} \quad \frac{\partial}{\partial t} V(x(t)) \leq 0, \tag{2.13a}$$

$$\text{DT:} \quad V(x(t+1)) - V(x(t)) \leq 0, \tag{2.13b}$$

for all $t \in \mathcal{T}$ and $x \in \pi_{\mathrm{x}} \mathfrak{B}_{\mathrm{w}}(w \equiv w_*)$. Moreover, the nonlinear system is asymptotically stable at $x_* \in \mathscr{X}$, if (2.13) holds, but with strict inequality except when $x(t) = x_*$.

The Lyapunov function V can be seen as representing the internal energy in the system, which for the system to be (asymptotically) stable always has to decrease over time, till it reaches zero, coinciding with the equilibrium point, see also Figure 1.2. As analyzing (2.13) directly based on the trajectories is generally infeasible, the check for Lyapunov stability is often rewritten to a test on a value set (Khalil 2002; Haddad and Chellaboina 2008).

Lemma 2.1 (Lyapunov stability using value set)**.** *If there exists a Lyapunov function $V : \mathcal{X} \to \mathbb{R}_0^+$ with $V \in \mathcal{C}_1$ and $V \in \mathcal{Q}_{x_*}$ such that*

$$\text{CT:} \quad \nabla V(x) f(x, w_*) \leq 0, \tag{2.14a}$$

$$\text{DT:} \quad V(f(x, w_*)) - V(x) \leq 0, \tag{2.14b}$$

for all $x \in \mathcal{X}$, then, the system is stable at $x_* \in \mathscr{X}$ with corresponding $w_* \in \mathscr{W}$. Similarly, if (2.14) holds, but with strict inequality for all $x \in \mathcal{X} \setminus \{x_*\}$, then the system is asymptotically stable at $x_* \in \mathscr{X}$.

2.3.2 Performance

Input-output properties of systems are important to analyze not only their stability, but also to quantify their performance. Popular performance metrics can be described by the gain from (input) signals in $\mathcal{L}_{pe}$ to (output) signals in $\mathcal{L}_{qe}$ for $p, q = 1, 2, \ldots, \infty$ in CT or from sequences in ℓ_{pe} to sequences in ℓ_{qe} in DT. These function spaces are defined as follows, adopted from (Van der Schaft 2017):

Definition 2.8 ($\mathcal{L}_p/\ell_p$ space)**.** *For $p \in \mathbb{N}$, the signal space $\mathcal{L}_p$ or sequence space ℓ_p is the space of functions $f : \mathbb{R}_0^+ \to \mathbb{R}^n$ or $f : \mathbb{N}_0 \to \mathbb{R}^n$, respectively, for which*

$$\mathcal{L}_p: \ \|f\|_p := \left(\int_0^\infty \|f(t)\|^p \, dt\right)^{\frac{1}{p}} < \infty, \qquad \ell_p: \ \left(\sum_{t=0}^{\infty} \|f(t)\|^p\right)^{\frac{1}{p}} < \infty, \tag{2.15}$$

where $\|\cdot\|$ is the Euclidian (vector) norm[2]*. The signal space $\mathcal{L}_\infty$ or sequence space ℓ_∞ is the space of functions $f : \mathbb{R}_0^+ \to \mathbb{R}^n$ or $f : \mathbb{N}_0 \to \mathbb{R}^n$, respectively, for which*

$$\mathcal{L}_\infty: \ \|f\|_\infty := \operatorname*{ess\,sup}_{t \in \mathbb{R}_0^+} \|f(t)\| < \infty, \qquad \ell_\infty: \ \|f\|_\infty := \sup_{t \in \mathbb{N}_0} \|f(t)\| < \infty. \tag{2.16}$$

Definition 2.9 (Truncation)**.** *For signals $f : \mathbb{R}_0^+ \to \mathbb{R}^n$ or sequences $f : \mathbb{N}_0 \to \mathbb{R}^n$, define $f_T : \mathbb{R}_0^+ \to \mathbb{R}^n$ with $T \in \mathbb{R}_0^+$ or $f_T : \mathbb{N}_0 \to \mathbb{R}^n$ with $T \in \mathbb{N}_0$, respectively, by*

$$f_T(t) := \begin{cases} f(t) & 0 \le t \le T \\ 0 & t > T \end{cases}, \tag{2.17}$$

which expresses the truncation of f on $[0, T]$.

Definition 2.10 (Extended $\mathcal{L}_p/\ell_p$ space)**.** *For $p = 1, 2, \ldots, \infty$, the extended $\mathcal{L}_p$ or ℓ_p space, denoted by $\mathcal{L}_{pe}$ or ℓ_{pe}, respectively, is the space of all functions $f : \mathbb{R}_0^+ \to \mathbb{R}^n$ or $f : \mathbb{N}_0 \to \mathbb{R}^n$ such that $f_T \in \mathcal{L}_p$ for all $T \in \mathbb{R}_0^+$ or $f_T \in \ell_p$ for all $T \in \mathbb{N}_0$, respectively. For $p \in \mathbb{N}$, the accompanying norms of these truncated signals or sequences*[3] *are denoted by*

$$\mathcal{L}_{pe}: \ \|f\|_{p,T} := \left(\int_0^T \|f(t)\|^p \, dt\right)^{\frac{1}{p}}, \qquad \ell_{pe}: \ \|f\|_{p,T} := \left(\sum_{t=0}^{T} \|f(t)\|^p\right)^{\frac{1}{p}}, \tag{2.18a}$$

and for $\mathcal{L}_{\infty e}$ and $\ell_{\infty e}$ by

$$\mathcal{L}_{\infty e}: \ \|f\|_{\infty,T} := \operatorname*{ess\,sup}_{t \in [0,\, T]} \|f(t)\|, \qquad \ell_{\infty e}: \ \|f\|_{\infty,T} := \sup_{t \in \mathbb{I}_0^T} \|f(t)\|. \tag{2.18b}$$

Having defined these signal spaces, we can then define the notion of $\mathcal{L}_p$-$\mathcal{L}_q$-gain of a system (2.1), adopted from (Van der Schaft 2017).

[2] One can also define these functions spaces for different vector norms. However, the Euclidian norm is used here as it will be beneficial in deriving analysis and controller synthesis results later in the thesis.

[3] Note that these truncated norms are defined on the full signal/sequence f, but can similarly be taken on the truncated version f_T, but these would not change their definition.

Definition 2.11 ($\mathcal{L}_p$-$\mathcal{L}_q$-gain)**.** *A CT nonlinear system given by (2.1) is said to have a finite $\mathcal{L}_p$-$\mathcal{L}_q$-gain, or in DT a finite ℓ_p-ℓ_q-gain, if there is a finite $\gamma \geq 0$ and function $\zeta : \mathcal{X} \to \mathbb{R}$ such that*

$$\|z\|_{q,T} \leq \gamma \|w\|_{p,T} + \zeta(x_0), \tag{2.19}$$

for all $T \geq 0$ and $(x, w, z) \in \mathfrak{B}$ with $w \in \mathcal{L}_{\mathrm{pe}}$ in CT or $w \in \ell_{\mathrm{pe}}$ in DT. The induced $\mathcal{L}_p$-$\mathcal{L}_q$-gain/ℓ_p-ℓ_q-gain of (2.1) is the infimum of γ such that (2.19) still holds. If $p = q$, we will refer to this as the (induced) $\mathcal{L}_p$-gain/ℓ_p-gain.

One of the most well known and widely used performance metrics of this type is the induced $\mathcal{L}_2$-gain, which can be seen as a bound on mapping input signals with finite energy to output signal with finite energy. For stable *Linear Time-Invariant* (LTI) systems, the induced $\mathcal{L}_2$-gain of the system is equal to the $\mathcal{H}_\infty$ norm of its corresponding transfer function representation. Other popular choices are peak-to-peak performance, through the induced $\mathcal{L}_\infty$-gain, and the generalized $\mathcal{H}_2$ nominal performance, through the induced $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain (Scherer and Weiland 2015).

Another popular performance notion, is that of passivity, adopted from (Van der Schaft 2017).

Definition 2.12 (Passivity)**.** *Assume that, for a nonlinear system given by (2.1), $n_\mathrm{w} = n_\mathrm{z}$, i.e., the number of inputs is equal to the number of outputs. The nonlinear system is said to be passive, if there exists a function $\zeta : \mathcal{X} \to \mathbb{R}$ such that*

$$\text{CT:} \quad \int_0^T z(t)^\top w(t)\, dt \geq \zeta(x_0), \tag{2.20a}$$

$$\text{DT:} \quad \sum_{t=0}^{T} z(t)^\top w(t) \geq \zeta(x_0), \tag{2.20b}$$

for all $T \geq 0$ and $(x, w, z) \in \mathfrak{B}$.

Passivity is often used for analysis of physical systems, e.g., electrical networks or mechanical systems, due it close connection to the power flow of the system. For LTI systems, passivity also corresponds to positive realness of its transfer function representation (Van der Schaft 2017).

While for LTI systems, the various discussed performance notions can generally be analyzed by analyzing properties of the transfer function representation, analyzing these performance notions for nonlinear systems directly based on their definitions is more difficult if not impossible. For nonlinear systems, performance and stability is often analyzed through the concept of dissipativity.

2.4 Dissipativity

2.4.1 Dissipativity of nonlinear systems

Dissipativity analysis allows for a joint framework for stability and performance analysis of systems, as was first formulated by Willems for LTI systems (Willems

1972). In this thesis, we use a similar definition of dissipativity, adopted from (Van der Schaft 2017; Brogliato et al. 2020).

Definition 2.13 (Classical dissipativity). *The nonlinear system given by (2.1) is dissipative w.r.t. a supply function $s : \mathcal{W} \times \mathcal{Z} \to \mathbb{R}$, if there exists a storage function $\mathcal{V} : \mathcal{X} \to \mathbb{R}_0^+$ with $\mathcal{V} \in \mathcal{C}_0$ and $\mathcal{V} \in \mathcal{Q}_0$, such that*

$$\text{CT:} \quad \mathcal{V}(x(t_1)) - \mathcal{V}(x(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t))\, dt, \tag{2.21a}$$

$$\text{DT:} \quad \mathcal{V}(x(t_1+1)) - \mathcal{V}(x(t_0)) \leq \sum_{t=t_0}^{t_1} s(w(t), z(t)), \tag{2.21b}$$

for all $t_0, t_1 \in \mathcal{T}$ with $t_1 \geq t_0$, and $(x, w, z) \in \mathfrak{B}$.

The storage function $\mathcal{V}$ can be interpreted as a representation of the stored 'energy' in the system w.r.t. a single point of neutral storage (energy minimum), where $\mathcal{V}$ is zero. The point of neutral storage can be any point in the state-space, however, it is often considered to be at the origin, i.e., $x = 0$, which we will also consider to be the case. The supply function s can be seen as representing the energy flowing in and out of the system. To distinguish the notion of dissipativity in Definition 2.13 from other dissipativty based concepts that will be introduced in later chapters of this thesis, from here on, we will refer to this form of dissipativity as *classical dissipativity.* Similar as for Lyapunov stability, we can also formulate time differentiated/difference versions of the classical dissipation inequality (2.21) and value set based conditions (Willems 1972; Brogliato et al. 2020):

Lemma 2.2 (Differentiated/Difference Dissipation Inequality). *If, for the nonlinear system given by (2.1) and the supply function $s : \mathcal{W} \times \mathcal{Z} \to \mathbb{R}$, there exists a storage function $\mathcal{V} : \mathcal{X} \to \mathbb{R}_0^+$ with $\mathcal{V} \in \mathcal{C}_1$ and $\mathcal{V} \in \mathcal{Q}_0$ such that it holds that*

$$\text{CT:} \quad \frac{\partial}{\partial t}\mathcal{V}(x(t)) \leq s(w(t), z(t)), \tag{2.22a}$$

$$\text{DT:} \quad \mathcal{V}(x(t+1)) - \mathcal{V}(x(t)) \leq s(w(t), z(t)), \tag{2.22b}$$

for all $t \in \mathcal{T}$ and $(x, w, z) \in \mathfrak{B}$, then, the nonlinear system is classically dissipative w.r.t. the supply function s.

Lemma 2.3 (Classical dissipativity condition). *If, for the nonlinear system given by (2.1) and the supply function $s : \mathcal{W} \times \mathcal{Z} \to \mathbb{R}$, there exists a storage function $\mathcal{V} : \mathcal{X} \to \mathbb{R}_0^+$ with $\mathcal{V} \in \mathcal{C}_1$ and $\mathcal{V} \in \mathcal{Q}_0$ such that it holds that*

$$\text{CT:} \quad \nabla\mathcal{V}(x) f(x, w) \leq s(w, h(x, w)), \tag{2.23a}$$

$$\text{DT:} \quad \mathcal{V}(f(x, w)) - \mathcal{V}(x) \leq s(w, h(x, w)), \tag{2.23b}$$

for all $(x, w) \in \mathcal{X} \times \mathcal{W}$, then, the nonlinear system is classically dissipative w.r.t. the supply function s.

Next, we will discuss how classical dissipativity implies stability and performance of a system.

2.4.2 Inducing stability

As discussed, the storage function can be seen as representing the internal energy of the system, similar to a Lyapunov function, and hence plays a crucial role in the connection between classical dissipativity and stability. The following theorem is adopted from (Van der Schaft 2017).

Theorem 2.3 (Stability from classical dissipativity)**.** *Assume the nonlinear system given by (2.1) is dissipative under a storage function $\mathcal{V} \in \mathcal{C}_1$ w.r.t. a supply function s that satisfies*

$$s(0, z) \leq 0, \tag{2.24}$$

for all $z \in \mathcal{Z}$, then, the nonlinear system (2.1) is stable. If the supply function satisfies (2.24), but with strict inequality when $z \neq 0$, and the system is observable, then the nonlinear system is asymptotically stable.

Remark 2.2. The observability requirement in Theorem 2.3 for asymptotic stability, can also be weakened to only require zero state observability or zero state detectability of (2.1), see e.g. (Moylan 2014; Nijmeijer and Van der Schaft 2016; Van der Schaft 2017; Brogliato et al. 2020) for more details.

2.4.3 Inducing performance

While the storage function connects classical dissipativity to stability, the supply function connects it to various performance notions, often used in practice for control design. A popular choice of supply function is the class of quadratic ones of the form

$$s(w, z) = \begin{bmatrix} w \\ z \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix}, \tag{2.25}$$

where $Q \in \mathbb{S}^{n_\mathrm{w}}$, $R \in \mathbb{S}^{n_\mathrm{z}}$, and $S \in \mathbb{R}^{n_\mathrm{w} \times n_\mathrm{z}}$, which connect to passivity and the norm based performance notions discussed in Section 2.3.2. Classical dissipativity w.r.t. a supply function of the form (2.25) is also often referred to as (classical) (Q, S, R) dissipativity. Moreover, not every choice of supply function (or triplet (Q, S, R)) is interesting, e.g., if $s(w, z) \geq 0$ for all $w \in \mathcal{W}$, $z \in \mathcal{Z}$ then every system is classically dissipative, and if $s(w, z) < 0$ then no system can be classically dissipative in the considered sense(Moylan 2014). Moreover, scaling a supply function by a positive scalar results in the same effective supply function, as we can scale the storage function by the same amount. Similarly, if we want to make the link to stability for (Q, S, R) supply functions, then, by Theorem 2.3, we need to ensure that $R \preceq 0$.

For $\mathcal{L}_2$-gain boundedness and passivity, we have the following connections to classical (Q, S, R) dissipativity of the system. See (Van der Schaft 2017) for more details.

Lemma 2.4 ($\mathcal{L}_2$-gain from classical dissipativity). *If the nonlinear system given by (2.1) is classically (Q, S, R) dissipative with*[4] $(Q, S, R) = (\gamma^2 I, 0, -I)$, *then the system has a bounded $\mathcal{L}_2$-gain of γ.*

Lemma 2.5 (Passivity from classical dissipativity). *If the nonlinear system given by (2.1) is classically (Q, S, R) dissipative with*[5] $(Q, S, R) = (0, I, 0)$, *then the system is passive.*

For $\mathcal{L}_\infty$ and generalized $\mathcal{H}_2$, i.e. $\mathcal{L}_2$-$\mathcal{L}_\infty$, performance, the corresponding supply functions are $(Q, S, R) = (\alpha I, 0, 0)$, where α is a slack variable, and $(Q, S, R) = (\gamma I, 0, 0)$, where γ is the corresponding $\mathcal{L}_2$-$\mathcal{L}_\infty$ bound, respectively. However, extra conditions are required, besides satisfying the classical dissipation inequality, in order to ensure these performance notions, see (Scherer and Weiland 2015) for the LTI case.

2.5 LPV Analysis and Control

2.5.1 LPV systems

As mentioned in Chapter 1, the LPV framework has become a popular choice for analysis and control of nonlinear systems as it builds upon the systematic and computational efficient analysis and controller synthesis results for LTI systems. In this thesis, we consider LPV systems described in terms of the state-space representation:

$$\begin{aligned} \xi x(t) &= A(p(t))x(t) + B(p(t))w(t); \\ z(t) &= C(p(t))x(t) + D(p(t))w(t); \end{aligned} \tag{2.26}$$

where $x(t) \in \mathbb{R}^{n_\mathrm{x}}$ is the state associated with (2.26) with initial condition $x(0) = x_0 \in \mathbb{R}^{n_\mathrm{x}}$, $w(t) \in \mathbb{R}^{n_\mathrm{w}}$ and $z(t) \in \mathbb{R}^{n_\mathrm{z}}$ are the input and output of the system, $p(t) \in \mathcal{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$ with $n_\mathrm{p} \in \mathbb{N}$ is the scheduling-variable, with $\mathcal{P}$ often taken to be a convex set, and $A : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$, $B : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{w}}$, $C : \mathcal{P} \to \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{x}}$, $D : \mathcal{P} \to \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{w}}$ are matrix functions. Like for nonlinear systems given by (2.1), we assume that the solutions of (2.26) are forward complete and unique. For a given scheduling trajectory $p \in \mathcal{P}^\mathcal{T}$, we define the behavior of (2.26) as

$$\mathfrak{B}_\mathrm{p}(p) := \left\{ (x, w, z) \in (\mathcal{X}, \mathcal{W}, \mathcal{Z})^\mathcal{T} \mid (x, w, z, p) \text{ satisfy } (2.26) \right\}, \tag{2.27}$$

moreover, we define the full behavior of (2.26), i.e., for all scheduling trajectories, as

$$\breve{\mathfrak{B}}_\mathrm{p} := \bigcup_{p \in \mathcal{P}^\mathcal{T}} \mathfrak{B}_\mathrm{p}(p). \tag{2.28}$$

[4]Or equivalently, $(Q, S, R) = (\gamma I, 0, -\gamma^{-1} I)$ for $\gamma > 0$.
[5]Or equivalently, $(Q, S, R) = (0, \frac{1}{2} I, 0)$.

The behavior of (2.26) for the zero input and a particular scheduling trajectory $p \in \mathcal{P}^{\mathcal{T}}$ is denoted by $\mathfrak{B}_{\mathrm{p},0}(p)$. We also define the state-transition map $\phi_{\mathrm{p,x}} : \mathcal{T} \times \mathcal{T} \times \mathcal{X} \times \mathcal{W}^{\mathcal{T}} \times \mathcal{P}^{\mathcal{T}} \to \mathcal{X}$ for (2.26), such that

$$x(t) = \phi_{\mathrm{p,x}}(t, 0, x(0), w, p). \tag{2.29}$$

Besides the state-space representation, there also exist other representations for LPV systems, such as the *Input-Output* (IO) and kernel representation (Tóth 2010). However, the majority of the available analysis and controller synthesis methods assume that the LPV system is given by a state-space representation (2.26). Hence, in this thesis, we will only focus on LPV systems described by the state-space representation (2.26).

As aforementioned, the main interest towards the LPV framework originates from analyzing and controlling nonlinear systems using LTI like tools. The idea of the LPV framework was strengthened when it was observed that it is possible to fully embed the solution set of a nonlinear system given by (2.1), i.e., the behavior $\mathfrak{B}$, into the solution set $\breve{\mathfrak{B}}_{\mathrm{p}}$ of an LPV representation. This so-called *LPV embedding* is achieved by expressing the scheduling-variable p as a function of the states and/or inputs of the nonlinear system, through a so-called *scheduling-map* η (Tóth 2010). Next, we will give a more concrete definition of this idea:

Definition 2.14 (Global LPV embedding)**.** *Consider the nonlinear system given by (2.1) and an LPV system given by (2.26). If there exists a so-called scheduling-map $\eta : \mathcal{X} \times \mathcal{W} \to \mathcal{P}$, such that*

$$\begin{aligned} f(x, w) &= A(\eta(x, w))x + B(\eta(x, w))w, \\ h(x, w) &= C(\eta(x, w))x + C(\eta(x, w))w, \end{aligned} \tag{2.30}$$

for all $(x, w) \in \mathcal{X} \times \mathcal{W} \subseteq \mathcal{X} \times \mathcal{W}$ and such that $\eta(\mathcal{X}, \mathcal{W}) \subseteq \mathcal{P}$, then, the LPV system is a so-called global LPV embedding of the nonlinear system on the region $\mathcal{X} \times \mathcal{W}$ with the scheduling-variable given by $p(t) = \eta(x(t), w(t))$.

Remark 2.3. Note that if $(0, 0) \in \mathcal{X} \times \mathcal{W}$, which is often the case, to be able to perform a global LPV embedding, we require that $f(0, 0) = 0$ and $h(0, 0) = 0$. This means that the origin is an equilibrium point of the system, see also (2.4). While for nonlinear systems this might not always be the case, we can always perform a coordinate transformation to ensure this.

It is clear that through the scheduling-map, the LPV embedding describes all the solutions of the nonlinear system (on the region $\mathcal{X} \times \mathcal{W}$). However, as for the behavior of the LPV system, the scheduling-variable is considered independent, we model a larger set of solutions, which does include the original nonlinear behavior (Abbas, Tóth, Petreczky, Meskin, Mohammadpour Velni, and Koelewijn 2021). This is formalized in the following lemma:

Lemma 2.6 (LPV behavioral embedding)**.** *Consider the nonlinear system given by (2.1) and LPV system given by (2.26). If the LPV system is a global embedding of* *the nonlinear system on the region $\mathcal{X} \times \mathcal{W} = \mathcal{X} \times \mathcal{W}$, then, the behavior of nonlinear system is included in that of the LPV system, i.e., $\mathfrak{B} \subseteq \breve{\mathfrak{B}}_{\mathrm{p}}$.*

Proof. See Appendix B.1. □

As the LPV representation describes a larger set of valid system trajectories than the nonlinear system, using an LPV embedding to describe the nonlinear system is inherently conservative. However, what we gain through the LPV embedding is that we can use all of the convex tools that are available for analysis and controller synthesis for LPV systems to analyze and design controllers for them.

As mentioned in Section 1.2.3, constructing a global LPV embedding for a given nonlinear system is non-unique. While there exist multiple approaches in order to perform this step (Kwiatkowski, Bol, et al. 2006; Tóth 2010; Hoffmann and Werner 2015b; Abbas, Tóth, Petreczky, Meskin, Mohammadpour Velni, and Koelewijn 2021), these techniques are still far from being widely applicable or being fully automated. Hence, in practice, constructing a global LPV embedding is generally still performed in an ad-hoc fashion. As also discussed in Section 1.2.3, besides global methods to construct the LPV embedding, one can also construct an LPV embedding through so-called *local* methods, usually based on linearizations. However, for local LPV embeddings, it does not necessarily hold that the behavior of the nonlinear system $\mathfrak{B}$ is included in the behavior $\breve{\mathfrak{B}}_{\mathrm{p}}$ of the resulting LPV representation. On the contrary, often they can only describe a subset of it. While, analyzing and designing controllers for nonlinear systems through local LPV embeddings can work in practice, there are consequently no formal guarantees. Hence, we will only consider global LPV embeddings when talking about embeddings of nonlinear systems (2.1) in an LPV representation.

Next, we will give an overview of analysis of LPV systems.

2.5.2 Analysis

Stability analysis

First, we will discuss stability analysis of LPV systems. Similar as for LTI systems, also for LPV systems (given by (2.26)), the origin of the considered state-space representation is always an equilibrium point, i.e., $(0, 0, 0) \in \mathscr{E}$ for all $p \in \mathcal{P}$. For LTI systems, due to the superposition principle, stability of the origin implies stability of all forced equilibrium points of the LTI system. Hence, for LTI systems, only stability of the origin is required to be analyzed. Due to the linearity of LPV systems, this property is also extends LPV systems under the assumption that the scheduling-variable is independent of the dynamics. Consequently, for stability analysis of LPV systems, generally only stability w.r.t. the origin is analyzed. Hence, when talking about stability of LPV systems, we often refer to stability of the origin, unless otherwise noted. However, as we will discuss in detail in Chapter 3, naive application of this concept can actually result in undesirable results when the LPV representation is used to analyze or ensure via control design the stability of an underlying nonlinear system.

We have the following formal definition of stability of an LPV system given by (2.26), adopted from (Briat 2015):

Definition 2.15 (Stability of LPV system)**.** *The LPV system (2.26) is*

- *stable, if for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that*
$$\|x(0)\| < \delta(\epsilon) \implies \|x(t)\| < \epsilon, \tag{2.31}$$
for all $t \in \mathcal{T}$, $x \in \pi_{\mathrm{x}} \mathfrak{B}_{\mathrm{p},0}(p)$, and $p \in \mathcal{P}^{\mathcal{T}}$.
- *asymptotically stable, if it is stable and attractive, i.e., there exists a $\delta > 0$ such that*
$$\|x(0)\| < \delta \implies \lim_{t\to\infty} \|\phi_{\mathrm{p,x}}(t, 0, x(0), 0, p)\| = 0, \tag{2.32}$$
for all $p \in \mathcal{P}^{\mathcal{T}}$.
- *unstable, if it is not stable.*

Like for nonlinear systems, also for LPV systems, Lyapunov stability theory is an important tool for stability analysis. In order to make these analysis problems tractable, one often considers quadratic functions V_{p} of the form:

$$V_{\mathrm{p}}(x, p) = x^\top M(p)x, \tag{2.33}$$

where $M : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$, $M(p) \succ 0$ for all $p \in \mathcal{P}$, and M is bounded, in the sense that there exist an $\alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1, \alpha_2 > 0$, s.t. $\alpha_1 I \preceq M(p) \preceq \alpha_2 I$ for all $p \in \mathcal{P}$. Note that for each $p \in \mathcal{P}$, the function $x \mapsto V_{\mathrm{p}}(x, p)$ satisfies the conditions for a Lyapunov function. Moreover, in CT, we assume that[6] for all $t \in \mathcal{T}$, $v(t) = \frac{d}{dt}p(t) \in \Pi$, while in DT, we assume that for all $t \in \mathcal{T}$, $v(t) = p(t + 1) - p(t) \in \Pi$. Here $\Pi \subseteq \mathbb{R}^{n_{\mathrm{p}}}$, which, similar to $\mathcal{P}$, is generally assumed to be convex.

Under these considerations, we can apply the Lyapunov stability theory to LPV systems to obtain the following results. Let us first denote by p_i and v_i the i'th elements of p and v, respectively. Moreover, for a matrix function $M : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$ with $M \in \mathcal{C}_1$, we define $\partial M : \mathcal{P} \times \Pi \to \mathbb{S}^{n_{\mathrm{x}}}$ as $\partial M(p, v) = \sum_{i=1}^{n_{\mathrm{p}}} \frac{\partial M(p)}{\partial p_i} v_i$.

Theorem 2.4 (Stability condition for LPV systems)**.** *The LPV system given by (2.26) is stable at the equilibrium point $x_* = 0$ with $w_* = 0$, if there exists a positive definite matrix function $M : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$ with $M \in \mathcal{C}_1$ such that, in CT, it holds that*

$$A(p)^\top M(p) + M(p)A(p) + \partial M(p, v) \preceq 0, \tag{2.34a}$$

for all $p \in \mathcal{P}$ and $v \in \Pi$, while, in DT, it holds that

$$A(p)^\top M(p + v)A(p) - M(p) \preceq 0, \tag{2.34b}$$

[6] In case that M is a constant matrix, i.e. $M \in \mathbb{S}^{n_{\mathrm{x}}}$, the assumptions on the 'velocity' of p are not required.

for all $p \in \mathcal{P}$ and $v \in \Pi$. If (2.34) holds, but with strict inequality, then the LPV system is asymptotically stable at $x_* = 0$.

See (G. S. Becker 1993; Wu 1995; Briat 2015) for proofs of this result. Note that the conditions in Theorem 2.4 correspond to the feasibility of an infinite set of *Linear Matrix Inequalities* (LMIs), which corresponds to an indefinite convex program. To turn this infinite-dimensional check into a finite one, several approaches exist such as polytopic techniques (Apkarian, Gahinet, and G. Becker 1995; De Caigny et al. 2012; Cox et al. 2018), grid-based techniques (Wu 1995; Apkarian and Adams 1998), and multiplier-based techniques (Scorletti and El Ghaoui 1998; Scherer 2001), see also (Hoffmann and Werner 2015a) for an overview. The resulting set of LMIs, corresponding to a *Semidefinite Program* (SDP), can then be solved efficiently using various SDP solvers, e.g., SDPT3 (Toh et al. 1999) or MOSEK (E. D. Andersen and K. D. Andersen 2000). Various software tools also exist for MATLAB through which the feasibility problem of a set of LMIs can be turned into a SDP, e.g., YALMIP (Löfberg 2004) or CVX (Grant and Boyd 2014).

Classical dissipativity analysis

Classical dissipativity is the backbone for the majority of the existing analysis and controller synthesis results that are available for LPV systems. Like when talking about stability of LPV systems, we consider an LPV system to be classically dissipative if the dissipation inequality (2.22) holds for every scheduling trajectory in the scheduling set, more formally:

Definition 2.16 (Classical dissipativity of LPV systems)**.** *The LPV system given by (2.26) is classically dissipative w.r.t. a supply function $s : \mathcal{W} \times \mathcal{Z} \to \mathbb{R}$, if there exists a so-called storage function $\mathcal{V}_\mathrm{p} : \mathcal{X} \times \mathcal{P} \to \mathbb{R}_0^+$ with $\mathcal{V}_\mathrm{p}(\cdot, p) \in \mathcal{C}_0$ and $\mathcal{V}_\mathrm{p}(\cdot, p) \in \mathcal{Q}_0$, $\forall\, p \in \mathcal{P}$, such that*

$$\text{CT:}\quad \mathcal{V}_\mathrm{p}(x(t_1), p(t_1)) - \mathcal{V}_\mathrm{p}(x(t_0), p(t_0)) \leq \int_{t_0}^{t_1} s(w(t), z(t))\, dt, \tag{2.35a}$$

$$\text{DT:}\quad \mathcal{V}_\mathrm{p}(x(t_1+1), p(t_1+1)) - \mathcal{V}_\mathrm{p}(x(t_0), p(t_0)) \leq \sum_{t=t_0}^{t_1} s(w(t), z(t)), \tag{2.35b}$$

for all $t_0, t_1 \in \mathcal{T}$ with $t_1 \geq t_0$, and $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ for all $p \in \mathcal{P}^\mathcal{T}$.

In the previous section, we have shown how stability analysis of an LPV system given by (2.26) can be formulated as a matrix inequality condition. Similarly, we will next present how dissipativity conditions can be formulated as matrix inequality conditions. For this purpose, classical (Q, S, R) dissipativity is considered (i.e., classical dissipativity w.r.t. a supply function of the form (2.25)), and the storage function is also considered to be of a quadratic form:

$$\mathcal{V}_\mathrm{p}(x, p) = x^\top M(p) x, \tag{2.36}$$

analogous to the Lyapunov function (2.33), with the same assumptions holding on M. This results in the following condition:

Theorem 2.5 (Classical (Q,S,R) dissipativity conditions for LPV systems)**.** *The LPV system given by (2.26) is classically (Q,S,R) dissipative with a storage function of the form (2.36), if there exists a positive definite matrix function $M : \mathcal{P} \to \mathbb{S}^{n_x}$ with $M \in \mathcal{C}_1$ such that, in CT, for all $p \in \mathcal{P}$ and $v \in \Pi$, it holds that*

$$(\star)^\top \begin{bmatrix} \partial M(p,v) & M(p) \\ \star & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A(p) & B(p) \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C(p) & D(p) \end{bmatrix} \preceq 0, \tag{2.37a}$$

and such that, in DT, it holds for all $p \in \mathcal{P}$ and $v \in \Pi$ that

$$(\star)^\top \begin{bmatrix} -M(p) & 0 \\ \star & M(p+v) \end{bmatrix} \begin{bmatrix} I & 0 \\ A(p) & B(p) \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C(p) & D(p) \end{bmatrix} \preceq 0. \tag{2.37b}$$

Proof. See Appendix B.1. □

Similar to the stability result, the condition for classical (Q,S,R) dissipativity of an LPV system in Theorem 2.5 corresponds to an infinite set of LMIs. As aforementioned, several techniques exists within the LPV framework to turn this into a finite set of LMIs, such as polytopic techniques (Apkarian, Gahinet, and G. Becker 1995; De Caigny et al. 2012; Cox et al. 2018), grid-based techniques (Wu 1995; Apkarian and Adams 1998), and multiplier-based techniques (Scorletti and El Ghaoui 1998; Scherer 2001).

Based on the result and proof of Theorem 2.5 we can obtain the following conditions for various performance notions.

Corollary 2.1 ($\mathcal{L}_2$-gain of an LPV system)**.** *The LPV system given by (2.26) has a bounded $\mathcal{L}_2$-gain of γ, if there exists a $\gamma \in \mathbb{R}_0^+$ and a positive definite matrix function $M : \mathcal{P} \to \mathbb{S}^{n_x}$ with $M \in \mathcal{C}_1$ such that, in CT, for all $p \in \mathcal{P}$ and $v \in \Pi$ it holds that*

$$\begin{bmatrix} A(p)^\top M(p) + (\star)^\top + \partial M(p,v) & M(p)B(p) & C(p)^\top \\ \star & -\gamma I & D(p)^\top \\ \star & \star & -\gamma I \end{bmatrix} \preceq 0, \tag{2.38a}$$

while, in DT, it holds for all $p \in \mathcal{P}$ and $v \in \Pi$ that

$$\begin{bmatrix} M(p+v) & A(p)M(p) & B(p) & 0 \\ \star & M(p) & 0 & M(p)C(p)^\top \\ \star & 0 & \gamma I & D(p)^\top \\ \star & \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{2.38b}$$

Corollary 2.2 (Passivity of an LPV system)**.** *The LPV system given by (2.26) is passive, if there exists a positive definite matrix function $M : \mathcal{P} \to \mathbb{S}^{n_x}$ with $M \in \mathcal{C}_1$*

such that, in CT, for all $p \in \mathcal{P}$ and $v \in \Pi$, it holds that

$$\begin{bmatrix} A(p)^\top M(p) + (\star)^\top + \partial M(p,v) & M(p)B(p) - C(p)^\top \\ \star & -D(p) + (\star)^\top \end{bmatrix} \preceq 0, \tag{2.39a}$$

while, in DT, it holds for all $p \in \mathcal{P}$ and $v \in \Pi$ that

$$\begin{bmatrix} M(p+v) & A(p)M(p) & B(p) \\ \star & M(p) & M(p)C(p)^\top \\ \star & \star & D(p) + (\star)^\top \end{bmatrix} \succeq 0. \tag{2.39b}$$

Corollary 2.3 ($\mathcal{L}_2$-$\mathcal{L}_\infty$-gain of an LPV system)**.** *The LPV system given by*[7] *(2.26) has a bounded $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain of γ, if there exists a $\gamma \in \mathbb{R}_0^+$ and a positive definite matrix function $M : \mathcal{P} \to \mathbb{S}^{n_\mathrm{x}}$ with $M \in \mathcal{C}_1$ such that, in CT, for all $p \in \mathcal{P}$ and $v \in \Pi$, it holds that*

$$\begin{bmatrix} A(p)^\top M(p) + (\star)^\top + \partial M(p,v) & M(p)B(p) \\ \star & -\gamma I \end{bmatrix} \preceq 0, \quad \begin{bmatrix} M(p) & C(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0. \tag{2.40a}$$

while, in DT, it holds for all $p \in \mathcal{P}$ and $v \in \Pi$ that

$$\begin{bmatrix} M(p+v) & A(p)M(p) & B(p) \\ \star & M(p) & 0 \\ \star & \star & \gamma I \end{bmatrix} \succeq 0, \quad \begin{bmatrix} M(p) & M(p)C(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0. \tag{2.40b}$$

Corollary 2.4 ($\mathcal{L}_\infty$-gain of an LPV system)**.** *The LPV system given by (2.26) has a bounded $\mathcal{L}_\infty$-gain of γ, if there exist $\alpha, \beta, \gamma \in \mathbb{R}_0^+$ and a positive definite matrix function $M : \mathcal{P} \to \mathbb{S}^{n_\mathrm{x}}$ with $M \in \mathcal{C}_1$ such that, in CT, for all $p \in \mathcal{P}$ and $v \in \Pi$, it holds that*

$$\begin{gathered} \begin{bmatrix} A(p)^\top M(p) + (\star)^\top + \beta M(p) + \partial M(p,v) & M(p)B(p) \\ \star & -\alpha I \end{bmatrix} \preceq 0, \\ \begin{bmatrix} \beta M(p) & 0 & C(p)^\top \\ \star & (\gamma - \alpha) I & D(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0. \end{gathered} \tag{2.41a}$$

while, in DT, it holds for all $p \in \mathcal{P}$ and $v \in \Pi$ that

$$\begin{gathered} \begin{bmatrix} M(p+v) & A(p)M(p) & B(p) \\ \star & (1-\beta)M(p) & 0 \\ \star & \star & \alpha I \end{bmatrix} \succeq 0, \\ \begin{bmatrix} \beta M(p) & 0 & M(p)C(p)^\top \\ \star & (\gamma - \alpha) I & D(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0. \end{gathered} \tag{2.41b}$$

The full derivations of the LMIs given in Corollaries 2.1 to 2.4 can be found in Appendix A.2, which are inspired by the derivations for LTI systems given in (Scherer and Weiland 2015).

[7] Under the condition that $D(p) = 0$ for all $p \in \mathcal{P}$.

Remark 2.4. Note that conditions in Corollaries 2.1 to 2.4 can also be formulated in terms of strict, instead of non-strict, matrix inequality conditions. In turn, these will guarantee asymptotic stability of the corresponding LPV system given by (2.26).

The infinite set of LMIs in Corollaries 2.1 to 2.4 can be turned into a finite one using the aforementioned approaches, such as polytopic techniques, grid-based techniques, and multiplier based techniques, see (Hoffmann and Werner 2015a) for an overview. For LPV systems given by (2.26) with an affine scheduling dependency, i.e., when $A(p) = A_0 + \sum_{i=1}^{n_\mathrm{p}} A_i p_i$, etc. in (2.26), the analysis results in Corollaries 2.1 to 2.4 are implemented in the LPVcore Toolbox (Boef et al. 2021). Also for general scheduling dependencies, using a grid-based approach, the analysis results are implemented in LPVcore. Other MATLAB toolboxes for LPV systems, such as (Hjartarson et al. 2015; Verbandt et al. 2018; Veenman 2022), generally only include $\mathcal{L}_2$-gain and sometimes (generalized) $\mathcal{H}_2$ based performance analysis algorithms. However, these also support other scheduling dependencies such as rational and polynomial ones.

2.5.3 Controller synthesis

Similar as for LTI systems, the analysis LMIs in Section 2.5.2 can be transformed to synthesis LMIs through a nonlinear transformation of variables. For LPV controller synthesis, the idea of the generalized plant concept from the LTI framework (Doyle 1983) is also used. The generalized plant concept provides a systematic and generic approach to consider different control problems, such as a four block mixed sensitivity design, model matching problems, and many others (Zhou et al. 1996), while also allowing to systematically shape the performance through the inclusion of weighting filters in the generalized plant.

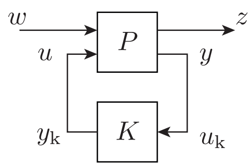


Figure 2.1: Closed-loop interconnection of a generalized plant P and the (to-be-designed) controller K.

More concretely, for the controller synthesis problem, we consider an interconnection of a generalized plant P and to-be-designed controller K, as depicted in Figure 2.1. We consider a generalized plant P of the form, omitting dependence on time for brevity,

$$\begin{bmatrix} \xi x \\ z \\ y \end{bmatrix} = \begin{bmatrix} A(p) & B_\mathrm{w}(p) & B_\mathrm{u}(p) \\ C_\mathrm{z}(p) & D_\mathrm{zw}(p) & D_\mathrm{zu}(p) \\ C_\mathrm{y}(p) & D_\mathrm{yw}(p) & D_\mathrm{yu}(p) \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix}, \tag{2.42}$$

where w and z now play the role of generalized disturbance (collecting references, disturbances, etc.) and generalized performance channel (collecting tracking errors, control efforts, etc.), respectively, and where $u(t) \in \mathcal{U} \subseteq \mathbb{R}^{n_\mathrm{u}}$ is the control input, and $y(t) \in \mathcal{Y} \subseteq \mathbb{R}^{n_\mathrm{y}}$ is the measured output. Moreover, $B_\mathrm{w} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{w}}$, $B_\mathrm{u} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{u}}$, $C_\mathrm{z} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{x}}$, $C_\mathrm{y} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{x}}$, $D_\mathrm{zw} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{w}}$, $D_\mathrm{zu} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{u}}$, $D_\mathrm{yw} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{w}}$, and $D_\mathrm{yu} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{u}}$ are matrix functions. For most synthesis algorithms, it is assumed for well-posedness that $D_\mathrm{yu}(p) = 0$, $\forall\, p \in \mathcal{P}$. In the remainder of this section, this will also be assumed.

The controller K, is considered in the form

$$\begin{bmatrix} \xi x_\mathrm{k} \\ y_\mathrm{k} \end{bmatrix} = \begin{bmatrix} A_\mathrm{k}(p) & B_\mathrm{k}(p) \\ C_\mathrm{k}(p) & D_\mathrm{k}(p) \end{bmatrix} \begin{bmatrix} x_\mathrm{k} \\ u_\mathrm{k} \end{bmatrix}, \tag{2.43}$$

where $x_\mathrm{k}(t) \in \mathcal{X}_\mathrm{k} \subseteq \mathbb{R}^{n_{\mathrm{x}_\mathrm{k}}}$ is the state[8] associated with (2.43), and where $u_\mathrm{k}(t) \in \mathcal{U}_\mathrm{k} \subseteq \mathbb{R}^{n_{\mathrm{u}_\mathrm{k}}}$ and $y_\mathrm{k}(t) \in \mathcal{Y}_\mathrm{k} \subseteq \mathbb{R}^{n_{\mathrm{y}_\mathrm{k}}}$ are the input and output of the controller, respectively. Furthermore, $A_\mathrm{k} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}_\mathrm{k}} \times n_{\mathrm{x}_\mathrm{k}}}$, $B_\mathrm{k} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}_\mathrm{k}} \times n_{\mathrm{u}_\mathrm{k}}}$, $C_\mathrm{k} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{y}_\mathrm{k}} \times n_{\mathrm{x}_\mathrm{k}}}$, and $D_\mathrm{k} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{y}_\mathrm{k}} \times n_{\mathrm{u}_\mathrm{k}}}$ are matrix functions.

The closed-loop interconnection of P and K, as depicted in Figure 2.1, i.e., such that $u_\mathrm{k} = y$ and $u = y_\mathrm{k}$ (meaning also that $n_{\mathrm{u}_\mathrm{k}} = n_\mathrm{y}$ and $n_\mathrm{u} = n_{\mathrm{y}_\mathrm{k}}$), is denoted by $\mathcal{F}_\mathrm{l}(P, K)$. The dynamics of the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P, K)$ are then given by

$$\begin{bmatrix} \xi x \\ \xi x_\mathrm{k} \\ \hline z \end{bmatrix} = \left[\begin{array}{c|c} A_\mathrm{cl}(p) & B_\mathrm{cl}(p) \\ \hline C_\mathrm{cl}(p) & D_\mathrm{cl}(p) \end{array}\right] \begin{bmatrix} x \\ x_\mathrm{k} \\ \hline w \end{bmatrix}, \tag{2.44}$$

where

$$A_\mathrm{cl}(p) := \begin{bmatrix} A(p) + B_\mathrm{u}(p) D_\mathrm{k}(p) C_\mathrm{y}(p) & B_\mathrm{u}(p) C_\mathrm{k}(p) \\ B_\mathrm{k}(p) C_\mathrm{y}(p) & A_\mathrm{k}(p) \end{bmatrix}, \tag{2.45a}$$

$$B_\mathrm{cl}(p) := \begin{bmatrix} B_\mathrm{w}(p) + B_\mathrm{u}(p) D_\mathrm{k}(p) D_\mathrm{yw}(p) \\ B_\mathrm{k}(p) D_\mathrm{yw}(p) \end{bmatrix}, \tag{2.45b}$$

$$C_\mathrm{cl}(p) := \begin{bmatrix} C_\mathrm{z}(p) + D_\mathrm{zu}(p) D_\mathrm{k}(p) C_\mathrm{y}(p) & D_\mathrm{zu}(p) C_\mathrm{k}(p) \end{bmatrix}, \tag{2.45c}$$

$$D_\mathrm{cl}(p) := D_\mathrm{zw}(p) + D_\mathrm{zu}(p) D_\mathrm{k}(p) D_\mathrm{yw}(p). \tag{2.45d}$$

Note that for P to be considered as a generalized plant, there should exist a controller K of the form (2.43) such that $\mathcal{F}_\mathrm{l}(P, K)$ is stable. For this to be the case, the pairs (A, B_u) and (A, C_y) should be stabilizable and detectable, respectively, according to the following definitions:

Definition 2.17 (LPV stabilizability)**.** *Consider the matrix functions $A : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$ and $B : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{u}}$. The pair (A, B) is called stabilizable if there exists a matrix function $F : \mathcal{P} \to \mathbb{R}^{n_\mathrm{u} \times n_\mathrm{x}}$ such that the LPV state-space representation*

$$\xi x(t) = \left(A(p) + B(p) F(p)\right) x(t), \tag{2.46}$$

is asymptotically stable.

Definition 2.18 (LPV detectability)**.** *Consider the matrix functions $A : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$ and $C : \mathcal{P} \to \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{x}}$. The pair (A, C) is called detectable if the pair $(A^\top, C^\top)$ is stabilizable.*

[8]For most state-space based LPV controller synthesis methods $n_{\mathrm{x}_\mathrm{k}} = n_\mathrm{x}$.

Moreover, note that $\mathcal{F}_\mathrm{l}(P,K)$ corresponds to an LPV system represented by (2.26). For controller synthesis, the objective is to find the controller matrix functions A_k, B_k, C_k, D_k such that $\mathcal{F}_\mathrm{l}(P,K)$ is classically (Q,S,R) dissipative (for a given tuple (Q,S,R)). Similar to the analysis problems, also the synthesis problem can be expressed as a feasibility condition of a matrix inequality (Apkarian and Adams 1998; De Caigny et al. 2012; Scherer and Weiland 2015):

Theorem 2.6. *There exists a controller K given by (2.43) such that the closed-loop interconnection of a generalized plant P given by (2.42) and K is classically (Q,S,R) dissipative, if, in CT, there exist a positive-define matrix $M_\mathrm{x} \in \mathbb{S}^{n_\mathrm{x}}$, positive-definite matrix function $M_\mathrm{y} : \mathcal{P} \to \mathbb{S}^{n_\mathrm{x}}$ with $M_\mathrm{y} \in \mathcal{C}_1$, and (transformed controller) matrix functions $\mathcal{A}_\mathrm{k} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{x}}$, $\mathcal{B}_\mathrm{k} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{y}}$, $\mathcal{C}_\mathrm{k} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{u}\times n_\mathrm{x}}$, and $\mathcal{D}_\mathrm{k} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{u}\times n_\mathrm{y}}$, such that for all $(p,v) \in \mathcal{P}\times\Pi$, it holds that*

$$\begin{bmatrix} \mathcal{A}_\mathrm{cl}(p,v) + (\star)^\top & \mathcal{B}_\mathrm{cl}(p) \\ \star & 0 \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ \mathcal{C}_\mathrm{cl}(p) & \mathcal{D}_\mathrm{cl}(p) \end{bmatrix} \preceq 0. \tag{2.47a}$$

$$\begin{bmatrix} M_\mathrm{y}(p) & I \\ \star & M_\mathrm{x} \end{bmatrix} \succ 0, \tag{2.47b}$$

where the closed-loop related matrices are

$$\begin{aligned} \mathcal{A}_\mathrm{cl}(p,v) = & \begin{bmatrix} A(p)M_\mathrm{y}(p) + B_\mathrm{u}(p)\mathcal{C}_\mathrm{k}(p) & A(p) + B_\mathrm{u}(p)\mathcal{D}_\mathrm{k}(p)C_\mathrm{y}(p) \\ \mathcal{A}_\mathrm{k}(p) & M_\mathrm{x}A(p) + \mathcal{B}_\mathrm{k}(p)C_\mathrm{y}(p) \end{bmatrix} + \\ & \begin{bmatrix} -\frac{1}{2}\partial M_\mathrm{y}(p,v) & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned} \tag{2.48a}$$

$$\mathcal{B}_\mathrm{cl}(p) = \begin{bmatrix} B_\mathrm{w}(p) + B_\mathrm{u}(p)D_\mathrm{k}(p)D_\mathrm{yw}(p) \\ M_\mathrm{x}B_\mathrm{w}(p) + \mathcal{B}_\mathrm{k}(p)D_\mathrm{yw}(p) \end{bmatrix}, \tag{2.48b}$$

$$\mathcal{C}_\mathrm{cl}(p) = \begin{bmatrix} C_\mathrm{z}(p)M_\mathrm{y}(p) + D_\mathrm{zu}(p)\mathcal{C}_\mathrm{k}(p) & C_\mathrm{z}(p) + D_\mathrm{zu}(p)\mathcal{D}_\mathrm{k}(p)C_\mathrm{y}(p) \end{bmatrix}, \tag{2.48c}$$

$$\mathcal{D}_\mathrm{cl}(p) = D_\mathrm{zw}(p) + D_\mathrm{zu}(p)\mathcal{D}_\mathrm{k}(p)D_\mathrm{yw}(p). \tag{2.48d}$$

The matrices of the LPV state-space representation of the controller K that achieves dissipativity are given by

$$\begin{aligned} & \begin{bmatrix} A_\mathrm{k}(p) & B_\mathrm{k}(p) \\ C_\mathrm{k}(p) & D_\mathrm{k}(p) \end{bmatrix} = \\ & \begin{bmatrix} U & M_\mathrm{x}B_\mathrm{u}(p) \\ 0 & I \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathcal{A}_\mathrm{k}(p) - M_\mathrm{x}A(p)M_\mathrm{y}(p) & \mathcal{B}_\mathrm{k}(p) \\ \mathcal{C}_\mathrm{k}(p) & \mathcal{D}_\mathrm{k}(p) \end{bmatrix} \right) \begin{bmatrix} V(p)^\top & 0 \\ C_\mathrm{y}(p)M_\mathrm{y}(p) & I \end{bmatrix}^{-1}, \end{aligned} \tag{2.49}$$

where U and V are arbitrary solutions to $M_\mathrm{x}M_\mathrm{y}(p) + UV(p)^\top = I$.
In DT, there exists a controller K such that $\mathcal{F}_\mathrm{l}(P,K)$ given by (2.44) is classically (Q,S,R) dissipative with[9] *$R \preceq 0$, if there exist a matrix $G_\mathrm{y} \in \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{x}}$, positive-define matrix functions $M_\mathrm{x} : \mathcal{P} \to \mathbb{S}^{n_\mathrm{x}}$, $M_\mathrm{z} : \mathcal{P} \to \mathbb{S}^{n_\mathrm{x}}$, and matrix functions $M_\mathrm{y} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{x}}$, $G_\mathrm{x} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{x}}$, $J : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{x}}$, $\mathcal{A}_\mathrm{k} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x}\times n_\mathrm{x}}$,*

[9] Note that $R \preceq 0$ is also required for $\mathcal{F}_\mathrm{l}(P,K)$ to be stable, see also Theorem 2.3.

$\mathcal{B}_\mathrm{k} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{y}}$, $\mathcal{C}_\mathrm{k} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{u} \times n_\mathrm{x}}$, *and* $\mathcal{D}_\mathrm{k} : \mathcal{P} \to \mathbb{R}^{n_\mathrm{u} \times n_\mathrm{y}}$, *such that for all* $(p, v) \in \mathcal{P} \times \Pi$, *it holds that*

$$\begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_\mathrm{cl}(p) & \mathcal{B}_\mathrm{cl}(p) \\ \star & \mathcal{G}(p) + (\star)^\top - \mathcal{M}(p) & 0 \\ \star & \star & 0 \end{bmatrix} + (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & \mathcal{C}_\mathrm{cl}(p) & \mathcal{D}_\mathrm{cl}(p) \end{bmatrix} \succeq 0, \tag{2.50a}$$

$$\mathcal{M}(p) \succ 0, \tag{2.50b}$$

where

$$\mathcal{A}_\mathrm{cl}(p) = \begin{bmatrix} A(p) G_\mathrm{x}(p) + B_\mathrm{u}(p) \mathcal{C}_\mathrm{k}(p) & A(p) + B_\mathrm{u}(p) \mathcal{D}_\mathrm{k}(p) C_\mathrm{y}(p) \\ \mathcal{A}_\mathrm{k}(p) & G_\mathrm{y} A(p) + \mathcal{B}_\mathrm{k}(p) C_\mathrm{y}(p) \end{bmatrix}, \tag{2.51a}$$

$$\mathcal{B}_\mathrm{cl}(p, v) = \begin{bmatrix} B_\mathrm{w}(p) + B_\mathrm{u}(p) \mathcal{D}_\mathrm{k}(p) D_\mathrm{yw}(p) \\ G_\mathrm{y} B_\mathrm{w}(p) + \mathcal{B}_\mathrm{k}(p) D_\mathrm{yw}(p) \end{bmatrix}, \tag{2.51b}$$

$$\mathcal{C}_\mathrm{cl}(p) = \begin{bmatrix} C_\mathrm{z}(p) G_\mathrm{x}(p) + D_\mathrm{zu}(p) \mathcal{C}_\mathrm{k}(p) & C_\mathrm{z}(p) + D_\mathrm{zu}(p) \mathcal{D}_\mathrm{k}(p) C_\mathrm{y}(p) \end{bmatrix}, \tag{2.51c}$$

$$\mathcal{D}_\mathrm{cl}(p) = D_\mathrm{zw}(p) + D_\mathrm{zu}(p) \mathcal{D}_\mathrm{k}(p) D_\mathrm{yw}(p) \tag{2.51d}$$

$$\mathcal{M}(p) = \begin{bmatrix} M_\mathrm{x}(p) & M_\mathrm{y}(p) \\ \star & M_\mathrm{z}(p) \end{bmatrix}, \tag{2.52}$$

$$\mathcal{G}(p) = \begin{bmatrix} G_\mathrm{x}(p) & I \\ J(p) & G_\mathrm{y} \end{bmatrix}. \tag{2.53}$$

The matrices of the LPV state-space representation of the controller K *that achieves dissipativity are given by*

$$\begin{bmatrix} A_\mathrm{k}(p) & B_\mathrm{k}(p) \\ C_\mathrm{k}(p) & D_\mathrm{k}(p) \end{bmatrix} =$$

$$\begin{bmatrix} V & G_\mathrm{y} B_\mathrm{u}(p) \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{A}_\mathrm{k}(p) - G_\mathrm{y} A(p) G_\mathrm{x}(p) & \mathcal{B}_\mathrm{k}(p) \\ \mathcal{C}_\mathrm{k}(p) & \mathcal{D}_\mathrm{k}(p) \end{bmatrix} \begin{bmatrix} U(p) & 0 \\ C_\mathrm{y}(p) G_\mathrm{x}(p) & I \end{bmatrix}^{-1}, \tag{2.54}$$

where U *and* V *are arbitrary solutions to* $J(p) = G_\mathrm{y} G_\mathrm{x}(p) + U(p) V$.

The full derivations of these conditions and more details can be found in Appendix A.3.

Note that the matrix functions in Theorem 2.6, such as M_y, $\mathcal{A}_\mathrm{k}$, $\mathcal{B}_\mathrm{k}$, etc., are generally assumed to be in the same function class, e.g., affine, polynomial, or rational, as the matrices of the LPV state-space representation of the generalized plant G.

Based on Theorem 2.6, we can obtain the following synthesis conditions for various performance notions.

Corollary 2.5 ($\mathcal{L}_2$-gain based LPV synthesis)**.** *There exists a controller* K *given by (2.43) such that the closed-loop interconnection of a generalized plant* P *given by (2.42) and* K *has a bounded* $\mathcal{L}_2$*-gain of* γ*, if, in CT, there exist a positive-define* *matrix* $M_{\mathrm{x}} \in \mathbb{S}^{n_{\mathrm{x}}}$*, a positive-definite matrix function* $M_{\mathrm{y}} : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$ *with* $M_{\mathrm{y}} \in \mathcal{C}_1$*, and matrix functions* $\mathcal{A}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $\mathcal{B}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{y}}}$*,* $\mathcal{C}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{x}}}$*, and* $\mathcal{D}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{y}}}$*, such that for all* $(p, v) \in \mathcal{P} \times \Pi$*, it holds that*

$$\begin{bmatrix} \mathcal{A}_{\mathrm{cl}}(p, v) + (\star)^\top & \mathcal{B}_{\mathrm{cl}}(p) - \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & -\mathcal{D}_{\mathrm{cl}}(p) + (\star)^\top \end{bmatrix} \preceq 0, \tag{2.55a}$$

along with (2.47b), where $\mathcal{A}_{\mathrm{cl}}, \dots \mathcal{D}_{\mathrm{cl}}$ *are given by (2.48). The matrices of the LPV state-space representation of the controller* K *are then given by (2.49), where* U *and* V *are obtained by finding a solution to* $M_{\mathrm{x}} M_{\mathrm{y}}(p) + UV(p)^\top = I$*.*

In DT, there exists a controller K *such that* $\mathcal{F}_{\mathrm{l}}(P, K)$ *given by (2.44) has a bounded* ℓ_2*-gain of* γ*, if there exist a matrix* $G_{\mathrm{y}} \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*, positive-define matrix functions* $M_{\mathrm{x}} : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$*,* $M_{\mathrm{z}} : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$*, and matrix functions* $M_{\mathrm{y}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $G_{\mathrm{x}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $J : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $\mathcal{A}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $\mathcal{B}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{y}}}$*,* $\mathcal{C}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{x}}}$*, and* $\mathcal{D}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{y}}}$*, such that for all* $(p, v) \in \mathcal{P} \times \Pi$*, it holds that*

$$\begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_{\mathrm{cl}}(p, v) & \mathcal{B}_{\mathrm{cl}}(p, v) & 0 \\ \star & \mathcal{G}(p) + (\star)^\top - \mathcal{M}(p) & 0 & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I & \mathcal{D}_{\mathrm{cl}}(p)^\top \\ \star & \star & \star & \gamma I \end{bmatrix} \succeq 0, \tag{2.55b}$$

along with (2.50b), where $\mathcal{A}_{\mathrm{cl}}, \dots \mathcal{D}_{\mathrm{cl}}$*,* $\mathcal{M}$*, and* $\mathcal{G}$ *are given by (2.51), (2.52), and (2.53), respectively. The matrices of the LPV state-space representation of the controller* K *are then given by (2.54), where* U *and* V *are obtained by finding a solution to* $J(p) = G_{\mathrm{y}} G_{\mathrm{x}}(p) + U(p)V$*.*

Corollary 2.6 (Passivity based LPV synthesis)**.** *There exists a controller* K *given by (2.43) such that the closed-loop interconnection of a generalized plant* P *given by (2.42) and* K *is passive, if, in CT, there exist a positive-define matrix* $M_{\mathrm{x}} \in \mathbb{S}^{n_{\mathrm{x}}}$*, a positive-definite matrix function* $M_{\mathrm{y}} : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$ *with* $M_{\mathrm{y}} \in \mathcal{C}_1$*, and matrix functions* $\mathcal{A}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $\mathcal{B}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{y}}}$*,* $\mathcal{C}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{x}}}$*, and* $\mathcal{D}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{y}}}$*, such that for all* $(p, v) \in \mathcal{P} \times \Pi$*, it holds that*

$$\begin{bmatrix} \mathcal{A}_{\mathrm{cl}}(p, v) + (\star)^\top & \mathcal{B}_{\mathrm{cl}}(p) - \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & -\mathcal{D}_{\mathrm{cl}}(p) + (\star)^\top \end{bmatrix} \preceq 0, \tag{2.56a}$$

along with (2.47b), where $\mathcal{A}_{\mathrm{cl}}, \dots \mathcal{D}_{\mathrm{cl}}$ *are given by (2.48). The matrices of the LPV state-space representation of the controller* K *are then given by (2.49), where* U *and* V *are obtained by finding a solution to* $M_{\mathrm{x}} M_{\mathrm{y}}(p) + UV(p)^\top = I$*.*

In DT, there exists a controller K *such that* $\mathcal{F}_{\mathrm{l}}(P, K)$ *given by (2.44) is passive, if there exists a matrix* $G_{\mathrm{y}} \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*, positive-define matrix functions* $M_{\mathrm{x}} : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$*,* $M_{\mathrm{z}} : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$*, and matrix functions* $M_{\mathrm{y}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $G_{\mathrm{x}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $J : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $\mathcal{A}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $\mathcal{B}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{y}}}$*,* $\mathcal{C}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{x}}}$*, and* $\mathcal{D}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{y}}}$*, such that for all* $(p, v) \in \mathcal{P} \times \Pi$*, it holds that*

$$\begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_{\mathrm{cl}}(p, v) & \mathcal{B}_{\mathrm{cl}}(p, v) \\ \star & \mathcal{G}(p) + (\star)^\top - \mathcal{M}(p) & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & \star & \mathcal{D}_{\mathrm{cl}} + (\star)^\top \end{bmatrix} \succeq 0, \tag{2.56b}$$

along with (2.50b), where $\mathcal{A}_{\rm cl}, \ldots \mathcal{D}_{\rm cl}$, $\mathcal{M}$, *and* $\mathcal{G}$ *are given by (2.51), (2.52), and (2.53), respectively. The matrices of the LPV state-space representation of the controller* K *are then given by (2.54), where* U *and* V *are obtained by finding a solution to* $J(p) = G_{\rm y}G_{\rm x}(p) + U(p)V$.

Corollary 2.7 ($\mathcal{L}_2$-$\mathcal{L}_\infty$-gain based LPV synthesis)**.** *There exists a controller* K *given by (2.43) such that the closed-loop interconnection of a generalized plant* P *given by*[10] *(2.42) and* K *has a bounded* $\mathcal{L}_2$-$\mathcal{L}_\infty$*-gain of* γ*, if, in CT, there exists a positive-define matrix* $M_{\rm x} \in \mathbb{S}^{n_{\rm x}}$*, a positive-definite matrix function* $M_{\rm y} : \mathcal{P} \to \mathbb{S}^{n_{\rm x}}$ *with* $M_{\rm y} \in \mathcal{C}_1$*, and matrix functions* $\mathcal{A}_{\rm k} : \mathcal{P} \to \mathbb{R}^{n_{\rm x}\times n_{\rm x}}$, $\mathcal{B}_{\rm k} : \mathcal{P} \to \mathbb{R}^{n_{\rm x}\times n_{\rm y}}$, $\mathcal{C}_{\rm k} : \mathcal{P} \to \mathbb{R}^{n_{\rm u}\times n_{\rm x}}$*, and* $\mathcal{D}_{\rm k} : \mathcal{P} \to \mathbb{R}^{n_{\rm u}\times n_{\rm y}}$*, such that for all* $(p, v) \in \mathcal{P} \times \Pi$*, it holds that*

$$\begin{bmatrix} \mathcal{A}_{\rm cl}(p,v) + (\star)^\top & B_{\rm cl}(p) \\ \star & -\gamma I \end{bmatrix} \preceq 0, \qquad \begin{bmatrix} \mathcal{M}(p) & \mathcal{C}_{\rm cl}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0, \tag{2.57a}$$

along with (2.47b), where $\mathcal{A}_{\rm cl}, \ldots \mathcal{D}_{\rm cl}$ *are given by (2.48). The matrices of the LPV state-space representation of the controller* K *are then given by (2.49), where* U *and* V *are obtained by finding a solution to* $M_{\rm x}M_{\rm y}(p) + UV(p)^\top = I$.

In DT, there exists a controller K *such that* $\mathcal{F}_{\rm l}(P, K)$ *given by (2.44) has a bounded* ℓ_2-ℓ_∞*-gain of* γ*, if there exists a matrix* $G_{\rm y} \in \mathbb{R}^{n_{\rm x}\times n_{\rm x}}$*, positive-define matrix functions* $M_{\rm x} : \mathcal{P} \to \mathbb{S}^{n_{\rm x}}$, $M_{\rm z} : \mathcal{P} \to \mathbb{S}^{n_{\rm x}}$*, and matrix functions* $M_{\rm y} : \mathcal{P} \to \mathbb{R}^{n_{\rm x}\times n_{\rm x}}$, $G_{\rm x} : \mathcal{P} \to \mathbb{R}^{n_{\rm x}\times n_{\rm x}}$, $J : \mathcal{P} \to \mathbb{R}^{n_{\rm x}\times n_{\rm x}}$, $\mathcal{A}_{\rm k} : \mathcal{P} \to \mathbb{R}^{n_{\rm x}\times n_{\rm x}}$, $\mathcal{B}_{\rm k} : \mathcal{P} \to \mathbb{R}^{n_{\rm x}\times n_{\rm y}}$, $\mathcal{C}_{\rm k} : \mathcal{P} \to \mathbb{R}^{n_{\rm u}\times n_{\rm x}}$*, and* $\mathcal{D}_{\rm k} : \mathcal{P} \to \mathbb{R}^{n_{\rm u}\times n_{\rm y}}$*, such that for all* $(p, v) \in \mathcal{P} \times \Pi$*, it holds that*

$$\begin{gathered} \begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_{\rm cl}(p,v) & \mathcal{B}_{\rm cl}(p,v) \\ \star & \mathcal{G}(p) + (\star) - \mathcal{M}(p) & 0 \\ \star & \star & \gamma I \end{bmatrix} \succeq 0, \\ \begin{bmatrix} \mathcal{G} + (\star)^\top - \mathcal{M}(p) & \mathcal{C}_{\rm cl}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0, \end{gathered} \tag{2.57b}$$

along with (2.50b), where $\mathcal{A}_{\rm cl}, \ldots \mathcal{D}_{\rm cl}$, $\mathcal{M}$, *and* $\mathcal{G}$ *are given by (2.51), (2.52), and (2.53), respectively. The matrices of the LPV state-space representation of the controller* K *are then given by (2.54), where* U *and* V *are obtained by finding a solution to* $J(p) = G_{\rm y}G_{\rm x}(p) + U(p)V$.

Corollary 2.8 ($\mathcal{L}_\infty$-gain based LPV synthesis)**.** *There exists a controller* K *given by (2.43) such that the closed-loop interconnection of a generalized plant* P *given by (2.42) and* K *has a bounded* $\mathcal{L}_\infty$*-gain of* γ*, if, in CT, if there exist scalars* $\alpha, \beta \in \mathbb{R}_0^+$*, a positive-define matrix* $M_{\rm x} \in \mathbb{S}^{n_{\rm x}}$*, a positive-definite matrix function* $M_{\rm y} : \mathcal{P} \to \mathbb{S}^{n_{\rm x}}$ *with* $M_{\rm y} \in \mathcal{C}_1$*, and matrix functions* $\mathcal{A}_{\rm k} : \mathcal{P} \to \mathbb{R}^{n_{\rm x}\times n_{\rm x}}$, $\mathcal{B}_{\rm k} : \mathcal{P} \to \mathbb{R}^{n_{\rm x}\times n_{\rm y}}$, $\mathcal{C}_{\rm k} : \mathcal{P} \to \mathbb{R}^{n_{\rm u}\times n_{\rm x}}$*, and* $\mathcal{D}_{\rm k} : \mathcal{P} \to \mathbb{R}^{n_{\rm u}\times n_{\rm y}}$*, such that for all* $(p, v) \in \mathcal{P} \times \Pi$*, it holds that*

$$\begin{bmatrix} \mathcal{A}_{\rm cl}(p,v) + (\star)^\top + \beta\mathcal{M}(p) & \mathcal{B}_{\rm cl}(p) \\ \star & -\alpha I \end{bmatrix} \preceq 0, \qquad \begin{bmatrix} \beta\mathcal{M}(p) & 0 & \mathcal{C}_{\rm cl}(p)^\top \\ \star & (\gamma-\alpha)I & \mathcal{D}_{\rm cl}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0, \tag{2.58a}$$

[10]Under the condition that $D_{\rm cl}(p) = 0$, which is the case when $D_{\rm zu}(p) = 0$ or $D_{\rm yw}(p) = 0$, along with $D_{\rm zw}(p) = 0$, for all $p \in \mathcal{P}$.

along with (2.47b), where $\mathcal{A}_{\mathrm{cl}}, \dots \mathcal{D}_{\mathrm{cl}}$ *are given by (2.48). The matrices of the LPV state-space representation of the controller* K *are then given by (2.49), where* U *and* V *are obtained by finding a solution to* $M_{\mathrm{x}} M_{\mathrm{y}}(p) + UV(p)^\top = I$.

In DT, there exists a controller K *such that* $\mathcal{F}_{\mathrm{l}}(P, K)$ *given by (2.44) has a bounded* ℓ_∞*-gain of* γ*, there exist scalar* $\alpha, \beta \in \mathbb{R}_0^+$*, a matrix* $G_{\mathrm{y}} \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*, positive-define matrix functions* $M_{\mathrm{x}} : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$*,* $M_{\mathrm{z}} : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$*, and matrix functions* $M_{\mathrm{y}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $G_{\mathrm{x}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $J : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $\mathcal{A}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$*,* $\mathcal{B}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{y}}}$*,* $\mathcal{C}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{x}}}$*, and* $\mathcal{D}_{\mathrm{k}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{u}} \times n_{\mathrm{y}}}$*, such that for all* $(p, v) \in \mathcal{P} \times \Pi$*, it holds that*

$$
\begin{gathered}
\begin{bmatrix}
\mathcal{M}(p+v) & \mathcal{A}_{\mathrm{cl}}(p,v) & \mathcal{B}_{\mathrm{cl}}(p,v) \\
\star & (1-\beta)(\mathcal{G}(p) + (\star)^\top - \mathcal{M}(p)) & 0 \\
\star & \star & \alpha I
\end{bmatrix} \succeq 0, \\
\begin{bmatrix}
\beta \cdot (\star)^\top (\mathcal{G}(p) + (\star)^\top - \mathcal{M}(p)) N(p) & 0 & \mathcal{C}_{\mathrm{cl}}(p)^\top \\
\star & (\gamma - \alpha) I & \mathcal{D}_{\mathrm{cl}}(p)^\top \\
\star & \star & \gamma I
\end{bmatrix} \succeq 0,
\end{gathered}
\tag{2.58b}
$$

along with (2.50b), where $\mathcal{A}_{\mathrm{cl}}, \dots \mathcal{D}_{\mathrm{cl}}$*,* $\mathcal{M}$*, and* $\mathcal{G}$ *are given by (2.51), (2.52), and (2.53), respectively. The matrices of the LPV state-space representation of the controller* K *are then given by (2.54), where* U *and* V *are obtained by finding a solution to* $J(p) = G_{\mathrm{y}} G_{\mathrm{x}}(p) + U(p) V$.

The full derivations of the LMIs given in Corollaries 2.5 to 2.8 can be found in Appendices A.3.2 to A.3.5 in Appendix A.3. Also for the synthesis results in Corollaries 2.5 to 2.8, strict LMI conditions can be formulated, which will ensure asymptotic stability of $\mathcal{F}_{\mathrm{l}}(P, K)$ (see also Remark 2.4).

Like for the analysis results, the infinite set of LMIs in Corollaries 2.5 to 2.8 can be turned into a finite one, using polytopic techniques, grid-based techniques, and multiplier based techniques, see (Hoffmann and Werner 2015a) for an overview. Also the synthesis results have been implemented in the LPVcore Toolbox (Boef et al. 2021) for CT and DT LPV systems (2.26) with general scheduling dependencies (using a grid-based approach) and for LPV systems with an affine scheduling dependency for the $\mathcal{L}_2$-gain, $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain, $\mathcal{L}_\infty$-gain, and passivity performance metrics. Like for the analysis algorithms, the other aforementioned toolboxes (Hjartarson et al. 2015; Verbandt et al. 2018; Veenman 2022) mostly focus only on $\mathcal{L}_2$-gain and $\mathcal{H}_2$ performance based controller synthesis, but have support for LPV systems with rational and/or polynomial scheduling dependency.

2.5.4 Application to analysis and control of nonlinear systems

In the previous sections, we have discussed how the analysis and controller synthesis problems for LPV systems can be turned into a finite set of LMIs which can be solved efficiently through various SDP solvers. If the LPV system given by (2.26) is a global LPV embedding of a nonlinear system given by (2.1), these results can be used to efficiently analyze and synthesize controllers for nonlinear systems. Namely, through the use of the Lemma 2.6 we have the following result.

Theorem 2.7 (Dissipativity of nonlinear systems through the LPV framework)**.** *Consider the nonlinear system given by (2.1) and LPV system given by (2.26). If the LPV system is a global embedding of the nonlinear system on the region* $\mathscr{X} \times \mathscr{W} = \mathcal{X} \times \mathcal{W}$*, and the LPV system is classically dissipative, then, the nonlinear system is classically dissipative.*

Theorem 2.7 trivially follows from using the result of Lemma 2.6, which shows that the behavior of the nonlinear system is included in its LPV embedding, i.e., $\mathfrak{B} \subseteq \breve{\mathfrak{B}}_{\mathrm{p}}$. Hence, if the LPV system is classically dissipative for all considered scheduling trajectories it also holds that, through the LPV embedding, the nonlinear system is classically dissipative along all solution trajectories. In practice, one often only considers an embedding region for which $\mathscr{X} \times \mathscr{W} \subset \mathcal{X} \times \mathcal{W}$. In this case, one can still conclude classical dissipativity of the underlying nonlinear system, but only for a subset of the behavior for which holds that $(x(t), w(t)) \in \mathscr{X} \times \mathscr{W}$ for all $t \in \mathcal{T}$.

Note that both the analysis and controller synthesis results for LPV systems, as shown in Sections 2.5.2 and 2.5.3, are based on classical dissipativity. Consequently, this implies that through Theorem 2.7, both stability and performance of nonlinear systems can be guaranteed by making use of LPV based analysis and controller synthesis. However, stability and performance w.r.t. the origin of an LPV embedding is only a sufficient condition for stability and performance w.r.t. the origin of the underlying nonlinear system and not a necessary condition. This is because a global LPV embedding of a nonlinear system is non-unique and the corresponding LPV representation also models a larger set of behavior than the nonlinear system. On the other hand, one can wonder how generalization of stability and dissipativity w.r.t. other forced equilibria of the nonlinear system stems from these results. This is what we will analyze next.

2.6 Summary

In this chapter, we have given an overview of stability and performance analysis of nonlinear and LPV systems given by a state-space representations. We have shown how through the concept of (classical) dissipativity, stability and performance could jointly be analyzed. For LPV systems we have also shown how the dissipativity concept can be used for controller synthesis and how the corresponding analysis and controller synthesis problems can be cast as convex optimization problems. Finally, we have shown how the LPV framework can be used in order to also analyze and design controllers for nonlinear systems through convex optimization problems. Nonetheless, in the next chapter, we will present how naive application of these results in their current form can lead to improper stability and performance guarantees on the underlying nonlinear system. After that, in the following chapters, we will develop a systematic framework for analysis and control of nonlinear systems to ensure global stability and performance by making use of approaches from LPV framework. The framework that will be developed will also address the shortcomings of the current LPV methods when applied to the analysis and control of nonlinear systems.

3

Pitfalls of LPV Analysis and Control of Nonlinear Systems

THE *Linear Parameter-Varying* (LPV) framework provides powerful methods for systematic analysis and controller synthesis for nonlinear systems. However, recently, a number of counter examples have surfaced where naive application of the LPV framework is unable to provide the desired guarantees. Namely, LPV controller synthesis applied to accomplish asymptotic output tracking and disturbance rejection for a nonlinear system can fail to achieve the desired asymptotic tracking and rejection behavior even when the scheduling variations remain in the bounded region considered during design. It has been observed that the controlled system may exhibit an oscillatory motion around the equilibrium point in the presence of a bounded constant input disturbance even if integral action is present. In this chapter, we investigate how and why the baseline Lyapunov stability notion, currently widely used in the LPV framework, fails to guarantee the desired system behavior. Specifically, it is shown that (asymptotic) stability analysis of a global LPV embedding of a nonlinear system using a quadratic Lyapunov function is only able to guarantee (asymptotic) stability of the origin of the nonlinear state-space representation, and there are no stability guarantees for other (non-zero) forced equilibria. Hence, under reference tracking and disturbance rejection controller design scenarios, the current LPV framework is insufficient to imply the desired guarantees for the underlying nonlinear system. The introduced concepts and the apparent pitfalls are demonstrated via a simulation example.

3.1 Introduction

As also highlighted in Chapter 1, the ever-growing industrial performance demands have resulted in increased system complexity, requiring tools for analysis and controller synthesis of systems beyond the *Linear Time-Invariant* (LTI) framework. This has resulted in various nonlinear analysis and control methods being developed. However, one of the drawbacks of these nonlinear methods is that they often lack the systematic controller design procedures and performance shaping approaches of the LTI framework. As an alternative, varying concepts using linear surrogate models have appeared and have extended the systematic analysis and synthesis tools of the LTI framework. Among these, the *Linear Parameter-Varying* (LPV) framework has become a popular approach (Tóth 2010).

As also described in Sections 1.2.3 and 2.5, LPV models are capable of describing nonlinear behavior in terms of a linear dynamical relation whose mathematical description depends on a time-varying parameter which can be measured, the so-called *scheduling-variable* p, that resides in an a priori known set $\mathcal{P}$. This allows for the convex analysis and control synthesis results with stability and performance guarantees from the LPV framework to be applied to nonlinear systems. These tools have reduced computational complexity and are more robustness when compared to other nonlinear methods. Due to these useful properties, many powerful LPV analysis and control synthesis methods have appeared and have been successfully applied to a wide range of industrial applications, see (Mohammadpour Velni and Scherer 2012; Hoffmann and Werner 2015a) and the references therein for more details.

However, as also highlighted in Section 1.3, it has been shown by counterexamples, e.g., in (Scorletti, Fromion, et al. 2015), that a controller ensuring asymptotic stability of the origin and $\mathcal{L}_2$-gain performance is not sufficient to guarantee asymptotic output tracking and disturbance rejection for nonlinear systems using LPV control methods. This will also be demonstrated in this chapter and we will offer solutions to the problem in the other chapters of this thesis. In the simulation study in (Scorletti, Fromion, et al. 2015), it has been shown that for a reference tracking application, the controlled system can exhibit oscillatory motion around an equilibrium point in the presence of a bounded constant input disturbance. The closed-loop displays this behavior even though integral action is present in the control loop. In fact, such a problem may also occur with other linear surrogate model based frameworks that build on the extension of the LTI framework, such as *Piecewise-Affine* (PWA) and *Takagi-Sugeno* (TS) fuzzy modeling and control design concepts.

Despite of the remedies that have been proposed in (Scorletti, Fromion, et al. 2015), no further analysis has been given why using 'standard' approaches for LPV controller design can in some cases fail to guarantee expected stability and performance requirements for nonlinear systems, while for LTI systems no such problems exist. In this chapter, we provide an analysis of this question from a nonlinear (Lyapunov) stability point of view for *Continuous-Time* (CT) systems. It is shown that the conditions for asymptotic stability guarantees for LPV representations with scheduling signals dependent on the state signals associated with the nonlinear system do not ensure the same guarantees for the represented nonlinear system for equilibrium points other than zero. Thus, naively using the usual standard LPV control methods to ensure stability and performance for a nonlinear system for reference tracking and/or disturbance rejection could result in unexpected behavior of the closed-loop system.

The chapter is structured as follows. In Section 3.2, we describe the problem of analyzing or synthesizing controllers for nonlinear systems using the LPV framework and an example is given to illustrate the issue with the current LPV results when applied to nonlinear systems. Section 3.3 describes the current stability analysis of LPV models and gives conditions when the current stability analysis results do hold and when they fail in their full extent for the underlying nonlinear system. In Section 3.4, the results of Section 3.3 are demonstrated on an example system. Finally, in Section 3.5, conclusions on the provided results are given.

3.2 Analysis and Control via the LPV Framework

3.2.1 LPV Embedding of Nonlinear Systems

We consider CT nonlinear dynamical systems, like we considered in Section 2.2, described by

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t)); \\ z(t) &= h(x(t), w(t)); \end{aligned} \tag{3.1}$$

where $x(t) \in \mathbb{R}^{n_\mathrm{x}}$ is the state, $w(t) \in \mathbb{R}^{n_\mathrm{w}}$ is the input to the system and $z(t) \in \mathbb{R}^{n_\mathrm{z}}$ the output of the system, and $t \in \mathbb{R}$ is time. The functions $f : \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}} \times \mathbb{R}^{n_\mathrm{u}} \to \mathbb{R}^{n_\mathrm{x}}$ and $h : \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}} \to \mathbb{R}^{n_\mathrm{y}}$ are assumed to be Lipschitz continuous.

This study focuses on the investigating the implied stability guarantees that are obtained when the (global) LPV embedding of the nonlinear system (3.1) is analyzed. As we considered in Section 2.5.1, here investigate LPV systems that can be represented in the form of

$$\begin{aligned} \dot{x}(t) &= A(p(t))x(t) + B(p(t))w(t); \\ z(t) &= C(p(t))x(t) + D(p(t))w(t); \end{aligned} \tag{3.2}$$

where $p(t) \in \mathcal{P} \subseteq \mathbb{R}^{n_\mathrm{p}}$, with $\mathcal{P}$ often taken to be a convex set. Moreover, $A : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$, $B : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{w}}$, $C : \mathcal{P} \to \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{x}}$, $D : \mathcal{P} \to \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{w}}$ are matrix functions. We assume that this LPV form is a global LPV embedding of the nonlinear system, see also Section 2.5 and specifically Definition 2.14 for more details. In short, this implies that there exists a scheduling-map $\eta : \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}} \to \mathbb{R}^{n_\mathrm{p}}$, such that $p(t) = \eta(x(t), w(t))$ and $f(x, w) = A(\eta(x, w))x + B(\eta(x, w))w$, $h(x, w) = C(\eta(x, w))x + C(\eta(x, w))w$ for all $(x, w) \in \mathcal{X} \times \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}}$, where $\mathcal{X} \times \mathcal{W}$ is a user chosen set on which the embedding is performed. See for example (Kwiatkowski, Bol, et al. 2006; Tóth 2010; Abbas, Tóth, Petreczky, Meskin, and Mohammadpour Velni 2014) for several procedures to embed the dynamics of nonlinear systems in an LPV model.

Systems of the form (3.1) includes cases when an LPV controller is designed/analyzed for a nonlinear system described by

$$\begin{aligned} \dot{x}(t) &= f(x(t), w(t), u(t)); \\ z(t) &= h_{\mathrm{z}}(x(t), w(t), u(t)); \\ y(t) &= h_{\mathrm{y}}(x(t), w(t)); \end{aligned} \tag{3.3}$$

where $u(t) \in \mathbb{R}^{n_{\mathrm{u}}}$ is the control input, $y(t) \in \mathbb{R}^{n_{\mathrm{y}}}$ is the measured output, and w and z now play the role of generalized disturbance, consisting of references, disturbances, etc., and generalized performance, consisting of tracking errors, control effort, etc., respectively. In this case, using the LPV framework, stability and performance guarantees are ensured with respect to the closed-loop behavior $w \to z$ in order to achieve, e.g., asymptotic output tracking and disturbance rejection. In which case (3.3) is embedded in an LPV representation of the form

$$\begin{aligned} \dot{x}(t) &= A(p(t))x(t) + B_{\mathrm{w}}(p(t))w(t) + B_{\mathrm{u}}(p(t))u(t); \\ z(t) &= C_{\mathrm{z}}(p(t))x(t) + D_{\mathrm{zw}}(p(t))w(t) + D_{\mathrm{zu}}(p(t))u(t); \\ y(t) &= C_{\mathrm{y}}(p(t))x(t) + D_{\mathrm{yw}}(p(t))w(t). \end{aligned} \tag{3.4}$$

Then, based on the LPV form (3.4), which serves as a proxy description of the nonlinear system (3.3), a controller is synthesized such that the interconnection of controller and the LPV system represented by (3.4) is asymptotically stable and the desired performance criteria on the performance channel from $w \to z$ are ensured for all $p(t) \in \mathcal{P}$. Powerful methods and performance shaping techniques exist to synthesize LPV controllers via convex optimization, see Section 2.5 or (Packard 1993; Apkarian, Gahinet, and G. Becker 1995; Wu 1995; Scherer 2001). In these cases, a dynamic output feedback controller of the form

$$\begin{aligned} \dot{x}_{\mathrm{k}}(t) &= A_{\mathrm{k}}(p(t))x_{\mathrm{k}}(t) + B_{\mathrm{k}}(p(t))u_{\mathrm{k}}(t); \\ y_{\mathrm{k}}(t) &= C_{\mathrm{k}}(p(t))x_{\mathrm{k}}(t) + D_{\mathrm{k}}(p(t))u_{\mathrm{k}}(t); \end{aligned} \tag{3.5}$$

is designed, where $x_{\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}}$ is the state, $u_{\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{u_k}}}$ the input and $y_{\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{y_k}}}$ the output of the controller, respectively. For $n_{\mathrm{u}} = n_{\mathrm{y_k}}$ and $n_{\mathrm{u_k}} = n_{\mathrm{y}}$, the closed-loop LPV system, defined by interconnecting the LPV controller (3.5) with (3.4), by taking $u = y_{\mathrm{k}}$ and $u_{\mathrm{k}} = y$, will be of the form (3.2). See also Section 2.5 for more details on LPV analysis and controller synthesis.

For simplicity, we will consider for our analysis two cases: (i) when $\eta : \mathbb{R}^{n_{\mathrm{x}}} \to \mathcal{P}$, i.e., $p(t) = \eta(x(t))$, which we will call the "dependent" scheduling-variable case, and (ii) when $\eta : \mathbb{R}^{n_{\mathrm{w}}} \to \mathcal{P}$, i.e., $p(t) = \eta(w(t))$, and hence p depends on an external independent signals, which we will call the "independent" case. For the dependent case, $A, \dots, D$ come from the decomposition of f and h from the nonlinear system (3.1), through the LPV embedding, hence, we have that

$$\begin{aligned} \dot{x}(t) &= \underbrace{A(\eta(x(t))x(t) + B(\eta(x(t))w(t)}_{f(x(t),w(t))}; \\ z(t) &= \underbrace{C(\eta(x(t))x(t) + D(\eta(x(t)))w(t)}_{h(x(t),w(t)}. \end{aligned} \tag{3.6}$$

As aforementioned, it has been observed recently in (Scorletti, Fromion, et al. 2015) that, when applying LPV controller synthesis techniques to nonlinear systems to guarantee reference tracking and disturbance rejection, the resulting closed-loop system can exhibit oscillations around the state equilibrium point in the presence of bounded input disturbances, even though asymptotic stability is guaranteed during synthesis, as $p(t) = \eta(x(t))$ resides in the set $\mathcal{P}$, and integral action is also present.

Hence, the question arises why the LPV controller is unable to achieve asymptotic reference tracking and disturbance rejection when interconnected to the nonlinear system, or more generally, why the stability and performance guarantees of the LPV representation are unable to imply the desired stability and performance guarantees of the nonlinear system.

3.2.2 When the Implied Stability Guarantee Fails

Before analyzing why guarantees on the LPV representation fail to imply the desired guarantees on the nonlinear system, we will first demonstrate this phenomenon by means of a simple example.

Example 3.1 *(A simple example)***.**

Control scenario

Consider the following nonlinear system

$$\begin{aligned} \dot{x}_\mathrm{g}(t) &= -x_\mathrm{g}(t) - x_\mathrm{g}^3(t) + u_\mathrm{g}(t); \\ y_\mathrm{g}(t) &= x_\mathrm{g}(t); \end{aligned} \tag{3.7}$$

with $x_\mathrm{g}(t)$, $u_\mathrm{g}(t)$, $y_\mathrm{g}(t) \in \mathbb{R}$. We aim to design an LPV controller in order to achieve reference tracking and disturbance rejection for this system. A possible closed-loop interconnection to achieve this objective is depicted in Figure 3.1. We define the generalized disturbance $w = \mathrm{col}(r, d)$, where r is the reference and d the input disturbance, and the generalized performance $z = e$, where $e = r - y_\mathrm{g}$ is the tracking error.

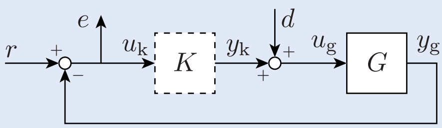


Figure 3.1: Closed-loop interconnection of plant G, given by (3.7), and controller K, given by (3.9).

In order to design an LPV controller for our plant and analyze the corresponding closed-loop interconnection, the plant (3.7) is embedded in an LPV model. A possible LPV embedding for (3.7) is

$$\begin{aligned} \dot{x}_\mathrm{g}(t) &= -(1 + p(t))x_\mathrm{g}(t) + u_\mathrm{g}(t); \\ y_\mathrm{g}(t) &= x_\mathrm{g}(t); \end{aligned} \tag{3.8}$$

where $p(t) \in \mathcal{P}$ is the scheduling-variable for which we consider $\mathcal{P} = [0,\, 9]$. The corresponding scheduling-map η is given by $p(t) = \eta(x_{\mathrm{g}}(t)) = x_{\mathrm{g}}^2(t) = y_{\mathrm{g}}^2(t)$.
In order to achieve our control objectives, we consider a PI-like LPV controller, given by

$$\begin{aligned} \dot{x}_{\mathrm{k}}(t) &= u_{\mathrm{k}}(t); \\ y_{\mathrm{k}}(t) &= (k_{11} + k_{12}p(t))x_{\mathrm{k}}(t) + k_{21}u_{\mathrm{k}}(t); \end{aligned} \tag{3.9}$$

where k_{11}, k_{12}, $k_{21} \in \mathbb{R}$ are parameters of the controller with $x_{\mathrm{k}}(t)$, $u_{\mathrm{k}}(t)$, $y_{\mathrm{k}}(t) \in \mathbb{R}$. For the (numerical) analysis to follow, the controller parameters are assumed to have the values: $k_{11} = 5$, $k_{12} = 2$ and $k_{21} = 1$. These controller parameters were chosen to demonstrate the stability issues.
The interconnection of (3.8) and (3.9), as depicted in Figure 3.1, results in an LPV model of the form (3.2) given by

$$\begin{aligned} \dot{x}_{\mathrm{g}}(t) &= -(1 + k_{21} + p(t))x_{\mathrm{g}}(t) + (k_{11} + k_{12}p(t))x_{\mathrm{k}}(t) + k_{21}r(t) + d(t); \\ \dot{x}_{\mathrm{k}}(t) &= -x_{\mathrm{g}}(t) + r(t); \\ e(t) &= -x_{\mathrm{g}}(t) + r(t); \end{aligned} \tag{3.10}$$

where $w = \mathrm{col}(r, d)$, $z = e$, $x = \mathrm{col}(x_{\mathrm{g}}, x_{\mathrm{k}})$, and $p = \eta(x) = x_{\mathrm{g}}^2$. By substituting the scheduling-map into (3.10), the corresponding nonlinear closed-loop interconnection is

$$\begin{aligned} \dot{x}_{\mathrm{g}}(t) &= -(1 + k_{21})x_{\mathrm{g}}(t) - x_{\mathrm{g}}^3(t) + (k_{11} + k_{12}x_{\mathrm{g}}^2(t))x_{\mathrm{k}}(t) + k_{21}r(t) + d(t); \\ \dot{x}_{\mathrm{k}}(t) &= -x_{\mathrm{g}}(t) + r(t); \\ e(t) &= -x_{\mathrm{g}}(t) + r(t). \end{aligned} \tag{3.11}$$

which is a model of the form (3.6).

$\mathcal{L}_2$-gain analysis via the LPV concept

The LPV framework allows for the calculation of an upper bound on the $\mathcal{L}_2$-gain of (3.11), by considering (3.10) and assuming $p \in \mathcal{P}$. Before computing the $\mathcal{L}_2$-gain, we connect weighting filters to the inputs and output of the interconnection (3.10) in order to incorporate the desired performance specification into our test. To r, we connect the weighting filter $W_{\mathrm{r}} = 1.5$ (expected magnitude of the reference), to d we connect $W_{\mathrm{d}} = 8$ (expected magnitude of the disturbance) and to e we connect $W_{\mathrm{e}}(s) = \frac{0.14(s+1)}{s+1\cdot 10^{-7}}$ (sensitivity shaping for integral action and 20% max overshoot). Computing the $\mathcal{L}_2$-gain of (3.10) with the weighting filters connected using the LPVcore Toolbox (Boef et al. 2021) results in an $\mathcal{L}_2$-gain of 0.98. Hence, we can conclude that (3.11) is asymptotically stable and has an $\mathcal{L}_2$-gain of at most 0.98, as long as $\eta(x(t)) \in \mathcal{P}$. As the $\mathcal{L}_2$-gain of the closed-loop system with weighting filters connected is ≤ 1, the closed-loop system should adhere to the performance specifications defined by the weighting filters. In order to get a sense of the performance of the closed-loop interconnection, the Bode magnitude plot of the sensitivity (i.e., from r to e) and process sensitivity (i.e., from d to e) for a number of frozen[1] values of the scheduling-variable (in $\mathcal{P}$) is displayed in Figure 3.2.

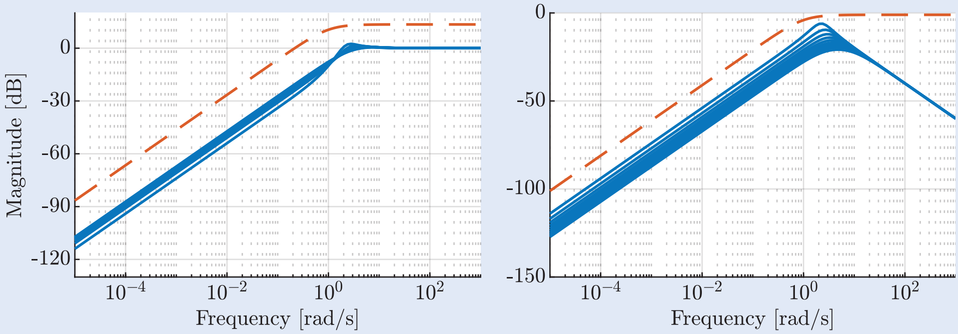


Figure 3.2: Sensitivity (left, —), i.e., $r \to e$, and process sensitivity (right, —), i.e., $d \to e$, Bode magnitude plots for frozen values of the scheduling-variable, including respective inverse weighting filters (- -).

In Figure 3.2, it can be seen that both the sensitivity and process sensitivity for frozen values of the scheduling-variable have magnitudes of zero for a frequency of zero. Hence, as the LPV representation is an LPV embedding of the nonlinear system, it would be reasonable to conclude from an LTI analysis point of view that, for constant reference and disturbance signals (for which still holds that $p \in \mathcal{P}$), the closed-loop system given by (3.11) has zero steady-state error. However, as will be shown next, this is not the case.

Nonlinear time-domain analysis

Simulating (3.11) for a constant reference $r \equiv 0.5$ and various constant disturbances d, results in the time responses displayed in Figure 3.3. From Figure 3.3, it is apparent that, for input disturbances closer to zero, the output converges to the reference (and we have zero steady-state error). However, applying $d(t) = -7$ or $d(t) = -8$ results in trajectories that converge to orbit-stable limit cycles around the target reference trajectory. Note that based on the trajectories of $y(t)$ in Figure 3.3, the corresponding scheduling trajectory stays within $\mathcal{P} = [0, 9]$ as $y(t) \in [-3, 3]$. Hence, while we adhere to the weighting filters and the scheduling-variable p stays within the set $\mathcal{P}$, we have shown that we do not obtain the expected desired behavior.

[1] Constant fixed trajectory of the scheduling-variable, i.e., $p \equiv \bar{p} \in \mathcal{P}$. Under such scheduling trajectory, (3.2) corresponds to an LTI system, for which a frequency response can be computed.

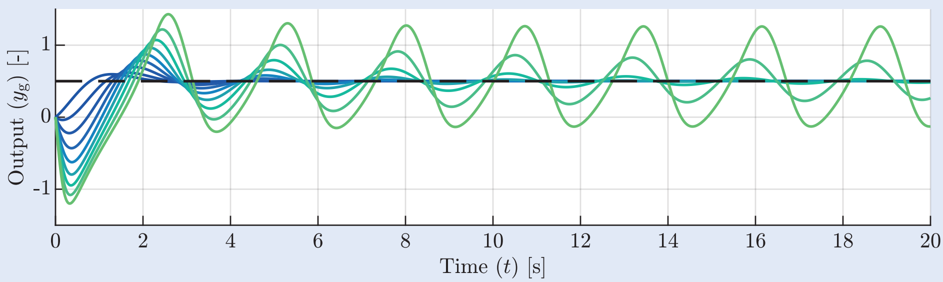


Figure 3.3: Time responses y_g of the closed-loop interconnection (3.11) along with the reference r (- -), for constant disturbances d ranging from 0 (—) to -8 (—).

Thus, by means of this simple example, we have demonstrated that the current $\mathcal{L}_2$-gain/asymptotic stability and performance analysis through the LPV framework is unfortunately inadequate to imply the tracking and rejection properties for nonlinear systems in general. Next, it will be analyzed why this is the case.

3.3 Behind the Scenes of Stability Analysis

3.3.1 Stability analysis for LPV systems

The tools for LPV analysis and controller synthesis, also used for Example 3.1, make use of (classical) dissipativity theory in order to simultaneously ensure stability and performance guarantees, see also Section 2.4 and Section 2.5 for more details. In this section, we will first investigate what the (Lyapunov) stability guarantees that are ensured on the LPV system imply in the "independent" case, i.e., when the scheduling is determined fully externally, meaning $p(t) = \eta(w(t))$.

We consider the LPV system given by (3.2) with $p(t) \in \mathcal{P}$. Define (x_*, w_*, p_*) to be an equilibrium point of (3.2). This means that for a given $w_* \in \mathscr{W} \subseteq \mathcal{W}$ and $p_* = \eta(w_*) \in \mathcal{P}$,

$$0 = \begin{bmatrix} A(p_*) & B(p_*) \end{bmatrix} \begin{bmatrix} x_* \\ w_* \end{bmatrix}. \tag{3.12}$$

To simplify the analysis, we assume that, for a given w_* and corresponding p_*, there is a unique x_* that satisfies (3.12).

Stability of the origin

It can be seen from (3.12) that the origin is an equilibrium point of the LPV system given by (3.2), by which we mean that $(x_*, w_*, p_*) = (0, 0, p_*)$ is an equilibrium point for all $p_* \in \mathcal{P}$. As we have shown in Section 2.5.2 using standard Lyapunov stability theory, the origin is a stable equilibrium for the LPV system given by (3.2), if there exists a quadratic Lyapunov function

$$V(x) = x^\top M x, \tag{3.13}$$

with $M \in \mathbb{S}^{n_x}$, where $M \succ 0$, such that, omitting dependence on time for brevity,

$$\dot{V}(x) = x^\top \left(A(p)^\top M + MA(p)\right) x \leq 0, \tag{3.14}$$

along all trajectories x and p of the unperturbed system, i.e., (3.2) with $w(t) = 0$. If (3.14) is only zero when $x = 0$, then (3.14) implies asymptotic stability of (3.2). Equivalently:

$$A(p)^\top M + MA(p) \prec 0, \quad \forall p \in \mathcal{P}. \tag{3.15}$$

As $\mathcal{P}$ is considered to be a compact convex set, the infinite dimensional LMI problem (3.15) can be reduced to a finite dimensional problem. Assuming $A(p)$ is a convex function, this problem can be solved efficiently using various semidefinite programming solvers, e.g., (Toh et al. 1999). This form of stability is also guaranteed when designing various controllers to ensure performance guarantees such as bounded $\mathcal{L}_2$-gain, passivity, etc.

Stability of non-zero equilibrium points

For quadratic stability of equilibrium points other than the origin, consider the quadratic Lyapunov function

$$V_{x_*}(x) = (x - x_*)^\top M (x - x_*), \tag{3.16}$$

with again $M \in \mathbb{S}^{n_x}$ and $M \succ 0$. Thus, based on (3.2) with $w(t) = w_*$, we obtain that

$$\begin{aligned}
\dot{V}_{x_*}(x) =& \dot{x}^\top M(x - x_*) + (x - x_*)^\top M\dot{x}, \\
=& \left(A(p)x + B(p)w_*\right)^\top M(x - x_*) + (x - x_*)^\top M \left(A(p)x + B(p)w_*\right), \\
=& 2(x - x_*)^\top MA(p)x + 2(x - x_*)^\top MB(p)w_*, \\
=& 2(x - x_*)^\top MA(p)(x - x_*) + 2(x - x_*)^\top MA(p)x_* + 2(x - x_*)^\top MB(p)w_*, \\
=& (x - x_*)^\top \underbrace{\left(A(p)^\top M + MA(p)\right)}_{Q(p)}(x - x_*) + 2(x - x_*)^\top M\underbrace{\left(A(p)x_* + B(p)w_*\right)}_{Z(p)}.
\end{aligned} \tag{3.17}$$

As we consider here the independent scheduling case, we have that $p(t) = \eta(w(t)) = \eta(w_*) = p_*$. Therefore, we have by (3.12) that $Z(p(t)) = Z(p_*) = 0$. Consequently, continuing from (3.17) with $Z(p(t)) = 0$, we obtain

$$\dot{V}_{x_*}(x) = (x - x_*)^\top Q(p)(x - x_*). \tag{3.18}$$

Therefore, if there exists $M \succ 0$ such that

$$Q(p) = A(p)^\top M + MA(p) \prec 0, \quad \forall p \in \mathcal{P}, \tag{3.19}$$

any equilibrium (x_*, w_*, p_*) is asymptotically stable.

This result is equivalent with (3.15), hence, for the scheduling independent case, this means asymptotic convergence to any equilibrium point (x_*, w_*, p_*) satisfying (3.12). Due to this property, which similarly holds in the LTI case, LPV stability analysis and performance analysis is needed to be accomplished with respect to the origin only (using (3.13) and (3.14) as a Lyapunov condition or (3.13) as a storage function) as it implies the same guarantees for any other equilibrium point.

3.3.2 Implying stability guarantees under state dependent scheduling

Stability of the origin

In the previous section, we have seen the stability guarantees for LPV systems in the scheduling independent case. Next, we will investigate the "dependent" scheduling case, i.e., when the LPV system describes a nonlinear system and $p(t) = \eta(x(t))$, see (3.6). Like for the scheduling independent case, we will first investigate stability of the origin, followed by stability of non-zero equilibrium points.

In the scheduling dependent case, the equilibrium at the origin of (3.6) corresponds to $(x_*, w_*) = (0, 0)$, that is equivalent with $(0, 0, \eta(0))$ in terms of (3.12). Performing the stability analysis for the origin of (3.6) using the quadratic Lyapunov function (3.13) gives

$$\dot{V}(x) = x^\top \left(A(\eta(x))^\top M + MA(\eta(x))\right) x. \tag{3.20}$$

If the following conditions are satisfied:

- For the LPV embedding (3.2) of (3.6), there exist an $M \succ 0$ such that (3.15) holds;
- $\eta(\mathcal{X}) \subseteq \mathcal{P}$ is satisfied, with $\mathcal{X}$ including the origin;

then (3.6) is asymptotically stable as these conditions will imply $\dot{V}(x(t)) < 0$ for all $x(t) \in \mathcal{X} \setminus \{0\}$.

Therefore, asymptotic stability of the origin of the LPV embedding implies asymptotic stability of the origin of the corresponding nonlinear system.

Stability of non-zero equilibrium points

Next, we consider non-zero equilibrium points of (3.6) given by (x_*, w_*), that is equivalent to $(x_*, w_*, \eta(x_*))$ in terms of (3.12). Performing the stability analysis for non-zero equilibrium points of (3.6) using the quadratic Lyapunov function (3.16) gives

$$\dot{V}_{x_*}(x) = (x - x_*)^\top Q(\eta(x))(x - x_*) + 2(x - x_*)^\top MZ(\eta(x)). \tag{3.21}$$

If we only analyze the LPV representation, one would only ensure that (3.19) holds, i.e, $Q(p) \prec 0$ for all $p \in \mathcal{P}$. However, this does not imply negativity of (3.21) along all trajectories $(x(t), \eta(x(t))$. In case $\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \eta(x(t)) = \eta(x_*)$, then $\lim_{t\to\infty} Z(\eta(x)) = Z(\eta(x_*)) = 0$. However, this is not imposed by (3.19). Continuing the analysis of (3.21) and taking $\Delta x = x - x_*$, (3.21) can be written as

$$\Delta x^\top Q(\eta(x))\Delta x + 2\Delta x^\top MZ(\eta(x)), \tag{3.22}$$

which for any fixed $x \in \mathcal{X}$ is a quadratic matrix polynomial. This quadratic form has as its global maximum at

$$(MZ(\eta(x)))^\top(-Q(\eta(x)))^{-1}(MZ(\eta(x))). \tag{3.23}$$

As we enforce by (3.19) that $Q(p) \prec 0$, the maximum of (3.23) will always be non-negative, hence, there will always be parts of the state-space where the Lyapunov function increases. Therefore, based on this analysis, no guarantees for (asymptotic) stability of the equilibrium point can be given in the general case if we rely on the results of the LPV test constructed Lyapunov function. In other words, based on the LPV test here, there is no guarantee that the corresponding nonlinear system will be asymptotically stable for an arbitrary equilibrium point, but only for the origin, where the asymptotic stability guarantees are ensured for any arbitrary trajectory of p in $\mathcal{P}$. This means, we have no stability guarantees when the LPV representation is used to ensure reference tracking and disturbance rejection of a nonlinear system, as in that case we do want to ensure stability of non-zero equilibrium points. This is in contrast to the scheduling independent case in Section 3.3.1, for which we showed that asymptotic stability is guaranteed for any equilibrium point (x_*, w_*, p_*), satisfying (3.12), if (3.19) holds.

While for the case of state dependent scheduling-variables, there are no general guarantees that the system is asymptotically stable when performing reference tracking and disturbance rejection, it could still be the case that for a subset of equilibrium points, (3.21) is strictly negative for a subregion of the state-space. Hence, as long as the trajectory stays within this subset of the state-space, asymptotic stability can still be guaranteed for the corresponding set of equilibrium points. This requires computing where (3.21) is negative or alternatively finding the roots of (3.21). However, even in the case that $\eta(x)$ is a linear or a polynomial mapping, (3.21) becomes a multivariable polynomial, for which it is difficult to find the roots, even for simple systems. Moreover, despite the loss of asymptotic stability, boundedness, as can be observed in Section 3.2.2, can still hold. However, this does not coincide with the expected outcome of the LPV analysis, nor would be a desired objective in synthesis.

Furthermore, this stability analysis is based on the Lyapunov function constructed in the LPV analysis step. Of course, for a given nonlinear system this does not mean that with an alternative method one could not find a Lyapunov function that actually shows stability. Here, we have only investigated the limitations of the currently widely used LPV stability concept when applied to analyze nonlinear systems.

3.4 Example

Based on the (asymptotic) Lyapunov stability guarantees given in Section 3.3, we aim to show for the example system (3.11) from Example 3.1 that forced equilibrium points exist for which asymptotic stability cannot be guaranteed while they are admissible in the considered scheduling range.

Example 3.2 *(Non-zero equilibrium analysis)***.** Using the LPVcore Toolbox, we have computed an upperbound for the $\mathcal{L}_2$-gain[2] of 1.78 for (3.10) and verified that it is asymptotically stable for $p \in \mathcal{P} = [0,\, 9]$, using a quadratic storage/Lyapunov function (3.13). The obtained matrix M of (3.13) is given by

$$M = \begin{bmatrix} 0.6240 & -0.6951 \\ -0.6951 & 3.1187 \end{bmatrix}. \tag{3.24}$$

As described in Section 3.3, due to the scheduling being dependent on the state, this result only implies asymptotic stability of the origin of (3.11). Next, we are interested for which set of equilibrium points asymptotic stability of the underlying nonlinear system can be guaranteed using the Lyapunov function (3.16) where M is given by (3.24). Computing the set of equilibrium points of (3.11) results in

$$\mathscr{E} = \{(x_*, w_*) \in \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}} \mid x_* = \Omega(w_*)\}\,, \tag{3.25}$$

where[3]

$$\Omega(w_*) := \begin{bmatrix} r_* & \frac{r_*^3 + r_* - d_*}{k_{12} r_*^2 + k_{11}} \end{bmatrix}^\top, \tag{3.26}$$

with $w_* = \begin{bmatrix} r_* & d_* \end{bmatrix}^\top$. Furthermore, we define the sets $\mathscr{W} := \pi_{\mathrm{w}_*} \mathscr{E}$ and $\mathscr{X} := \pi_{\mathrm{x}_*} \mathscr{E}$, denoting the projections of $\mathscr{E}$ on w_* and x_*, respectively. Due to the assumption of $\mathcal{P}$ being a convex and compact set, we only consider a part of the state-space for the analysis. We consider $x(t) \in \mathcal{X} \subset \mathbb{R}^{n_\mathrm{x}}$, with

$$\mathcal{X} = \{x = \mathrm{col}(x_\mathrm{g}, x_\mathrm{k}) \in \mathbb{R}^{n_\mathrm{x}} \mid \eta(x) \in \mathcal{P}\}. \tag{3.27}$$

For each element $w_* \in \mathscr{W}$, the subset of $\mathcal{X}$ is computed where $\dot{V}_{x_* = \Omega(w_*)}(x) < 0$, i.e.,

$$\mathcal{S}_{w_*} := \left\{x \mid x \in \mathcal{X},\, \dot{V}_{\Omega(w_*)}(x) < 0,\, w_* \in \mathscr{W}\right\}. \tag{3.28}$$

When we consider only a subset of possible reference and disturbance values $\hat{\mathscr{W}} \subseteq \mathscr{W}$, the intersection of the corresponding $\mathcal{S}_{w_*}$ sets gives

$$\hat{\mathcal{S}} := \bigcap_{w_* \in \hat{\mathscr{W}}} \mathcal{S}_{w_*}. \tag{3.29}$$

Hence, as long as $x(t) \in \hat{\mathcal{S}}$ and $w_* \in \hat{\mathscr{W}}$, the trajectory is guaranteed to converge towards a corresponding x_*. By computing the largest invariant set (reachability set)

[2] Note that no weighting filters are considered in this case.

[3] Assuming that $k_{11}, k_{12} > 0$ or $k_{11}, k_{12} < 0$.

$\mathcal{R} \subseteq \hat{\mathcal{S}}$ over inputs w, with $w(t) \in \hat{\mathscr{W}}$, the nonlinear system is asymptotically stable under any initial condition $x_0 \in \mathcal{R}$. As commented on before, analytically computing $\hat{\mathcal{S}}$ would be difficult, even for this example with only two states and a polynomial scheduling-map. Hence, the computation is performed by gridding $\hat{\mathscr{W}}$ and $\mathcal{X}$. For this example we consider[4] $\hat{\mathscr{W}} = [-2, 2] \times [-8, 8]$ and $x \in \mathcal{X} = [-3, 3] \times [-3, 3]$. Furthermore, in order to get an understanding of the range of disturbances $w \equiv w_*$ for which the system is still asymptotically stable, several (gridded) subsets of $\hat{\mathscr{W}}$ are considered given by $\hat{\mathscr{W}}_\alpha = \alpha\hat{\mathscr{W}}$ where $\alpha \in [0, 1]$. The set $\mathcal{R}$ is approximated by simulating (3.11) for a wide range of inputs with $w(t) \in \hat{\mathscr{W}}_\alpha$.

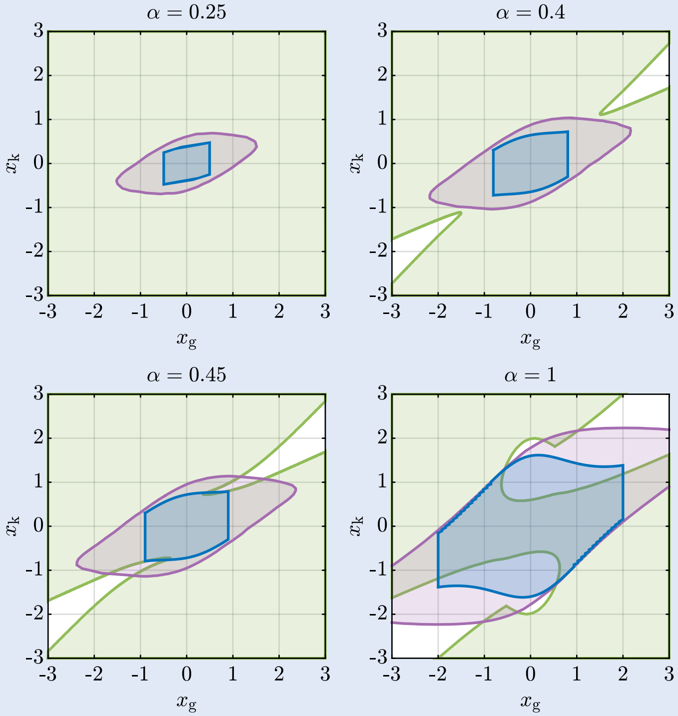


Figure 3.4: The sets $\hat{\mathcal{S}}$ (□), $\mathscr{X}$ (□), and $\mathcal{R}$ (□), considering $\hat{\mathscr{W}}_\alpha$ for different values of α.

In Figure 3.4, the resulting sets $\hat{\mathcal{S}}$, $\mathscr{X}$, and $\mathcal{R}$ are displayed under $\hat{\mathscr{W}}_\alpha$ for different values of α. In the figure, it can be observed that only for approximately $\alpha \leq 0.4$, $\mathcal{R} \subseteq \hat{\mathcal{S}}$. Hence, based on this analysis, we can only conclude asymptotic stability of all forced equilibria of the system corresponding to $w \equiv w_* \in \hat{\mathscr{W}}_\alpha$ with $\alpha \leq 0.4$. Based on the computed $\mathcal{R}$, this would correspond to $x_g(t) \in [-2.2, 2.2]$. This is in contrast to the $\mathcal{L}_2$-gain guarantee of the LPV model (3.10), which holds for all $p(t) \in [0, 9]$, corresponding to $x_g(t) \in [-3, 3]$, and all generalized disturbances

[4]Note, the specific $\mathcal{X}$ taken here is consistent with the considered scheduling set $\mathcal{P}$, as $p = \eta(x) = x_g^2$ and $p \in \mathcal{P} = [0, 9]$.

$w(t) \in \mathbb{R}^2$, hence, also for all $w(t) \in \mathscr{W}$. However, as mentioned in Section 3.3, if the scheduling-variable is not independent of the system dynamics, there will always exist regions for which asymptotic stability cannot be guaranteed, as it can be observed from Figure 3.4 for this example. Whether or not these regions will fall into the reachability set where the system is operated is unpredictable from the viewpoint of LPV analysis and controller synthesis, and hence may or may not endanger the closed-loop operation of the system.

3.5 Conclusions

The LPV framework provides attractive convex methods to analyze (asymptotic) stability and performance of nonlinear systems through embedding them in an LPV representation. This way of guaranteeing asymptotic stability is also heavily used in synthesizing controllers for nonlinear systems through the LPV framework. However, the underlying stability test only ensures asymptotic stability of the origin, which is then argued in the LPV framework to extend to all equilibrium points due to the linearity of the system. In this chapter, we have shown that this fails to hold in case the scheduling-map is a function of the state. Hence, for such LPV embeddings of nonlinear systems, asymptotic stability cannot be guaranteed for equilibrium points other than the origin. This means that when applying LPV control methods for a nonlinear system with state dependent scheduling, there are no actual rigorous guarantees when the operating condition changes, e.g., when tracking and rejection is considered.

This highlights the importance of moving towards global, equilibrium free, stability and performance concepts in order to analyze and synthesize controllers for nonlinear systems. As these concepts *are* able to give the rigorous guarantees for multiple equilibrium points or target trajectories, compared to the single equilibrium point oriented Lyapunov concept. In the following chapters, we will develop analysis and controller synthesis methods using two types of equilibrium free concepts, namely, universal shifted stability and performance in Chapter 4 and incremental stability and performance in Chapters 5 and 6.

4

Universal Shifted Dissipativity based Analysis and Control

SYSTEMATIC and computationally efficient analysis and control of nonlinear systems has become increasingly important as discussed in Chapter 1. A key ingredient to achieve this are global stability and performance concepts, as these will allow us to achieve analysis and controller synthesis that is independent of particular equilibrium points or trajectories. In this chapter we focus on universal shifted stability and performance, which is such a global concept, and which ensures stability and performance w.r.t. each forced equilibrium point of the system. Therefore, this concept is especially beneficial for control problems that require the tracking and rejection of constant signals. In this chapter, we show how universal shifted stability and performance can be analyzed through analysis of the time-differentiated dynamics of a system. It is also shown how, through the application of *Linear Parameter-Varying* (LPV) methods, this analysis can then be done computationally efficiently. Moreover, based on the analysis results, a controller synthesis method is developed, which makes use of the LPV framework, in order to ensure universal shifted stability and performance. The proposed controller design is verified in a simulation and experimental study. Moreover, we also compare the proposed controller to standard LPV controller designs, demonstrating the improved stability and performance guarantees of the proposed approach.

4.1 Introduction

The analysis and control of nonlinear systems becomes increasingly important as we discussed in Chapter 1. However, *Linear Time-Invariant* (LTI) methods are still widely used, as for LTI systems there is an extensive, systematic, and computationally efficient framework for analysis and control design that allows to ensure and shape stability and performance of the closed-loop system. While there exists a multitude of analysis and controller synthesis methods for nonlinear systems, so far, a systematic framework for analysis and controller synthesis for nonlinear systems has not been introduced like is available for LTI systems. Therefore, in this thesis, we set out to develop a systematic and computationally efficient framework for nonlinear systems to ensure and shape global stability and performance of the (closed-loop) system. While approaches such as the *Linear Parameter-Varying* (LPV) framework have aimed to achieve this, as we have shown in Chapter 3, they are unable to do so, as the stability and performance guarantees of the current state-of-the-art LPV methods are dependent on the choice of equilibrium point.

While standard stability is sufficient if we only want to analyze stability and/or dissipativity w.r.t. a single (forced) equilibrium point of the system, it becomes troublesome to use if we want to ensure stability/dissipativity w.r.t. all (forced) equilibrium points of the system. This is especially relevant in cases when one wants to track constant references and/or reject constant (unknown) disturbances. Consequently, global stability and performance concepts are highly important in order to arrive at a systematic framework for nonlinear analysis and controller synthesis. In the literature, notions such as so-called *shifted* stability/dissipativity (Van der Schaft 2017) and *equilibrium independent* stability/dissipativity (Jayawardhana 2006; Hines et al. 2011; Simpson-Porco 2019) have been introduced, whereby stability/dissipativity w.r.t. a particular (non-zero) (forced) equilibrium point is ensured, or w.r.t. all forced equilibrium points of the system, respectively. In literature, equilibrium independent dissipativity has also been referred to as *constant incremental dissipativity* (Jayawardhana 2006). In order to not confuse this notion of stability/dissipativity with other notions we will discuss later in this thesis, we will refer to this notion as *universal shifted* stability/dissipativity.

In literature, the analysis of non-zero equilibrium points of *Continuous-Time* (CT) nonlinear systems has also been investigated through its time-differentiated dynamics. In gain-scheduling and the LPV framework, this has it roots in the so-called velocity-based scheduling technique (Kaminer et al. 1995; Leith and Leithead 1998b; Leith and Leithead 1999; Tóth 2010). However, these results are based on the argument that locally around an equilibrium point, the dynamics of velocity form coincide with the linearization of the nonlinear system at the equilibrium point (see e.g. (1.2)). Consequently, these results are only able to provide local guarantees in a neighborhood around the equilibrium points, which severely hampers their viability. From a nonlinear perspective, the time-differentiated dynamics also connect to the so-called Krasovskii method for the construction of a Lyapunov function in order to show stability (Khalil 2002). This has also been explored more recently in connection to non-zero equilibrium point stability and/or performance properties, in (Kosaraju et al. 2019; Kawano et al. 2021) and (Schweidel and Arcak

2022). The work in (Kawano et al. 2021) uses the time-differentiated properties to ensure universal shifted stability and universal shifted passivity of the system. However, they restrict the output map of nonlinear state-space representation to a particular form and use properties of the passivity supply function in order to prove their implications. Due to the reliance on properties of passivity for these results, they have not been extended to other (quadratic) performance notions. The resulting controller design also requires manual design choices to be made by the user, requiring expert knowledge. The work in (Schweidel and Arcak 2022) strictly focusses on the analysis of the network interconnection of systems, in which they show that (unique) equilibrium points of interconnections of velocity dissipative systems are stable. However, no connection to performance is made in this work or how these results could be used for controller design.

Consequently, as the main contribution of this chapter, we will show how analysis of the time-differentiated dynamics can be used in order imply both universal shifted stability and universal shifted performance of CT nonlinear systems. Moreover, we will also show how the analysis of the time-differentiated dynamics can be performed through the LPV framework in order to systematically and computationally efficiently analyze universal shifted stability and performance. Finally, as an additional contribution, we will present a procedure to systemically and computationally efficiently synthesize controllers in order to ensure universal shifted stability and shape universal shifted performance. With these contributions, we build up key parts of our intended framework for systematic and computationally efficient analysis and controller design for nonlinear systems to ensure global stability and performance guarantees, which is the objective of this thesis. Later, in Chapter 8, we also extend the results of this chapter to *Discrete-Time* (DT) nonlinear systems.

In Section 4.2, we will introduce the concept of universal shifted stability, performance, and dissipativity. Next, in Section 4.3, we introduce velocity based analysis, i.e., analysis of the time-differentiated dynamics of the system, and show how it can be used to imply universal shifted stability and performance. In Section 4.4, we then show how velocity based analysis can be performed through the LPV framework. Section 4.5, shows how the analysis results in the previous section can then be used in order to synthesize controllers to ensure universal shifted stability and performance. The performance and properties of the developed universal shifted controller synthesis method are demonstrated in Section 4.6, first in a simulation study and then via experiments on a unbalanced disk system. Finally, in Section 4.7, conclusions on the developed results and capabilities of the established synthesis and analysis toolchain are drawn.

4.2 Universal Shifted Stability and Performance

4.2.1 Nonlinear system

Similar to Section 2.2, we consider CT nonlinear dynamical systems given by

$$\dot{x}(t) = f(x(t), w(t)); \tag{4.1a}$$
$$z(t) = h(x(t), w(t)); \tag{4.1b}$$

where $t \in \mathbb{R}_0^+$ is time, $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$ is the state with initial condition $x(0) = x_0 \in \mathbb{R}^{n_\mathrm{x}}$, $w(t) \in \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{w}}$ is the input of the system, and $z(t) \in \mathcal{Z} \subseteq \mathbb{R}^{n_\mathrm{z}}$ is the output of the system. Moreover, the functions $f : \mathcal{X} \times \mathcal{W} \to \mathbb{R}^{n_\mathrm{x}}$ and $h : \mathcal{X} \times \mathcal{W} \to \mathcal{Z}$ are assumed to be in $\mathcal{C}_1$, i.e., $f, h \in \mathcal{C}_1$. We define the set of solutions of (4.1) as

$$\mathfrak{B} := \{(x, w, z) \in (\mathcal{X} \times \mathcal{W} \times \mathcal{Z})^{\mathbb{R}_0^+} \mid x \in \mathcal{C}_1 \text{ and } (x, w, z) \text{ satisfy (4.1)}\}, \tag{4.2}$$

and the behavior of (4.1) for a specific input trajectory $\bar{w} \in \mathcal{W}^{\mathbb{R}_0^+}$, by

$$\mathfrak{B}_\mathrm{w}(\bar{w}) := \{(x, w, z) \in \mathfrak{B} \mid w = \bar{w} \in \mathcal{W}^{\mathbb{R}_0^+}\}. \tag{4.3}$$

For the nonlinear system given (4.1), the equilibrium points satisfy

$$0 = f(x_*, w_*); \tag{4.4a}$$
$$z_* = h(x_*, w_*); \tag{4.4b}$$

where $x_* \in \mathcal{X}$, $w_* \in \mathcal{W}$ and $z_* \in \mathcal{Z}$. The set of equilibrium points is then defined as

$$\mathscr{E} := \{(x_*, w_*, z_*) \in \mathcal{X} \times \mathcal{W} \times \mathcal{Z} \mid (x_*, w_*, z_*) \text{ satisfy (4.4)}\}. \tag{4.5}$$

Define $\mathscr{X} := \pi_{\mathrm{x}_*} \mathscr{E}$, $\mathscr{W} := \pi_{\mathrm{w}_*} \mathscr{E}$, and $\mathscr{Z} := \pi_{\mathrm{z}_*} \mathscr{E}$. Throughout this chapter, we make the following assumption:

Assumption 4.1. *For the nonlinear system given by (4.1) with equilibrium points $(x_*, w_*, z_*) \in \mathscr{E}$, we assume that there exists a bijective map $\kappa : \mathscr{W} \to \mathscr{X}$ such that $x_* = \kappa(w_*)$, for all $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$. This means that for each $w_* \in \mathscr{W}$ there is a unique corresponding $x_* \in \mathscr{X}$, and vice versa, for each $x_* \in \mathscr{X}$ there is a unique corresponding $w_* \in \mathscr{W}$.*

This assumption on the uniqueness of the equilibrium points is taken for convenience, in order to not overcomplicate the discussion.

4.2.2 Universal shifted stability

As mentioned in Section 4.1, universal shifted stability is stability w.r.t. all forced equilibrium points of the system. Defined more formally:

Definition 4.1 (Universal shifted stability)**.** *The nonlinear system given by (4.1) is universally shifted (asymptotically) stable, if it is (asymptotically) stable at each $x_* \in \mathscr{X}$ with corresponding $w_* \in \mathscr{W}$, i.e., $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$.*

Note that this definition is nothing more than the 'standard' stability definition, i.e., Definition 2.3, required to hold for each equilibrium point of the system. In literature, the set of all asymptotically stable equilibrium points is sometimes also referred as the positive limit set or ω-limit set (Haddad and Chellaboina 2008; Mei and Bullo 2017). We can extend the standard Lyapunov condition to analyze stability of an equilibrium point, see Theorem 2.2, to a condition to analyze universal shifted stability.

Theorem 4.1 (Universal shifted Lyapunov stability)**.** *The nonlinear system given by (4.1) is universally shifted stable, if there exists a function* $V_{\mathrm{s}} : \mathcal{X} \times \mathscr{W} \to \mathbb{R}_0^+$ *with* $V_{\mathrm{s}}(\cdot, w_*) \in \mathcal{C}_1$ *and* $V_{\mathrm{s}}(\cdot, w_*) \in \mathcal{Q}_{x_*}$ *for every* $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*}\mathscr{E}$*, such that for every* $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*}\mathscr{E}$*, it holds that*

$$\frac{\partial}{\partial t} V_{\mathrm{s}}(x(t), w_*) \leq 0, \tag{4.6}$$

for all $t \in \mathbb{R}_0^+$ *and* $x \in \pi_{\mathrm{x}}\mathfrak{B}_{\mathrm{w}}(w \equiv w_*)$*. If (4.6) holds, but with strict inequality except when* $x(t) = x_*$*, then the system is universally shifted asymptotically stable.*

Proof. See Appendix B.2. □

4.2.3 Universal shifted dissipativity

Classical dissipativity as developed in (Willems 1972), also discussed Section 2.4, is an important concept for stability and performance analysis of nonlinear systems. The theory has also resulted in various controller design methods in order to ensure dissipativity of the (closed-loop) system, see e.g. (Van der Schaft 2017; Brogliato et al. 2020). Nonetheless, as also discussed in Chapter 1, a downside of classical dissipativity is that it only considers a single point for neutral storage, which is often the origin. This means that only stability and performance guarantees w.r.t. the origin of the system can be given, which is limiting in cases where one wants to ensure stability and performance w.r.t. more or even all equilibrium points of the system.

Universal shifted dissipativity is a stronger dissipativity notion aimed at overcoming this shortcoming of classical dissipativity. Namely, universal shifted dissipativity considers dissipativity w.r.t. to each (forced) equilibrium point of the system. In literature, this has also been referred to as equilibrium independent dissipativity (Hines et al. 2011; Simpson-Porco 2019) or constant incremental dissipativity (Jayawardhana 2006). This allows for analyzing the (dissipated) energy flow between trajectories and equilibrium points of the system.

More concretely, we take the following definition for universal shifted dissipativity, adopted from (Simpson-Porco 2019):

Definition 4.2 (Universal shifted dissipativity)**.** *The nonlinear system given by (4.1) is universally shifted dissipative w.r.t. the supply function* $s_{\mathrm{s}} : \mathcal{W} \times \mathscr{W} \times \mathcal{Z} \times \mathscr{Z}$*,* *if there exists a storage function* $\mathcal{V}_{\mathrm{s}} : \mathcal{X} \times \mathscr{W} \to \mathbb{R}_0^+$ *with* $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{C}_0$ *and* $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{Q}_{x_*}$ *for every* $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*}\mathscr{E}$*, such that*

$$\mathcal{V}_{\mathrm{s}}(x(t_1), w_*) - \mathcal{V}_{\mathrm{s}}(x(t_0), w_*) \leq \int_{t_0}^{t_1} s_{\mathrm{s}}(w(t), w_*, z(t), z_*)\, dt, \tag{4.7}$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ *with* $t_1 \geq t_0$ *and* $(x, w, z) \in \mathfrak{B}$*.*

Based on this definition, we can also formulate the following condition for universal shifted dissipativity of a system given by (4.1):

Lemma 4.1 (Condition for universal shifted dissipativity)**.** *If there exists a storage function* $\mathcal{V}_{\mathrm{s}}$ *with* $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{C}_1$ *and* $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{Q}_{x_*}$ *for every* $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*}\mathscr{E}$*, such that*

$$\nabla_x \mathcal{V}_{\mathrm{s}}(x, w_*) f(x, w) \leq s_{\mathrm{s}}(w, w_*, h(x, w), z_*), \tag{4.8}$$

for all $x \in \mathcal{X}$ *and* $w \in \mathcal{W}$*, then, the nonlinear system given by (4.1) is universally shifted dissipative w.r.t. the supply function* s_{s}*.*

Proof. See Appendix B.2. □

For classical dissipativity, we can connect to many well-known performance metrics, such as the $\mathcal{L}_2$-gain and passivity, by considering (Q, S, R) supply functions, see Section 2.4.3. Similarly, also for universal shifted dissipativity, we will mostly focus our attention on quadratic supply functions s_{s} of the form

$$s_{\mathrm{s}}(w, w_*, z, z_*) = \begin{bmatrix} w - w_* \\ z - z_* \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w - w_* \\ z - z_* \end{bmatrix}, \tag{4.9}$$

where $Q \in \mathbb{S}^{n_{\mathrm{w}}}$, $R \in \mathbb{S}^{n_{\mathrm{z}}}$, and $S \in \mathbb{R}^{n_{\mathrm{w}} \times n_{\mathrm{z}}}$. If a system given by (4.1) is universally shifted dissipative w.r.t. to a supply function of the form (4.9), we will also refer to it being universally shifted (Q, S, R) dissipative. Like how classical (Q, S, R) dissipativity implies quadratic performance notions of the system (see Section 2.4.3), such as induced $\mathcal{L}_2$-gain and passivity, we will also show how universal shifted (Q, S, R) dissipativity implies universal shifted versions of these quadratic performance notions.

As universal shifted dissipativity is a stronger notion than classical dissipativity, we can easily show the following result:

Theorem 4.2 (Induced classical dissipativity)**.** *If a nonlinear system given by (4.1) is universally shifted* (Q, S, R) *dissipative and* $(0, 0, 0) \in \mathscr{E}$*, then, the system is also classically* (Q, S, R) *dissipative for the same tuple* (Q, S, R)*.*

Proof. See Appendix B.2. □

4.2.4 Induced universal shifted performance

As aforementioned, classical (Q, S, R) dissipativity links back to many popular performance metrics such as the $\mathcal{L}_2$-gain and passivity. Next, we will show how universal shifted (Q, S, R) dissipativity connects to universal shifted versions of these performance metrics. Before giving this results, we will first give a definition for the universal shifted extension of the $\mathcal{L}_p$-$\mathcal{L}_q$-gain (see also Definition 2.11):

Definition 4.3 (Universal shifted $\mathcal{L}_p$-$\mathcal{L}_q$-gain)**.** *A nonlinear system given by (4.1) is said to have a finite universal shifted $\mathcal{L}_p$-$\mathcal{L}_q$-gain, if there is a finite $\gamma \geq 0$ and function $\zeta_\mathrm{s} : \mathcal{X} \times \mathscr{X} \to \mathbb{R}$ such that for every $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$ it holds that*[1]

$$\|z - z_*\|_{q,T} \leq \gamma \|w - w_*\|_{p,T} + \zeta_\mathrm{s}(x_0, x_*), \tag{4.10}$$

for all $T \geq 0$ and $(x, w, z) \in \mathfrak{B}$ with[2] *$w \in \mathcal{L}_{pe}$. The induced universal shifted $\mathcal{L}_p$-$\mathcal{L}_q$-gain of (4.1), denoted as $\mathcal{L}_{\mathrm{s}p}$-$\mathcal{L}_{\mathrm{s}q}$-gain, is the infimum of γ such that (4.10) still holds. If $p = q$, we will refer to this as the (induced) universal shifted $\mathcal{L}_p$-gain, denoted as $\mathcal{L}_{\mathrm{s}p}$-gain.*

Using this definition, we directly have universal shifted extensions of the well-known $\mathcal{L}_2$-gain, $\mathcal{L}_\infty$-gain, and $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain through the $\mathcal{L}_{\mathrm{s}2}$-gain, $\mathcal{L}_{\mathrm{s}\infty}$-gain, and $\mathcal{L}_{\mathrm{s}2}$-$\mathcal{L}_{\mathrm{s}\infty}$-gain, respectively. We can then connect universal shifted (Q, S, R) dissipativity to $\mathcal{L}_{\mathrm{s}2}$-gain performance through the following lemma:

Lemma 4.2 ($\mathcal{L}_{\mathrm{s}2}$-gain based on universal shifted dissipativity)**.** *If the nonlinear system given by (4.1) is universally shifted (Q, S, R) dissipative with $(Q, S, R) = (\gamma^2 I, 0, -I)$, then, the $\mathcal{L}_{\mathrm{s}2}$-gain of the system is bounded by γ.*

Proof. See Appendix B.2. □

For universal shifted passivity, we have adopted the following definition inspired by the shifted passivity definition in (Van der Schaft 2017):

Definition 4.4. *A nonlinear system given by (4.1) is said to be universally shifted passive, if it is universally shifted dissipative w.r.t. to the supply function*

$$s_\mathrm{s}(w, w_*, z, z_*) = (w - w_*)^\top (z - z_*) + (z - z_*)^\top (w - w_*). \tag{4.11}$$

This also corresponds to the system being universally shifted (Q, S, R) dissipative w.r.t. to the tuple $(Q, S, R) = (0, I, 0)$.

Through this definition, we directly link universal shifted passivity to universal shifted dissipativity. Note that if $(0, 0, 0) \in \mathscr{E}$, then, the universal shifted performance notions also imply their standard counterparts in terms of the $\mathcal{L}_p$-$\mathcal{L}_q$-gain (see Definition 2.11) and passivity (see Definition 2.12).

[1] Note that w and z are signals, while w_* and z_* are constant values, so $(w - w_*)$ and $(z - z_*)$ are understood as the signals $(t \mapsto (w(t) - w_*))$ and $(t \mapsto (z(t) - z_*))$, respectively.

[2] Note that this implies $(w - w_*) \in \mathcal{L}_{pe}$.

4.2.5 Induced universal shifted stability

Classical dissipativity of a nonlinear system implies stability (at the origin) of the system if the supply function satisfies a negativity condition, see Theorem 2.3. We will show how a similar condition on the universal shifted supply function s_s can be formulated in order to link universal shifted dissipativity and universal shifted stability.

Theorem 4.3 (Universal shifted stability from universal shifted dissipativity)**.** *If the nonlinear system given by (4.1) is universally shifted dissipative under a storage function $\mathcal{V}_\mathrm{s}$ with $\mathcal{V}_\mathrm{s}(\cdot, w_*) \in \mathcal{C}_1$ for all $w_* \in \mathscr{W}$, w.r.t. a supply function s_s that satisfies for every $(x_*, w_*, z_*) \in \mathscr{E}$ that*

$$s_\mathrm{s}(w_*, w_*, z, z_*) \leq 0, \tag{4.12}$$

for all $z \in \mathcal{Z}$, then, the nonlinear system is universally shifted stable. If the supply function satisfies (4.12), but with strict inequality for all $z \neq z_*$, and the system is observable (see Definition 2.2), then the nonlinear system is universally shifted asymptotically stable.

Proof. See Appendix B.2. □

Note that similar to classical (Q, S, R) dissipativity, the condition in Theorem 4.3 is satisfied for universal shifted (Q, S, R) supply functions for which $R \preceq 0$ (with $R \prec 0$ implying universal shifted asymptotic stability, such as is the case for the $\mathcal{L}_{\mathrm{s}2}$-gain).

As previously mentioned, in this chapter we are interested in efficiently analyzing and ensuring universal shifted stability and performance of nonlinear systems. We have seen in this section and in Section 4.2.4 how universal shifted dissipativity allows us to simultaneously analyze both universal shifted stability and performance. While Definition 4.2 and Lemma 4.1 give us conditions to analyze universal shifted dissipativity, these conditions require to hold for all state and input trajectories/values and for every equilibrium point $(x_*, w_*, z_*) \in \mathscr{E}$. Hence, checking universal shifted dissipativity directly through these conditions can be a difficult if not an infeasible task, even when assuming a particular class of storage functions $\mathcal{V}_\mathrm{s}$. Next, as one of the main results of this chapter, we will show how analysis of the time-differentiated dynamics of the system will allow us to simplify universal shifted stability and performance analysis of nonlinear systems.

4.3 Velocity based Analysis

4.3.1 The velocity form

Let us first define the following restriction of the solution set of (4.1) by [3]

$$\mathfrak{B}_{\mathrm{c}} := \{(x, w, z) \in \mathfrak{B} \mid x \in \mathcal{C}_2,\ w, z \in \mathcal{C}_1\}, \tag{4.13}$$

i.e., the solutions in $\mathfrak{B}$ that are differentiable. We also define $\mathfrak{B}_{\mathrm{c,w}}(w) := \mathfrak{B}_{\mathrm{w}}(w) \cap \mathfrak{B}_{\mathrm{c}}$ for a $w \in \mathcal{W}^{\mathbb{R}_0^+}$. Furthermore, define the operator ∂ for these sets such that

$$\partial\mathfrak{B}_{\mathrm{c}} = \big\{(\dot{x}, \dot{w}, \dot{z}) \in (\mathbb{R}^{n_{\mathrm{x}}} \times \mathbb{R}^{n_{\mathrm{w}}} \times \mathbb{R}^{n_{\mathrm{z}}})^{\mathbb{R}_0^+} \mid \dot{x}(t) = \tfrac{d}{dt}x(t),$$
$$\dot{w}(t) = \tfrac{d}{dt}w(t), \dot{z}(t) = \tfrac{d}{dt}z(t), \forall t \in \mathbb{R}_0^+, (x, w, z) \in \mathfrak{B}_{\mathrm{c}}\big\}. \tag{4.14}$$

For a nonlinear system given by (4.1), we call the time-differentiated dynamics of (4.1) to be the velocity form of the system.

Definition 4.5 (Velocity form)**.** *For a nonlinear system given by (4.1), the velocity form is*

$$\ddot{x}(t) = A_{\mathrm{v}}(x(t), w(t))\dot{x}(t) + B_{\mathrm{v}}(x(t), w(t))\dot{w}(t); \tag{4.15a}$$
$$\dot{z}(t) = C_{\mathrm{v}}(x(t), w(t))\dot{x}(t) + D_{\mathrm{v}}(x(t), w(t))\dot{w}(t); \tag{4.15b}$$

where $A_{\mathrm{v}} = \frac{\partial f}{\partial x}$, $B_{\mathrm{v}} = \frac{\partial f}{\partial w}$, $C_{\mathrm{v}} = \frac{\partial h}{\partial x}$, $D_{\mathrm{v}} = \frac{\partial h}{\partial w}$, *and* $(x, w, z) \in \mathfrak{B}_{\mathrm{c}}$.

The solution set of (4.15) is given by $\mathfrak{B}_{\mathrm{v}} := \partial\mathfrak{B}_{\mathrm{c}}$, and we define $\mathfrak{B}_{\mathrm{v,w}}(w) := \partial\mathfrak{B}_{\mathrm{c,w}}(w)$ for a $w \in \mathcal{W}^{\mathbb{R}_0^+}$.

In this context, the velocity form (4.15) is nothing more than the dynamics of the time-differentiated (i.e., velocity) dynamics of our original system given by (4.1). We will refer to the original system given by (4.1) as the primal form of the system. As aforementioned, the analysis of the time-differentiated dynamics has been investigated in connection to gain scheduling (Leith and Leithead 1998b; Leith and Leithead 1999), the construction of (local) LPV embeddings (Tóth 2010), and more recently also in connection with (universal) shifted stability/dissipativity (Kosaraju et al. 2019; Kawano et al. 2021; Schweidel and Arcak 2022). However, as previously mentioned, the existing works on this topic are limited as they are only able to provide local guarantees, assume severe restrictions to the output dynamics, and/or only focus on stability. Next, we will show how the velocity form can be used to provide stability and performance guarantees in the form of universal shifted stability and performance.

4.3.2 Velocity dissipativity

Before connecting analysis of the velocity form to universal shifted analysis of the primal form, we will first define the following dissipativity notion based on the velocity form:

[3] As solutions are defined on $\mathbb{R}_0^+$, we assume they are also continuously differentiable at $t = 0$.

Definition 4.6 (Velocity dissipativity)**.** *The nonlinear system given by (4.1) is velocity dissipative w.r.t. the supply function* s_{v}*, if there exists a storage function* $\mathcal{V}_{\mathrm{v}} : \mathbb{R}^{n_{\mathrm{x}}} \to \mathbb{R}_0^+$ *with* $\mathcal{V}_{\mathrm{v}} \in \mathcal{C}_1$ *and* $\mathcal{V}_{\mathrm{v}} \in \mathcal{Q}_0$*, such that, for all* $t_0, t_1 \in \mathbb{R}_0^+$ *with* $t_1 \geq t_0$*,*

$$\mathcal{V}_{\mathrm{v}}(\dot{x}(t_1)) - \mathcal{V}_{\mathrm{v}}(\dot{x}(t_0)) \leq \int_{t_0}^{t_1} s_{\mathrm{v}}(\dot{w}(t), \dot{z}(t))\, dt, \tag{4.16}$$

for all $(\dot{x}, \dot{w}, \dot{z}) \in \mathfrak{B}_{\mathrm{v}}$*.*

Note that in this sense, velocity dissipativity can be seen as 'classical dissipativity' (see Definition 2.13) of the velocity form (4.15) of the system. In literature, similar notions have also been introduced, such as *Krasovskii passivity* (Kosaraju et al. 2019; Kawano et al. 2021) and *delta dissipativity* (Schweidel and Arcak 2022), which in these works is also connected to non-zero equilibrium point properties of the original system.

Similarly to the universal shifted (Q, S, R) supply function, also for velocity dissipativity, we focus on quadratic supply functions of the form

$$s_{\mathrm{v}}(\dot{w}, \dot{z}) = \begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} \dot{w} \\ \dot{z} \end{bmatrix}, \tag{4.17}$$

where again $Q \in \mathbb{S}^{n_{\mathrm{w}}}$, $S \in \mathbb{R}^{n_{\mathrm{w}} \times n_{\mathrm{z}}}$, and $R \in \mathbb{S}^{n_{\mathrm{z}}}$. If a system is velocity dissipative w.r.t. a supply function of the form (4.17), we will refer to it being *velocity* (Q, S, R) *dissipative.*

Similar to the work in (Kawano et al. 2021; Schweidel and Arcak 2022), based on our definition of velocity dissipativity, we can derive the following condition:

Lemma 4.3 (Condition for velocity dissipativity)**.** *If there exists a storage function* $\mathcal{V}_{\mathrm{v}} : \mathbb{R}^{n_{\mathrm{x}}} \to \mathbb{R}_0^+$ *with* $\mathcal{V}_{\mathrm{v}} \in \mathcal{C}_1$ *and* $\mathcal{V}_{\mathrm{v}} \in \mathcal{Q}_0$*, such that, for all values* $w_{\mathrm{v}} \in \mathbb{R}^{n_{\mathrm{w}}}$*,* $x \in \mathcal{X}$*, and* $w \in \mathcal{W}$*,*

$$\begin{aligned} &\nabla \mathcal{V}_{\mathrm{v}}(f(x, w))\, (A_{\mathrm{v}}(x, w) f(x, w) + B_{\mathrm{v}}(x, w) w_{\mathrm{v}}) \leq \\ &\qquad\qquad s_{\mathrm{v}}(w_{\mathrm{v}}, C_{\mathrm{v}}(x, w) f(x, w) + D_{\mathrm{v}}(x, w) w_{\mathrm{v}}), \end{aligned} \tag{4.18}$$

then the nonlinear system given by (4.1) is velocity dissipative w.r.t. the supply function s_{v}*.*

Proof. See Appendix B.2. □

Using the results of Lemma 4.3, for a (Q, S, R) supply function (4.17) and quadratic storage function:

$$\mathcal{V}_{\mathrm{v}}(\dot{x}) = \dot{x}^\top M \dot{x}, \tag{4.19}$$

where $M \in \mathbb{S}^{n_{\mathrm{x}}}$ with $M \succ 0$, we can derive the following sufficient condition for velocity dissipativity:

Theorem 4.4 (Velocity (Q, S, R) dissipativity condition)**.** *The system given by (4.1) is velocity (Q, S, R) dissipative, if there exists an $M \in \mathbb{S}^{n_\mathrm{x}}$ with $M \succ 0$, such that, for all $(x, w) \in \mathcal{X} \times \mathcal{W}$,*

$$(\star)^\top \begin{bmatrix} 0 & M \\ \star & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\mathrm{v}(x, w) & B_\mathrm{v}(x, w) \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\mathrm{v}(x, w) & D_\mathrm{v}(x, w) \end{bmatrix} \preceq 0. \tag{4.20}$$

Proof. See Appendix B.2. □

Note, what is compelling about the condition given in Theorem 4.4 is that it corresponds to a feasibility check of an infinite dimensional set of *Linear Matrix Inequalities* (LMIs), as for a fixed $(x, w) \in \mathcal{X} \times \mathcal{W}$, (4.20) becomes an LMI. Later, in Section 4.4, we will see how we can use tools from the LPV framework to reduce this infinite dimensional set of LMIs to a finite dimensional set, which can computationally efficiently be verified. This will then gives us computationally efficient tools to analyze velocity (Q, S, R) dissipativity of a system.

4.3.3 Induced universal shifted stability

Having introduced the velocity form and velocity dissipativity, we will first present results on how these notions connect to universal shifted stability of the system.

Let us first introduce the set $\mathfrak{B}_{\mathrm{v},\mathscr{W}} := \bigcup_{w_* \in \mathscr{W}} \mathfrak{B}_{\mathrm{v,w}}(w \equiv w_*)$, i.e., the behavior of the velocity form for which the input is $w(t) = w_* \in \mathscr{W}$, and hence $\dot{w}(t) = 0$, for all $t \in \mathbb{R}_0^+$.

Theorem 4.5 (Implied universal shifted stability)**.** *The nonlinear system given by (4.1), with solutions in $\mathfrak{B}_\mathrm{c}$, is universally shifted stable, if there exists a function $V_\mathrm{v} : \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}_0^+$ with $V_\mathrm{v} \in \mathcal{C}_1$ and $\mathcal{V}_\mathrm{v} \in \mathcal{Q}_0$, such that*

$$\frac{d}{dt} V_\mathrm{v}(\dot{x}(t)) \leq 0, \tag{4.21}$$

for all $t \in \mathbb{R}_0^+$ and $\dot{x} \in \pi_{\dot{\mathrm{x}}} \mathfrak{B}_{\mathrm{v},\mathscr{W}}$. If (4.21) holds, but with strict inequality except when $\dot{x}(t) = 0$, then the system is universally shifted asymptotically stable.

Proof. See Appendix B.2. □

The proof of Theorem 4.5 relies on a construction of a universally shifted Lyapunov function based on V_v. This construction is based on the so-called Krasovskii method for 'standard' stability (Khalil 2002), and construction of the storage/Lyapunov function in a similar manner is also used for the results in (Kawano et al. 2021) and (Schweidel and Arcak 2022).

Note that the condition in Theorem 4.5 can be interpreted as the velocity form being stable (w.r.t. $\dot{x} = 0$), similar to how the condition for velocity dissipativity could be seen as the velocity form being classically dissipative. Note that conceptually this is also intuitive, as $\dot{x}(t) = 0$ corresponds to an equilibrium point of the system (see (4.4)). Hence, (in the asymptotic stability case) for $t \to \infty$, we have that $\dot{x}(t) \to 0$, meaning the state approaches an equilibrium point, i.e., $x(t) \to x_* \in \mathscr{X}$.

Through the result of Theorem 4.5, we can also connect velocity dissipativity of a system to universal shifted stability. Similar to how universal shifted dissipativity is connected to universal shifted stability through restrictions on the supply function s_s, we can do the same by restricting the velocity supply function s_v.

Theorem 4.6 (Universal shifted stability from velocity dissipativity)**.** *Assume the nonlinear system given by (4.1) is velocity dissipative under a storage function $\mathcal{V}_\mathrm{v} \in \mathcal{C}_1$ w.r.t. a supply function s_v that satisfies*

$$s_\mathrm{v}(0, z_\mathrm{v}) \le 0, \tag{4.22}$$

for all $z_\mathrm{v} \in \mathbb{R}^{n_\mathrm{z}}$, then, the nonlinear system is universally shifted stable. If the supply function satisfies (4.22), but with strict inequality when $z_\mathrm{v} \neq 0$, and the system is observable, then the nonlinear system is universally shifted asymptotically stable.

Proof. See Appendix B.2. □

Again, in a similar fashion as for classical (Q, S, R) dissipativity and universal shifted (Q, S, R) dissipativity, also for velocity (Q, S, R) dissipativity the condition in Theorem 4.6 corresponds to $R \preceq 0$.

4.3.4 Induced universal shifted dissipativity

In the previous section, we have seen how velocity (Q, S, R) dissipativity implies universal shifted stability. Next, we are interested if velocity (Q, S, R) dissipativity also implies universal (Q, S, R) dissipativity. Therefore, we formulate the following proposition:

Proposition 4.1 (Induced (Q, S, R) universal shifted dissipativity)**.** *If a nonlinear system given by (4.1) is velocity (Q, S, R) dissipative, then, it is also universally shifted (Q, S, R) dissipative for the same tuple (Q, S, R).*

While we have not able been to proof Proposition 4.1 completely, we will next present results which link velocity (Q, S, R) dissipativity and universal shifted (Q, S, R) dissipativity under some assumptions.

Instead of considering nonlinear systems of the form (4.1), we restrict ourselves to nonlinear systems that can be represented as

$$\dot{x}(t) = f(x(t)) + Bw(t); \tag{4.23a}$$
$$y(t) = Cx(t). \tag{4.23b}$$

This restricted class of nonlinear systems is only assumed for the results that follow in this subsection. Note that at the cost of increasing state dimension, e.g., by appending the system with appropriate input-output filters (see Appendix C.2.1), we can always transform nonlinear systems given by (4.1) to the form (4.23). Besides considering systems of the form (4.23), we will assume in this subsection that $\mathcal{X}$, i.e., the state set, is convex and compact.

For the nonlinear system given by (4.23), the equilibrium points $(x_*, w_*, z_*) \in \mathscr{E}$ satisfy

$$0 = f(x_*) + Bw_*; \tag{4.24a}$$
$$y_* = Cx_*; \tag{4.24b}$$

and its velocity form is given by

$$\ddot{x}(t) = A_\mathrm{v}(x(t))\dot{x}(t) + B\dot{w}(t); \tag{4.25a}$$
$$\dot{z}(t) = C\dot{x}(t). \tag{4.25b}$$

Under these considerations, we will next connect velocity (Q, S, R) dissipativity for (Q, S, R) tuples for which $S = 0$, $Q \succeq 0$, and $R \preceq 0$ to universal shifted performance notions that can be characterized by a similar universal shifted (Q, S, R) supply function. Before presenting these results, we will first introduce the following assumptions:

Assumption 4.2. *For the nonlinear system given by (4.23) assume that $CB = 0$.*

Note that systems of the form (4.1) can be converted to the form (4.23), satisfying Assumption 4.2, by connecting appropriate filters, see Appendix C.2.1.

Assumption 4.3. *Given a matrix $T \in \mathbb{S}^{n_\mathrm{z}}$ with $T \preceq 0$, assume that there exists an $\alpha \in \mathbb{R}^+$ such that for all $x_* \in \mathscr{X}$ and $x \in \mathcal{X}$*

$$(x - x_*)^\top \bar{A}(x, x_*)^\top C^\top T C \bar{A}(x, x_*)(x - x_*) \leq \alpha^{-1}(x - x_*)^\top C^\top T C(x - x_*), \tag{4.26}$$

where $\bar{A}(x, x_*) = \int_0^1 A_\mathrm{v}(x_* + \lambda(x - x_*))\, d\lambda$.

In case A_v is bounded, there will always exists an α such that Assumption 4.3 holds, as $\mathcal{X}$ is considered compact.

Assumption 4.4. *For a given $(x_*, w_*, z_*) \in \mathscr{E}$, assume that w is generated by the exosystem*

$$\dot{w}(t) = A_\mathrm{w}(w(t) - w_*), \tag{4.27}$$

where $A_\mathrm{w} \in \mathbb{R}^{n_\mathrm{w} \times n_\mathrm{w}}$ is Hurwitz and $\|A_\mathrm{w}\| \leq \beta$. Define the corresponding behavior as

$$\mathfrak{W} := \left\{ w \in \mathcal{W}^{\mathbb{R}_0^+} \mid \dot{w} \in \mathcal{C}_1,\ w \text{ satisfies } (4.27) \right\}. \tag{4.28}$$

Note that constant and decaying (towards w_*) disturbances satisfy the behavior considered in Assumption 4.4.

Under these assumptions, we can formulate the following result to link velocity dissipativity to universal shifted performance:

Theorem 4.7 (Universal shifted performance from velocity dissipativity)**.** *If a nonlinear system given by (4.23) is velocity (Q, S, R) dissipative for the (Q, S, R) tuple where $S = 0$, $Q \succeq 0$, and $R \preceq 0$, where R satisfies Assumption 4.3, Assumptions 4.2 and 4.4 hold for every $(x_*, w_*, z_*) \in \mathscr{E}$, and $\dot{x}(0) = 0$, then for every $(x_*, w_*, z_*) \in \mathscr{E}$, it holds that*

$$\int_0^T \beta^2(\star)^\top Q(w(t) - w_*) + \alpha^{-1}(\star)^\top R(z(t) - z_*)\, dt > 0, \tag{4.29}$$

for all $T > 0$ and $(w, z) \in \pi_{\mathrm{w,z}} \mathfrak{B}_{\mathrm{c}}$ with $w \in \mathfrak{W}$.

Proof. See Appendix B.2. □

Note that the assumption that $\dot{x}(0) = 0$ in Theorem 4.7 does *not* mean we stay in an equilibrium point, as we have an input $w \in \mathfrak{W}$ which acts on the system.

Applying the result of Theorem 4.7 to the (Q, S, R) tuple $(Q, S, R) = (\gamma^2 I, 0, -I)$, corresponding to the (universal shifted) $\mathcal{L}_2$-gain, we obtain the following corollary:

Corollary 4.1 (Bounded $\mathcal{L}_{\mathrm{s}2}$-gain from velocity dissipativity)**.** *If a nonlinear system given by (4.23) is velocity (Q, S, R) dissipative for $(Q, S, R) = (\gamma^2 I, 0, -I)$, where $R = -I$ satisfies Assumption 4.3, Assumptions 4.2 and 4.4 hold for every $(x_*, w_*, z_*) \in \mathscr{E}$, and $\dot{x}(0) = 0$, then the system has an $\mathcal{L}_{\mathrm{s}2}$-gain bound of $\tilde{\gamma} = \sqrt{\alpha\beta^2\gamma^2}$.*

Proof. See Appendix B.2. □

Due to technicalities, Theorem 4.7 and Corollary 4.1 are what we can prove in terms of the connection between velocity (Q, S, R) dissipativity and universal shifted (Q, S, R) dissipativity. Compared to Proposition 4.1, these results form a direct upperbound. Moreover, based on some of our other technical results on analyzing this problem (see Appendix C.3) and the results when these concepts are used for controller synthesis (see Section 4.6), there are strong indications that Proposition 4.1 holds true.

4.4 Convex Universal Shifted Analysis

In Section 4.3, we have shown how the velocity form in conjunction with the concept of velocity dissipativity can be used to analyze universal shifted stability and performance. Moreover, in Theorem 4.4, we have shown how velocity (Q, S, R) dissipativity, considering a quadratic storage function $\mathcal{V}_{\mathrm{v}}$ of the form (4.19), can be analyzed through a feasibility check of an infinite dimensional set of LMIs. In this section, we will discuss how we can make the analysis computationally feasible and efficient through the use of methods from the LPV framework.

As the state-space matrices of the velocity form vary with (x, w), we obtain an infinite dimensional set of LMIs for velocity dissipativity analysis. This is similar to the LPV case, where the (LPV) state-space matrices vary with the scheduling-variable p, which also results in an infinite dimensional set of LMIs for classical dissipativity analysis of LPV systems, see also Section 2.5.2. In the LPV framework, various methods exist in order to turn the infinite dimensional problem into a finite dimensional one, such as through polytopic, grid-based, and multiplier based techniques (Hoffmann and Werner 2015a). Inspired by the connection between the velocity form and LPV representations, we propose the use of the LPV analysis results to make the velocity dissipativity analysis problem of nonlinear systems computationally feasible and efficient.

For that purpose, we introduce the following so-called *Velocity Parameter-Varying* (VPV) embedding, which is a embedding of the velocity form (4.15) in an LPV representation:

Definition 4.7 (VPV embedding). *Consider a system given by (4.1) and its velocity form (4.15). Furthermore, consider the LPV state-space representation*

$$\dot{x}_\mathrm{v}(t) = A(p(t))x_\mathrm{v}(t) + B(p(t))w_\mathrm{v}(t), \tag{4.30a}$$

$$z_\mathrm{v}(t) = C(p(t))x_\mathrm{v}(t) + D(p(t))w_\mathrm{v}(t), \tag{4.30b}$$

where $x_\mathrm{v}(t) \in \mathbb{R}^{n_\mathrm{x}}$ is the state, $w_\mathrm{v}(t) \in \mathbb{R}^{n_\mathrm{w}}$ the input, and $z_\mathrm{v} \in \mathbb{R}^{n_\mathrm{z}}$ the output of the LPV representation, with $p(t) \in \mathcal{P} \subset \mathbb{R}^{n_\mathrm{p}}$ being the scheduling-variable, and matrix functions $A, \ldots, D$ being of appropriate size and belonging to a given class of functions $\mathfrak{A}$ (e.g., affine or rational). The LPV representation (4.30) is a so-called VPV embedding of (4.1) on the region $\mathcal{X} \times \mathcal{W} \subseteq \mathcal{X} \times \mathcal{W}$, if there exists a function $\eta : \mathcal{X} \times \mathcal{W} \to \mathcal{P}$, the so-called scheduling-map, with $p = \eta(x, w)$, and $\mathcal{P} \supseteq \eta(\mathcal{X}, \mathcal{W})$, such that:

$$\begin{aligned} A(\eta(x,w)) &= A_\mathrm{v}(x,w), & B(\eta(x,w)) &= B_\mathrm{v}(x,w), \\ C(\eta(x,w)) &= C_\mathrm{v}(x,w), & D(\eta(x,w)) &= D_\mathrm{v}(x,w), \end{aligned} \tag{4.31}$$

for all $(x, w) \in \mathcal{X} \times \mathcal{W}$, where $A, \ldots, D$ belong to a given function class (affine, polynomial, etc.).

The behavior of a VPV embedding given by (4.30) for a $p \in \mathcal{P}^{\mathbb{R}_0^+}$ is[4]

$$\mathfrak{B}_\mathrm{p}(p) := \{(x_\mathrm{v}, w_\mathrm{v}, z_\mathrm{v}) \in (\mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}} \times \mathbb{R}^{n_\mathrm{z}})^{\mathbb{R}_0^+} \mid \\ x_\mathrm{v} \in \mathcal{C}_2,\ w_\mathrm{v}, z_\mathrm{v} \in \mathcal{C}_1 \text{ and } (x_\mathrm{v}, w_\mathrm{v}, z_\mathrm{v}, p) \text{ satisfy (4.30)}\}, \tag{4.32}$$

with $\breve{\mathfrak{B}}_\mathrm{p} := \bigcup_{p \in \mathcal{P}^{\mathbb{R}_0^+}} \mathfrak{B}_\mathrm{p}(p)$ being the full behavior (i.e., for all scheduling trajectories).

As the VPV embedding is an LPV representation, the various methods available in the LPV framework to reduce the conservatism of the embedding, for a given dependency class of $A, \ldots, D$ (e.g., affine, polynomial, rational, etc.), can also

[4]Note that we assume for the behavior that $w_\mathrm{v}, z_\mathrm{v} \in \mathcal{C}_1$. The function class $\mathfrak{A}$ and trajectories of p that are considered are implicitly restricted such that this is the case.

be used to reduce the conservatism of the VPV embedding, see e.g., (Tóth 2010; Sadeghzadeh and Tóth 2020). Later, in Chapter 9, we will also present a data-based method to reduce the number of scheduling-variables of an LPV representation, to allow for reduced complexity of LPV analysis and synthesis algorithms.

Through the construction of the VPV embedding, we can then describe the behavior of the velocity form of a nonlinear system given by (4.1). This is similar to how the behavior of a nonlinear system can also be described by an LPV representation, see Lemma 2.6. More formally, we have the following lemma:

Lemma 4.4 (VPV behavioral embedding)**.** *Consider a nonlinear system given by (4.1) with a velocity form given by (4.15). If the LPV representation (4.30) is a VPV embedding of the nonlinear system on the region $\mathscr{X} \times \mathscr{W} = \mathcal{X} \times \mathcal{W}$, then, the behavior of velocity form is included in that of the LPV representation, i.e.,* $\mathfrak{B}_{\mathrm{v}} \subseteq \breve{\mathfrak{B}}_{\mathrm{p}}$.

Proof. See Appendix B.2. □

Remark 4.1. In the case our VPV embedding only considers part of the state-space, i.e., (4.30) is a VPV embedding of the nonlinear system given by (4.1) on the region $\mathscr{X} \times \mathscr{W} \subset \mathcal{X} \times \mathcal{W}$, we can still describe part of the behavior of the velocity form (4.15). However, one has to consider a smaller set of behavior of the nonlinear system given by (4.1) (and correspondingly of the velocity form (4.15)) for which $(x(t), w(t)) \in \mathscr{X} \times \mathscr{W}$ for all $t \in \mathbb{R}_0^+$.

As aforementioned, velocity dissipativity of a nonlinear system, see Definition 4.6, can be seen as 'classical dissipativity' of the velocity form. Moreover, through the VPV embedding, which is an LPV representation, we can describe the behavior of the velocity form, by Lemma 4.4. Combining these two results then allows us to cast the velocity dissipativity analysis problem as a classical dissipativity analysis problem of an LPV representation. This is also evident when we compare the condition for velocity (Q, S, R) dissipativity given in Theorem 4.4 and the condition for classical (Q, S, R) dissipativity of an LPV representation given in Theorem 2.5. This then also allows us to formulate the following theorem:

Theorem 4.8 (Velocity dissipativity analysis through the LPV framework)**.** *Consider the nonlinear system given by (4.1) for which the LPV representation (4.30) is a VPV embedding of the system on the region $\mathscr{X} \times \mathscr{W} = \mathcal{X} \times \mathcal{W}$. If the LPV representation (4.30) is classically dissipative, then the nonlinear system is velocity dissipative.*

Proof. See Appendix B.2. □

With Theorem 4.8, we now have a powerful tool to analyze velocity dissipativity of nonlinear systems, as we can cast it as a classical dissipativity analysis problem of an LPV representation, for which their exists many results, see also Section 2.5.2. In

Section 4.3, we have shown how universal shifted stability and quadratic universal shifted performance notions can be analyzed through velocity (Q, S, R) dissipativity. Consequently, combining these two results, we are now able to analyze universal shifted stability and performance of nonlinear systems through the use of the LPV framework. This then gives us a systematic, and computationally efficient tool to analyze global stability and performance in the form of universal shifted stability and performance.

In the next section, we will use these analysis tools to also develop a systematic controllers synthesis method in order to ensure and shape universal shifted stability and performance.

4.5 Convex Universal Shifted Controller Synthesis

4.5.1 Controller synthesis problem

For LTI and also for (standard) LPV systems, as discussed in Section 2.5.3, the generalized plant concept (Doyle 1983; Apkarian, Gahinet, and G. Becker 1995) is used in order to provide a systematic approach to controller synthesis. Through the generalized plant concept, various control configurations can be described and through the inclusion of weighting filters in the generalized plant, the closed-loop behavior of the plant and controller can be shaped. Due to systematic and intuitive nature of the generalized plant concept in the LTI and LPV framework, we will consider a notion similar to it for our controller synthesis problem. This will allow us to achieve our goal of systematic controller synthesis for nonlinear systems to ensure and shape universal shifted stability and performance.

Therefore, in this section, we will consider nonlinear systems P of the form

$$\dot{x}(t) = f(x(t), u(t)) + B_{\mathrm{w}} w(t); \tag{4.33a}$$

$$z(t) = h_{\mathrm{z}}(x(t), u(t)) + D_{\mathrm{zw}} w(t); \tag{4.33b}$$

$$y(t) = h_{\mathrm{y}}(x(t), u(t)) + D_{\mathrm{yw}} w(t); \tag{4.33c}$$

where again $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_{\mathrm{x}}}$ is the state with $x \in \mathcal{C}_1$ and initial condition $x(0) = x_0 \in \mathbb{R}^{n_{\mathrm{x}}}$ and where now $w(t) \in \mathcal{W} \subseteq \mathbb{R}^{n_{\mathrm{w}}}$ and $z(t) \in \mathcal{Z} \subseteq \mathbb{R}^{n_{\mathrm{z}}}$ are called the generalized disturbance (consisting of references, disturbances, etc.) and generalized performance (consisting of tracking errors, control efforts, etc.) channels, respectively. Moreover, we introduce the channels $u(t) \in \mathcal{U} \subseteq \mathbb{R}^{n_{\mathrm{u}}}$ and $y(t) \in \mathcal{Y} \subseteq \mathbb{R}^{n_{\mathrm{y}}}$, denoting the control input and measured output channel, through which the to-be-designed controller will interact with the plant. Furthermore, $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^{n_{\mathrm{x}}}$, $h_{\mathrm{z}} : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^{n_{\mathrm{z}}}$ and $h_{\mathrm{y}} : \mathcal{X} \to \mathbb{R}^{n_{\mathrm{y}}}$ are assumed to be in $\mathcal{C}_1$ and $B_{\mathrm{w}} \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{w}}}$, $D_{\mathrm{zw}} \in \mathbb{R}^{n_{\mathrm{z}} \times n_{\mathrm{w}}}$, and $D_{\mathrm{yw}} \in \mathbb{R}^{n_{\mathrm{y}} \times n_{\mathrm{w}}}$. In Figure 4.1, an example of such a plant P interconnected with a controller K is given.

The to-be-designed controller K for our plant P is considered to be of the form

$$\dot{x}_{\mathrm{k}}(t) = f_{\mathrm{k}}(x_{\mathrm{k}}(t), u_{\mathrm{k}}(t)); \tag{4.34a}$$

$$y_{\mathrm{k}}(t) = h_{\mathrm{k}}(x_{\mathrm{k}}(t), u_{\mathrm{k}}(t)); \tag{4.34b}$$

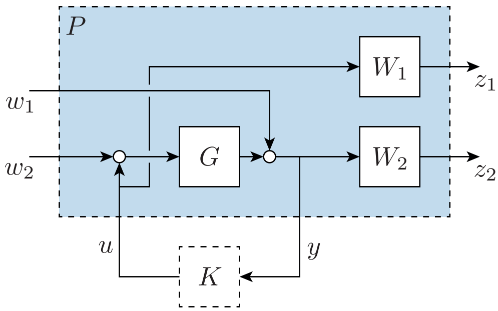


Figure 4.1: Example of a control configuration in terms of a closed-loop connection of the controller K with a plant P, consisting of a nonlinear system G and weighting filters W_1 and W_2.

where $x_\mathrm{k}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}}$ is the state, $u_\mathrm{k}(t) \in \mathbb{R}^{n_{\mathrm{u_k}}}$ is the input, and $y_\mathrm{k}(t) \in \mathbb{R}^{n_{\mathrm{y_k}}}$ is the output of the controller. Furthermore, $f_\mathrm{k} : \mathbb{R}^{n_{\mathrm{x_k}}} \times \mathbb{R}^{n_{\mathrm{u_k}}} \to \mathbb{R}^{n_{\mathrm{x_k}}}$ and $h_\mathrm{k} : \mathbb{R}^{n_{\mathrm{x_k}}} \times \mathbb{R}^{n_{\mathrm{u_k}}} \to \mathbb{R}^{n_{\mathrm{y_k}}}$.

The closed-loop interconnection of P and K for which $u_\mathrm{k} = y$ and $u = y_\mathrm{k}$ (hence, $n_{\mathrm{u_k}} = n_\mathrm{y}$ and $n_{\mathrm{y_k}} = n_\mathrm{u}$) will be denoted by $\mathcal{F}_\mathrm{l}(P, K)$. Note that this closed-loop interconnection will be a system of the form (4.1), which has as input w and as output z. Furthermore, the output $z \in \mathcal{Z}^{\mathbb{R}_0^+}$ of $\mathcal{F}_\mathrm{l}(P, K)$ for an input $w \in \mathcal{W}^{\mathbb{R}_0^+}$ and initial condition $x_{\mathrm{cl},0} = \mathrm{col}(x(0), x_\mathrm{k}(0)) \in \mathcal{X} \times \mathbb{R}^{n_{\mathrm{x_k}}}$, will be denoted by $\mathcal{F}_\mathrm{l}(P, K)(w, x_{\mathrm{cl},0}) = z \in \mathcal{Z}^{\mathbb{R}_0^+}$.

Considering this closed-loop interconnection $\mathcal{F}_\mathrm{l}(P, K)$, our objective then is to synthesize the controller K such that $\mathcal{F}_\mathrm{l}(P, K)$ is universally shifted stable and satisfies a (desired) universal shifted performance criteria. To simplify the discussion, we will consider the $\mathcal{L}_{\mathrm{s}2}$-gain, see (4.10), as our desired performance metric, which we will aim to minimize. More concretely, we are hence interested in synthesizing a controller K for our generalized plant P s.t. the $\mathcal{L}_{\mathrm{s}2}$-gain γ of our closed-loop interconnection $\mathcal{F}_\mathrm{l}(P, K)$ from w to z is minimized, i.e., synthesize a K such that there exists a function $\zeta_\mathrm{s} : \mathcal{X} \times \mathscr{X} \to \mathbb{R}$ and a $\gamma \geq 0$, for which

$$\|\mathcal{F}_\mathrm{l}(P, K)(w, x_{\mathrm{cl},0}) - z_*\|_{2,T} \leq \gamma \, \|w - w_*\|_{2,T} + \zeta_\mathrm{s}(x_{\mathrm{cl},0}, x_{\mathrm{cl},*}), \tag{4.35}$$

for all $T \geq 0$ and $(x_\mathrm{cl}, w, z) \in \mathfrak{B}_\mathrm{cl}$ with $w \in \mathcal{L}_{2\mathrm{e}}$ and for every $(x_{\mathrm{cl},*}, w_*, z_*) \in \mathscr{E}_\mathrm{cl}$, where γ is minimal. In (4.35), $x_\mathrm{cl} = \mathrm{col}(x, x_\mathrm{k})$ is the state associated with the state-space representation of the closed-loop $\mathcal{F}_\mathrm{l}(P, K)$ and $\mathfrak{B}_\mathrm{cl}$ and $\mathscr{E}_\mathrm{cl}$ are the behavior and set of equilibrium points associated with $\mathcal{F}_\mathrm{l}(P, K)$.

To ensure that the above given synthesis problem is feasible with a finite γ, we require P to be a generalized plant in the following sense:

Definition 4.8 (Generalized plant for universal shifted synthesis)**.** *P, given by (4.33), is a generalized plant, if there exists a controller K of the form (4.34) such that the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P, K)$ is universally shifted stable.*

Proposition 4.2. *P, given by (4.33), is a generalized plant in the sense of Definition 4.8, if $\left(\frac{\partial f}{\partial x}(x,u), \frac{\partial f}{\partial u}(x,u)\right)$ is stabilizable and $\left(\frac{\partial f}{\partial x}(x,u), \frac{\partial h_{\mathrm{y}}}{\partial x}(x,u)\right)$ is detectable over $\mathcal{X} \times \mathcal{W}$, see (Pavlov, Van de Wouw, et al. 2006, Section 5.3.2).*

Note that Proposition 4.2 can be interpreted as the velocity form of (4.33) being stabilizable and detectable w.r.t. the input channel $\dot{u}$ and output channel $\dot{y}$, respectively, along all trajectories of nonlinear system given by (4.33). This is similar to condition that is required for standard LTI and LPV controller synthesis using the generalized plant concept, see also Section 2.5.3 and Definitions 2.17 and 2.18.

4.5.2 Universal shifted controller synthesis procedure

Overview

As aforementioned, we will achieve our previously discussed control objective through extension of the analysis tools for universal shifted stability and performance that were developed in Sections 4.2 and 4.3. Namely, we again make use of the velocity form, velocity dissipativity, and VPV embeddings in order to also simplify our controller synthesis problem. To simplify the discussion for controller synthesis, we will consider that Proposition 4.1 is true, meaning we will consider that velocity (Q, S, R) dissipativity of a system implies universal shifted (Q, S, R) dissipativity of that system for the same tuple (Q, S, R).

Based on this, we then propose the following novel procedure in order to synthesize a controller K for our generalized plant P s.t. the closed-loop interconnection $\mathcal{F}_{\mathrm{l}}(P, K)$ is universally shifted stable and has a bounded $\mathcal{L}_{\mathrm{s}2}$-gain:

1. *VPV embedding step:* For a nonlinear generalized plant P given by (4.33), compute its velocity form P_{v}. Construct a VPV embedding P_{vpv} of P based on P_{v}.

2. *Velocity controller synthesis step:* For the VPV embedding P_{vpv} an LPV controller K_{v} is synthesized, ensuring a minimal closed-loop $\mathcal{L}_2$-gain γ. For this, we use standard LPV controller synthesis methods, see also Section 2.5.3.

3. *Universal shifted controller realization step:* The synthesized controller K_{v} of Step 2, which is in the velocity domain, is realized to a nonlinear controller K in the primal form (4.34) to be used with the primal form of the generalized plant P, to ensure the closed-loop $\mathcal{L}_{\mathrm{s}2}$-gain γ.

By considering Proposition 4.1 to be true, all typical (Q, S, R) performance metrics can also be considered in the velocity controller synthesis in Step 2 to induce various universal shifted (Q, S, R) performance notions of the closed-loop interconnection.

Before detailing the individual steps of the above procedure, we will first show that the velocity form of the closed-loop interconnection is equal to the closed-loop interconnection of the velocity form of the plant an velocity form of the controller.

This simplifies the controller design procedure, as it allows us to independently 'transform' the plant and controller between their primal and velocity forms.

Theorem 4.9 (Closed-loop velocity form)**.** *The velocity form of the closed-loop system $\mathcal{F}_\mathrm{l}(P, K)$ is equal to the closed-loop interconnection of P_v and K_v, i.e., $\mathcal{F}_\mathrm{l}(P_\mathrm{v}, K_\mathrm{v})$, if the interconnection of P and K is well-posed, i.e., there exists a $\mathcal{C}_1$ function $\breve{h}$ such that $u = h_\mathrm{k}(x_\mathrm{k}, h_\mathrm{y}(x, u))$ can be expressed as*[5] $u = \breve{h}(x, x_\mathrm{k})$.

Proof. See Appendix B.2. □

VPV embedding step

The first step of our universal shifted controller synthesis procedure consists of embedding the generalized plant P given by (4.33) in a VPV embedding. From here on, we will denote the behavior of (4.33) by

$$\mathfrak{B} := \Big\{(x, u, w, z, y) \in (\mathcal{X} \times \mathcal{U} \times \mathcal{W} \times \mathcal{Z} \times \mathcal{Y})^{\mathbb{R}_0^+} \mid x \in \mathcal{C}_1, \\ (x, u, w, z, y) \text{ satisfies } (4.33)\Big\}. \tag{4.36}$$

Moreover, we define the set of all differentiable solutions $\mathfrak{B}$, by

$$\mathfrak{B}_\mathrm{c} := \{(x, u, w, z, y) \in \mathfrak{B} \mid x \in \mathcal{C}_2,\ u, w, z, y \in \mathcal{C}_1\}. \tag{4.37}$$

First, we compute the velocity form of P given by (4.15), resulting in P_v given by

$$\ddot{x}(t) = A_\mathrm{v}(x(t), u(t))\dot{x}(t) + B_\mathrm{w}\dot{w}(t) + B_\mathrm{v,u}(x(t), u(t))\dot{u}(t); \tag{4.38a}$$
$$\dot{z}(t) = C_\mathrm{v,z}(x(t), u(t))\dot{x}(t) + D_\mathrm{zw}\dot{w}(t) + D_\mathrm{v,zu}(x(t), u(t))\dot{u}(t); \tag{4.38b}$$
$$\dot{y}(t) = C_\mathrm{v,y}(x(t), u(t))\dot{x}(t) + D_\mathrm{yw}\dot{w}(t) + D_\mathrm{v,yu}(x(t), u(t))\dot{u}(t); \tag{4.38c}$$

where $A_\mathrm{v} = \frac{\partial f}{\partial x}$, $B_\mathrm{v} = \frac{\partial f}{\partial u}$, $C_\mathrm{v,z} = \frac{\partial h_\mathrm{z}}{\partial x}$, $D_\mathrm{v,zu} = \frac{\partial h_\mathrm{z}}{\partial u}$, $C_\mathrm{v,y} = \frac{\partial h_\mathrm{y}}{\partial x}$, and $D_\mathrm{v,yu} = \frac{\partial h_\mathrm{y}}{\partial u}$. The behavior of (4.38) is then denoted by $\mathfrak{B}_\mathrm{v} := \partial\mathfrak{B}_\mathrm{c}$, see also Section 4.3.1.

We then embed P_v, given by (4.38), in an LPV representation in order to construct a VPV embedding of P (4.33). Based on Definition 4.7 of the VPV embedding, we then construct a VPV embedding of P given by (4.33) on the compact region[6] $\mathscr{X} \times \mathscr{U} \subseteq \mathcal{X} \times \mathcal{U}$, which we denote by P_vpv and is given by

$$\dot{x}_\mathrm{v}(t) = A(p(t))x_\mathrm{v}(t) + B_\mathrm{w}w_\mathrm{v}(t) + B_\mathrm{u}(p(t))u_\mathrm{v}(t); \tag{4.39a}$$
$$z_\mathrm{v}(t) = C_\mathrm{z}(p(t))x_\mathrm{v}(t) + D_\mathrm{zw}w_\mathrm{v}(t) + D_\mathrm{zu}(p(t))u_\mathrm{v}(t); \tag{4.39b}$$
$$y_\mathrm{v}(t) = C_\mathrm{y}(p(t))x_\mathrm{v}(t) + D_\mathrm{yw}w_\mathrm{v}(t) + D_\mathrm{yu}(p(t))u_\mathrm{v}(t); \tag{4.39c}$$

[5] In the proof, we give conditions for the existence of this function.

[6] Note that as w enters into (4.33) linearly, the scheduling-map η of the VPV embedding will not depend on it. Hence, the embedding region w.r.t. w can be taken equal to or as any subset of $\mathcal{W}$, i.e., the complete value set of w. Therefore, we omit it when talking about the VPV embedding region of (4.33).

with scheduling-variable $p(t) \in \mathcal{P} \subset \mathbb{R}^{n_\mathrm{p}}$, where $\mathcal{P}$ is assumed to be convex, and the accompanying scheduling-map η, s.t. $p(t) = \eta(x(t), u(t))$ and $\mathcal{P} \supseteq \eta(\mathcal{X}, \mathcal{U})$. We will also assume that $\eta \in \mathcal{C}_1$. Moreover, $x_\mathrm{v}(t) \in \mathbb{R}^{n_\mathrm{x}}$, $w_\mathrm{v}(t) \in \mathbb{R}^{n_\mathrm{w}}$, $u_\mathrm{v}(t) \in \mathbb{R}^{n_\mathrm{u}}$, $z_\mathrm{v}(t) \in \mathbb{R}^{n_\mathrm{z}}$, and $y_\mathrm{v}(t) \in \mathbb{R}^{n_\mathrm{y}}$. The accompanying behavior of the VPV embedding (4.39) for a scheduling trajectory $p \in \eta^{\mathbb{R}_0^+}$ is denoted by

$$\begin{aligned}\mathfrak{B}_\mathrm{p}(p) := \{&(x_\mathrm{v}, u_\mathrm{v}, w_\mathrm{v}, z_\mathrm{v}, y_\mathrm{v}) \in (\mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{u}} \times \mathbb{R}^{n_\mathrm{w}} \times \mathbb{R}^{n_\mathrm{z}} \times \mathbb{R}^{n_\mathrm{y}})^{\mathbb{R}_0^+} \mid \\ &x_\mathrm{v} \in \mathcal{C}_2,\, u_\mathrm{v}, w_\mathrm{v}, z_\mathrm{v}, y_\mathrm{v} \in \mathcal{C}_1 \text{ and } (x_\mathrm{v}, u_\mathrm{v}, w_\mathrm{v}, z_\mathrm{v}, y_\mathrm{v}, p) \text{ satisfy } (4.39)\},\end{aligned} \tag{4.40}$$

with $\breve{\mathfrak{B}}_\mathrm{p} := \bigcup_{p \in \mathcal{P}^{\mathbb{R}_0^+}} \mathfrak{B}_\mathrm{p}(p)$ being the full behavior of (4.39).

Moreover, we will denote the restriction of the state and control input solutions of P to $\mathcal{X}$ and $\mathcal{U}$, respectively, by $\mathfrak{B}_{\mathrm{c},\mathcal{XU}} := \{(x, u, w, z, y) \in \mathfrak{B}_\mathrm{c} \mid (x(t), u(t)) \in \mathcal{X} \times \mathcal{U}\}$ and the corresponding solution set for the velocity form P_v by $\mathfrak{B}_{\mathrm{v},\mathcal{XU}} := \partial\mathfrak{B}_{\mathrm{c},\mathcal{XU}}$. Through the VPV behavioral embedding principle, given by Lemma 4.4, we have $\mathfrak{B}_{\mathrm{v},\mathcal{XU}} \subseteq \breve{\mathfrak{B}}_\mathrm{p}$. This means that through P_vpv given by (4.39), we can describe the behavior of P_v given by (4.38) for which $(x(t), u(t)) \in \mathcal{X} \times \mathcal{U}$.

Remark 4.2. Note that for the controller synthesis problem to be feasible, one has to make sure that for the constructed VPV embedding given by (4.39), the pairs (A, B_u) and (A, C_y) are stabilizable and detectable[7], respectively. This has to be done in order to preserve the stabilizability and detectability properties of the underlying velocity form according to Proposition 4.2.

Velocity controller synthesis step

Having constructed a VPV embedding P_vpv for our generalized plant P, we will use it in order to synthesize a controller K_v s.t. $\mathcal{F}_\mathrm{l}(P_\mathrm{vpv}, K_\mathrm{v})$ has a minimal $\mathcal{L}_2$-gain, which will ensure that the $\mathcal{L}_2$-gain of $\mathcal{F}_\mathrm{l}(P_\mathrm{v}, K_\mathrm{v})$ is minimized. Note again that velocity (Q, S, R) dissipativity will imply universal shifted (Q, S, R) dissipativity, and that velocity dissipativity can be seen as 'classical dissipativity' of the velocity form. Therefore, minimizing the $\mathcal{L}_2$-gain of $\mathcal{F}_\mathrm{l}(P_\mathrm{v}, K_\mathrm{v})$ will minimize the $\mathcal{L}_{\mathrm{s}2}$-gain of $\mathcal{F}_\mathrm{l}(P, K)$, where K will be realized in the next subsection s.t. its velocity form will correspond to K_v.

As P_vpv is an LPV representation, we make use of the synthesis algorithms in the LPV framework in order to synthesize K_v s.t. $\mathcal{F}_\mathrm{l}(P_\mathrm{vpv}, K_\mathrm{v})$ has a minimal $\mathcal{L}_2$-gain. Note that this is just a standard LPV synthesis problem, hence, we can apply one of the various available controller synthesis techniques that ensure the closed-loop interconnection is classically dissipative with a minimal $\mathcal{L}_2$-gain bound, see e.g. (Packard 1993; Apkarian, Gahinet, and G. Becker 1995; Wu 1995; Scherer 2001) and see also Section 2.5.3. Concretely, we consider K_v, which we will refer to as velocity controller, in the form

$$\dot{x}_{\mathrm{v,k}}(t) = A_\mathrm{k}(p(t))x_{\mathrm{v,k}}(t) + B_\mathrm{k}(p(t))u_{\mathrm{v,k}}(t); \tag{4.41a}$$
$$y_{\mathrm{v,k}}(t) = C_\mathrm{k}(p(t))x_{\mathrm{v,k}}(t) + D_\mathrm{k}(p(t))u_{\mathrm{v,k}}(t); \tag{4.41b}$$

[7] See Definitions 2.17 and 2.18.

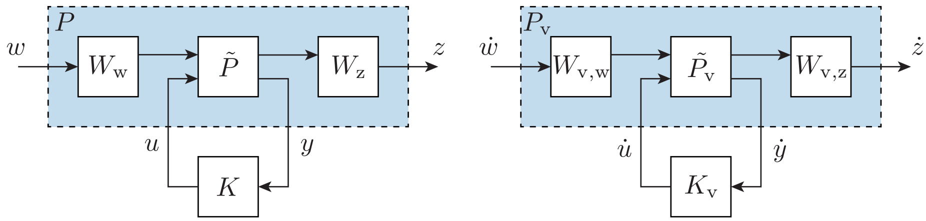
where $x_{\mathrm{v,k}}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}}$ is the state, $u_{\mathrm{v,k}}(t) \in \mathbb{R}^{n_{\mathrm{u_k}}}$ is the input, and $y_{\mathrm{v,k}}(t) \in \mathbb{R}^{n_{\mathrm{y_k}}}$ is the output of the controller, respectively, and $A_{\mathrm{k}}, \dots, D_{\mathrm{k}} \in \mathfrak{A}$ are matrix functions with appropriate dimensions. Note that when we connect K_{v} to P_{v} (to obtain $\mathcal{F}_{\mathrm{l}}(P_{\mathrm{v}}, K_{\mathrm{v}})$), we have that $u_{\mathrm{v,k}} = \dot{y}$ and $y_{\mathrm{v,k}} = \dot{u}$, and $p = \eta(x, u)$. Moreover, as $x, u \in \mathcal{C}_1$ and as $\eta \in \mathcal{C}_1$, we have that $p \in \mathcal{C}_1$.

Based on this, we can formulate the following theorem:

Theorem 4.10 (Velocity closed-loop $\mathcal{L}_2$-gain)**.** *If controller K_{v} of the form (4.41) ensures classical dissipativity and a bounded $\mathcal{L}_2$-gain γ of the closed-loop interconnection $\mathcal{F}_{\mathrm{l}}(P_{\mathrm{vpv}}, K_{\mathrm{v}})$ for all $(x_{\mathrm{v}}, u_{\mathrm{v}}) \in \pi_{x_{\mathrm{v}}, u_{\mathrm{v}}} \breve{\mathfrak{B}}_{\mathrm{p}}$, then, $\mathcal{F}_{\mathrm{l}}(P_{\mathrm{v}}, K_{\mathrm{v}})$ with $p = \eta(x, u)$ is classically dissipative and has an $\mathcal{L}_2$-gain bound $\leq \gamma$ for all $(\dot{x}, \dot{u}) \in \pi_{\dot{\mathrm{x}}, \dot{\mathrm{u}}} \mathfrak{B}_{\mathrm{v}, \mathcal{XU}}$.*

Proof. See Appendix B.2. □

Remark 4.3. By applying shaping filters on P that consequently appear in P_{v}, we can shape the closed-loop performance of $\mathcal{F}_{\mathrm{l}}(P, K)$, see Figure 4.2a. If the weighting filters included in P are LTI, then, as depicted in Figure 4.2, the input-output behavior of W_{w} and W_{z} is equivalent to that of $W_{\mathrm{v,w}}$ and $W_{\mathrm{v,z}}$. This is because the transfer function representation of the velocity form of an LTI system is given by the same transfer function as its primal form. This results in a one to one correspondence between the performance shaping of the primal form $\mathcal{F}_{\mathrm{l}}(P, K)$ (see Figure 4.2a) and performance shaping of the velocity form $\mathcal{F}_{\mathrm{l}}(P_{\mathrm{v}}, K_{\mathrm{v}})$ (see Figure 4.2b). This significantly simplifies the controller design, as shaping can be directly performed through the velocity form P_{v} and hence also through the VPV embedding P_{vpv}.



(a) Primal form. (b) Velocity form.

Figure 4.2: Shaping the closed-loop behavior of the primal and the velocity form by the use of weighting filters W_{w} and W_{z}.

Universal shifted controller realization step

Finally, we will describe the last step of the proposed synthesis procedure. For the last step, we realize the controller K to be used with the primal form of the generalized plant P based on the velocity controller K_{v} of the previous step, such that closed-loop universal shifted stability and performance is ensured.

To do this, we exploit the properties of the velocity form. As in the velocity form the inputs and outputs of our system are time differentiated versions of the inputs and outputs of the primal form of the system. This allows us to formulate the following theorem:

Theorem 4.11 (Velocity behavior inclusion)**.** *Consider a nonlinear system given by (4.1) with behavior $\mathfrak{B}_\mathrm{c}$ (see (4.13)). The nonlinear system with its inputs (time) integrated and its outputs (time) differentiated is equal to its velocity form (4.15).*

Proof. See Appendix B.2. □

Next, we exploit the result of Theorem 4.11 for our controller realization. Namely, we concatenate an integrator to the output of the velocity controller K_v and concatenate a differentiator to its input. We then absorb the differentiator and integrator into the dynamics of the controller using realization theory. Let us denote the i'th element of p by p_i.

Theorem 4.12 (Universal shifted controller realization)**.** *Consider the velocity controller K_v given by (4.41). Furthermore, consider the nonlinear time-invariant controller K given by*

$$\dot{\breve{x}}_\mathrm{k}(t) = \breve{A}_\mathrm{k}(p(t))\breve{x}_\mathrm{k} + \breve{B}_\mathrm{k}(p(t), \dot{p}(t))u_\mathrm{k}(t); \tag{4.42a}$$

$$y_\mathrm{k}(t) = \breve{C}_\mathrm{k}\breve{x}_\mathrm{k} + \breve{D}_\mathrm{k}(p(t))u_\mathrm{k}(t); \tag{4.42b}$$

where $\breve{x}_\mathrm{k}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}+n_{\mathrm{y_k}}}$ is the state of the controller, and where

$$\begin{aligned} \breve{A}_\mathrm{k}(p) &= \begin{bmatrix} A_\mathrm{k}(p) & 0 \\ C_\mathrm{k}(p) & 0 \end{bmatrix}, & \breve{B}_\mathrm{k}(p,\dot{p}) &= \begin{bmatrix} A_\mathrm{k}(p)B_\mathrm{k}(p) - \partial B_\mathrm{k}(p,\dot{p}) \\ C_\mathrm{k}(p)B_\mathrm{k}(p) - \partial D_\mathrm{k}(p,\dot{p}) \end{bmatrix}, \\ \breve{C}_\mathrm{k} &= \begin{bmatrix} 0 & I \end{bmatrix}, & \breve{D}_\mathrm{k}(p) &= D_\mathrm{k}(p), \end{aligned} \tag{4.43}$$

with $p(t) = \eta(x(t), u(t))$, $\partial B_\mathrm{k}(p(t), \dot{p}(t)) = \sum_{i=1}^{n_\mathrm{p}} \frac{\partial B_\mathrm{k}(p(t))}{\partial p_i}\dot{p}_i(t)$, and $\partial D_\mathrm{k}(p(t), \dot{p}(t)) = \sum_{i=1}^{n_\mathrm{p}} \frac{\partial D_\mathrm{k}(p(t))}{\partial p_i}\dot{p}_i(t)$. The controller K in (4.42) is the primal form of K_v (4.41) and the velocity form of K is K_v. Hence, K is called the primal realization of K_v.

Proof. See Appendix B.2. □

We will refer to the controller K in (4.42) as the universal shifted controller. For the realization of this controller, we require the derivative of the scheduling-variable, $\dot{p}$. Note that $\dot{p}$ exists, as $p \in \mathcal{C}_1$, and is bounded as the VPV embedding region $\mathcal{X} \times \mathcal{U}$ is compact. Moreover, also note that the dependency on $\dot{p}$ in (4.42) drops out when B_k and D_k of the velocity controller (4.41) are constant matrices. This then has to be ensured in Step 2 of the controller synthesis procedure. However, this might limit the achievable closed-loop performance that can be obtained, corresponding to a complexity trade-off. Due to the definition of the state of the controller through the proposed realization, there are no restrictions on the initial condition of the state of the controller. Nonetheless, we generally assume the initial condition to be zero. An interpretation of this controller realization procedure is also depicted in Figure 4.3.

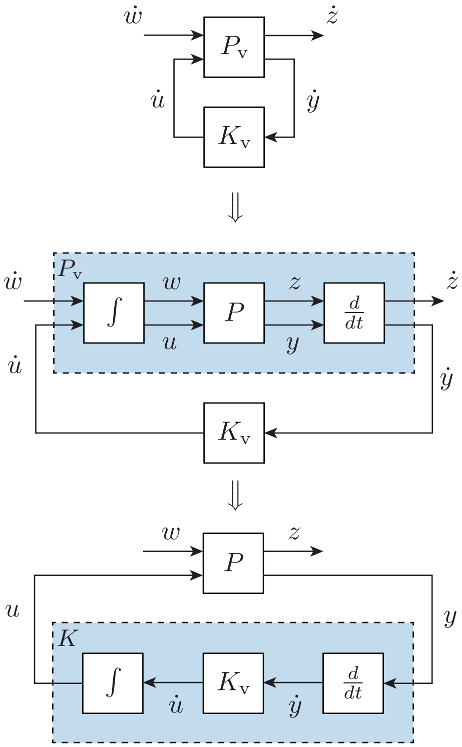


Figure 4.3: Universal shifted controller realization.

Remark 4.4. Similar control structures as (4.42) have been proposed in literature, see e.g (Kaminer et al. 1995; Mehendale and Grigoriadis 2006). Compared to these works, we connect the proposed controller design to universal shifted stability and performance, which, to the author's knowledge, has not been made so far in literature.

4.5.3 Closed-loop universal shifted stability and performance

Based on the proposed universal shifted controller realization of Section 4.5.2, we can show that the closed-loop interconnection $\mathcal{F}_{\mathrm{l}}(P,K)$ is universally shifted stable and has a bounded $\mathcal{L}_{\mathrm{s}2}$-gain. Before showing this result, we first introduce the following definition:

Definition 4.9 (Invariance)**.** *For a system*[8] *given by (4.1) with behavior* $\mathfrak{B}$*, we call* $\tilde{\mathcal{X}} \subseteq \mathcal{X}$ *to be invariant under a given* $\tilde{\mathcal{W}} \subseteq \mathcal{W}$*, if* $x(t) = \phi_{\mathrm{x}}(t, t_0, x_0, w) \in \tilde{\mathcal{X}}$

[8]Note that $\mathcal{F}_{\mathrm{l}}(P,K)$ is of this form.

for all $t \in \mathbb{R}_0^+$, $x_0 \in \tilde{\mathcal{X}}$ *and* $w \in \mathcal{W}^{\mathbb{R}_0^+}$. *The corresponding behavior is denoted by* $\mathfrak{B}_{\tilde{\mathcal{X}}\tilde{\mathcal{W}}} := \mathfrak{B} \cap \{(x, w, z) \in \mathfrak{B} \mid (x(t), w(t)) \in \tilde{\mathcal{X}} \times \tilde{\mathcal{W}},\ \forall t \in \mathbb{R}_0^+\}$.

Theorem 4.13 (Closed-loop universal shifted stability and performance). *Let* K_v, *given in (4.41), be an LPV controller, synthesized for the velocity form* P_v *in (4.38) of a nonlinear system given by (4.33) with behavior* $\mathfrak{B}_\mathrm{c}$, *which ensures classical dissipativity and a bounded* $\mathcal{L}_2$*-gain of* γ *of the closed-loop* $\mathcal{F}_\mathrm{l}(P_\mathrm{v}, K_\mathrm{v})$ *on* $\mathcal{X} \times \mathcal{U}$. *Consider the set* $\tilde{\mathcal{W}} \subseteq \mathcal{W}$, *such that there is an open and bounded* $\mathcal{X}_\mathrm{k} \subseteq \mathbb{R}^{n_{\mathrm{x_k}}}$ *for which* $\mathcal{X}_\mathrm{cl} = \mathcal{X} \times \mathcal{X}_\mathrm{k}$ *is invariant in the sense of Definition 4.9. Then, the controller* K, *given by (4.42), ensures universal shifted (asymptotic) stability and a bounded* $\mathcal{L}_{\mathrm{s}2}$*-gain of* $\leq \gamma$ *of the closed-loop* $\mathcal{F}_\mathrm{l}(P, K)$ *for all* $w \in \tilde{\mathcal{W}}^{\mathbb{R}_0^+} \cap \mathcal{L}_{2\mathrm{e}}$ *and any* $w_* \in \mathscr{W} \cap \tilde{\mathcal{W}}$.

Proof. See Appendix B.2. □

Note that considering $w \in \tilde{\mathcal{W}}^{\mathbb{R}_0^+} \cap \mathcal{L}_{2\mathrm{e}}$ in Theorem 4.13 ensures that the trajectories stay in the VPV embedding region considered during synthesis of the controller. However, computing $\tilde{\mathcal{W}}$ is a difficult problem, which is related to reachability analysis or invariant set computation. Nonetheless, there are numerical tools that can be employed for this purpose, see e.g. (Althoff 2013; Maidens and Arcak 2015).

4.5.4 Reference tracking and disturbance rejection

The previously presented universal shifted controller design makes use of the velocity form and velocity dissipativity to ensure universal shifted stability and performance of the closed-loop. This has as advantage that explicit knowledge of the (closed-loop) equilibrium points is not required, making the design procedure feasible. However, for reference tracking purposes and disturbance rejection purposes it is important that the controller is designed in such a way that the equilibrium points of the closed-loop system correspond to the to-be-followed (constant) reference trajectories.

In order to achieve this for the universal shifted controller design described in this chapter, we propose the following solution. For reference tracking and disturbance rejection purposes, frequently a generalized plant is considered where the generalized disturbance is assumed to be in the form $w = \mathrm{col}(w_1, w_2)$, where w_1 contains the reference signals and w_2 contains external disturbances. Moreover, corresponding to this, the measured output of the plant is then assumed of the form $y = \mathrm{col}(y_1, y_2)$, where y_1 contains signals to be tracked (such that w_1 and y_1 have the same dimension) and y_2 contains other to be controlled variables. Similar to controller design for LTI systems, in order to track the to-be-followed constant reference, an integrator is required to be included in the dynamics of the controller K corresponding to the y_1 channel. A simple way to achieve explicit integral action is by including a bi-proper filter with integrator(s) in the loop, or approximate integral action can be achieved by appropriate choice of weighting filters, see (Zhou et al. 1996, Section 17.4). Including an explicit integral filter results in the interconnection depicted in

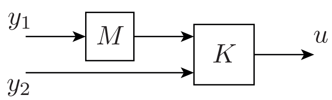


Figure 4.4: Controller interconnection for reference and disturbance rejection, with controller K and integral filter M.

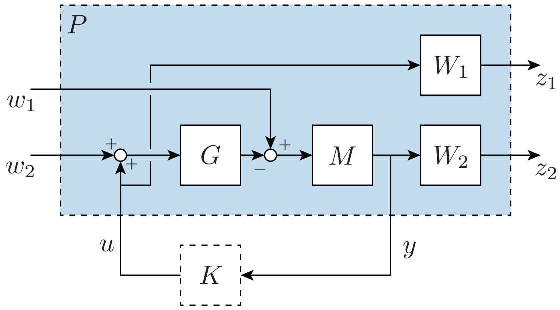


Figure 4.5: Example closed-loop control configuration, similar to Figure 4.1, with explicit integral filter M in the loop.

Figure 4.4. See also Figure 4.5, where a generalized plant P is depicted with an explicit integral filter in the loop (where y_2 is empty, i.e., $y_1 = y$).

The inclusion of explicit integrators in the loop also allows for state reduction of the interconnection of integrator (filters) and primal controller, as the differentiators used for realization of the controller (see Section 4.5.2) and the integrator (filters) in the loop will partially cancel out. This gives us the following corollary:

Corollary 4.2 (Universal shifted realization with integral action)**.** *Consider a generalized plant which includes an explicit integrator filter of the form $M(s) = \frac{s+\alpha}{s}$ (where s is the Laplace variable and $\alpha > 0$) in the loop, such that the (to-be-designed) controller K and M are connected as depicted in Figure 4.4 where y_2 is empty, i.e., $y_1 = y$ (see also see Figure 4.5). For K_v given by (4.41), the interconnection of the primal realization of the controller K and M can be expressed as (4.42) where $\breve{A}_\mathrm{k}$, $\breve{C}_\mathrm{k}$, and $\breve{D}_\mathrm{k}$ are given as in (4.43), and $\breve{B}_\mathrm{k}$ is given by*

$$\breve{B}_\mathrm{k}(t) = \begin{bmatrix} A_\mathrm{k}(p(t))B_\mathrm{k}(p(t)) + B_\mathrm{k}(p)\alpha I - \partial B_\mathrm{k}(p(t), \dot{p}(t)) \\ C_\mathrm{k}(p(t))B_\mathrm{k}(p(t)) + D_\mathrm{k}(p)\alpha I - \partial D_\mathrm{k}(p(t), \dot{p}(t)) \end{bmatrix}. \tag{4.44}$$

Proof. See Appendix B.2. □

4.6 Examples

In this section, we will demonstrate through examples that the universal shifted controller design guarantees closed-loop universal shifted stability and $\mathcal{L}_{\mathrm{s}2}$-gain performance. The results will be demonstrated through a simulation study and also on an experimental setup. Moreover, we compare the universal shifted controller design to a standard LPV controller design, which is only able to guarantee stability w.r.t. the origin (as discussed in Chapter 3) and standard $\mathcal{L}_2$-gain performance, showing the benefits of the universal shifted design.

Example 4.1 *(Duffing oscillator)***.** First, constant reference tracking and disturbance rejection for a Duffing oscillator is investigated. The system is described by the following differential equations:

$$\begin{aligned}\dot{q}(t) &= v(t);\\ \dot{v}(t) &= -\frac{k_1}{m}q(t) - \frac{k_2}{m}\left(q(t)\right)^3 - \frac{d}{m}v(t) + \frac{1}{m}F(t);\end{aligned} \tag{4.45}$$

where, q [m] is the position, v [m $\cdot$ s^{-1}] the velocity and F [N] is the (input) force acting on the mass. Furthermore, $m = 1$ [kg], $k_1 = 0.5$ [N $\cdot$ m^{-1}], $k_2 = 5$ [N $\cdot$ m^{-3}] and $d = 0.2$ [N $\cdot$ s $\cdot$ m^{-1}]. We assume that only the position q can be measured and hence it is considered to be the only output of the plant.
The generalized plant P that is considered for synthesis is depicted in Figure 4.6, where G is the system given by (4.45), K is the controller, $w = \mathrm{col}\,(r, d_\mathrm{i})$ is the generalized disturbance, with r the reference and d_i being an input disturbance. The performance channel consists of z_1 (tracking error) and z_2 (control effort). The considered LTI weighting filters $\{W_i\}_{i=1}^3$ are chosen as the transfer functions $W_1(s) = \frac{0.501(s+3)}{s+2\pi}$, $W_2(s) = \frac{10(s+50)}{s+5\cdot 10^4}$ and $W_3 = 1.5$. Furthermore, integral action is enforced by the filter $M(s) = \frac{s+2\pi}{s}$. The resulting sensitivity weight $W_1(s)M(s)$ has guaranteed 20 dB/dec roll-off at low frequencies in order to ensure good tracking performance, while $W_2(s)$ has high-pass characteristics in order to ensure proper roll-off at high frequencies.

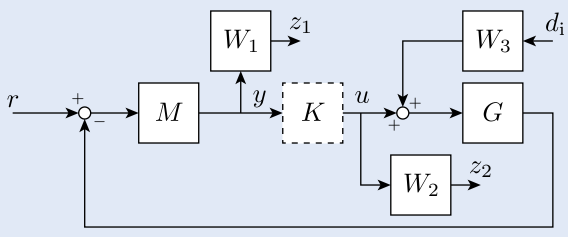


Figure 4.6: Generalized plant considered for Duffing oscillator.

As the Duffing Oscillator is the only nonlinear system in the generalized plant, we only require the computation of the velocity form of (4.45) for the VPV embedding, as all the LTI filters will have the same dynamics in their velocity forms. The following VPV embedding for (4.45) is constructed[9], where dependence on time is

[9]Note that variables with subscript v correspond to the time-derivatives, i.e., q_v corresponds to $\dot{q}$ etc. See also Definition 4.7.

omitted for brevity:

$$\begin{aligned} \dot{q}_{\mathrm{v}} &= v_{\mathrm{v}}; \\ \dot{v}_{\mathrm{v}} &= \left(-\frac{k_1}{m} - 3\frac{k_2}{m}p\right) q_{\mathrm{v}} - \frac{d}{m}v_{\mathrm{v}} + \frac{1}{m}F_{\mathrm{v}}; \end{aligned} \tag{4.46}$$

where the scheduling $p(t) = q^2(t)$ and $\mathcal{P} = [0,\, 2]$ is chosen to allow for a relatively large operating range. Consequently the resulting VPV embedding region is $\mathcal{X} = [-\sqrt{2},\, \sqrt{2}] \times \mathbb{R}$.

As the VPV embedding (4.46) (and hence, the resulting generalized plant) has affine scheduling dependency, polytopic $\mathcal{L}_2$-gain synthesis based on (Apkarian, Gahinet, and G. Becker 1995; Apkarian and Adams 1998) is used in the velocity controller synthesis (i.e., second) step of the universal shifted controller design procedure Section 4.5.2. This synthesis algorithm has been implemented in the LPVcore Toolbox (Boef et al. 2021), which has been used to synthesize the controller. For the velocity controller synthesis step, B_{k} and D_{k} are assumed to be constant[10], which gives us an $\mathcal{L}_{\mathrm{s}2}$-gain of $\gamma = 1.2$. Based on this structural restriction, the resulting universal shifted controller has affine dependence on $p(t)$ without dependence on $\dot{p}(t)$, see (4.42). Moreover, as an integration filter M is included in the loop, we make use of the result of Corollary 4.2 for the primal realization to obtain the universal shifted controller. Based on this, the closed-loop is universally shifted stable has an $\mathcal{L}_{\mathrm{s}2}$-gain of $\gamma \leq 1.2$ (under the consideration that Proposition 4.1 is true) for $(q(t), v(t)) \in [-\sqrt{2},\, \sqrt{2}] \times \mathbb{R}$.

For comparison, also a standard LPV controller design is done to ensure $\mathcal{L}_2$-gain performance and closed-loop stability (of the origin). For the design of the standard LPV controller, the primal form of the system given by (4.45) is embedded in an LPV representation:

$$\begin{aligned} \dot{q} &= v; \\ \dot{v} &= \left(-\frac{k_1}{m} - \frac{k_2}{m}p_{\mathrm{s}}\right) q - \frac{d}{m}v + \frac{1}{m}F; \end{aligned} \tag{4.47}$$

where $p_{\mathrm{s}}(t) = q^2(t)$ is the scheduling-variable. Here we will denote with subscript 's' this 'standard' concept of LPV embedding and control design. Note that p_{s} is the same as p, i.e., we are able to create an LPV embedding with the same scheduling-map. Consequently for comparison, we also consider $p_{\mathrm{s}}(t) \in \mathcal{P}_{\mathrm{s}} = [0,\, 2]$. The same generalized plant structure as for the universal shifted design is considered, see Figure 4.6, and we also use the polytopic controller synthesis method implemented in the LPVcore toolbox in order to synthesize the standard LPV controller. This then results in an $\mathcal{L}_2$-gain for the standard LPV controller design of $\gamma = 0.94$.

[10] This might reduce the achievable performance as the parameter dependency of the controller is partly fixed, which results in less freedom during synthesis.

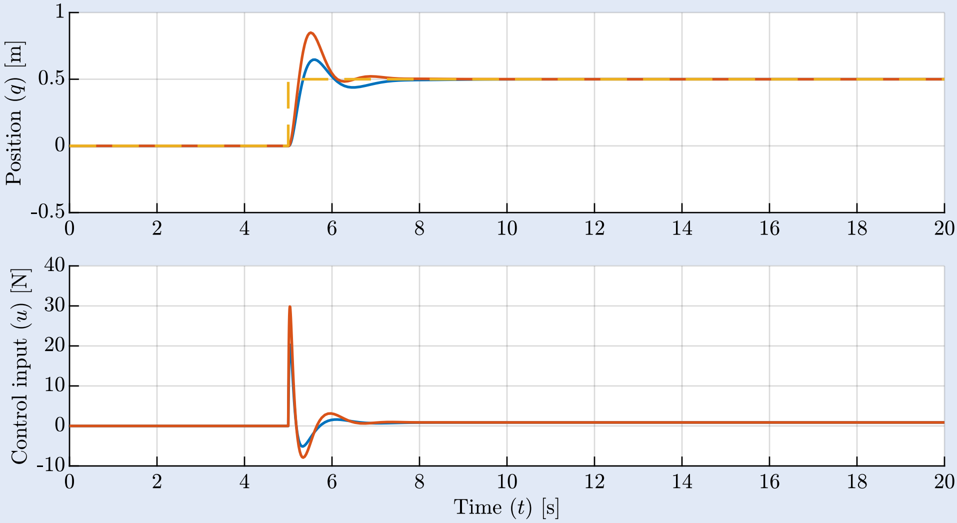


Figure 4.7: Position of the Duffing oscillator (top) in closed-loop with the standard LPV (—) and the universal shifted (—) controllers under reference (- -) and no input disturbance, together with the generated control inputs (bottom) by the controllers.

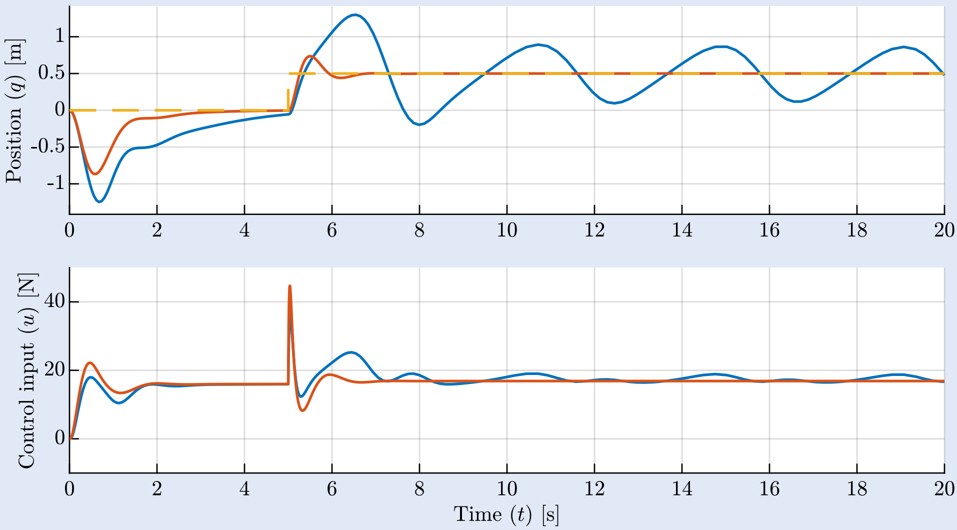


Figure 4.8: Position of the Duffing oscillator (top) in closed-loop with the standard LPV (—) and the incremental (—) controllers under reference (- -) and input disturbance, together with the generated control inputs (bottom) by the controllers.

In simulation, the resulting outputs of the system using the standard LPV controller and the universal shifted controller in closed-loop are depicted without and with input disturbance in Figures 4.7 and 4.8, respectively. In both cases, a step signal is taken as a reference trajectory which changes from zero to 0.5 at $t = 5$ seconds. For the simulation results in Figure 4.8, a constant input disturbance $d_\mathrm{i} \equiv -10\frac{2}{3}$ (corresponding to $-10\frac{2}{3} \cdot W_3 = -16$ [N]) is applied. Note that this reference signal is not continuously differentiable, due to the change in value at 5 seconds. Nonetheless, as can be seen, the proposed universal shifted controller works for non-differentiable references as well.

Comparing the results of the standard LPV controller and the universal shifted controller in Figure 4.7 shows that both controllers have similar performance when no input disturbance is present. The universal shifted controller has slightly more overshoot, but a lower settling time for this example. However, under constant input disturbance, it can be seen in Figure 4.8 that the standard LPV controller has a significant performance loss with oscillatory behavior, whereas the universal shifted controller preserves its constant reference tracking property. Note, that in both cases, the scheduling-variable p never leaves the set for which the controllers have been designed, i.e., $q(t) \in [-\sqrt{2}, \sqrt{2}]$.

In the next example, we demonstrate the universal shifted controller design on an experimental setup. Like for the duffing oscillator example, we consider a constant reference tracking disturbance rejection problem and also compare the achieved performance to a standard LPV controller design.

Example 4.2 *(Unbalanced disk system)***.**

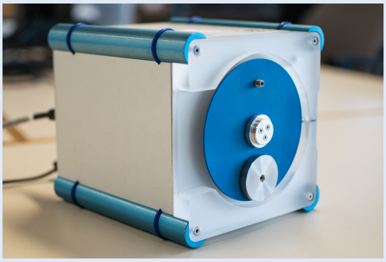


Figure 4.9: Unbalanced disk setup.

By neglecting the fast electrical dynamics of the motor, the motion of the unbalanced disk can be described as

$$\dot{\theta}(t) = \omega(t); \tag{4.48a}$$

$$\dot{\omega}(t) = \frac{Mgl}{J}\sin(\theta(t)) - \frac{1}{\tau}\omega(t) + \frac{K_m}{\tau}V(t); \tag{4.48b}$$

where θ [rad] is the angle of the disk, ω [rad $\cdot$ s^{-1}] its angular velocity, V [V] is the input voltage to the motor, g is the gravitational acceleration, l the length of the pendulum, J the inertia of the disk, and K_m and τ are the motor constant and time constant respectively. The angle of the disk θ is considered to be the output of the plant. The physical parameters, estimated based on measurement data, are given in Table 4.1.

Table 4.1: Physical parameters of the unbalanced disk.

Parameter	Value	Unit
g	9.8	m·s^{-2}
J	$2.4 \cdot 10^{-4}$	kg·m^2
K_m	$1.1 \cdot 10^{1}$	rad·s^{-1}·V^{-1}
l	$4.1 \cdot 10^{-2}$	m
M	$7.6 \cdot 10^{-2}$	kg
τ	$4.0 \cdot 10^{-1}$	s

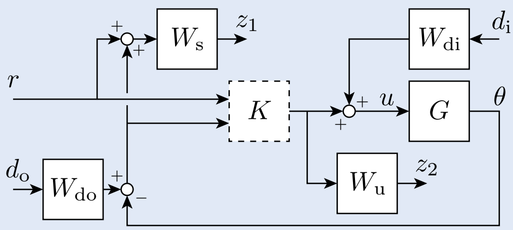


Figure 4.10: Generalized plant considered for the unbalanced disk.

A generalized plant structure is used as depicted in Figure 4.10, where G is the system given by (4.48), K is the to-be-designed controller, $w = \mathrm{col}\,(r, d_\mathrm{i}, d_\mathrm{o})$ is the generalized disturbance, with r the reference, d_i being an input disturbance, and d_o an output disturbance. In this case, the controller K has a two-degree of freedom structure, meaning the tracking error and reference trajectory are separate inputs to the controller. The weighting filters are chosen as

$$\begin{aligned} W_\mathrm{s}(s) &= \frac{0.5012s + 2.005}{s + 0.02005}, & W_\mathrm{u}(s) &= \frac{s+40}{s+4000}, \\ W_\mathrm{di} &= 0.5, & W_\mathrm{do} &= 0.1. \end{aligned} \tag{4.49}$$

Note that the integral action is approximate in this case, due to the choice of W_s, as W_s includes a real pole close the origin. While this means that we are not able to use Corollary 4.2 for the realization step of the universal shifted controller design, we will see that this choice will still result in good tracking and rejection behavior of the closed-loop.

As G corresponding to the unbalanced disk (4.48) is the only nonlinear system in the generalized plant, like in Example 4.1, only a VPV embedding of (4.48) has to be constructed. We use the following VPV embedding for (4.48):

$$\begin{aligned} \dot{\theta}_\mathrm{v}(t) &= \omega_\mathrm{v}(t); \\ \dot{\omega}_\mathrm{v}(t) &= \left(\tfrac{Mgl}{J}p(t)\right)\theta_\mathrm{v}(t) - \tfrac{1}{\tau}\omega_\mathrm{v}(t) + \tfrac{K_m}{\tau}V_\mathrm{v}(t); \end{aligned} \tag{4.50}$$

where $p(t) = \eta(\theta(t)) = \cos(\theta(t))$ is the scheduling-variable which is assumed to be in $\mathcal{P} = [-1, 1]$. This corresponds to the VPV embedding region of (4.48) being

$\mathcal{X} = \mathbb{R} \times \mathbb{R}$. Note that $\dot{p}(t) = -\sin(\theta(t))\omega(t)$ for which no bounds are explicitly assumed.
Like for the previous example, we again have an affine dependency on the scheduling-variable for the VPV embedding, hence, we again use $\mathcal{L}_2$-gain polytopic LPV controller synthesis to synthesize the velocity controller, resulting in an $\mathcal{L}_2$-gain of $\gamma = 0.56$ for the velocity form of the closed-loop. The resulting realized universal shifted controller then ensures that the closed-loop is universally shifted stable and has an $\mathcal{L}_{\text{s}2}$-gain of $\gamma \leq 0.56$ (under the consideration that Proposition 4.1 is true).
Also for the unbalanced disk, a standard LPV controller is designed for comparison. For this, the primal form of the nonlinear system (4.48) embedded in an LPV representation, which results in

$$\begin{aligned} \dot{\theta}(t) &= \omega(t); \\ \dot{\omega}(t) &= \left(\tfrac{Mgl}{J} p_{\text{s}}(t))\right) \theta(t) - \tfrac{1}{\tau}\omega(t) + \tfrac{K_m}{\tau} V(t); \end{aligned} \tag{4.51}$$

where $p_{\text{s}}(t) = \eta_{\text{s}}(\theta(t)) = \frac{\sin(\theta(t))}{\theta(t)} = \text{sinc}(\theta(t))$. $\mathcal{P}_{\text{s}}$ is chosen[11] as $[-0.22, 1]$. The same generalized plant structure as for the universal shifted design is considered, see Figure 4.10, and we also use the polytopic controller synthesis method in order to synthesize the standard LPV controller. This then results in $\mathcal{L}_2$-gain for the standard LPV controller design of $\gamma = 0.56$.

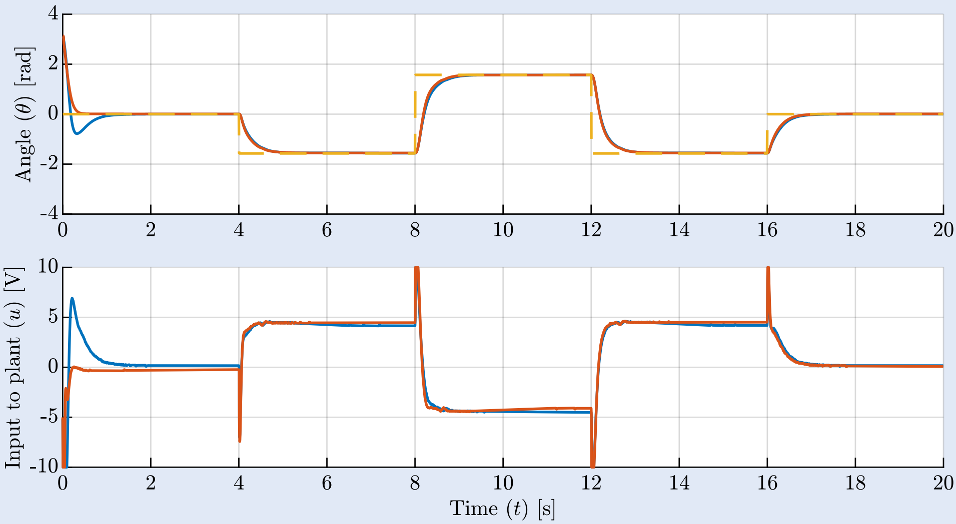


Figure 4.11: Measured angle of the unbalanced disk system (top) in closed-loop with the standard LPV (—) and the universal shifted (—) controllers under reference (- -) and no input disturbance, together with inputs to the plant (bottom) generated by the controllers.

[11] Note that $\eta_{\text{s}}(0) = 1$ as $\lim_{x\to 0} \text{sinc}(x) = 1$.

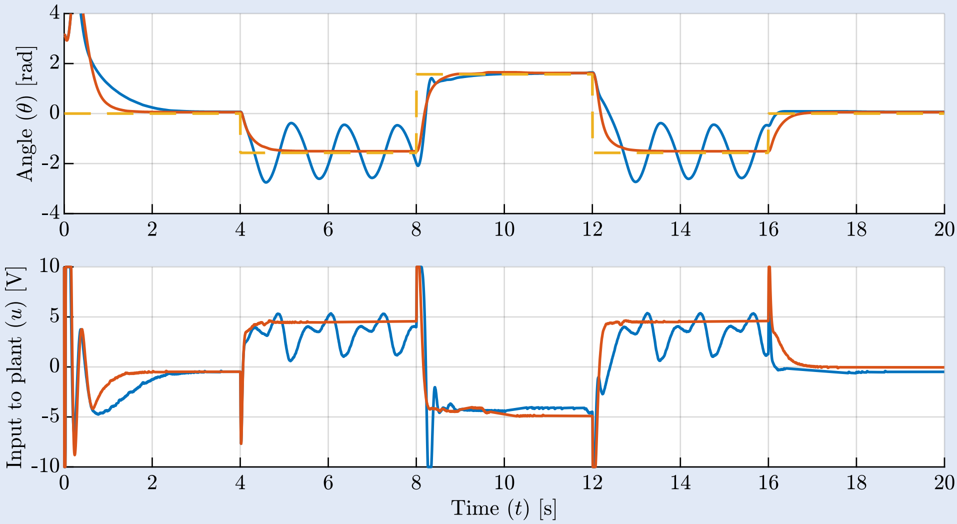


Figure 4.12: Measured angle of the unbalanced disk system (top) in closed-loop with the standard LPV (—) and the universal shifted (—) controller under reference (- -) and input disturbance, together with corresponding inputs to the plant (bottom) generated by the controllers.

Both the universal shifted controller and LPV controller are then implemented on the experimental setup, whereby, for safety, the input voltage to the system was saturated between ± 10 V. In Figure 4.11, the trajectory of the angle on the experimental setup is depicted along with the input to the plant (i.e. u) for a piece-wise constant reference signal. Note that on the experimental setup, the disk starts in the downward position, which is why the initial angle is at π radians. In Figure 4.12, the same reference trajectory is used, but a constant input disturbance of $d_\mathrm{i} = 60\,\mathrm{V}$ is introduced (which is implemented by adding 60 V to the control input that is sent to the plant before saturation). For this input disturbance, the standard LPV controller performs much worse, compared to universal shifted controller design, which has similar performance to the case when no input disturbance is applied. Both the LPV controller universal shifted controller are able to compensate the 60 V input disturbance, as visible in the control input that is sent to the plant (i.e. u in Figure 4.10), see bottom graph in Figure 4.12. However, while the control input that is sent to the plant is nearly identical for the universal shifted controller in both cases, see Figures 4.11 and 4.12, this is clearly not the case for the LPV controller, as oscillations in the signal are present when the input disturbance is applied which causes unwanted oscillation of the disk angle. While an input disturbance of 60 V is extraordinarily high for this system, and will likely never occur on the real setup, it still shows that there are inherent issues when using standard LPV controller for reference tracking and disturbance rejection.

4.7 Conclusions

As objective of our thesis, as we have discussed in Chapter 1, we intend to develop a systematic and computationally efficient framework for analysis and control of nonlinear systems to guarantee global stability and performance. In this chapter, we have build a key part of this intended framework, by developing a novel systematic and computationally efficient framework for analysis and control of nonlinear systems to guarantee universal shifted stability and performance. We have first shown how the concepts of the velocity form of a system and the corresponding notion of velocity dissipativity can be used in order to analyze universal shifted stability and performance. Namely, we have shown that velocity (Q, S, R) dissipativity implies universal shifted stability and, under assumptions, implies quadratic universal shifted performance. Through the so-called VPV embedding, we have also shown how the velocity (Q, S, R) dissipativity problem can be cast as a standard LPV analysis problem. This then allows us to analyze universal shifted stability and performance using the LPV framework. These analysis results have also been used in order to develop a novel controller synthesis method which is able to guarantee and shape universal shifted stability and performance, again making use of the LPV analysis and synthesis approaches. We have shown the controller synthesis procedure considering a universal shifted $\mathcal{L}_2$-gain performance objective, but we can apply the same procedure to also consider other universal shifted quadratic performance concepts. The benefits of the universal shifted controller design for reference tracking and disturbance rejection have also been demonstrated through a simulation and experimental study.

5

Incremental Dissipativity based Analysis

EFFICIENTLY computable (global) stability and performance analysis of nonlinear systems becomes increasingly more important in practical applications, as discussed in Chapter 1. Global stability and performance concepts allow to analyze properties of the system independent of its particular equilibrium points. In Chapter 4, we have seen how universal shifted stability and performance can be used to provide such stronger guarantees and ensure them through synthesis. In this chapter, we focus on the even stronger global concept of incremental stability and performance analysis through incremental dissipativity, whereby stability and performance w.r.t. arbitrary trajectories of the system is considered. We investigate how incremental dissipativity of nonlinear systems is linked to differential dissipativity of the system, i.e., the dissipativity of the variations along trajectories of the system. We also provide how these concepts link to other dissipativity concepts such as universal shifted and classical dissipativity. Moreover, we show how, through the *Linear Parameter-Varying* (LPV) framework, we can formulate tests for incremental dissipativity as a matrix inequalities-based conditions, which can be verified computationally efficiently. Finally, we also show how these results lead to the incremental extensions of the $\mathcal{L}_2$-gain, the generalized $\mathcal{H}_2$-norm, the $\mathcal{L}_\infty$-gain, and passivity of nonlinear systems.

5.1 Introduction

As discussed in Chapter 1, the *Linear Time-Invariant* (LTI) framework has been a systematic and easy-to-use approach for modeling, identification and control of physical systems for many years. However, growing performance demands in terms of accuracy, response speed and energy efficiency, together with increasing complexity of systems to accommodate such expectations, are pushing beyond the modeling and control capabilities of the LTI framework. Therefore, stability and performance analysis of *nonlinear* systems becomes increasingly more important.

While there are many existing tools available for analysis of nonlinear systems, see Section 1.2.2, they often require cumbersome computations and restrictive assumptions, and unlike the LTI case, they have not lead to systematic performance analysis and shaping methods. While dissipativity theory in principle allows for analysis of nonlinear systems, current results are not computationally attractive. Furthermore, they only provide local stability and performance guarantees, i.e., only w.r.t. a single point of natural storage (usually the origin), which is undesirable for disturbance rejection and reference tracking. Hence, there is need for a computationally efficient analysis tool for *global* stability and performance properties of nonlinear systems. Such analysis tools will also be key for developing synthesis results in order to achieve the goal of this thesis, see Section 1.4, of developing a systematic, computationally efficient, framework for nonlinear analysis and control with global stability and performance guarantees.

In Chapter 4, we have already seen the how one such global stability concept, universal shifted stability and performance, can be used to provide guarantees w.r.t. all equilibrium points of the system, instead of w.r.t. only a single equilibrium point. Moreover, we have also seen how, through the velocity form and the *Linear Parameter-Varying* (LPV) framework, we could convexify the resulting analysis problem. The results on universal shifted stability and performance have proved to be especially useful in analysis and control problems where tracking and rejection of constant signals is required. However, in many applications we do not only want to ensure stability and performance w.r.t. constant trajectories, i.e., equilibrium points, of the system, but also w.r.t. other trajectories of the system (i.e., w.r.t. time-varying signals). As already briefly discussed in Section 1.2.2, incremental stability (and performance) is one such global concept which allows to analyze this. Namely, incremental stability (Angeli 2002) analyzes stability of a system w.r.t. arbitrary trajectories of the system. Similar stability notions have also been developed, such as contraction (Lohmiller and Slotine 1998; Manchester and Slotine 2018) and convergence theory (Pavlov, Pogromsky, et al. 2004) with strong connections to incremental stability theory (Rüffer et al. 2013). Similar notions for performance have also been introduced such as incremental $\mathcal{L}_2$-gain (Fromion, Monaco, et al. 2001) and passivity (Jayawardhana 2006; Pavlov and Marconi 2008). With further extensions also towards global dissipativity analysis through differential dissipativity (Forni and Sepulchre 2013; Forni, Sepulchre, and Van der Schaft 2013; Van der Schaft 2013) and incremental dissipativity (Pavlov and Marconi 2008). However, they do not provide computationally efficient methods to verify these dissipativity notions. Current works discussing differential and incremental dissipativity only focus on passivity based performance. How the various dissipativity notions are linked to universal shifted dissipativity or classical dissipativity is generally not discussed.

To address these shortcomings, the main contributions in this chapter are (i) conditions on general quadratic performance analysis using incremental dissipativity for *Continuous-Time* (CT) nonlinear systems, (ii) establishing the missing link between the various dissipativity concepts for CT nonlinear systems, and (iii) computationally efficient convex tools to analyze incremental stability and performance of CT nonlinear systems. This is achieved by developing a general incremental dissipativity framework that connects differential, incremental, universal shifted, and classical dissipativity. Through these results, incremental notions of the $\mathcal{L}_2$-gain, the generalized $\mathcal{H}_2$-norm, the $\mathcal{L}_\infty$-gain and passivity are systematically introduced, also recovering some of the existing results on these concepts. Finally, convex analysis tools to compute the resulting conditions for differential and incremental dissipativity are derived using the LPV framework. Later, in Chapter 7, we also extend the results of this chapter to *Discrete-Time* (DT) nonlinear systems.

In Section 5.2, some preliminaries are introduced and a formal problem statement is given. Section 5.3 gives the main results on differential and incremental dissipativity and their connection to universal shifted dissipativity and classical dissipativity. In Section 5.4, the incremental extensions of well-known performance measures are derived and the use of the LPV framework is discussed to obtain convex computation methods to analyze these performance measures. The introduced concepts and methods are demonstrated on two academic examples in Section 5.5. Finally, in Section 5.6, conclusions are drawn from the presented results.

5.2 Incremental and Differential Dissipativity

Similar to Section 2.2, in this chapter, we consider nonlinear, time-invariant, CT systems given by

$$\dot{x}(t) = f(x(t), w(t)); \tag{5.1a}$$
$$z(t) = h(x(t), w(t)); \tag{5.1b}$$

where $t \in \mathbb{R}_0^+$ is time, $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$ is the state, $w(t) \in \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{w}}$ is the input, and $z(t) \in \mathcal{Z} \subseteq \mathbb{R}^{n_\mathrm{z}}$ is the output of the system. We consider the sets $\mathcal{X}$, $\mathcal{W}$ and $\mathcal{Z}$ to be open sets containing the origin, with $\mathcal{X}$, $\mathcal{W}$ being convex, and the mappings $f : \mathcal{X} \times \mathcal{W} \to \mathbb{R}^{n_\mathrm{x}}$ and $h : \mathcal{X} \times \mathcal{W} \to \mathcal{Z}$ to be in $\mathcal{C}_1$. The state-transition map is again $\phi_\mathrm{x} : \mathbb{R} \times \mathbb{R} \times \mathcal{X} \times \mathcal{W}^{\mathbb{R}_0^+} \to \mathcal{X}$, describing the evolution of the state such that

$$x(t) = \phi_\mathrm{x}(t, 0, x_0, w), \tag{5.2}$$

with $x_0 = x(0)$. The behavior of the system, i.e., the set of all possible solutions, is denoted by

$$\mathfrak{B} := \{(x, w, z) \in (\mathcal{X} \times \mathcal{W} \times \mathcal{Z})^{\mathbb{R}_0^+} \mid x \in \mathcal{C}_1 \text{ and } (x, w, z) \text{ satisfies (5.1)}\}. \tag{5.3}$$

Note that $\mathfrak{B} \subseteq \mathcal{B}^{\mathbb{R}_0^+}$, where $\mathcal{B} = \mathcal{X} \times \mathcal{W} \times \mathcal{Z}$ is called the signal value set.

Like in the previous chapter, the form presented in (5.1) will be referred to as the *primal form* of the nonlinear system. As we have discussed in Section 2.4, for the primal form, an extensive dissipativity theory has been developed over the years, with its roots in (Willems 1972). From the classical dissipativity notion many system properties can be derived, such as performance characteristics and stability, as well as a link with the physical interpretation of the system. Therefore, dissipativity is an important fundament in nonlinear system theory. However, a shortcoming of classical dissipativity theory that it is always considers the energy w.r.t. a single point of neutral storage, often being the origin of the state-space representation. If the system is nonlinear, this analysis is different for each considered neutral storage point and unlike in the LTI case, this difference cannot be eliminated by a coordinate transformation. This means that performance and stability analysis through classical dissipativity is equilibrium point *dependent*.

An extension to this concept is *incremental dissipativity*, i.e., analysis of the (dissipated) energy flow between any two system trajectories. We give an extension of the definition of incremental passivity in (Van der Schaft 2017, Def. 4.7.1). Let us first give the following definition:

Definition 5.1 ($\mathcal{Q}_\mathrm{i}$-space)**.** *A function $V_\mathrm{i} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_0^+$ is in $\mathcal{Q}_\mathrm{i}$, with $\mathcal{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$, if $V_\mathrm{i}(\cdot, \tilde{x}) \in \mathcal{Q}_{\tilde{x}}$ for all $\tilde{x} \in \mathcal{X}$ and $V_\mathrm{i}(x, \cdot) \in \mathcal{Q}_x$ for all $x \in \mathcal{X}$. See also Definition 2.7.*

Definition 5.2 (Incremental Dissipativity)**.** *The system (5.1) is called incrementally dissipative w.r.t. the supply function $s_\mathrm{i} : \mathcal{W} \times \mathcal{W} \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$, if there exists a storage function $\mathcal{V}_\mathrm{i} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_0^+$ with $\mathcal{V}_\mathrm{i} \in \mathcal{C}_0$ and $\mathcal{V}_\mathrm{i} \in \mathcal{Q}_\mathrm{i}$, such that for any two trajectories $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$,*

$$\mathcal{V}_\mathrm{i}\big(x(t_1), \tilde{x}(t_1)\big) - \mathcal{V}_\mathrm{i}\big(x(t_0), \tilde{x}(t_0)\big) \le \int_{t_0}^{t_1} s_\mathrm{i}\big(w(t), \tilde{w}(t), z(t), \tilde{z}(t)\big)\, dt, \tag{5.4}$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_0 \le t_1$.

Besides analyzing the difference between two trajectories, it is also possible to analyze *infinitesimal* variations of trajectories. First, define the set of paths

$$\Gamma_\Phi(\varphi, \tilde{\varphi}) := \{\bar{\varphi} \in \Phi^{[0,1]} \mid \bar{\varphi} \in \mathcal{C}_1,\, \bar{\varphi}(0) = \tilde{\varphi},\, \bar{\varphi}(1) = \varphi\}, \tag{5.5}$$

corresponding to all smooth paths along $\lambda \in [0, 1]$, between the points $\varphi \in \Phi \subseteq \mathbb{R}^\mathrm{n}$ and $\tilde{\varphi} \in \Phi \subseteq \mathbb{R}^\mathrm{n}$. Consider any two trajectories of (5.1): $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$. As $\mathcal{X}$, $\mathcal{W}$ are considered to be convex, any trajectory *between* these can be parametrized with $\bar{x}_0 \in \Gamma_\mathcal{X}(x_0, \tilde{x}_0)$ and $\bar{w}(t) \in \Gamma_\mathcal{W}(w(t), \tilde{w}(t))$, e.g., $\bar{x}_0(\lambda) = \tilde{x}_0 + \lambda(x_0 - \tilde{x}_0) \in \mathcal{X}$ and $\bar{w}(t, \lambda) = \tilde{w}(t) + \lambda(w(t) - \tilde{w}(t)) \in \mathcal{W}$, resulting in the state transition map $\bar{x}(t, \lambda) = \phi_\mathrm{x}(t, t_0, \bar{x}_0(\lambda), \bar{w}(\lambda)) \in \mathcal{X}$. Note that this parametrization covers transitions between all possible solutions in $\mathfrak{B}$. Given a λ, it holds that

$$\dot{\bar{x}}(t, \lambda) = f(\bar{x}(t, \lambda), \bar{w}(t, \lambda)); \tag{5.6a}$$

$$\bar{z}(t, \lambda) = h(\bar{x}(t, \lambda), \bar{w}(t, \lambda)), \tag{5.6b}$$

with $\lambda \in [0,1]$ and the resulting collection of trajectories $(\bar{x}(\lambda), \bar{w}(\lambda), \bar{z}(\lambda)) \in \mathfrak{B}$, also depicted in Figure 5.1. Note that λ here is a constant that parametrizes a collection of solution trajectories that represent a transition from one particular solution trajectory (x, w, z) to another trajectory $(\tilde{x}, \tilde{w}, \tilde{z})$.

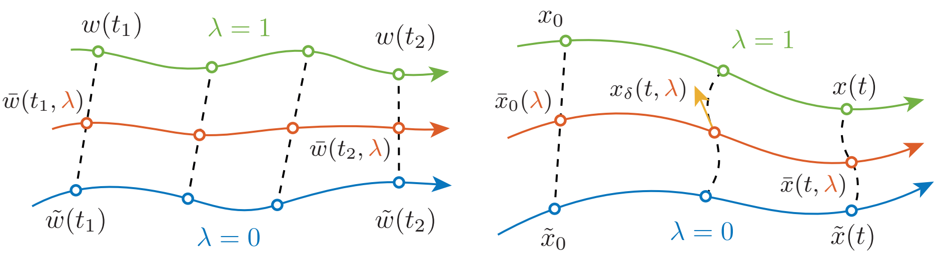


Figure 5.1: Collection of trajectories $(\bar{x}(\lambda), \bar{w}(\lambda), \bar{z}(\lambda)) \in \mathfrak{B}$ parametrized in $\lambda \in [0,1]$ between two arbitrary state and input trajectories, where $\bar{w}(t,\lambda) = \tilde{w}(t) + \lambda(w(t) - \tilde{w}(t))$ and $\bar{x}_0(\lambda) = \tilde{x}_0 + \lambda(x_0 - \tilde{x}_0)$ with the variation of $\bar{x}$ along λ given by x_δ.

As $f, h \in \mathcal{C}_1$, taking the derivative of (5.6) w.r.t. λ, the infinitesimal variation of the original trajectories, can be analyzed. A similar concept has been introduced in (Crouch and Van der Schaft 1987; Forni, Sepulchre, and Van der Schaft 2013; Reyes-Báez 2019) as *variational dynamics*[1]. Differentiation of (5.6) w.r.t. λ yields the variational system

$$\dot{x}_\delta(t,\lambda) = A_\delta(\bar{x}(t,\lambda), \bar{w}(t,\lambda))x_\delta(t,\lambda) + B_\delta(\bar{x}(t,\lambda), \bar{w}(t,\lambda))w_\delta(t,\lambda); \tag{5.7a}$$
$$z_\delta(t,\lambda) = C_\delta(\bar{x}(t,\lambda), \bar{w}(t,\lambda))x_\delta(t,\lambda) + D_\delta(\bar{x}(t,\lambda), \bar{w}(t,\lambda))w_\delta(t,\lambda); \tag{5.7b}$$

with $x_\delta(t,\lambda) = \frac{\partial \bar{x}}{\partial \lambda}(t,\lambda) \in \mathbb{R}^{n_\mathrm{x}}$, $w_\delta(t,\lambda) = \frac{\partial \bar{w}}{\partial \lambda}(t,\lambda) \in \mathbb{R}^{n_\mathrm{w}}$, $z_\delta(t,\lambda) = \frac{\partial \bar{z}}{\partial \lambda}(t,\lambda) \in \mathbb{R}^{n_\mathrm{z}}$, and

$$A_\delta = \frac{\partial f}{\partial x}, \quad B_\delta = \frac{\partial f}{\partial w}, \quad C_\delta = \frac{\partial h}{\partial x}, \quad D_\delta = \frac{\partial h}{\partial w}, \tag{5.8}$$

where $(\bar{x}(\lambda), \bar{w}(\lambda)) \in \pi_{\mathrm{x,w}}\mathfrak{B}$ for all $\lambda \in [0,\, 1]$. Note that for a trajectory in $\mathfrak{B}$, λ is fixed for all time. For a trajectory $x(\lambda)$, corresponding to a given $\lambda \in [0,1]$, its variations are captured in $x_\delta(\lambda)$. In the sequel, we will generally omit λ for brevity, e.g., $x_\delta(t) = x_\delta(t,\lambda)$ (which holds for any $\lambda \in [0,\, 1]$). In this chapter, we will refer to (5.7) as the *differential form* of the nonlinear system (5.1). Let us denote the behavior of (5.7) for a particular $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,w}}\mathfrak{B}$ by (omitting dependency on λ)

$$\mathfrak{B}_\delta(\bar{x}, \bar{w}) := \Big\{(x_\delta, w_\delta, z_\delta) \in (\mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}} \times \mathbb{R}^{n_\mathrm{z}})^{\mathbb{R}_0^+} \mid x_\delta \in \mathcal{C}_1, \\ (x_\delta, w_\delta, z_\delta) \text{ satisfies (5.7) along } (\bar{x}, \bar{w})\Big\}, \tag{5.9}$$

and the full behavior by $\breve{\mathfrak{B}}_\delta = \bigcup_{(\bar{x},\bar{w}) \in \pi_{\mathrm{x,w}}\mathfrak{B}} \mathfrak{B}_\delta(\bar{x}, \bar{w})$.

[1] In fact, we can obtain a variational system for any smooth (x, w) parametrization (see (Reyes-Báez 2019) for an alternative approach).

With the differential form of a system defined, we can define the notion of *differential dissipativity*, interpreted as the 'energy' dissipation of variations of the system trajectory that are not forced by the input. If the energy of these variations in the system trajectories decreases over time, the trajectory will eventually only be determined by the input of the system. Hence, the primal form of the system will converge to a *steady-state* solution, which is not necessary a forced equilibrium point, e.g., it can be a periodic orbit. We use the definition adopted from (Forni and Sepulchre 2013).

Definition 5.3 (Differential dissipativity). *Consider a system (5.1) and its differential form (5.7). The system is differentially dissipative w.r.t. a supply function* $s_\delta : \mathbb{R}^{n_\mathrm{w}} \times \mathbb{R}^{n_\mathrm{z}} \to \mathbb{R}$, *if there exists a storage function* $\mathcal{V}_\delta : \mathcal{X} \times \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}_0^+$ *with* $\mathcal{V}_\delta \in \mathcal{C}_0$ *and* $\mathcal{V}_\delta(\bar{x}, \cdot) \in \mathcal{Q}_0$, $\forall\, \bar{x} \in \mathcal{X}$, *such that*

$$\mathcal{V}_\delta\big(\bar{x}(t_1), x_\delta(t_1)\big) - \mathcal{V}_\delta\big(\bar{x}(t_0), x_\delta(t_0)\big) \leq \int_{t_0}^{t_1} s_\delta\big(w_\delta(t), z_\delta(t)\big)\, dt, \tag{5.10}$$

for all $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,w}} \mathfrak{B}$ *and for all* $t_0, t_1 \in \mathbb{R}_0^+$, *with* $t_0 \leq t_1$.

Differential *passivity* definitions can be found in (Forni, Sepulchre, and Van der Schaft 2013; Van der Schaft 2013). The condition for differential dissipativity (5.10) can be interpreted as the differential form of the system (5.7) being classically dissipative (along all solutions of the system (5.1)), see also Definition 2.13.

Remark 5.1. Note that when the incremental and differential storage functions $\mathcal{V}_\mathrm{i}$ and $\mathcal{V}_\delta$ are differentiable, we can also define equivalent conditions in terms of the (time) *differentiated* forms of (5.4) and (5.10). Similar as for classical dissipativity in (2.22).

Despite the interest in classical dissipativity, universal shifted dissipativity, incremental dissipativity, and differential dissipativity, the underlying connection between these notions have not been explored in the literature yet. We will establish this connection in case of quadratic supply functions in the next section based on which performance analysis of nonlinear systems is achieved. Furthermore, we will discuss implications of these dissipativity notions on stability as well.

5.3 Global Stability and Performance Analysis

5.3.1 Differential (Q, S, R) dissipativity

In this section, we present results on how differential dissipativity can be used in order to analyze global stability and performance of systems. We will first examine differential dissipativity with quadratic supply functions.

Consider the differential form (5.7) of a nonlinear system, which describes the variation of the system over a trajectory $(\bar{x}, \bar{w}, \bar{z}) \in \mathfrak{B}$. Note that this system always exists if the mappings f and h are in $\mathcal{C}_1$. To formulate our results for differential dissipativity, we consider a quadratic storage function of the form

$$\mathcal{V}_\delta(\bar{x}, x_\delta) = x_\delta^\top M(\bar{x}) x_\delta. \tag{5.11}$$

For $\mathcal{V}_\delta$ of the form (5.11) to be a differential storage function (see Definition 5.3), we have the following Condition:

Condition 5.1. *The matrix function $M : \mathcal{X} \to \mathbb{S}^{n_\mathrm{x}}$ with $M \in \mathcal{C}_1$ is real, symmetric, bounded and positive definite, i.e., $\exists\, \alpha_1, \alpha_2 \in \mathbb{R}^+$, such that for all $\bar{x} \in \mathcal{X}$, $\alpha_1 I \preceq M(\bar{x}) \preceq \alpha_2 I$.*

This storage function represents a measure of the energy of the variation along the state trajectory $\bar{x}$. We consider the following quadratic supply function,

$$s_\delta(w_\delta,\, z_\delta) = \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix}, \tag{5.12}$$

with matrices $Q \in \mathbb{S}^{n_\mathrm{w}}$, $S \in \mathbb{R}^{n_\mathrm{w} \times n_\mathrm{z}}$, and $R \in \mathbb{S}^{n_\mathrm{z}}$. We will refer to differentially dissipativity w.r.t. the quadratic supply function (5.12) as differential (Q, S, R) dissipativity. For the system in primal form given by (5.1), let us also consider the set $\mathcal{D}$ for which holds that $\frac{d}{dt} x(t) = \dot{x}(t) \in \mathcal{D}$ for all $t \in \mathbb{R}_0^+$. Based on these considerations, we can formulate the following theorem.

Theorem 5.1 (Differential (Q, S, R) dissipativity condition)**.** *The system in primal form (5.1) is differentially (Q, S, R) dissipative, under a quadratic storage function (5.11), if there exits a matrix function $M : \mathcal{X} \to \mathbb{S}^{n_\mathrm{x}}$ satisfying Condition 5.1, such that for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$ and $\bar{x}_\mathrm{v} \in \mathcal{D}$ it holds that*

$$(\star)^\top \begin{bmatrix} \partial M(\bar{x}, \bar{x}_\mathrm{v}) & M(\bar{x}) \\ \star & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\delta(\bar{x}, \bar{w}) & B_\delta(\bar{x}, \bar{w}) \end{bmatrix} - \\ (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\delta(\bar{x}, \bar{w}) & D_\delta(\bar{x}, \bar{w}) \end{bmatrix} \preceq 0, \tag{5.13}$$

where $\partial M(\bar{x}, \bar{x}_\mathrm{v}) = \sum_{i=1}^{n_\mathrm{x}} \frac{\partial M(\bar{x})}{\partial \bar{x}_i} \bar{x}_{\mathrm{v},i}$ ($\bar{x}_i$ and $\bar{x}_{\mathrm{v},i}$ denote the i'th element of $\bar{x}$ and $\bar{x}_\mathrm{v}$, respectively) and $A_\delta, \ldots, D_\delta$ are given as in (5.8).

Proof. See Appendix B.3. □

5.3.2 Inducing incremental dissipativity

First, we show that the property of differential dissipativity under supply function (5.12) and storage function (5.11) implies the property of incremental dissipativity with supply function

$$s_\mathrm{i}(w, \tilde{w}, z, \tilde{z}) = \begin{bmatrix} w - \tilde{w} \\ z - \tilde{z} \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w - \tilde{w} \\ z - \tilde{z} \end{bmatrix}. \tag{5.14}$$

We will refer to incremental dissipativity w.r.t. the supply function (5.14) as *incremental* (Q, S, R) *dissipativity.* Secondly, we give a computable condition to analyze incremental dissipativity. The following result is the core of our contribution of this chapter.

Theorem 5.2 (Induced incremental dissipativity)**.** *When the system in primal form (5.1) is differentially* (Q, S, R) *dissipative for a tuple* (Q, S, R) *for which* $R \preceq 0$*, under a storage function* $\mathcal{V}_\delta$ *of the form (5.11), then, there exists a storage function* $\mathcal{V}_\mathrm{i}$ *such that the system is incrementally* (Q, S, R) *dissipative for the same tuple* (Q, S, R)*.*

Proof. See Appendix B.3. □

Remark 5.2 (Restricted R). The restriction $R \preceq 0$ is a technical necessity in the proof of Theorem 5.2. In case of $R \succ 0$ or R being indefinite, validity of Theorem 5.2 is an open question. However, for supply functions of the form (5.12), $R \preceq 0$ is also required to infer incremental stability from incremental dissipativity, see Remark 5.3. Hence, this assumption is not restrictive.

Comparing Theorem 5.2 to existing results in this context, we want to highlight that (Waitman et al. 2016a; Waitman et al. 2016b) also give some results on incremental dissipativity. However, these works only focus on a specific and restrictive form of the supply function. Moreover, the technical result of (Waitman et al. 2016b) refers to a proof in a paper that has never appeared to the author's knowledge.

From Theorem 5.2, we have the following (trivial) result:

Corollary 5.1 (Incremental (Q, S, R) dissipativity condition)**.** *The system in primal form (5.1) is incrementally* (Q, S, R) *dissipative with* $R \preceq 0$*, if (5.13) holds for all* $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$ *and* $\bar{x}_\mathrm{v} \in \mathcal{D}$ *with* M *satisfying Condition 5.1.*

Corollary 5.1 gives a sufficient condition to verify incremental (Q, S, R) dissipativity of a general nonlinear system. Note that by this result, if the matrix inequality (5.13) holds for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$ and $\bar{x}_\mathrm{v} \in \mathcal{D}$, then we know that there exist a valid storage function $\mathcal{V}_\mathrm{i}$ (of the form (B.75), see the proof of Theorem 5.2 in Appendix B.3) such that (5.4) is satisfied. However, calculating this function in an explicit form might be difficult (see Section 5.3.3). If no positive definite M can be found to satisfy (5.13), then it does *not* necessarily mean that the system is not differentially or incrementally dissipative. Inequality (5.13) might hold for a non-quadratic $\mathcal{V}_\delta$, or a more complex M.

5.3.3 Explicit incremental storage function

Even if deriving an explicit form of $\mathcal{V}_\mathrm{i}$ is challenging in general, we can take an extra assumption for the quadratic form (5.11) to give an explicit construction:

Assumption 5.1. $M(\bar{x})$ *can be decomposed as* $M(\bar{x}) = N(\bar{x})^\top P N(\bar{x})$*,* $P \in \mathbb{S}^{n_\mathrm{x}}$ *with* $P \succ 0$ *and* $N(\bar{x}) \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$ *s.t.* $\exists\, \mu : \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}^{n_\mathrm{x}}$ *s.t.* $\frac{\partial \mu(\bar{x})}{\partial \bar{x}} = N(\bar{x})$*.*

While this decomposition of $M(\bar{x})$ is always possible if it satisfies Condition 5.1, see Appendix C.5, existence of μ such that $\frac{\partial \mu(\bar{x})}{\partial \bar{x}} = N(\bar{x})$ is not guaranteed for any $M(\bar{x})$. This illustrates well the challenges for obtaining an explicit construction of $\mathcal{V}_{\mathrm{i}}$. For the sake of simplicity, we assume in the remainder of this subsection that $\mathcal{X} = \mathbb{R}^{n_{\mathrm{x}}}$.

Lemma 5.1 (Induced incremental storage function)**.** *If the system in primal form (5.1) is differentially dissipative with a storage function*

$$\mathcal{V}_\delta(\bar{x}, x_\delta) = x_\delta^\top M(\bar{x}) x_\delta,$$

where M satisfies Condition 5.1 and Assumption 5.1, then, the incremental storage function $\mathcal{V}_{\mathrm{i}}$ in Theorem 5.2 is given by

$$\mathcal{V}_{\mathrm{i}}(x, \tilde{x}) = (\mu(x) - \mu(\tilde{x}))^\top P(\mu(x) - \mu(\tilde{x})). \tag{5.15}$$

Additionally, if $M(\bar{x}) = M \in \mathbb{S}^{n_{\mathrm{x}}}$ for all $\bar{x} \in \mathcal{X}$ with $M \succ 0$, then, the incremental storage function simplifies to

$$\mathcal{V}_{\mathrm{i}}(x, \tilde{x}) = (x - \tilde{x})^\top M(x - \tilde{x}). \tag{5.16}$$

Proof. See Appendix B.3. □

In case $\mathcal{X}$ is a bounded convex set, Lemma 5.1 can be also shown to hold true, if either beyond Assumption 5.1 it holds that $\mu(\mathcal{X})$ is also convex, or if M is a constant matrix.

5.3.4 Relation to other dissipativity notions

We now show that incremental dissipativity implies also the previously discussed dissipativity notions in this thesis. Namely, we will show incremental dissipativity implies universal shifted dissipativity, see Definition 4.2, and also classical dissipativity, Definition 2.13, assuming the origin is an equilibrium point of the system, with corresponding zero input and output.

Theorem 5.3 (Induced dissipativity)**.** *Consider a nonlinear system in its primal form (5.1). If the system is incrementally (Q, S, R) dissipative and satisfies*[2] *Assumption 4.1, then it is universally shifted (Q, S, R) dissipative, see Definition 4.2. If*[3] $(0, 0, 0) \in \mathscr{E}$, *the system is also classically (Q, S, R) dissipative.*

Proof. See Appendix B.3. □

[2]In case the system is also incrementally asymptotically stable (see Definition 5.4 and Theorem 5.5), this assumption can be dropped.

[3]Meaning the origin is an equilibrium point of the system, corresponding to zero input and output, see also (2.5).

By this last result, we have obtained a *chain of implications*, which connect the various dissipativity notions. Moreover, we have given a condition (matrix inequality (5.13)) that then allows to examine differential, incremental, universal shifted, and classical dissipativity and thus examine global stability and performance of a nonlinear system. This chain of implications is summarized in Figure 5.2. A result similar to Theorem 5.3 is given in (Liu et al. 2014) for single-input-single-output networked nonlinear systems and only considering the connection between incremental and classical dissipativity. However, note that Theorem 5.3 is more general, as it holds for general nonlinear multi-input-multi-output systems of the form (5.1) and also highlights the connection to universal shifted dissipativity.

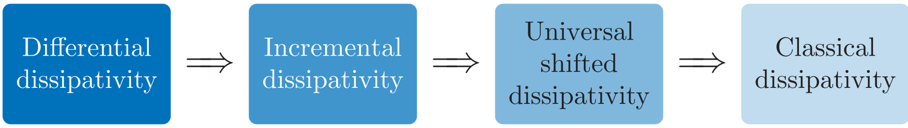


Figure 5.2: Chain of implications with the dissipativity notions for supply functions of the form (5.12).

For LTI systems (1.3), it is also trivial to see how these different dissipativity concepts are equivalent to each other. Namely, as for the differential form of an LTI state-space system (1.3) we have that $A_\delta(\bar{x}, \bar{w}) = A, \ldots, D_\delta(\bar{x}, \bar{w}) = D$. Consequently, the condition for differential dissipativity in Definition 5.3 is in that case equivalent to the classical dissipativity condition in Definition 2.13. This also further highlights why global stability and performance concepts do not have to be considered for analysis and control of LTI systems, as the standard/classical stability and performance concepts already guarantee them in that case.

5.3.5 Relation to stability notions

For classical dissipativity, it is well known how it connects to stability of the system (at the origin of the state-space representation), see also Section 2.4.2. Similarly, we will show how incremental dissipativity connects to an incremental notion of stability.

Before providing these results, we will first introduce some (existing) results on incremental stability. While definitions of incremental stability are often being given in terms of comparison functions, see e.g. (Angeli 2002; Tran et al. 2016), we will here use a definition more along the lines of 'standard' stability given in Definition 2.3, which is also consistent with the definitions in literature.

Definition 5.4 (Incremental stability). *The nonlinear system (5.1) is*

- *incrementally stable, if for each $\epsilon > 0$, there exists a $\delta(\epsilon) > 0$ such that*

$$\|x(0) - \tilde{x}(0)\| < \delta(\epsilon) \implies \|x(t) - \tilde{x}(t)\| < \epsilon; \tag{5.17}$$

for all $t \in \mathbb{R}$, $x, \tilde{x} \in \mathfrak{B}_\mathrm{w}(w)$, and measurable and bounded functions $w \in \mathcal{W}^{\mathbb{R}_0^+}$.

- *incrementally asymptotically stable, if it is stable and attractive, i.e., there exists a* $\delta > 0$ *such that*

$$\|x(0) - \tilde{x}(0)\| < \delta \implies \lim_{t\to\infty} \|\phi_{\mathrm{x}}(t, 0, x(0), w) - \phi_{\mathrm{x}}(t, 0, \tilde{x}(0), w)\| = 0; \tag{5.18}$$

for all $x, \tilde{x} \in \mathfrak{B}_{\mathrm{w}}(w)$ *and measurable and bounded functions* $w \in \mathcal{W}^{\mathbb{R}_0^+}$.

By taking one trajectory equal to an equilibrium point of the system (for the corresponding constant input), one can easily see that by this definition incremental stability also implies universal shifted stability, see Definition 4.1.

Similar to standard stability, see Section 2.3.1, we can use (incremental) Lyapunov functions to analyze (incremental) stability of the system. The following is adapted from (Angeli 2002):

Theorem 5.4 (Incremental Lyapunov stability)**.** *The nonlinear system given by (5.1) is incrementally stable, if there exists a positive definite, so-called incremental Lyapunov function* $V_{\mathrm{i}} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_0^+$ *with* $V_{\mathrm{i}} \in \mathcal{C}_1$ *and* $V_{\mathrm{i}} \in \mathcal{Q}_{\mathrm{i}}$*, such that*

$$\frac{d}{dt} V_{\mathrm{i}}(x(t), \tilde{x}(t)) \leq 0, \tag{5.19}$$

for all $t \in \mathbb{R}_0^+$ *and* $x, \tilde{x} \in \pi_{\mathrm{x}}\, \mathfrak{B}_{\mathrm{w}}(w)$ *under all measurable and bounded functions* $w \in \mathcal{W}^{\mathbb{R}_0^+}$*. Moreover, the nonlinear system is incrementally asymptotically stable, if (5.19) holds, but with strict inequality except when* $x(t) = \tilde{x}(t)$*.*

Having introduced these results on incremental stability, we will next show how it connects to incremental dissipativity.

Theorem 5.5 (Incremental stability implied by incremental dissipativity)**.** *Assume the nonlinear system given by (5.1) is incrementally dissipative under a storage function* $V_{\mathrm{i}} \in \mathcal{C}_1$ *w.r.t. a supply function* s *that satisfies*

$$s_{\mathrm{i}}(w, w, z, \tilde{z}) \leq 0, \tag{5.20}$$

for all $w \in \mathcal{W}$ *and all* $z, \tilde{z} \in \mathcal{Z}$*, then, the nonlinear system is incrementally stable. If the supply function satisfies (5.20), but with strict inequality when* $z \neq \tilde{z}$*, and the system is observable (see Definition 2.2), then the nonlinear system is incrementally asymptotically stable.*

Proof. See Appendix B.3. □

Remark 5.3. For quadratic supply functions of the form (5.14) the conditions for incremental stability of Theorem 5.5 correspond to $R \preceq 0$ (or to $R \prec 0$ in order to imply incremental asymptotic stability). Note, that this condition on R is also required for Theorem 5.2. See also Remark 5.2.

Similar to analyzing incremental dissipativity through the differential form, results have also been derived in literature to analyze incremental *stability* using the differential form (Lohmiller and Slotine 1998; Pavlov, Pogromsky, et al. 2004; Jouffroy and Fossen 2010; Rüffer et al. 2013; Forni and Sepulchre 2014):

Theorem 5.6 (Induced incremental stability)**.** *The system given by (5.1), with differential form (5.7), is incrementally stable, if there exists a quadratic positive definite function* $V_\delta : \mathcal{X} \times \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}_0^+$ *of the form (5.11), such that*

$$\frac{d}{dt} V_\delta\big(\bar{x}(t), x_\delta(t)\big) \leq 0, \tag{5.21}$$

for all[4] $\bar{x} \in \pi_\mathrm{x} \mathfrak{B}_\mathrm{w}(w)$ *under all measurable and bounded* $w \in \mathcal{W}^{\mathbb{R}_0^+}$ *and for all* $t \in \mathbb{R}_0^+$*. If (5.21) holds, but with strict inequality except when*[5] $x(t) = \tilde{x}(t)$*, then the system (5.7) is incrementally asymptotically stable.*

The condition given in (5.21) can be interpreted as the differential form being (asymptotically) stable (along all trajectories of the system).

The result of Theorem 5.6 can then also be used to derive a matrix inequality condition for incremental stability, as has been reported in (Lohmiller and Slotine 1998; Pavlov, Pogromsky, et al. 2004; Jouffroy and Fossen 2010; Rüffer et al. 2013). For the system in primal form (5.1), we consider again the set $\mathcal{D}$ for which holds that $\frac{d}{dt}x(t) = \dot{x}(t) \in \mathcal{D}$ for all $t \in \mathbb{R}_0^+$.

Lemma 5.2 (Incremental stability condition)**.** *The system given by (5.1) is incrementally stable, if*

$$A_\delta(\bar{x}, \bar{w})^\top M(\bar{x}) + M(\bar{x}) A_\delta(\bar{x}, \bar{w}) + \partial M(\bar{x}, \bar{x}_\mathrm{v}) \preceq 0, \tag{5.22}$$

for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$*,* $\bar{x}_\mathrm{v} \in \mathcal{D}$ *and* M *satisfying Condition 5.1. If (5.22) holds, but with strict inequality, then, the system is incrementally asymptotically stable.*

Note that the above results also gives us a similar chain of implications as discussed in Section 5.3.5, where now we have that the condition in Lemma 5.2 implies 'stability of the differential form', implying incremental stability, which in turn implies universal shifted stability, which finally implies standard stability (under the condition that the origin is an equilibrium point). Therefore, like for dissipativity, we have one condition (the matrix inequality in Lemma 5.2) that allows us to examine incremental, universal shifted, and standard stability.

5.3.6 Relation to velocity dissipativity

In Chapter 4, we have also discussed how universal shifted stability and performance can be analyzed through the velocity form and corresponding notion of velocity

[4] Note that this corresponds to $\bar{w}(\lambda) = w \in \pi_\mathrm{w} \mathfrak{B}$, $\forall\, \lambda \in [0, 1]$, hence, $w_\delta = 0$.
[5] Corresponding to $x_\delta(t) = 0$.

dissipativity, specifically see Section 4.3.

For a nonlinear system given by (5.1) its velocity form, see Definition 4.5, is given by

$$\ddot{x}(t) = A_{\mathrm{v}}(x(t), w(t))\dot{x}(t) + B_{\mathrm{v}}(x(t), w(t))\dot{w}(t); \tag{5.23a}$$
$$\dot{z}(t) = C_{\mathrm{v}}(x(t), w(t))\dot{x}(t) + D_{\mathrm{v}}(x(t), w(t))\dot{w}(t); \tag{5.23b}$$

where $A_{\mathrm{v}} = \frac{\partial f}{\partial x}$, $B_{\mathrm{v}} = \frac{\partial f}{\partial w}$, $C_{\mathrm{v}} = \frac{\partial h}{\partial x}$, $D_{\mathrm{v}} = \frac{\partial h}{\partial w}$, and $(x, w, z) \in \mathfrak{B}_{\mathrm{c}}$ (see (4.13)). Moreover, the solution set of (5.23) is given by

$$\begin{aligned}\mathfrak{B}_{\mathrm{v}} = \big\{(\dot{x}, \dot{w}, \dot{z}) \in (\mathbb{R}^{n_{\mathrm{x}}} \times \mathbb{R}^{n_{\mathrm{w}}} \times \mathbb{R}^{n_{\mathrm{z}}})^{\mathbb{R}_0^+} \mid \dot{x}(t) = \tfrac{d}{dt}x(t),\\ \dot{w}(t) = \tfrac{d}{dt}w(t), \dot{z}(t) = \tfrac{d}{dt}z(t), \forall t \in \mathbb{R}_0^+, (x, w, z) \in \mathfrak{B}_{\mathrm{c}}\big\},\end{aligned} \tag{5.24}$$

see also (4.14). When we compare the velocity form (5.23) with the differential form (5.7), note that $A_{\mathrm{v}} = A_\delta, \ldots, D_{\mathrm{v}} = D_\delta$. Moreover, due the definition of the solution set of the velocity form, $\mathfrak{B}_{\mathrm{v}}$, it is clear that any solution of the velocity form is also a solution of the differential form, i.e., for a $(\dot{x}, \dot{w}, \dot{z}) \in \mathfrak{B}_{\mathrm{v}}$ with corresponding $(x, w) \in \pi_{\mathrm{x,w}} \mathfrak{B}_{\mathrm{c}}$, it also holds that $(\dot{x}, \dot{w}, \dot{z}) \in \mathfrak{B}_\delta(x, w)$ (see (5.9)). However, not any solution of the differential form is also a solution of the velocity form, as solution for the velocity form are directly connected to the solutions of the primal form through time-differentiation. Consequently, we have that the solution set of the velocity form is included in that of the differential form, i.e., $\mathfrak{B}_{\mathrm{v}} \subset \breve{\mathfrak{B}}_\delta$.

Furthermore, remember that velocity dissipativity, see Definition 4.6, can be seen as classical dissipativity of the velocity form, similar how differential dissipativity can be seen as classical dissipativity of the differential form. Combined with the previous observation that the solution set of the velocity form is included in that of the differential form, i.e., $\mathfrak{B}_{\mathrm{v}} \subset \mathfrak{B}_\delta$, we therefore have that differential dissipativity also implies velocity dissipativity. This connection has also been reported in (Kosaraju et al. 2019; Kawano et al. 2021), however, only in case of a passivity based supply function.

That differential dissipativity implies velocity dissipativity also becomes clear when we compare the matrix inequality condition for velocity (Q, S, R) dissipativity given in Theorem 4.4, with the condition for differential (Q, S, R) dissipativity given in Theorem 5.1. Comparing the two conditions, it is evident that in case M in (5.13) is a constant matrix, that condition (5.13) is equivalent with condition (4.20). This means that with condition (4.20) for velocity (Q, S, R) dissipativity in Theorem 4.4, we actually imply the stronger notion of differential (Q, S, R) dissipativity. However, note that the velocity form and velocity dissipativity are still useful, as we can exploit their properties for controller synthesis in order to ensure closed-loop universal shifted stability and performance, as we have shown in Section 4.5.

5.4 Convex Incremental Performance Analysis

5.4.1 Incremental performance

We now use the dissipativity results of Section 5.3 to recover incremental notions of well-known performance indicators ($\mathcal{L}_2$-gain, $\mathcal{L}_\infty$-gain, passivity and the generalized $\mathcal{H}_2$-norm) and propose a method, through the LPV framework, that allows for global, convex performance analysis of nonlinear systems. This will also serve as a stepping stone for the formulation of incremental controller synthesis in Chapter 6.

In Section 2.3.2, we have introduced the performance metric given by the gain from input signals in $\mathcal{L}_{pe}$ to output signals in $\mathcal{L}_{qe}$, see Definition 2.11. This has given rise to the popular performance notion of $\mathcal{L}_2$-gain, and to performance notions such as generalized $\mathcal{H}_2$ nominal performance, through the $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain, and peak-to-peak performance, through the $\mathcal{L}_\infty$-gain. All these performance notions, along with stability of the system, can be characterized through classical dissipativity of the system, for a particular choice of supply function, see Section 2.4.3.

In the incremental case, most work in the literature has focussed on the incremental $\mathcal{L}_2$-gain (Fromion, Scorletti, and Ferreres 1999; Van der Schaft 2017), with some extensions to a general incremental $\mathcal{L}_p$-gain notion (Romanchuk and James 1996). Next we will give generalization of these results to an incremental version of $\mathcal{L}_p$-$\mathcal{L}_q$-gain for CT systems.

Definition 5.5 (Incremental $\mathcal{L}_p$-$\mathcal{L}_q$-gain)**.** *A (CT) nonlinear system given by (5.1) is said to have a finite incremental $\mathcal{L}_p$-$\mathcal{L}_q$-gain, if there is a finite $\gamma \geq 0$ and function $\zeta_\mathrm{i} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that*

$$\|z - \tilde{z}\|_{q,T} \leq \gamma \|w - \tilde{w}\|_{p,T} + \zeta_\mathrm{i}(x_0, \tilde{x}_0), \tag{5.25}$$

for all $T \geq 0$ and $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$ with[6] *$w, \tilde{w} \in \mathcal{L}_{\mathrm{pe}}$. The induced incremental $\mathcal{L}_p$-$\mathcal{L}_q$-gain of (5.1), denoted as $\mathcal{L}_{\mathrm{i}p}$-$\mathcal{L}_{\mathrm{i}q}$-gain, is the infimum of γ such that (5.25) still holds. If $p = q$, we will refer to this as the (induced) incremental $\mathcal{L}_p$-gain, denoted as $\mathcal{L}_{\mathrm{i}p}$-gain.*

Next, we will present conditions for the incremental performance notions considering storage functions of the form of (5.16). It is trivial to extend these results to the case when a matrix function $M(\bar{x})$ is considered.

Incremental $\mathcal{L}_2$-gain

As aforementioned, the $\mathcal{L}_2$-gain is one of the more popular performance notions for analysis and controller synthesis. Next, we give a results for its incremental extension, the $\mathcal{L}_{\mathrm{i}2}$-gain. Using Definition 5.5 to define the $\mathcal{L}_{\mathrm{i}2}$-gain and together with Corollary 5.1 lead to the following result:

[6] Which also implies that $(w - \tilde{w}) \in \mathcal{L}_{pe}$.

Corollary 5.2 ($\mathcal{L}_{\mathrm{i}2}$-gain bound). *Consider the system given by (5.1) with differential form given by (5.7), and let $\gamma \in \mathbb{R}^+$. If there exists an $M \in \mathbb{S}^{n_\mathrm{x}}$ with $M \succ 0$, s.t. for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$,*

$$\begin{bmatrix} A_\delta(\bar{x},\bar{w})^\top M + (\star) & M B_\delta(\bar{x},\bar{w}) & C_\delta(\bar{x},\bar{w})^\top \\ \star & -\gamma I & D_\delta(\bar{x},\bar{w})^\top \\ \star & \star & -\gamma I \end{bmatrix} \preceq 0, \tag{5.26}$$

then, (5.1) has an $\mathcal{L}_{\mathrm{i}2}$-gain bound of γ.

Proof. See Appendix B.3. □

In (Fromion and Scorletti 2003), it is shown that the primal form (5.1) has an incremental $\mathcal{L}_2$-gain of γ *if and only if* the differential form (5.7) has a bounded $\mathcal{L}_2$-gain of γ. It is an interesting (open) question how necessity can also be established via Theorem 5.2 in this case. Additionally, note that (5.26) is linear, i.e., convex, in M and γ, but it is an infinite semi-definite problem. We will discuss in Section 5.4.2 how to turn the condition in Corollary 5.2 into a finite number of *Linear Matrix Inequality* (LMI)-based convex optimization problem.

Incremental $\mathcal{L}_\infty$-gain

The well-known $\mathcal{L}_1$-norm is defined for stable LTI systems that map inputs with bounded amplitude to outputs with bounded amplitude. For LTI systems, the $\mathcal{L}_1$-norm is equivalent with the induced $\mathcal{L}_\infty$-norm, i.e., the peak-to-peak gain of a system. Similar to results for the $\mathcal{L}_{\mathrm{i}2}$-gain, we extend this to the incremental setting using Definition 5.5 to define the $\mathcal{L}_{\mathrm{i}\infty}$-gain, which, together with Corollary 5.1, as extension of (Scherer 2000, Sec. 10.3) and (Scherer and Weiland 2015, Sec. 3.3.5), gives us the following result:

Corollary 5.3 ($\mathcal{L}_{\mathrm{i}\infty}$-gain bound). *Consider the system given by (5.1), with differential form (5.7), and let $\gamma \in \mathbb{R}^+$. If there exists an $M \in \mathbb{S}^{n_\mathrm{x}}$ with $M \succ 0$, $\alpha > 0$, and $\beta > 0$, s.t. for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$,*

$$\begin{bmatrix} A_\delta(\bar{x},\bar{w})^\top M + (\star) + \beta M & M B_\delta(\bar{x},\bar{w}) \\ \star & -\alpha I \end{bmatrix} \preceq 0, \tag{5.27a}$$

$$\begin{bmatrix} \beta M & 0 & C_\delta(\bar{x},\bar{w})^\top \\ \star & (\gamma-\alpha) I & D_\delta(\bar{x},\bar{w})^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0, \tag{5.27b}$$

then, (5.1) has a $\mathcal{L}_{\mathrm{i}\infty}$-gain bound of γ for all $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$.

Proof. See Appendix B.3. □

Despite of the fact that (5.27a) is not convex in β and M due to their multiplicative relation, by fixing β and performing a line-search over it, (5.27a) again corresponds to an infinite semi-definite program.

Incremental passivity

Passivity is a widely studied system property and it has been extended towards the incremental setting (Pavlov and Marconi 2008; Van der Schaft 2017) and the differential setting (Forni and Sepulchre 2013; Forni, Sepulchre, and Van der Schaft 2013; Van der Schaft 2013). In (Kawano et al. 2021), the connection between differential and incremental passivity has been established for a storage function (5.11) with constant M. That work might serve as a parallel proof for Theorem 5.2, when focusing only on passivity.

A system is said to be passive, if it is dissipative w.r.t. to the supply rate $s(w, z) = w^\top z + z^\top w$. Based on (Van der Schaft 2017), the definition of incremental passivity is as follows:

Definition 5.6 (Incremental passivity)**.** *A system of the form (5.1) for which* $n_\mathrm{w} = n_\mathrm{z}$ *is incrementally passive, if it is incrementally dissipative w.r.t. the supply*

$$s_\mathrm{i}(w, \tilde{w}, z, \tilde{z}) = (w - \tilde{w})^\top (z - \tilde{z}) + (z - \tilde{z})^\top (w - \tilde{w}). \tag{5.28}$$

Based on Corollary 5.1, the following result holds:

Corollary 5.4 (Incremental passivity condition)**.** *The system given by (5.1) with* $n_\mathrm{w} = n_\mathrm{z}$ *is incrementally passive, if there exists an* $M \in \mathbb{S}^{n_\mathrm{x}}$ *with* $M \succ 0$*, such that for all* $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$

$$\begin{bmatrix} A_\delta(\bar{x}, \bar{w})^\top M + (\star) & M B_\delta(\bar{x}, \bar{w}) - C_\delta(\bar{x}, \bar{w})^\top \\ \star & -D_\delta(\bar{x}, \bar{w}) + (\star) \end{bmatrix} \preceq 0. \tag{5.29}$$

The proof simply follows by direct application of Corollary 5.1 with $Q = R = 0$ and $S = I$.

Comparing Corollary 5.4 to (Kawano et al. 2021) and (Van der Schaft 2013), these papers give results on differential passivity for a combined primal and differential system formulation (a prolonged system (Crouch and Van der Schaft 1987)) using a specific form of storage function. The result depends on equality constrains, which serve as a decoupling condition between the differential storage and the primal storage, while in this chapter the differential storage and the primal storage have the same structure (quadratic form with the same M), not requiring such equality constraints.

Incremental generalized $\mathcal{H}_2$ performance

There are several extensions of the $\mathcal{H}_2$-norm for nonlinear systems embedded as LPV systems (De Souza et al. 2003; Xie 2005; Bouali et al. 2008). In this chapter, we extend the notion of generalized $\mathcal{H}_2$ performance, corresponding to the $\mathcal{L}_2$-$\mathcal{L}_\infty$ gain, to the incremental setting through the $\mathcal{L}_{\mathrm{i}2}$-$\mathcal{L}_{\mathrm{i}\infty}$-gain, consistent with Definition 5.5. Note for the $\mathcal{L}_{\mathrm{i}2}$-$\mathcal{L}_{\mathrm{i}\infty}$-gain to be bounded, the assumption is required that $\frac{\partial h}{\partial \bar{w}}(\bar{x}, \bar{w}) = D_\delta(\bar{x}, \bar{w}) = 0$ for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$. As an extension of (Scherer and Weiland 2015, Sec. 3.3.4), the following result characterizes an upper bound γ on the $\mathcal{L}_{\mathrm{i}2}$-$\mathcal{L}_{\mathrm{i}\infty}$-gain.

Corollary 5.5 ($\mathcal{L}_{\mathrm{i}2}$-$\mathcal{L}_{\mathrm{i}\infty}$-gain bound)**.** *Consider the system given by (5.1) with differential form given by (5.7) for which* $\frac{\partial h}{\partial \bar{w}}(\bar{x},\bar{w}) = D_\delta(\bar{x},\bar{w}) = 0$ *for all* $(\bar{x},\bar{w}) \in \mathcal{X}\times\mathcal{W}$*, and let* $\gamma \in \mathbb{R}^+$*. If there exists an* $M \in \mathbb{S}^{n_\mathrm{x}}$*, s.t. for all* $(\bar{x},\bar{w}) \in \mathcal{X}\times\mathcal{W}$*,*

$$\begin{bmatrix} A(\bar{\eta})^\top M + MA(\bar{\eta}) & MB(\bar{\eta}) \\ B(\bar{\eta})^\top M & -\gamma I \end{bmatrix} \preceq 0, \tag{5.30a}$$

$$\begin{bmatrix} M & C(\bar{\eta})^\top \\ C(\bar{\eta}) & \gamma I \end{bmatrix} \succeq 0, \tag{5.30b}$$

then, (5.1) has an $\mathcal{L}_{\mathrm{i}2}$-$\mathcal{L}_{\mathrm{i}\infty}$*-gain bound of* γ *for all* $(x,w,z),(\tilde{x},\tilde{w},\tilde{z}) \in \mathfrak{B}$*.*

Proof. See Appendix B.3. □

Remark 5.4. Note that the performance measures discussed in this section have a specific (Q,S,R)-triplet associated with them. Specifically, $(Q,S,R) = (\gamma^2 I, 0, -I)$ for $\mathcal{L}_{\mathrm{i}2}$-gain, $(Q,S,R) = (\alpha I, 0, 0)$ for $\mathcal{L}_{\mathrm{i}\infty}$-gain, $(Q,S,R) = (0, I, 0)$ for incremental passivity and $(Q,S,R) = (\gamma I, 0, 0)$ for the $\mathcal{L}_{\mathrm{i}2}$-$\mathcal{L}_{\mathrm{i}\infty}$-gain.

Remark 5.5. Note that the results in Corollaries 5.2 to 5.5 are given in terms of non-strict (matrix) inequalities, while in literature their non-incremental counterparts are often given in terms of strict inequalities. The strict versions of these results can easily be retrieved by changing the dissipation inequality for differential dissipativity in Definition 5.3 to a strict inequality, from which the rest will follow.

5.4.2 Convex computation through the LPV framework

So far, the obtained results have yielded matrix inequalities that correspond to infinite dimensional *semidefinite programs* (SDPs). This problem is similar to what is encountered when analyzing LPV systems. This section presents an approach to recast these problems as standard LPV analysis problems by embedding of the differential form of the system in an LPV representation, which we will refer to as a *Differential Parameter-Varying* (DPV) embedding. As discussed in Chapters 1 and 2, there exists various approaches for LPV framework to cast the infinite dimensional LMIs to a finite dimensional problem. Inspired by (Tóth 2010; R. Wang, Tóth, et al. 2020), we define the DPV embedding of (5.1) as follows.

Definition 5.7 (DPV embedding)**.** *Consider a system (5.1) and its differential form (5.7). The LPV state-space representation*

$$\dot{x}_\delta(t) = A(p(t))x_\delta(t) + B(p(t))w_\delta(t), \tag{5.31a}$$

$$z_\delta(t) = C(p(t))x_\delta(t) + D(p(t))w_\delta(t), \tag{5.31b}$$

with $p(t) \in \mathcal{P} \subset \mathbb{R}^{n_\mathrm{p}}$ *being the scheduling-variable, is a so-called DPV embedding of (5.1) on the region* $\mathcal{X}\times\mathcal{W} \subseteq \mathcal{X}\times\mathcal{W}$*, if there exists a function* $\eta : \mathcal{X}\times\mathcal{W} \to \mathcal{P}$*, the so-called scheduling-map, with* $p = \eta(\bar{x},\bar{w})$*, and* $\mathcal{P} \supseteq \eta(\mathcal{X},\mathcal{W})$*, such that for all* $(\bar{x},\bar{w}) \in \mathcal{X}\times\mathcal{W}$*:*

$$\begin{aligned} A(\eta(\bar{x},\bar{w})) &= A_\delta(\bar{x},\bar{w}), & B(\eta(\bar{x},\bar{w})) &= B_\delta(\bar{x},\bar{w}), \\ C(\eta(\bar{x},\bar{w})) &= C_\delta(\bar{x},\bar{w}), & D(\eta(\bar{x},\bar{w})) &= D_\delta(\bar{x},\bar{w}), \end{aligned} \tag{5.32}$$

where A*, ...,* D *belong to a given function class (affine, polynomial, etc.).*

The convex set $\mathcal{P}$ is usually a superset of the η-projected values of possible state and input trajectories (even if $\mathscr{X}$, $\mathscr{W}$ are convex), hence, the DPV embedding of a nonlinear system introduces conservatism. However, similar to standard LPV analysis of nonlinear systems, this is considered to be the trade-off for efficiently computable stability and performance analysis of nonlinear systems. In case that $\mathcal{X} \times \mathcal{W}$ is unbounded, the DPV embedding is often realized on a convex subset $\mathscr{X} \times \mathscr{W} \subseteq \mathcal{X} \times \mathcal{W}$, such that there exists a compact and convex $\mathcal{P} \supseteq \eta(\mathscr{X} \times \mathscr{W})$. In this case, one either requires to add an extra condition of invariance of the system on $\mathscr{X} \times \mathscr{W}$ or assume it, which may introduce conservatism in the analysis, as not the full behavior of the original primal system is considered. Note that existence of a compact and convex $\mathcal{P}$, in case of unbounded $\mathcal{X} \times \mathcal{W}$, follows when $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial w}$, $\frac{\partial h}{\partial x}$, $\frac{\partial h}{\partial w}$ are bounded matrix functions, e.g., if $\frac{\partial h}{\partial x} = \sin(x)$, with $x \in \mathbb{R}$, we can take $p = \eta(x) = \sin(x) \in [-1, 1]$.

As the DPV embedding is an LPV representation, we can make use of various approaches from the LPV framework to reduce the conservatism of the DPV embedding (5.31) for a given preferred dependency class of A, B, C, D (e.g. affine, polynomial, rational). For example there exists methods whereby we can optimize η (with minimal n_p) such that $\mathrm{co}\,(\eta(\mathscr{X}, \mathscr{W})) \setminus \eta(\mathscr{X}, \mathscr{W})$ has minimal volume, see e.g. (Tóth 2010; Sadeghzadeh and Tóth 2020). In Chapter 9, we will also discuss a data based method to reduce scheduling dimension in order to improve the computational efficiency for more complex systems, requiring a large number of scheduling-variables for their LPV or DPV embedding.

Note that the DPV embedding serves as an important tool to *convexify* the variation of the matrix inequalities in the analysis. Through the DPV embedding, we can capture all the behavior of the differential form (5.7). Let us define the behavior of (5.31) for a particular $p \in \mathcal{P}^{\mathbb{R}_0^+}$ by $\mathfrak{B}_\mathrm{p}(p)$ and the full behavior by (i.e., for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$) by $\breve{\mathfrak{B}}_\mathrm{p}$, see also (2.27) and (2.28).

Lemma 5.3 (DPV behavioral embedding)**.** *Consider the nonlinear system given by (5.1) and the LPV system given by (5.31). If the LPV system is a DPV embedding of the nonlinear system on the region $\mathscr{X} \times \mathscr{W} = \mathcal{X} \times \mathcal{W}$, then, the behavior of differential form is included in that of the LPV system, i.e., $\breve{\mathfrak{B}}_\delta \subseteq \breve{\mathfrak{B}}_\mathrm{p}$.*

Proof. See Appendix B.3. □

Remark 5.6. If the DPV embedding is considered on the region $\mathscr{X} \times \mathscr{W} \subset \mathcal{X} \times \mathcal{W}$, i.e., only part of the state-space is considered, then we can still describe (part of) the behavior of the differential form using the DPV embedding. In this case one requires the assumption that the states and inputs stay in the embedding region, i.e., $(x(t), w(t)) \in \mathscr{X} \times \mathscr{W}$ for all $t \in \mathbb{R}_0^+$.

Note again that differential dissipativity can be seen as classical dissipativity of the differential form, which we have shown allows us to imply incremental dissipativity. Therefore, as we can represent the differential form by the DPV embedding, we can analyze classical dissipativity of the DPV embedding in order to analyze incremental dissipativity. As the DPV embedding is an LPV representation, we hence can cast the incremental analysis problem as a standard LPV analysis problem. This is formalized in the following theorem:

Theorem 5.7 (Incremental dissipativity through the LPV framework)**.** *Consider the nonlinear system given by (5.1), for which the LPV representation given by (5.31) is a DPV embedding of the system on the region* $\mathcal{X} \times \mathcal{W} = \mathcal{X} \times \mathcal{W}$*. If the LPV representation given by (5.31) is classically* (Q, S, R) *dissipative with* $R \preceq 0$*, then, the nonlinear system is incrementally* (Q, S, R) *dissipative.*

Proof. See Appendix B.3. □

Note that it is also evident that we can analyze incremental dissipativity through the LPV framework, if we compare the conditions for the various incremental performance notions in Section 5.4.1 to the conditions in Section 2.5.2. Namely, the conditions given in Section 5.4.1 become then equal to the LPV analysis conditions discussed in Section 2.5.2, when the LPV representation is given by the DPV embedding (5.31).

Similarly, the DPV embedding can also be used for incremental stability analysis, using the result of Lemma 5.2, which then becomes a standard LPV stability analysis problem, again, see also Section 2.5.2. In turn, this allows the proposed incremental stability and performance analysis to make use of all the computationally efficient methods constructed to analyze LPV representations (Hoffmann and Werner 2015a).

Note that these results also connect again to the velocity based results to analyze universal shifted stability and performance in Chapter 4, as we have also discussed in Section 5.3.6. Specifically, in Section 4.4, we have also proposed the use the LPV framework in order to efficiently analyze velocity dissipativity, by embedding the velocity form in an LPV representation, which we refer to as the *Velocity Parameter-Varying* (VPV) embedding. This is similar to how we use the DPV embedding to efficiently analyze differential dissipativity, and hence, incremental dissipativity. In Section 5.3.6, we have already shown the similarities between the differential and velocity forms. Due to these similarities, also the DPV embedding and VPV embedding of a system are closely related. In fact, an LPV representation can simultaneously be both a DPV embedding and VPV embedding of a system. This is especially evident when we compare the conditions that the LPV matrices $A, \ldots, D$ of an LPV representation (along with the scheduling-map η) need to satisfy in order to be a DPV embedding, see (5.32), and in order to be a VPV embedding, see (4.31). However, as aforementioned, despite these similarities, we can still exploit the different properties of the differential form and the velocity form through the DPV and VPV embedding, respectively, for controller synthesis. Like we have shown in Section 4.5 w.r.t. the VPV embedding, and which we will show w.r.t. the DPV embedding in Chapter 6.

5.5 Examples

This section demonstrates the developed notions of incremental dissipativity theory and the analysis tools on two example systems.

Example 5.1 *(Incremental dissipativity analysis of a Duffing oscillator)***.** Consider a second-order Duffing oscillator given in a state-space form by

$$\begin{aligned} \dot{x}_1(t) &= x_2(t); \\ \dot{x}_2(t) &= -a\,x_2(t) - \big(b + c\,x_1^2(t)\big)\,x_1(t) + w(t); \\ z(t) &= x_1(t), \end{aligned} \tag{5.33}$$

where a and b represent the linear damping and stiffness, respectively, and c represents the nonlinear stiffness component. The differential form of (5.33) is given by

$$\begin{aligned} \dot{x}_\delta(t) &= \begin{bmatrix} 0 & 1 \\ -b - 3\,c\,x_1^2(t) & -a \end{bmatrix} x_\delta(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_\delta(t); \\ z_\delta(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_\delta(t). \end{aligned} \tag{5.34}$$

Moreover, we assume for this system that $(x_1(t), x_2(t)) \in \mathcal{X}$ for all $t \in \mathbb{R}_0^+$, where $\mathcal{X} = [-\sqrt{2},\, \sqrt{2}] \times \mathbb{R}$, and

$$w \in \{\mathbb{R}^{\mathbb{R}_0^+} \mid (5.33) \text{ holds and } (x_1(t), x_2(t)) \in \mathcal{X},\, \forall\, t \in \mathbb{R}_0^+\}. \tag{5.35}$$

By choosing $a = 3.3$, $b = 7.9$, $c = 1$, (5.33) yields a system with finite $\mathcal{L}_{\text{i2}}$-gain. In this example, we determine the $\mathcal{L}_{\text{i2}}$-gain of the system, using Corollary 5.2. Note that the nonlinearity $x_1^2(t)$ in (5.34) can be captured by using a DPV inclusion $p(t) = \eta(x_1(t)) = x_1^2(t) \in [0,\, 2]$. The resulting DPV embedding is then given by

$$\begin{aligned} \dot{x}_\delta(t) &= \begin{bmatrix} 0 & 1 \\ -b - 3\,c\,p(t) & -a \end{bmatrix} x_\delta(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_\delta(t); \\ z_\delta(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} x_\delta(t), \end{aligned} \tag{5.36}$$

where $p(t) = \eta(x_1(t)) = x_1^2(t) \in [0,\, 2]$, which has an affine scheduling dependency. Computing an $\mathcal{L}_2$-gain upperbound for (5.36), corresponding to the $\mathcal{L}_{\text{i2}}$-gain upperbound for (5.33), using the LPVcore Toolbox (Boef et al. 2021), yields $M = \begin{bmatrix} 0.592 & 0.0896 \\ 0.0896 & 0.0543 \end{bmatrix} \succ 0$ and $\gamma = 0.155$. Hence, within less than a second, we know that the nonlinear system is differentially, incrementally, universally shifted, and classically dissipative[7] on $\mathcal{X}$ for which w satisfies (5.35), w.r.t. the supply function (5.12) with $Q = 0.155^2$, $R = -1$ and $S = 0$, corresponding to an $\mathcal{L}_{\text{i2}}$-gain upperbound of 0.155. Next, the system is simulated with two different input signals

$$w_1(t) = 3e^{-0.2t}\cos(\pi t)\,, \tag{5.37a}$$

$$w_2(t) = -2e^{-0.1t}\sin\left(0.6\pi t + \tfrac{\pi}{4}\right), \tag{5.37b}$$

[7]Note that (5.33) satisfies $(0, 0, 0) \in \mathscr{E}$ as per Theorem 5.2.

for which holds that w_1, w_2 satisfy (5.35) and $w_1, w_2 \in \mathcal{L}_2$. The inputs and the state trajectories are shown in Figure 5.3, which shows that the states stay within the defined state-space $\mathcal{X}$.

To visualize differential dissipativity of the system, the signals of (5.34) are substituted in the dissipation inequality for the differential form (5.10). The left- and right-hand side of the dissipation inequality (5.10) are plotted in Figure 5.4a corresponding to the system trajectories of Figure 5.3. As can be seen in Figure 5.4a, the stored energy in the system is always less than the supplied energy plus the initial stored energy, hence the system is differentially dissipative w.r.t. the considered $\mathcal{L}_2$-gain supply.

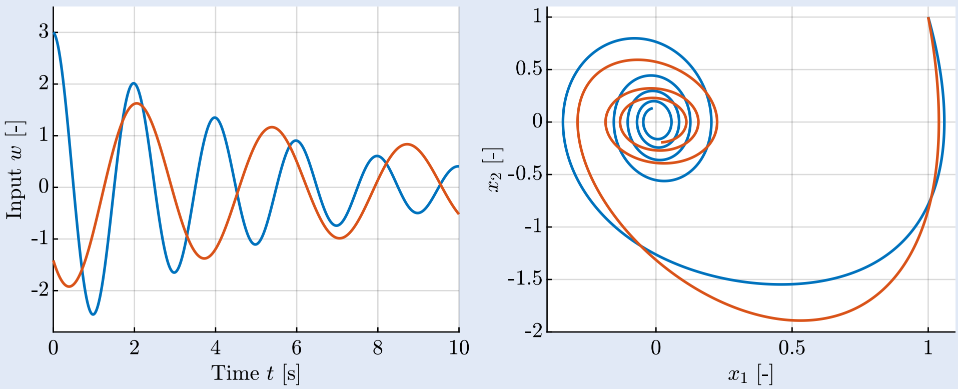
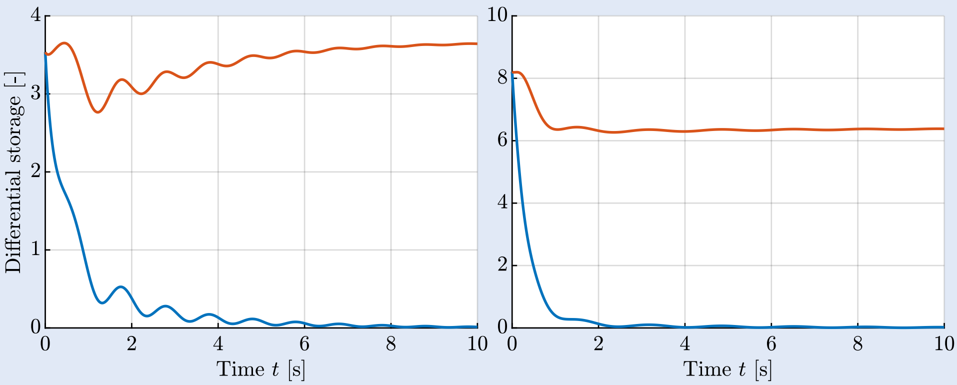
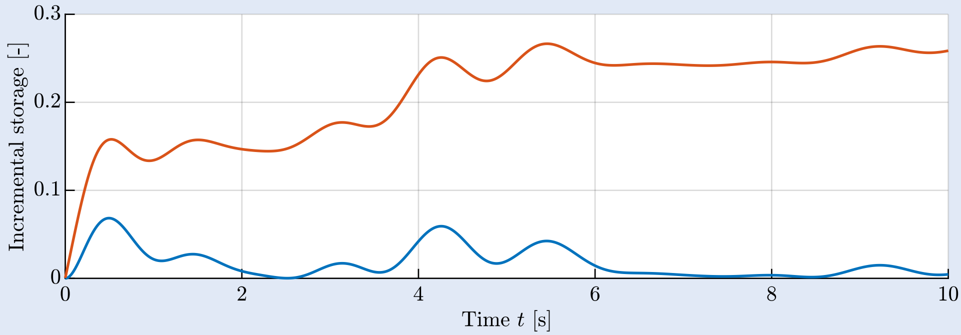


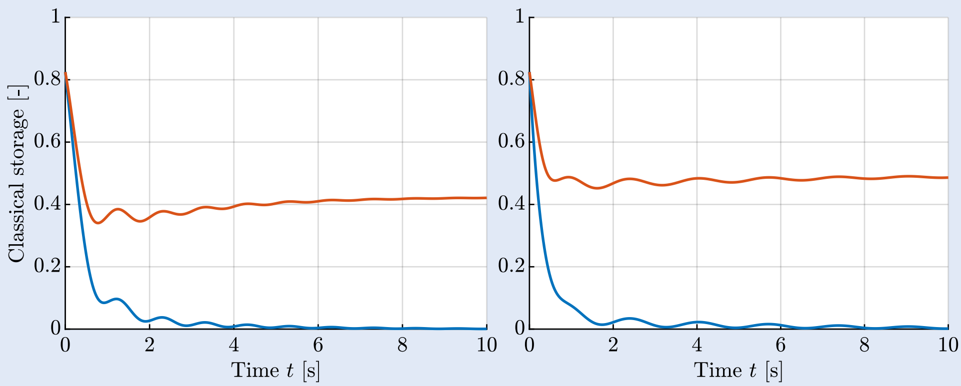
Figure 5.3: The applied inputs, w_1 ( ) and w_2 ( ), (left) and the resulting state trajectories (right). Both trajectories start at $\big(x_1(0), x_2(0)\big) = (1, 1)$.



(a) Differential dissipativity of the system trajectories, $[\mathcal{V}_\delta(\bar{x}(t), x_\delta(t))]$ ( ) and $\left[\int_{t_0}^{t_1} s_\delta(w_\delta(\tau), z_\delta(\tau))\, d\tau + \mathcal{V}_\delta(\bar{x}(0), x_\delta(0))\right]$ ( ), with $w_1(t)$ as input (left) and $w_2(t)$ as input (right).



(b) Incremental dissipativity based on the system trajectories, $[\mathcal{V}_{\mathrm{i}}(x(t), \tilde{x}(t))]$ (—) and $\left[\int_{t_0}^{t_1} s_{\mathrm{i}}(w(\tau), \tilde{w}(\tau), z(\tau), \tilde{z}(\tau))\, d\tau + \mathcal{V}_{\mathrm{i}}(x(0), \tilde{x}(0))\right]$ (—), with $w(t) = w_1(t)$ and $\tilde{w}(t) = w_2(t)$ as input.



(c) Classical dissipativity of the system trajectories, $[\mathcal{V}(x(t))]$ (—) and $\left[\int_{t_0}^{t_1} s(w(\tau), z(\tau))\, d\tau + \mathcal{V}_{\mathrm{i}}(x(0))\right]$ (—), with $w_1(t)$ as input (left) and $w_2(t)$ as input (right).

Figure 5.4: Simulation results for the different notions of dissipativity for a Duffing oscillator w.r.t. a (Q, S, R) supply function with $Q = \gamma^2 I$, $S = 0$, and $R = -I$, corresponding to the $\mathcal{L}_{\mathrm{i}2}$-gain.

Since the system is differentially dissipative it is also incrementally dissipative. Figure 5.4b shows the incremental dissipation inequality, i.e., the stored energy and the supplied energy between the two trajectories in Figure 5.3. As can be observed in Figure 5.4b, the stored energy between two trajectories is always less than the supplied energy between two trajectories. Hence, considering these trajectories, the system is incrementally dissipative. Therefore, we can state (based on these two trajectories) that these results correspond to the developed theory. Furthermore, because the supply function is parametrized such that it represents the $\mathcal{L}_{\mathrm{i}2}$-gain of a system, $\gamma = 0.155$ is an upper bound for the $\mathcal{L}_{\mathrm{i}2}$-gain of the system (5.33).
Moreover, by Theorem 5.3, incremental dissipativity implies universal shifted dissi-

pativity and classical dissipativity of the original system (5.33). Figure 5.4c gives the storage and supply function evolution over time for the two considered trajectories, showing that the original system is dissipative, since the stored energy is always less than the supplied energy.

The next example shows that incremental dissipativity is a *stronger* notion than classical dissipativity, if the same type of storage function is considered.

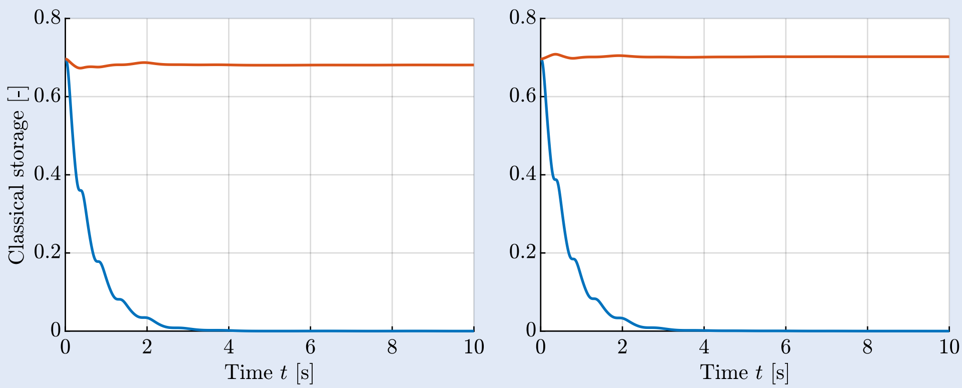
Example 5.2 *(Comparison between incremental and classical dissipativity on the Duffing oscillator)***.** This example again uses a Duffing oscillator, now with the output equation given by $z(t) = x_2(t)$. With this small modification compared to (5.33), the Duffing oscillator can be written as a port-Hamiltonian system. Based on (Molero et al. 2013), we take the Hamiltonian function as

$$H(x) = \tfrac{1}{2}x_2^2 + \tfrac{1}{2}b\,x_1^2 + \tfrac{1}{4}c\,x_1^4. \tag{5.38}$$

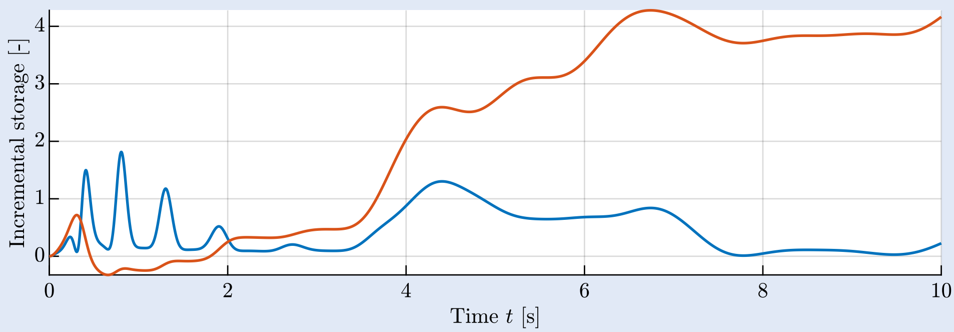
The resulting port-Hamiltonian form of this system is

$$\begin{aligned} \dot{x}(t) &= \underbrace{\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}\right)}_{(J(x)-R(x))} \underbrace{\begin{bmatrix} \frac{\partial H}{\partial x_1} \\ \frac{\partial H}{\partial x_2} \end{bmatrix}}_{\nabla H(x)} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B(x)} w(t); \\ z(t) &= B(x)^\top \nabla H(x). \end{aligned}$$

Since a port-Hamiltonian system is always dissipative w.r.t. the supply function $s(w, z) = 2w^\top z$, we know that the dissipation inequality holds for all trajectories.



(a) Classical dissipativity of the system trajectories, $[\mathcal{V}(x(t))]$ (—) and $\left[\int_{t_0}^{t_1} s(w(\tau), z(\tau))\, d\tau + \mathcal{V}_\mathrm{i}(x(0))\right]$ (—), with $w_1(t)$ as input (left) and $w_2(t)$ as input (right), here the y-axis is normalized unitarily.



(b) Incremental dissipativity based on the system trajectories, $[\mathcal{V}_\mathrm{i}(x(t), \tilde{x}(t))]$ (—) and $\left[\int_{t_0}^{t_1} s_\mathrm{i}(w(\tau), \tilde{w}(\tau), z(\tau), \tilde{z}(\tau))\, d\tau + \mathcal{V}_\mathrm{i}(x(0), \tilde{x}(0))\right]$ (—), with $w_1(t)$ and $w_2(t)$ as input.

Figure 5.5: Simulation results for a Duffing oscillator which is passive, but not incrementally passive, when the same storage function (i.e., the Hamiltonian function) is used: $H(x)$ for general passivity and $H(x - \tilde{x})$ for incremental passivity.

Moreover, this supply function indicates passivity, hence the port-Hamiltonian system is passive. By choosing $a = 1.3$, $b = 7.9$, $c = 3$, the storage function based on the Hamiltonian allows us to show the system is passive, however, we cannot conclude if the system is incrementally passive based on a storage function which uses the same Hamiltonian. The two plots in Figure 5.5a show the (normalized) classical dissipation inequality for two arbitrary inputs, and indeed the energy in the system, based on the Hamiltonian as measure, is less than the supplied energy to the system. Hence, the system is passive. However, when incremental dissipativity is examined by subtracting both trajectories, the plot in Figure 5.5b is obtained. For some time-interval, the energy in the system is more than the energy supplied to the system, hence the system is not incrementally passive w.r.t. the supply function $s_\mathrm{i}(w, \tilde{w}, z, \tilde{z}) = 2(w - \tilde{w})^\top (z - \tilde{z})$ and storage function $H(x - \tilde{x})$. This shows that incremental dissipativity is a stronger notion than classical dissipativity, when the storage function has the same complexity. Note that the system might be incrementally dissipative for some different storage function.

5.6 Conclusions

In this chapter, we have established the link between various dissipativity notions, namely, we have shown that differential dissipativity implies incremental dissipativity, universal shifted dissipativity, as discussed in Chapter 4, and classical dissipativity of nonlinear systems. Moreover, we have given results on general quadratic incremental performance notions and how, through the introduced concept of DPV embeddings, we can use the LPV framework to analyze the different notions of dissipativity computationally efficiently. The links we have established give us a generic framework to analyze global stability and performance notions of nonlinear systems, irrespective of a particular equilibrium point or trajectory. Finally, the computation tools we have presented allow to efficiently analyze global stability and performance of a rather general class of nonlinear systems. In the next chapter, Chapter 6, we will show how these results open up the possibility to establish computationally efficient controller synthesis based on the DPV embedding, such that we can synthesize (nonlinear) controllers for nonlinear systems with incremental stability and performance guarantees of the closed-loop behavior.

6

Incremental Dissipativity based Control

THE constant push for higher performance requirements in practical systems has resulted in that nonlinear behavior is becoming increasingly more dominant in applications, as we have discussed in Chapter 1. This has resulted in computationally efficient and systematic control of nonlinear systems growing more and more important for practical applications. While the *Linear Parameter-Varying* (LPV) framework aimed to provide systematic and computationally efficient tools for nonlinear systems, we have seen in Chapter 3 that the current results are not always capable of providing the desired guarantees in terms of stability and performance for nonlinear systems. Namely, they can only provide guarantees w.r.t. a single equilibrium point of the nonlinear system. Consequently, to achieve the goal of systematic control of nonlinear systems, we require the use of global stability and performance concepts, which are independent of a particular point or trajectory. In Chapter 5, we have shown how the global concept of incremental stability and performance through incremental dissipativity analysis could computationally efficiently be analyzed through the use of the LPV framework. In this chapter, for controller synthesis, we build on top of these results to develop a novel, computationally efficient, systematic controller design method which is able to ensure incremental dissipativity. The proposed control method is verified through simulation studies and on an experimental setup. Moreover, it is also compared to standard LPV controller designs, showing significant performance improvements.

6.1 Introduction

The control of nonlinear systems has been an intense, ongoing field of research since the early 1970's, and it is still to this date, as we also discussed in Chapter 1. So far, no systematic way has been found to perform controller synthesis for general nonlinear systems with performance shaping, compared to the class of *Linear Time-Invariant* (LTI) systems where several systematic approaches exist to design or synthesize controllers.

Hence, in this thesis, we set out to develop a computationally efficient, systematic framework for analysis and control of nonlinear systems to guarantee and shape global stability and performance requirements. In Chapter 3, we have first shown how and why the existing results for the *Linear Parameter-Varying* (LPV) framework, which promises to provide a systematic framework for nonlinear analysis and control, can fail to provide the desired guarantees. Namely, the standard results for the LPV framework can only ensure stability and performance w.r.t. a single point, and not the global, equilibrium point independent, notions that are required for proper systematic nonlinear analysis and control. In Chapter 4, we have already seen how we can use the LPV framework to analyze and design controllers to ensure the global notion of universal shifted stability and performance. However, this notion is limited to ensuring stability and performance w.r.t. constant trajectories only, i.e., equilibrium points, and not w.r.t. varying trajectories. In Chapter 5, we have seen how through the stronger global notion of incremental stability and performance we can analyze stability and performance w.r.t. arbitrary trajectories of the system. Moreover, we have also shown how incremental dissipativity can be efficiently analyzed through the use of the LPV framework.

A first attempt to use the concept of incremental stability and performance to perform controller synthesis in conjunction with the LPV framework was first made in (Scorletti, Fromion, et al. 2015). Using the results from (De Hillerin et al. 2011), this work has provided a control synthesis method where the controller itself is restricted to be *Linear Time-Invariant* (LTI), with an extra input being the scheduling-variable. However, the lack of a multiplicative relationship between the state and the scheduling-variable in the control structure is a heavy limitation compared to standard LPV control. Despite this restriction, the general benefits of the alternative design have been clearly visible from the results.

Besides the incremental stability concept, similar stability concepts such as *contraction* (Lohmiller and Slotine 1998) and *convergence* (Pavlov, Van de Wouw, et al. 2006) have also been employed to design controllers to ensure a global form of stability. However, these results often rely on complex procedures for controller design, and often have no explicit guarantees on performance. More recently, a convex synthesis framework has been introduced for state feedback design to achieve contraction (Manchester and Slotine 2018; R. Wang, Tóth, et al. 2020).

In this chapter, our contribution is the development of a systematic output feedback controller synthesis framework to ensure incremental stability and dissipativity based performance for nonlinear systems on the basis of the results of Chapter 5. This is achieved through three key sub-contributions: (i) proposing a methodology and a performance shaping framework to synthesize an output feedback controller for the differential form of the system by exploiting computationally efficient LPV methods, (ii) introducing a realization method for the controller designed for the differential form of the system to get a nonlinear controller that can be implemented for regulating the (original) target system, (iii) rigorous proofs that the obtained controller ensures closed-loop incremental stability and dissipativity based performance specs with the system.

Compared to previous work, we extend the results in (Manchester and Slotine 2018) which use state feedback to ensure $\mathcal{L}_2$-gain performance to *output* feedback design under *general* quadratic performance specifications. Moreover, we present how the LPV framework can be used effectively to synthesize the output feedback controller in a computationally efficient manner. Compared to (Scorletti, Fromion, et al. 2015), in which the resulting controller is limited to an LTI structure, our proposed controller has full multiplicative relationship between the controller state and scheduling-variable, similar to a standard LPV controller, hence, potentially allowing to achieve better performance. The overall capabilities of the design approach are demonstrated on simulation examples and via experimental studies.

The chapter is structured as follows. In Section 6.2, a formal definition of the incremental controller synthesis problem is given. Section 6.3 describes the proposed framework used to synthesize nonlinear controllers ensuring incremental stability and dissipativity based performance via convex optimization. In Section 6.5, examples are given on the application of the developed control method. Finally, in Section 6.6, conclusions on the presented results are drawn.

6.2 Incremental Controller Synthesis Problem

Similar as we considered for universal shifted controller synthesis in Section 4.5, in this chapter, we consider the problem of control synthesis for a rather wide class of nonlinear control configurations, described by so-called *generalized plants* P (Apkarian, Gahinet, and G. Becker 1995). The objective is to solve the synthesis problem by a novel LPV approach that, via exploiting differential dissipativity, can ensure global stability and performance guarantees for tracking and rejection. As described in Section 6.1, see also Chapter 3, current LPV synthesis methods cannot provide such guarantees in general. A wide range of control structures from feedback and feedforward control to observer design for nonlinear systems can be expressed in the form of the plant P, given by

$$\dot{x}(t) = f\left(x(t), u(t)\right) + B_{\mathrm{w}} w(t); \tag{6.1a}$$

$$z(t) = h_{\mathrm{z}}(x(t), u(t)) + D_{\mathrm{zw}} w(t); \tag{6.1b}$$

$$y(t) = h_{\mathrm{y}}(x(t), u(t)) + D_{\mathrm{yw}} w(t); \tag{6.1c}$$

where $t \in \mathbb{R}_0^+$ is time, $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_{\mathrm{x}}}$ is the state, with $x \in \mathcal{C}_1$ and initial condition $x(0) = x_0 \in \mathbb{R}^{n_{\mathrm{x}}}$, $w \in \mathcal{W}^{\mathbb{R}_0^+}$ with $\mathcal{W} \subseteq \mathbb{R}^{n_{\mathrm{w}}}$ correspond to references, external disturbances, etc., collectively called as *generalized disturbances*, while elements of $z \in \mathcal{Z}^{\mathbb{R}_0^+}$ with $\mathcal{Z} \subseteq \mathbb{R}^{n_{\mathrm{z}}}$ characterize the *generalized performance* (e.g. tracking error, control effort, etc.). Furthermore, the channels u and y where $u \in \mathcal{U}^{\mathbb{R}_0^+}$ with $\mathcal{U} \subseteq \mathbb{R}^{n_\mathrm{u}}$ is the control input and $y \in \mathcal{Y}^{\mathbb{R}_0^+}$ with $\mathcal{Y} \subseteq \mathbb{R}^{n_\mathrm{y}}$ is the measured output. These represent the channels on which the controller K interacts with P. Additionally, $f : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^{n_\mathrm{x}}$, $h_\mathrm{z} : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^{n_\mathrm{z}}$ and $h_\mathrm{y} : \mathcal{X} \times \mathcal{U} \to \mathbb{R}^{n_\mathrm{y}}$ are assumed to be in $\mathcal{C}_1$, while $B_\mathrm{w} \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{w}}$, $D_\mathrm{zw} \in \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{w}}$, and $D_\mathrm{yw} \in \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{w}}$.

The controller K for a given plant (i.e., control configuration) P is considered in the form

$$\dot{x}_\mathrm{k}(t) = f_\mathrm{k}(x_\mathrm{k}(t), u_\mathrm{k}(t)); \tag{6.2a}$$

$$y_\mathrm{k}(t) = h_\mathrm{k}(x_\mathrm{k}(t), u_\mathrm{k}(t)); \tag{6.2b}$$

where $x_\mathrm{k}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}}$ is the state, $u_\mathrm{k}(t) \in \mathbb{R}^{n_{\mathrm{u_k}}}$ is the input, and $y_\mathrm{k}(t) \in \mathbb{R}^{n_{\mathrm{y_k}}}$ is the output of the controller. Furthermore, $f_\mathrm{k} : \mathbb{R}^{n_{\mathrm{x_k}}} \times \mathbb{R}^{n_{\mathrm{u_k}}} \to \mathbb{R}^{n_{\mathrm{x_k}}}$ and $h_\mathrm{k} : \mathbb{R}^{n_{\mathrm{x_k}}} \times \mathbb{R}^{n_{\mathrm{u_k}}} \to \mathbb{R}^{n_{\mathrm{y_k}}}$. The closed-loop interconnection of P and K through $u_\mathrm{k} = y$ and $u = y_\mathrm{k}$ (hence, $n_{\mathrm{u_k}} = n_\mathrm{y}$ and $n_{\mathrm{y_k}} = n_\mathrm{u}$) will be denoted by $\mathcal{F}_\mathrm{l}(P, K)$, which has as input w and as output z. The output $z \in \mathcal{Z}^{\mathbb{R}_0^+}$ of $\mathcal{F}_\mathrm{l}(P, K)$ for an input $w \in \mathcal{W}^{\mathbb{R}_0^+}$ and initial condition $x_{\mathrm{cl},0} = \mathrm{col}(x(0), x_\mathrm{k}(0)) \in \mathcal{X} \times \mathbb{R}^{n_{\mathrm{x_k}}}$, is also denoted by $\mathcal{F}_\mathrm{l}(P, K)(w, x_{\mathrm{cl},0}) = z \in \mathcal{Z}^{\mathbb{R}_0^+}$.

Our objective in this chapter is to synthesize K for a given plant P, such that the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P, K)$ is incrementally dissipative in terms of Definition 5.2 under a given (incremental) (Q, S, R) supply function s_i, i.e., of the form (5.14), for which $R \preceq 0$, implying closed-loop incremental stability (see Remark 5.3). However, for the sake of compactness of the discussion, we will exemplify the theoretical toolchain only via the incremental $\mathcal{L}_2$-gain, although the overall machinery can be easily extended to the other incremental performance concepts discussed in Chapter 5. This leads to the following problem statement:

Problem Statement 1. *For a given plant P, synthesize K such that the $\mathcal{L}_{\mathrm{i}2}$-gain γ from w to z of the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P, K)$ is minimized, i.e., synthesize a K such that there exists a function $\zeta_\mathrm{i} : \mathcal{X}_\mathrm{cl} \times \mathcal{X}_\mathrm{cl} \to \mathbb{R}$ and a $\gamma \geq 0$, for which*

$$\|\mathcal{F}_\mathrm{l}(P, K)(w, x_{\mathrm{cl},0}) - \mathcal{F}_\mathrm{l}(P, K)(\tilde{w}, \tilde{x}_{\mathrm{cl},0})\|_{2,T} \leq \gamma \|w - \tilde{w}\|_{2,T} + \zeta_\mathrm{i}(x_{\mathrm{cl},0}, \tilde{x}_{\mathrm{cl},0}), \tag{6.3}$$

for all $T \geq 0$, $x_{\mathrm{cl},0}, \tilde{x}_{\mathrm{cl},0} \in \mathcal{X}_\mathrm{cl} \subseteq \mathbb{R}^{n_\mathrm{x} + n_{\mathrm{x_k}}}$ and $w, \tilde{w} \in \mathcal{W}^{\mathbb{R}_0^+}$ with $w, \tilde{w} \in \mathcal{L}_2$, where γ is minimal. In (6.3), $x_\mathrm{cl}(t) = \mathrm{col}(x(t), x_\mathrm{k}(t)) \in \mathcal{X}_\mathrm{cl}$ is the state associated with the state-space representation of the closed-loop $\mathcal{F}_\mathrm{l}(P, K)$.

Similar as for universal shifted synthesis in Section 4.5, to ensure that the above given synthesis problem is feasible with a finite γ, we require P to be a generalized plant in the following sense

Definition 6.1 (Generalized plant for incremental synthesis)**.** *P, given by (6.1), is a generalized plant, if there exists a controller K of the form (6.2) such that the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P, K)$ is incrementally stable.*

Proposition 6.1. *P, given by (6.1), is a generalized plant in the sense of Definition 6.1, if $\left(\frac{\partial f}{\partial x}(x, u), \frac{\partial f}{\partial u}(x, u)\right)$ is stabilizable and $\left(\frac{\partial f}{\partial x}(x, u), \frac{\partial h_\mathrm{y}}{\partial x}(x, u)\right)$ is detectable over $\mathcal{X} \times \mathcal{W}$, see (Pavlov, Van de Wouw, et al. 2006, Section 5.3.2).*

Note that the condition in Proposition 4.2 for universal shifted synthesis is equivalent to the ones in Proposition 6.1. This is because both the differential form and velocity form have a similar structure, as we discussed in Section 5.3.6. Therefore, analogous to Proposition 4.2, Proposition 6.1 can be interpreted as the differential form of (6.1) being stabilizable and detectable w.r.t. the (differential versions of the) control input channel u and measured output channel y, respectively, along all trajectories of nonlinear system given by (6.1).

To further simplify our discussion, we will assume that (6.1) can be transformed to the form

$$\dot{x}(t) = f(x(t)) + B_{\mathrm{w}}w(t) + B_{\mathrm{u}}u(t); \tag{6.4a}$$
$$z(t) = h_{\mathrm{z}}(x(t)) + D_{\mathrm{zw}}w(t) + D_{\mathrm{zu}}u(t); \tag{6.4b}$$
$$y(t) = C_{\mathrm{y}}x(t) + D_{\mathrm{yw}}w(t); \tag{6.4c}$$

where now $f : \mathcal{X} \to \mathbb{R}^{n_{\mathrm{x}}}$ and $h_{\mathrm{z}} : \mathcal{X} \to \mathbb{R}^{n_{\mathrm{z}}}$ with $f, h_{\mathrm{z}} \in \mathcal{C}_1$, and where $B_{\mathrm{u}} \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{u}}}$ and $D_{\mathrm{zu}} \in \mathbb{R}^{n_{\mathrm{z}} \times n_{\mathrm{u}}}$. While (6.4) may seem restrictive, (6.1) can be always expressed as (6.4) at cost of increasing the state dimension and requiring the input u to be (piecewise) differentiable, see e.g. (Nijmeijer and Van der Schaft 2016) or the procedure discussed in Appendix C.2.2. We will see that P in the form of (6.4) is advantageous to provide a realization of K after synthesis.

6.3 Convex Incremental Controller Synthesis

6.3.1 Main concept

To solve Problem Statement 1, we propose a novel procedure to synthesize a nonlinear controller K that ensures $\mathcal{L}_{\mathrm{i2}}$-gain stability and performance of $\mathcal{F}_{\mathrm{l}}(P, K)$. The main steps of the method are summarized as follows:

1. *Differential embedding* step: Given a generalized plant P, its differential form P_δ is computed. Based on the differential form, a *Differential Parameter-Varying* (DPV) embedding[1] P_{dpv} is then constructed to represent the resulting P_δ.

2. *Differential controller synthesis* step: For the DPV embedding P_{dpv}, an LPV controller K_δ is synthesized, ensuring a minimal closed-loop $\mathcal{L}_2$-gain γ. This synthesis is accomplished using standard methods of the LPV framework, as discussed in Section 2.5.3.

3. *Incremental controller realization* step: The synthesized controller K_δ is realized as a primal nonlinear controller K in the form of (6.2) to be used with the original nonlinear system P to ensure the closed-loop $\mathcal{L}_{\mathrm{i2}}$-gain γ.

[1] See also Definition 5.7.

Our key contributions in the above proposed controller synthesis scheme is the controller realization procedure (Theorem 6.3) and proving that the resulting K solves Problem Statement 1, i.e., performance and stability guarantees obtained in the differential controller synthesis step do hold in the incremental sense on $\mathcal{F}_\mathrm{l}(P, K)$ (see Theorems 6.1 to 6.4).

Note that the same procedure can be applied in order to ensure different performance specifications by changing the used performance notion in the differential controller synthesis step, e.g., in order to ensure incremental passivity one would synthesize an LPV controller for the differential form of the generalized plant such that closed-loop passivity is ensured.

Remark 6.1. Note that the steps proposed above for incremental controller synthesis are similar to the steps for universal shifted controller synthesis as we have presented in Section 4.5.2. For the first step, instead of a *Velocity Parameter-Varying* (VPV) embedding, we use the for the incremental procedure a DPV embedding, which as discussed (see the end of Section 5.4.2) can in fact be given by the same LPV representation. Moreover, in the second step of both the universal shifted and incremental controller synthesis procedures, the LPV framework is used to synthesize a controller for the VPV embedding or DPV embedding of the generalized plant, respectively. In the third and final step of both the universal shifted and incremental controller procedure, we realize the controller to be used with the original nonlinear system. However, the realization procedure in the third step is where the universal shifted and incremental controller design greatly differ. For the universal shifted controller design, we exploit properties of the velocity form, while, as we will show, for the incremental controller design, we will exploit properties of the differential form. While the velocity form and differential form share similarities, they are different distinct representations, as we have discussed in Section 5.3.6. In the end, this results in the primal forms of the universal shifted controller design and incremental controller design having vastly different structures. We will also compare the controller designs in more detail in Section 6.4.

6.3.2 Separability in the differential domain

The procedure relies on the results of Chapter 5, specifically Theorem 5.2, which shows that 'classical (Q, S, R) dissipativity' of the differential form (i.e., differential (Q, S, R) dissipativity) implies incremental (Q, S, R) dissipativity of the primal form (for the same tuple (Q, S, R)). Hence, to solve Problem Statement 1, we can equivalently minimize the $\mathcal{L}_2$-gain of the differential form of $\mathcal{F}_\mathrm{l}(P, K)$. Before discussing the steps of the proposed procedure, we will first show that the differential form of $\mathcal{F}_\mathrm{l}(P, K)$ is equal to $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$. This significantly simplifies the synthesis procedure, as it allows for independently 'transforming' P and K between the primal and differential domains. This is similar to the result we provide for the velocity form in Theorem 4.9.

Theorem 6.1 (Closed-loop differential form)**.** *The differential form of the closed-loop system $\mathcal{F}_\mathrm{l}(P, K)$ is equal to the closed-loop interconnection of P_δ and K_δ, i.e.,*

$\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$, if the interconnection of P and K is well-posed, i.e., there exists a $\mathcal{C}_1$ function $\breve{h}$ such that $u = h_\mathrm{k}(x_\mathrm{k}, h_\mathrm{y}(x, u))$ can be expressed as $u = \breve{h}(x, x_\mathrm{k})$.

Proof. See Appendix B.4. □

6.3.3 Differential embedding

In the first step of the synthesis procedure, the differential form of the generalized plant P is computed, and the result is embedded in an LPV representation, resulting in a DPV embedding.

Let us introduce the solution set of (6.4), defined as follows

$$\mathfrak{B} := \Big\{(x, u, w, z, y) \in (\mathcal{X} \times \mathcal{U} \times \mathcal{W} \times \mathcal{Z} \times \mathcal{Y})^{\mathbb{R}_0^+} \mid x \in \mathcal{C}_1, \\ (x, u, w, z, y) \text{ satisfies (6.4)}\Big\}. \tag{6.5}$$

Then, computing the differential form of P given in (6.4), results in P_δ, given by[2]

$$\dot{x}_\delta(t) = A_\delta(\bar{x}(t))x_\delta(t) + B_\mathrm{w} w_\delta(t) + B_\mathrm{u} u_\delta(t); \tag{6.6a}$$
$$z_\delta(t) = C_{\delta,\mathrm{z}}(\bar{x}(t))x_\delta(t) + D_\mathrm{zw} w_\delta(t) + D_\mathrm{zu} u_\delta(t); \tag{6.6b}$$
$$y_\delta(t) = C_\mathrm{y} x_\delta(t) + D_\mathrm{yw} w_\delta(t); \tag{6.6c}$$

where $A_\delta = \frac{\partial f}{\partial x}$ and $C_{\delta,\mathrm{z}} = \frac{\partial h_\mathrm{z}}{\partial x}$ with $\bar{x} \in \pi_\mathrm{x} \mathfrak{B}$, $x_\delta \in \mathcal{C}_1$ and $x_\delta(t) \in \mathbb{R}^{n_\mathrm{x}}$ with $x_\delta(0) = x_{\delta,0} \in \mathbb{R}^{n_\mathrm{x}}$, $u_\delta(t) \in \mathbb{R}^{n_\mathrm{u}}$, $w_\delta(t) \in \mathbb{R}^{n_\mathrm{w}}$, $z_\delta(t) \in \mathbb{R}^{n_\mathrm{z}}$ and $y_\delta(t) \in \mathbb{R}^{n_\mathrm{y}}$. Along a $\bar{x} \in \pi_\mathrm{x} \mathfrak{B}$ solution of (6.4), the set of solutions of (6.6) is

$$\mathfrak{B}_\delta(\bar{x}) := \Big\{(x_\delta, u_\delta, w_\delta, z_\delta, y_\delta) \in (\mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{u}} \times \mathbb{R}^{n_\mathrm{w}} \times \mathbb{R}^{n_\mathrm{z}} \times \mathbb{R}^{n_\mathrm{y}})^{\mathbb{R}_0^+} \mid x_\delta \in \mathcal{C}_1, \\ (x_\delta, u_\delta, w_\delta, z_\delta, y_\delta) \text{ satisfies (6.6) along } \bar{x}\Big\}. \tag{6.7}$$

Then $\breve{\mathfrak{B}}_\delta = \bigcup\limits_{\bar{x} \in \pi_\mathrm{x} \mathfrak{B}} \mathfrak{B}_\delta(\bar{x})$ gives the complete solution set of (6.6).

Next, we embed (6.6) in an LPV representation, i.e., we construct a DPV embedding of (6.4):

Definition 6.2 (Generalized plant DPV embedding)**.** *Consider a nonlinear system with primal form (6.4) and differential form (6.6). The LPV state-space representation*

$$\dot{x}_\delta(t) = A(p(t))x_\delta(t) + B_\mathrm{w} w_\delta(t) + B_\mathrm{u} u_\delta(t); \tag{6.8a}$$
$$z_\delta(t) = C_\mathrm{z}(p(t))x_\delta(t) + D_\mathrm{zw} w_\delta(t) + D_\mathrm{zu} u_\delta(t); \tag{6.8b}$$
$$y_\delta(t) = C_\mathrm{y} x_\delta(t) + D_\mathrm{yw} w_\delta(t); \tag{6.8c}$$

[2]Similar to the notation in Chapter 5, we drop the dependency on λ for brevity.

with A, C_{z} belonging to a given class of functions $\mathfrak{A}$ (e.g., affine or rational functions) and $p(t) \in \mathcal{P}$ being the scheduling-variable with a compact and convex $\mathcal{P} \subset \mathbb{R}^{n_{\mathrm{p}}}$, is called a DPV embedding of (6.4) on the region $\mathcal{X} \subseteq \mathcal{X}$, if there is a function $\eta : \mathcal{X} \to \mathcal{P}$, called the scheduling-map, with $\eta \in \mathcal{C}_1$ and $\eta(\mathcal{X}) \subseteq \mathcal{P}$, such that $A \circ \eta = A_\delta$ (i.e., $A(\eta(\bar{x})) = A_\delta(\bar{x})$) and $C_{\mathrm{z}} \circ \eta = C_{\delta,\mathrm{z}}$ for all $\bar{x} \in \mathcal{X}$.

While Definition 6.2 is consistent with the general DPV embedding definition given in Definition 5.7, it is given for clarity of the procedure.

Remark 6.2. For tractable controller synthesis, later in Section 6.3.4, the function η is must be chosen such that the resulting dependence of A and C_{z} on p, i.e., the class $\mathfrak{A}$, is either affine, polynomial or rational and n_{p} is minimal. Furthermore, $\mathcal{P}$ needs to be chosen such that for the LPV representation (6.8), (A, B_{u}) is stabilizable and (A, C_{y}) is detectable[3]. This will ensure that the properties of the generalized plant in Proposition 6.1 are preserved when constructing the DPV embedding. Moreover, $\mathcal{P}$ is also chosen such that it is the smallest convex set in a given complexity class (n-vertex polytope, hyper-ellipsoid, etc.) such that $\eta(\mathcal{X}) \subseteq \mathcal{P}$, in order to minimize the conservativeness of the LPV representation in describing the differential form. See (Kwiatkowski and Werner 2008; Hoffmann 2016; Sadeghzadeh, Sharif, et al. 2020) for approaches to fulfill these properties.

In accordance with Definition 6.2, we assume that a DPV embedding, denoted by P_{dpv}, of (6.4) is constructed in terms of (6.8) on the region $\mathcal{X} \subseteq \mathcal{X}$, where $\mathcal{X}$ is compact. This means that i.e., we assume we embed part of the state-space. For this DPV embedding, let us denote for a given $p \in \mathcal{P}^{\mathbb{R}_0^+}$ the solution set by $\mathfrak{B}_{\mathrm{p}}(p)$ and the full behavior by (i.e., for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$) by $\breve{\mathfrak{B}}_{\mathrm{p}}$, see also (2.27) and (2.28). Moreover, denote the restriction of state solutions of P (6.4) to $\mathcal{X}$ by $\mathfrak{B}_{\mathcal{X}} := \{(x, u, w, z, y) \in \mathfrak{B} \mid x(t) \in \mathcal{X}\}$ and the corresponding set for the differential form P_δ by $\breve{\mathfrak{B}}_{\delta,\mathcal{X}} := \cup_{(\bar{x} \in \pi_{\mathrm{x}} \mathfrak{B}_{\mathcal{X}})} \mathfrak{B}_\delta(\bar{x})$. Through Lemma 5.3, we then have that $\breve{\mathfrak{B}}_{\delta,\mathcal{X}} \subseteq \breve{\mathfrak{B}}_{\mathrm{p}}$, i.e., through the DPV embedding (6.8) we can describe the behavior of the differential form (6.6) under $\bar{x}(t) \in \mathcal{X}$. Through this DPV embedding P_{dpv}, we can use the LPV framework in order to synthesize a controller for P_δ, which will be the next step in the procedure.

6.3.4 Differential synthesis

As aforementioned, we want to synthesize a controller K in order to minimize the $\mathcal{L}_{\mathrm{i}2}$-gain of $\mathcal{F}_{\mathrm{l}}(P, K)$. This is done by first synthesizing a differential controller K_δ such that the $\mathcal{L}_2$-gain of $\mathcal{F}_{\mathrm{l}}(P_\delta, K_\delta)$ is minimized. Then, later in Section 6.3.5, a primal form K of the controller K_δ is realized that preserves the achieved closed-loop properties of $\mathcal{F}_{\mathrm{l}}(P_\delta, K_\delta)$. In order to perform controller synthesis for the differential form P_δ, the LPV framework is used. More concretely, we synthesize a controller for the LPV embedding of the differential form P_{dpv}, given in (6.8), which has been constructed in the differential embedding step in the previous subsection. To achieve this, we can apply our standard $\mathcal{L}_2$-gain LPV synthesis techniques on (6.8) such as polytopic or LFT-based LPV synthesis methods, e.g.

[3] See Definitions 2.17 and 2.18.

(Packard 1993; Apkarian, Gahinet, and G. Becker 1995; Wu 1995; Scherer 2001), see also Section 2.5.3, to synthesize a controller K_δ and ensure $\mathcal{L}_2$-gain stability[4] of the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P_\mathrm{dpv}, K_\delta)$, for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$. This synthesized controller K_δ is considered to be of the following form

$$\dot{x}_{\delta,\mathrm{k}}(t) = A_\mathrm{k}(p(t))x_{\delta,\mathrm{k}}(t) + B_\mathrm{k}(p(t))u_{\delta,\mathrm{k}}(t); \tag{6.9a}$$

$$y_{\delta,\mathrm{k}}(t) = C_\mathrm{k}(p(t))x_{\delta,\mathrm{k}}(t) + D_\mathrm{k}(p(t))u_{\delta,\mathrm{k}}(t); \tag{6.9b}$$

which we will refer to as the differential controller, where $x_{\delta,\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}}$ is the state, $u_{\delta,\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{u_k}}}$ is the input, and $y_{\delta,\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{y_k}}}$ is the output of the controller, respectively and $A_\mathrm{k}, \ldots, D_\mathrm{k} \in \mathfrak{A}$ are matrix functions with appropriate dimensions.

Theorem 6.2 (Differential closed-loop $\mathcal{L}_2$-gain)**.** *If controller K_δ of the form (6.9) ensures bounded $\mathcal{L}_2$-gain γ of the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P_\mathrm{dpv}, K_\delta)$ for all $p \in \mathcal{P}^{\mathbb{R}}$, then $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$ with $p = \eta(\bar{x})$ is also $\mathcal{L}_2$-gain stable with an $\mathcal{L}_2$-gain $\leq \gamma$ for all $\bar{x} \in \pi_\mathrm{x}\mathfrak{B}_\mathcal{X}$.*

Proof. See Appendix B.4. □

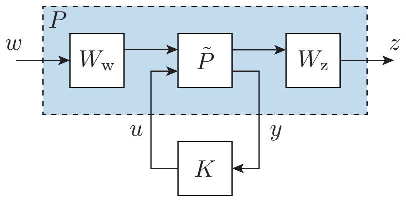
Assumption 6.1. *We assume that the controller synthesis has been solved such that $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$ is dissipative with a quadratic storage function, i.e., the corresponding (differential) storage function is assumed to be of the form $\mathcal{V}_\delta(x_\mathrm{cl}, x_{\delta,\mathrm{cl}}) = x_{\delta,\mathrm{cl}}^\top M x_{\delta,\mathrm{cl}}$, where $M \succ 0$, i.e., a quadratic $\mathcal{V}_\delta$ which is independent of x_cl. This is required for the proposed controller realization procedure in Section 6.3.5.*

Remark 6.3. Note that similar as for the universal shifted controller design, see Remark 4.3, if the weighting filters included in P are LTI, then, as depicted in Figure 6.1, the input-output behaviors of W_w and W_z are equivalent to that of $W_{\delta,\mathrm{w}}$ and $W_{\delta,\mathrm{z}}$. This is the case because the dynamics of the differential form of an LTI system are equivalent to the dynamics of its primal form. Therefore, there in a one to one correspondence between the performance shaping of the primal form $\mathcal{F}_\mathrm{l}(P, K)$ (see Figure 6.1a) and performance shaping of the differential form $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$ (see Figure 6.1b). This significantly simplifies the controller design, as shaping can be directly performed through the differential form P_δ and hence also through the DPV embedding P_dpv.

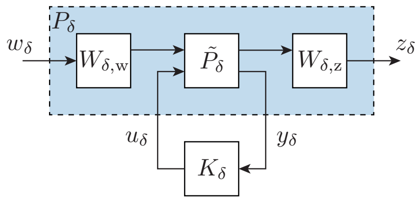
6.3.5 Controller realization

We will now describe how to realize the primal form K of the controller for the nonlinear system such that the differential form of K is given by K_δ in (6.9) and incremental dissipativity of the closed-loop is ensured. Similar to the approach in (Manchester and Slotine 2018), we take a path integral based realization. In Chapter 5 we have seen that to obtain the differential form we differentiate over the variation of trajectories, in terms of λ, hence, to go back to the primal form we

[4] By which we mean the system is classically dissipative with a supply function corresponding to a bounded $\mathcal{L}_2$-gain, also implying stability of the system.



(a) Primal form.



(b) Differential form.

Figure 6.1: Shaping the closed-loop behavior of the primal and the differential form by the use of weighting filters W_{w} and W_{z}.

integrate over the variation λ. This lets us converge from the current trajectory towards a known desired (feasible) steady-state trajectory. Namely, to guarantee incremental stability and performance, we consider $\vartheta \triangleq (x^*, u^*, w^*, z^*, y^*) \in \mathfrak{B}_{\mathcal{X}}$ of P (6.4) to be a *known* trajectory, towards which we want to converge. Let us denote by $x_{\delta,\mathrm{cl}}(t) \in \mathbb{R}^{n_{\mathrm{x_{cl}}}}$ the state associated with $\mathcal{F}_{\mathrm{l}}(P_\delta, K_\delta)$, analogous to the state x_{cl} of the primal form of the closed-loop interconnection $\mathcal{F}_{\mathrm{l}}(P, K)$.

Theorem 6.3 (Incremental controller realization)**.** *Consider a differential controller K_δ in the form of (6.9) that ensures closed-loop $\mathcal{L}_2$-gain stability of $\mathcal{F}_{\mathrm{l}}(P_\delta, K_\delta)$ satisfying Assumption 6.1. Let $(x^*, u^*, y^*) = \pi_{\mathrm{x,u,y}}\,\vartheta \in \pi_{\mathrm{x,u,y}}\,\mathfrak{B}_{\mathcal{X}}$ be the (desired) steady-state trajectory of P and consider the nonlinear controller K, omitting dependence on time for brevity, given by*

$$\dot{x}_{\Delta,\mathrm{k}} = \bar{A}_{\mathrm{k}}(x, x^*)x_{\Delta,\mathrm{k}} + \bar{B}_{\mathrm{k}}(x, x^*)u_{\Delta,\mathrm{k}}; \tag{6.10a}$$

$$y_{\mathrm{k}} = y_{\mathrm{k}}^* + \bar{C}_{\mathrm{k}}(x, x^*)x_{\Delta,\mathrm{k}} + \bar{D}_{\mathrm{k}}(x, x^*)u_{\Delta,\mathrm{k}}; \tag{6.10b}$$

with $(y_{\mathrm{k}}^*, u_{\mathrm{k}}^*) = (u^*, y^*)$, $x_{\Delta,\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}}$, $u_{\Delta,\mathrm{k}} := u_{\mathrm{k}} - u_{\mathrm{k}}^*$, and

$$\begin{aligned}\bar{A}_{\mathrm{k}}(x,x^*)&=\int_0^1 A_{\mathrm{k}}\Big(\eta\big(x^*+\lambda(x-x^*)\big)\Big)d\lambda, & \bar{B}_{\mathrm{k}}(x,x^*)&=\int_0^1 B_{\mathrm{k}}\Big(\eta\big(x^*+\lambda(x-x^*)\big)\Big)d\lambda,\\ \bar{C}_{\mathrm{k}}(x,x^*)&=\int_0^1 C_{\mathrm{k}}\Big(\eta\big(x^*+\lambda(x-x^*)\big)\Big)d\lambda, & \bar{D}_{\mathrm{k}}(x,x^*)&=\int_0^1 D_{\mathrm{k}}\Big(\eta\big(x^*+\lambda(x-x^*)\big)\Big)d\lambda.\end{aligned} \tag{6.11}$$

The controller K in (6.10) is the primal form of K_δ (6.9) and the differential form of K is K_δ. Hence, K is called the primal realization of K_δ.

Proof. See Appendix B.4. □

We will refer to the proposed controller K as the *incremental controller*. Even if K_δ is an LPV controller, realization by Theorem 6.3 results in a nonlinear controller through the resubstitution and integration over the scheduling-map η.

Remark 6.4. Note that the dependency on x and x^* of the incremental controller K given by (6.10) is only through the scheduling-map η. This means that depending on differential form of the generalized plant, P_δ, it could be that the scheduling-map η only depends on some of the states when constructing a DPV embedding P_{dpv} in first step of the controller design procedure (Section 6.3.3). Therefore, in that case, K also does not depend on the full x and x^*.

Note that the resulting controller consists of both a direct feedforward action $y_{\mathrm{k}}^* = u^*$, corresponding to the desired steady-state trajectory ϑ, and a feedback action on the measured deviation from the desired steady-state output y^*. Therefore, the controller has a structure as depicted in Figure 6.2, where K_Δ (based on (6.10)) is given by

$$\dot{x}_{\Delta,\mathrm{k}}(t) = \bar{A}_{\mathrm{k}}\big(x(t), x^*(t)\big)x_{\Delta,\mathrm{k}}(t) + \bar{B}_{\mathrm{k}}\big(x(t), x^*(t)\big)u_{\Delta,\mathrm{k}}(t); \tag{6.12a}$$

$$y_{\Delta,\mathrm{k}}(t) = \bar{C}_{\mathrm{k}}\big(x(t), x^*(t)\big)x_{\Delta,\mathrm{k}}(t) + \bar{D}_{\mathrm{k}}\big(x(t), x^*(t)\big)u_{\Delta,\mathrm{k}}(t); \tag{6.12b}$$

with $\bar{A}_{\mathrm{k}}, \ldots, \bar{D}_{\mathrm{k}}$ given by (6.11). Note that this structure is similar to gain-scheduling or LPV controller design with trimming (through u_{k}^* and y_{k}^*) in order to ensure guarantees w.r.t. a desired operating point. However, gain-scheduling and LPV controllers which use such a trimming approach, generally do not have stability and performance guarantees w.r.t. to any arbitrary trimming point. This is in contrast to the incremental controller design, which does ensure stability and performance w.r.t. any 'trimming point'. For the incremental controller, the 'trimming point' corresponds to steady state-trajectory ϑ, which can be any (feasible, varying or constant) trajectory of the system. Due to the closed-loop incremental stability performance guarantees of the incremental controller, we therefore ensure stability and performance w.r.t. any trajectory.

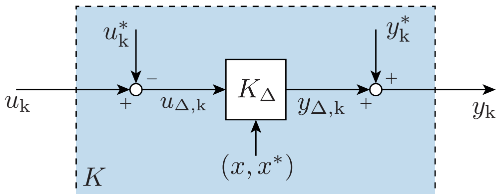


Figure 6.2: Structure of the realized incremental controller K, where K_Δ is given by (6.12). Note that K_Δ does not necessarily depend on the full x and x^*, see Remark 6.4.

Note that this means that for implementation we require knowledge of ϑ. This also implies we require knowledge of w^*, which is part of ϑ, either through direct measurement or estimation. This will be discussed in more detail in Section 6.3.7.

For implementation of the incremental controller, we also require computation of the integrals for $\bar{A}_{\mathrm{k}}, \ldots, \bar{D}_{\mathrm{k}}$ in (6.11). Analytically computing these integrals might be difficult in some cases, however, they can reliably calculated (online) through numerical computation (Atkinson 1989). Moreover, if the scheduling-dependency of K_δ is affine, (6.11) can be further simplified:

Corollary 6.1. *Assume that $A_\mathrm{k}, \ldots, D_\mathrm{k}$ in (6.9), characterizing K_δ, are affine in p, i.e., $A_\mathrm{k}(p) = A_{\mathrm{k},0} + \sum_{i=1}^{n_\mathrm{p}} A_{\mathrm{k},i}\, p_i$. Then, the matrix functions in (6.10) are given as*

$$\begin{aligned} \bar{A}_\mathrm{k}(x, x^*) &= \int_0^1 A_\mathrm{k}\Big(\eta\big(x^* + \lambda(x - x^*)\big)\Big)\, d\lambda, \\ &= A_{\mathrm{k},0} + \sum_{i=1}^{n_\mathrm{p}} A_{\mathrm{k},i}\, \rho_i = A_\mathrm{k}(\rho), \end{aligned} \tag{6.13}$$

where $\rho = \int_0^1 \eta\big(x^* + \lambda(x - x^*)\big)\, d\lambda$, and similarly $\bar{B}_\mathrm{k}(x, x^*) = B_\mathrm{k}(\rho)$, $\bar{C}_\mathrm{k}(x, x^*) = C_\mathrm{k}(\rho)$ and $\bar{D}_\mathrm{k}(x, x^*) = D_\mathrm{k}(\rho)$.

Therefore, in case of an affine scheduling dependency, we only require integration of the scheduling-map, instead of requiring integration of the matrix function $A_\mathrm{k}, \ldots, D_\mathrm{k}$, which simplifies implementation.

6.3.6 Closed-loop incremental stability and performance

Next, we will show that the proposed controller K ensures closed-loop $\mathcal{L}_{\mathrm{i}2}$-gain stability of $\mathcal{F}_\mathrm{l}(P, K)$.

Theorem 6.4 (Closed-loop $\mathcal{L}_{\mathrm{i}2}$-gain stability)**.** *Let K_δ, given in (6.9), be an LPV controller, synthesized for P_δ given in (6.6) of a nonlinear system (6.4), which ensures $\mathcal{L}_2$-gain stability of the closed-loop $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$ with a bounded $\mathcal{L}_2$-gain of γ on $\mathcal{X}$ under Assumption 6.1. Consider the set $\tilde{\mathcal{W}} \subseteq \mathcal{W}$, such that there is an open and bounded $\mathcal{X}_\mathrm{k} \subseteq \mathbb{R}^{n_{\mathrm{x_k}}}$ for which $\mathcal{X}_\mathrm{cl} = \mathcal{X} \times \mathcal{X}_\mathrm{k}$ is invariant, in the sense of Definition 4.9. Then, the controller K, given by (6.10), ensures $\mathcal{L}_{\mathrm{i}2}$-gain stability of the closed-loop $\mathcal{F}_\mathrm{l}(P, K)$ with a bounded $\mathcal{L}_{\mathrm{i}2}$-gain of γ in the following sense: there exists a function $\zeta_\mathrm{i} : \mathcal{X}_\mathrm{cl} \times \mathcal{X}_\mathrm{cl} \to \mathbb{R}$ s.t.*

$$\|\mathcal{F}_\mathrm{l}(P, K)(w, x_{\mathrm{cl},0}) - \mathcal{F}_\mathrm{l}(P, K)(\tilde{w}, \tilde{x}_{\mathrm{cl},0})\|_{2,T} \leq \gamma\, \|w - \tilde{w}\|_{2,T} + \zeta_\mathrm{i}(x_{\mathrm{cl},0}, \tilde{x}_{\mathrm{cl},0}), \tag{6.14}$$

for all $T \geq 0$, $x_{\mathrm{cl},0}, \tilde{x}_{\mathrm{cl},0} \in \mathcal{X}_\mathrm{cl}$ and any $w, \tilde{w} \in \tilde{\mathcal{W}}^{\mathbb{R}_0^+}$ with $w - w^* \in \mathcal{L}_{2\mathrm{e}}$.

Proof. See Appendix B.4. □

Remark 6.5. The value set of the generalized disturbance signals $\tilde{\mathcal{W}}$, considered in Theorem 6.4, i.e., $w(t) \in \tilde{\mathcal{W}}$, ensures that only generalized disturbances are considered such that $x(t) \in \mathcal{X}$, which corresponds to the set for which $\mathcal{L}_2$-gain stability was verified of the closed-loop differential form, see Section 6.3.4. Computing this set is a difficult problem which is related to reachability analysis or invariant set computation, however there are numerical tools that can be employed for this purpose, see e.g. (Althoff 2013; Maidens and Arcak 2015).

6.3.7 Steady-state solution

Estimating the steady-state solution

The realized controller K in terms of (6.10) consists of a feedforward and a feedback part, see also Figure 6.2. The feedforward part $u^* = y_{\mathrm{k}}^*$ corresponds to the steady-state trajectory $\vartheta = (x^*, u^*, w^*, z^*, y^*) \in \mathfrak{B}_{\mathcal{X}}$. This trajectory is chosen a-priori by the user, based on the desired reference the system needs to follow, similar to trajectory planning in robotics. Computation of ϑ can be accomplished by using trajectory planning algorithms (Gasparetto et al. 2015) or in some cases by computation of the analytic solution of the system equations.

Note that the generalized disturbances w also influence the steady-state trajectory. The generalized disturbance consists of disturbances w_{m} that are known or can be measured, such as references, and consists of disturbances w_{u} that are not known and/or cannot be measured, such as measurement noise and load variation, composing $w = \mathrm{col}(w_{\mathrm{m}}, w_{\mathrm{u}})$. Hence, to guarantee convergence towards the designed desired steady-state trajectory, we also require knowledge of the asymptotic behavior of the unknown part w_{u} of w^*. We only need knowledge on the asymptotic behavior of w_{u} as only that influences the steady-state trajectory ϑ, e.g. a zero mean measurement noise does not need to be estimated as it does not influence the steady-state trajectory, while estimation of a constant load is required as it directly influences it. Next, we will present a solution using a disturbance observer in order to estimate the unknown elements w_{u}^* of the steady-state generalized disturbance w^*.

Disturbance observers have been widely used to estimate and compensate for the effect of unknown disturbances (Chen et al. 2016). Often they work on the basis of the internal model principle, whereby the (assumed) dynamics of the disturbance are included in the design (Chen 2004). In our case, we assume w_{u} can be modeled in terms of a disturbance model:

Assumption 6.2 (Disturbance model)**.** *Given the generalized plant (6.1), assume that the unknown disturbances w_{u}, can be modeled by the disturbance generator*

$$\dot{x}_{\mathrm{w}}(t) = f_{\mathrm{w}}(x_{\mathrm{w}}(t)); \tag{6.15a}$$

$$w_{\mathrm{u}}(t) = C_{\mathrm{w}} x_{\mathrm{w}}(t); \tag{6.15b}$$

with $x_{\mathrm{w}}(t) \in \mathcal{X}_{\mathrm{w}}$.

Given Assumption 6.2, the state dynamics of the combined generalized plant (6.1) and disturbance model are given by

$$\begin{aligned}\dot{x}_{\mathrm{e}}(t) = \begin{bmatrix} \dot{x}(t) \\ \dot{x}_{\mathrm{w}}(t) \end{bmatrix} &= \begin{bmatrix} f(x(t), u(t)) + B_{\mathrm{w}} \begin{bmatrix} w_{\mathrm{m}}(t) \\ C_{\mathrm{w}} x_{\mathrm{w}}(t) \end{bmatrix} \\ f_{\mathrm{w}}(x_{\mathrm{w}}(t)) \end{bmatrix}, \\ &= f_{\mathrm{e}}(x_{\mathrm{e}}(t), u(t), w_{\mathrm{m}}(t)),\end{aligned} \tag{6.16a}$$

where $x_{\mathrm{e}}(t) \in \mathcal{X} \times \mathcal{X}_{\mathrm{w}} = \mathcal{X}_{\mathrm{e}} \subseteq \mathbb{R}^{n_{\mathrm{x_e}}}$, and the measured output is given by

$$\begin{aligned} y &= C_{\mathrm{y}} x(t) + D_{\mathrm{yw}} \begin{bmatrix} w_{\mathrm{m}}(t) \\ C_{\mathrm{w}} x_{\mathrm{w}}(t) \end{bmatrix}, \\ &= C_{\mathrm{e}} x_{\mathrm{e}}(t) + D_{\mathrm{yw}} w_{\mathrm{m}}(t)), \end{aligned} \tag{6.16b}$$

where $C_{\mathrm{e}} = \begin{bmatrix} C_{\mathrm{y}} & D_{\mathrm{yw}} C_{\mathrm{w}} \end{bmatrix}$.

Remark 6.6. Note that the disturbance model in Assumption 6.2 has a different purpose than the disturbance model that is included in the generalized plant P given by (6.4), which is modeled in terms of weighted norm relation. Namely, the former is used to model disturbances that are acting on the system and influence the asymptotic behavior of the steady-state trajectory (and hence influence w^*), while the latter can be seen as modeling the difference/increment between the steady-state disturbance and other possible disturbances (i.e. $w - w^*$) which do not influence the asymptotic behavior of the steady-state trajectory, e.g., measurement noise.

Assumption 6.3. *In order for the construction of the disturbance observer on the basis of (6.16) to be feasible, we assume that $(\frac{\partial f_{\mathrm{e}}}{\partial x_{\mathrm{e}}}, C_{\mathrm{e}})$ is detectable over $x_{\mathrm{e}} \in \mathcal{X}_{\mathrm{e}}$ (Pavlov, Van de Wouw, et al. 2006, Section 5.3.2).*

Theorem 6.5 (Nonlinear observer)**.** *Under Assumption 6.3, the state observer for (6.16), given by*

$$\dot{\hat{x}}_{\mathrm{e}}(t) = f_{\mathrm{e}}(\hat{x}_{\mathrm{e}}(t), u(t), w_{\mathrm{m}}(t)) + L(y(t) - \hat{y}(t)); \tag{6.17a}$$
$$\hat{y}(t) = C_{\mathrm{e}} \hat{x}_{\mathrm{e}}(t) + D_{\mathrm{yw}} w_{\mathrm{m}}(t)); \tag{6.17b}$$

with $\hat{x}_{\mathrm{e}} \in \mathbb{R}^{n_{\mathrm{x_e}}}$ and $\hat{y}(t) \in \mathbb{R}^{n_{\mathrm{y}}}$ ensures that for $t \to \infty$, $\hat{x}_{\mathrm{e}}(t) \to x_{\mathrm{e}}(t)$, if there exists an $P \in \mathbb{S}^{n_{\mathrm{x_e}}}$ with $P \succ 0$, and an $F \in \mathbb{R}^{n_{\mathrm{x_e}} \times n_{\mathrm{y}}}$ such that

$$A_{\delta,\mathrm{e}}(x_{\mathrm{e}}, u, w_{\mathrm{m}})^\top P - F C_{\mathrm{e}} + (\star)^\top \prec 0, \tag{6.18}$$

for all $(x_{\mathrm{e}}, u, w_{\mathrm{m}}) \in \mathcal{X}_{\mathrm{e}} \times \mathcal{U} \times \pi_{\mathrm{w_m}} \mathcal{W}$, where $A_{\delta,\mathrm{e}} = \frac{\partial f_{\mathrm{e}}}{\partial x_{\mathrm{e}}}$, and $L = P F^{-1}$.

Proof. See Appendix B.4. □

Remark 6.7. Similar to controller design, also for observer design the LPV framework can be used by embedding the differential form of the system in an LPV representation. This allows to use convex optimization for the computation of the observer gain L in Theorem 6.5.

Applying the nonlinear observer (6.17) gives us a disturbance observer for the combined generalized plant and disturbance model (6.16), which then allows us to estimate the unknown disturbances on the system by taking $w_{\mathrm{u}}^*(t) = C_{\mathrm{w}} \hat{x}_{\mathrm{w}}(t)$, where $\hat{x}_{\mathrm{w}}$ is the estimated state of the disturbance model. This can then be used to compute the steady-state control input trajectory $u^* = y_{\mathrm{k}}^*$, used by the realized controller K (6.10), corresponding to the steady-state output trajectory y^*. Note that co-design of the controller and the observer under (6.17) can also be accomplished.

Unknown steady-state solution

In case w^* cannot be measured or estimated, we cannot guarantee that $w^*(t) \to w(t)$ as $t \to \infty$. Consequently, we cannot ensure convergence towards our desired steady-state solution ϑ. However, in this case, the controller still ensures an $\mathcal{L}_2$-gain bound[5] of γ from $w - w^*$ to $z - z^*$, i.e., the steady state trajectory will remain close in an $\mathcal{L}_2$ sense to the desired reference and the controller will still ensure stability of the closed-loop system. This weaker performance guarantee is also referred to as universal $\mathcal{L}_2$-gain performance, see also (Manchester and Slotine 2018).

6.4 Comparison to Universal Shifted and LPV Controllers

6.4.1 Universal shifted controller design

Both the universal shifted controller design, proposed in Chapter 4, and the incremental controller design proposed in this chapter are able to ensure global stability and performance guarantees. In this section we briefly compare the two controller designs.

In Section 5.3.4, specifically Theorem 5.3, we discussed how incremental dissipativity also guarantees universal shifted dissipativity, considering (Q, S, R) supply functions. This means that using the proposed incremental controller in this chapter, we can also guarantee universal shifted stability and performance of the closed-loop system $\mathcal{F}_\mathrm{l}(P, K)$, similar how the universal shifted controller design of Section 4.5 was able to ensure this. In case the incremental controller design is applied in order to ensure universal shifted dissipativity, the desired trajectory ϑ becomes a constant trajectory, equal to an equilibrium point of P (6.4). More concretely, the equilibrium points of (6.4) satisfy that

$$0 = f(x_*) + B_\mathrm{w} w_* + B_\mathrm{u} u_*; \tag{6.19a}$$

$$z_* = h_\mathrm{z}(x_*) + D_\mathrm{zw} w_* + D_\mathrm{zu} u_*; \tag{6.19b}$$

$$y_* = C_\mathrm{y} x_* + D_\mathrm{yw} w_*; \tag{6.19c}$$

with corresponding set

$$\mathscr{E} := \big\{(x_*, u_*, w_*, z_*, y_*) \in \mathcal{X} \times \mathcal{U} \times \mathcal{W} \times \mathcal{Z} \times \mathcal{Y} \mid (x_*, u_*, w_*, z_*, y_*) \text{ satisfy } (6.19)\big\}. \tag{6.20}$$

[5] Assuming that $w - w^* \in \mathcal{L}_{2\mathrm{e}}$.

For universal shifted dissipativity we then have

$$\vartheta \in \big\{(x,u,w,z,y) \in \mathfrak{B}_X \mid (x(t),u(t),w(t),z(t),y(t)) = (x_*,u_*,w_*,z_*,y_*) \in \mathscr{E},\, \forall\, t \in \mathbb{R}_0^+\big\}. \quad (6.21)$$

As for the proposed incremental design we require knowledge of the desired trajectory ϑ, we hence require explicit knowledge of the equilibrium points to ensure universal shifted dissipativity. Compared to the universal shifted controller design we propose in Chapter 4, this is disadvantageous, as for the universal shifted design, *no* explicit knowledge of the equilibrium points is required (and hence also not of any (constant) unmeasured disturbances). However, as the knowledge of the equilibrium point is also incorporated in the incremental controller as feedforward action, the incremental controller will likely converge faster to the desired equilibrium point compared to the universal shifted controller (for the same closed-loop performance metric). This can be seen as an advantage of the incremental controller design compared to the universal shifted controller design, although a feedforward action or controller can also be added or designed for the universal shifted controller.

Another disadvantage of using the incremental controller design to ensure universal shifted stability and performance compared to using the universal shifted controller design is the (computational) complexity of the controller implementation. Namely, for the incremental controller design, one must compute integrals of the differential controller matrices, see (6.11). As aforementioned, analytically computing these integrals might be difficult, in which case they need to computed (online) using numerical integration procedures. While there exists fast numerical integration, procedures, they can be costly on when the controller is implemented on an embedded platform. On the other hand, the universal shifted controller implementation only requires evaluation of the matrices of the velocity(/differential) controller, see (4.43). Therefore, the computational cost for the universal shifted controller is lower. However, a disadvantage of the universal shifted controller is that it does depend on the derivative of the considered scheduling-variable. Therefore, for implementation of the universal shifted controller, one would need to be able to measure the derivative of the scheduling-variable or obtain the derivative it through filtering. Although when obtaining the derivative through filtering, one needs to be careful as this might amplify high frequent measurement noise, which can lead to an endangerment of the closed-loop stability and performance guarantees. Nonetheless, as discussed in Section 4.5.2, one can avoid the dependency on the derivative of the scheduling-variable by ensuring that B and D matrix of the synthesized velocity controller are constant. Although this might limit the achievable closed-loop performance that can be obtained, which is then a trade-off.

6.4.2 Standard LPV controller design

As incremental dissipativity also implies classical dissipativity (assuming that the origin is an equilibrium point of the system), we can similarly also ensure classical dissipativity using the incremental controller design, similar how a standard LPV controller ensure this.

Applying the incremental controller to ensure classical dissipativity, we need that $(0,0,0,0,0) \in \mathscr{E}$. In this case,

$$\vartheta = t \mapsto (0,0,0,0,0). \tag{6.22}$$

For classical dissipativity this then means that the 'feedforward terms' in the controller, i.e., u_{k}^* and y_{k}^*, and the dependency on x^*, drop out, as they are equal to zero, i.e., $u_{\mathrm{k}}^*(t) = 0$, $y_{\mathrm{k}}^*(t) = 0$, and $x^* = 0$. The resulting incremental controller then becomes a 'standard' LPV controller. This becomes especially clear in case the differential controller K_δ given by (6.9) has an affine scheduling dependency. In that case, through the result of Corollary 6.1, we then have that (6.10) becomes

$$\dot{x}_{\mathrm{k}}(t) = A_{\mathrm{k}}(\rho(t))x_{\mathrm{k}}(t) + B_{\mathrm{k}}(\rho(t))u_{\mathrm{k}}(t); \tag{6.23a}$$

$$y_{\mathrm{k}}(t) = C_{\mathrm{k}}(\rho(t))x_{\mathrm{k}}(t) + D_{\mathrm{k}}(\rho(t))u_{\mathrm{k}}(t); \tag{6.23b}$$

where $x_{\mathrm{k}} = x_{\Delta,\mathrm{k}}$ and $\rho(t) = \int_0^1 \eta(\lambda x(t))\, d\lambda$. In fact, the LPV controller (6.23) corresponds to specific standard (affine) LPV embedding of the generalized plant P given by (6.4). Namely, we have that for the DPV embedding of P, given by (6.8), it holds that

$$A(\eta(\bar{x})) = A_\delta(\bar{x}), \tag{6.24}$$

for all $\bar{x} \in \mathcal{X}$. Assuming that $\mathcal{X}$ is convex, we therefore have by Lemma C.1.1, as $f(0,0) = 0$, that

$$\begin{aligned}
f(x) &= \int_0^1 \frac{\partial f}{\partial x}(\lambda x)\, d\lambda, && \text{(6.25a)}\\
&= \int_0^1 A_\delta(\lambda x)\, d\lambda, \quad \text{(by definition of } A_\delta\text{)}, && \text{(6.25b)}\\
&= \int_0^1 A(\eta(\lambda x))\, d\lambda, \quad \text{(by (6.24))}, && \text{(6.25c)}\\
&= A\left(\int_0^1 \eta(\lambda x)\, d\lambda\right), \quad \text{(as } A \text{ has affine scheduling dependency)}, && \text{(6.25d)}\\
&= A(\rho), && \text{(6.25e)}
\end{aligned}$$

for all $x \in \mathcal{X}$ where again $\rho = \int_0^1 \eta(\lambda x)\, d\lambda$. We can derive a similar result to obtain that $h_{\mathrm{z}}(x) = C_{\mathrm{z}}(\rho)$ with $\rho = \int_0^1 \eta(\lambda x)\, d\lambda$ for all $x \in \mathcal{X}$. Therefore, by Definition 2.14, the LPV representation:

$$\dot{x}(t) = A(\rho(t)) + B_{\mathrm{w}}w(t) + B_{\mathrm{u}}u(t); \tag{6.26a}$$

$$z(t) = C_{\mathrm{z}}(\rho(t)) + D_{\mathrm{zw}}w(t) + D_{\mathrm{zu}}u(t); \tag{6.26b}$$

$$y(t) = C_{\mathrm{y}}x(t) + D_{\mathrm{yw}}w(t); \tag{6.26c}$$

with $\rho = \int_0^1 \eta(\lambda x)\, d\lambda$, is a (standard) global LPV embedding of the generalized plant P given by (6.4) on the region $\mathcal{X}$. The LPV controller (6.23) than ensures classical dissipativity of (6.26) and consequently of (6.4) on the region $\mathcal{X}$. We discuss this specific global LPV embedding of a nonlinear system also in more detail in Appendix C.6 and also discuss further applications of this type of embedding.

6.5 Examples

In this section, the proposed incremental controller will be demonstrated through examples. Moreover, it will also be compared to standard LPV controller designs. We will also demonstrate using simulation and experimental results that standard LPV control can fail to result in the desired behavior, while the proposed LPV synthesis resulting in an incremental controller can reliably achieve it.

Example 6.1 *(Duffing oscillator)***.** First, we apply the incremental controller design to the Duffing oscillator that was also considered for universal shifted control in Example 4.1. The Duffing oscillator is described by the following differential equations:

$$\begin{aligned} \dot{q}(t) &= v(t); \\ \dot{v}(t) &= -\frac{k_1}{m}q(t) - \frac{k_2}{m}\left(q(t)\right)^3 - \frac{d}{m}v(t) + \frac{1}{m}F(t); \end{aligned} \tag{6.27}$$

where, q [m] is the position, v [$\mathrm{m} \cdot \mathrm{s}^{-1}$] the velocity and F [N] is the (input) force acting on the mass. Furthermore, we consider the same parameters for the Duffing oscillator as in Example 4.1, i.e., $m = 1$ [kg], $k_1 = 0.5$ [$\mathrm{N} \cdot \mathrm{m}^{-1}$], $k_2 = 5$ [$\mathrm{N} \cdot \mathrm{m}^{-3}$] and $d = 0.2$ [$\mathrm{N} \cdot \mathrm{s} \cdot \mathrm{m}^{-1}$]. We again assume that the position q is measured, and hence it is considered to be the only output of the plant.
For the incremental synthesis, the differential form of (6.27) is calculated and a DPV embedding is constructed, resulting in

$$\begin{aligned} \dot{q}_\delta &= v_\delta; \\ \dot{v}_\delta &= \left(-\frac{k_1}{m} - 3\frac{k_2}{m}p\right)q_\delta - \frac{d}{m}v_\delta + \frac{1}{m}F_\delta; \end{aligned} \tag{6.28}$$

where the scheduling $p(t) = q^2(t) \in \mathcal{P}$ where we consider $\mathcal{P} = [0,\, 2]$. Note that this is the same region that has been considered in Example 4.1 for the universal shifted controller design.
We also consider the same generalized plant that has been considered in the universal shifted example. This generalized plant P is also depicted in Figure 6.3, where G is the system given by (6.27), K is the controller, $w = \mathrm{col}\,(r, d_\mathrm{i})$ is the generalized disturbance, with r the reference and d_i being an input disturbance. The performance channel consists of z_1 (tracking error) and z_2 (control effort). We also consider the same LTI weighting filters as in Example 4.1, i.e., $W_1(s) = \frac{0.501(s+3)}{s+2\pi}$, $W_2(s) = \frac{10(s+50)}{s+5\cdot 10^4}$, $W_3 = 1.5$, and $M(s) = \frac{s+2\pi}{s}$, where s corresponds to the complex frequency. The resulting sensitivity weight $W_1(s)M(s)$ has guaranteed 20 dB/dec roll-off at low frequencies in order to ensure good tracking performance, while $W_2(s)$ has high-pass characteristics in order to ensure proper roll-off at high frequencies. Because the system (6.28) and the corresponding generalized plant have affine dependency on the scheduling-variable, polytopic $\mathcal{L}_2$-gain synthesis based on (Apkarian, Gahinet, and G. Becker 1995; Apkarian and Adams 1998) has been used in the design the incremental controller to ensure closed-loop $\mathcal{L}_{\mathrm{i}2}$-gain stability and performance. This synthesis algorithm has been implemented in the LPVcore Toolbox (Boef et al. 2021), which has been used to synthesize the controllers.

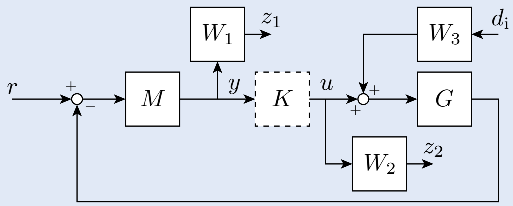


Figure 6.3: Generalized plant used in the examples.

Using the given weighting filters, synthesizing the LPV controller for the differential plant results in an $\mathcal{L}_{i2}$-gain of $\gamma \approx 1.2$. As the synthesis provides the controller with affine parameter dependence, we can use the result of Corollary 6.1 in order to compute $\rho(t) = \int_0^1 \eta(\bar{x}(t, \lambda))\, d\lambda = q^2(t) + q(t)q^*(t) + (q^*(t))^2$ and realize the incremental controller K. As we assume the presence of an (unknown) input disturbance d_i, a disturbance observer is constructed for the incremental controller as described in Section 6.3.7. It is assumed the input disturbance will be constant, hence, the following disturbance model is used for the disturbance observer design

$$\begin{aligned} \dot{x}_d(t) &= 0; \\ d_i(t) &= x_d(t). \end{aligned} \tag{6.29}$$

The LPV controller which we will use for comparison is the same one that is used for comparison in Example 4.1. This standard LPV controller uses the same generalized plant with weighting filters as the incremental design and achieves an $\mathcal{L}_2$-gain of $\gamma \approx 0.94$.

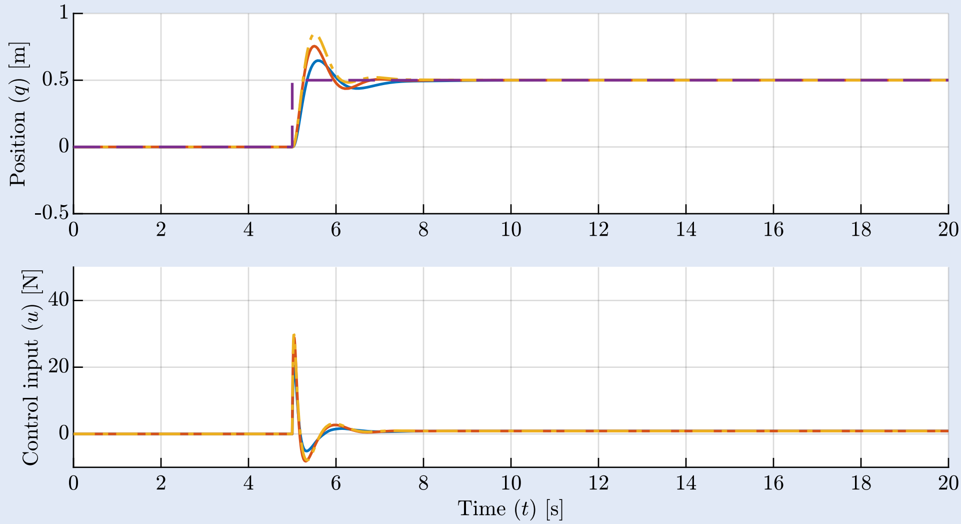


Figure 6.4: Position of the Duffing oscillator (top) in closed-loop with the standard LPV (—), the incremental (—), and universal shifted (-·-) controller under reference (- -) and no input disturbance, together with the generated control inputs (bottom) by the controllers.

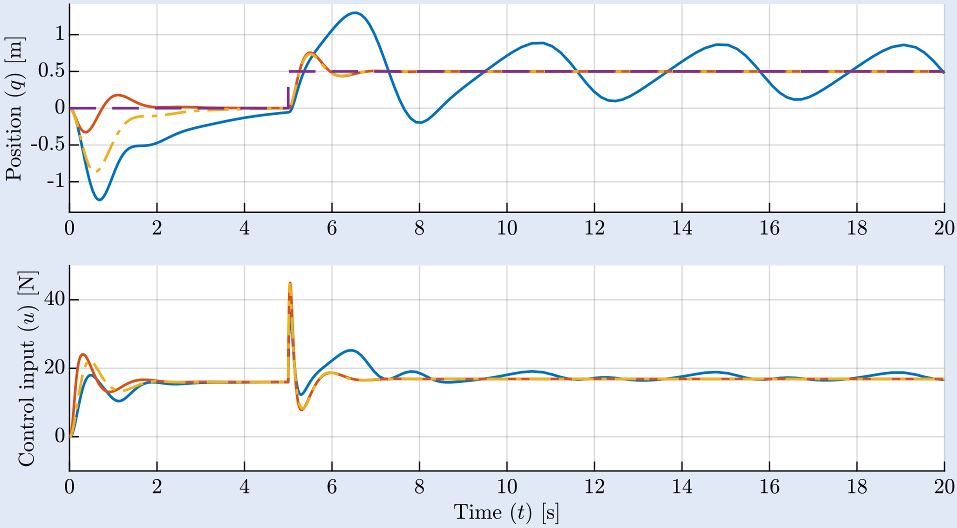


Figure 6.5: Position of the Duffing oscillator (top) in closed-loop with the standard LPV ( ), the incremental ( ), and universal shifted ( ) controller under reference ( ) and input disturbance, together with the generated control inputs (bottom) by the controllers.

In simulation, the resulting outputs of the system using the standard LPV controller and the incremental controller in closed-loop are depicted without and with input disturbance in Figures 6.4 and 6.5, respectively. Moreover, also the results of the universal shifted controller design of Example 4.1 are depicted in Figures 6.4 and 6.5, specifically the results that are also in Figures 4.7 and 4.8. The universal shifted controller of Example 4.1 uses the same generalized plant structure and weighting filters. For all controllers, a step signal is taken as a reference trajectory which changes from zero to 0.5 at $t = 5$ seconds. For the incremental controller, the reference r corresponds to q^* with $u^*(t) = k_1 q^*(t) + k_2(q^*(t))^3 - W_3 d_i^*(t)$ (as the trajectory of q^* is piecewise constant), where q^* is known, and d_i^* is estimated using a disturbance observer. For the simulation results in Figure 6.5, a constant input disturbance $d_\mathrm{i} \equiv -10\frac{2}{3}$ (corresponding to $-10\frac{2}{3} \cdot W_3 = -16$ [N]) is applied. Comparing the results of the standard and the incremental controllers in Figure 6.4 shows that both controllers have similar performance when no input disturbance is present. The incremental controller has slightly more overshoot, but a lower settling time for this example. However, under constant input disturbance, it can be seen in Figure 6.5 that the standard LPV controller has a significant performance loss with oscillatory behavior, whereas the incremental controller preserves its smooth reference tracking property. Note, that in both cases, the scheduling-variable p never leaves the set for which the controllers have been designed, i.e., $q^2(t) \in [0, 2]$. Compared to the universal shifted controller design, the incremental controller design has similar performance for the same constant reference trajectory and constant disturbance. Both the universal shifted controller and incremental controller have smooth reference tracking and disturbance rejection properties as they both guarantee closed-loop global stability and performance.

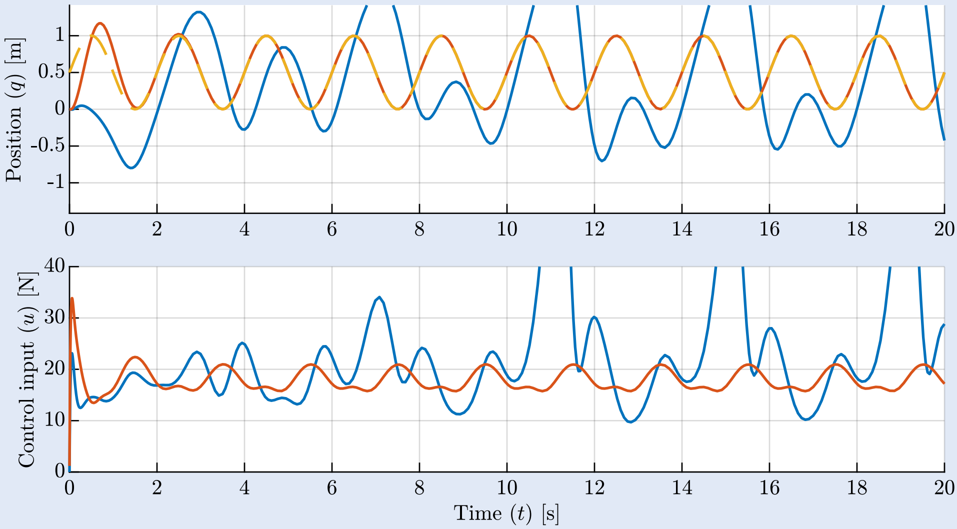


Figure 6.6: Position of the Duffing oscillator (top) in closed-loop with the standard LPV (—) and the incremental (—) controllers under varying reference (- -) and constant input disturbance, together with the generated control inputs (bottom) by the controllers.

Next, the system in closed-loop with the incremental controller and standard LPV controller is simulated with a varying reference $r(t) = q^*(t) = 0.5\sin(\pi t) + 0.5$, and again the constant input disturbance $d_\mathrm{i} \equiv -16$, see Figure 6.6. For the incremental controller the corresponding u^* is computed based on the dynamics of the plant (6.27), which is not given due to its complexity. As shown in Figure 6.6, the incremental controller also in this case converges toward the desired reference trajectory, while the standard LPV controller fails to do so, also leaving the scheduling set it was designed for in the process.

For the next example, the proposed incremental control method is experimentally verified on an unbalanced disk setup for a reference tracking and disturbance rejection control problem that was also considered in Example 4.2. Like for the previous example it is also compared to a standard LPV controller.

Example 6.2 *(Unbalanced disk)*.

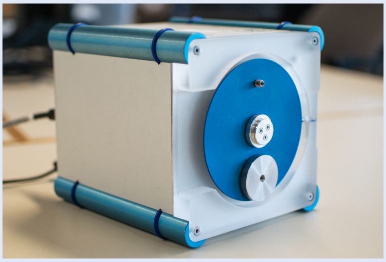


Figure 6.7: Unbalanced disk setup.

We consider again the model for the unbalanced disk as given in Example 4.2, which is given by

$$\dot{\theta}(t) = \omega(t); \tag{6.30a}$$
$$\dot{\omega}(t) = \tfrac{Mgl}{J}\sin(\theta(t)) - \tfrac{1}{\tau}\omega(t) + \tfrac{K_m}{\tau}V(t); \tag{6.30b}$$

where again θ [rad] is the angle of the disk, ω [$\mathrm{rad}\cdot\mathrm{s}^{-1}$] its angular velocity, V [V] is the input voltage to the motor, g is the gravitational acceleration, l the length of the pendulum, J the inertia of the disk, and K_m and τ are the motor constant and time constant respectively. The angle of the disk θ is considered to be the output of the plant. The values of the physical parameters are given in Table 4.1.
We construct a DPV embedding of (6.30), which is given by

$$\begin{aligned} \dot{\theta}_\delta(t) &= \omega_\delta(t); \\ \dot{\omega}_\delta(t) &= \left(\tfrac{Mgl}{J}p(t)\right)\theta_\delta(t) - \tfrac{1}{\tau}\omega_\delta(t) + \tfrac{K_m}{\tau}V_\delta(t); \end{aligned} \tag{6.31}$$

where $p(t) = \eta(\theta(t)) = \cos(\theta(t))$ is the scheduling-variable which is assumed to be in $\mathcal{P} = [-1, 1]$.
As in the previous example, the primal form of the nonlinear system (6.30) is also embedded in an LPV representation to construct a standard LPV controller for the sake of comparison. This results in

$$\begin{aligned} \dot{\theta}(t) &= \omega(t); \\ \dot{\omega}(t) &= \left(\tfrac{Mgl}{J}p_\mathrm{s}(t)\right)\theta(t) - \tfrac{1}{\tau}\omega(t) + \tfrac{K_m}{\tau}V(t); \end{aligned} \tag{6.32}$$

where $p_\mathrm{s}(t) = \eta_\mathrm{s}(\theta(t)) = \frac{\sin(\theta(t))}{\theta(t)} = \mathrm{sinc}(\theta(t)) \in \mathcal{P}_\mathrm{s}$, where $\mathcal{P}_\mathrm{s}$ is chosen[6] as $[-0.22, 1]$. The used generalized plant structure is depicted in Figure 6.3. Note that this is different from the structure we considered in Example 4.2 for the unbalanced disk. The weighting filters are chosen as $W_1(s) = \frac{0.5012(s+4)}{s+\pi}$, $M(s) = \frac{s+\pi}{s}$, $W_2(s) = \frac{s+40}{s+4000}$ and $W_3 = 0.5$. Synthesizing the controllers, using the same approach as for the duffing oscillator in Example 6.1, results in an $\mathcal{L}_2$-gain of $\gamma \approx 1.1$ and

[6]Note that $\eta_\mathrm{s}(0) = 1$ as $\lim_{x\to 0}\mathrm{sinc}(x) = 1$.

$\mathcal{L}_{i2}$-gain of $\gamma \approx 1.2$. As the LPV controller resulting for the DPV embedding has an affine dependency, we can use Corollary 6.1 to compute[7] $\rho(t) = \int_0^1 \eta(\bar{x}(t,\lambda))\, d\lambda = \frac{\sin(\theta(t))-\sin(\theta^*(t))}{\theta(t)-\theta^*(t)}$. For the resulting incremental controller, a disturbance observer is also designed to estimate the (unknown) disturbances d_i. As d_i is assumed to be constant, therefore (6.29) is consider for the design.
Both the standard LPV controller and incremental controller are implemented on the experimental setup. On the experimental setup, for safety, the input voltage to the system is saturated between ± 10 [V]. The results of the experiments are depicted in Figures 6.8 to 6.10.

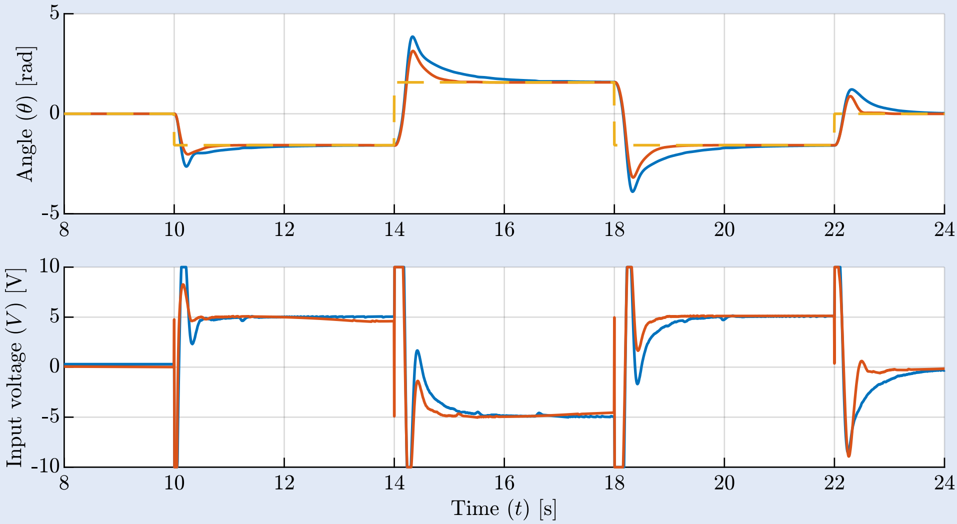


Figure 6.8: Measured angle of the unbalanced disk system (top) in closed-loop with the standard LPV ( ) and the incremental ( ) controllers under reference ( ) and no input disturbance, together with inputs to the plant (bottom) generated by the controllers.

[7]Note that $\lim_{\theta\to\theta^*} \frac{\sin(\theta)-\sin(\theta^*)}{\theta-\theta^*} = \cos(\theta) = \cos(\theta^*)$.

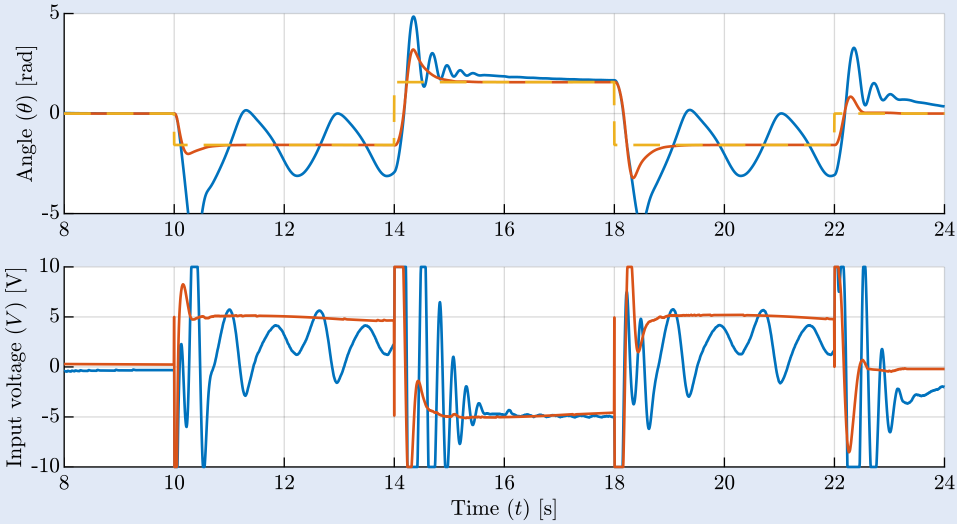


Figure 6.9: Measured angle of the unbalanced disk system (top) in closed-loop with the standard LPV ( ) and the incremental ( ) controller under reference ( ) and input disturbance, together with corresponding inputs to the plant (bottom) generated by the controllers.

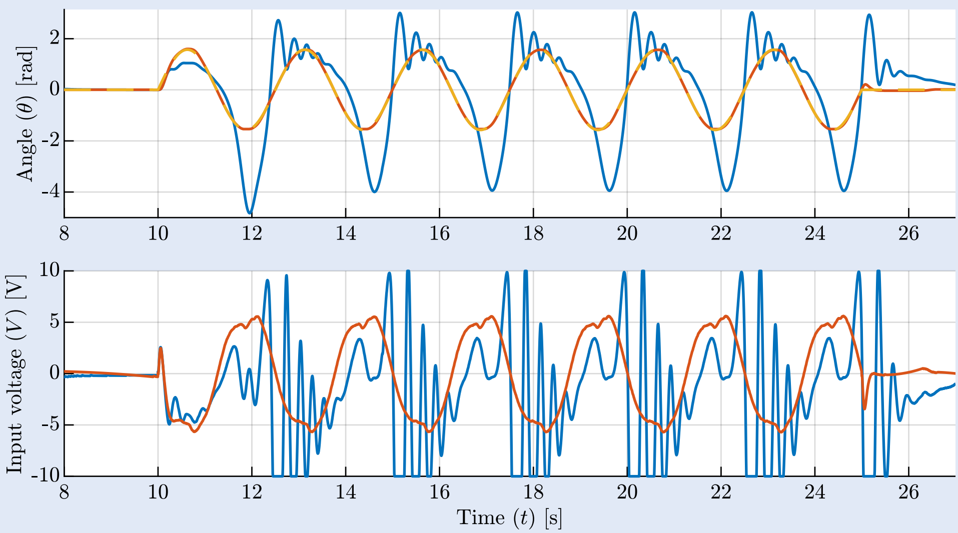


Figure 6.10: Measured angle of the unbalanced disk system (top) in closed-loop with the standard LPV ( ) and the incremental ( ) controller under reference ( ) and input disturbance, together with corresponding inputs to the plant (bottom) generated by the controllers.

In Figure 6.8, the measured angular response of the disk during the experiments is depicted along with the input to the setup (i.e., V). The reference trajectory r

switches between 0 and $\pm\frac{\pi}{2}$ rad/s. For the incremental controller, r corresponds to θ^* with $u^*(t) = -\frac{Mgl\tau}{JK_m}\sin(\theta^*(t)) - W_3 d_{\mathrm{i}}^*(t)$ (as the trajectory is piecewise constant). In Figure 6.9, the same reference trajectory is used, but a constant input disturbance of $d_{\mathrm{i}} = 100$, corresponding to $100 \cdot W_3 = 50$ [V], is introduced (which is implemented by adding 50 V to the control input that is sent to the plant before saturation). Note that the reference only starts at 10s to give the controllers time to compensate for the input disturbance. The standard LPV controller performs much worse when a constant input disturbance is present, compared to the incremental controller, which has similar performance to the case when no input disturbance is applied. Both the standard LPV and the incremental controllers are able to compensate the 50 [V] input disturbance, as visible in the total received input by the plant (i.e., $V = u + W_3 d_{\mathrm{i}}$), see bottom graph in Figure 6.9. However, while the input that is sent to the plant is nearly identical for the incremental controller in both cases, see Figures 6.8 and 6.9, this is clearly not the case for the standard LPV controller. For the latter, oscillations in the input signal are present when the input disturbance is applied which causes unwanted oscillation of the disk angle. While an input disturbance of 50 [V] is extraordinarily high for this system, and will likely never occur on the real setup, it still shows that there are inherent issues when using standard LPV controllers.
Finally, in Figure 6.10, experimental results are shown under a sinusoidal reference and a constant input disturbance. For this experiment, the reference r is taken to be $r(t) = \frac{\pi}{2}\sin(5\pi t)$ and again an input disturbance $d_{\mathrm{i}} = 100$ is used. For the incremental controller r again corresponds to θ^*, and for this reference the corresponding u^* is equal to $u^*(t) = \frac{\tau}{K_m}\ddot{\theta}^*(t) - \frac{Mgl\tau}{JK_m}\sin(\theta^*(t)) + \frac{1}{K_m}\dot{\theta}^*(t) - W_3 d_{\mathrm{i}}^*(t)$. Again, the reference only starts at 10s, to give the controllers time to compensate for the input disturbance, and we stop the reference at 25s. Moreover, for this experiment u^* is also used as a feedforward trajectory for the LPV controller, i.e., the total input sent to the system is $u^* + u$ (not considering disturbances), where u is generated by the standard LPV controller. From Figure 6.10 it is evident that also for this sinusoidal reference and under the disturbance, the incremental controller is able to smoothly track the reference. On the other hand, the standard LPV controller is not able to track the reference and again oscillates it, even with the feedforward action added.

Finally, we compare the results of our method with the results from (Scorletti, Fromion, et al. 2015).

Example 6.3 *(Scorletti et al. example)***.** The example system in (Scorletti, Fromion, et al. 2015) is described by the following state-space representation

$$\dot{x}_{\mathrm{g},1}(t) = -100\varphi(x_{\mathrm{g},1}(t)) - 70x_{\mathrm{g},2}(t) + 300u_{\mathrm{g}}(t); \tag{6.33a}$$

$$\dot{x}_{\mathrm{g},2}(t) = 70x_{\mathrm{g},2}(t) - 14x_{\mathrm{g},2}(t); \tag{6.33b}$$

$$y_{\mathrm{g}}(t) = x_{\mathrm{g},1}(t); \tag{6.33c}$$

where

$$\varphi(x) = \begin{cases} 0.9x^3 - 2|x|x + 1.2x, & \text{for } |x| < \frac{5}{3}; \\ 2x - 2.72, & \text{for } x \geq \frac{5}{3}; \\ 2x + 2.72, & \text{for } x \leq -\frac{5}{3}. \end{cases} \tag{6.34}$$

Next, we construct a DPV embedding of (6.33), which is given by (and is also considered in (Scorletti, Fromion, et al. 2015)):

$$\dot{x}_{\delta,\mathrm{g},1}(t) = (-100p(t))\, x_{\delta,\mathrm{g},1} - 70x_{\delta,\mathrm{g},2}(t) + 300u_{\delta,\mathrm{g}}(t); \tag{6.35a}$$
$$\dot{x}_{\delta,\mathrm{g},2}(t) = 70x_{\delta,\mathrm{g},1}(t) - 14x_{\delta,\mathrm{g},2}(t); \tag{6.35b}$$
$$y_{\delta,\mathrm{g}}(t) = x_{\delta,\mathrm{g},1}(t); \tag{6.35c}$$

where $p(t) = \varphi_\delta(x_{\mathrm{g},1}(t))$, is the scheduling-variable, which is assumed to be in $\mathcal{P} \in [-0.3,\, 2]$, where

$$\varphi_\delta(x) = \begin{cases} 2.7x^2 - 4x + 1.2, & \text{for } 0 \leq x < \frac{5}{3}; \\ 2.7x^2 + 4x + 1.2, & \text{for } -\frac{5}{3} < x < 0; \\ 2, & \text{for } x \geq \frac{5}{3} \wedge x \leq -\frac{5}{3}. \end{cases} \tag{6.36}$$

A generalized plant structure is taken as in Figure 6.3, where $W_1(s) = \frac{50}{s+2\pi}$, $W_2(s) = \frac{10(s+10)}{s+1000}$, $W_3 = 0.1$ and $M(s) = \frac{s+2\pi}{s}$. Which is similar to the generalized plant and weighting filters taken in (Scorletti, Fromion, et al. 2015). On the basis of this, an incremental controller for (6.33) is synthesized, using (6.35). Like for previous examples, LPVcore is used to synthesize an affine LPV controller. This results in the incremental controller achieving an $\mathcal{L}_{\mathrm{i}2}$-gain of $\gamma \approx 1.0$, similar to the incremental gain obtained in (Scorletti, Fromion, et al. 2015), where an $\mathcal{L}_{\mathrm{i}2}$-gain of $\gamma \approx 1$ is reported. As the LPV controller resulting for the DPV embedding of the generalized plant has affine dependency we can, like was done for the previous examples, use the result of Corollary 6.1 in order to compute[8] $\rho(t) = \int_0^1 \bar{p}(t,\lambda))\, d\lambda = \frac{\varphi(x_{\mathrm{g},1}(t)) - \varphi(x^*_{\mathrm{g},1}(t))}{x_{\mathrm{g},1}(t) - x^*_{\mathrm{g},1}(t)}$. Furthermore, like for the previous examples, a disturbance observer is used for the incremental controller for which the disturbance model (6.29) is also used. In order to compute the feasible steady-state trajectory used by the incremental controller, the reference r is filtered by the lowpass filter $F(s) = \frac{1000}{s+1000}$ which then corresponds to $x^*_{\mathrm{g},1}$, this trajectory is then used to compute the corresponding control input $u^*(t)$ (which due to its complexity is not given).

[8] Note that $\lim_{x_1 \to x_1^*} \frac{\varphi(x_{\mathrm{g},1}) - \varphi(x^*_{\mathrm{g},1})}{x_{\mathrm{g},1} - x^*_{\mathrm{g},1}} = \varphi_\delta(x_{\mathrm{g},1}) = \varphi_\delta(x^*_{\mathrm{g},1})$.

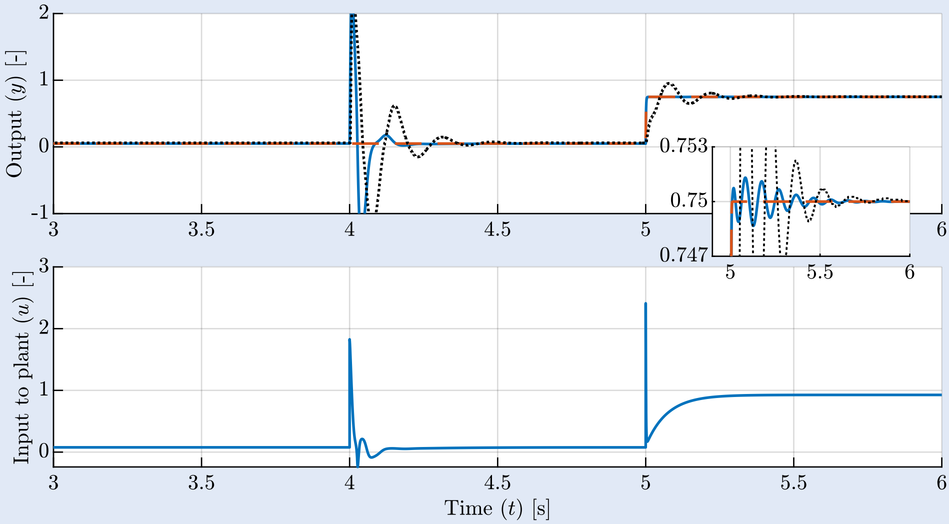


Figure 6.11: Output of the nonlinear system (6.33) (top) in closed-loop with the proposed incremental controller (—), as well as the results from (Scorletti, Fromion, et al. 2015, Fig. 7) (···) under the reference trajectory (- -), together with the input to the plant (bottom).

Figure 6.11 shows the output of the nonlinear system from (Scorletti, Fromion, et al. 2015) in closed-loop with the proposed incremental controller alongside the results from (Scorletti, Fromion, et al. 2015, Fig. 7), as well as the input to the plant, i.e. u_{g}, for the proposed incremental controller. It can be observed that our proposed incremental control design performs better in this example than the one proposed in (Scorletti, Fromion, et al. 2015), using the same performance and stability requirements set by the weighting filters. This is likely due to the fact that (i) our proposed controller has a more flexible dependency structure for the scheduling-variable, compared to the linear structure of the controller proposed in (Scorletti, Fromion, et al. 2015); (ii) our proposed controller contains besides a feedback part also a feedforward component, corresponding to the steady-state trajectory, which the method in (Scorletti, Fromion, et al. 2015) does not have.

6.6 Conclusions

In this chapter, we proposed a novel systematic dynamic output feedback controller synthesis method for nonlinear systems under general controller parameterization which provides incremental stability and performance guarantees of the achieved closed-loop behavior. The proposed incremental controller synthesis method is based on two key ingredients: (i) LPV controller synthesis on the differential form of the nonlinear plant to be controlled and (ii) realization of the resulting LPV controller as an implementable incremental nonlinear controller with closed-loop stability and performance guarantees. A key advantage of the method is that it facilitates systematic controller design for nonlinear plants by convex synthesis and enables the use of powerful performance shaping concepts available for linear controller design. Although a large variety of incremental (Q, S, R) dissipativity notions can be ensured by the proposed methodology, we chose to exemplify the approach with incremental $\mathcal{L}_2$-gain and performance. As it is demonstrated through simulation and experimental studies, the proposed approach successfully achieves desired closed-loop stability and performance requirements for tracking and rejection problems and overcomes issues of standard LPV controller synthesis methods.

7

Discrete-Time Extension of the Incremental Theory

In this chapter, we discus the extension of the *Continuous-Time* (CT) results on incremental analysis and controller synthesis which we have presented in Chapters 5 and 6, respectively, to *Discrete-Time* (DT) nonlinear systems. Similar to the CT results, we show that we can use differential (Q, S, R) dissipativity in order to analyze incremental (Q, S, R) dissipativity. Furthermore, analogous to the CT results, we show that also in DT, the differential (Q, S, R) dissipativity analysis problem can be cast as a classical (Q, S, R) dissipativity of an *Linear Parameter-Varying* (LPV) representation. Based on these analysis results, we show that for DT incremental controller synthesis, an equivalent procedure as in CT can be used, with the resulting DT incremental controller having the same structure as the CT incremental controller. Both the capabilities of the DT incremental analysis and controller synthesis frameworks are investigated and analyzed through simulation studies, which show the achieved global stability and performance guarantees.

7.1 Introduction

In Chapters 5 and 6, we have presented our results on incremental dissipativity based analysis and control of *Continuous-Time* (CT) systems. For analysis, we have seen that by analyzing stability and dissipativity properties of the differential form of the system, representing variations along trajectories, we can infer incremental stability and dissipativity properties of the primal form of the system, i.e., the original nonlinear dynamics. Moreover, we have shown that by embedding the differential form in a *Linear Parameter-Varying* (LPV) representation, which we refer to as a so-called *Differential Parameter-Varying* (DPV) embedding of the system, we can then use the analysis tools of the LPV framework to efficiently analyze incremental properties of CT nonlinear systems. Based on these analysis results, we have shown in Chapter 6 how the DPV embedding and properties of the differential form could be exploited for computationally efficient and systematic incremental controller synthesis.

For controller synthesis, CT methods have as advantage that shaping the closed-loop behavior is intuitive due to the frequency domain interpretation of *Linear Time-Invariant* (LTI) weighting filters. Nonetheless, the designed controller needs to be implemented digitally in order to be used. This requires discretization of the CT controller which can lead to deteriorated performance or even loss of closed-loop stability and performance guarantees. Therefore, the analysis and control of *Discrete-Time* (DT) systems is also of high importance. Moreover, the recent resurgence in data and learning based methods for analysis and control of nonlinear systems also rely on DT systems analysis.

While there have been some results on incremental and contraction based stability of DT systems, see e.g. (Tran et al. 2016; Tran et al. 2018), similar extensions to incremental dissipativity have not yet been made to the author's knowledge. Hence, in this chapter we will propose an extension of the CT incremental dissipativity results of Chapter 5 to the DT systems. We will show that in DT, we obtain analogous results to the CT case, namely, that we can analyze incremental dissipativity of the primal form by analyzing differential dissipativity . Moreover, we show that we can again use DPV embeddings to analyze differential dissipativity in a computationally efficient manner, making use of the LPV framework.

With respect to controller synthesis to ensure incremental properties, there are even less results in existing literature, with only a few works that ensure incremental stability (like properties) through state-feedback (Wei et al. 2021) and through *Nonlinear Model Predictive Control* (NMPC) (Köhler et al. 2020). Comprehensive results for DT output-feedback synthesis to ensure closed-loop incremental dissipativity remains an open problem. Hence, in this chapter, we also develop an incremental dissipativity based convex output-feedback controller synthesis method for DT nonlinear systems, based on an extension of the CT results in Chapter 6.

This chapter is structured as follows, in Section 7.2 we present the results for incremental stability and performance analysis for DT nonlinear systems. Next, in Section 7.3, we show how the incremental analysis conditions presented in Section 7.2 can efficiently be verified through the LPV framework. These analysis results are then also demonstrated through an example in Section 7.4. In Section 7.5, we present the results for incremental controller synthesis of DT systems. Finally, in Section 7.7, conclusions are drawn.

7.2 Incremental Stability and Performance Analysis

7.2.1 Incremental dissipativity

Like we considered in Section 2.2, we consider DT nonlinear systems given by

$$x(t+1) = f(x(t), w(t)); \tag{7.1a}$$

$$z(t) = h(x(t), w(t)); \tag{7.1b}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$ is the state with initial condition $x_0 \in \mathcal{X}$, $w(t) \in \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{w}}$ is the input, $z(t) \in \mathcal{Z} \subseteq \mathbb{R}^{n_\mathrm{z}}$ is the output, and $t \in \mathbb{N}_0$ is the discrete-time instant. The sets $\mathcal{X}$, $\mathcal{W}$ and $\mathcal{Z}$ are open and convex, containing the origin. The functions $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ and $h : \mathcal{X} \times \mathcal{W} \to \mathcal{Z}$ are assumed to be $\mathcal{C}_1$, i.e., $f, h \in \mathcal{C}_1$. We define the set of solutions of (7.1) as

$$\mathfrak{B} := \Big\{ (x, w, z) \in (\mathcal{X} \times \mathcal{W} \times \mathcal{Z})^{\mathbb{N}_0} \mid (x, w, z) \text{ satisfies (7.1)} \Big\}. \tag{7.2}$$

Furthermore, we define the state transition map $\phi_\mathrm{x} : \mathbb{N}_0 \times \mathbb{N}_0 \times \mathcal{X} \times \mathcal{W}^{\mathbb{N}_0} \to \mathcal{X}$, such that

$$x(t) = \phi_\mathrm{x}(t, 0, x(0), w), \tag{7.3}$$

which is the state $x(t) \in \mathcal{X}$ at discrete-time instant $t \in \mathbb{N}_0$, when the system is driven from $x(0) \in \mathcal{X}$ by input signal $w \in \mathcal{W}^{\mathbb{N}_0}$.

In Section 2.4, we have already seen how classical dissipativity can be used to simultaneously analyze stability and performance of nonlinear systems. However, as mentioned in Chapter 5, the classical dissipativity framework only analyzes the internal energy of the system with respect to a single point of neutral storage, often taken as the origin of the state-space associated with the nonlinear representation. However, it is often of interesest to analyze a set of equilibrium points/trajectories, e.g., in the case of reference tracking or disturbance rejection, which is cumbersome to be performed with the classical dissipativity results. Hence, there is a need for global dissipativity notions such as incremental dissipativity, as they allow to handle these cases efficiently. Incremental dissipativity is an extension of the classical dissipativity results which takes into account multiple trajectories of a system and can be thought of as analyzing the energy flow between trajectories.

Similar to the incremental dissipativity definition for CT systems in Definition 5.2, we define incremental dissipativity of DT nonlinear systems as follows:

Definition 7.1 (DT incremental dissipativity). *The system given by (7.1) is called incrementally dissipative w.r.t. the supply function* $s_\mathrm{i} : \mathcal{W} \times \mathcal{W} \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{R}$, *if* *there exists a storage function* $\mathcal{V}_\mathrm{i} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_0^+$ *with* $\mathcal{V}_\mathrm{i} \in \mathcal{C}_0$ *and* $\mathcal{V}_\mathrm{i} \in \mathcal{Q}_\mathrm{i}$*, such that, for any two trajectories* $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$*,*

$$\mathcal{V}_\mathrm{i}\big(x(t_1+1), \tilde{x}(t_1+1)\big) - \mathcal{V}_\mathrm{i}\big(x(t_0), \tilde{x}(t_0)\big) \leq \sum_{t=t_0}^{t_1} s_\mathrm{i}\big(w(t), \tilde{w}(t), z(t), \tilde{z}(t)\big), \quad (7.4)$$

for all $t_0, t_1 \in \mathbb{N}_0$ *with* $t_0 \leq t_1$*.*

Again, we will focus on quadratic supply functions of the form

$$s_\mathrm{i}(w, \tilde{w}, z, \tilde{z}) = \begin{bmatrix} w - \tilde{w} \\ z - \tilde{z} \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w - \tilde{w} \\ z - \tilde{z} \end{bmatrix}, \quad (7.5)$$

where $Q \in \mathbb{S}^{n_\mathrm{w}}$, $R \in \mathbb{S}^{n_\mathrm{z}}$ and $S \in \mathbb{R}^{n_\mathrm{w} \times n_\mathrm{z}}$. We focus on this particular family, often referred to as (incremental) (Q, S, R) supply functions, as they allow formulation of many useful performance notions, such as incremental versions of ℓ_2-gain performance and passivity.

7.2.2 Differential Dissipativity

Analogous to the CT results, we are interested in analyzing incremental dissipativity by analyzing the dynamics of the variations along the trajectories, captured through the differential form of the system.

Like it has been done in Section 5.2, we consider two arbitrary trajectories of the system (7.1): $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$. Where we parameterize any two trajectories between them by $\bar{x}_0 \in \Gamma_\mathcal{X}(x_0, \tilde{x}_0)$ and $\bar{w}(t) \in \Gamma_\mathcal{W}(w(t), \tilde{w}(t))$, resulting in the state transition map $\bar{x}(t, \lambda) = \phi_\mathrm{x}(t, t_0, \bar{x}_0(\lambda), \bar{w}(\lambda)) \in \mathcal{X}$, where $\Gamma_\mathcal{A}(a_1, a_2)$ denotes the set of paths between $a_1, a_2 \in \mathcal{A}$, see (5.5) for the exact definition. This results in that for any $\lambda \in [0,\, 1]$ and all $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$, it holds that

$$\bar{x}(t+1, \lambda) = f(\bar{x}(t, \lambda), \bar{w}(t, \lambda)); \quad (7.6a)$$

$$\bar{z}(t, \lambda) = h(\bar{x}(t, \lambda), \bar{w}(t, \lambda)); \quad (7.6b)$$

where $(\bar{x}(\lambda), \bar{w}(\lambda), \bar{z}(\lambda)) \in \mathfrak{B}$. Differentiating the parameterized dynamics w.r.t. λ results in the differential form of (7.1), given by

$$x_\delta(t+1, \lambda) = A_\delta(\bar{x}(t, \lambda), \bar{w}(t, \lambda))x_\delta(t, \lambda) + B_\delta(\bar{x}(t, \lambda), \bar{w}(t, \lambda))w_\delta(t, \lambda); \quad (7.7a)$$

$$z_\delta(t, \lambda) = C_\delta(\bar{x}(t, \lambda), \bar{w}(t, \lambda))x_\delta(t, \lambda) + D_\delta(\bar{x}(t, \lambda), \bar{w}(t, \lambda))w_\delta(t, \lambda); \quad (7.7b)$$

with $x_\delta(t, \lambda) = \frac{\partial \bar{x}}{\partial \lambda}(t, \lambda) \in \mathbb{R}^{n_\mathrm{x}}$, $w_\delta(t, \lambda) = \frac{\partial \bar{w}}{\partial \lambda}(t, \lambda) \in \mathbb{R}^{n_\mathrm{w}}$, $z_\delta(t, \lambda) = \frac{\partial \bar{z}}{\partial \lambda}(t, \lambda) \in \mathbb{R}^{n_\mathrm{z}}$, and

$$A_\delta = \frac{\partial f}{\partial x}, \quad B_\delta = \frac{\partial f}{\partial w}, \quad C_\delta = \frac{\partial h}{\partial x}, \quad D_\delta = \frac{\partial h}{\partial w}, \quad (7.8)$$

where $(\bar{x}(\lambda), \bar{w}(\lambda)) \in \pi_{\mathrm{x,w}}\,\mathfrak{B}$ for all $\lambda \in [0,\, 1]$. Note that this analogous to the CT case, with the differential form given by (5.7). We also similarly omit the dependence on λ if it holds for any $\lambda \in [0, 1]$. This then allows us to state a DT definition for differential dissipativity:

Definition 7.2 (DT differential dissipativity)**.** *Consider the system given by (7.1) and its differential form (7.7). The system is differentially dissipative w.r.t. a supply function* $s_\delta : \mathbb{R}^{n_\mathrm{w}} \times \mathbb{R}^{n_\mathrm{z}} \to \mathbb{R}$*, if there exists a storage function* $\mathcal{V}_\delta : \mathcal{X} \times \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}_0^+$ *with* $\mathcal{V}_\delta \in \mathcal{C}_0$ *and* $\mathcal{V}_\delta(\bar{x}, \cdot) \in \mathcal{Q}_0$*,* $\forall\, \bar{x} \in \mathcal{X}$*, such that*

$$\mathcal{V}_\delta\big(\bar{x}(t_1+1), x_\delta(t_1+1)\big) - \mathcal{V}_\delta\big(\bar{x}(t_0), x_\delta(t_0)\big) \le \sum_{t=t_0}^{t_1} s_\delta\big(w_\delta(t), z_\delta(t)\big), \tag{7.9}$$

for all $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,u}} \mathfrak{B}$ *and for all* $t_0, t_1 \in \mathbb{N}_0$*, with* $t_0 \le t_1$*.*

In correspondence with the incremental quadratic, (Q, S, R), supply function, we will also focus on quadratic supply functions for differential dissipativity. Namely, we will consider supply functions of the form

$$s_\delta(w_\delta,\, z_\delta) = \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w_\delta \\ z_\delta \end{bmatrix}. \tag{7.10}$$

Note that like for CT systems, checking differential (Q, S, R) dissipativity of the (primal form of the) system (7.1) can be seen as checking 'classical (Q, S, R) dissipativity' of the differential form of the system (7.7).

In CT, we have the result of Theorem 5.1, which shows how we can analyze differential (Q, S, R) dissipativity through a feasibility check of a (infinite dimensional) set of *Linear Matrix Inequalities* (LMIs). Under the same considerations, we will also show that likewise for DT systems we can formulate a matrix inequality condition to analyze differential dissipativity, which also corresponds to a feasibility check of a(n) (infinite dimensional) set of LMIs. To obtain the results, we consider, like in CT, storage functions of a quadratic form given by

$$\mathcal{V}_\delta(\bar{x}, x_\delta) = x_\delta^\top M(\bar{x}) x_\delta, \tag{7.11}$$

with M satisfying Condition 5.1. Moreover, for the system in primal form (7.1), let us also consider the set $\mathcal{D}$ for which holds that $(x(t+1) - x(t)) \in \mathcal{D}$ for all $t \in \mathbb{N}_0$. This allows us to obtain the following result to analyze differential dissipativity in DT:

Theorem 7.1 (DT differential (Q, S, R) dissipativity condition)**.** *A nonlinear system given by (7.1) is differentially (Q, S, R) dissipative, if there exists a storage function (7.11) with* M *satisfying Condition 5.1, such that for all* $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$ *and* $x_\mathrm{v} \in \mathcal{D}$

$$(\star)^\top \begin{bmatrix} -M(\bar{x}) & 0 \\ 0 & M(\bar{x} + \bar{x}_\mathrm{v}) \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\delta(\bar{x}, \bar{w}) & B_\delta(\bar{x}, \bar{w}) \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\delta(\bar{x}, \bar{w}) & D_\delta(\bar{x}, \bar{w}) \end{bmatrix} \preceq 0. \tag{7.12}$$

Proof. See Appendix B.5. □

With this theorem, we now have analogous conditions in CT and DT in order to analyze differential dissipativity of nonlinear systems through a feasibility check of an infinite dimensional set of LMIs. Later, in Section 7.3, we will discuss how we can make use of the results in LPV framework in order to make the check computationally feasible, similar to the CT results in Section 5.4.2.

7.2.3 Induced incremental dissipativity

Next, we will present how we can use differential (Q, S, R) dissipativity in order to analyze incremental (Q, S, R) dissipativity of DT nonlinear systems. In Chapter 5, we have shown for CT systems how differential (Q, S, R) dissipativity implies incremental (Q, S, R) dissipativity, for (Q, S, R) supplies for which $R \preceq 0$, see Theorem 5.2. Combined with the matrix inequality condition resulting from Theorem 5.1, this gave us a very powerful tool to check incremental dissipativity of CT nonlinear systems. Next, we will similarly show, how also in DT differential (Q, S, R) dissipativity of a DT nonlinear system (7.1) implies incremental (Q, S, R) dissipativity.

Theorem 7.2 (Induced DT incremental dissipativity)**.** *If the nonlinear system given by (7.1) is differentially* (Q, S, R) *dissipative, i.e., w.r.t. the supply function (7.10), with* $R \preceq 0$*, under a storage function* $\mathcal{V}_\delta$ *of the form (7.11), then the system is incrementally* (Q, S, R) *dissipative for the same tuple* (Q, S, R)*.*

Proof. See Appendix B.5. □

This shows, similar to the CT results in Theorem 5.2, that also in DT, differential (Q, S, R) dissipativity implies incremental (Q, S, R) dissipativity under a quadratic storage function of the form (7.11) and with $R \preceq 0$.

We can then combine the results of Theorems 7.1 and 7.2 to arrive at the following corollary:

Corollary 7.1 (Incremental dissipativity condition)**.** *The nonlinear system given by (7.1) is incrementally* (Q, S, R) *dissipative with* $R \preceq 0$*, if (7.12) holds for all* $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$ *and* $x_{\mathrm{v}} \in \mathcal{D}$ *with* M *satisfying Condition 5.1.*

Remark 7.1. Note that the results of Lemma 5.1 and Theorem 5.3 also hold in the DT case, as they can trivially be extended through the use Theorem 7.2.

With these results, we have a powerful tool, through the matrix inequality condition in Theorem 7.1, to check incremental (Q, S, R) dissipativity of DT systems. Similarly, with these results, we have in DT the same chain of implications which connects differential, incremental, universal shifted, and classical dissipativity of the system. This gives us a systematic way to analyze global dissipativity concepts of both CT and DT nonlinear systems.

Next, we will discuss how these results connect to incremental performance notions for DT systems.

7.2.4 Induced incremental performance

Using classical (Q, S, R) dissipativity, many useful performance notions can be retrieved such as ℓ_2-gain performance and passivity. In Section 5.4, we have shown how also in the incremental case in CT, we can retrieve incremental versions of these performance notions, such as the incremental $\mathcal{L}_2$-gain and incremental passivity. Next, we will use the previously derived results to also connect DT incremental dissipativity to incremental performance notions.

Before deriving these results, we will first give definition of incremental the ℓ_p-ℓ_q-gain of a DT system, analogous to the CT version in Definition 5.5.

Definition 7.3 (Incremental ℓ_p-ℓ_q-gain)**.** *A DT nonlinear system given by (7.1) is said to have a finite incremental ℓ_p-ℓ_q-gain, if there is a finite $\gamma \geq 0$ and function $\zeta_\mathrm{i} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that*

$$\|z - \tilde{z}\|_{q,T} \leq \gamma \|w - \tilde{w}\|_{p,T} + \zeta_\mathrm{i}(x_0, \tilde{x}_0), \tag{7.13}$$

for all $T \geq 0$ and $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$ with $w, \tilde{w} \in \ell_\mathrm{pe}$ in CT. The induced incremental ℓ_p-ℓ_q-gain of (7.1), denoted as ℓ_{ip}-ℓ_{iq}-gain, is the infimum of γ such that (7.13) still holds. If p = q, we will refer to this as the (induced) incremental ℓ_p-gain, denoted as ℓ_{ip}-gain.

Using this definition, we will now show how the $\ell_{\mathrm{i}2}$-gain of a nonlinear system (7.1) can be analyzed using the results of Corollary 7.1.

Corollary 7.2 ($\ell_{\mathrm{i}2}$-gain analysis)**.** *A nonlinear system given by (7.1) has a finite $\ell_{\mathrm{i}2}$-gain of γ if there exists a matrix $M \in \mathbb{S}^{n_\mathrm{x}}$ with $M \succ 0$ such that for all $\bar{x} \in \mathcal{X}$ and $\bar{w} \in \mathcal{W}$*

$$\begin{bmatrix} M & A_\delta(\bar{x}, \bar{w})M & B_\delta(\bar{x}, \bar{w}) & 0 \\ \star & M & 0 & MC_\delta(\bar{x}, \bar{w})^\top \\ \star & \star & \gamma I & D_\delta(\bar{x}, \bar{w})^\top \\ \star & \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{7.14}$$

Proof. See Appendix B.5. □

Next, we will look into DT incremental passivity. Similar to Definition 5.6, we define DT incremental passivity as follows:

Definition 7.4 (Incremental passivity)**.** *A nonlinear system given by (7.1) for which $n_\mathrm{w} = n_\mathrm{z}$ is incrementally passive, if it is incrementally dissipative w.r.t. the supply*

$$s_\mathrm{i}(w, \tilde{w}, z, \tilde{z}) = (w - \tilde{w})^\top (z - \tilde{z}) + (z - \tilde{z})^\top (w - \tilde{w}). \tag{7.15}$$

Using this definition, we can then use Corollary 7.1 to obtain the following result:

Corollary 7.3 (DT incremental passivity analysis)**.** *A nonlinear system given by (7.1) is incrementally passive, if there exists a matrix $M \in \mathbb{S}^{n_\mathrm{x}}$ with $M \succ 0$, such that for all $\bar{x} \in \mathcal{X}$ and $\bar{w} \in \mathcal{W}$*

$$\begin{bmatrix} M & A_\delta(\bar{x}, \bar{w})M & B_\delta(\bar{x}, \bar{w}) \\ \star & M & MC_\delta(\bar{x}, \bar{w})^\top \\ \star & \star & D_\delta(\bar{x}, \bar{w}) + (\star) \end{bmatrix} \succeq 0. \tag{7.16}$$

Proof. See Appendix B.5. □

Remark 7.2. Note that the obtained conditions for $\ell_{\mathrm{i}2}$-gain and incremental passivity analysis result in checking positive semi-definiteness of a matrix, while in literature these, or similar conditions, are often found as positive definiteness checks. The positive definite versions of the conditions can simply retrieved by making the incremental dissipativity check strict, i.e., changing $\leq$ to $<$ in (7.9) and (7.4), which then imply the strict versions of the conditions found in this section.

Remark 7.3. In Chapter 5, conditions for CT systems for $\mathcal{L}_{\mathrm{i}\infty}$ performance and incremental generalized $\mathcal{H}_2$ performance, corresponding to the $\mathcal{L}_{\mathrm{i}2}$-$\mathcal{L}_{\mathrm{i}\infty}$-gain, are also given besides the $\mathcal{L}_{\mathrm{i}2}$-gain and passivity conditions. For DT systems, similar results can also be derived along the same lines as the CT proofs given in Appendix B.3 in order to obtain $\ell_{\mathrm{i}\infty}$ and $\ell_{\mathrm{i}2}$-$\ell_{\mathrm{i}\infty}$-gain conditions.

Again note that the results of Corollaries 7.2 and 7.3 are analogous to their CT counterparts in Corollaries 5.2 and 5.4, respectively. With these results, we have now shown that both in CT and DT differential dissipativity analysis can be done through a feasibility check of a matrix inequality, which can then be used in order to achieve systematic incremental performance analysis of nonlinear systems. Next, we will discuss how in DT, like in CT, these results link to incremental stability analysis of nonlinear systems.

7.2.5 Relation to incremental stability

In Section 5.3.5, we extensively discussed for the CT case how incremental dissipativity connects to incremental stability of nonlinear systems. A key result to show this connection is the incremental extension to Lyapunov functions in CT from (Angeli 2002), see Theorem 5.4. Similarly, in DT, incremental to Lyapunov stability theory have also been made in (Tran et al. 2016; Tran et al. 2018):

Theorem 7.3 (DT Incremental Lyapunov stability)**.** *The nonlinear system given by (7.1) is incrementally stable, if there exists an incremental Lyapunov function $V_{\mathrm{i}} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}_0^+$ with $V_{\mathrm{i}} \in \mathcal{C}_1$ and $V_{\mathrm{i}} \in \mathcal{Q}_{\mathrm{i}}$, such that*

$$V_{\mathrm{i}}(x(t+1), \tilde{x}(t+1)) - V_{\mathrm{i}}(x(t), \tilde{x}(t)) \leq 0, \tag{7.17}$$

for all $t \in \mathbb{R}_0^+$ and $x, \tilde{x} \in \pi_{\mathrm{x}} \mathfrak{B}_{\mathrm{w}}(w)$ under all measurable and bounded functions $w \in \mathcal{W}^{\mathbb{R}_0^+}$. Moreover, the nonlinear system is asymptotically stable, if (7.17) holds, but with strict inequality except when $x(t) = \tilde{x}(t)$.

This also allows Theorem 5.5 to be trivially extended to DT and is therefore not given. Moreover, using the differential form, results analogous to Lemma 5.2 have also been derived in literature in order to arrive at a matrix inequality condition for incremental stability (Tran et al. 2016; Tran et al. 2018):

Lemma 7.1 (DT incremental stability condition)**.** *The nonlinear system given by (7.1) is incrementally stable, if*

$$A_\delta(\bar{x}, \bar{w})^\top M(\bar{x} + \bar{x}_\mathrm{v}) A_\delta(\bar{x}, \bar{w}) - M(\bar{x}) \preceq 0 \tag{7.18}$$

for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$, $\bar{x}_\mathrm{v} \in \mathcal{D}$ *and* M *satisfying Condition 5.1. If (7.18) holds with strict inequality, then, the system is incrementally asymptotically stable.*

7.3 Convex Incremental Analysis

In the previous section, we have shown how incremental stability and performance in DT, similar as in CT, can be analyzed in terms of feasibility check of an infinite dimensional set of LMIs, see Corollary 7.1. The question then arises how we can computationally efficiently verify this infinite dimensional set of LMIs. Like we discussed in Section 5.4.2 in the CT case, this problem is similar to the analysis problem of LPV systems, where due to the scheduling-variable we obtain an infinite dimensional set of LMIs. Hence, in a similar manner as has been done in the CT case, we make once again use of the LPV framework to make the proposed check for DT incremental dissipativity computationally feasible.

As we have shown in Section 7.2, the resulting incremental dissipativity conditions for a system given by (7.1) are related to standard dissipativity of its differential form (7.7). Hence, we embed the differential form of the nonlinear system in an LPV representation, which we call a DPV embedding of the nonlinear system (7.1).

Definition 7.5 (DT DPV embedding)**.** *Assume we have a nonlinear system given by (7.1) with differential form given by (7.7). The LPV representation given by*

$$x_\delta(t+1) = A(p(t))x_\delta(t) + B(p(t))w_\delta(t), \tag{7.19a}$$

$$z_\delta(t) = C(p(t))x_\delta(t) + D(p(t))w_\delta(t), \tag{7.19b}$$

where $p(t) \in \mathcal{P} \subset \mathbb{R}^{n_\mathrm{p}}$ *is the scheduling-variable, is a DPV embedding of (7.1) on the compact convex region* $\mathscr{X} \times \mathscr{W} \subseteq \mathcal{X} \times \mathcal{W}$ *if there exists a function, called the scheduling-map,* $\eta : \mathscr{X} \times \mathscr{W} \to \mathcal{P}$ *such that under a given choice of function class for* $A, \ldots, D$*, e.g. affine, polynomial, etc.,* $A(\eta(\bar{x}, \bar{w})) = A_\delta(\bar{x}, \bar{w}), \ldots, D(\eta(\bar{x}, \bar{w})) = D_\delta(\bar{x}, \bar{w})$ *for all* $(\bar{x}, \bar{w}) \in \mathscr{X} \times \mathscr{W}$ *and* $\eta(\mathscr{X}, \mathscr{W}) \subseteq \mathcal{P}$*.*

Through the DPV embedding of a system, like in the CT case in Section 5.4.2, we can embed the behavior of the differential form of the system. Hence, also in DT we can use the results of Lemma 5.3 and Theorem 5.7. This then allows us to make use of the LPV analysis results for DT systems, see also Section 2.5, in order to computationally efficiently check the condition in Theorem 7.1. In fact, like in the CT case, the resulting analysis problems are standard LPV analysis problems, i.e., incremental (Q, S, R) dissipativity of the primal form (7.1) can be checked through checking 'classical (Q, S, R) dissipativity' of the differential form

(7.7), which we then check through checking classical (Q, S, R) dissipativity of the DPV embedding (7.19). As the proposed analysis results can be cast as classical (Q, S, R) dissipativity analysis problem of an LPV representation, all the techniques to reduce the evaluation of an infinite set of LMIs to only checking a finite set of LMIs from the LPV framework can be used. Often for this, $A, \ldots, D$ are needed to be restricted to an affine function in the embedding (7.19). The most common techniques are polytopic, multiplier or gridding-based approaches, see (Hoffmann and Werner 2015a) for an overview. Although the same tools from the LPV framework can be used for checking incremental dissipativity and classical dissipativity of nonlinear systems, we would like to remind again that the underlying dissipativity and stability concepts are very different. Namely, using the incremental dissipativity tools developed in this thesis, global stability and performance guarantees, i.e., irrespective of a particular equilibrium point or trajectory, can be given for the nonlinear system, while standard dissipativity tools can only provide performance and stability analysis with respect to single equilibrium point, often the origin of the state-space representation of the nonlinear system.

7.4 Analysis Example

In this section, we apply the results of the previous sections in order to analyze incremental dissipativity of a controlled unbalanced disk.

Example 7.1 *(Analysis of a controlled unbalanced disk)*.

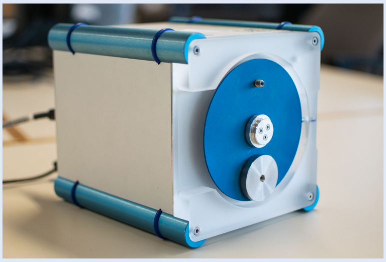


Figure 7.1: Unbalanced disk setup.

We consider the (CT) model for an unbalanced disk, as also used in Examples 4.2 and 6.2, which is given by

$$\dot{\theta}(t) = \omega(t); \tag{7.20a}$$

$$\dot{\omega}(t) = \tfrac{Mgl}{J}\sin(\theta(t)) - \tfrac{1}{\tau}\omega(t) + \tfrac{K_m}{\tau}V(t); \tag{7.20b}$$

where θ [rad] is the angle of the disk, ω [$\mathrm{rad}\cdot\mathrm{s}^{-1}$] its angular velocity, V [V] is the input voltage to the motor, g is the gravitational acceleration, l the length of the pendulum, J, the inertia of the disk and K_m and τ are the motor constant and time constant respectively. The values of the physical parameters of the system are given in Table 4.1.

We discretize equation (7.20) using a fourth order Runge-Kutta (RK4) method, where the control input is assumed to be constant over the sampling period. More specifically, assuming the CT dynamics are $\dot{x}_\mathrm{g}(t) = f_\mathrm{c}(x_\mathrm{g}(t), u_\mathrm{g}(t))$, we have the RK4 discretized dynamics given by

$$x_\mathrm{g}(t+1) = x_\mathrm{g}(t) + \frac{T_\mathrm{s}}{6}(\varphi_1(t) + 2\varphi_2(t) + 2\varphi_3(t) + \varphi_4(t)), \tag{7.21}$$

where T_s is the sample time, $t \in \mathbb{N}_0$ is now the (discrete) time-instant, and

$$\begin{aligned} \varphi_1(t) &= f_\mathrm{c}(x_\mathrm{g}(t), u_\mathrm{g}(t)), && (7.22a)\\ \varphi_2(t) &= f_\mathrm{c}\left(x_\mathrm{g}(t) + \tfrac{T_\mathrm{s}}{2}\varphi_1(t), u_\mathrm{g}(t)\right), && (7.22b)\\ \varphi_3(t) &= f_\mathrm{c}\left(x_\mathrm{g}(t) + \tfrac{T_\mathrm{s}}{2}\varphi_2(t), u_\mathrm{g}(t)\right), && (7.22c)\\ \varphi_4(t) &= f_\mathrm{c}\left(x_\mathrm{g}(t) + T_\mathrm{s}\varphi_3(t), u_\mathrm{g}(t)\right). && (7.22d) \end{aligned}$$

Applying this method to the CT dynamics of the unbalanced disk given by (7.20), with a sample time $T_\mathrm{s} = \frac{1}{20}$ second, results in a DT nonlinear state-space representation of the form

$$x_\mathrm{g}(t+1) = f(x_\mathrm{g}(t), u_\mathrm{g}(t)), \tag{7.23}$$

where $x_\mathrm{g} = \mathrm{col}(\theta, \omega)$ and $u_\mathrm{g} = V$. For the discretized version of the unbalanced disk given by (7.23), a (robust) DT LTI controller is heuristically designed in order to achieve reference tracking. This controller is given by

$$\begin{aligned} x_\mathrm{k}(t+1) &= x_\mathrm{k}(t) + B_\mathrm{k}u_\mathrm{k}(t); && (7.24a)\\ y_\mathrm{k}(t) &= C_\mathrm{k}x_\mathrm{k}(t) + D_\mathrm{k}u_\mathrm{k}(t); && (7.24b) \end{aligned}$$

where x_k is the state, u_k is the input and y_k is the output of the controller. For the LTI controller, $B_\mathrm{k} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $C_\mathrm{k} = -0.5$ and $D_\mathrm{k} = \begin{bmatrix} -10 & -1 \end{bmatrix}$ are chosen, corresponding to a PID controller. The closed-loop interconnection of plant and controller is given in Figure 7.2, where K is the DT LTI controller given by (7.24), G is the discretized unbalanced disk dynamics given by (7.23), w is the input disturbance, and z is the generalized performance channel. The controller K in this configuration can be thought of as a PID controller for regulation of the disk angle at zero and rejection of constant input disturbances. The closed-loop interconnection results in a system of the form (7.1).

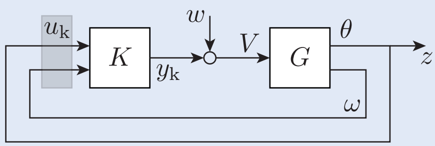


Figure 7.2: Closed-loop interconnection of DT controller K and discretized dynamics of the unbalanced disk G.

Using Definition 7.5, the differential form of the closed-loop dynamics of the DT LTI controller and discretized unbalanced disk dynamics is computed and a DPV

embedding is constructed on the compact region $\theta(t) \in [-\pi, \pi]$, $\omega(t) \in [-10, 10]$ and $V(t) \in [-10, 10]$, with scheduling-variable $p = \mathrm{col}(p_1, p_2, p_3) = \mathrm{col}(\theta, \omega, V)$. Next, an upper-bound for the induced $\ell_{\mathrm{i}2}$-gain of the closed-loop interconnection on the compact region is computed using the results of Corollary 7.2. Due to the complexity of the DPV embedding (as we use the discretized plant), we use a gridding-based LPV analysis method, which has been implemented in the LPVcore Toolbox (Boef et al. 2021). For the grid-based method, we consider equidistant set of grid points on the embedding region with 11 grid-points in each dimension, resulting in a total of 1331 grid-points. Computing the $\mathcal{L}_2$-gain of the DPV embedding, results in an ℓ_2-gain of $\gamma = 0.220$. Consequently, the upper-bound for the induced $\ell_{\mathrm{i}2}$-gain of the closed-loop interconnection on the compact region is given by $\gamma = 0.220$. In order to compute the closed-loop ℓ_2-gain of the closed-loop interconnection, the DT primal form in the plant (7.23) is embedded in a grid-based LPV model using the technique described Appendix C.6 on the aforementioned equidistant grid. The closed-loop interconnection of the LTI controller and primal form of the plant obtains an upper-bound for the ℓ_2-gain[1] of $\gamma_{\ell_2} = 0.219$.
For comparison, an LPV version of the controller is also heuristically designed, where B is taken the same as for the LTI controller (7.24), but C_k and D_k are made parameter-varying by taking $C_\mathrm{k}(p) = -0.5 - \frac{1}{20}\sin(p_1)$ and $D_\mathrm{k}(p) = \begin{bmatrix}-10 - 2\cos(p_1) & -1\end{bmatrix}$ (hence, they only depend on $p_1 = \theta$). For the closed-loop interconnection of the LPV controller and the primal form of the plant, an upper-bound for the ℓ_2-gain is computed using a standard grid-based LPV method. This results in $\gamma_{\ell_2} = 0.179$, which is better than the closed-loop interconnection with the LTI controller. However, unlike for the closed-loop with the LTI controller, we cannot compute an upperbound for the $\ell_{\mathrm{i}2}$-gain for the closed-loop with the LPV controller using the results of Corollary 7.2. While this does not immediately mean that the closed-loop does not have a bounded $\ell_{\mathrm{i}2}$-gain, as Corollary 7.2 is only a sufficient condition, we will see by means of simulation that the closed-loop using the LPV controller is indeed not incrementally stable.
In Figure 7.3, simulation results of trajectory of the angle of the disk for both the interconnection of the discrete-time plant with the LTI controller and with the LPV controller for different input disturbances w are shown[2]. When $w(t) = 0$, the closed-loop with the LPV controller has a faster response and less overshoot compared to the closed-loop with LTI controller. This is also to be expected, as the closed-loop ℓ_2-gain with LPV controller ($\gamma_{\ell_2} = 0.179$) is lower than that of the closed-loop system with the LTI controller ($\gamma_{\ell_2} = 0.219$). However, when $w(t) = -\min(t, 70)$, it can be seen that while the LTI controller is still able to reject the disturbance when it becomes constant (at $t = 70$), the closed-loop with the LPV controller ends up in a limit cycle and is not able to reject the disturbance, which is a results of the system not being incrementally stable/dissipative. This behavior is similar to what we have seen in the CT case. These observations highlight the importance of analyzing stability and performance of nonlinear systems using global dissipativity notions instead of using only classical dissipativity based notions.

[1] Note that the ℓ_2-gain is smaller than the $\ell_{\mathrm{i}2}$-gain, as the $\ell_{\mathrm{i}2}$-gain is a stronger notion.
[2] Note that during simulation all the scheduling-variables stayed within the compact-set.

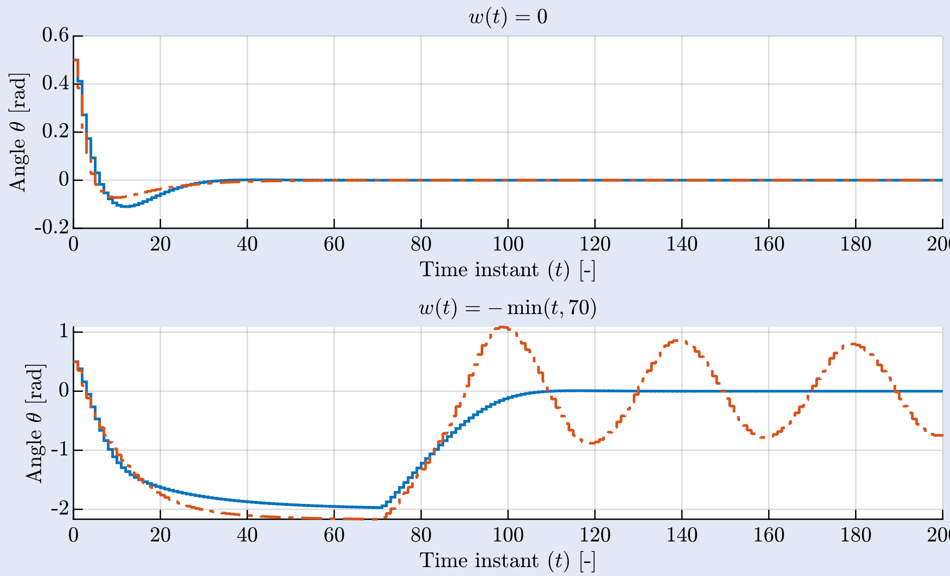


Figure 7.3: Angle of the disk in closed-loop with the LTI controller (▬) and LPV controller (-·-) for different input disturbances w.

7.5 Convex Incremental Synthesis

7.5.1 Incremental synthesis problem

In this section Section 7.5, we will present how we can formulate DT incremental controller synthesis results in a similar manner as the CT controller synthesis results of Chapter 6.

We consider a similar setup as is considered in the CT case that is discussed in Chapter 6. Namely, we consider a DT nonlinear system, being the generalized plant (see also Definition 6.1), of the form

$$x(t+1) = f(x(t)) + B_{\mathrm{w}}w(t) + B_{\mathrm{u}}u(t); \tag{7.25a}$$

$$z(t) = h_{\mathrm{z}}(x(t)) + D_{\mathrm{zw}}w(t) + D_{\mathrm{zu}}u(t); \tag{7.25b}$$

$$y(t) = C_{\mathrm{y}}x(t) + D_{\mathrm{yw}}w(t); \tag{7.25c}$$

where $t \in \mathbb{N}_0$ is time, $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_{\mathrm{x}}}$, $w(t) \in \mathcal{W} \subseteq \mathbb{R}^{n_{\mathrm{w}}}$ and $z(t) \in \mathcal{Z} \subseteq \mathbb{R}^{n_{\mathrm{z}}}$ are the state, generalized disturbance and generalized performance signals of the plant, respectively, and where $u(t) \in \mathcal{U} \subseteq \mathbb{R}^{n_{\mathrm{u}}}$ is the control input and $y(t) \in \mathcal{Y} \subseteq \mathbb{R}^{n_{\mathrm{y}}}$ is the measured output. The sets $\mathcal{X}$, $\mathcal{W}$, $\mathcal{U}$, $\mathcal{Z}$ and $\mathcal{Y}$ are open and convex, containing the origin. The solutions of (7.25) satisfy (7.25) in the ordinary sense and are restricted to $t \in \mathbb{N}_0$. The functions $f : \mathcal{X} \to \mathcal{X}$ and $h_{\mathrm{z}} : \mathcal{X} \to \mathcal{Z}$ are assumed to be in $\mathcal{C}_1$, i.e. $f, h_{\mathrm{z}} \in \mathcal{C}_1$. Furthermore, $B_{\mathrm{w}} \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{w}}}$, $B_{\mathrm{u}} \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{u}}}$, $D_{\mathrm{zw}} \in \mathbb{R}^{n_{\mathrm{z}} \times n_{\mathrm{w}}}$,

$D_{\mathrm{zu}} \in \mathbb{R}^{n_\mathrm{z} \times n_\mathrm{u}}$, and $D_{\mathrm{yw}} \in \mathbb{R}^{n_\mathrm{y} \times n_\mathrm{w}}$ are matrices. The solution set of (7.25) is defined as

$$\mathfrak{B} := \Big\{ (x, w, u, z, y) \in (\mathcal{X} \times \mathcal{W} \times \mathcal{U} \times \mathcal{Z} \times \mathcal{Y})^{\mathbb{N}_0} \mid \\ (x, w, u, z, y) \text{ satisfies } (7.25) \Big\}. \tag{7.26}$$

Remark 7.4. Note that while (7.25) might seem restrictive, a larger class of nonlinear generalized plants of the form

$$x(t+1) = f(x(t), u(t)) + B_\mathrm{w} w(t); \tag{7.27a}$$
$$z(t) = h_\mathrm{z}(x(t), u(t)) + D_{\mathrm{zw}} w(t); \tag{7.27b}$$
$$y(t) = h_\mathrm{y}(x(t)) + D_{\mathrm{yw}} w(t); \tag{7.27c}$$

where $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, $h_\mathrm{z} : \mathcal{X} \times \mathcal{U} \to \mathcal{Z}$ and $h_\mathrm{y} : \mathcal{X} \to \mathcal{Y}$ with $f, h_\mathrm{z}, h_\mathrm{y} \in \mathcal{C}_1$, can be written in the form (7.25) by interconnecting appropriate filters to u and y of (7.27). See Appendix C.2.2 for an example of such a procedure in CT.

The to-be-designed controller for the generalized plant given by (7.25) is considered in the form

$$x_\mathrm{k}(t+1) = f_\mathrm{k}(x_\mathrm{k}(t), u_\mathrm{k}(t)); \tag{7.28a}$$
$$y_\mathrm{k}(t) = h_\mathrm{k}(x_\mathrm{k}(t), u_\mathrm{k}(t)); \tag{7.28b}$$

where $x_\mathrm{k}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}}$ is the state of the controller, $u_\mathrm{k}(t) \in \mathbb{R}^{n_{\mathrm{u_k}}}$ its input, and $y_\mathrm{k}(t) \in \mathbb{R}^{n_{\mathrm{y_k}}}$ its output. Moreover, $f_\mathrm{k} : \mathbb{R}^{n_{\mathrm{x_k}}} \times \mathbb{R}^{n_{\mathrm{u_k}}} \to \mathbb{R}^{n_{\mathrm{x_k}}}$ and $h_\mathrm{k} : \mathbb{R}^{n_{\mathrm{x_k}}} \times \mathbb{R}^{n_{\mathrm{u_k}}} \to \mathbb{R}^{n_{\mathrm{y_k}}}$. The closed-loop interconnection of a generalized plant P given by (7.25) and a controller K given by (7.28) with $u_\mathrm{k} = y$ and $u = y_\mathrm{k}$ is denoted by $\mathcal{F}_\mathrm{l}(P, K)$, which is assumed to be well-posed and hence in the form (7.1). In this section we propose a DT convex controller synthesis method such that the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P, K)$ is incrementally (Q, S, R) dissipative. Consequently, we assume that (7.25) is a generalized plant in the sense that there exists a controller K such that the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P, K)$ is incrementally stable, see also Definition 6.1 and Proposition 6.1.

Like for the CT results in Chapter 6, the procedure will be exemplified by ensuring $\ell_{\mathrm{i}2}$-gain stability[3] with minimal $\ell_{\mathrm{i}2}$-gain of the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P, K)$. However, the results also hold for the other aforementioned incremental (Q, S, R) performance notions.

7.5.2 Controller synthesis method

Overview

In this section, the proposed controller synthesis method is discussed. The proposed procedure follows along the same lines as the CT version discussed in Chapter 6.

[3] By which we mean the system is incrementally dissipative w.r.t. the $\ell_{\mathrm{i}2}$-gain supply function, implying incremental stability.

This is possible because like in the CT case, it also holds for DT systems that 'classical (Q, S, R) dissipativity' of the differential form (i.e., differential (Q, S, R) dissipativity) implies incremental (Q, S, R) dissipativity of the primal form (for the same tuple (Q, S, R)), see Theorem 7.2. Consequently, to ensure a minimal $\ell_{\mathrm{i}2}$-gain of $\mathcal{F}_{\mathrm{l}}(P, K)$, we can equivalently ensure a minimal ℓ_2-gain of the differential form of $\mathcal{F}_{\mathrm{l}}(P, K)$.

In order to obtain a DT controller ensuring closed-loop $\ell_{\mathrm{i}2}$-gain stability and performance the following procedure is proposed:

1. For the generalized plant given by (7.25), its differential form is computed, which is then used to construct a DPV embedding (see Definition 7.5) for (7.25).

2. For the DPV embedding of the generalized plant, an LPV controller is synthesized such that the closed-loop interconnection is ℓ_2-gain stable with minimal ℓ_2-gain γ. This controller will be referred to as the differential (LPV) controller.

3. The differential controller designed in Step 2 is realized into a primal form. The resulting closed-loop interconnection of the primal form of the generalized plant and realized primal form of the controller then is $\ell_{\mathrm{i}2}$-gain stable with an $\ell_{\mathrm{i}2}$-gain bounded by γ.

Similar to the CT case, also in DT we can make use of the separability of system interconnections when transforming to and from the differential domain, see Theorem 6.1. Note that in DT Theorem 6.1 also holds as the interconnection structure is the same as in CT, hence, the proof for DT trivially follows. Consequently, this also allows us in DT for independently 'transforming' the generalized plant and controller between the primal and differential domains.

Next, we will discuss each step in more detail.

DPV embedding of the generalized plant

Similar as in CT in Section 6.3.3, as a first step in our proposed controller synthesis procedure, the differential form of the generalized plant is computed and embedded in an LPV representation. The differential form of (7.25) is given by

$$x_\delta(t+1) = A_\delta(\bar{x}(t))x_\delta(t) + B_{\mathrm{w}} z_\delta(t) + B_{\mathrm{u}} u_\delta(t); \tag{7.29a}$$

$$z_\delta(t) = C_{\delta,\mathrm{z}}(\bar{x}(t))x_\delta(t) + D_{\mathrm{zw}} z_\delta(t) + D_{\mathrm{zu}} u_\delta(t); \tag{7.29b}$$

$$y_\delta(t) = C_{\mathrm{y}} x_\delta(t) + D_{\mathrm{yw}} z_\delta(t); \tag{7.29c}$$

where $A_\delta = \frac{\partial f}{\partial x}$, $C_{\delta,\mathrm{z}} = \frac{\partial h_{\mathrm{z}}}{\partial x}$ with $\bar{x} \in \pi_{\mathrm{x}} \mathfrak{B}$, $x_\delta(t) \in \mathbb{R}^{n_{\mathrm{x}}}$, $u_\delta(t) \in \mathbb{R}^{n_{\mathrm{u}}}$, $w_\delta(t) \in \mathbb{R}^{n_{\mathrm{w}}}$, $z_\delta(t) \in \mathbb{R}^{n_{\mathrm{z}}}$ and $y_\delta(t) \in \mathbb{R}^{n_{\mathrm{y}}}$. Analogous as for (6.6), we denote the set of solutions of (7.29) along a $\bar{x} \in \pi_{\mathrm{x}} \mathfrak{B}$ solution of (7.25) by $\mathfrak{B}_\delta(\bar{x})$ (see also (6.7)) and we denote the complete solution set by $\breve{\mathfrak{B}}_\delta = \bigcup_{\bar{x} \in \pi_{\mathrm{x}} \mathfrak{B}} \mathfrak{B}_\delta(\bar{x})$.

The differential form of the generalized plant (7.29) is then embedded in an LPV representation, such that we obtain a DPV embedding of (7.25), see Definition 7.5 and also Definition 6.2. We assume that this DPV embedding of (7.25) is constructed on the region $X \subseteq \mathcal{X}$, where X is compact, and is given by

$$x_\delta(t+1) = A(p(t))x_\delta(t) + B_{\mathrm{w}} z_\delta(t) + B_{\mathrm{u}} u_\delta(t); \tag{7.30a}$$
$$z_\delta(t) = C_{\mathrm{z}}(p(t))x_\delta(t) + D_{\mathrm{zw}} z_\delta(t) + D_{\mathrm{zu}} u_\delta(t); \tag{7.30b}$$
$$y_\delta(t) = C_{\mathrm{y}} x_\delta(t) + D_{\mathrm{yw}} z_\delta(t); \tag{7.30c}$$

where $p(t) \in \mathcal{P} \subset \mathbb{R}^{n_{\mathrm{p}}}$ is the scheduling-variable with corresponding scheduling-map $\eta : X \to \mathcal{P}$, s.t. $p(t) = \eta(\bar{x}(t))$, and A, C_{z} are assumed to be a given function class $\mathfrak{A}$ (e.g., affine, polynomial, rational, etc.). Similar as in CT, for the controller synthesis problem to be feasible, we assume that for the LPV representation given by (7.30) that (A, B_{u}) is stabilizable and (A, C_{y}) is detectable, see Definitions 2.17 and 2.18.

We denote the behavior of the DPV embedding (7.30) for a scheduling trajectory $p \in \mathcal{P}^{\mathbb{N}_0}$ by $\mathfrak{B}_{\mathrm{p}}(p)$ and the full behavior (i.e., for all $p \in \mathcal{P}^{\mathbb{N}_0}$) by $\breve{\mathfrak{B}}_{\mathrm{p}}$, similar as for LPV systems, see (2.27) and (2.28). As the DPV embedding is considered on $X \subseteq \mathcal{X}$, we denote the restriction of the state trajectories of the generalized plant representation (7.25) to X by $\mathfrak{B}_X := \{(x, u, w, z, y) \in \mathfrak{B} \mid x(t) \in X\}$. The corresponding restriction of the solution set of the differential form (7.29) is denoted by $\breve{\mathfrak{B}}_{\delta,X} := \cup_{(\bar{x} \in \pi_{\mathrm{x}} \mathfrak{B}_X)} \mathfrak{B}_\delta(\bar{x})$. Using these behaviors, we then have, through Lemma 5.3 that $\breve{\mathfrak{B}}_{\delta,X} \subseteq \breve{\mathfrak{B}}_{\mathrm{p}}$, i.e., the behavior of the differential form of the plant (7.29) is included in the behavior of the DPV embedding (7.30).

Next, we use the DPV embedding (7.30) in order to be able to use convex controller synthesis through the LPV framework.

Differential controller synthesis

In second step of our procedure, a controller for the differential form of the generalized plant (7.29) is synthesized such that the closed-loop interconnection is ℓ_2-gain stable with minimal ℓ_2-gain. To convexify this problem, the LPV framework is used to perform this step. Hence, we synthesize an LPV controller for the DPV embedding of the generalized plant (7.30), obtained in the previous step. The LPV controller is considered to be of the form

$$x_{\delta,\mathrm{k}}(t+1) = A_{\mathrm{k}}(p(t))x_{\delta,\mathrm{k}}(t) + B_{\mathrm{k}}(p(t))u_{\delta,\mathrm{k}}(t); \tag{7.31a}$$
$$y_{\delta,\mathrm{k}}(t) = C_{\mathrm{k}}(p(t))x_{\delta,\mathrm{k}}(t) + D_{\mathrm{k}}(p(t))u_{\delta,\mathrm{k}}(t); \tag{7.31b}$$

where $x_{\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}}$ is the state, $u_{\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{y}}}$ is the input and $y_{\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{u}}}$ is the output of the controller and $A_{\mathrm{k}}, \ldots, D_{\mathrm{k}} \in \mathfrak{A}$. We will refer to (7.31) as the differential controller. As aforementioned, various methods exist to obtain an LPV controller (7.31), i.e., to synthesize a DT LPV controller minimizing the ℓ_2-gain of the closed-loop system, see e.g. (Apkarian and Gahinet 1995; M. Ali and Werner 2011; De Caigny et al. 2012) or Section 2.5.3.

More formally, we can formulate the following theorem:

Theorem 7.4 (Differential closed-loop ℓ_2-gain)**.** *The closed-loop interconnection of the differential form of the generalized plant P_δ given by (7.29) and the controller K_δ given by (7.31), denoted by $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$, is ℓ_2-gain stable and has an ℓ_2-gain bounded by γ for all $\bar{x} \in \pi_{\bar{\mathrm{x}}} \mathfrak{B}_\mathcal{X}$, if the closed-loop interconnection of the DPV embedding of the generalized plant P_dpv given by (7.30) and K_δ, denoted by $\mathcal{F}_\mathrm{l}(P_\mathrm{dpv}, K_\delta)$, is ℓ_2-gain stable and with a bounded ℓ_2-gain of γ for all $p \in \mathcal{P}^{\mathbb{N}_0}$.*

Proof. See Appendix B.5. □

Like in CT case, we assume Assumption 6.1 holds. This means we assume that the controller synthesis has been solved such that $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$ is classicaly dissipative with a quadratic (differential) storage function of the form $\mathcal{V}_\delta(x_\mathrm{cl}, x_{\delta,\mathrm{cl}}) = x_{\delta,\mathrm{cl}}^\top M x_{\delta,\mathrm{cl}}$, where $M \succ 0$. Here, $x_\mathrm{cl} = \mathrm{col}(x, x_\mathrm{k})$ and $x_{\delta,\mathrm{cl}} = \mathrm{col}(x_\delta, x_{\delta,\mathrm{k}})$.

Note that we can also consider other (Q, S, R) performance notions in this step, similar as in CT, which will ensure different closed-loop incremental (Q, S, R) performance notions. For example, if the differential controller is synthesized such that its closed-loop interconnection with the differential form of the generalized plant is passive, then, after realization of the controller, incremental passivity of the primal form of the closed-loop interconnection is ensured.

Next, it is shown how to realize the primal form of the controller based on synthesized differential controller (7.31) such that the closed-loop interconnection of the primal form of the controller and primal form of the generalized plant (7.25) is $\ell_{\mathrm{i}2}$-gain stable.

Primal controller realization

Inspired by realization procedure for CT systems in Section 6.3.5, we make use of a path integral based realization to obtain the primal form of the controller that enforces convergence of the plant response from our current trajectory towards a desired steady-state response $\vartheta = (x^*, w^*, u^*, z^*, y^*) \in \mathfrak{B}_\mathcal{X}$.

Theorem 7.5 (Primal controller realization)**.** *Consider a differential controller K_δ given by (7.31) synthesized for P_δ given by (7.29) such that $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$ is ℓ_2-gain stable under Assumption 6.1. Moreover, let $(x^*, u^*, y^*) = \pi_{\mathrm{x,u,y}} \vartheta \in \pi_{\mathrm{x,u,y}} \mathfrak{B}_\mathcal{X}$ be the (desired) steady-state trajectory of P and consider the nonlinear controller K, omitting dependence on time for brevity, given by,*

$$q\, x_{\Delta,\mathrm{k}} = \bar{A}_\mathrm{k}(x, x^*) x_{\Delta,\mathrm{k}} + \bar{B}_\mathrm{k}(x, x^*) u_{\Delta,\mathrm{k}}; \tag{7.32a}$$

$$y_\mathrm{k} = y_\mathrm{k}^* + \bar{C}_\mathrm{k}(x, x^*) x_{\Delta,\mathrm{k}} + \bar{D}_\mathrm{k}(x, x^*) u_{\Delta,\mathrm{k}}; \tag{7.32b}$$

where q is the discrete time-shift operator, $(y_\mathrm{k}^*, u_\mathrm{k}^*) = (u^*, y^*)$, $x_{\Delta,\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}}$,

$u_{\Delta,\mathrm{k}} := u_{\mathrm{k}} - u^*_{\mathrm{k}}$, and

$$\bar{A}_{\mathrm{k}}(x,x^*)=\int_0^1 A_{\mathrm{k}}\Big(\eta\big(x^*+\lambda(x-x^*)\big)\Big)d\lambda,\ \ \bar{B}_{\mathrm{k}}(x,x^*)=\int_0^1 B_{\mathrm{k}}\Big(\eta\big(x^*+\lambda(x-x^*)\big)\Big)d\lambda,$$
$$\bar{C}_{\mathrm{k}}(x,x^*)=\int_0^1 C_{\mathrm{k}}\Big(\eta\big(x^*+\lambda(x-x^*)\big)\Big)d\lambda,\ \ \bar{D}_{\mathrm{k}}(x,x^*)=\int_0^1 D_{\mathrm{k}}\Big(\eta\big(x^*+\lambda(x-x^*)\big)\Big)d\lambda. \tag{7.33}$$

The controller K in (7.32) is the primal form of K_δ (7.31) and the differential form of K is K_δ. Hence, K is called the primal realization of K_δ.

Note that the structure of the controller (7.32) is equivalent to the CT version (6.10). Therefore, it will also have a structure as is depicted in Figure 6.2. Accordingly, the proof for Theorem 7.5 also follows along the same line as the proof of Theorem 6.3 and is therefore not repeated here. Also here, in DT, we will refer to (7.32) as the incremental controller. As also discussed in Chapter 6, the incremental controller consists of a feedback part, to converge towards the steady-state trajectory, and a feedforward part, corresponding to the steady-state trajectory. Using the controller K given by (7.32) we can then formulate the following theorem:

Theorem 7.6 (Closed-loop $\ell_{\mathrm{i}2}$-gain)**.** *The closed-loop interconnection $\mathcal{F}_{\mathrm{l}}(P,K)$ of a generalized plant P given by (7.25) and controller K given by (7.32) is $\ell_{\mathrm{i}2}$-gain stable and its $\ell_{\mathrm{i}2}$-gain is bounded by $\gamma$for all $x,\tilde{x}\in\mathfrak{B}_{\mathcal{X}}$, if the closed-loop $\mathcal{F}_{\mathrm{l}}(P_{\mathrm{dpv}},K_\delta)$ of the DPV embedding P_{dpv} (7.30) and the LPV controller K_δ (7.31) is ℓ_2-gain stable and has a bounded ℓ_2-gain of γ for all $p\in\mathcal{P}^{\mathbb{N}_0}$ with a storage function satisfying Assumption 6.1.*

Proof. See Appendix B.5. □

Remark 7.5. Theorem 7.6 holds for $x,\tilde{x}\in\mathfrak{B}_{\mathcal{X}}$. Similar to Theorem 6.4, we can consider a set $\tilde{\mathcal{W}}\subseteq\mathcal{W}$ s.t. for $w,\tilde{w}\in\tilde{\mathcal{W}}^{\mathbb{N}_0}$ we have that $x,\tilde{x}\in\mathfrak{B}_{\mathcal{X}}$. See Definition 4.9 and Theorem 6.4 for more details.

Remark 7.6. Note that as in the CT case, also in DT, the proposed incremental controller explicitly depends on $(u^*,y^*)\in\pi_{\mathrm{u,y}}\,\mathfrak{B}_{\mathcal{X}}$ corresponding to $(x^*,w^*,u^*,z^*,y^*)\in\mathfrak{B}_{\mathcal{X}}$, hence, explicit knowledge of w^* is required. As w^* in the generalized plant framework can contain besides known disturbances, e.g., references, also unknown disturbances, a disturbance observer is required in order to estimate the unknown entries of w^*. For this, we can employ similar design considerations as in CT, including observer design, see the discussion in Section 6.3.7 for more details.

Remark 7.7. Analogous to the CT incremental controller, also for the DT incremental controller presented in this section, we can ensure universal shifted and classical dissipativity of the closed-loop system, see also Section 6.4. The corresponding results are not repeated here.

With the results we have now presented in this chapter, we have the same tools in DT as we have in CT for incremental dissipativity based analysis and control of nonlinear systems. Moreover, to obtain these results, we have shown that we can take the same considerations in DT as in CT.

7.6 Controller Synthesis Example

In this section we demonstrate the proposed incremental controller synthesis method on a simulation example. For comparison, a standard LPV controller, ensuring ℓ_2-gain stability, will also be designed.

Example 7.2 *(DT incremental control)***.** Consider the following DT nonlinear system

$$x_{\mathrm{g},1}(t+1) = 0.1x_{\mathrm{g},1}(t) - x_{\mathrm{g},2}(t); \tag{7.34a}$$
$$x_{\mathrm{g},2}(t+1) = 0.9\sin(x_{\mathrm{g},1}) + x_{\mathrm{g},2} + u_{\mathrm{g}}(t); \tag{7.34b}$$
$$y_{\mathrm{g}}(t) = x_{\mathrm{g},1}(t). \tag{7.34c}$$

For this plant, we want to design a controller which achieves reference tracking.

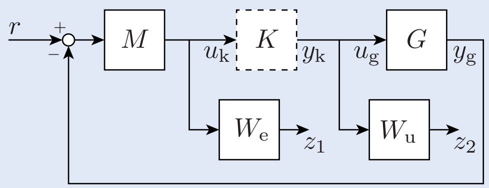


Figure 7.4: Generalized plant.

The generalized plant structure that is considered in order to achieve this objective is depicted in Figure 7.4, where G is the system given by (7.34), K is the to-be-synthesized controller, r is the reference, $W_\mathrm{e}(q) = \frac{0.2(q-0.5)}{q+\alpha}$, $M(q) = \frac{q+\alpha}{q-1}$, and $W_\mathrm{u} = 0.2$, where $\alpha = \frac{1}{\pi}$.
For the synthesis of the incremental controller using the procedure described in Section 7.5.2, we require a DPV embedding of the generalized plant. As the system given by (7.34) is the only nonlinear system in the generalized plant (the weighting filters are linear), we only require computation of the differential form of (7.34) (as the dynamics of differential form of an LTI system are equivalent to its primal form). The following DPV embedding on the region $\mathcal{X}$ of (7.34) is taken:

$$x_{\delta,\mathrm{g},1}(t+1) = 0.1x_{\delta,\mathrm{g},1}(t) - x_{\delta,\mathrm{g},2}(t); \tag{7.35a}$$
$$x_{\delta,\mathrm{g},1}(t+1) = 0.9p(t)x_{\delta,\mathrm{g},1}(t) + x_{\delta,\mathrm{g},2}(t) + u_{\delta,\mathrm{g}}(t); \tag{7.35b}$$
$$y_{\delta,\mathrm{g}}(t) = x_{\delta,\mathrm{g},1}(t); \tag{7.35c}$$

where $p(t) = \cos(x_{\mathrm{g},1}(t)) \in \mathcal{P} = [-1, 1]$ such that $\eta(x_\mathrm{g}) = \cos(x_{\mathrm{g},1})$ with $x_\mathrm{g}(t) \in \mathcal{X} = \mathbb{R}^2$. Note that (7.35) has an affine scheduling dependency. Under these consideration, we synthesize an incremental controller for (7.34) using the synthesis procedure described in Section 7.5.2. For the synthesis of the differential controller

in Step 2 of the synthesis method, see Section 7.5.2, we use a polytopic $\mathcal{L}_2$-gain LPV controller synthesis method implemented in the LPVcore Toolbox (Boef et al. 2021). This synthesis procedure results in a closed-loop $\ell_{\mathrm{i}2}$-gain bound of 1.1.
For comparison, a standard LPV controller is also synthesized in order to achieve a bounded closed-loop ℓ_2-gain. For the synthesis of the standard LPV controller, the same generalized plant structure as depicted in Figure 7.4 is considered. To perform standard LPV synthesis, the primal form of the system given by (7.34) is embedded in an LPV representation on the region $\hat{\mathcal{X}}$, given by

$$x_{\mathrm{g},1}(t+1) = 0.1x_{\mathrm{g},1}(t) - x_{\mathrm{g},2}(t); \tag{7.36a}$$

$$x_{\mathrm{g},2}(t+1) = 0.9p_{\mathrm{s}}(t) + x_{\mathrm{g},2} + u_{\mathrm{g}}(t); \tag{7.36b}$$

$$y_{\mathrm{g}}(t) = x_{\mathrm{g},1}(t). \tag{7.36c}$$

where $p_{\mathrm{s}} = \mathrm{sinc}(x_{\mathrm{g},1}(t)) \in [-0.22, 1]$ such that $\eta_{\mathrm{s}}(x_{\mathrm{g}}) = \mathrm{sinc}(x_{\mathrm{g},1})$ with $x_{\mathrm{g}}(t) \in \hat{\mathcal{X}} = \mathbb{R}^2$. For synthesis of the standard LPV controller, the LPVcore Toolbox is also used (however, applied to (7.36)), which results in a closed-loop ℓ_2-gain bound of 0.80.
The closed-loop systems with incremental controller and standard LPV controller are both simulated for a reference $r \equiv 1$ and $r \equiv 2$. For the incremental controller, this corresponds to the steady-state trajectory $x^*_{\mathrm{g},1}(t) = r(t)$ with $u^*_{\mathrm{g}}(t) = y^*_{\mathrm{k}}(t) = 0.9\sin(x^*_{\mathrm{g},1}(t))$. The trajectories of the closed-loop systems for both of these controllers can be found in the top two graphs in Figure 7.5. From the figure, it can be seen that for both references, the incremental controller achieves similar tracking behavior and it asymptotically converges towards the references. However, with the standard LPV controller, the output of the plant ends up in a limit cycle around the reference when $r \equiv 2$. The closed-loop system with standard LPV controller displays similar issues as have been observed in CT.
The incremental controller also allows to track and guarantee convergence towards more complex reference trajectories. In the bottom graph in Figure 7.5, the reference $r(t) = \sin(\frac{\pi}{8}t) + 2.5$ is used. For this reference, the corresponding feedforward trajectory y^*_{k} (which is not given due to its complexity) is also added to the output of the standard LPV controller. However, it can again be seen that also for this reference, the standard LPV controller is not able to guarantee convergence, even using feedforward.

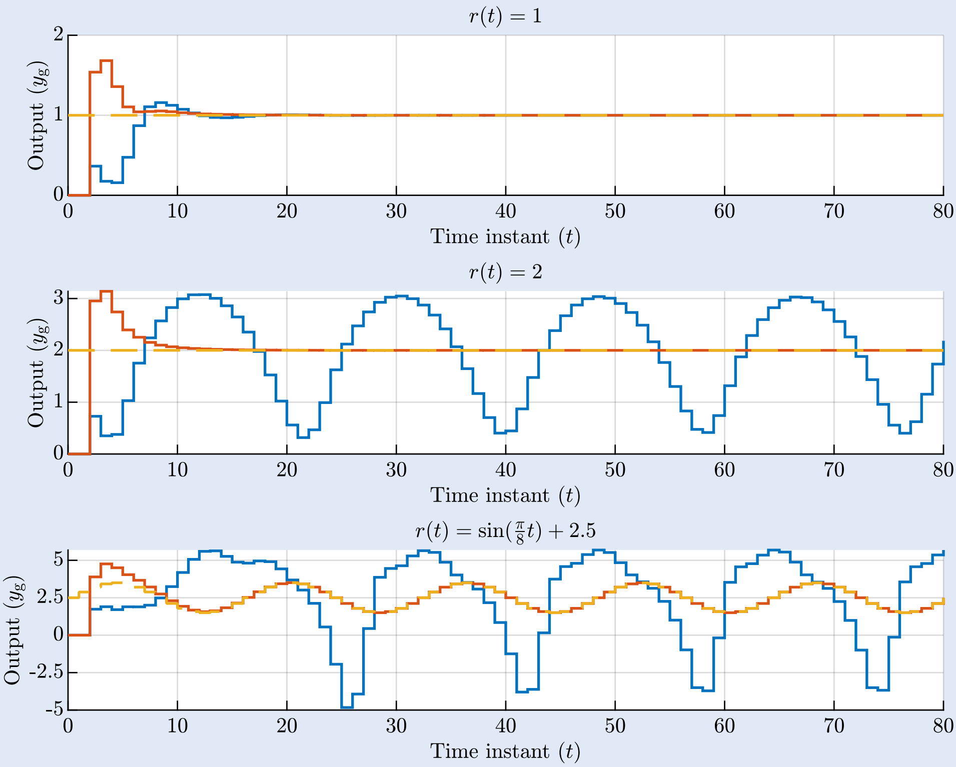


Figure 7.5: Output response of the closed-loop of the plant with standard LPV controller (—) and the incremental controller (—) for the reference trajectory (- -).

7.7 Conclusions

In this chapter, extensions of the CT incremental dissipativity based analysis and controller synthesis results to DT nonlinear systems have been proposed. The obtained DT analysis results, similar to the CT results, use the LPV framework for efficient computation of the various incremental performance notions. Moreover, also analogous to the CT results, we show that DT incremental (Q, S, R) dissipativity of nonlinear systems can be analyzed by analyzing 'classical (Q, S, R) dissipativity' of their differential form. Through the DPV embedding and by making use of the LPV framework, this analysis problem can then be casted as a classical (Q, S, R) dissipativity check of an LPV representation. This allows for the many computational techniques of the LPV framework to be used to efficiently solve incremental stability performance analysis problems using convex optimization. This then gives us a systematic and computationally efficient tool to analyze incremental stability and performance of both CT and DT nonlinear systems. Moreover, we have also proposed a systematic, computationally efficient output-feedback controller synthesis method to ensure incremental dissipativity of DT nonlinear systems. This is achieved through combining the results we have presented on incremental dissipativity analysis of DT nonlinear systems using the LPV framework and extending the CT incremental controller synthesis results from Chapter 6. We show that an equivalent controller synthesis procedure as in CT can be used, and that the resulting incremental controller has the same structure as the CT incremental controller. Taken together, these results give us a systematic and computationally efficient framework to ensure and shape closed-loop incremental dissipativity of both CT and DT nonlinear systems, with the CT case and DT case having the same considerations.

8

Discrete-Time Extension of the Universal Shifted Theory

THIS chapter discusses the extension of the *Continuous-Time* (CT) universal shifted results what we have presented in Chapter 4 to *Discrete-Time* (DT) systems. We show that analogous to the CT results, we can use a velocity representation of the system in order to analyze universal shifted stability and performance. In DT, this velocity representation corresponds to the *time-difference dynamics* of the system. Moreover, like in CT, we show how the resulting analysis problem of the DT velocity representation can be cast as a standard *Linear Parameter-Varying* (LPV) analysis problem. We also show how these results can be used in order to construct a systematic controller synthesis method, which achieves closed-loop universal shifted stability and performance. Finally, we investigate and analyze the capabilities of the controller synthesis method through a simulation study, which shows stronger and stability and performance guarantees that we achieve, which are also advantageous compared to a standard LPV controller design.

8.1 Introduction

In Chapter 7, we have already presented how we can extend our *Continuous-Time* (CT) incremental results of Chapters 5 and 6 to *Discrete-Time* (DT) systems. In this chapter, we will show how we can also extend the CT universal shifted results of Chapter 4 to DT. In CT, we have shown how we can use the velocity form and analysis and synthesis results of the *Linear Parameter-Varying* (LPV) framework to arrive at systematic and computationally efficient methods for analysis and controller synthesis in order to guarantee universal shifted stability and performance.

While the notion of universal shifted dissipativity for DT system is shortly discussed in (Simpson-Porco 2019) in order to analyze stability and performance w.r.t. all equilibrium points of the system, there are to the author's knowledge no results available in literature which also try to ensure this notion through control. As aforementioned, in CT, we have seen how the velocity form, i.e., the time-differentiated dynamics, imply stability and performance properties w.r.t. (non-zero) equilibrium points, which also has been investigated in several other works (Leith and Leithead 1998a; Kosaraju et al. 2019; Schweidel and Arcak 2022) In DT, the time-difference dynamics, analogous to the time-differentiated dynamics in CT, have primarily received attention in the context (nonlinear) *Model Predictive Control* (MPC) methods. In the context of MPC, the time-difference dynamics have been used in order to penalize or constraint the differences between consecutive samples and have also been use in the context of offset free tracking (Ferramosca et al. 2009; Cisneros et al. 2016). However, to the author's knowledge, there have not been results in literature which connect the time-difference dynamics to stability and performance guarantees w.r.t. equilibrium points in the nonlinear context.

Therefore, as a contribution of this chapter, we will present how the time-difference dynamics can be used in order to analyze universal shifted stability and performance of the system, analogous to the CT results in Chapter 4. Moreover, we will also show how the analysis of time-difference dynamics can be performed through the LPV framework and how we can formulate controller synthesis results on the basis of it to ensure closed-loop universal shifted stability and performance.

First, in Section 8.2, we will discuss universal shifted stability and performance in the context of DT systems. Next, in Section 8.3, we will present how the DT velocity based analysis, i.e., analysis based on the time-difference dynamics, can be connected to universal shifted stability and performance properties of the system. In Section 8.4, we show how velocity based analysis can in DT also be performed through the LPV framework. In Section 8.5, we present a systematic controller synthesis method to guarantee universal shifted stability and performance on the basis of the time-difference dynamics and the LPV framework. Next, in Section 8.6, we demonstrate the proposed controller synthesis method through a simulation study. Finally, in Section 8.7, conclusions are drawn on the presented results.

8.2 Universal Shifted Stability and Performance

Like we have considered in Section 2.2, we consider DT nonlinear dynamical systems given by

$$x(t+1) = f(x(t), w(t)); \tag{8.1a}$$
$$z(t) = h(x(t), w(t)); \tag{8.1b}$$

where $t \in \mathbb{N}_0$ is the discrete-time instant, $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$ the state with initial condition $x(0) = x_0 \in \mathbb{R}^{n_\mathrm{x}}$, $w(t) \in \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{w}}$ is the input of the system, and $z(t) \in \mathcal{Z} \subseteq \mathbb{R}^{n_\mathrm{z}}$ is the output of the system. The sets $\mathcal{X}$ and $\mathcal{W}$ are assumed to be open and convex, containing the origin. Moreover, the functions $f : \mathcal{X} \times \mathcal{W} \to \mathcal{X}$ and $h : \mathcal{X} \times \mathcal{W} \to \mathcal{Z}$ are considered to be in $\mathcal{C}_1$, i.e., $f, h \in \mathcal{C}_1$. We define the set of solutions of (8.1) as

$$\mathfrak{B} := \{(x, w, z) \in (\mathcal{X} \times \mathcal{W} \times \mathcal{Z})^{\mathbb{N}_0} \mid (x, w, z) \text{ satisfy } (8.1)\}, \tag{8.2}$$

and the behavior of (8.1) for a specific input trajectory $\bar{w} \in \mathcal{W}^{\mathbb{N}_0}$, by

$$\mathfrak{B}_\mathrm{w}(\bar{w}) := \{(x, w, z) \in \mathfrak{B} \mid w = \bar{w} \in \mathcal{W}^{\mathbb{N}_0}\}. \tag{8.3}$$

For the nonlinear system given by (8.1), the equilibrium points satisfy

$$x_* = f(x_*, w_*); \tag{8.4a}$$
$$z_* = h(x_*, w_*); \tag{8.4b}$$

where $x_* \in \mathcal{X}$, $w_* \in \mathcal{W}$ and $z_* \in \mathcal{Z}$. The set of equilibrium points is then defined as

$$\mathscr{E} := \{(x_*, w_*, z_*) \in \mathcal{X} \times \mathcal{W} \times \mathcal{Z} \mid (x_*, w_*, z_*) \text{ satisfy } (8.4)\}. \tag{8.5}$$

Define $\mathscr{X} := \pi_{\mathrm{x}_*} \mathscr{E}$, $\mathscr{W} := \pi_{\mathrm{w}_*} \mathscr{E}$, and $\mathscr{Z} := \pi_{\mathrm{z}_*} \mathscr{E}$. Similar as we did for CT systems, we assume Assumption 4.1 is satisfied.

In DT, we consider the same definition for universal shifted stability as has been considered in CT, i.e., a system given by (8.1) is universally shifted stable if it is stable w.r.t. to all its forced equilibrium points, see again Definition 4.1. This then allows for a trivial extension of the results of Theorem 4.1 to discrete time systems:

Theorem 8.1 (Universal shifted Lyapunov stability in DT)**.** *The nonlinear system given by (8.1) is universally shifted stable, if there exists a function* $V_\mathrm{s} : \mathcal{X} \times \mathscr{W} \to \mathbb{R}_0^+$ *with* $V_\mathrm{s}(\cdot, w_*) \in \mathcal{C}_1$ *and* $V_\mathrm{s}(\cdot, w_*) \in \mathcal{Q}_{x_*}$ *for every* $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$*, such that, for every* $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$*, it holds that*

$$V_\mathrm{s}(x(t+1), w_*) - V_\mathrm{s}(x(t), w_*) \leq 0, \tag{8.6}$$

for all $t \in \mathbb{N}_0$ *and* $x \in \pi_\mathrm{x} \mathfrak{B}_\mathrm{w}(w \equiv w_*)$*. If (8.6) holds, but with strict inequality except when* $x(t) = x_*$*, then the system is universally shifted asymptotically stable.*

Proof. The proofs follows similarly as the proof of the CT version (i.e., Theorem 4.1) in Appendix B.2 and therefore it is not repeated. □

For universal shifted dissipativity, we can also formulate the extension of Definition 4.2 to DT, similar to the DT version of classical dissipativity (see Definition 2.13), which gives us the following definition:

Definition 8.1 (Universal shifted dissipativity of DT systems)**.** *The nonlinear system given by (8.1) is universally shifted dissipative w.r.t. the supply function* $s_{\mathrm{s}} : \mathcal{W} \times \mathscr{W} \times \mathcal{Z} \times \mathscr{Z}$, *if there exists a storage function* $\mathcal{V}_{\mathrm{s}} : \mathcal{X} \times \mathscr{W} \to \mathbb{R}_0^+$ *with* $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{C}_0$ *and* $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{Q}_{x_*}$ *for every* $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$, *such that*

$$\mathcal{V}_{\mathrm{s}}(x(t_1+1), w_*) - \mathcal{V}_{\mathrm{s}}(x(t_0), w_*) \leq \sum_{t=t_0}^{t_1} s_{\mathrm{s}}(w(t), w_*, z(t), z_*), \tag{8.7}$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ *with* $t_1 \geq t_0$ *and* $(x, w, z) \in \mathfrak{B}$.

As the universal shifted stability and dissipativity conditions in DT are similar to those in CT, we can trivially extend a few of the CT results in Section 4.2 to DT systems. Namely, we can trivially extend Theorem 4.2 to DT, which shows that universal shifted (Q, S, R) dissipativity implies classical (Q, S, R) dissipativity of the system (considering $(0, 0, 0) \in \mathscr{E}$). Furthermore, Theorem 4.3 can trivially be extended to DT, which shows how universal shifted dissipativity implies universal shifted stability under restrictions of the supply function. As both these results are trivially extended to DT, they are not repeated.

The CT universal shifted performance notions and results we have discussed in Section 4.2.4 can also easily be extended to DT. Namely, for the universal shifted extension to the ℓ_p-ℓ_q-gain (see also Definition 2.11), we consider the following definition:

Definition 8.2 (Universal shifted $\mathcal{L}_p$-$\mathcal{L}_q$-gain)**.** *A nonlinear system given by (8.1) is said to have a finite universal shifted* ℓ_p-ℓ_q*-gain, if there is a finite* $\gamma \geq 0$ *and function* $\zeta_{\mathrm{s}} : \mathcal{X} \times \mathscr{X} \to \mathbb{R}$ *such that for every* $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$ *it holds that*

$$\|z - z_*\|_{q,T} \leq \gamma \|w - w_*\|_{p,T} + \zeta_{\mathrm{s}}(x_0, x_*), \tag{8.8}$$

for all $T \geq 0$ *and* $(x, w, z) \in \mathfrak{B}$ *with* $w \in \ell_{\mathrm{pe}}$. *The induced universal shifted* ℓ_p-ℓ_q*-gain of (8.1), denoted as* $\ell_{\mathrm{s}p}$-$\ell_{\mathrm{s}q}$*-gain, is the infimum of* γ *such that (8.8) still holds. If* $p = q$, *we will refer to this as the (induced) universal shifted* ℓ_p*-gain, denoted as* $\ell_{\mathrm{s}p}$*-gain.*

Using this definition, the result of Lemma 4.2 can easily be extended to DT to show that universal shifted (Q, S, R) dissipativity of a DT nonlinear system given by (8.1) with $(Q, S, R) = (\gamma^2, 0, -I)$ implies that the system has a $\ell_{\mathrm{s}2}$-gain bound of γ.

The CT version of the definition of universal shifted passivity given in Definition 4.4 can also directly be applied to DT systems. This means that the system given by (8.1) is universally shifted passive is it is universally shifted (Q, S, R) dissipative with $(Q, S, R) = (0, I, 0)$.

8.3 Discrete-Time Velocity based Analysis

8.3.1 The DT velocity form and velocity dissipativity

For CT systems in Section 4.3, we have shown how the universal shifted stability and performance properties of nonlinear systems could be analyzed through the time-differentiated dynamics, i.e., velocity form of the system. In DT, the counterpart to time-differentiation is taking difference of the dynamics in time. By doing this, we get a new representation which we will show can be used to imply universal shifted stability and performance properties in DT.

Let us introduce $x_\Delta(t) := x(t+1)-x(t) \in \mathbb{R}^{n_\mathrm{x}}$, $w_\Delta(t) := w(t+1)-w(t) \in \mathbb{R}^{n_\mathrm{w}}$, and $z_\Delta(t) := z(t+1)-z(t) \in \mathbb{R}^{n_\mathrm{z}}$. Based on these variables, we have that time-difference dynamics of (8.1) can be expressed as

$$x_\Delta(t+1) = f(x(t+1), w(t+1)) - f(x(t), w(t)); \tag{8.9a}$$
$$z_\Delta(t) = h(x(t+1), w(t+1)) - h(x(t), w(t)). \tag{8.9b}$$

By Lemma C.1.1, we can then equivalently write the time-difference dynamics (8.9) in an alternative form, which we will refer to as the (DT) velocity form:

Definition 8.3 (Discrete-time velocity form)**.** *For a nonlinear system given by (8.1), the velocity form is*

$$x_\Delta(t+1) = \bar{A}_\mathrm{v}\big(\varsigma(t+1), \varsigma(t)\big)x_\Delta(t) + \bar{B}_\mathrm{v}\big(\varsigma(t+1), \varsigma(t)\big)w_\Delta(t); \tag{8.10a}$$
$$z_\Delta(t) = \bar{C}_\mathrm{v}\big(\varsigma(t+1), \varsigma(t)\big)x_\Delta(t) + \bar{D}_\mathrm{v}\big(\varsigma(t+1), \varsigma(t)\big)w_\Delta(t); \tag{8.10b}$$

where $(x, w, z) \in \mathfrak{B}$, $\varsigma = \mathrm{col}(x, w)$, *and*

$$\begin{aligned}
\bar{A}_\mathrm{v}(x_+, w_+, x, w) &= \int_0^1 \frac{\partial f}{\partial x}(\bar{x}(\lambda), \bar{w}(\lambda))\, d\lambda,\\
\bar{B}_\mathrm{v}(x_+, w_+, x, w) &= \int_0^1 \frac{\partial f}{\partial w}(\bar{x}(\lambda), \bar{w}(\lambda))\, d\lambda,\\
\bar{C}_\mathrm{v}(x_+, w_+, x, w) &= \int_0^1 \frac{\partial h}{\partial x}(\bar{x}(\lambda), \bar{w}(\lambda))\, d\lambda,\\
\bar{D}_\mathrm{v}(x_+, w_+, x, w) &= \int_0^1 \frac{\partial h}{\partial w}(\bar{x}(\lambda), \bar{w}(\lambda))\, d\lambda,
\end{aligned} \tag{8.11}$$

with $\bar{x}(\lambda) = x + \lambda(x_+ - x)$, $\bar{w}(\lambda) = w + \lambda(w_+ - w)$.

Let us define the operator Δ for the behavior $\mathfrak{B}$ of (8.1), given by (8.2), such that

$$\begin{aligned}
\Delta\mathfrak{B} = \big\{&(x_\Delta, w_\Delta, z_\Delta) \in (\mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}} \times \mathbb{R}^{n_\mathrm{z}})^{\mathbb{R}_0^+} \mid \\
&x_\Delta(t) = x(t+1) - x(t),\ w_\Delta(t) = w(t+1) - w(t),\\
&z_\Delta(t) := z(t+1) - z(t),\ \forall\, t \in \mathbb{N}_0,\ (x, w, z) \in \mathfrak{B}\big\}.
\end{aligned} \tag{8.12}$$

The solution set of (8.10) is then given by $\mathfrak{B}_\mathrm{v} := \Delta\mathfrak{B}$, and we can also define $\mathfrak{B}_{\mathrm{v},\mathrm{w}}(w) := \Delta\mathfrak{B}_\mathrm{w}(w)$ for a $w \in \mathcal{W}^{\mathbb{N}_0}$.

The DT velocity form represents the dynamics of the change between consecutive time-instances of the original dynamics. This is analogous to the CT velocity form, whereby the velocity form represents the dynamics of the instantaneous change in time (i.e., time derivative) of the original dynamics. Similar to the CT results, we will show that the DT velocity form also connects to universal shifted stability and performance. Before presenting this connection, we will first present our analysis results based on the DT velocity form.

Based on the DT velocity form, we formulate the following definition for DT velocity dissipativity, analogous to the CT definition in Definition 4.6:

Definition 8.4 (DT velocity dissipativity)**.** *The nonlinear system given by (8.1) is velocity dissipative w.r.t. the supply function s_v, if there exists a storage function $\mathcal{V}_\mathrm{v} : \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}_0^+$ with $\mathcal{V}_\mathrm{v} \in \mathcal{C}_1$ and $\mathcal{V}_\mathrm{v} \in \mathcal{Q}_0$, such that, for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_1 \geq t_0$,*

$$\mathcal{V}_\mathrm{v}(x_\Delta(t_1+1)) - \mathcal{V}_\mathrm{v}(x_\Delta(t_0)) \leq \sum_{t=t_0}^{t_1} s_\mathrm{v}(w_\Delta(t), z_\Delta(t)), \tag{8.13}$$

for all $(x_\Delta, w_\Delta, z_\Delta) \in \mathfrak{B}_\mathrm{v}$.

Similar to the CT case, the DT version of velocity dissipativity can be seen as 'classical dissipativity' (see Definition 2.13) of the velocity form of the system. Moreover, analogous to the CT condition in Lemma 4.3, we can also derive a condition for DT velocity dissipativity.

Lemma 8.1 (Condition for DT velocity dissipativity)**.** *If there exists a storage function $\mathcal{V}_\mathrm{v} : \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}_0^+$ with $\mathcal{V}_\mathrm{v} \in \mathcal{C}_1$ and $\mathcal{V}_\mathrm{v} \in \mathcal{Q}_0$, such that, for all values $w_+, w \in \mathcal{W}$ and $x \in \mathcal{X}$,*

$$\begin{aligned}\mathcal{V}_\mathrm{v}\left(\bar{A}_\mathrm{v}(x_+, x, w_+, w)(x_+ - x) + \bar{B}_\mathrm{v}(x_+, x, w_+, w)(w_+ - w)\right) - \mathcal{V}_\mathrm{v}(x_+ - x) \leq \\ s_\mathrm{v}(w_+ - w, \bar{C}_\mathrm{v}(x_+, x, w_+, w)(x_+ - x) + \bar{D}_\mathrm{v}(x_+, x, w_+, w)(w_+ - w)),\end{aligned} \tag{8.14}$$

where $x_+ = f(x, w)$, then the nonlinear system given by (8.1) is velocity dissipative w.r.t. the supply function s_v.

Proof. See Appendix B.6. □

In CT, we could formulate an (infinite dimensional) *Linear Matrix Inequality* (LMI) feasibility condition for velocity dissipativity by considering a quadratic storage function and a quadratic (Q, S, R) supply function, see Theorem 4.4. We will show that in DT, we can formulate a similar condition. For this, we will consider quadratic (Q, S, R) supply functions for velocity dissipativity of the form

$$s_\mathrm{v}(w_\Delta, z_\Delta) = \begin{bmatrix} w_\Delta \\ z_\Delta \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w_\Delta \\ z_\Delta \end{bmatrix}, \tag{8.15}$$

where again $Q \in \mathbb{S}^{n_\mathrm{w}}$, $S \in \mathbb{R}^{n_\mathrm{w} \times n_\mathrm{z}}$, and $R \in \mathbb{S}^{n_\mathrm{z}}$. Moreover, we also consider the storage function $\mathcal{V}_\mathrm{v}$ to be quadratic:

$$\mathcal{V}_\mathrm{v}(x_\Delta) = x_\Delta^\top M x_\Delta \tag{8.16}$$

where $M \in \mathbb{S}^{n_\mathrm{x}}$ with $M \succ 0$. Under these considerations, we can derive the following theorem:

Theorem 8.2 (DT velocity (Q, S, R) dissipativity condition)**.** *The system given by (8.1) is velocity (Q, S, R) dissipative where $R \preceq 0$, if there exists an $M \in \mathbb{S}^{n_\mathrm{x}}$ with $M \succ 0$, such that for all $(x, w) \in \mathcal{X} \times \mathcal{W}$, it holds that*

$$(\star)^\top \begin{bmatrix} -M & 0 \\ \star & M \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\mathrm{v}(x, w) & B_\mathrm{v}(x, w) \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\mathrm{v}(x, w) & D_\mathrm{v}(x, w) \end{bmatrix} \preceq 0, \tag{8.17}$$

where $A_\mathrm{v} = \frac{\partial f}{\partial x}$, $B_\mathrm{v} = \frac{\partial f}{\partial w}$, $C_\mathrm{v} = \frac{\partial h}{\partial x}$, $D_\mathrm{v} = \frac{\partial h}{\partial w}$.

Proof. See Appendix B.6. □

From the proof of Theorem 8.2, it follows that the condition for velocity (Q, S, R) dissipativity in DT, can be expressed in terms of the matrix functions $A_\mathrm{v}, \ldots, D_\mathrm{v}$ instead of the matrix functions $\bar{A}_\mathrm{v}, \ldots, \bar{D}_\mathrm{v}$ of the DT velocity form[1] (8.10). This results in a similar condition for velocity (Q, S, R) dissipativity as in CT in Theorem 4.4, which also uses $A_\mathrm{v}, \ldots, D_\mathrm{v}$. Expressing the condition for velocity (Q, S, R) dissipativity in DT in terms of $A_\mathrm{v}, \ldots, D_\mathrm{v}$ instead of $\bar{A}_\mathrm{v}, \ldots, \bar{D}_\mathrm{v}$ simplifies the condition. Namely, $A_\mathrm{v}, \ldots, D_\mathrm{v}$ only depend on two arguments, which results in the condition needing to be verified at all $(x, w) \in \mathcal{X} \times \mathcal{W}$. On the other hand, a condition using $\bar{A}_\mathrm{v}, \ldots, \bar{D}_\mathrm{v}$, which take four arguments, would need to be verified at all $(x, w) \in \mathcal{X} \times \mathcal{W}$ *and* all $(x_+, w_+) \in \mathcal{X} \times \mathcal{W}$.

Analogous to the CT condition in Theorem 4.4, we have that the condition in Theorem 8.2 corresponds to a feasibility check of an infinite dimensional set of LMIs, as for a fixed $(x, w) \in \mathcal{X} \times \mathcal{W}$ (8.17) becomes an LMI. Later, in Section 8.4, we will see how we can use tools from LPV analysis to reduce this infinite dimensional set of LMIs to a finite dimensional set, which can computationally efficiently be verified. This will then gives us computationally efficient tools to analyze velocity (Q, S, R) dissipativity of a system, like the ones we have developed for CT systems in Section 4.4.

8.3.2 Induced universal shifted stability

In Section 4.3.3, we have shown how the velocity form in CT can be used to formulate a condition to imply universal shifted stability of a system. Likewise, we will show that also in DT, we can formulate a condition for universal shifted stability of a system using the DT velocity form. Before doing so, let us first introduce the

[1] Do note that $\bar{A}_\mathrm{v}, \ldots, \bar{D}_\mathrm{v}$ are integrated versions (over the time difference) of $A_\mathrm{v}, \ldots, D_\mathrm{v}$.

behavior $\mathfrak{B}_{\mathrm{v},\mathscr{W}} := \bigcup_{w_* \in \mathscr{W}} \mathfrak{B}_{\mathrm{v,w}}(w \equiv w_*)$, i.e., the behavior of the velocity form for which the input $w(t) = w_* \in \mathscr{W}$, and hence $w_\Delta(t) = 0$, for all $t \in \mathbb{N}_0$.

Theorem 8.3 (Implied universal shifted stability)**.** *The nonlinear system given by (8.1) is universally shifted stable, if there exists a function $V_\mathrm{v} : \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}_0^+$ with $V_\mathrm{v} \in \mathcal{C}_1$ and $V_\mathrm{v} \in \mathcal{Q}_0$, such that*

$$V_\mathrm{v}(x_\Delta(t+1)) - V_\mathrm{v}(x_\Delta(t)) \leq 0, \tag{8.18}$$

for all $t \in \mathbb{N}_0$ and $x_\Delta \in \pi_{\mathrm{x}_\Delta} \mathfrak{B}_{\mathrm{v},\mathscr{W}}$. If (8.18) holds, but with strict inequality except when $x_\Delta(t) = 0$, then the system is universally shifted asymptotically stable.

Proof. See Appendix B.6. □

Like the proof for Theorem 4.5, also the proof for Theorem 8.3 relies on the construction of the universally shifted Lyapunov function based on V_v. While in CT, a similar construction is often referred to as the Krasovskii method (Khalil 2002; Kawano et al. 2021), to the author's knowledge, an equivalent result in DT like we have presented here is not available in literature. Moreover, to the author's knowledge, this is also the first time that properties of the time-difference dynamics have been connected to universal shifted stability.

Analogous to the CT condition, we can also interpret the DT condition (8.18) as the velocity form being (asymptotically) stable.

In CT, we have shown in Theorem 4.6 how velocity dissipativity implies universal shifted stability, under restriction of s_v. Similarly, we can trivially extend the results of Theorem 4.6 to DT systems for which the same condition on the supply function holds (see (4.22)), therefore it is not repeated.

With these results, we have shown so far that DT versions of the velocity form and velocity dissipativity can also be used in order to imply universal shifted stability of nonlinear systems, analogous to our CT results in Section 4.3.3.

8.3.3 Induced universal shifted dissipativity

Next, we are interested if velocity (Q, S, R) dissipativity also implies universal shifted (Q, S, R) dissipativity. In CT, in Section 4.3.4, we have not been able to proof completely that velocity (Q, S, R) dissipativity implies universal shifted (Q, S, R) dissipativity, i.e., Proposition 4.1. Nevertheless, we have shown results in Section 4.3.4 that link these dissipativity concepts under some assumptions. While also in DT, we are not able to completely proof Proposition 4.1, we will present results that link the two, which are analogous to the CT results in Section 4.3.4.

Like we considered in CT, instead of considering nonlinear system given by (8.1), we will also restrict ourselves here to nonlinear systems that can be represented as

$$x(t+1) = f(x(t)) + Bw(t); \tag{8.19a}$$

$$y(t) = Cx(t). \tag{8.19b}$$

Also for the DT system given by (8.1), we can always transform (8.1) to the form (8.19) at the cost of increasing state dimension, e.g., by appending appropriate filters (see the CT result in Appendix C.2.1). For (8.19), we will also assume in this section that $\mathcal{X}$, besides being convex, is compact.

For a nonlinear system given by (8.19), the equilibrium points $(x_*, w_*, z_*) \in \mathscr{E}$ satisfy

$$x_* = f(x_*) + Bw_*; \tag{8.20a}$$
$$y_* = Cx_*; \tag{8.20b}$$

and the velocity form of (8.19) is given by

$$x_\Delta(t) = \bar{A}_\mathrm{v}(x(t+1), x(t))x_\Delta(t) + Bw_\Delta(t); \tag{8.21a}$$
$$z_\Delta(t) = Cx_\Delta(t); \tag{8.21b}$$

for which now $\bar{A}_\mathrm{v}(x_+, x) = \int_0^1 \frac{\partial f}{\partial x}(x + \lambda(x_+ - x))\, d\lambda$.

Analogous to the CT results, we will next connect velocity (Q, S, R) dissipativity for (Q, S, R) tuples for which $S = 0$, $Q \succeq 0$, and $R \preceq 0$ to universal shifted performance notions that can be characterized by a similar (Q, S, R) universal shifted supply function. Like in CT, we assume that Assumption 4.2 holds for (8.19), i.e., for (8.19), we assume $CB = 0$. Moreover, we take the following assumptions, analogous to Assumptions 4.3 and 4.4 in CT:

Assumption 8.1. *Given a matrix $T \in \mathbb{S}^{n_\mathrm{z}}$ with $T \preceq 0$, assume that there exists an $\alpha \in \mathbb{R}^+$ such that for all $x_* \in \mathscr{X}$ and $x \in \mathcal{X}$*

$$(\star)^\top TC\left(\bar{A}_\mathrm{v}(x, x_*) - I\right)(x - x_*) \leq \alpha^{-1}(\star)^\top TC(x - x_*). \tag{8.22}$$

Assumption 8.2. *For a given $(x_*, w_*, z_*) \in \mathscr{E}$, assume that w is generated by the exosystem*

$$w(t+1) = A_\mathrm{w}(w(t) - w_*) + w_*, \tag{8.23}$$

where $A_\mathrm{w} \in \mathbb{R}^{n_\mathrm{w} \times n_\mathrm{w}}$ is Schur and $\|A_\mathrm{w} - I\| \leq \beta$. Define the corresponding behavior as

$$\mathfrak{W} := \left\{w \in \mathcal{W}^{\mathbb{N}_0} \mid w \text{ satisfies } (8.23)\right\}. \tag{8.24}$$

Under these assumptions, we can show the following result, similar to Theorem 4.7:

Theorem 8.4 (Universal shifted performance from velocity dissipativity)**.** *If a nonlinear system given by (8.19) is velocity (Q, S, R) dissipative for the (Q, S, R) tuple where $S = 0$, $Q \succeq 0$, and $R \preceq 0$, , where R satisfies Assumption 8.1, Assumptions 4.2 and 8.2 hold for every $(x_*, w_*, z_*) \in \mathscr{E}$, and $x_\Delta(0) = 0$, then for every $(x_*, w_*, z_*) \in \mathscr{E}$, it holds that*

$$\sum_{t=0}^{T} \beta^2(\star)^\top Q(z(t) - w_*) + \alpha^{-1}(\star)^\top R(z(t) - z_*) > 0, \tag{8.25}$$

for all $T > 0$ and $(w, z) \in \pi_{\mathrm{w},\mathrm{z}}\,\mathfrak{B}$ with $w \in \mathfrak{W}$.

Proof. See Appendix B.6. □

Similarly, applying the result of Theorem 8.4 to the (Q, S, R) tuple $(Q, S, R) = (\gamma^2 I, 0, -I)$, corresponding the (universal shifted) $\mathcal{L}_2$-gain, we obtain the following corollary:

Corollary 8.1 (Bounded $\mathcal{L}_{\mathrm{s}2}$-gain from velocity dissipativity)**.** *If a nonlinear system given by (8.19) is velocity (Q, S, R) dissipative for $(Q, S, R) = (\gamma^2 I, 0, -I)$, where $R = -I$ satisfies Assumption 8.1, Assumptions 4.2 and 8.2 hold for every $(x_*, w_*, z_*) \in \mathscr{E}$, and $x_\Delta(0) = 0$, then the system has an $\mathcal{L}_{\mathrm{s}2}$-gain bound of $\tilde{\gamma} = \sqrt{\alpha\beta^2\gamma^2}$.*

Proof. The proof follows in a similar manner as the proof of Corollary 4.1 in Appendix B.2. □

Like in CT, due to technicalities, Theorem 8.4 and Corollary 8.1 are what we can prove in terms of the connection between velocity (Q, S, R) dissipativity and universal shifted (Q, S, R) dissipativity. Nonetheless, based on the results that we obtain for controller synthesis (see Section 8.6), there are also strong indications for DT nonlinear systems that Proposition 4.1 holds true.

8.4 Convex Universal Shifted Analysis

In the previous section, we have shown how the DT velocity form (8.10) can be used to formulate the concept of velocity dissipativity which can then be used in order to imply universal shifted stability and performance properties of the primal form of the system. This is similar to what we have shown for CT systems in Chapter 4. Furthermore, we have shown in Theorem 8.2 how velocity (Q, S, R) dissipativity in DT could be analyzed through a feasibility check of an infinite dimensional set of LMIs. Next, we will discuss how we can make the analysis computationally feasible through the use of methods from the LPV framework.

In Section 4.4, we have shown in the CT case that the CT velocity form could be embedded in an LPV representation, obtaining a so-called *Velocity Parameter-Varying* (VPV) embedding. Through the VPV embedding, we can cast the velocity (Q, S, R) dissipativity analysis problem as a classical (Q, S, R) dissipativity problem of the VPV embedding. This is also directly connected to the matrix inequality condition (4.20) in Theorem 4.4. Namely, through the VPV embedding, the matrix inequality condition (4.20) becomes equivalent to the LPV based matrix inequality condition (2.37).

In the DT case, the matrix inequality condition for velocity (Q, S, R) dissipativity given in Theorem 8.2 uses the matrix functions $A_\mathrm{v}, \ldots, D_\mathrm{v}$ instead of matrix functions $\bar{A}_\mathrm{v}, \ldots, \bar{D}_\mathrm{v}$ corresponding to the (DT) velocity form (8.10) (which are integrated versions of $A_\mathrm{v}, \ldots, D_\mathrm{v}$). While, as aforementioned, a similar condition could be formulated that uses $\bar{A}_\mathrm{v}, \ldots, \bar{D}_\mathrm{v}$, it would significantly increase the complexity of the condition, as the resulting condition would need to hold for all $(x, w) \in \mathcal{X} \times \mathcal{W}$

and all $(x_+, w_+) \in \mathcal{X} \times \mathcal{W}$ (corresponding to all the arguments of $\bar{A}_\mathrm{v}, \ldots, \bar{D}_\mathrm{v}$). The current condition (8.17) only needs to hold at all $(x, w) \in \mathcal{X} \times \mathcal{W}$. While we could also construct an LPV embedding of the DT velocity form (8.10), this would result in a more complex embedding procedure due the increased complexity of $\bar{A}_\mathrm{v}, \ldots, \bar{D}_\mathrm{v}$ (compared to $A_\mathrm{v}, \ldots, D_\mathrm{v}$).

Nonetheless, despite this difference between the matrix inequality condition and the velocity form in DT, we can still use the LPV framework to arrive at an LPV based analysis condition. Namely, instead of embedding the DT velocity form in an LPV representation, we just 'embed' the matrix functions $A_\mathrm{v}, \ldots, D_\mathrm{v}$ (used in (8.17)) in LPV matrix functions $A, \ldots, D$. More concretely, for a region $\mathscr{X} \times \mathscr{W} \subseteq \mathcal{X} \times \mathcal{W}$, we construct the LPV matrix functions A, B, C, D, belonging to a given class of functions $\mathfrak{A}$ (e.g., affine or rational), and function $\eta : \mathscr{X} \times \mathscr{W} \to \mathcal{P}$ with $p = \eta(x, w)$ and $\mathcal{P} \supseteq \eta(\mathscr{X}, \mathscr{W})$, such that, $\forall\, (x, w) \in \mathscr{X} \times \mathscr{W}$:

$$\begin{aligned} A(\eta(x,w)) &= A_\mathrm{v}(x,w), & B(\eta(x,w)) &= B_\mathrm{v}(x,w), \\ C(\eta(x,w)) &= C_\mathrm{v}(x,w), & D(\eta(x,w)) &= D_\mathrm{v}(x,w). \end{aligned} \tag{8.26}$$

If $\mathscr{X} \times \mathscr{W} = \mathcal{X} \times \mathcal{W}$, then condition (8.17) can equivalently be verified by ensuring

$$(\star)^\top \begin{bmatrix} -M & 0 \\ \star & M \end{bmatrix} \begin{bmatrix} I & 0 \\ A(p) & B(p) \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C(p) & D(p) \end{bmatrix} \preceq 0, \tag{8.27}$$

for all $p \in \mathcal{P}$, which is equivalent to the condition for classical (Q, S, R) dissipativity of an LPV state-space representation in (2.37) (considering M in (2.37) is a constant matrix).

Therefore, also in DT, we can still check velocity (Q, S, R) dissipativity through the LPV framework. Combined with the results of Section 8.3, we can also analyze universal shifted stability and performance through the LPV framework. This also give us in DT a systematic and computationally efficient tool for universal shifted stability and performance analysis of nonlinear systems, similar to what we have shown for CT systems.

While we have shown in Chapter 7 that the analysis results for incremental stability and performance in CT and DT are equivalent, with the results presented in this chapter, we have now seen that for universal shifted stability and performance there are some differences between the CT and DT results for analysis. Namely, while in both CT and DT we can formulate a velocity form to analyze universal shifted properties, the velocity forms represent the time-differentiated dynamics in CT, while it represents the time-difference dynamics in DT. Due to these differences, slightly different assumptions have to be considered for the results presented in Section 8.3.3. Similarly, in this section, we have also shown how we do not directly use a VPV embedding in DT to make the velocity based analysis results computationally feasible. Nevertheless, in the end, we can still come to analogous results in CT and DT for universal shifted stability and performance analysis based on the velocity form.

In the next section, we will show how we can also develop a systematic controller synthesis method in order to ensure and shape universal shifted stability and performance through the use of the DT velocity form.

8.5 Convex Universal Shifted Synthesis

8.5.1 Controller synthesis problem

Like we considered for the universal shifted (and incremental) controller synthesis problem in CT, we again make use of the generalized plant concept in order systematically and intuitively ensure and shape universal shifted stability and performance requirements. In this case, we consider a DT nonlinear generalized plant of the form

$$x(t+1) = f(x(t)) + B_{\mathrm{w}}w(t) + B_{\mathrm{u}}u(t); \tag{8.28a}$$
$$z(t) = h_{\mathrm{z}}(x(t)) + D_{\mathrm{zw}}w(t) + D_{\mathrm{zu}}u(t); \tag{8.28b}$$
$$y(t) = C_{\mathrm{y}}x(t) + D_{\mathrm{yw}}w(t); \tag{8.28c}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^{n_{\mathrm{x}}}$, $w(t) \in \mathcal{W} \subseteq \mathbb{R}^{n_{\mathrm{w}}}$ and $z(t) \in \mathcal{Z} \subseteq \mathbb{R}^{n_{\mathrm{z}}}$ are the state, generalized disturbance and generalized performance signals of the plant, respectively, and where $u(t) \in \mathcal{U} \subseteq \mathbb{R}^{n_{\mathrm{u}}}$ is the control input and $y(t) \in \mathcal{Y} \subseteq \mathbb{R}^{n_{\mathrm{y}}}$ is the measured output. The sets $\mathcal{X}$, $\mathcal{W}$, $\mathcal{U}$, $\mathcal{Z}$ and $\mathcal{Y}$ are open and convex, containing the origin. The functions $f : \mathcal{X} \to \mathcal{X}$ and $h_{\mathrm{z}} : \mathcal{X} \to \mathcal{Z}$ are assumed to be in $\mathcal{C}_1$, i.e. $f, h_{\mathrm{z}} \in \mathcal{C}_1$. Furthermore, $B_{\mathrm{w}} \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{w}}}$, $B_{\mathrm{u}} \in \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{u}}}$, $D_{\mathrm{zw}} \in \mathbb{R}^{n_{\mathrm{z}} \times n_{\mathrm{w}}}$, $D_{\mathrm{zu}} \in \mathbb{R}^{n_{\mathrm{z}} \times n_{\mathrm{u}}}$, and $D_{\mathrm{yw}} \in \mathbb{R}^{n_{\mathrm{y}} \times n_{\mathrm{w}}}$. The solution set of (8.28) will from here on be denoted by $\mathfrak{B}$ and given by

$$\mathfrak{B} := \Big\{ (x, w, u, z, y) \in (\mathcal{X} \times \mathcal{W} \times \mathcal{U} \times \mathcal{Z} \times \mathcal{Y})^{\mathbb{N}_0} \mid (x, w, u, z, y) \text{ satisfies } (8.28) \Big\}. \tag{8.29}$$

Remark 8.1. Note that while (8.28) might seem restrictive, a larger class of nonlinear generalized plants of the form

$$x(t+1) = f(x(t), u(t)) + B_{\mathrm{w}}w(t); \tag{8.30a}$$
$$z(t) = h_{\mathrm{z}}(x(t), u(t)) + D_{\mathrm{zw}}w(t); \tag{8.30b}$$
$$y(t) = h_{\mathrm{y}}(x(t)) + D_{\mathrm{yw}}w(t); \tag{8.30c}$$

where $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$, $h_{\mathrm{z}} : \mathcal{X} \times \mathcal{U} \to \mathcal{Z}$ and $h_{\mathrm{y}} : \mathcal{X} \to \mathcal{Y}$ with $f, h_{\mathrm{z}}, h_{\mathrm{y}} \in \mathcal{C}_1$, can be written in the form (8.28) by interconnecting appropriate filters to u and y of (8.30). See Appendix C.2.2 for an example of such a procedure in CT.

The to-be-designed controller K for the generalized plant P (8.28) is considered to be of the form

$$x_{\mathrm{k}}(t+1) = f_{\mathrm{k}}(x_{\mathrm{k}}(t), u_{\mathrm{k}}(t)); \tag{8.31a}$$
$$y_{\mathrm{k}}(t) = h_{\mathrm{k}}(x_{\mathrm{k}}(t), u_{\mathrm{k}}(t)); \tag{8.31b}$$

where $x_{\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}}$ is the state of the controller, $u_{\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{u_k}}}$ is its input, and $y_{\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{y_k}}}$ is its output. Moreover, $f_{\mathrm{k}} : \mathbb{R}^{n_{\mathrm{x_k}}} \times \mathbb{R}^{n_{\mathrm{u_k}}} \to \mathbb{R}^{n_{\mathrm{x_k}}}$ and $h_{\mathrm{k}} : \mathbb{R}^{n_{\mathrm{x_k}}} \times \mathbb{R}^{n_{\mathrm{u_k}}} \to \mathbb{R}^{n_{\mathrm{y_k}}}$.

The closed-loop interconnection of P given by (8.28) and K given by (8.31) for which $u_{\mathrm{k}} = y$ and $u = y_{\mathrm{k}}$ is denoted by $\mathcal{F}_{\mathrm{l}}(P, K)$. Note that this closed-loop interconnection will be a system of the form (8.1) with input w and output z. Moreover, we assume that (8.28) is a generalized plant in the sense that there exists a controller K such that the closed-loop interconnection $\mathcal{F}_{\mathrm{l}}(P, K)$ is universally shifted stable, see also Definition 4.8 and Proposition 4.2.

Our objective for controller synthesis is then to synthesize a controller K such that $\mathcal{F}_{\mathrm{l}}(P, K)$ is universally shifted stable and satisfies a (desired) universal shifted performance criteria. Like for the CT results in Chapter 4, the procedure will be exemplified by considering the $\ell_{\mathrm{s}2}$-gain and universal shifted performance criteria, which we will aim to minimize. However, the procedure can also be used when other (Q, S, R) performance notions are considered.

8.5.2 Universal shifted controller synthesis procedure

Overview

In this section, we will discuss the proposed controller synthesis method to achieve universal shifted stability and performance in DT. As aforementioned, for this we will exploit of the DT velocity form, similar to the CT case. Like in CT, to simplify the discussion, we will also assume Proposition 4.1 to be true for DT system, meaning we will consider that velocity (Q, S, R) dissipativity of a system implies universal shifted (Q, S, R) dissipativity of that system for the same tuple (Q, S, R).

In CT, we have shown how we can embed the velocity form in an LPV representation to obtain a VPV embedding. This has been used to accommodate the controller synthesis procedure, see Section 4.5.2. However, as has been discussed in Section 8.4, doing the same in DT leads to a more complex embedding procedure. Next, we will show that we can actually use the differential form and DT incremental controller synthesis results that have been presented in Section 7.5 to ensure universal shifted stability and performance. While incremental (Q, S, R) dissipativity implies universal shifted (Q, S, R) dissipativity, see Theorems 5.3 and 7.2, we will show that under some restrictions the DT incremental controller design can be simplified. Under these restrictions, we cannot ensure incremental (Q, S, R) dissipativity anymore, however, they will imply DT velocity (Q, S, R) dissipativity. Through the presented velocity (Q, S, R) dissipativity results (Section 8.3), we will then ensure universal shifted stability and performance of the closed-loop.

Next, we will give an overview of the proposed procedure, after which more details are given on the specifics of the procedure and how the aforementioned implications will follow. While the procedure will use the DT incremental controller synthesis results that have been presented in Section 7.5, we will present for clarity all the keys steps and results that are required for the proposed DT universal shifted controller design.

Similar to the CT universal shifted and incremental controller synthesis methods, we propose a three step procedure in order to obtain a DT controller which ensures universal shifted stability and a bounded ℓ_{s2}-gain of the closed-loop system:

1. For the generalized plant given by (8.28), its differential form is computed, which is then used to construct a *Differential Parameter-Varying* (DPV) embedding of (8.28).
2. For the DPV embedding of the generalized plant, an LPV controller is synthesized to ensure a minimal closed-loop $\mathcal{L}_2$-gain γ.
3. The previously synthesized LPV controller is realized into a primal form such that the resulting closed-loop interconnection of the primal form of the generalized plant and realized primal form of the controller is universally shifted stable and its ℓ_{s2}-gain is bounded by γ.

The first two steps of this proposed approach are equivalent to the first two steps of the proposed DT incremental controller design approach in Section 7.5. However, in the third and final step, we make restrictions to the realization procedure. As aforementioned, the restrictions will then be used to simplify the resulting realization procedure. Under these simplification, we will still be able to imply DT velocity (Q, S, R) dissipativity and consequently universal shifted stability and performance of the closed-loop.

DPV embedding of the generalized plant

For the first step in the procedure, the differential form of (8.28) is computed and embedded in an LPV representation, resulting in a DPV embedding of (8.28). The differential form of (8.28) is given by

$$x_\delta(t+1) = A_\delta(\bar{x}(t))x_\delta(t) + B_\mathrm{w}z_\delta(t) + B_\mathrm{u}u_\delta(t); \tag{8.32a}$$
$$z_\delta(t) = C_{\delta,\mathrm{z}}(\bar{x}(t))x_\delta(t) + D_\mathrm{zw}z_\delta(t) + D_\mathrm{zu}u_\delta(t); \tag{8.32b}$$
$$y_\delta(t) = C_\mathrm{y}x_\delta(t) + D_\mathrm{yw}z_\delta(t); \tag{8.32c}$$

where $A_\delta = \frac{\partial f}{\partial x}$, $C_{\delta,\mathrm{z}} = \frac{\partial h_\mathrm{z}}{\partial x}$ with $\bar{x} \in \pi_\mathrm{x}\,\mathfrak{B}$, $x_\delta(t) \in \mathbb{R}^{n_\mathrm{x}}$, $u_\delta(t) \in \mathbb{R}^{n_\mathrm{u}}$, $w_\delta(t) \in \mathbb{R}^{n_\mathrm{w}}$, $z_\delta(t) \in \mathbb{R}^{n_\mathrm{z}}$ and $y_\delta(t) \in \mathbb{R}^{n_\mathrm{y}}$. We denote the set of solutions of (8.32) along a $\bar{x} \in \pi_\mathrm{x}\,\mathfrak{B}$ solution of (8.28) by $\mathfrak{B}_\delta(\bar{x})$ (see also (6.7)) with the complete solution set denoted by $\breve{\mathfrak{B}}_\delta = \bigcup_{\bar{x}\in\pi_\mathrm{x}\,\mathfrak{B}} \mathfrak{B}_\delta(\bar{x})$, see also (6.6) and (6.7).

The differential form, given by (8.32), is then embedded in an LPV representation such that we obtain a DPV embedding of (8.28), see also Definitions 6.2 and 7.5. We assume that the DPV embedding is constructed on the region $\mathcal{X} \subseteq \mathcal{X}$, where $\mathcal{X}$ is compact, and is given by

$$x_\delta(t+1) = A(p(t))x_\delta(t) + B_\mathrm{w}z_\delta(t) + B_\mathrm{u}u_\delta(t); \tag{8.33a}$$
$$z_\delta(t) = C_\mathrm{z}(p(t))x_\delta(t) + D_\mathrm{zw}z_\delta(t) + D_\mathrm{zu}u_\delta(t); \tag{8.33b}$$
$$y_\delta(t) = C_\mathrm{y}x_\delta(t) + D_\mathrm{yw}z_\delta(t); \tag{8.33c}$$

where $p(t) \in \mathcal{P} \subset \mathbb{R}^{n_\mathrm{p}}$ is the scheduling-variable with corresponding scheduling-map $\eta : \mathcal{X} \to \mathcal{P}$, s.t. $p(t) = \eta(\bar{x}(t))$. Moreover, the matrix functions A, C_z are assumed to be of a user chosen function class $\mathfrak{A}$, such as affine, polynomial, rational, etc. Furthermore, for the controller synthesis problem to be feasible, we assume that for (8.33), (A, B_u) is stabilizable and (A, C_y) is detectable, see Definitions 2.17 and 2.18.

The behavior of the DPV embedding (8.33) for a given scheduling trajectory $p \in \mathcal{P}^{\mathbb{N}_0}$ is denoted by $\mathfrak{B}_\mathrm{p}(p)$ and the full behavior of the DPV embedding is denoted by (i.e., for all $p \in \mathcal{P}^{\mathbb{N}_0}$) $\breve{\mathfrak{B}}_\mathrm{p}$, similar as for LPV systems, see (2.27) and (2.28). As the DPV embedding is considered on $\mathcal{X} \subseteq \mathcal{X}$, we will denote the restriction of the state solutions of the generalized plant (8.28) to $\mathcal{X}$ by $\mathfrak{B}_\mathcal{X} := \{(x, u, w, z, y) \in \mathfrak{B} \mid x(t) \in \mathcal{X}\}$. The corresponding restriction of the solution set of the differential form (8.32) is denoted by $\breve{\mathfrak{B}}_{\delta,\mathcal{X}} := \cup_{(\bar{x} \in \pi_\mathrm{x} \mathfrak{B}_\mathcal{X})} \mathfrak{B}_\delta(\bar{x})$. Using these behaviors, we have through Lemma 5.3 (which also holds in DT, see Section 7.3) that $\breve{\mathfrak{B}}_{\delta,\mathcal{X}} \subseteq \breve{\mathfrak{B}}_\mathrm{p}$. This means that the behavior of the differential form of the plant (8.32) is included in the behavior of the DPV embedding (8.33).

Next, we use the DPV embedding (8.33) in order to be able to use the convex controller synthesis results of the LPV framework.

Controller synthesis for the DPV embedding

In the second step of the procedure, we use the DPV embedding (8.33) to be able to use the convex synthesis results of the LPV framework to synthesize a controller. More concretely, we synthesize an LPV controller for (8.33) in order to ensure the closed-loop has a bounded (and minimal) ℓ_2-gain. The synthesized LPV controller is considered to be of the form:

$$x_{\delta,\mathrm{k}}(t+1) = A_\mathrm{k}(p(t))x_{\delta,\mathrm{k}}(t) + B_\mathrm{k}(p(t))u_{\delta,\mathrm{k}}(t); \tag{8.34a}$$
$$y_{\delta,\mathrm{k}}(t) = C_\mathrm{k}(p(t))x_{\delta,\mathrm{k}}(t) + D_\mathrm{k}(p(t))u_{\delta,\mathrm{k}}(t); \tag{8.34b}$$

where $x_\mathrm{k}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}}$ is the state, $u_\mathrm{k}(t) \in \mathbb{R}^{n_\mathrm{y}}$ is the input and $y_\mathrm{k}(t) \in \mathbb{R}^{n_\mathrm{u}}$ is the output of the controller and $A_\mathrm{k}, \ldots, D_\mathrm{k} \in \mathfrak{A}$. As also mentioned in Section 7.5.2, various methods exist to synthesize a controller (8.34) such that the closed-loop ℓ_2-gain is minimized, see e.g. (Apkarian and Gahinet 1995; M. Ali and Werner 2011; De Caigny et al. 2012) and also Section 2.5.3. The synthesized controller then gives us the following result (also given in Theorem 7.4):

Theorem 8.5 (Differential closed-loop ℓ_2-gain)**.** *The closed-loop interconnection of the differential form of the generalized plant P_δ given by (8.32) and the controller K_δ given by (8.34), denoted by $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$, is classically dissipative and has an ℓ_2-gain bounded by γ for all $\bar{x} \in \pi_{\bar{\mathrm{x}}} \mathfrak{B}_\mathcal{X}$, if the closed-loop interconnection of the DPV embedding of the generalized plant P_dpv given by (8.33) and K_δ, denoted by $\mathcal{F}_\mathrm{l}(P_\mathrm{dpv}, K_\delta)$, is classically dissipative with a bounded ℓ_2-gain of γ for all $p \in \mathcal{P}^{\mathbb{N}_0}$.*

Proof. See the proof of Theorem 7.4 in Appendix B.5. □

Similar to the CT and DT incremental controller synthesis procedures, we assume that Assumption 6.1 holds in order to facilitate the realization procedure in the next step. Consequently, we assume that the synthesis problem has been solved such that $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$ is classically dissipative with a quadratic (differential) storage function of the form $\mathcal{V}_\delta(x_\mathrm{cl}, x_{\delta,\mathrm{cl}}) = x_{\delta,\mathrm{cl}}^\top M x_{\delta,\mathrm{cl}}$, where $M \succ 0$, $x_\mathrm{cl} = \mathrm{col}(x, x_\mathrm{k})$, and $x_{\delta,\mathrm{cl}} = \mathrm{col}(x_\delta, x_{\delta,\mathrm{k}})$.

DT universal shifted controller realization

In the final step of the procedure, we use the synthesized LPV controller (8.34) of the previous step in order to realize our universal shifted controller that ensures closed-loop universal shifted stability and performance.

This is achieved using the following idea. For the incremental controller realization in CT in Section 6.3.5 and similarly in DT in Section 7.5.2, a path integral realization is used to converge towards a desired steady-state response $\vartheta = (x^*, w^*, u^*, z^*, y^*) \in \mathfrak{B}_\mathcal{X}$. As ϑ is any trajectory in $\mathfrak{B}_\mathcal{X}$ in the incremental realization, we obtain our incremental stability and incremental performance guarantees. For the DT universal shifted controller realization instead of considering any $\vartheta \in \mathfrak{B}_\mathcal{X}$, we restrict ourselves to a single specific trajectory. Namely, we consider ϑ to be the current trajectory shifted one time instance back. Based on this restriction, we will show that we can simplify the incremental controller realization (7.32) (which is equivalent to the CT version (6.10)). Moreover, as ϑ is now restricted to the current trajectory shifted one time instance back, we have stability and performance guarantees w.r.t. this time shifted trajectory. This can then be connected to the DT velocity form, which expresses the time-difference dynamics, and therefore we can connect it to DT velocity dissipativity. Through velocity dissipativity, we can then establish the connection to universal shifted stability and performance through the results in Sections 8.3.2 and 8.3.3. This idea is formalized in the following theorem:

Theorem 8.6 (DT universal shifted controller realization)**.** *Consider the controller K_δ, given by (8.34), synthesized for P_δ, given by (8.32), such that the closed-loop is classically dissipative and has a bounded ℓ_2-gain of γ on $\mathcal{X}$, satisfying Assumption 6.1. Moreover, consider the nonlinear controller K, omitting dependence on time for brevity, given by*

$$q\,\breve{x}_\mathrm{k} = \breve{A}_\mathrm{k}(x, x_-)\breve{x}_\mathrm{k} + \breve{B}_\mathrm{k}(x, x_-)u_\mathrm{k}; \tag{8.35a}$$

$$y_\mathrm{k} = \breve{C}_\mathrm{k}(x, x_-)\breve{x}_\mathrm{k} + \breve{D}_\mathrm{k}(x, x_-)u_\mathrm{k}; \tag{8.35b}$$

where q is the discrete time-shift operator, $x_- = x(t-1)$, $\breve{x}_\mathrm{k}(t) \in \mathbb{R}^{n_{\mathrm{u}_\mathrm{k}}+n_{\mathrm{x}_\mathrm{k}}+n_{\mathrm{y}_\mathrm{k}}}$ is the state of the controller, and where

$$\breve{A}_\mathrm{k}(x, x_-) = \begin{bmatrix} 0 & 0 & 0 \\ -\bar{B}_\mathrm{k}(x, x_-) & \bar{A}_\mathrm{k}(x, x_-) & 0 \\ -\bar{D}_\mathrm{k}(x, x_-) & \bar{C}_\mathrm{k}(x, x_-) & I \end{bmatrix}, \qquad \breve{B}_\mathrm{k}(x, x_-) = \begin{bmatrix} I \\ \bar{B}_\mathrm{k}(x, x_-) \\ \bar{D}_\mathrm{k}(x, x_-) \end{bmatrix},$$
$$\breve{C}_\mathrm{k}(x, x_-) = \begin{bmatrix} -\bar{D}_\mathrm{k}(x, x_-) & \bar{C}_\mathrm{k}(x, x_-) & I \end{bmatrix}, \qquad \breve{D}_\mathrm{k}(x, x_-) = \bar{D}_\mathrm{k}(x, x_-), \tag{8.36}$$

with

$$\begin{aligned}
\bar{A}_{\mathrm{k}}(x, x_-) &= \int_0^1 A_{\mathrm{k}}\Big(\eta\big(x_- + \lambda(x - x_-)\big)\Big) d\lambda, \\
\bar{B}_{\mathrm{k}}(x, x_-) &= \int_0^1 B_{\mathrm{k}}\Big(\eta\big(x_- + \lambda(x - x_-)\big)\Big) d\lambda, \\
\bar{C}_{\mathrm{k}}(x, x_-) &= \int_0^1 C_{\mathrm{k}}\Big(\eta\big(x_- + \lambda(x - x_-)\big)\Big) d\lambda, \\
\bar{D}_{\mathrm{k}}(x, x_-) &= \int_0^1 D_{\mathrm{k}}\Big(\eta\big(x_- + \lambda(x - x_-)\big)\Big) d\lambda.
\end{aligned} \tag{8.37}$$

The controller K given by (8.35) ensures for the generalized plant P given by (8.28) that the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P, K)$ is velocity (Q, S, R) dissipative for $(Q, S, R) = (\gamma^2 I, 0, -I)$ on $\mathcal{X}$.

Proof. See Appendix B.6. □

Through Theorem 8.6, we have that the controller given by (8.35) ensures closed-loop (Q, S, R) velocity dissipativity with $(Q, S, R) = (\gamma^2 I, 0, -I)$. From the results presented in Sections 8.3.2 and 8.3.3, this implies universal shifted stability and quadratic universal shifted performance of the closed-loop. Therefore, we will refer to the controller given by (8.35) as the (DT) universal shifted controller. Moreover, under Proposition 4.1, we can imply universal shifted (Q, S, R) dissipativity of the closed-loop system with $(Q, S, R) = (\gamma^2 I, 0, -I)$. This then implies a closed-loop $\mathcal{L}_{\mathrm{s}2}$-gain bound of γ. As aforementioned, for the controller in Theorem 8.6, other (Q, S, R) performance metrics can also be considered. For the universal shifted controller, we require knowledge of the state[2] at the previous time instant (x_- in (8.35)). At initialization, the previous time instant of the state is not known. Therefore, for implementation of the controller, the previous time instant of the state is taken equal to its current value.

Note that similar as for CT universal shifted and incremental results, see Sections 4.5.3 and 6.3.6, one can consider a set $\tilde{\mathcal{W}}$ such that for $w \in \tilde{\mathcal{W}}^{\mathbb{N}_0} \cap \ell_{2\mathrm{e}}$ the state $x(t)$ stays in the region $\mathcal{X}$ for all $t \in \mathbb{N}_0$, i.e., $\mathcal{X}$ is invariant. Meaning that we stay in the set on which the controller was designed (specifically the set that is considered during the second step, where an LPV controller for the DPV embedding is synthesized). However, as aforementioned, computing the set $\tilde{\mathcal{W}}$ is challenging.

The realization of the universal shifted controller K given by (8.35) based on the LPV controller K_δ given by (8.34) can also be interpreted as connecting a time-difference operator and time-summation operator to the input and output of K_δ, respectively. This is similar to the interpretation of the CT universal shifted realization whereby time differentiation and time integration operators are connected, see Figure 4.3.

As mentioned in Section 8.4, for universal shifted analysis using the velocity form, there are a few differences between the CT and DT case due to the differences

[2]Only the part of the state that the scheduling-map η depends on is required to be measured.

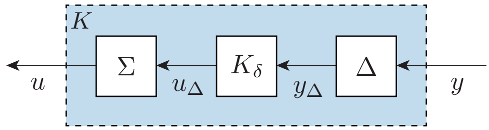


Figure 8.1: Universal shifted controller realization K in DT through time difference and time summation operations Δ and Σ, respectively.

between the CT and DT velocity forms. Similarly, as we have seen in this section, the different velocity forms in CT and DT also result in a different structure for the resulting universal shifted controller (see Theorem 4.12 and Theorem 8.6). While for the incremental dissipativity based controller synthesis results, the CT and DT incremental controller have the same structure, this is not the case for the CT and DT universal shifted controller. Nonetheless, we still have analogous results and similar steps for the universal shifted controller design procedures in CT and DT.

8.5.3 Reference tracking and disturbance rejection

The presented DT universal shifted controller design, analogous to the CT design, makes use of the velocity form and velocity dissipativity to ensure universal shifted stability and performance of the closed-loop. Like in CT, this has as an advantage that explicit knowledge of the (closed-loop) equilibrium points is not required. However, it is still important that the controller is designed in such a way that we converge toward the desired point, especially in reference tracking applications.

In Section 4.5.4, we have proposed the use of explicit integral action in the controller, which has also resulted in a simplified realization procedure. Here, in DT, we propose a similar solution. Like mentioned in Section 4.5.4, for reference tracking and disturbance rejection purposes, we assume the measured output of the plant to be of the form $y = \mathrm{col}(y_1, y_2)$, where y_1 contains signals to be tracked and y_2 contains other to-be-controlled variables. In order to achieve (constant) reference tracking for y_1, we can include a DT (integral) filter in the loop (of the generalized) plant, like we have depicted in Figure 4.4.

In DT, an integration filter corresponds to summation over time. Therefore, we can simplify the realization of the controller in Theorem 8.6, as it can be interpreted as appending time summation and time difference operators to the synthesized LPV controller K_δ, see Figure 8.1. Similar to the CT result in Corollary 4.2, we can then formulate the following corollary:

Corollary 8.2 (Universal shifted realization with integral action)**.** *Consider a generalized plant which includes an explicit DT integrator filter of the form* $M(q) = \frac{q+\alpha}{q-1}$ *(where q is the discrete time-shift operator and $-1 < \alpha < 1$) in the loop, such that the (to-be-designed) controller K and M connect such as depicted in Figure 4.4 where y_2 is empty, i.e., $y_1 = y$, see also Figure 4.5. For K_δ given by (8.34), the interconnection of the primal realization of the controller K and M can be expressed* *as (8.35) where* $\breve{B}_\mathrm{k}$ *and* $\breve{D}_\mathrm{k}$ *are given as in (8.36), and* $\breve{A}_\mathrm{k}$ *and* $\breve{C}_\mathrm{k}$ *are given by*

$$\begin{aligned}\breve{A}_\mathrm{k}(x,x_-) &= \begin{bmatrix} 0 & 0 & 0 \\ \alpha\bar{B}_\mathrm{k}(x,x_-) & \bar{A}_\mathrm{k}(x,x_-) & 0 \\ \alpha\bar{D}_\mathrm{k}(x,x_-) & \bar{C}_\mathrm{k}(x,x_-) & I \end{bmatrix}, \\ \breve{C}_\mathrm{k}(x,x_-) &= \begin{bmatrix} \alpha\bar{D}_\mathrm{k}(x,x_-) & \bar{C}_\mathrm{k}(x,x_-) & I \end{bmatrix},\end{aligned} \tag{8.38}$$

where again $\bar{A}_\mathrm{k}$, $\bar{B}_\mathrm{k}$, $\bar{C}_\mathrm{k}$, *and* $\bar{D}_\mathrm{k}$ *are given as in (8.37).*

Proof. See Appendix B.6. □

8.6 Example

In this section, we will demonstrate the proposed DT universal shifted controller design through a simulation study. Moreover, the resulting controller will also be compared to a standard LPV controller design.

Example 8.1 *(DT universal shifted control)***.** We consider the DT nonlinear system that has also been considered in Example 7.2. This DT nonlinear system is given by

$$x_{\mathrm{g},1}(t+1) = 0.1x_{\mathrm{g},1}(t) - x_{\mathrm{g},2}(t); \tag{8.39a}$$
$$x_{\mathrm{g},2}(t+1) = 0.9\sin(x_{\mathrm{g},1}) + x_{\mathrm{g},2} + u_\mathrm{g}(t); \tag{8.39b}$$
$$y_\mathrm{g}(t) = x_{\mathrm{g},1}(t). \tag{8.39c}$$

For this plant we want to design a controller which achieves constant reference tracking.

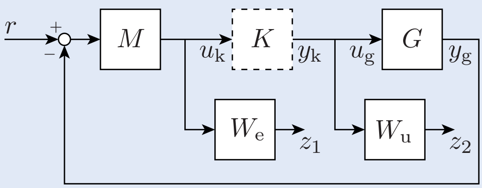


Figure 8.2: Generalized plant.

We consider the same generalized plant structure and weightings filters that have been considered in Example 7.2 for this system. This generalized plant structure is depicted in Figure 8.2, where G is the system given by (8.39), K is the to-be-synthesized controller, r is the reference, $W_\mathrm{e}(q) = \frac{0.2(q-0.5)}{q+\alpha}$, $M(q) = \frac{q+\alpha}{q-1}$, and $W_\mathrm{u} = 0.2$, where $\alpha = \frac{1}{\pi}$.
For universal shifted controller synthesis procedure described in Section 8.5, we require a DPV embedding of the generalized plant. As the plant (8.39) is the only nonlinear system in the generalized plant (the weighting filters are linear), we only require computation of the differential form of (8.39) (as the dynamics of differential

form of an LTI system are equivalent to its primal form). The following DPV embedding on the region $\mathcal{X}$ of (8.39) is taken:

$$x_{\delta,\mathrm{g},1}(t+1) = 0.1x_{\delta,\mathrm{g},1}(t) - x_{\delta,\mathrm{g},2}(t); \tag{8.40a}$$

$$x_{\delta,\mathrm{g},1}(t+1) = 0.9p(t)x_{\delta,\mathrm{g},1}(t) + x_{\delta,\mathrm{g},2}(t) + u_{\delta,\mathrm{g}}(t); \tag{8.40b}$$

$$y_{\delta,\mathrm{g}}(t) = x_{\delta,\mathrm{g},1}(t); \tag{8.40c}$$

where $p(t) = \cos(x_{\mathrm{g},1}(t)) \in [-1, 1]$ such that $\eta(x_\mathrm{g}) = \cos(x_{\mathrm{g},1})$ with $x_\mathrm{g}(t) \in \mathcal{X} = \mathbb{R}^2$. Note that this is the same DPV embedding that is considered in Example 7.2. For the synthesis of the $\mathcal{L}_2$-gain optimal LPV controller in the second step of the procedure, the polytopic LPV synthesis method in the LPVcore Toolbox (Boef et al. 2021) is used. The resulting closed-loop of the DPV embedding of the generalized plant and LPV controller achieves an $\mathcal{L}_2$-gain of 1.1. This means, under Proposition 4.1, that the primal closed-loop achieves an $\mathcal{L}_{\mathrm{s}2}$-gain of 1.1. As the generalized plant contains the integral weighting filter M, we also use the result of Corollary 8.2 to simplify the interconnection of the realized universal shifted controller and the filter M.

For comparison, we will also consider the standard LPV controller and incremental controller that are designed in Example 7.2 for the same system. These controllers consider the same generalized plant structure and weighting filters as the universal shifted controller design. The standard LPV controller achieves a closed-loop ℓ_2-gain bound of 0.80, while the incremental controller achieves a closed-loop $\mathcal{L}_{\mathrm{i}2}$-gain of 1.1.

The closed-loop systems with universal shifted controller, standard LPV controller, and incremental controller are simulated for a reference $r \equiv 1$ and $r \equiv 2$. The trajectories of the closed-loop systems for these controllers can be found in Figure 8.3. From the figure, it can be seen that while the universal shifted controller has slower settling time than the LPV controller, for both references the universal shifted controller achieves similar tracking behavior and it asymptotically converges towards the reference. On the other hand, the output of the plant under the LPV controller ends up in a limit cycle for $r \equiv 2$.

Compared to the incremental controller design, the universal shifted controller displays quite different behavior, as can be seen in Figure 8.3. Namely, for the same constant reference, the incremental controller overshoots the to-be-followed reference, while the universal shifted controller does not have any overshoot. The universal shifted controller also has slightly higher settling time than the incremental controller. This difference is partly due to the feedforward action that is inherent to the incremental controller design, which is not present in the universal shifted controller design. Despite these differences, both the incremental controller and the universal shifted controller achieve there desired global stability and performance guarantees.

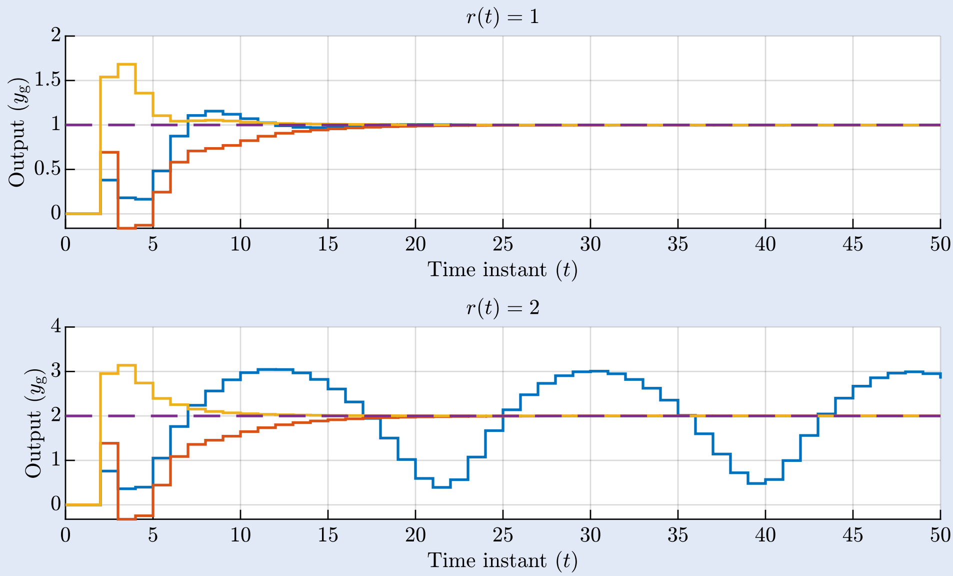


Figure 8.3: Output response of the closed-loop of the plant with standard LPV controller (—), the universal shifted controller (—), and incremental controller (—) for the reference trajectory (- -).

8.7 Conclusions

In this chapter, we have presented extensions of the CT universal shifted analysis and controller synthesis methods to DT systems. We have seen that similar to the CT results, we could analyze universal shifted stability and performance through analysis of the velocity form of the system, which in DT represents the time-difference dynamics. Analogous to the CT case, we have shown that also in DT velocity (Q, S, R) dissipativity implies universal shifted stability and under assumptions also quadratic universal shifted performance. Moreover, we have also shown how the analysis of velocity (Q, S, R) dissipativity can be cast as an LPV analysis problem, similar to what we have shown in CT. Based on the DT velocity form and using the DT incremental controller synthesis results, we have also shown that we could then obtain a systematic and computationally efficient universal shifted controller synthesis procedure for DT nonlinear systems, also making use of LPV methods. Combined with the CT results of Chapter 4, this gives us a systematic and computationally efficient framework to ensure and shape closed-loop universal shifted stability and performance of both CT and DT nonlinear systems.

9

Scheduling Dimension Reduction of LPV Models

SCHEDULING *Dimension Reduction* (SDR) methods allow for a reduction of the number of scheduling-variables of an *Linear Parameter-Varying* (LPV) model based on a given data set. This allows for reduced complexity when the resulting system is used for analysis and or controller synthesis and also reducing conservatism by taking into account the expected behavior of the underlying system through the data set. In this chapter, the existing SDR methods are reviewed and a *Deep (Artifical) Neural Network* (DNN) approach is developed that achieves higher model accuracy under scheduling dimension reduction. Moreover, it is shortly discussed how the DNN method can be used in order to automatically obtain an affine LPV or *Differential Parameter-Varying* (DPV) embedding of the nonlinear system based on a given data set. The proposed DNN method and existing SDR methods are compared on a two-link robotic manipulator example, both in terms of model accuracy and performance of controllers synthesized with the reduced models. Compared to existing methods, the DNN method achieves improvements to modeling error and closed-loop performance when used for controller synthesis compared to existing methods.

9.1 Introduction

In the previous chapters, we have developed an extensive framework to analyze and synthesize controllers for nonlinear systems in order to guarantee global stability and performance requirements, making use of the tools of the *Linear Parameter-Varying* (LPV) framework. While the powerful tools for analysis and synthesis of LPV representations have matured, the embedding of nonlinear systems in LPV representations is still underdeveloped and depends on the expertise of the user. Some automated procedures exists for direct embedding of the nonlinear system, see e.g. (Shamma and Cloutier 1993; Kwiatkowski, Bol, et al. 2006; Casella et al. 2009; Tóth 2010), but it is generally true that these procedures and heuristic methods can easily lead to an LPV model with a large amount of scheduling-variables. For the methods discussed in the previous chapters, the *velocity* and *differential form* of the system are used, which are embedded in LPV representations. These embeddings are called the *Velocity Parameter-Varying* (VPV) and *Differential Parameter-Varying* (DPV) embedding of a nonlinear system, respectively. Constructing a VPV or DPV embedding is easier compared to embedding the primal form in an LPV representation. This is because the velocity and the differential form are already in a factorized form. Nonetheless, the construction of a VPV or DPV embedding under a given scheduling dependency is still done heuristically. Therefore, this can still be challenging and lead to a large amount of scheduling-variables.

A large amount of scheduling-variables is an issue when the embedding is used for analysis or controller synthesis because the complexity of these procedures is proportional to the number of scheduling-variables. Limiting the number of scheduling-variables in the embedding is therefore highly important for tractable LPV analysis and controller synthesis. Moreover, during the embedding procedure, control objectives are not taken into account, hence the LPV embedding can be conservative for the required control objectives. Hence, the objective of *Scheduling Dimension Reduction* (SDR) is to reduce this conservativeness, which can be achieved by taking into account a set of scheduling-variable trajectories associated with the expected (closed-loop) behavior of the system. In the literature, based on this concept, several data-based methods have been proposed to reduce the scheduling dimension of LPV models. These methods only focus on the reduction of the number of scheduling-variables and not on the reduction of the amount of states (referred to as model reduction), for which also several methods exist, see e.g. (Wood et al. 1996; Tóth et al. 2012; Theis et al. 2018). Methods for reducing the amount of scheduling-variables based on data include methods based on *Principal Component Analysis* (PCA) (Kwiatkowski and Werner 2008; Sadeghzadeh, Sharif, et al. 2020), *Kernel Principal Component Analysis* (KPCA) (Rizvi, Mohammadpour Velni, et al. 2016) and *Autoencoders* (AEs) (Rizvi, Abbasi, et al. 2018). Other methods for dimensionality reduction for LPV models, not relying on data, exist, where the LPV model is represented by a *Linear Fractional Representation* (LFR), see e.g. (Varga et al. 1998; Hecker and Varga 2005; Beck 2006). These methods were not considered in this work as they aim only to the reduction of the extracted Δ-block using a controllability/observability argument, not the reduction of the number of scheduling-variables.

While KPCA and the AE method use nonlinear mappings to construct the new scheduling-variable, significantly improving the PCA method, they require an extra optimization step in order to synthesize an inverse transformation that enables the realization of the reduced LPV state-space model with affine dependency. However, due to this extra optimization step, one can quickly lose advantage over the PCA method, for which this is not required. Hence, the PCA method remains the most reliable approach in practice. Therefore, in this chapter, a novel method is developed, which like the AE method, proposed in (Rizvi, Abbasi, et al. 2018), uses an *Artifical Neural Network* (ANN) in order to perform SDR, but from which the matrices of the reduced LPV state-space model can directly be extracted. This avoids the use of a second optimization step, hence, leads to better results. Moreover, the addition of hidden layers in the encoding layer is proposed to handle more complex scheduling mappings. The chapter is structured as follows, in Section 9.2, a mathematical problem definition of the SDR problem will be given. Section 9.3 gives an overview of the existing SDR methods. In Section 9.4, the developed *Deep (Artifical) Neural Network* (DNN) approach is explained. An application of SDR methods to the automated affine LPV embedding of a nonlinear system in primal form and velocity/differential form is discussed in Section 9.5. A comparison of the DNN and existing methods is given in Section 9.6 on a two-link planar robot manipulator example. On this example, we compare the modeling error as a result of the different SDR methods and the achieved performance of the controller that is synthesized based on the reduced model. Finally, in Section 9.7, conclusions on the given results are drawn.

9.2 Scheduling Dimension Reduction Problem

Consider a nonlinear dynamical system with a state-space representation given by

$$\begin{aligned} \xi x(t) &= f(x(t), u(t)); \\ y(t) &= h(x(t), u(t)); \end{aligned} \tag{9.1}$$

where $x(t) \in \mathbb{R}^{n_\mathrm{x}}$ is the state variable associated with the considered state-space representation of the system, $u(t) \in \mathbb{R}^{n_\mathrm{u}}$ is the input, $y(t) \in \mathbb{R}^{n_\mathrm{y}}$ is the output of the system, and $t \in \mathcal{T}$ is time. In the *Continuous-Time* (CT) case, $\xi = \frac{d}{dt}$ and $\mathcal{T} = \mathbb{R}_0^+$, and in the *Discrete-Time* (DT) case, $\xi = q$ (i.e., the forward time shift operator) and $\mathcal{T} = \mathbb{N}_0$. The functions f and h are defined as $f : \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{u}} \to \mathbb{R}^{n_\mathrm{x}}$ and $h : \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{u}} \to \mathbb{R}^{n_\mathrm{y}}$ and assumed to be Lipschitz continuous. As considered in the LPV literature and as described in Section 2.5.1, the embedding of the nonlinear system (9.1) in an LPV representation, corresponds to constructing

$$\begin{aligned} \xi x(t) &= A(p(t))x(t) + B(p(t))u(t); \\ y(t) &= C(p(t))x(t) + D(p(t))u(t); \end{aligned} \tag{9.2}$$

where $p(t) \in \mathbb{R}^{n_\mathrm{p}}$ is the scheduling-variable and there existing a function $\eta : \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{u}} \to \mathbb{R}^{n_\mathrm{p}}$, called the scheduling-map, such that $\eta(x(t), u(t)) = p(t)$. This proxy representation of (9.1) is used during control synthesis by confining $p(t)$ into a compact convex set $\mathcal{P} \subset \mathbb{R}^{n_\mathrm{p}}$ and synthesizing a controller that ensures stability and performance of (9.2) under all possible variations of $p(t) \in \mathcal{P}$. Therefore the embedding is constructed on the compact sets $\mathcal{X}$, $\mathcal{U}$, where $x \in \mathcal{X}$ and $u \in \mathcal{U}$, such that $\mathcal{P} \supseteq \eta(\mathcal{X}, \mathcal{U})$, often taken as the convex hull of $\eta(\mathcal{X}, \mathcal{U})$. Moreover, unless specified otherwise, it is assumed that the embedding of (9.1) is performed such that the resulting LPV representation has an affine scheduling dependency, meaning that

$$L(p) = \begin{bmatrix} A(p) & B(p) \\ C(p) & D(p) \end{bmatrix} = \underbrace{\begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix}}_{L_0} + \sum_{i=1}^{n_\mathrm{p}} \underbrace{\begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix}}_{L_i} p_i, \tag{9.3}$$

where $p = \mathrm{col}(p_1, \ldots, p_{n_\mathrm{p}})$ and $L_i \in \mathbb{R}^{m \times n}$, $\forall\, i \in \mathbb{I}_1^{n_\mathrm{p}}$ with $m = n_\mathrm{x} + n_\mathrm{y}$, $n = n_\mathrm{x} + n_\mathrm{u}$.

By restricting the scheduling dependency to being affine, the scheduling-map η can contain many nonlinear functions that are dependent on the same elements of x and u. Hence, the variation of scheduling-variables are not independent from each other, which contributes to the conservativeness of the LPV model (Shamma and Athans 1992). This can have dramatic effects on the capability of the LPV synthesis to find a stabilizing controller for (9.1) via (9.2) with acceptable performance. Therefore, reducing conservativeness through SDR can help with attaining performance requirements. Given an LPV embedding (9.2) of (9.1) and a set of nominal scheduling-variable trajectories of (9.2), denoted by $\mathfrak{D}$ that correspond to typical expected behavior of (9.1) through η, find an LPV embedding given by

$$\begin{aligned} \xi x(t) &= \hat{A}(\varphi(t))x(t) + \hat{B}(\varphi(t))u(t); \\ y(t) &= \hat{C}(\varphi(t))x(t) + \hat{D}(\varphi(t))u(t); \end{aligned} \tag{9.4}$$

which approximates (9.2) under all trajectories of $p \in \mathfrak{D}$, where $\varphi(t) \in \Phi \subset \mathbb{R}^{n_\varphi}$ is the (reduced) scheduling-variable with $n_\varphi \leq n_\mathrm{p}$ and (9.4) has an affine parameter dependency, i.e., a structure like (9.3), with $\varphi = \mathrm{col}(\varphi_1, \cdots, \varphi_{n_\varphi})$, where Φ is compact and convex. Here, approximation is considered in the sense that $\varphi = \mu \circ \eta$, with $\mu : \mathbb{R}^{n_\mathrm{p}} \to \mathbb{R}^{n_\varphi}$, where $\mu(p(t)) = \varphi(t)$ is chosen such that for all $p \in \mathfrak{D}$

$$\hat{L}(\varphi) = \begin{bmatrix} \hat{A}(\varphi) & \hat{B}(\varphi) \\ \hat{C}(\varphi) & \hat{D}(\varphi) \end{bmatrix} \approx L(p). \tag{9.5}$$

We also aim to find the inverse mapping μ^{-1}, for which $\mu^{-1}(\varphi(t)) = \hat{p}(t)$, where $\hat{p}(t)$ is an approximation of the original scheduling-variable $p(t)$, such that $\hat{L}(\varphi) = L(\hat{p})$. The SDR problem is solved if (9.4) is a satisfactory approximation of (9.2). A satisfactory approximation is achieved, e.g., when, for a given a user defined ε, $\|L(p) - \hat{L}(\mu(p))\|_\mathrm{F} < \varepsilon$ for all $p \in \mathfrak{D}$, where $\|\cdot\|_\mathrm{F}$ is corresponds to the Frobenius norm, i.e., for a matrix $A \in \mathbb{R}^{n \times m}$, $\|A\|_\mathrm{F} = \sqrt{\mathrm{trace}(AA^\top)}$.

9.3 Overview of SDR Techniques

9.3.1 Preliminaries

In this section, we briefly explain the procedures of the existing data-based SDR methods in the literature. Before giving an overview of these methods, we discuss some preliminaries that all the methods use.

Assume a nonlinear system given by (9.1) is embedded in an LPV representation given by (9.2). Furthermore, assume that we have generated a set of nominal trajectories of the scheduling-variable $p(t) = \eta(x(t), u(t))$, based on the trajectories of the nonlinear system. For CT systems, these trajectories are sampled at time instances $t = kT_\mathrm{s}$, $k = 0, \ldots, N-1$, with sample-time T_s, resulting in

$$\Gamma = \begin{bmatrix} p(0) & \cdots & p((N-1)T_\mathrm{s}) \end{bmatrix}, \tag{9.6}$$

with $\Gamma \in \mathbb{R}^{n_\mathrm{p} \times N}$ and $\mathfrak{D} = \{p(kT_\mathrm{s})\}_{k=0}^{N}$. Moreover we introduce the shorthand notation $p_{(i)} := p((i-1)T_\mathrm{s}) = \Gamma_{*,i}$ for $i \in \mathbb{I}_1^N$, where $\Gamma_{*,i}$ denotes the i'th column of Γ. Similarly, in the DT case, we get

$$\Gamma = \begin{bmatrix} p(0) & \cdots & p(N-1) \end{bmatrix}, \tag{9.7}$$

$\mathfrak{D} = \{p(k)\}_{k=0}^{N}$, and $p_{(i)} := p(i-1) = \Gamma_{*,i}$ for $i \in \mathbb{I}_1^N$.

The trajectories in Γ are then normalized by an affine function $\mathcal{N}$, e.g., such that each row of the data matrix varies in $[-1, 1]$, which results in a normalized data matrix $\Gamma_\mathrm{n} = \mathcal{N}(\Gamma)$. For all the discussed algorithms (also for the DNN method in Section 9.4), it is assumed[1] that the data matrix, and therefore also $p_{(i)}$, is normalized, meaning that $\Gamma \equiv \Gamma_\mathrm{n}$. Next, the algorithms of the considered SDR methods will be discussed.

9.3.2 Principal component analysis

One of the earliest works to perform SDR based on trajectory data makes use of PCA (Kwiatkowski and Werner 2008). The core idea of the PCA method is to extract the most significant directions, principal components, of the scheduling data Γ. For the PCA method, a *Singular Value Decomposition* (SVD) is performed on Γ in order to obtain the principal components, such that

$$\Gamma = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} S_1 & 0 & 0 \\ 0 & S_2 & 0 \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \end{bmatrix}, \tag{9.8}$$

where $U_1 \in \mathbb{R}^{n_\mathrm{p} \times n_\varphi}$, $V_1 \in \mathbb{R}^{N \times n_\varphi}$ are unitary and $S_1 \in \mathbb{R}^{n_\varphi \times n_\varphi}$ is positive diagonal, containing the highest n_φ singular values of Γ. The complexity of the reduced model is governed by the user chosen n_φ. The corresponding approximation of

[1]While not explicitly written down for the discussed methods, the normalization transformation *does* have to be taken into account for the mapping μ (and μ^{-1}) and during construction of $\hat{L}(\varphi)$.

the data matrix, denoted by $\hat{\Gamma}$, is given by $\hat{\Gamma} = U_1 S_1 V_1^\top \approx \Gamma$. The new reduced scheduling-variable φ is then given by

$$\varphi(t) = \mu(p(t)) = U_1^\top p(t), \tag{9.9}$$

and an approximation of the old scheduling-variable p is obtained via

$$\hat{p}(t) = \mu^{-1}(\varphi(t)) = U_1 \varphi(t). \tag{9.10}$$

The matrices of the (scheduling dimension) reduced LPV model can then be constructed by using the relation $\hat{L}(\varphi) = L(\hat{p})$ and (9.10). For a more in-depth overview of the PCA method for SDR, see (Kwiatkowski and Werner 2008).

9.3.3 Kernel PCA

The next method discussed is the KPCA method (Rizvi, Mohammadpour Velni, et al. 2016). A disadvantage of the PCA method is that it only seeks a linear mapping for μ. The KPCA method extends the PCA method by allowing a nonlinear mapping. In the KPCA method, the data is nonlinearly mapped to a higher dimensional, so called, feature space on which normal PCA is applied. This mapping into the feature space is denoted by Θ. The covariance matrix is then given by

$$\bar{C} := \frac{1}{N} \sum_{j=1}^{N} \Theta\left(p_{(j)}\right) \Theta^\top \left(p_{(j)}\right). \tag{9.11}$$

The principal components are then computed such that $\lambda_l v_l = \bar{C} v_l$ holds, where λ_l is the l'th eigenvalue and v_l is the l'th eigenvector. By this method, we seek for a mapping to an appropriate feature space where PCA will result in the smallest number of components. The mapping Θ is a priori not known. In KPCA, the idea is to characterize the inner product of Θ with an a priori chosen kernel function, resulting in the kernel matrix

$$K_{ij} = (\Theta(p_{(i)})^\top \Theta(p_{(j)})) = k(p_{(i)}, p_{(j)}), \tag{9.12}$$

where $K \in \mathbb{R}^{N \times N}$ and $k(\cdot,\cdot)$ is a nonlinear kernel function. Examples of the kernel function include, the sigmoid kernel $k(p_{(i)}, p_{(j)}) = \tanh(\kappa(p_{(i)}^\top p_{(j)}) + \iota)$, the radial basis function $k(p_{(i)}, p_{(j)}) = \exp(-\left\|p_{(i)} - p_{(j)})\right\|^2 / \kappa^2)$, and polynomial kernel $k(p_{(i)}, p_{(j)}) = (p_{(i)}^\top p_{(j)} + \iota)^\kappa$, where κ and ι are hyperparameters chosen such that PCA can be accomplished with the lowest number of components given the structure of k. The data in the feature space is assumed to be centered, which is not always the case, therefore the centered kernel matrix is constructed as follows

$$K_\mathrm{c} = K - 1_\mathrm{N} K - K 1_\mathrm{N} + 1_\mathrm{N} K 1_\mathrm{N}, \tag{9.13}$$

where $1_\mathrm{N} \in \mathbb{R}^{N \times N}$ denotes the matrix with each element being $\frac{1}{N}$. The principal components of K_c are then computed instead of $\bar{C}$. Resulting, for non-zero eigenvalues, in

$$\lambda_l \alpha_l = K_\mathrm{c} \alpha_l, \tag{9.14}$$

where $\alpha_l = \mathrm{col}(\alpha_{l,1} \ldots \alpha_{l,N})$, with $v_l = \sum_{i=1}^{N} \alpha_{l,i}\Theta(p_{(i)})$ and with α_l (for $l \in \mathbb{I}_1^N$) normalized by $1/\sqrt{\lambda_l}$. The new scheduling-variable φ is then

$$\varphi_l(t) = \sum_{i=1}^{N} \alpha_{l,i} k(p_{(i)}, p(t)) \quad \text{for } l \in \mathbb{I}_1^{n_\varphi}. \tag{9.15}$$

For KPCA, the inverse mapping μ^{-1} cannot analytically be constructed in general and hence a further optimization step is required. We refer the reader to (Rizvi, Mohammadpour Velni, et al. 2014) for the details. Due to the difficulty of constructing the inverse mapping and due to this mapping being nonlinear, the affine scheduling-variable dependent matrices $\hat{A}, \ldots, \hat{D}$ are constructed by solving the following optimization problem

$$\min_{\{\hat{L}_i\}_{i=1}^{n_\varphi}} \frac{1}{N} \sum_{i=1}^{N} \left\| L\left(p_{(i)}\right) - \hat{L}\left(\varphi_{(i)}\right) \right\|_{\mathrm{F}}^2, \tag{9.16}$$

where $\varphi_{(i)} := \varphi((i-1)T_{\mathrm{s}})$ in CT or $\varphi_{(i)} := \varphi(i-1)$ for $i \in \mathbb{I}_1^N$ in DT, computed through (9.15). The reduced scheduling-variable φ is not directly constructed in order to minimize (9.16), therefore it is difficult to attach any guarantees to the overall outcome of the method. See (Rizvi, Mohammadpour Velni, et al. 2016) for a more in-depth overview of the KPCA method for SDR.

9.3.4 Autoencoder

The final method that will be discussed is the AE method (Rizvi, Abbasi, et al. 2018). The AE method like KPCA uses nonlinear functions for the mapping μ. This method makes use of an AE ANN, which is trained on the data set to construct the mappings μ and μ^{-1}. Therefore it has as an advantage over KPCA that μ^{-1} is co-synthesized with μ. Again, assume we have an LPV model given by (9.2), and a data matrix given by (9.6). The data matrix is fed into a two-layer[2] AE ANN, which consist of a single encoding layer (from p to φ) and a decoding layer (from φ to $\hat{p}$). On the first layer (the encoding layer), we have the following relation

$$\varphi_{(i)} = \sigma^{[1]}\left(W^{[1]} p_{(i)} + b^{[1]}\right) \quad \text{for } i \in \mathbb{I}_1^N, \tag{9.17}$$

where σ is called the activation function, with the weight matrix $W^{[1]} \in \mathbb{R}^{n_\varphi \times n_\mathrm{p}}$, and bias vector $b^{[1]} \in \mathbb{R}^{n_\varphi}$. Examples of activation functions include the logistic sigmoid (logsig) function, hyperbolic function (tanh), and *Rectified Linear Unit* (ReLU).

On the second layer, on the decoding side, we have the following relation

$$\hat{p}_{(i)} = \sigma^{[2]}\left(W^{[2]} \varphi_{(i)} + b^{[2]}\right) \quad \text{for } i \in \mathbb{I}_1^N, \tag{9.18}$$

[2]In general, the AE can have more than two layers, but as in (Rizvi, Abbasi, et al. 2018), this is not considered here.

where $W^{[2]} \in \mathbb{R}^{n_\mathrm{p} \times n_\varphi}$ and $b^{[2]} \in \mathbb{R}^{n_\mathrm{p}}$. To train the AE, an optimization procedure is used to minimize the error between $p_{(i)}$ and $\hat{p}_{(i)}$ for $i \in \mathbb{I}_1^N$. This is done by solving the following optimization problem

$$\min_{W^{[k]}, b^{[k]},\, k=1,2} \quad \frac{1}{N} \sum_{i=1}^{n_\mathrm{p}} \sum_{j=1}^{N} \left(\Gamma_{i,j} - \hat{\Gamma}_{i,j} \right)^2 . \tag{9.19}$$

Additional terms can be incorporated into the cost function in order to reduce overfitting such as weight and/or sparsity regularization, see (Rizvi, Abbasi, et al. 2018) for more details. After optimization of (9.19) the new scheduling-variable can be expressed as

$$\varphi(t) = \mu(p(t)) = \sigma^{[1]} \left(W^{[1]} p(t) + b^{[1]} \right), \tag{9.20}$$

with the inverse mapping

$$\hat{p}(t) = \mu^{-1}(\varphi(t)) = \sigma^{[2]} \left(W^{[2]} \varphi(t) + b^{[2]} \right). \tag{9.21}$$

However, due to the nonlinear inverse mapping (9.21), a separate optimization procedure is required to obtain the affine scheduling-variable dependent state-space matrices $\hat{A}, \ldots, \hat{D}$ of the reduced model. This is similar to the KPCA method, see Section 9.3.3, specifically (9.16). See (Rizvi, Abbasi, et al. 2018) for a detailed explanation of the AE method for SDR.

9.4 A Deep Neural Network Approach

While the AE and KPCA approaches have the benefit of using nonlinear mappings for the reduction, they still require an extra optimization step in order to obtain the matrices of the reduced LPV state-space model. However, the impact of this second optimization step, characterizing the model approximation error, is not taken into account in the synthesis of $\varphi(t)$. Hence, as the contribution of this chapter, a new method is developed which, like the AE method, uses an ANN in order to construct the mapping $\varphi = \mu(p)$. However, unlike the AE method, (multiple) hidden layers are used in the encoding layer of the network, which allows to capture more complex mappings. Furthermore, a vectorized form of $\hat{L}(\varphi)$ is used as output of the network in order to directly construct the state-space matrices of the reduced LPV model. In this way, the synthesis of $\varphi(t)$ and the state-space matrices associated with it are co-determined under optimal approximation of $L(p)$ on $\mathcal{D}$. The idea of using the state-space matrix data instead of the scheduling data for SDR has been recently also adopted in (Sadeghzadeh, Sharif, et al. 2020), whereby instead of a DNN, PCA was used to perform the scheduling reduction. However, the use of PCA in (Sadeghzadeh, Sharif, et al. 2020) has two downsides, namely, (i) the reduced scheduling-map is dependent on the original LPV state-space matrix functions, i.e., $A, \ldots, D$, which might be complex, and (ii) PCA searches for a linear mapping for the reduction, while the DNN method allows for a nonlinear mapping, which allows us to achieve a reduced model approximation error.

Assume we have an LPV model given by (9.2) and a data matrix given by (9.6), which we will call the input data matrix. Moreover, assume we have another data matrix, the output data matrix, given by

$$\Lambda = \begin{bmatrix} L_{\mathrm{v}}(p_{(1)}) & \cdots & L_{\mathrm{v}}(p_{(N)}) \end{bmatrix}, \tag{9.22}$$

where $\Lambda \in \mathbb{R}^{\nu \times N}$, with $\nu = m \cdot n$, and where

$$L_{\mathrm{v}}(p) = \mathrm{vec}\left(\sum_{i=1}^{n_{\mathrm{p}}} L_i(p)\right) = \sum_{i=1}^{n_{\mathrm{p}}} \mathrm{vec}\,(L_i)\, p_i, \tag{9.23}$$

with $\mathrm{vec}(\cdot)$ denoting the column stacked vectorized form of a matrix, i.e., for the matrix $L \in \mathbb{R}^{m \times n}$, $\mathrm{vec}(L) \in \mathbb{R}^{(m \cdot n)}$, so a vector with $m \cdot n$ elements. Note that we assume that the output data matrix does not contain (a vectorized form of) the parameter independent term L_0 of our affine LPV model, see (9.3), as it does not play a role in the SDR problem. Also note that in this way, we are not limited to reducing LPV models (9.2) with only affine scheduling dependency. Namely, any dependency can be considered as long as we can evaluate the state-space matrices of the LPV model at scheduling points $p_{(i)}$, similar to what is presented in (Sadeghzadeh, Sharif, et al. 2020).

The input data matrix is fed into the input layer of the ANN, for which it holds that

$$l_{(i)}^{[1]} = \sigma^{[1]}\left(W^{[1]} p_{(i)} + b^{[1]}\right) \quad \text{for } i \in \mathbb{I}_1^N, \tag{9.24}$$

where $l^{[1]} \in \mathbb{R}^{n_{1,1}}$ is the 'output' of the first hidden layer, with the weight matrix $W^{[1]} \in \mathbb{R}^{n_{1,1} \times n_{\mathrm{p}}}$ and bias vector $b^{[1]} \in \mathbb{R}^{n_{1,1}}$. For the second hidden layer till the $(n_{\mathrm{h}} - 1)$'th hidden layer (where n_{h} denotes the number of hidden layers), we have the relation

$$l_{(i)}^{[j]} = \sigma^{[j]}\left(W^{[j]} l_{(i)}^{[j-1]} + b^{[j]}\right) \text{ for } j \in \mathbb{I}_1^{n_{\mathrm{h}}-1}, \text{ and } i \in \mathbb{I}_1^N, \tag{9.25}$$

where $l^{[j]} \in \mathbb{R}^{n_{1,j}}$ is the 'output' of the j'th hidden layer, with the weight matrix $W^{[j]} \in \mathbb{R}^{n_{1,j} \times n_{1,j-1}}$, and bias vector $b^{[j]} \in \mathbb{R}^{n_{1,j}}$. For the n_{h}'th hidden layer, i.e., the final hidden layer, we have

$$\varphi_{(i)} = \sigma^{[n_{\mathrm{h}}]}\left(W^{[n_{\mathrm{h}}]} l_{(i)}^{[n_{\mathrm{h}}-1]} + b^{[n_{\mathrm{h}}]}\right) \quad \text{for } i \in \mathbb{I}_1^N, \tag{9.26}$$

with weight matrix $W^{[n_{\mathrm{h}}]} \in \mathbb{R}^{n_\varphi \times n_{1,(n_{\mathrm{h}}-1)}}$ and bias vector $b^{[n_{\mathrm{h}}]} \in \mathbb{R}^{n_\varphi}$. Together, equations (9.24)-(9.26) make up the encoding layer of the ANN and hence make up the mapping $\mu : \mathbb{R}^{n_{\mathrm{p}}} \to \mathbb{R}^{n_\varphi}$, i.e.,

$$\mu(p) = \sigma^{[n_{\mathrm{h}}]}\left(W^{[n_{\mathrm{h}}]}\left(\cdots \sigma^{[1]}\left(W^{[1]} p + b^{[1]}\right) \cdots\right) + b^{[n_{\mathrm{h}}]}\right). \tag{9.27}$$

While many different activation functions (as in the AE case) can be used in the encoding layer, we propose to use ReLU activation functions, given by $\sigma(x) = \max(0, x)$. DNNs using ReLU functions have been proven to be universal function approximators (Hanin 2019) with favorable benefits during training (stable non-vanishing gradient propagation). Moreover, ReLU functions have computational benefits, as their expression can be evaluated quickly and because they introduce sparsity in the network (as some neurons in a particular layer can be zero).

Finally, we have the matrix mapping layer, which is the final layer in the network. Before giving the relation of the final layer, we first define the vector

$$\hat{L}_{\mathrm{v}}(\varphi) = \operatorname{vec}\left(\sum_{i=1}^{n_\varphi} \hat{L}_i(\varphi)\right) = \sum_{i=1}^{n_\varphi} \operatorname{vec}\left(\hat{L}_i\right)\varphi_i, \tag{9.28}$$

where $\hat{L}_{\mathrm{v}}(\varphi) = \operatorname{col}\left(\hat{L}_{\mathrm{v},1}(\varphi), \hat{L}_{\mathrm{v},2}(\varphi), \ldots, \hat{L}_{\mathrm{v},\nu}(\varphi)\right) \in \mathbb{R}^\nu$. The final layer is then given by

$$\hat{L}_{\mathrm{v}}(\varphi_{(i)}) = W^{[n_{\mathrm{h}}+1]}\varphi_{(i)} + b^{[n_{\mathrm{h}}+1]} \quad \text{for } i \in \mathbb{I}_1^N, \tag{9.29}$$

where $W^{[n_{\mathrm{h}}+1]} \in \mathbb{R}^{\nu \times n_\varphi}$ with bias vector $b^{[n_{\mathrm{h}}+1]} \in \mathbb{R}^\nu$. Note that the matrix mapping layer does *not* use any activation functions, this is done such that the required affine relation between the reduced scheduling-variables and the (vectorized) reduced LPV model matrices is obtained. The encoding layer, (9.24)-(9.26), together with the matrix mapping layer, (9.29), make up the full ANN. The full ANN structure for the DNN approach is depicted in Figure 9.1.

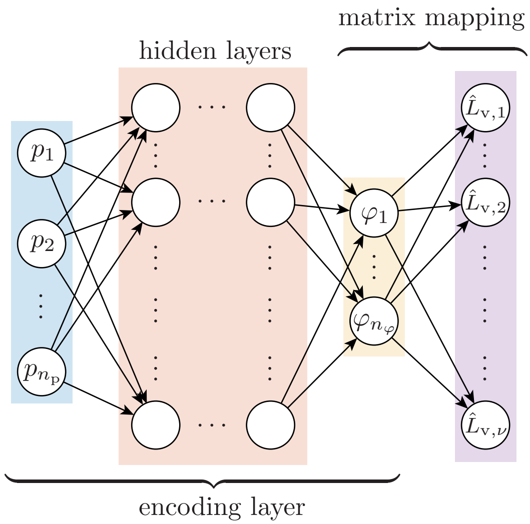


Figure 9.1: DNN architecture.

To reconstruct the state-space matrices of the reduced LPV model, we can use (9.28) and (9.29) to obtain the relations

$$\begin{aligned} \operatorname{vec}\hat{L}_0 &= \operatorname{vec}(L_0) + b^{[n_{\mathrm{h}}+1]}; \\ \begin{bmatrix}\operatorname{vec}\left(\hat{L}_1\right) & \cdots & \operatorname{vec}\left(\hat{L}_{n_\varphi}\right)\end{bmatrix} &= W^{[n_{\mathrm{h}}+1]}. \end{aligned} \tag{9.30}$$

The weights and biases of the layers of the DNN are trained by solving the following optimization problem

$$\min_{W^{[j]},b^{[j]},\, j=1,\ldots,n_{\mathrm{h}}+1} \quad \frac{1}{N}\sum_{i=1}^{N}\left\|L\left(p_{(i)}\right)-\hat{L}\left(\mu(p_{(i)})\right)\right\|_{\mathrm{F}}^{2}, \tag{9.31}$$

with μ given by (9.27). This optimization problem can then be solved by means of a backpropagation algorithm (Goodfellow et al. 2016) in combination with a gradient decent algorithm, such as Stochastic Gradient Decent (SGD), Adam (Kingma and Ba 2014), or AdaBound (Luo et al. 2019). Like for the AE method, in order to reduce overfitting, regularization techniques can be used such as L_1 or L_2 weight regularization (Goodfellow et al. 2016). Moreover, if multiple hidden layers are used in the encoding layer, regularization techniques such as dropout (Srivastava et al. 2014) can be used to also reduce overfitting.

9.5 Application to Automated Affine LPV Embedding

9.5.1 Embedding of nonlinear systems

As highlighted in Chapter 1, one challenge of the existing LPV framework is embedding a given (primal form of the) nonlinear system of the form (9.1) in an LPV representation (9.2) with a certain scheduling dependency, such as affine or rational. Next, we will discuss how the DNN SDR method can be used to obtain an (approximate) affine LPV embedding of a nonlinear system using a given data set of state and input trajectories.

We assume we have a nonlinear system given by (9.1), for which both f and h are continuously differentiable, i.e., $f, h \in \mathcal{C}_1$, and such that $f(0,0) = 0$ and $h(0,0) = 0$. Moreover, we assume we have a data set $\mathfrak{D}_{\mathrm{xu}}$ of N state $x \in \mathcal{X}$ and input $u \in \mathcal{U}$ samples, corresponding to nominal trajectories or operating range on which we which we want to perform the embedding. More concretely,

$$\mathfrak{D}_{\mathrm{xu}} := \{(x_{(i)}, u_{(i)}) \in \mathcal{X} \times \mathcal{U},\, i = 1, \ldots N\}, \tag{9.32}$$

where $x_{(i)}$ and $u_{(i)}$ denote the i'th sample of the state and input in $\mathfrak{D}_{\mathrm{xu}}$, respectively.

Based on this data set, we can use Theorem C.6.1 (see Appendix C.6) to construct a grid-based LPV model. In short, we can embed the nonlinear system (9.1) in an LPV representation given by (9.2) where

$$\begin{aligned} A(p) &= \bar{A}(x,u), \qquad & B(p) &= \bar{B}(x,u), \\ C(p) &= \bar{C}(x,u), \qquad & D(p) &= \bar{D}(x,u), \end{aligned} \tag{9.33}$$

with $p = \mathrm{col}(x, u)$, meaning that η is the identity map, and,

$$\begin{aligned} \bar{A}(x,u) &= \int_0^1 \frac{\partial f}{\partial x}(\lambda x, \lambda u)\, d\lambda, & \bar{B}(x,u) &= \int_0^1 \frac{\partial f}{\partial u}(\lambda x, \lambda u)\, d\lambda, \\ \bar{C}(x,u) &= \int_0^1 \frac{\partial h}{\partial x}(\lambda x, \lambda u)\, d\lambda, & \bar{D}(x,u) &= \int_0^1 \frac{\partial h}{\partial u}(\lambda x, \lambda u)\, d\lambda. \end{aligned} \tag{9.34}$$

Note that, as mentioned in Appendix C.6, analytical computation of the integrals of the Jacobian is not required, as *Automatic Differentiation* (AD) and numerical integration techniques can be used to compute the values of $\bar{A}(x,u)$, ..., $\bar{D}(x,u)$ at points $(x, u) \in \mathfrak{D}_{\mathrm{xu}}$. Note that as $p = \mathrm{col}(x, u)$, we have for SDR that $\mathfrak{D} = \mathfrak{D}_{\mathrm{xu}}$, and

$$L(p) = L(x,u) = \begin{bmatrix} \bar{A}(x,u) & \bar{B}(x,u) \\ \bar{D}(x,u) & \bar{D}(x,u) \end{bmatrix}. \tag{9.35}$$

Note that as the scheduling-map η is the identity map, our scheduling-map for the reduced LPV model will become equal to μ, i.e., $\varphi(t) = \mu(\eta(x(t), u(t))) = \mu(x(t), u(t))$. Hence, based on just a data set $\mathfrak{D}_{\mathrm{xu}}$ and the expressions of f and h, we can automatically construct the matrices Γ (9.6) and Λ (9.22) for the DNN method, which can then be used to construct an (approximate) affine LPV embedding of the of the nonlinear system.

9.5.2 Embedding of differential and velocity forms

For the proposed universal shifted and incremental based analysis and synthesis approaches discussed in Chapters 4 to 8, the LPV framework is also used. Namely, the LPV framework is used in order to embed the velocity or differential form of a nonlinear system in an LPV representation, giving us a VPV or DPV embedding of the system, respectively. As the velocity and differential forms of a nonlinear system are already in a factorized form, the embedding procedure is partially simplified. However, embedding them in an LPV representation for a given scheduling dependency is still done heuristically and can therefore still be challenging. Next, we will discuss how we can also use the DNN SDR method in order to obtain an (approximate) affine VPV or DPV embedding of a nonlinear system based a given data set of state and input trajectories. As the VPV and DPV embedding have identical structures (see also the discussion at the end of Section 5.4.2), we will only present the procedure for construction of an affine DPV embedding.

For a nonlinear system given by (9.1), its differential form (see Chapter 5 for more details) is given by

$$\begin{aligned} \xi x_\delta(t) &= A_\delta(x,u)x_\delta(t) + B_\delta(x,u)u_\delta(t); \\ y_\delta(t) &= C_\delta(x,u)x_\delta(t) + D_\delta(x,u)u_\delta(t); \end{aligned} \tag{9.36}$$

where $A_\delta = \frac{\partial f}{\partial x}$, $B_\delta = \frac{\partial f}{\partial u}$, $C_\delta = \frac{\partial h}{\partial x}$, and $D_\delta = \frac{\partial h}{\partial u}$. As the differential form is already in a factorized form, we can take as LPV embedding (9.2), where

$$\begin{aligned} A(p) &= A_\delta(x,u), & B(p) &= B_\delta(x,u), \\ C(p) &= C_\delta(x,u), & D(p) &= D_\delta(x,u), \end{aligned} \tag{9.37}$$

$p = \mathrm{col}(x, u)$, meaning that η is the identity map. Similar as in Section 9.5.1, we only need to know the values of $A_\delta(x, u), \ldots, D_\delta(x, u)$ at points $(x, u) \in \mathfrak{D}_{\mathrm{xu}}$. Hence, we can use AD techniques in order to compute and evaluate the Jacobians, for which implementations exists in various software tools such as in Python (Maclaurin et al. 2019) and Julia (Revels et al. 2016). As $p = \mathrm{col}(x, u)$, we have for SDR that $\mathfrak{D} = \mathfrak{D}_{\mathrm{xu}}$, and

$$L(p) = L(x, u) = \begin{bmatrix} A_\delta(x, u) & B_\delta(x, u) \\ C_\delta(x, u) & D_\delta(x, u) \end{bmatrix}. \tag{9.38}$$

Similarly as in Section 9.5.1, note that the scheduling-map η is the identity map, hence, our scheduling-map for the reduced LPV model will become equal to μ, i.e., $\varphi(t) = \mu(\eta(x(t), u(t))) = \mu(x(t), u(t))$. Therefore, based the DNN SDR approach, we can automatically construct an (approximate) affine LPV embedding of the differential form, i.e., a DPV embedding of the nonlinear system.

In Section 10.3, we will also demonstrate the application this approach in order to construct an affine DPV embedding of a *Generic Parafoil Return Vehicle* (GPRV), which is then used for the design of an incremental controller for this system.

9.6 Example

In this section the methods discussed from Section 9.3 and Section 9.4 are compared on the LPV modeling and control design problem of a two-link planar robot manipulator (Kwiatkowski and Werner 2005), see also Figure 9.2.

Example 9.1 *(Two-link robot manipulator)*.

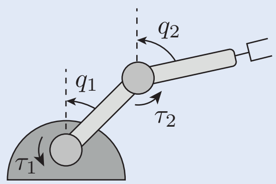


Figure 9.2: Diagram of the two-link planar robot manipulator.

Nonlinear Model

The robot manipulator can be described by the following equation of motion

$$M(q(t))\ddot{q}(t) + C(q(t), \dot{q}(t)) + g(q(t)) = n\tau(t), \tag{9.39}$$

where $q(t) = \mathrm{col}(q_1(t), q_2(t))$ are the angles, $\tau(t) = \mathrm{col}(\tau_1(t), \tau_2(t))$ the motor torques and

$$M(q) = \begin{bmatrix} a & b\cos(q_1 - q_2)) \\ b\cos(q_1 - q_2) & c \end{bmatrix}, \quad g(q) = \begin{bmatrix} -d\sin(q_1) \\ -e\sin(q_2) \end{bmatrix},$$
$$C(q, \dot{q}) = \begin{bmatrix} b\sin(q_1 - q_2)\dot{q}_2^2 + f\dot{q}_1 \\ -b\sin(q_1 - q_2)\dot{q}_1^2 + f(\dot{q}_2 - \dot{q}_1) \end{bmatrix}.$$

The values of the (physical) parameters of the robot manipulator are given in Table 9.1. The equations of motion (9.39) can be rewritten to a CT nonlinear state-space representation, of the form (9.1), given by

$$\begin{aligned} \dot{x}(t) &= \begin{bmatrix} x_2(t) \\ M\big(x_1(t)\big)^{-1}\Big(nu(t) - C\big(x_1(t), x_2(t)\big) - g\big(x_1(t)\big)\Big) \end{bmatrix}; \\ y(t) &= \begin{bmatrix} I_2 & 0 \end{bmatrix} x(t); \end{aligned} \tag{9.40}$$

where $x = \mathrm{col}(x_1,\, x_2) = \mathrm{col}(q,\, \dot{q})$, and $u = \mathrm{col}(\tau_1,\, \tau_2)$.

Table 9.1: Physical parameters of the robot manipulator.

Parameter	Value
a	5.6794
b	1.473
c	1.7985
d	$4 \cdot 10^{-1}$
e	$4 \cdot 10^{-1}$
f	2
n	1

LPV model

For the (manual) LPV embedding of the nonlinear model of the robot manipulator in an affine LPV representation, we take the approach given in (Kwiatkowski and Werner 2005), which gives

$$\begin{aligned} \dot{x}(t) &= A(p(t))x(t) + B(p(t))u(t); \\ y(t) &= Cx(t) + Du(t); \end{aligned} \tag{9.41}$$

where

$$\begin{aligned} A(p) &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ cdp_3 & -bep_4 & p_5 & bp_6 \\ -bdp_7 & aep_8 & p_9 & p_{10} \end{bmatrix}, \quad C = \begin{bmatrix} I_2 & 0 \end{bmatrix}, \\ B(p) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ cnp_1 & -bnp_2 \\ -bnp_2 & anp_1 \end{bmatrix}, \qquad D = 0. \end{aligned} \tag{9.42}$$

The corresponding scheduling-map (only depending on the states) $\eta : \mathbb{R}^4 \to \mathbb{R}^{10}$ is given by

$$\eta(x) = \begin{bmatrix} h \\ \cos(x_1 - x_2)h \\ \operatorname{sinc}(x_1)h \\ \cos(x_1 - x_2)\operatorname{sinc}(x_2)h \\ (-b^2 \sin(x_1 - x_2)\cos(x_1 - x_2)x_3 - (c + b\cos(x_1 - x_2))f)h \\ (-c\sin(x_1 - x_2)x_4 + \cos(x_1 - x_2)f)h \\ \cos(x_1 - x_2)\operatorname{sinc}(x_1)h \\ \operatorname{sinc}(x_2)h \\ (ab\sin(x_1 - x_2)x_3 + f(a + b\cos(x_1 - x_2)))h \\ (b^2 \sin(x_1 - x_2)\cos(x_1 - x_2)x_4 - af)h \end{bmatrix}, \tag{9.43}$$

where $h = \frac{1}{ac - b^2 \cos(x_1 - x_2)^2}$.

Scheduling dimension reduction

In order to perform SDR on (9.41), a data set $\mathfrak{D}$ is required containing a set of typical trajectories of the scheduling-variables. As the reduced LPV models will be used for controller synthesis and as the objective of the controller will be reference tracking, the to be followed reference trajectory is used as a data set, see 'Reference 1' in Figure 9.5. To obtain the trajectories of the scheduling-variables, the scheduling-map η (9.43) is used. Note, that while it is not displayed in Figure 9.5, the angular joint velocities corresponding to the reference trajectory are also required and used to obtain the data set.

Regarding the hyperparameters of the SDR methods, for the KPCA method, a sigmoid kernel is chosen with hyperparameters $\kappa = 0.1$ and $\iota = 0.1$. For the AE method, logsig activation functions are used for both the decoding and encoding layer. Furthermore, L_2 weight regularization is added with a coefficient of $1 \cdot 10^{-5}$. The implementation and training of the AE based method is done using the `Autoencoder` class from the MATLAB Deep Learning Toolbox. For the DNN approach, the training of the ANN was done in Python using Keras (Chollet et al. 2015). As an optimization method, in terms of gradient descent, AdaBound (Luo et al. 2019) is used with the default learning rates. For the encoding layer of the DNN approach, one hidden layer is used with 5 neurons, and, as mentioned, ReLU functions are used as activation functions in the encoding layer. Moreover, for the DNN method, L_2 weight regularization is added with a coefficient of $1 \cdot 10^{-6}$. The hyperparameters for each method were chosen such that their respective cost function were minimal.

Modeling error

The approximation error, in terms of average squared Frobenius norm (see (9.16)), of the reduced LPV models is compared for various reductions to a scheduling size n_φ. These results are given in Figure 9.3. From the results, it can be concluded that the new developed DNN method results overall in the best performance for all the considered scheduling sizes. A lower cost can be achieved using the DNN methods because it directly optimizes the average squared Frobenius cost instead of needing an extra optimization step like for the KPCA or AE method. Moreover, the scaling to larger scheduling dimensions is also better with the developed DNN method compared to the other methods.

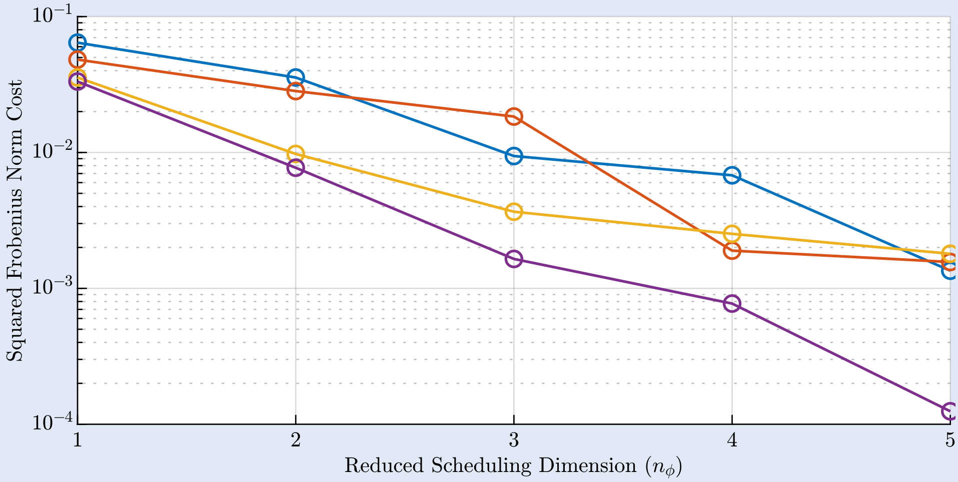


Figure 9.3: Average squared Frobenius norm cost model error for different scheduling sizes (n_φ) using PCA ( ), KPCA ( ), AE ( ), DNN ( ).

LPV controller design

For the controller design with the obtained models, a generalized plant is constructed, equivalent with the one used in (Rizvi, Mohammadpour Velni, et al. 2016), in order to achieve reference tracking of x_1 and x_2 (i.e., the two angles, q_1 and q_2). The generalized plant is shown in Figure 9.4, where r is the reference signal, d the input disturbance, z_1 and z_2 are performance channels, K is the to-be designed controller, G is the robot manipulator model, and W_* are weighting filters. The weighting filter W_1 is chosen to include low-pass characteristics on both channels in order to ensure good tracking performance at low frequencies, while W_2 is chosen as a constant gain for both channels in order to limit the motor torques. The weighting filter W_u is chosen as a true low-pass filter on both channels[3]. The exact transfer functions applied in Figure 9.4 are given in[4] Table 9.2. Based on this (weighted) generalized plant, an affine ($\mathcal{L}_2$-gain optimal) LPV controller is synthesized, using the LPVcore Toolbox (Boef et al. 2021), which minimizes the $\mathcal{L}_2$-gain from disturbance to performance channel. This synthesis procedure is performed using the scheduling reduced plants (G in Figure 9.4) resulting from the various SDR techniques. The resulting $\mathcal{L}_2$-gains can be found in Table 9.3.

[3] W_u is included because the synthesis procedure requires the relation from the control input to the state of the generalized plant to be independent of the scheduling-variable.

[4] All the weighting filters are block diagonal with the same transfer function for both channels.

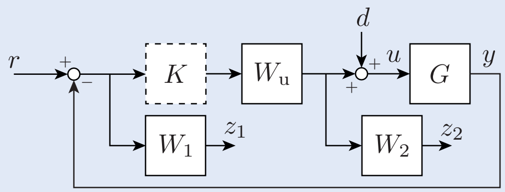


Figure 9.4: Generalized plant for controller synthesis.

Table 9.2: Weighting filters of the generalized plant where s is the complex frequency.

Weighting filter	Transfer function
$W_1(s)$	$\frac{(5\cdot10^{-1})s+5}{s+(5\cdot10^{-5})}$
$W_1(s)$	$3\cdot10^{-3}$
$W_\mathrm{u}(s)$	$\frac{1\cdot10^{3}}{s+(1\cdot10^{3})}$

Table 9.3: Closed-loop $\mathcal{L}_2$-gains obtained using the models resulting from the SDR methods for $n_\varphi = 1$.

SDR method	$\mathcal{L}_2$-gain
PCA	1.25
KPCA	1.26
AE	1.25
DNN	1.24

LPV controller performance

The controllers resulting from the synthesis on the reduced LPV models (with $n_\varphi = 1$) are interconnected with the nonlinear model of the robot arm. The tracking performance of the various resulting controllers is evaluated for three reference trajectories. The first reference trajectory (reference 1) is the reference trajectory also used to construct the data set. The second reference trajectory (reference 2) is constructed to have similar 'behavior' as reference 1, while still being different from the first reference. Reference 2 is used to analyze if the methods overfit. Finally, the third reference trajectory (reference 3) consists of a square wave trajectory for q_1 while q_2 is kept at zero. Reference trajectory 3 has different 'behavior' compared to reference 1 and 2 (hence not present in the data used for SDR) and is used to analyze the robustness of the methods.

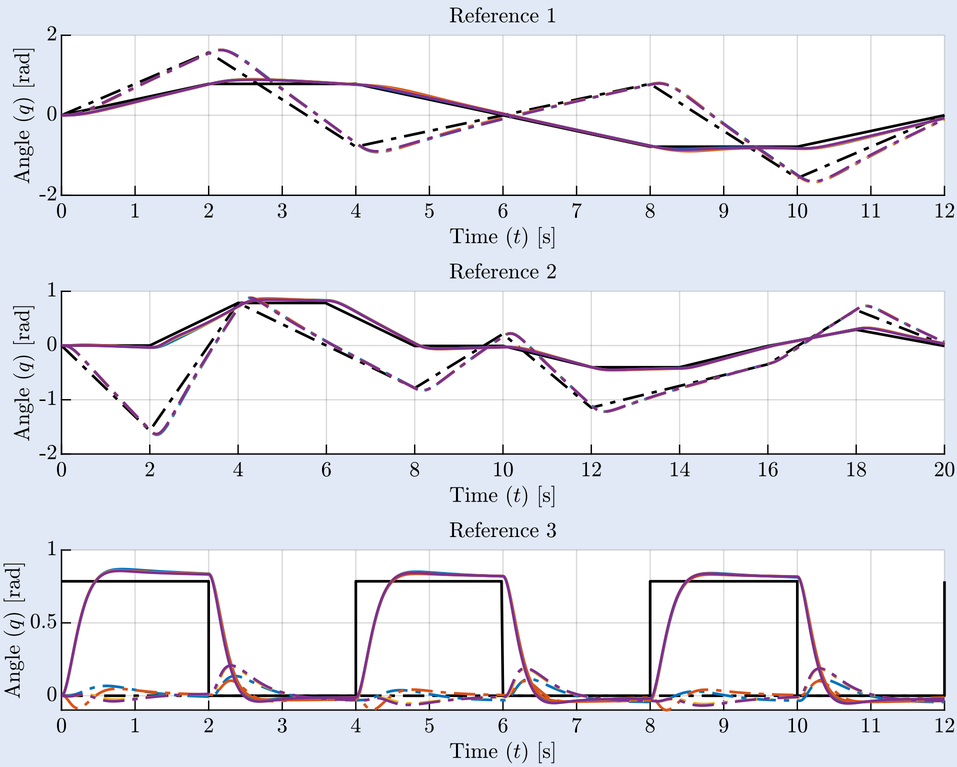


Figure 9.5: Tracking performance of LPV controllers w.r.t. reference q_1 (—) and q_2 (- -) using the SDR techniques with $n_\varphi = 1$, using PCA (—), KPCA (—), AE (—), DNN (—).

Shown in Figure 9.5 are the simulation results for the considered reference trajectories using the resulting controllers (i.e., using the various SDR techniques). It is apparent from these figures that for this example the methods all result in trajectories that are close together. Therefore, the closed-loop $\mathcal{L}_2$-gains γ using the different controllers is also computed/approximated based on the simulation data, i.e.,

$$\gamma = \frac{\sqrt{\int_0^\infty \|z(t)\|^2 \, dt}}{\sqrt{\int_0^\infty \|w(t)\|^2 \, dt}} \approx \frac{\sqrt{\sum_{k=0}^{T} \|z(kT_\mathrm{s})\|^2 \, T_\mathrm{s}}}{\sqrt{\sum_{k=0}^{T} \|w(kT_\mathrm{s})\|^2 \, T_\mathrm{s}}}, \tag{9.44}$$

where $z = \mathrm{col}(z_1, z_2)$, $w = \mathrm{col}(r, d)$ (see Figure 9.4) and T is the end-time of the simulation. The $\mathcal{L}_2$-gains based on the simulation data, computed using (9.44), for each of the different reference trajectories are given in Table 9.4. From the computed $\mathcal{L}_2$-gains it is clear that the DNN method provides overall the best controller performance. An additional benefit of the proposed DNN method is that due to the use of ReLU activation functions, the mapping μ from p to φ can be computed faster compared to the AE method proposed in (Rizvi, Abbasi, et al.

2018) and the KPCA methods, which use more complex nonlinear functions. This makes the DNN method also better suited for realtime implementation.

Table 9.4: $\mathcal{L}_2$-gain performance of LPV controllers in simulation based on the different SDR techniques with $n_\varphi = 1$.

Reference	SDR method			
	PCA	KPCA	AE	DNN
1	0.9466	1.008	0.9532	**0.9062**
2	0.9564	0.9109	0.9088	**0.8834**
3	2.044	**1.333**	1.511	1.500

9.7 Conclusions

In this chapter, a novel SDR method that uses a DNN in order to perform the scheduling reduction has been introduced. In this way, the number of scheduling-variables is reduced which allows for reduced computational complexity of the LPV analysis and controller synthesis methods. Moreover, through the SDR method, conservatism of the resulting LPV model w.r.t. a certain set of trajectories is also reduced. The DNN method has as a benefit that it directly optimizes the approximation error (squared Frobenius norm in this case) and directly gives the state-space matrices of the reduced LPV representation, whereas for the KPCA and AE method, an additional optimization step is required. Furthermore, as suggested in this chapter, using a deep encoding layer with ReLU functions in the DNN method allows for better approximation of more complex scheduling-maps while still being able to compute the output of the encoding layer rapidly and efficiently. Moreover, we show how the DNN or other SDR methods can be used in order to automatically construct an affine LPV, VPV, or DPV embedding of the nonlinear system based on a given data set.

Based on the results of the two-link robotic manipulator example, it can also be concluded that the developed DNN method results in a improved representation (of the matrix variations) of the original model (in terms of average Frobenius norm squared) compared to the current SDR methods. Moreover, the DNN method also gives improved tracking performance when the (reduced) LPV model is used for synthesis compared to the other methods.

The DNN SDR method, together with the analysis and synthesis results of previous chapters, allows for the computational efficient analysis and controller synthesis for nonlinear systems in order to guarantee global stability and performance requirements, even as the complexity of these systems increases.

10

Applications

In this chapter, we demonstrate the capabilities of the developed framework that has been presented in the previous chapters on two realistic applications for controller design in order to ensure global stability and performance guarantees. Namely, we consider a *Control Momement Gyroscope* (CMG) lab setup, to which our universal shifted controller design, presented in Chapter 4, is applied in an experimental study for tracking piece-wise constant reference signals. Next to that, a simulation study is presented, whereby our incremental controller design, presented in Chapter 6, is used to achieve reference tracking for a *Generic Parafoil Return Vehicle* (GPRV). For both applications, the benefits of the proposed controller design approaches are demonstrated in terms of a systematic controller design process and their ability to guarantee the desired stability performance properties for the underlying nonlinear systems. Moreover, the obtained controllers are compared to standard *Linear Parameter-Varying* (LPV) controller designs, which share a similar design complexity. Compared to these controller designs, the universal shifted and incremental controllers achieve significantly improved closed-loop performance, as they are able to guarantee global stability and performance requirements.

10.1 Introduction

In the previous chapters, we have presented a framework for analysis and control of nonlinear systems in order to ensure and shape global stability and performance guarantees. So far, we have demonstrated the developed tools on various (academic) examples and small scale experimental studies in the individual chapters. In this chapter, we will apply our developed universal shifted controller design (of Chapter 4) and incremental controller design (of Chapter 6), to more complex and realistic problems. More concretely, we will show experimental results of the application of our universal shifted controller design to a *Control Momement Gyroscope* (CMG) lab setup, and we will present a simulation study of our incremental controller design applied to a *Generic Parafoil Return Vehicle* (GPRV).

This chapter is structured as follows. First, in Section 10.2, we present a universal shifted control design and its implementation for a CMG. Next, in Section 10.3, we present an incremental flight controller design for the final descent phase of a GPRV. Finally, in Section 10.4, conclusions are drawn based on the obtained results that have been achieved on the presented applications.

10.2 Universal Shifted Control of a Control Moment Gyroscope

10.2.1 Introduction

In this section, we discuss the application of the proposed universal shifted controller design from Chapter 4 in order to achieve constant reference tracking control for a CMG. CMGs are used in various applications such as attitude control of satellites and stabilization of ships (Townsend and Shenoi 2011). Moreover, they are regularly used as a test bed for nonlinear control applications due to the challenging coupled rotational nonlinear dynamics (Reyhanoglu and Van de Loo 2006). This has also made them a popular choice to demonstrate the efficacy of various standard *Linear Parameter-Varying* (LPV) control based methods (Abbas, A. Ali, et al. 2014; Hoffmann and Werner 2015b; Koelewijn, Cisneros, et al. 2018).

Next, we will first give an overview of the dynamics and the considered model of the CMG in Section 10.2.2. Secondly, we will describe in Section 10.2.3 the considered universal shifted controller design along with a standard LPV controller approach for comparison purposes. In Section 10.2.4, we show experimental results of the control designs applied to the CMG setup and we analyze them to evaluate the obtained closed-loop performance.

10.2.2 Dynamical model of the CMG

We consider a four *Degree of Freedom* (DOF) CMG as displayed in Figure 10.1, with a schematic overview depicted in Figure 10.2. The three gimbals and disk of the

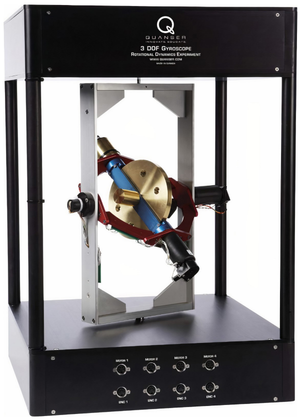


Figure 10.1: Photo of the CMG setup.

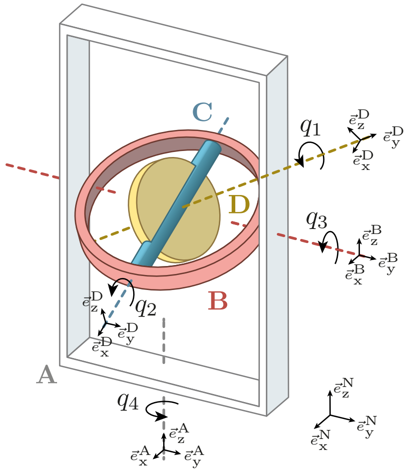


Figure 10.2: Schematic overview of the CMG.

CMG are modeled as four bodies, denoted by A, B, C, D, respectively. The angular rotations of D, C, B, and A are denoted by q_1, q_2, q_3, q_4 (in radian), respectively. Whereas, the current of the motors actuating D, C, B, and A are denoted by i_1, i_2, i_3, i_4 (in Ampere), respectively. Through Euler-Lagrange equations, the dynamical model of the CMG can be derived, see (Bloemers and Tóth 2019) for more details, resulting in

$$M(q(t))\ddot{q}(t) + \left(C(q(t), \dot{q}(t)) + F_{\mathrm{v}}\right)\dot{q}(t) = K_{\mathrm{m}} i(t) \tag{10.1}$$

$q = \mathrm{col}(q_1, q_2, q_3, q_4)$ (where $q(t) \in \mathbb{R}^4$), $i = \mathrm{col}(i_1, i_2, i_3, i_4)$ (where $i(t) \in \mathbb{R}^4$), M is the inertia matrix, C the Coriolis matrix, F_{v} viscous friction matrix, and K_{m} the motor constant matrix. The inertia matrix is given by $M(q) = M_{\mathrm{A}} + M_{\mathrm{B}}(q) + M_{\mathrm{C}}(q) + M_{\mathrm{D}}(q)$ with

$$M_{\mathrm{A}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathcal{K}_{\mathrm{A}} \end{bmatrix}, \tag{10.2a}$$

$$M_{\mathrm{B}}(q) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{J}_{\mathrm{B}} & 0 \\ 0 & 0 & 0 & \mathcal{I}_{\mathrm{B}} s_3^2 + \mathcal{K}_{\mathrm{B}} c_3^2 \end{bmatrix}, \tag{10.2b}$$

$$M_{\mathrm{C}}(q) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathcal{I}_{\mathrm{C}} & 0 & -\mathcal{I}_{\mathrm{C}} s_3 \\ 0 & 0 & \mathcal{J}_{\mathrm{C}} c_2^2 + \mathcal{K}_{\mathrm{C}} s_2^2 & \alpha_1 s_2 c_2 c_3 \\ 0 & -\mathcal{I}_{\mathrm{C}} s_3 & \alpha_1 s_2 c_2 c_3 & \mathcal{I}_{\mathrm{C}} s_3^2 + \left(\mathcal{J}_{\mathrm{C}} s_2^2 + \mathcal{K}_{\mathrm{C}} c_2^2\right) c_3^2 \end{bmatrix}, \tag{10.2c}$$

$$M_{\mathrm{D}}(q) = \begin{bmatrix} \mathcal{J}_{\mathrm{D}} & 0 & \mathcal{J}_{\mathrm{D}} c_2 & \mathcal{J}_{\mathrm{D}} s_2 c_3 \\ 0 & \mathcal{I}_{\mathrm{D}} & 0 & -\mathcal{I}_{\mathrm{D}} s_3 \\ \mathcal{J}_{\mathrm{D}} c_2 & 0 & \mathcal{I}_{\mathrm{D}} s_2^2 + \mathcal{J}_{\mathrm{D}} c_2^2 & \alpha_2 s_2 c_2 c_3 \\ \mathcal{J}_{\mathrm{D}} s_2 c_3 & -\mathcal{I}_{\mathrm{D}} s_3 & \alpha_2 s_2 c_2 c_3 & \mathcal{I}_{\mathrm{D}} s_3^2 + \left(\mathcal{I}_{\mathrm{D}} c_2^2 + \mathcal{J}_{\mathrm{D}} s_2^2\right) c_3^2 \end{bmatrix}, \tag{10.2d}$$

with s_i and c_i being shorthand for $\sin(q_i)$ and $\cos(q_i)$, respectively. The Coriolis matrix is given by

$$C(q,\dot{q}) = \begin{bmatrix} \dot{q}^\top & 0 & 0 & 0 \\ 0 & \dot{q}^\top & 0 & 0 \\ 0 & 0 & \dot{q}^\top & 0 \\ 0 & 0 & 0 & \dot{q}^\top \end{bmatrix} \begin{bmatrix} \Gamma_1(q) \\ \Gamma_2(q) \\ \Gamma_3(q) \\ \Gamma_4(q) \end{bmatrix}, \tag{10.3}$$

with

$$\Gamma_1(q) = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ \star & 0 & -\mathcal{J}_\mathrm{D} s_2 & \mathcal{J}_\mathrm{D} c_2 c_3 \\ \star & \star & 0 & -\mathcal{J}_\mathrm{D} s_2 s_3 \\ \star & \star & \star & 0 \end{bmatrix}, \tag{10.4a}$$

$$\Gamma_2(q) = \frac{1}{2}\begin{bmatrix} 0 & 0 & \mathcal{J}_\mathrm{D} s_2 & -\mathcal{J}_\mathrm{D} c_2 c_3 \\ \star & 0 & 0 & 0 \\ \star & \star & -2\alpha_3 s_2 c_2 & \alpha_3\left(c_2^2 c_3 - s_2^2 c_3\right) - \alpha_4 c_3 \\ \star & \star & \star & 2\alpha_3 c_2 c_3^2 s_2 \end{bmatrix}, \tag{10.4b}$$

$$\Gamma_3(q) = \frac{1}{2}\begin{bmatrix} 0 & -\mathcal{J}_\mathrm{D} s_2 & 0 & \mathcal{J}_\mathrm{D} s_2 s_3 \\ \star & 0 & 2\alpha_3 s_2 c_2 & \alpha_4 c_3 + \alpha_3\left(c_3 s_2^2 - c_2^2 c_3\right) \\ \star & \star & 0 & 0 \\ \star & \star & \star & -2\left(\alpha_5 + \alpha_3 s_2^2\right) c_3 s_3 \end{bmatrix}, \tag{10.4c}$$

$$\Gamma_4(q) = \frac{1}{2}\begin{bmatrix} 0 & \mathcal{J}_\mathrm{D} c_2 c_3 & -\mathcal{J}_\mathrm{D} s_2 s_3 & 0 \\ \star & 0 & \alpha_3\left(c_3 s_2^2 - c_2^2 c_3\right) - \alpha_4 c_3 & -2\alpha_3 c_2 c_3^2 s_2 \\ \star & \star & 2\alpha_3 c_2 s_2 s_3 & 2\left(\alpha_5 + \alpha_3 s_2^2\right) c_3 s_3 \\ \star & \star & \star & 0 \end{bmatrix}. \tag{10.4d}$$

The friction matrix is given by $F_\mathrm{v} = \mathrm{diag}(f_{\mathrm{v},1}, f_{\mathrm{v},2}, f_{\mathrm{v},3}, f_{\mathrm{v},4})$ and motor constant matrix is given by $K_\mathrm{m} = \mathrm{diag}(k_{\mathrm{m},1}, k_{\mathrm{m},2}, k_{\mathrm{m},3}, k_{\mathrm{m},4})$. Moreover,

$$\begin{aligned} \alpha_1 &= \mathcal{J}_\mathrm{C} - \mathcal{K}_\mathrm{C}, & \alpha_2 &= \mathcal{J}_\mathrm{D} - \mathcal{I}_\mathrm{D}, \\ \alpha_3 &= \mathcal{I}_\mathrm{D} - \mathcal{J}_\mathrm{C} - \mathcal{J}_\mathrm{D} + \mathcal{K}_\mathrm{C}, & \alpha_4 &= \mathcal{I}_\mathrm{B} + \mathcal{I}_\mathrm{C} - \mathcal{K}_\mathrm{B} - \mathcal{K}_\mathrm{C}, \end{aligned} \tag{10.5}$$

and the physical parameters of the system are given in Tables 10.1 and 10.2.

Table 10.1: Moments of inertia of the gimbals of the CMG.

Gimbal	Moments of inertia [kg·m^2]		
i	$\mathcal{I}_i$	$\mathcal{J}_i$	$\mathcal{K}_i$
A	$9.02 \cdot 10^{-2}$	$5.34 \cdot 10^{-2}$	$3.74 \cdot 10^{-2}$
B	$3.88 \cdot 10^{-3}$	$1.62 \cdot 10^{-3}$	$2.00 \cdot 10^{-3}$
C	$9.21 \cdot 10^{-4}$	$1.62 \cdot 10^{-3}$	$2.55 \cdot 10^{-3}$
D	$3.01 \cdot 10^{-3}$	$5.50 \cdot 10^{-3}$	$3.01 \cdot 10^{-3}$

The model (10.1) can be written in the form

$$\mathcal{E}(x_\mathrm{f}(t))\dot{x}_\mathrm{f}(t) = \mathcal{A}(x_\mathrm{f}(t))x_\mathrm{f}(t) + B_\mathrm{g} u_\mathrm{f}(t), \tag{10.6}$$

Table 10.2: Other physical parameters of the CMG.

j	$f_{v,j}$ [N·m· s]	$\mathcal{K}_{m,j}$ [N·m· A^{-1}]
1	$7.11 \cdot 10^{-5}$	$6.80 \cdot 10^{-2}$
2	$1.24 \cdot 10^{-5}$	$1.01 \cdot 10^{-1}$
3	$1.41 \cdot 10^{-2}$	$1.05 \cdot 10^{-1}$
4	$3.72 \cdot 10^{-2}$	$6.06 \cdot 10^{-2}$

where $x_\mathrm{f} = \mathrm{col}(q, \dot{q})$, $u_\mathrm{f} = i$, and

$$\mathcal{E}(x_\mathrm{f}) = \begin{bmatrix} I & 0 \\ 0 & M(q) \end{bmatrix}, \qquad \mathcal{A}(x_\mathrm{f}) = \begin{bmatrix} 0 & I \\ 0 & -C(q, \dot{q}) - F_\mathrm{v} \end{bmatrix}, \qquad B_\mathrm{g} = \begin{bmatrix} 0 \\ K_\mathrm{m} \end{bmatrix}. \tag{10.7}$$

As control objective, we consider a special operating mode of the disk, where only inputs i_1 and i_2 are available for control and gimbal B is locked at $q_3 = 0$, meaning that $\dot{q}_3 = 0$. In this operating mode our objective will be the control of the outer (silver) gimbal and the disk speed, i.e., q_4 and $\dot{q}_1$, respectively. This can be seen as simple form of pointing control for satellites. Taking into account these restrictions, we can obtain the resulting dynamical model from (10.6) by removing the 3th and 7th row and column form $\mathcal{E}$ and $\mathcal{A}$, corresponding to q_3 and $\dot{q}_3$, and the 3th and 4th column of B_g, corresponding to i_3 and i_4. Moreover, also the 1st and 2nd rows and columns of $\mathcal{E}$ and $\mathcal{A}$ are removed, corresponding to q_1 and q_2, as these will not considered in our control objectives (see also Section 10.2.3). Note that this can be done as the bottom left (block) entry of $\mathcal{E}$ and $\mathcal{A}$ is zero, meaning the angular velocities have no linear dependency on q_1 and q_2 in our model. The resulting model will be of the form

$$\mathcal{E}_\mathrm{r}(q_2(t))\dot{x}_\mathrm{g}(t) = \mathcal{A}_\mathrm{r}(x_\mathrm{g}(t), q_2(t))x_\mathrm{g}(t) + B_\mathrm{r}u_\mathrm{g}(t), \tag{10.8}$$

where $x_\mathrm{g} = \mathrm{col}(q_4, \dot{q}_1, \dot{q}_2, \dot{q}_4)$ and $u_\mathrm{g} = \mathrm{col}(i_1, i_2)$, and where $\mathcal{E}_\mathrm{r}$, $\mathcal{A}_\mathrm{r}$, and B_r are the $\mathcal{E}$, $\mathcal{A}$, and B_g matrices/matrix functions from (10.7) with the discussed rows and columns removed, respectively. Note that the model is still dependent on q_2, due to its dependency in the $\mathcal{E}_\mathrm{r}$ and $\mathcal{A}_\mathrm{r}$ matrix. However, it is not a free parameter, due to its relation to $\dot{q}_2$. We assume that q_4, $\dot{q}_1$, and $\dot{q}_4$ can be measured, and these are therefore considered to be the outputs of the system, i.e., $y_\mathrm{g} = \mathrm{col}(q_4, \dot{q}_1, \dot{q}_4)$.

Finally, (10.8) can be written in a nonlinear state-space form

$$\dot{x}_\mathrm{g}(t) = f_\mathrm{g}(x_\mathrm{g}(t), u_\mathrm{g}(t), q_2(t)), \tag{10.9a}$$
$$y_\mathrm{g}(t) = C_\mathrm{g}x_\mathrm{g}(t), \tag{10.9b}$$

with $x_\mathrm{g}(t) \in \mathbb{R}^4$, $u_\mathrm{g}(t) \in \mathbb{R}^2$, $y_\mathrm{g}(t) \in \mathbb{R}^3$, and

$$f_\mathrm{g}(x_\mathrm{g}, u_\mathrm{g}, q_2) = \mathcal{E}_\mathrm{r}(q_2)^{-1}\mathcal{A}_\mathrm{r}(x_\mathrm{g}, q_2)x_\mathrm{g} + \mathcal{E}_\mathrm{r}(q_2)^{-1}B_\mathrm{r}u_\mathrm{g}, \tag{10.10}$$
$$C_\mathrm{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{10.11}$$

Note that f_g is well-posed as $\mathcal{E}_\mathrm{r}$ is invertible (due to invertibility of the inertia matrix M).

10.2.3 Controller design

Universal shifted controller

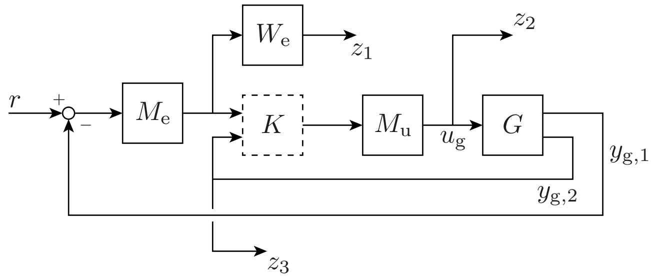


Figure 10.3: Generalized plant considered for the control design for the CMG.

For the universal shifted controller design, in order to achieve constant reference tracking for q_4 and $\dot{q}_1$, we consider a generalized plant structure as is depicted in Figure 10.3. In Figure 10.3, r is the reference signal, z_i are the generalized performance channels, G is the CMG given by (10.9) where $y_{g,1} = \mathrm{col}(q_4, \dot{q}_1)$ and $y_{g,2} = \dot{q}_4$ (such that $y_g = \mathrm{col}(y_{g,1}, y_{g,2})$), K is our to-be-designed controller, and the weighting filters are *Linear Time-Invariant* (LTI), given as (in transfer function representation):

$$M_e(s) = \begin{bmatrix} \frac{s+\pi}{s} & 0 \\ 0 & \frac{s+\pi}{s} \end{bmatrix}, \qquad M_u(s) = \begin{bmatrix} \frac{20\pi}{s+20\pi} & 0 \\ 0 & \frac{20\pi}{s+20\pi} \end{bmatrix},$$
$$W_e(s) = \begin{bmatrix} 10^{\left(-\frac{5}{20}\right)} \frac{s+2}{s+\pi} & 0 \\ 0 & 10^{\left(-\frac{10}{20}\right)} \frac{s+4}{s+\pi} \end{bmatrix}. \tag{10.12}$$

The combinations of the weighting filters $W_e(s)M_e(s)$ ensures good tracking performance for (piece-wise) constant reference signals. Moreover, W_u is used to low pass filter the signals entering the plant G. Additionally, $y_{g,2} = \dot{q}_4$ is fed back to the controller and added as generalized performance channel in order to penalize high velocities of q_4 to ensure smooth tracking behavior of q_4. Moreover, the plant G given by (10.9) is pre-scaled, such that signals of unit magnitude correspond to the maximum values of the signals. More concretely, instead of considering G directly in the generalized plant, we consider

$$\tilde{G} = S_y^{-1} G S_u, \tag{10.13}$$

where $S_y = \mathrm{diag}(\frac{\pi}{4}, 10, 3)$ and $S_u = \mathrm{diag}(3, 5)$. Later, when the controller K is implemented on the setup, this scaling is also taken into account.

Based on this generalized plant, we then perform the three-step-procedure, as discussed in Section 4.5.2, to design the universal shifted controller.

In the first step, we require the construction of a *Velocity Parameter-Varying* (VPV) embedding of our generalized plant. In the generalized plant, the CMG, G, given by

(10.9) is the only nonlinear system as the weightings filters are LTI. Consequently, we only require construction of a VPV embedding of (10.9).

For construction of the VPV embedding of (10.9), we first compute its velocity form, which results in (omitting dependence on time for brevity):

$$\ddot{x}_{\mathrm{g}} = A_{\mathrm{v}}(x_{\mathrm{g}}, q_2)\dot{x}_{\mathrm{g}} + B_{\mathrm{v}}(q_2)\dot{u}_{\mathrm{g}} + \frac{\partial f_{\mathrm{g}}}{\partial q_2}(x_{\mathrm{g}}, u_{\mathrm{g}}, q_2)\dot{q}_2, \tag{10.14a}$$

$$\dot{y}_{\mathrm{g}} = C_{\mathrm{g}}\dot{x}_{\mathrm{g}}, \tag{10.14b}$$

where $A_{\mathrm{v}} = \frac{\partial f_{\mathrm{g}}}{\partial x_{\mathrm{g}}}$ and $B_{\mathrm{v}} = \frac{\partial f_{\mathrm{g}}}{\partial u_{\mathrm{g}}}$. Note that A_{v} only depends on x_{g} and q_2 and B_{v} only on q_2 due to the structure of f_{g}, see (10.10). As the nonlinear system (10.9) also depends on q_2 (not part of x_{g}), we also get the extra term $\frac{\partial f_{\mathrm{g}}}{\partial q_2}(x_{\mathrm{g}}, u_{\mathrm{g}}, q_2)\dot{q}_2$ in the velocity form. We will omit this term as part of the considered velocity form, and consider it to be a disturbance/unmodeled dynamics, which through the controller will be compensated. Hence, we will consider the following velocity form for (10.9):

$$\ddot{x}_{\mathrm{g}} = A_{\mathrm{v}}(x_{\mathrm{g}}, u_{\mathrm{g}}, q_2)\dot{x}_{\mathrm{g}} + B_{\mathrm{v}}(q_2)\dot{u}_{\mathrm{g}}, \tag{10.15a}$$

$$\dot{y}_{\mathrm{g}} = C_{\mathrm{g}}\dot{x}_{\mathrm{g}}. \tag{10.15b}$$

Based on the velocity form (10.15), we construct a VPV embedding of (10.9) (see also Definition 4.7), which is given by

$$\dot{x}_{\mathrm{v}} = A(p)x_{\mathrm{v}} + B(p)u_{\mathrm{v}}, \tag{10.16a}$$

$$y_{\mathrm{v}} = C_{\mathrm{g}}u_{\mathrm{v}}, \tag{10.16b}$$

where $x_{\mathrm{v}}(t) \in \mathbb{R}^4$, $u_{\mathrm{v}}(t) \in \mathbb{R}^2$, $y_{\mathrm{v}}(t) \in \mathbb{R}^3$, with scheduling-variable $p(t) \in \mathcal{P}$ and scheduling-map η such that $p(t) = \eta(x_{\mathrm{g}}(t), q_2(t))$ where $\eta(x_{\mathrm{g}}, q_2) = [q_2 \;\; \dot{q}_1 \;\; \dot{q}_2 \;\; \dot{q}_4]^\top = [p_1 \;\; p_2 \;\; p_3 \;\; p_4]^\top$, note that the scheduling-map also depends on q_2. As the scheduling set, we consider (the hypercube) $\mathcal{P} = [-0.8,\, 0.8] \times [30,\, 50] \times [-4,\, 4] \times [-3,\, 3]$, corresponding to the (VPV) embedding region $\mathcal{X} = \mathbb{R} \times [30,\, 50] \times [-4,\, 4] \times [-3,\, 3]$ of (10.9). The VPV embedding of (10.9), given by (10.16), together with the LTI weighting filters (for which the dynamics of their velocity forms are equivalent to their primal form dynamics) M_{e}, M_{u}, and W_{e}, as interconnected in Figure 10.3, then give us a VPV embedding of the generalized plant.

For the second step in the universal shifted controller synthesis procedure, we synthesize an $\mathcal{L}_2$-gain optimal LPV controller (of the form (4.41)) for our VPV embedding of the generalized plant. Due to the complex scheduling-dependency, we use a grid-based LPV controller synthesis approach (Wu 1995), which has been implemented in the LPVcore Toolbox (Boef et al. 2021). For the grid-based synthesis, we consider the grid points $p_1 \in \{-0.8, 0, 0.8\}$, $p_2 \in \{30, 40, 50\}$, $p_3 \in \{-4, 0, 40\}$, $p_4 \in \{-3, 0, 3\}$, i.e., three grid points for each scheduling-variable, for a total of 81 grid points. For the synthesis, a parameter independent quadratic storage function is considered (i.e., of the form (2.36), where M is a constant positive definite matrix). We also restrict the B and D matrix of the state-space controller (i.e., B_{k} and D_{k} in (4.41)) to be parameter independent, s.t. later on the resulting universal shifted controller does not depend on $\dot{p}$. Moreover, a *Linear Matrix Inequality* (LMI) region constraint is added (also implemented in the LPVcore Toolbox), s.t. the closed-loop eigenvalues of the generalized plant and controller at each grid point have a real part larger than -500. Under these consideration, the resulting LPV controller, i.e., the velocity controller, obtains a closed-loop $\mathcal{L}_2$-gain of 1.14 for the VPV embedding of the generalized plant. The Bode magnitude plot of the velocity form of the closed-loop system is given in Figure 10.4 at the considered grid points.

Finally, based on the synthesized velocity controller, we realize the universal shifted controller as the last step of the universal shifted controller synthesis procedure. As our generalized plant includes integral filters, we make use of the realization result in Corollary 4.2, see also Section 4.5.4 and Theorem 4.12. Note, as aforementioned, the velocity controller has been synthesized such that its B and D matrices are constant, hence, our universal shifted controller does not depend on $\dot{p}$. The resulting universal shifted controller (of the form (4.42)) achieves universal shifted (asymptotic) stability and quadratic universal shifted performance of the closed-loop (under Proposition 4.1 this is with an $\mathcal{L}_{\mathrm{s}2}$-gain of 1.14).

Before we show the results of this universal shifted controller design applied to the experimental CMG setup, we first shortly discuss the design of a standard LPV controller, which will be used for comparison.

Standard LPV controller

For the standard LPV controller design, we use the same generalized plant structure[1] as for our universal shifted controller design, depicted in Figure 10.3, and we use the same weighting filters (see (10.12)). Consequently, we only require construction of a global LPV embedding of (10.9), as it is the only nonlinear system in the interconnection that makes up the generalized plant. The LPV embedding of (10.9) we construct is given by

$$\dot{x}_{\mathrm{g}} = A_{\mathrm{s}}(p_{\mathrm{s}})x_{\mathrm{g}} + B_{\mathrm{s}}(p_{\mathrm{s}})u_{\mathrm{g}}, \tag{10.17a}$$

$$y_{\mathrm{g}} = C_{\mathrm{g}}u_{\mathrm{g}}, \tag{10.17b}$$

(where subscript 's' is used to denote the standard LPV embedding variables), where $p_{\mathrm{s}} \in \mathcal{P}_{\mathrm{s}}$ is the scheduling-variable and for the scheduling-map η_{s} we consider $p_{\mathrm{s}} = \eta_{\mathrm{s}}(x_{\mathrm{g}}, q_2)$ s.t. $\eta_{\mathrm{s}}(x_{\mathrm{g}}, q_2) = [q_2\ \dot{q}_1\ \dot{q}_2\ \dot{q}_4]^\top = [p_{\mathrm{s},1}\ p_{\mathrm{s},2}\ p_{\mathrm{s},3}\ p_{\mathrm{s},4}]^\top$. As f_{g} can be written in a factorized form, see (10.10), the LPV matrix functions A_{s} and B_{s} are chosen as

$$A_{\mathrm{s}}(\eta_{\mathrm{s}}(x_{\mathrm{g}}, q_2)) = \mathcal{E}_{\mathrm{r}}(q_2)^{-1}\mathcal{A}_{\mathrm{r}}(x_{\mathrm{g}}, q_2), \qquad B_{\mathrm{s}}(\eta_{\mathrm{s}}(x_{\mathrm{g}}, q_2)) = \mathcal{E}_{\mathrm{r}}(q_2)^{-1}B_{\mathrm{r}}. \tag{10.18}$$

Note that the scheduling-map of our global LPV embedding is equal to the scheduling-map of the VPV embedding for the universal shifted controller design, i.e., $\eta_{\mathrm{s}}(x_{\mathrm{g}}, q_2) = \eta(x_{\mathrm{g}}, q_2) = \begin{bmatrix} q_2 & \dot{q}_1 & \dot{q}_2 & \dot{q}_4 \end{bmatrix}^\top$, consequently $p(t) = p_{\mathrm{s}}(t)$. Hence, for a fair comparison between the universal shifted and standard LPV controller designs, we consider the same scheduling set, meaning, $\mathcal{P}_{\mathrm{s}} = [-0.8,\ 0.8] \times [30,\ 50] \times [-4,\ 4] \times [-3,\ 3]$.

[1] With also the same pre-scaling applied as in (10.13).

Similarly, we also synthesize an $\mathcal{L}_2$-gain LPV controller for the generalized plant consisting of (10.17) using a grid-based synthesis approach. We take the same considerations for the standard LPV (grid-based) controller synthesis as were taken for the (grid-based) velocity controller synthesis, i.e., same grid points, same form of quadratic parameter independent storage function, restricting the B and D matrices to be parameter independent in the controller, and the same LMI region constraint. This results in an LPV controller that achieves an $\mathcal{L}_2$-gain bound of 1.07 for its closed-loop interconnection with the nonlinear generalized plant.

Next, we compare both the universal shifted controller design and the standard LPV controller design on the experimental CMG setup.

10.2.4 Experimental results

For the experimental study, both the standard LPV controller and universal shifted controller designs are implemented on the CMG setup. On the experimental setup, the control inputs, corresponding to the currents sent to the motors, are saturated between $\pm$ 3 [A] for i_1 and $\pm$ 5 [A] for i_2. For the experiment, the disk is first sped up to a velocity of 40 rad/s before switching on either the standard LPV controller or the universal shifted controller. As reference signals, a piecewise-constant signal is chosen which makes steps of $\frac{\pi}{4}$ rad every 5 seconds for q_4, while the reference for $\dot{q}_1$ is taken to be constant and equal to 40 rad/s. Under this reference, the resulting closed-loop behavior of the setup with the standard LPV controller design and with the universal shifted controller is depicted in Figures 10.5 to 10.7.

In Figure 10.5, the angle q_4 and angular velocity $\dot{q}_1$ are depicted, along with their to-be-followed reference signals. It can be seen that as soon as the reference for q_4 switches for the first time at $t = 5$ seconds, the standard LPV controller starts oscillating around the reference. This is likely due to (unmodeled) Coulomb friction which affects the experimental setup, which acts as a disturbance. This once again shows, as we have also demonstrated in the other chapters, that the standard LPV controller is not able to ensure the desired stability and performance guarantees for reference tracking and disturbance rejection. On the other hand, the universal shifted controller is able to smoothly track the piece-wise constant reference signal for q_4 and $\dot{q}_1$ without any oscillations, as it ensures the stronger stability and performance guarantees in the form of universal shifted stability and performance. The differences in performance between the two controllers are also clearly visible in Figures 10.6 and 10.7, where the motor currents (generated by the controller) and scheduling trajectories (corresponding to q_2 and angular velocities of the CMG) are depicted. The induced oscillations due of the LPV controller even result in the motor currents becoming saturated, while again, the universal controller stays well within the saturation limits and varies smoothly. Note that the scheduling-variables also stay within the assumed set $\mathcal{P} = \mathcal{P}_\mathrm{s}$ (only p_1 for the LPV controller briefly leaves the scheduling set), see Figure 10.7.

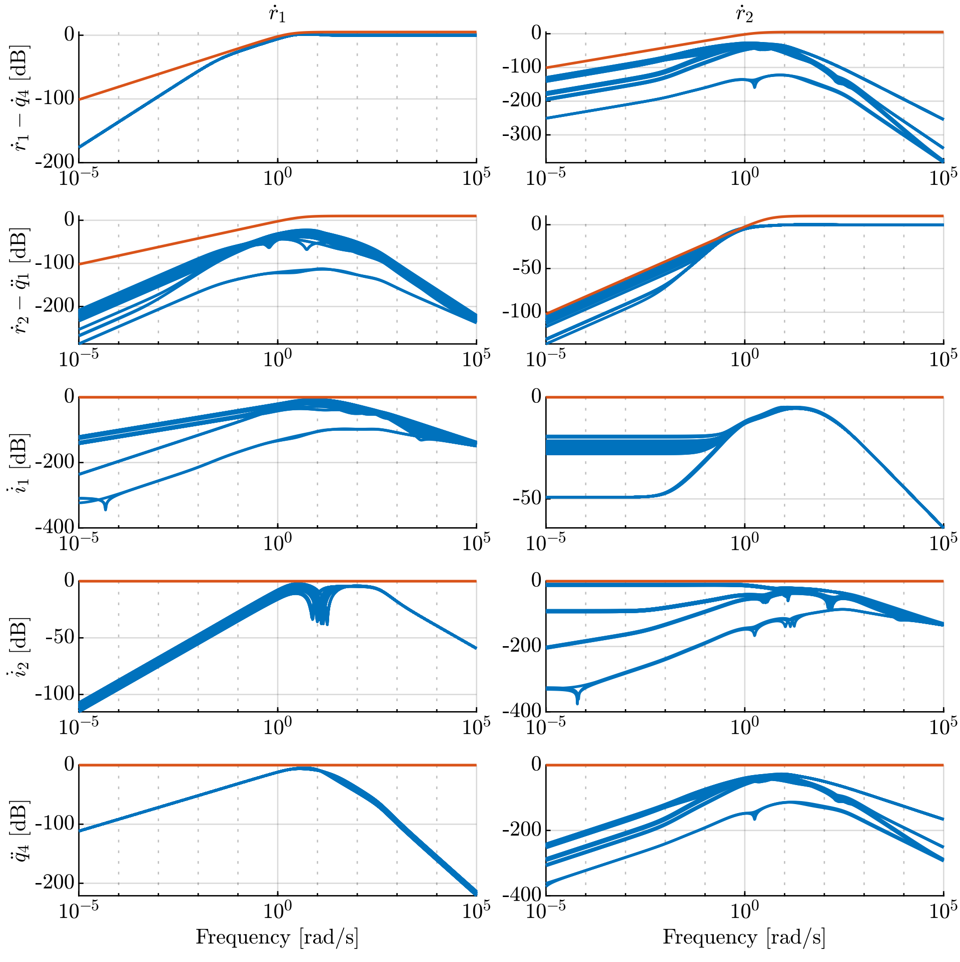


Figure 10.4: Bode magnitude plot of velocity form of closed-loop system at the considered grid points (—), along with the corresponding inverse weighting filters (—).

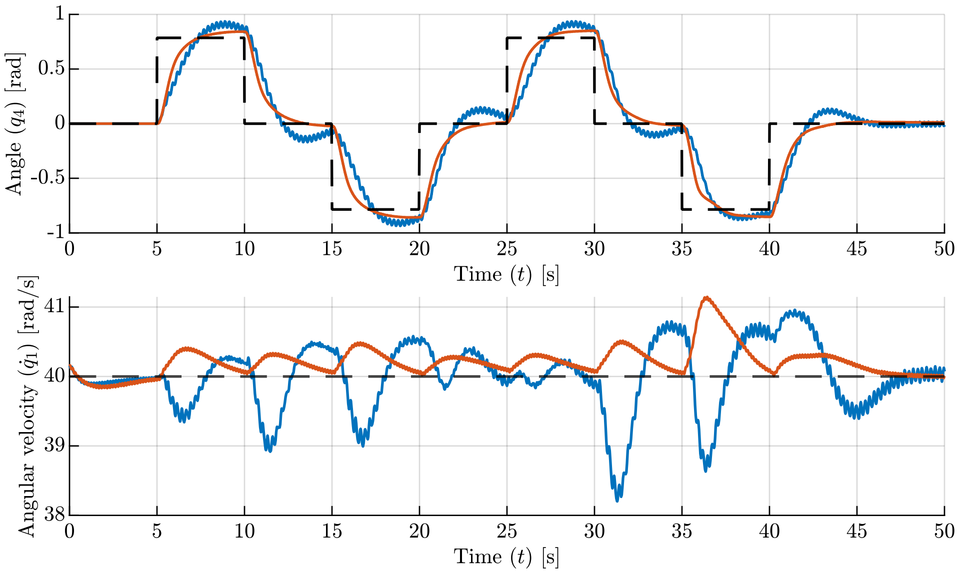


Figure 10.5: Angle q_4 (top) and angular velocity $\dot{q}_1$ (bottom) of the CMG in closed-loop with the standard LPV ( ) and the universal shifted ( ) controllers under reference ( ).

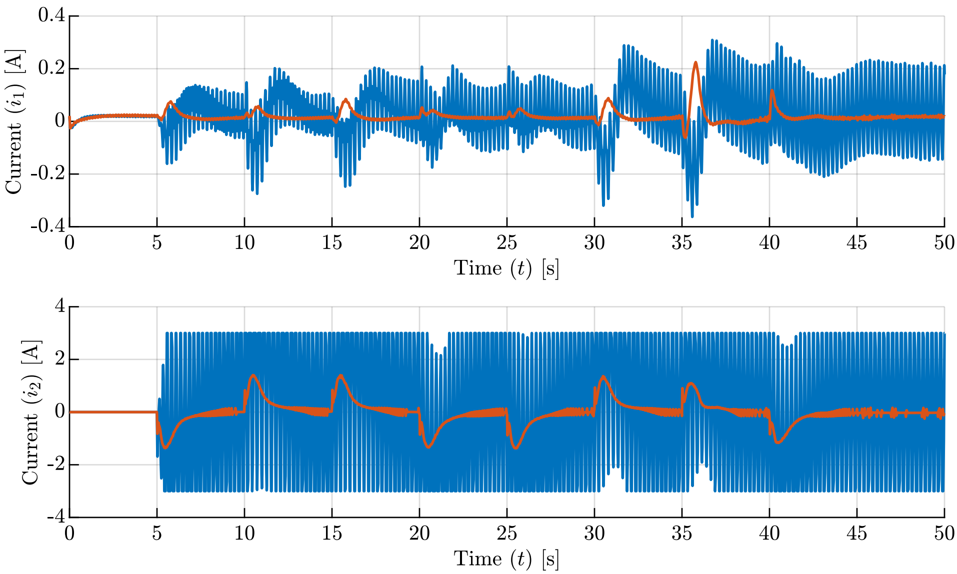


Figure 10.6: Motor currents i_1 (top) and i_2 (bottom) sent to the CMG in closed-loop with the standard LPV ( ) and the universal shifted ( ) controllers.

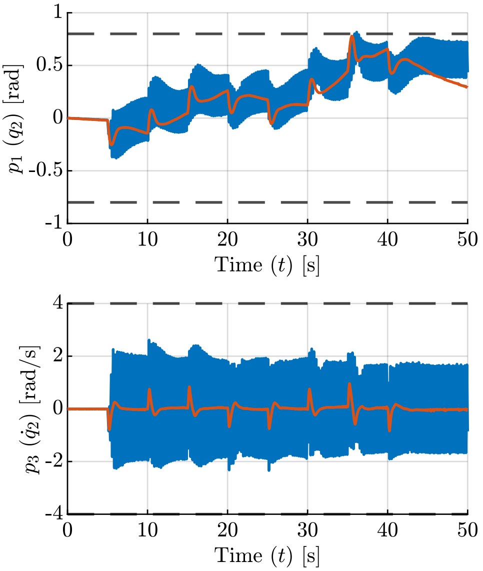


Figure 10.7: Scheduling trajectories corresponding to the CMG in closed-loop with the standard LPV ( ) and the universal shifted ( ) controllers, along with the considered scheduling limits ( ).

10.3 Incremental Control of a Generic Parafoil Return Vehicle

10.3.1 Introduction

In this section, we present the application of the incremental controller design we have proposed in Chapter 6 to a GPRV for reference tracking purposes during atmospheric flight. GPRVs are used as reusable space transportation systems, in order to perform missions in low earth orbit. In the past, they have been explored to be used as crew return vehicles, such as the NASA X-38 (Shin et al. 2001), see Figure 10.8. More recently, the European Space Agency has been developing a GPRV, the Space Rider, see Figure 10.9. Its primary intended use will be to perform research in microgravity and as demonstration platform for robotic and surveillance applications (Cacciatore et al. 2019; De Lange 2021).

In this application example, we will specifically focus on GPRVs during their so-called terminal guidance phase. During this phase, the craft has entered the Earth's atmosphere and needs to be steered in order to land at a designated location on the surface.

The rest of the section is structured as follows. In Section 10.3.2, we give a description of the dynamical model of the GPRV that will be considered. Next, in Section 10.3.3, we discuss the incremental controller design that is developed for the GPRV. Finally, in Section 10.3.4, we show results of a simulation study that is performed in order to evaluate the performance of the incremental controller design. Moreover, in the simulation study the incremental controller design is also compared to a (local) LPV controller design.

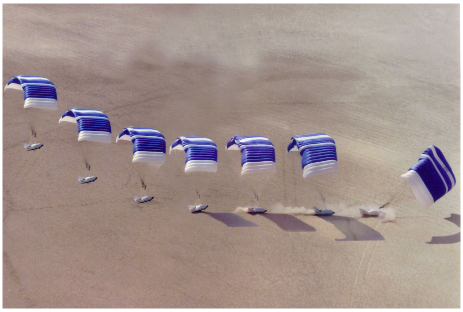


Figure 10.8: Photo (sequence) of NASA X-38 CRV prototype.

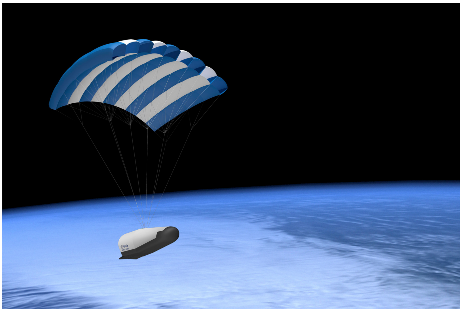


Figure 10.9: Artist's impression of ESA Space Rider.

10.3.2 Dynamical model of the GPRV

The dynamics of a GPRV in atmospheric flight are complex due to aerodynamical effects, governed by fluid dynamics, and the interactions between spacecraft body and the parafoil (De Lange 2021). This makes navigation of the vehicle also challenging from a control perspective, especially as there is no active propulsion and the system can only be actuated through steering the parafoil. Here, we consider a simplified dynamical model describing a GPRV. Namely, we consider a 3 DOF model, whereby the combination of the spacecraft body and the parafoil is modeled as a point mass which moves in 3D space affected by aerodynamical effects such as lift and drag.

More concretely, we consider the model of the GPRV, adopted from (De Lange 2022), given by the following set of differential equations

$$\dot{r}_{\mathrm{x}}(t) = V(t)\cos(\alpha(t))\cos(\psi(t)); \tag{10.19a}$$
$$\dot{r}_{\mathrm{y}}(t) = V(t)\cos(\alpha(t))\sin(\psi(t)); \tag{10.19b}$$
$$\dot{r}_{\mathrm{z}}(t) = V(t)\sin(\alpha(t)); \tag{10.19c}$$
$$\dot{V}(t) = -\tfrac{1}{m}D(V(t),\epsilon(t)) - g\sin(\alpha(t)); \tag{10.19d}$$
$$\dot{\alpha}(t) = \frac{\tfrac{1}{m}L(V(t),\epsilon(t))\cos(\sigma(t)) - g\cos(\alpha(t))}{V(t)}; \tag{10.19e}$$
$$\dot{\psi}(t) = \frac{L(V(t),\epsilon(t))\sin(\sigma(t))}{mV(t)\cos(\alpha(t))}; \tag{10.19f}$$
$$\dot{\epsilon}(t) = \frac{\delta_{\mathrm{s}}(t) - \epsilon(t)}{\tau_\epsilon}; \tag{10.19g}$$
$$\dot{\sigma}(t) = \frac{\delta_{\mathrm{a}}(t) - \sigma(t)}{\tau_\sigma}; \tag{10.19h}$$

where t [s] is time, r_{x} [m], r_{y} [m] and r_{z} [m] is the position of the GPRV in x (north), y (east), z (up), direction (in a left-handed coordinate system) in the Earth frame, respectively, V [m·s^{-1}] is its absolute velocity, α [rad] is the flight path angle, and ψ [rad] is the heading angle. δ_{s} and δ_{a} are the (normalized) symmetric and asymmetric deflections (and hence are dimensionless), respectively, which are the inputs of the system. Furthermore, ϵ and σ represent delayed versions of the symmetric and asymmetric deflection, respectively. The mass of the system is given by m [kg], the gravitational acceleration is g [m·s^{-2}] (and is considered to be constant along the atmospheric flight), and τ_ϵ and τ_σ are time constants. The aerodynamic drag D and aerodynamic lift L are given by

$$D(V(t),\epsilon(t)) = \frac{1}{2}\rho V(t)^2 S C_{\mathrm{D}}(\epsilon(t)), \tag{10.20a}$$
$$L(V(t),\epsilon(t)) = \frac{1}{2}\rho V(t)^2 S C_{\mathrm{L}}(\epsilon(t)), \tag{10.20b}$$

where ρ is the air-density [kg·m^{-3}] (assumed to be constant), S is the aerodynamic surface area [m^2], C_{D} and C_{L} are aerodynamic drag and lift coefficients, respectively, given as lookup tables. The considered physical parameters are given in Table 10.3. A schematic representation of the GPRV based on this model is given in Figure 10.10.

As aforementioned, we consider control of the GPRV during the terminal guidance phase. During the terminal guidance phase, the vehicle needs to follow a predetermined path which is expressed in terms of a heading profile that needs to be followed. Therefore, for this model of the GPRV, we are interested in regulating the

Table 10.3: Physical parameters of the GPRV model.

Parameter	Value	Unit
g	9.81	m·s^{-2}
m	$2.35 \cdot 10^3$	kg
ρ	1.23	kg·m^{-3}
S	$2.79 \cdot 10^2$	m^2
τ_ϵ	$2 \cdot 10^{-2}$	-
τ_σ	$2 \cdot 10^{-2}$	-

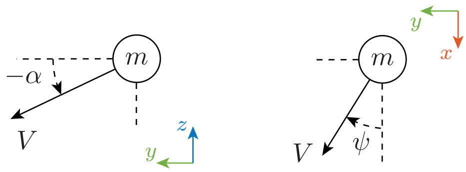


Figure 10.10: Schematic representation of the GPRV based on (10.19).

flight-path angle α and tracking a desired heading angle ψ, consequently we consider them to be outputs of the system which can be measured. Based on this, we can rewrite the dynamics of the GPRV, given by (10.19), in the following nonlinear state-space representation

$$\dot{x}_\mathrm{g}(t) = f_\mathrm{g}(x_\mathrm{g}(t)) + B_\mathrm{g} u_\mathrm{g}(t)), \tag{10.21a}$$
$$y_\mathrm{g}(t) = C_\mathrm{g} x_\mathrm{g}(t), \tag{10.21b}$$

where $x_\mathrm{g}(t) = \mathrm{col}(V(t), \alpha(t), \psi(t), \epsilon(t), \sigma(t)) \in \mathbb{R}^5$, $u_\mathrm{g}(t) = \mathrm{col}(\delta_\mathrm{s}(t), \delta_\mathrm{a}(t)) \in [-1,\, 1] \times [-1,\, 1]$, and where

$$f_\mathrm{g}(x_\mathrm{g}) = \begin{bmatrix} -\frac{1}{m} D(V,\epsilon) - g\sin(\alpha) \\ \frac{\frac{1}{m} L(V,\epsilon)\cos(\sigma) - g\cos(\alpha)}{V} \\ \frac{L(V,\epsilon)\sin(\sigma)}{mV\cos(\alpha)} \\ \frac{-\epsilon}{\tau_\epsilon} \\ \frac{-\sigma}{\tau_\sigma} \end{bmatrix}, \quad B_\mathrm{g} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C_\mathrm{g} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}. \tag{10.22}$$

Note that we have omitted the positional dynamics, given by (10.19a)–(10.19c), as the dynamics of x_g do not depend on it.

10.3.3 Controller design

Generalized plant

In order to achieve our control objective of regulating the flight-path angle α and tracking a desired heading angle ψ, we consider a generalized plant structure as depicted in Figure 10.11. In Figure 10.11, r is the to-be-followed reference and d is

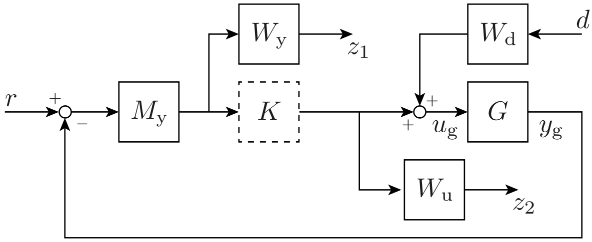


Figure 10.11: Generalized plant considered for the flight control design of the GPRV.

the disturbance, which collectively are the generalized disturbance channel w (i.e., $w = \mathrm{col}(r, d)$), z_1 and z_2 together are the generalized performance channel, G is system given by (10.21), K is the to-be-designed controller, M_y, W_y, W_u, and W_d are (LTI) weighting filters given (in transfer function representation by):

$$\begin{gathered} M_\mathrm{y}(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1.12(s^2+0.71s+0.18)}{s^2} \end{bmatrix}, \qquad W_\mathrm{y}(s) = \begin{bmatrix} 1 & 0 \\ 0 & \frac{0.45(s+0.63)^2}{s^2+0.71s+0.18} \end{bmatrix}, \\ W_\mathrm{u} = \begin{bmatrix} 0.32 & 0 \\ 0 & 0.32 \end{bmatrix}, \qquad W_\mathrm{d} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}. \end{gathered} \tag{10.23}$$

The combinations of the weighting filters $W_\mathrm{y}(s)M_\mathrm{y}(s)$ ensures regulation of α and good tracking performance of ψ, especially for constant and ramp references due the -40 dB/dec slope of the filter. The constant weight W_u ensures that too large control inputs are penalized. The constant W_d is used to ensure robustness against disturbances. Like for the universal shifted controller design for the CMG in Section 10.2.3, we pre-scale G using (10.13). In this case, for the GPRV, the pre-scale weights are given by $S_\mathrm{u} = I$ and $S_\mathrm{y} = \mathrm{diag}(0.5, \frac{\pi}{2})$.

Considering this generalized plant, we then use the three-step-design procedure to perform an incremental controller design as has been presented in Section 6.3. Note that G given by (10.21) has linear dependence w.r.t. u_g and has a linear output map, hence, we have that the generalized plant is of the form (6.4).

DPV embedding

For the first step in the procedure, we need to compute the differential form of the generalized plant and construct a *Differential Parameter-Varying* (DPV) embedding based on it. The system given by (10.21) is the only nonlinear system in the generalized plant, hence, we only require construction of a DPV embedding of (10.21). As the dynamics of (10.21) depend on the aerodynamic drag and lift coefficients, C_D and C_L, which are given by lookup tables, we cannot analytically compute the differential form of (10.21). Therefore, we use the *Deep (Artifical) Neural Network* (DNN) based *Scheduling Dimension Reduction* (SDR) method of Chapter 9, in order to construct an (approximate version of the) DPV embedding of (10.21) based on a data set, see also Section 9.5.2. More concretely, the differential form of (10.21) is given by

$$x_{\delta,\mathrm{g}}(t) = A_{\delta,\mathrm{g}}(x_\mathrm{g}(t)) + B_\mathrm{g} u_{\delta,\mathrm{g}}(t)), \tag{10.24a}$$
$$y_{\delta,\mathrm{g}}(t) = C_\mathrm{g} x_{\delta,\mathrm{g}}(t), \tag{10.24b}$$

where $A_{\delta,\mathrm{g}} = \frac{\partial f_\mathrm{g}}{\partial x_\mathrm{g}}$. While we cannot analytically compute $A_{\delta,\mathrm{g}}$, we can use *Automatic Differentiation* (AD) methods in order to evaluate it at different points x_g in a data set $\mathfrak{D}$, corresponding to the considered embedding region $\mathcal{X}$. For the DPV embedding region, we consider[2] $\mathcal{X} = [10,\, 21] \times [-0.35,\, -0.21] \times \mathbb{R} \times [-1,\, 1] \times [-1,\, 1]$. As data set $\mathfrak{D}$, we consider an equidistant grid on $\mathcal{X}$ with 9 grid points in each dimension (for a total of data 6561 points). For the DNN SDR method, the output data matrix Λ of the DNN, see also (9.22), is given by

$$\Lambda = \left[\mathrm{vec}\left(A_{\delta,\mathrm{g}}(x_{\mathrm{g},(1)})\right) \quad \ldots \mathrm{vec}\left(A_{\delta,\mathrm{g}}(x_{\mathrm{g},(6561)})\right)\right] \tag{10.25}$$

where $x_{\mathrm{g},(i)}$ is the i'th element in $\mathfrak{D}$. Instead of considering x_g directly as an input to the DNN, we create the extended vector θ that is considered to be the input

$$\theta = \mathrm{col}(V, \sin(\alpha), \sin(\sigma), \cos(\alpha), \cos(\sigma), \epsilon). \tag{10.26}$$

This extended vector is chosen as it will wrap the angle variation in the interval $[-1,\, 1]$ in order to help training of the DNN. The input data matrix Γ of the DNN, see also (9.6), is then given by

$$\Gamma = \left[\theta_{(1)} \quad \ldots \quad \theta_{(6561)}\right], \tag{10.27}$$

where $\theta_{(i)}$ is the value of θ corresponding to $x_{\mathrm{g},(i)}$. The normalized version of the data matrices Γ and Λ are then used as the input data and output data, respectively, to train the DNN.

For the considered architecture of the DNN, see also Figure 9.1, the size of the (reduced) scheduling-variable φ is taken equal to 3, i.e., $n_\varphi = 3$. This was chosen as it gave a good trade-off between modeling error and scheduling size. Moreover, for the encoding layer, two hidden layers with 32 neurons each using *Rectified Linear Unit* (ReLU) activation functions were used. A higher number of hidden layers and/or neurons did not significantly improve the modeling error for considered scheduling size.

For training of the DNN, the gradient descent algorithm Adam (Kingma and Ba 2014) is used with a learning rate of $1 \cdot 10^{-4}$. Moreover, L_2 weight regularization is considered with a coefficient of $1 \cdot 10^{-3}$. The DNN is then trained for a 1000 epochs, resulting in a final squared Frobenius norm cost (see also (9.31)) of $\sim 6 \cdot 10^2$. Based on the DNN network we can then construct the LPV representation

$$x_{\delta,\mathrm{g}}(t) = \hat{A}(p(t)) + B_\mathrm{g} u_{\delta,\mathrm{g}}(t)), \tag{10.28a}$$
$$y_{\delta,\mathrm{g}}(t) = C_\mathrm{g} x_{\delta,\mathrm{g}}(t), \tag{10.28b}$$

[2]Note that f_g, given in (10.22), does not depend on ψ, consequently, $A_{\delta,\mathrm{g}}$ does not depend on it as well. This means that we can consider $\mathbb{R}$ as the embedding region of ψ.

where $p(t) \in \mathcal{P} \subset \mathbb{R}^3$ is the scheduling-variable, with scheduling-map

$$\eta(x_{\mathrm{g}}) = \mu(V, \sin(\alpha), \sin(\sigma), \cos(\alpha), \cos(\sigma), \epsilon), \tag{10.29}$$

where μ is the encoding layer of the DNN (with normalization taken into account), and $\hat{A}$ is constructed based on the matrix mapping layer of the DNN. Due to the DNN method, (10.28) has an affine scheduling dependency. The scheduling set $\mathcal{P}$ is then constructed by evaluating $\eta(x_{\mathrm{g}})$ for $x_{\mathrm{g}} \in \mathfrak{D}$, and computing the minimal volume bounding box of the resulting data, see also (Sadeghzadeh, Sharif, et al. 2020, Section 4.1)). The vertices of the resulting minimal volume bounding box are then used to define the convex set $\mathcal{P}$. The LPV representation (10.28) is then used as a DPV embedding of (10.21) on the region $\mathcal{X}$. The DPV embedding (10.28) interconnected with the weighting filters, as displayed in Figure 10.11, then gives us a DPV embedding of the generalized plant.

Differential controller synthesis

The DPV embedding of the generalized plant is then used in the second step of the incremental controller synthesis procedure. For the DPV embedding, an $\mathcal{L}_2$-gain optimal LPV (of the form (6.9)) is synthesized, which will be the differential controller. After realization of the controller, this will imply an $\mathcal{L}_{\mathrm{i}2}$-gain performance bound of the primal form of the closed-loop. As the DPV embedding (10.28) has affine scheduling dependency, we make use of polytopic $\mathcal{L}_2$-gain optimal LPV controller synthesis (Apkarian, Gahinet, and G. Becker 1995), implemented in the LPVcore Toolbox (Boef et al. 2021). Moreover, in order to enforce roll-off at high frequencies of the controller, we set the D-matrix of the controller (i.e., D_{k} of (4.41)) to zero and add an LMI region constraint s.t. for constant values of the scheduling-variable, the closed-loop eigenvalues of the interconnection have a real part larger than -60. Synthesizing the differential controller under these considerations, achieves a closed-loop $\mathcal{L}_2$-gain of 1.96. The Bode magnitude plot of the differential form of the closed-loop system is given in Figure 10.12 .

Incremental control realization

For the final step of the incremental controller design procedure, we use the result of Theorem 6.3. This results in an incremental controller of the form (6.10). Moreover, as the differential controller has affine scheduling dependency, we make use of Corollary 6.1. This means that we only require integration of the scheduling-map η, instead of integration of the matrix functions of the differential controller, for realization of the incremental controller. In this case, the scheduling-map η consists of a DNN. Due to the DNN in the scheduling-map, computing the analytical solution of the integral is difficult. Therefore, the integration of the scheduling-map is numerically computed in MATLAB. For fast execution, C-code of the resulting MATLAB function is generated, using MATLAB code generation. On a modern Intel Core i5 processor with a (boost) frequency of 3.8GHz, the resulting function is able to execute in ~ 0.43 ms, making it feasible for real-time deployment.

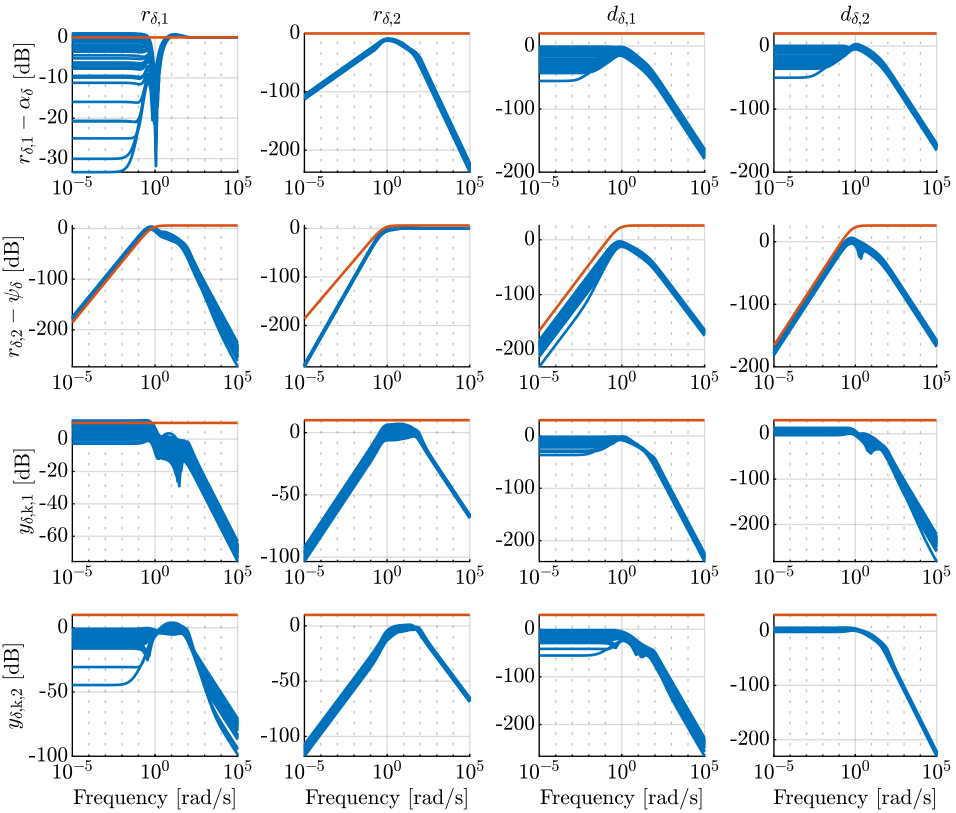


Figure 10.12: Bode magnitude plot of differential form of closed-loop system at frozen values of $x_\mathrm{g} \in \mathcal{X}$ (—), along with the corresponding inverse weighting filters (—).

Standard LPV controller design

In industry, the linearization of a nonlinear system is often used for standard (local) LPV controller design. While it is known that these type of controllers do not have explicit guarantees (Shamma and Athans 1990), they are still regularly used in industry as they are able to provide satisfactory performance. The differential form of a system can be interpreted as the linearization of the system along a certain trajectory. Therefore, the synthesized differential controller, obtained in the second step of the incremental controller design procedure, can also be seen as a local LPV controller design for the system if we do not compute the realization step. Therefore, we will compare this local LPV controller to our proposed incremental controller design.

Next, we will present simulations results when the incremental controller and the local LPV controller are applied to the GPRV model (10.19).

10.3.4 Simulation results

For our simulation study, we will compare the tracking performance of the designed incremental controller and local LPV controller. As reference trajectory we consider a (simulated) open loop trajectory of the GPRV model, which consist of constant heading and constant heading rate sections, see Figure 10.13 and also the reference in Figure 10.14.

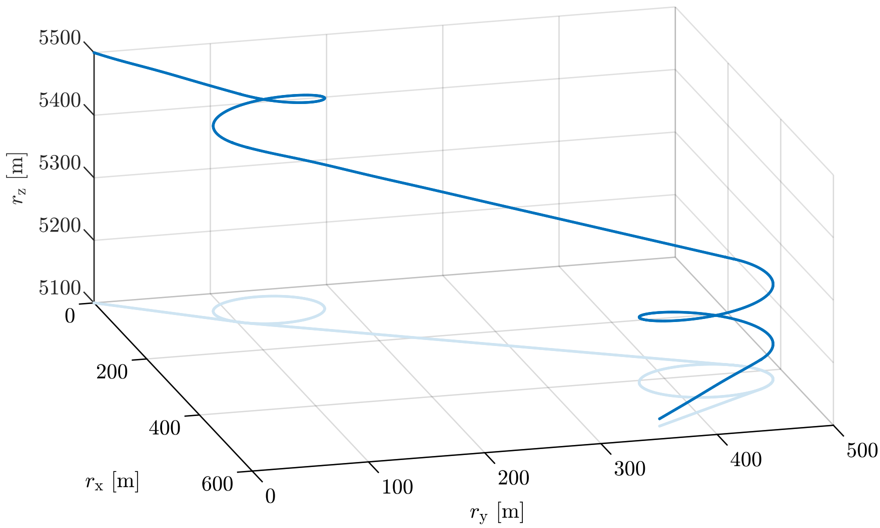


Figure 10.13: Position of the considered reference trajectory (▬) and its projection on r_x and r_y (▬).

For the incremental controller, this trajectory is also used as the desired steady-state trajectory, i.e., it is used for $(y_\mathrm{k}^*, u_\mathrm{k}^*) = (u^*, y^*)$ in (6.10). Moreover, as the symmetric and asymmetric deflection are physically restricted to be between ± 1, the control actions generated by the controllers are saturated between ± 1. The resulting simulation results under these considerations is displayed in Figures 10.14 to 10.16.

In Figure 10.14, the flight path angle α and heading angle ψ are depicted for the system in closed-loop with the local LPV and incremental controller, along with their to-be-followed reference signals. It can be seen that using the incremental controller, both the flight path and heading angle converge quickly to their corresponding reference trajectories. While the local LPV controller is able to track the heading angle fairly well, the behavior of the flight path angle is more erratic. This can also been seen from Figure 10.15, where the corresponding tracking errors are displayed. From Figure 10.15, it is evident that the incremental controller achieves much better tracking performance for the heading angle ψ. Finally, in Figure 10.16, the control inputs of both controllers are depicted in terms of the symmetric deflection δ_s and asymmetric deflection δ_a. From the behavior of the control inputs, it can also be seen that the incremental controller generates much smoother control signals compared to the local LPV controller. In fact, due to the design of the incremental controller, the control signals converge to the ones of the steady-state trajectory (also used as reference). While the local LPV controller is able to achieve relatively good performance, it has, as aforementioned, no strict stability and performance guarantees. On the other hand, our incremental controller design achieves better tracking performance compared to the local LPV controller, while at the same time it also has guarantees in terms of incremental stability and performance.

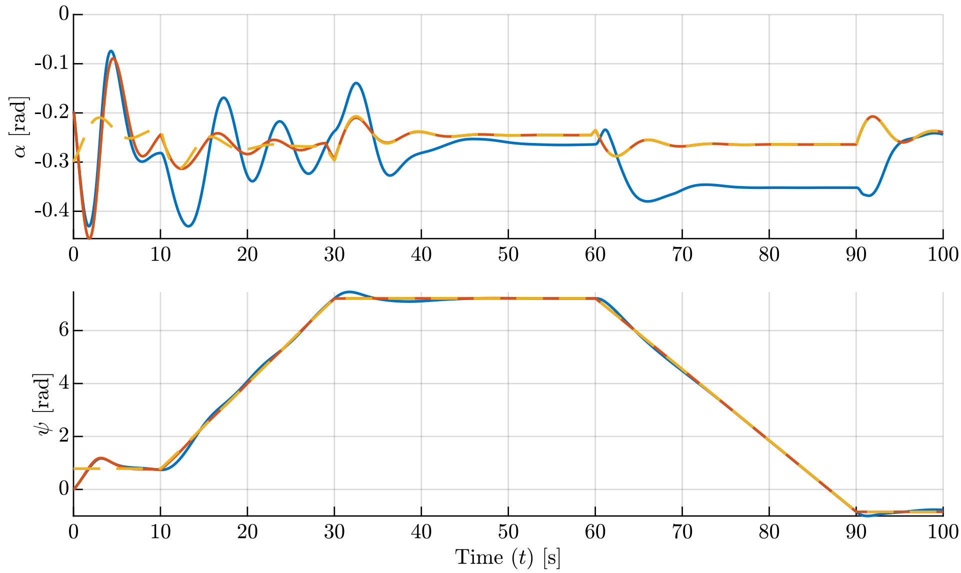


Figure 10.14: Flight path angle α (top) and heading angle ψ (bottom) of the GPRV in closed-loop with the local LPV ( ) and the incremental ( ) controllers under reference ( ).

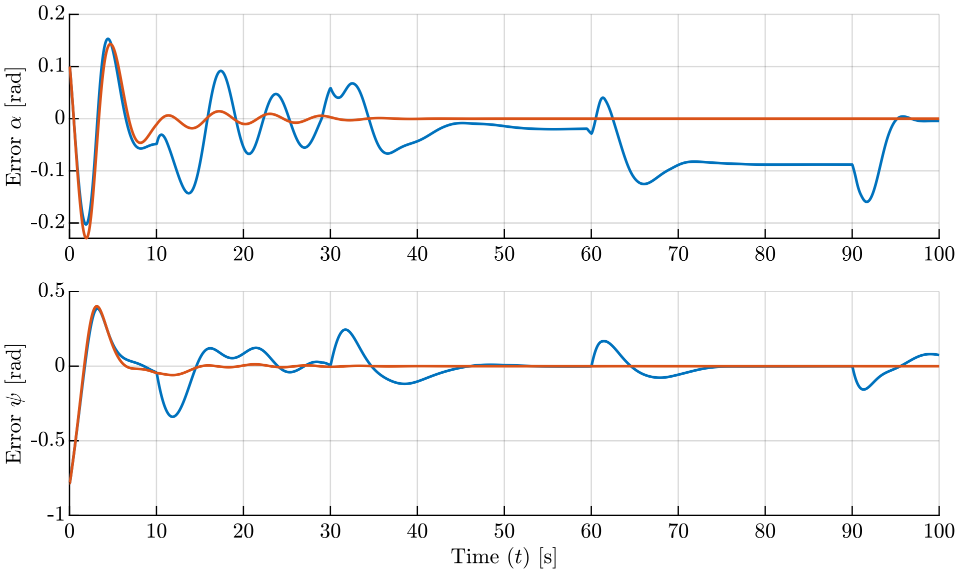


Figure 10.15: Tracking errors of the flight path angle α (top) and heading angle ψ (bottom) using the local LPV (▬) and the incremental (▬) controllers.

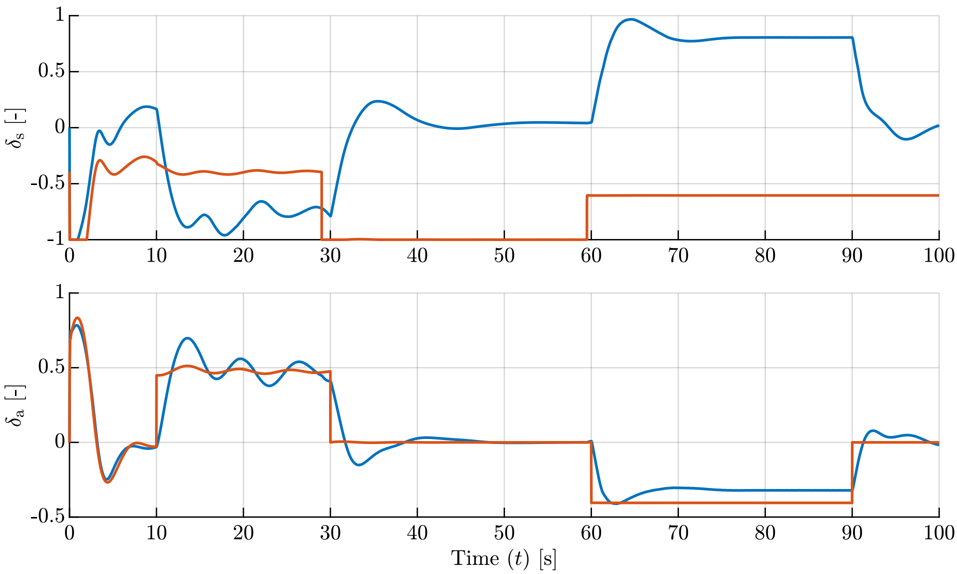


Figure 10.16: Symmetric deflection δ_s (top) and asymmetric deflection δ_a (bottom) sent to the GPRV in closed-loop with the local LPV (▬) and the incremental (▬) controllers.

10.4 Conclusions

In this chapter, we applied the developed universal shifted controller design and the incremental controller design to reference tracking control problems for a CMG and GPRV, respectively. For both these realistic applications, the results show that both controller designs are able to successfully achieve their desired control objectives and guarantee global stability and performance properties of the closed-loop system. Moreover, they achieve improved performance compared to standard LPV controller designs, while having a comparable design complexity. These results once again highlight that we can achieve systematic and computationally efficient control of nonlinear systems in order to ensure and shape global stability and performance guarantees through developed framework in this thesis.

11 Conclusions

In this chapter, we reflect on the objective of this thesis to develop a systematic framework for the analysis and control of nonlinear systems. Firstly, the main results of the thesis are summarized and it is discussed to which extend the research questions have been addressed in order to achieve the set out research objective. Secondly, we highlight the main contributions of each chapter. Finally, we give some recommendations for future research directions.

11.1 Overview and Summary

To cope with the increasing complexity and higher performance requirements of systems nowadays, there is a growing need to move beyond the *Linear Time-Invariant* (LTI) framework in order to analyze and design controllers for these systems. As discussed in Chapter 1, specifically Section 1.3, while there is a vast array nonlinear methods to handle analysis and control of systems, they are often cumbersome to use due to a lack of a systematic and computationally efficient framework for analysis and controller synthesis, like in the case of LTI systems. While approaches such as the *Linear Parameter-Varying* (LPV) framework, which build on top of the LTI results, have systematic and computationally efficient results, they lack results for guaranteeing global stability and performance of nonlinear systems. This can result in undesirable behavior when the current LPV methods are used to analyze or design controllers for nonlinear systems, as we have shown in Chapter 3. Global stability and performance guarantees are crucial for systematically analyzing and designing controllers for nonlinear systems, as they are able to provide guarantees independent of a particular equilibrium point or trajectory of the system, similar to the stability and performance guarantees for LTI systems. Therefore, in this thesis, we set out to achieve the research objective of developing a framework for systematic, computationally efficient analysis and control of nonlinear systems to ensure and shape global stability and performance guarantees.

To achieve this objective, we have specifically focussed on the global notions of *universal shifted* and *incremental* dissipativity. As we have shown in the thesis, these notions allow us to guarantee stability and performance w.r.t. all equilibria and/or all feasible trajectories of the systems at once, respectively. Under these considerations, we formulated three crucial Research Questions to be answered in Section 1.4. Next, we discuss in detail our findings in answering the three Research Questions.

Research Question 1 – Analysis

For Research Question 1, we have investigated “How to systematically and computationally efficiently analyze universal shifted and incremental dissipativity of *Continuous-Time* (CT) and *Discrete-Time* (DT) nonlinear systems?”

First, in Chapter 4, we have focused on universal shifted stability and performance, which considers stability and performance w.r.t. all (forced) equilibrium points of the system. We have shown how classical dissipativity of the so-called velocity form, i.e., the time-differentiated dynamics in CT and time-difference dynamics in DT, is able to imply universal shifted stability and performance for CT systems. We furthermore have shown how the velocity form can naturally be represented by an LPV representation, which we refer to as a *Velocity Parameter-Varying* (VPV) embedding of the nonlinear system. This crucially allows us to cast the universal shifted stability and performance analysis problem of a nonlinear system as a standard stability and performance analysis problem of an LPV representation. This allows for all the computationally efficient analysis methods that are available to

LPV systems to be applied to analyzing universal shifted stability and performance of nonlinear systems. Which gives us a systematic and computationally efficient framework for global stability and performance analysis, in terms of universal shifted stability and performance.

In Chapter 5, we have focused on incremental dissipativity analysis of nonlinear systems. Incremental dissipativity is a stronger notion than universal shifted dissipativity as it considers dissipativity w.r.t. any arbitrary trajectory of the system, instead of w.r.t. only the equilibrium points. In Chapter 5, we have shown how for CT systems, incremental dissipativity can be implied through ensuring classical dissipativity of the so-called differential form, which represents the dynamics of the variations along trajectories of the system. We show that also the differential form, like the velocity form for universal shifted dissipativity analysis, can naturally be embedded in an LPV representation, which we refer to as a *Differential Parameter-Varying* (DPV) embedding. Through the DPV embedding, we can also cast the incremental dissipativity analysis problem of a nonlinear system as a classical dissipativity analysis problem of an LPV representation. Therefore, similar to the universal shifted analysis results, we can use the efficient tools available for LPV systems, in order analyze incremental dissipativity. This also gives us a systematic and computationally efficient framework for global stability and performance analysis, but now in terms of incremental stability and performance.

In fact, due to the similarities between the velocity form and the differential form of a system and the corresponding VPV and DPV embeddings, a single LPV representation can represent both the velocity and differential form of a system. Through classical dissipativity analysis of the LPV representation, both incremental, universal shifted, and also classical dissipativity can then be concluded.

Moreover, in Chapters 7 and 8, we have similarly shown how these results for universal shifted and incremental stability and performance analysis of CT systems can be extended to DT systems. This gives us also in DT a systematic and computationally efficient framework for global stability and performance analysis.

Taken all together, we have the following procedure in order to systematically and computationally efficiently analyze global stability and performance of nonlinear systems:

1. For the to-be-analyzed nonlinear system compute its velocity form or differential form;

2. Embed the velocity or differential form in an LPV representation, resulting in a VPV or DPV embedding, respectively;

3. Analyze classical dissipativity of the VPV or DPV embedding, for which standard LPV analysis methods can be used.

4. If the VPV or DPV embedding is classically dissipative, universal shifted or incremental stability and performance of the nonlinear system is guaranteed, respectively.

Again, these results give us an overarching systematic procedure for global stability and performance analysis of nonlinear systems. Importantly, we show that this check can be done computationally efficiently, as it can be cast as a standard LPV analysis problem, for which a wide variety of existing tools can be used. With this framework for analysis, we address Research Question 1 of this thesis.

Research Question 2 – Synthesis

Next, for Research Question 2, we have investigated "How to systematically and computationally efficiently design and shape controllers for CT and DT nonlinear systems such that universal shifted and incremental performance requirements and stability are ensured?". To answer this research question, we have again tackled the considered global dissipativity notions separately. First, in Chapter 4, Section 4.5, we have addressed the question of synthesizing a controller to ensure universal shifted stability and performance of CT systems. In this chapter, we have developed a novel convex controller synthesis method in order to guarantee closed-loop universal shifted stability and performance, by exploiting properties of the velocity form and using through the use VPV embeddings. Similarly, in Chapter 6, we have shown how to address the controller synthesis problem to ensure incremental stability and performance of CT systems. In this chapter, we have proposed a novel convex controller synthesis method, by exploiting properties of the differential form and using DPV embeddings, in order to ensure the desired incremental stability and performance guarantees.

Moreover, in Chapter 7, we have shown how also in DT, we can construct a convex controller synthesis procedure to guarantee closed-loop incremental stability and performance. We show that this resulting procedure is equivalent to the procedure in CT. In Chapter 8, we have proposed a procedure for DT controller synthesis in order to ensure closed-loop universal shifted stability and performance. To achieve this, we have shown, we can use part of the DT incremental controller synthesis results due to specific structure of the DT velocity form. Due to the properties of the DT velocity form, the controller structure can be simplified. While the DT and CT universal shifted controllers do not have the same structure, they are analogous to each other. Namely, they both are able to guarantee universal shifted stability and performance by exploiting properties of the velocity form. Together, these results give us a systematic and computationally efficient framework for controller synthesis in order to ensure and shape global stability and performance requirements.

Taken all together, we can formulate the following procedure in order to systematically and computationally efficiently synthesize controllers for nonlinear systems in order to ensure and shape global stability and performance of nonlinear systems:

1. Formulate a (weighted) nonlinear generalized plant, based on the desired closed-loop performance requirements;
2. Embed the nonlinear generalized plant in a VPV or DPV embedding;
3. Synthesize a standard LPV output-feedback controller for the VPV or DPV embedding of the generalized plant.

4. Based on the synthesized LPV controller for the VPV or DPV embedding, realize the controller using the proposed universal shifted controller or incremental controller realization procedure, respectively. Consequently, we obtain a universal shifted or incremental controller, respectively.

5. The universal shifted controller and incremental controller in closed-loop with the nonlinear generalized plant ensure universal shifted and incremental stability and performance, respectively.

An important result we discuss w.r.t. shaping is that when LTI weighting filters are used to construct the generalized plant, the shaping can also be done directly on the VPV or DPV embedding using the *same* weights, which significantly simplifies the shaping procedure.

A key takeaway here is that we can use existing synthesis methods from the LPV framework in the third step of the above procedure, giving us a convex controller synthesis procedure in order to guarantee closed-loop global stability and performance. With this framework for synthesis, we adress the second research question of this thesis.

Research Question 3 – Complexity

Finally, for Research Question 3, we have investigated "How to reduce the complexity for the to-be-developed analysis and controller synthesis methods to address complex systems?". A key part of the analysis and synthesis framework we have developed has been the use of LPV methods combined with the proposed VPV and DPV embeddings of systems. For increasingly complex systems, more state and input dependent terms enter the velocity or differential form, which makes it increasingly difficult to embed them in LPV representations. In order to cope with this increasing complexity, we have proposed in Chapter 9 a data-based method to obtain an affine LPV embedding of the velocity or differential form with a user defined number of scheduling-variables. This was achieved by the development of a *Deep (Artifical) Neural Network* (DNN) based *Scheduling Dimension Reduction* (SDR) method, whereby the scheduling-map and state-space matrices of the LPV model are parametrized through an DNN. Compared to existing methods, the proposed method achieves lower modeling error and improved performance when used for controller synthesis. The proposed DNN method allows us to reduce the number of scheduling-variables of a given VPV or DPV embedding. In this way, we can handle more complex systems. Moreover, we have also shown that DNN method allows us to automatically construct a VPV or DPV embedding based on a given data set of typical trajectories or operating points of the system. To achieve this, when combined with *Automatic Differentiation* (AD) techniques to evaluate the velocity or differential form, one would only need the primal form of the system and a given data set to construct a VPV or DPV embedding. This VPV or DPV embedding can then be used for either the proposed global analysis or controller synthesis procedures. This approach has been demonstrated for an incremental flight controller design for a *Generic Parafoil Return Vehicle* (GPRV) in Chapter 10.

In conclusion, the proposed DNN SDR method gives us a tool to reduce the complexity for the developed analysis and controller synthesis methods to also deal with complex systems. This addresses the final, and third research question in this thesis.

The Complete Framework

The combined results of answering Research Questions 1–3 gives us the systematic, computationally efficient framework for analysis and control of nonlinear systems to ensure and shape stability and performance. In Figure 11.1, an overview of the proposed framework is summarized and displayed in terms of a flow chart. In the figure, ‘Analysis’ refers to the approaches we have developed for addressing Research Question 1, ‘Synthesis’ refers to the approaches we have developed for addressing Research Question 2, and ‘Complexity’ to the approach we have developed for addressing Research Question 3.

In Chapter 10, we have demonstrated the capabilities of the proposed framework through an experimental study of an universal shifted controller design for a *Control Momement Gyroscope* (CMG) lab setup and through a simulation study of an incremental flight controller design for a GPRV. For both applications, the benefits of the proposed controller design approaches are demonstrated in terms of a systematic controller design process and their ability to guarantee the desired global stability performance properties for the underlying nonlinear systems. Moreover, the proposed controllers achieve significantly improved performance compared to standard LPV controller design for these systems.

11.2 Main Contributions

Next, we highlight the main contributions of each chapter individually.

- In Chapter 3, we show that the current use of the LPV framework is insufficient to properly analyze stability and performance of nonlinear systems. Namely, we show that using the LPV framework in its current form, we can only give guarantees for stability and performance for the underlying nonlinear system w.r.t. the origin of the particular state-space representation. This means that the current state-of-the-art LPV analysis and controller synthesis results are insufficient when they are applied to nonlinear systems in order to ensure stability and performance w.r.t. multiple non-zero equilibrium points or trajectories. This demonstrates the need to move to global notions for stability and performance analysis of nonlinear systems.
- In Chapter 4, we show how that the time-differentiated dynamics can be used in order imply both universal shifted stability and universal shifted performance of nonlinear systems. Moreover, we show how the analysis of the time-differentiated dynamics can be performed by using LPV analysis methods. Together, these results show how to systematically and computationally

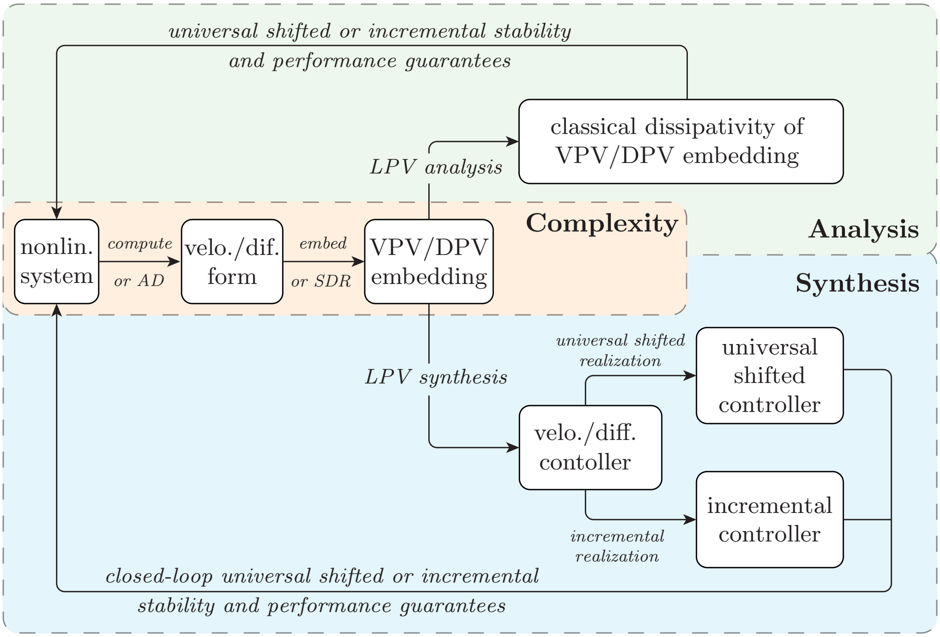


Figure 11.1: Overview of the steps in the framework for analysis, related to Research Question 1; synthesis, related to Research Question 2; and complexity, related to Research Question 3.

efficiently analyze universal shifted stability and performance of nonlinear systems. Moreover, as additional contribution, we show how these analysis results can be used in order to systemically and computationally efficiently synthesize controllers to ensure closed-loop universal shifted stability and shape universal shifted performance.

- In Chapter 5, we show how the differential dissipativity of a nonlinear system implies incremental, universal shifted, and classical dissipativity. Moreover, we show how differential dissipativity can computationally efficiently be analyzed by using analysis methods from the LPV framework.

- In Chapter 6, we propose a novel convex output feedback controller synthesis method in order to ensure and shape closed-loop incremental dissipativity. This is achieved by synthesizing an LPV controller for the differential form of the plant and developping a novel realization procedure which uses the synthesized LPV controller to realize an implementable incremental controller.

- In Chapter 7, we extend the CT results for incremental dissipativity based analysis and controller synthesis of Chapters 5 and 6 to DT nonlinear systems. We show that the same results in terms of analysis and controller synthesis procedure can be obtained as in CT.

- In Chapter 8, we extend the CT results for universal shifted based analysis and controller design of Chapter 4 to DT nonlinear systems. We show that time-difference dynamics, analogous to the time-differentiated dynamics in CT, can be used to imply universal shifted stability and performance properties. Moreover, we propose a novel convex output feedback controller synthesis method in order to ensure and shape closed-loop universal shifted stability and performance, which uses the DT incremental controller results of Chapter 7.

- In Chapter 9, we present an DNN based SDR method for LPV systems. The proposed method uses a single *Artifical Neural Network* (ANN) to model both the (reduced) scheduling-map and LPV state-space matrices. The proposed method achieves improved model representation and results in improved closed-loop performance when the reduced model is used for controller synthesis compared to existing SDR methods. Moreover, we show how the DNN method can be used for the automated construction of LPV, VPV, and DPV embeddings.

- In Chapter 10, we demonstrate the capabilities of the proposed analysis and controller synthesis framework on a lab setup and on a real-life example. We demonstrate that the proposed methods allow us to systematically design controllers to ensure and shape the desired global stability and performance requirements. Moreover, the improvements of the proposed methods compared to standard LPV controller designs are shown in terms of improved stability and performance properties.

11.3 Recommendations

While the results for the developed framework are fairly general, there are still open problems and challenges to be solved, e.g., as a results of assumptions taken along the way or interesting directions to extend the framework into. Next, we will discuss some of these open problems and challenges.

- For both the analysis and synthesis results, assumptions are taken on the class of nonlinear systems that is considered. Namely, for both analysis and synthesis, the velocity and differential forms require the state transition map f and output map h of the nonlinear system to be in $\mathcal{C}_1$. However, common nonlinearities such as saturations, dead-zones, etc. do not satisfy this. Hence, it is worth investigating if the extensions can be made to nonlinearities that are differentiable almost everywhere, $\mathcal{C}_0$, or even discontinues, while still allowing them to be used for universal shifted and incremental dissipativity analysis. There exist extensions of the Jacobian to non-smooth (but locally Lipschitz) functions, such as in the form of the Clarke's generalized Jacobian (Clarke 1983). However, it is still an open question if a velocity or differential form formulated using these will still be able to imply incremental or universal shifted stability and performance.

  Furthermore, for both synthesis procedures, i.e., universal shifted and incremental controller designs, various restrictions on the class of nonlinear systems are considered. While we show how through concatenation of simple filters more general classes of nonlinear systems can also be addressed, it is still of interest to investigate extensions to larger system classes.

- In Chapters 4 and 8, we have shown how velocity (Q, S, R) dissipativity implies universal shifted stability of nonlinear systems. However, we have not been able to completely proof that velocity (Q, S, R) dissipativity also implies universal shifted (Q, S, R) dissipativity. While we have presented results that strongly indicate that this is the case, it is still an open problem to formally proof this.

- The velocity form and differential form, while similar, are distinct objects describing different system representations. While for controllers synthesis, explicit properties of the velocity form are exploited to ensure universal shifted stability and performance, for analysis this is not the case. In fact, as we have discussed in Sections 5.3.6 and 5.4.2, analysis of the VPV embedding can be seen as analysis of an DPV embedding, which actually analyzes the stronger notion of incremental stability and performance. Hence, analysis of universal shifted stability and performance through the velocity form is inherently conservative in the developed framework. Therefore, the question arises if this conservatism in the analysis can be reduced by exploiting other properties of the velocity form specifically for universal shifted stability and performance analysis.

- For the proposed incremental controller design, knowledge of the steady state trajectory to which we want to converge is required for realization of the controller. While some results exist which do not require knowledge of this steady state trajectory in order to realize the controller and ensure the desire incremental stability and performance guarantees, they only consider an LTI controller (Scorletti, Fromion, et al. 2015). Therefore, this is still an open problem in the case a nonlinear controller is considered.

- We have shown that the velocity and differential form can naturally be represented by an LPV representation. However, for a given scheduling dependency, the corresponding construction of the VPV or DPV embedding has to be performed manually and is generally non-unique. Hence, the question arises how to perform this embedding automatically and what the most ‘optimal’ VPV or DPV embedding is for a given scheduling class, e.g. in terms of best controller performance. We have shown we can use the developed DNN SDR method to automatically obtain an affine VPV or DPV embedding based on data. However, this requires gathering of data and the resulting LPV representation is only an approximation of the underlying velocity or differential form due to the scheduling reduction. Some results exist in the literature related to automated and/or optimal embedding construction. However, these results are only w.r.t. embedding the primal form of the nonlinear systems in an LPV representation (Kwiatkowski, Bol, et al. 2006; Tóth 2010; Abbas, Tóth, Petreczky, Meskin, Mohammadpour Velni, and Koelewijn 2021). Therefore, it is still an open question how to perform the VPV or DPV embedding of a nonlinear system ‘optimally’ and automatically.

- With respect to controller synthesis, we have shown how we can intuitively shape the velocity/differential form to shape shifted/incremental properties of the closed-loop system through the use of LTI weighting filters. However, for LTI systems there is also a strong connection of the weighting filters to the frequency domain behavior of the expected closed-loop dynamics. While for notions related to incremental dissipativity there are some results to frequency domain behavior of the nonlinear system (Pavlov, Van de Wouw, et al. 2007), it is still an open question how these results connect to properties of the differential form, and if and how these results can be used effectively to shape the frequency domain behavior of the nonlinear system.

- For standard LPV controller design, besides using LTI weighting filters, it is also common to use parameter-dependent weighting filters in order to shape the closed-loop performance differently for different operating/scheduling points. For the proposed universal shifted and incremental controller designs, we assume that LTI weighting filters are used, as this ensures that shaping the velocity/differential form is equivalent to shaping universal shifted/incremental properties of the primal form. While, parameter-varying weights could in theory be used to shape the, LPV embedded, velocity/differential form, this breaks the interpretation of how these weights will shape universal shifted/incremental performance of the primal form. Alternatively, one could also interconnect nonlinear weighting filters in the primal form to shape universal shifted/incremental properties. However, this will result in the velocity/differential form also depending on the nonlinearities coming from these weighting filters, which will make effective shaping more difficult. Hence, it is of interest to investigate how parameter-dependent or nonlinear weighting filters can efficiently and intuitively be incorporated in the shaping procedure for the universal shifted and incremental controller designs.

- Using the developed incremental controller synthesis technique, we have stability and performance guarantees w.r.t. any trajectory of the system. In most cases, we only require these guarantees w.r.t. a subset of the behavior of the system. This can then be exploited by making use of a so-called virtual systems, which can be seen a general nonlinear embedding of the nonlinear system. By enforcing differential dissipativity of the virtual systems rather than that of the original nonlinear system, stability and performance w.r.t. the trajectories that are in the behavior of both the virtual system and nonlinear system can be guaranteed. On this front, there have already been some developments for CT systems using state-feedback controllers (R. Wang, Tóth, et al. 2020; R. Wang, Koelewijn, et al. 2021). However, extensions to the output-feedback case and DT systems is still an open problem.

- As discussed in Chapter 1, one of the more popular and successful nonlinear control methods is *Nonlinear Model Predictive Control* (NMPC). While there are some results on using incremental properties for NMPC (Köhler et al. 2022), these methods still require expensive, non-convex, optimization problems to be solved at each sampling instant. On the other hand, there also exist LPV based methods (Cisneros et al. 2016; Hanema et al. 2017) which are solved by (a series of) linear or quadratic programs making them more efficient than nonlinear methods, however, these generally lack convergence guarantees in reference tracking applications when applied to nonlinear systems. Hence, extending the results of the proposed framework in this thesis to NMPC based methods would be of great interest.

- The developed framework relies on the fact that a (state-space) model of the nonlinear system is available. The increasing complexity of systems over the last decades has lead to a great research interest in learning based methods for analysis and control design. Some methods have already been developed for system identification of nonlinear systems with built-in incremental stability and performance guarantees (Revay et al. 2021). However, the various learning based analysis (Boffi et al. 2020), control (Tsukamoto et al. 2021), and control synthesis techniques (Junnarkar et al. 2022), rely on just ensuring incremental stability and do not focus on performance.

  Moreover, in the last few years, increasing attention has been devoted towards analysis and synthesis of systems directly based on data with stability and performance guarantees. However, the data-based results with explicit stability and performance guarantees are still limited to LTI (Van Waarde et al. 2022) and LPV (Verhoek, Tóth, et al. 2021) systems.

  Hence, extensions of these various data and learning based methods to analyze and ensure universal shifted or incremental performance and/or dissipativity guarantees is still an open problem. However, the systematic framework we have developed in this thesis can help pave the way for this.

- To use the results of the developed systematic framework for analysis and control, there are still a few gaps in the software tooling that need to be overcome. An important part of the proposed framework is the use of the LPV analysis and synthesis methods, for which tools are readily available, such as the LPVcore Toolbox (Boef et al. 2021) for MATLAB. However, computation of the velocity or differential form and their embedding in an LPV representation still needs to be done by hand or by using different toolboxes and/or software packages. Hence, while individual components exist, a complete end-to-end toolbox for the proposed analysis and synthesis framework is still required to be developed.

A

Derivations of LPV Analysis and Synthesis Results

A.1 Introduction

In this appendix, we will present derivations for the analysis and controllers synthesis conditions for *Linear Parameter-Varying* (LPV) systems in state-space representation that are given in Sections 2.5.2 and 2.5.3. Some of the results that we will present on analysis and controller synthesis for LPV systems are already available in literature, specifically those for $\mathcal{L}_2$-gain performance, see e.g. (Apkarian, Gahinet, and G. Becker 1995; Apkarian and Adams 1998; M. Ali and Werner 2011; De Caigny et al. 2012). Nonetheless, results for other the performance notions we discuss in this thesis, such as passivity, $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain, and $\mathcal{L}_\infty$-gain, are not widely available or do not exist in literature to the author's knowledge. Therefore, in this appendix, we will present derivations for the various conditions, in terms of *Linear Matrix Inequalities* (LMIs), in order to analyze LPV systems or synthesize LPV controllers for them.

The derivations in this appendix are inspired by the derivations and results in (Scherer and Weiland 2015) in *Continuous-Time* (CT) and by (De Caigny et al. 2012) in *Discrete-Time* (DT).

First, in Section A.2, we will present derivations for the $\mathcal{L}_2$-gain, passivity, $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain, and $\mathcal{L}_\infty$-gain based analysis conditions for both CT and DT LPV systems. In Section A.3, we will present derivations for $\mathcal{L}_2$-gain, passivity, $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain, and $\mathcal{L}_\infty$-gain based LPV controller synthesis conditions for both CT and DT LPV systems.

A.2 Derivations of Analysis Results

A.2.1 $\mathcal{L}_2$-gain

Continuous-time

We first proof that a CT LPV system given by (2.26) has a bounded $\mathcal{L}_2$-gain of γ, if it is classically (Q,S,R) dissipative with $(Q,S,R)=(\gamma^2 I,0,-I)$ (see also Lemma 2.4). Consequently, it then holds that there exists a storage function $\mathcal{V}_\mathrm{p}$ such that

$$\mathcal{V}_\mathrm{p}(x(t_1),p(t_1))-\mathcal{V}_\mathrm{p}(x(t_0),p(t_0))\leq\int_{t_0}^{t_1}\gamma^2(\star)^\top w(t)-(\star)^\top z(t)\,dt, \tag{A.1}$$

for all $t_0,t_1\in\mathbb{R}_0^+$ with $t_1\geq t_0$ and $(x,w,z)\in\mathfrak{B}_\mathrm{p}(p)$ for all $p\in\mathcal{P}^{\mathbb{R}_0^+}$. Hence, it also holds that

$$0\leq\mathcal{V}_\mathrm{p}(x(T),p(T))\leq\int_0^T\gamma^2(\star)^\top w(t)-(\star)^\top z(t)\,dt+\mathcal{V}_\mathrm{p}(x(0),p(0)), \tag{A.2}$$

for all $T\geq 0$ and $(x,w,z)\in\mathfrak{B}_\mathrm{p}(p)$ for all $p\in\mathcal{P}^{\mathbb{R}_0^+}$. This implies that, using the (extended) $\mathcal{L}_2$-norm definition in Definition 2.10, that

$$\gamma^2\left\|w\right\|_{2,T}^2-\left\|z\right\|_{2,T}^2+\mathcal{V}_\mathrm{p}(x(0),p(0))\geq 0, \tag{A.3}$$

for all $(x,w,z)\in\mathfrak{B}_\mathrm{p}(p)$ and $p\in\mathcal{P}^{\mathbb{R}_0^+}$ with $w\in\mathcal{L}_{2\mathrm{e}}$. Therefore[1],

$$\left\|z\right\|_{2,T}\leq\sqrt{\gamma^2\left\|w\right\|_{2,T}^2+\mathcal{V}_\mathrm{p}(x(0),p(0))}\leq\gamma\left\|w\right\|_{2,T}+\sqrt{\mathcal{V}_\mathrm{p}(x(0),p(0))}, \tag{A.4}$$

for all $(x,w,z)\in\mathfrak{B}_\mathrm{p}(p)$ and $p\in\mathcal{P}^{\mathbb{R}_0^+}$ with $w\in\mathcal{L}_{2\mathrm{e}}$, which is the $\mathcal{L}_2$-gain definition (see Definition 2.11), where $\zeta(x_0)=\sqrt{\mathcal{V}_\mathrm{p}(x_0,p(0))}$.

Note that similarly, classical (Q,S,R) dissipativity with $(Q,S,R)=(\gamma I,0,-\gamma^{-1}I)$ (for $\gamma>0$) also ensures a bounded $\mathcal{L}_2$-gain. This follows from multiplying (A.1) by $\gamma^{-1}>0$ and following the same steps above.

Next, we show the derivation to obtain the LMI condition given in (2.38a). Based on the result of Theorem 2.5 for classical (Q,S,R) dissipativity, we have that a CT LPV system given by (2.26) is classically (Q,S,R) dissipative with $(Q,S,R)=(\gamma,0,-\gamma^{-1}I)$, if for all $p\in\mathcal{P}$ and $v\in\Pi$, it holds that

$$(\star)^\top\begin{bmatrix}\partial M(p,v) & M(p)\\ \star & 0\end{bmatrix}\begin{bmatrix}I & 0\\ A(p) & B(p)\end{bmatrix}-(\star)^\top\begin{bmatrix}\gamma I & 0\\ \star & -\gamma^{-1}I\end{bmatrix}\begin{bmatrix}0 & I\\ C(p) & D(p)\end{bmatrix}\preceq 0, \tag{A.5}$$

which can be rewritten as

$$\begin{bmatrix}A(p)^\top M(p)+(\star)^\top+\partial M(p,v) & M(p)B(p)\\ \star & -\gamma I\end{bmatrix}-(\star)^\top(-\gamma^{-1}I)\begin{bmatrix}C(p) & D(p)\end{bmatrix}\preceq 0. \tag{A.6}$$

[1] Using that $\sqrt{a+b}\leq\sqrt{a}+\sqrt{b}$ for $a,b\geq 0$.

Through a Schur complement, this is equivalent to

$$\begin{bmatrix} A(p)^\top M(p) + (\star)^\top + \partial M(p,v) & M(p)B(p) & C(p)^\top \\ \star & -\gamma I & D(p)^\top \\ \star & \star & -\gamma I \end{bmatrix} \preceq 0, \tag{A.7}$$

which is equivalent to the condition in (2.38a).

Discrete-time

Similar as for the CT result, we can also show in DT that an LPV system given by (2.26) has a bounded ℓ_2-gain of γ, if it is classically (Q,S,R) dissipative with $(Q,S,R) = (\gamma^2, 0, -I)$ (or $(Q,S,R) = (\gamma I, 0, -\gamma^{-1}I)$). This proof will not be repeated as it follows in a similar manner as in CT, see (A.1)–(A.4), but by using the DT dissipation inequality (2.35b).

Next, we show the derivation to obtain the LMI condition given in (2.38b). Based on the result of Theorem 2.5 for classical (Q,S,R) dissipativity, we have that a DT LPV system given by (2.26) is classically (Q,S,R) dissipative with $(Q,S,R) = (\gamma I, 0, -\gamma^{-1}I)$, if for all $p \in \mathcal{P}$ and $v \in \Pi$, it holds that

$$(\star)^\top \begin{bmatrix} -M(p) & 0 \\ \star & M(p+v) \end{bmatrix} \begin{bmatrix} I & 0 \\ A(p) & B(p) \end{bmatrix} - (\star)^\top \begin{bmatrix} \gamma I & 0 \\ \star & -\gamma^{-1} I \end{bmatrix} \begin{bmatrix} 0 & I \\ C(p) & D(p) \end{bmatrix} \preceq 0, \tag{A.8}$$

which can be rewritten as

$$\begin{bmatrix} (\star)^\top M(p+v)A(p) - M(p) & A(p)^\top M(p+v)B(p) \\ \star & (\star)^\top M(p+v)B(p) - \gamma I \end{bmatrix} - \\ (\star)^\top(-\gamma^{-1}I)\begin{bmatrix} C(p) & D(p) \end{bmatrix} \preceq 0, \tag{A.9}$$

$$\begin{bmatrix} -M(p) & 0 \\ \star & -\gamma \end{bmatrix} + (\star)^\top \begin{bmatrix} M(p+v)^{-1} & 0 \\ \star & \gamma^{-1} I \end{bmatrix} \begin{bmatrix} M(p+v)A(p) & M(p+v)B(p) \\ C(p) & D(p) \end{bmatrix} \preceq 0, \tag{A.10}$$

and hence,

$$\begin{bmatrix} M(p) & 0 \\ \star & \gamma \end{bmatrix} - (\star)^\top \begin{bmatrix} M(p+v)^{-1} & 0 \\ \star & \gamma^{-1} \end{bmatrix} \begin{bmatrix} M(p+v)A(p) & M(p+v)B(p) \\ C(p) & D(p) \end{bmatrix} \succeq 0. \tag{A.11}$$

Through a Schur complement, this is equivalent to

$$\begin{bmatrix} M(p) & 0 & A(p)^\top M(p+v) & C(p)^\top \\ \star & \gamma & B(p)^\top M(p+v) & D(p)^\top \\ \star & \star & M(p+v) & 0 \\ \star & \star & \star & \gamma \end{bmatrix} \succeq 0. \tag{A.12}$$

Using a congruence transformation, this is equivalent to

$$(\star)^\top \begin{bmatrix} M(p) & 0 & A(p)^\top M(p+v) & C(p)^\top \\ \star & \gamma & B(p)^\top M(p+v) & D(p)^\top \\ \star & \star & M(p+v) & 0 \\ \star & \star & \star & \gamma \end{bmatrix} \begin{bmatrix} 0 & M(p)^{-1} & 0 & 0 \\ 0 & 0 & I & 0 \\ M(p+v)^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \succeq 0, \tag{A.13}$$

resulting in

$$\begin{bmatrix} \bar{M}(p+v) & A(p)\bar{M}(p) & B(p) & 0 \\ \star & \bar{M}(p) & 0 & \bar{M}(p)C(p)^\top \\ \star & \star & \gamma I & D(p)^\top \\ \star & \star & \star & \gamma I \end{bmatrix} \succeq 0, \tag{A.14}$$

where $\bar{M}(p) := M(p)^{-1} \succ 0$ and $\bar{M}(p+v) := M(p+v)^{-1} \succ 0$, which is equivalent to the condition in (2.38b).

A.2.2 Passivity

Continuous-time

We first proof that a CT LPV system given by (2.26) is passive, if it is classically (Q, S, R) dissipative with $(Q, S, R) = (0, I, 0)$ (see also Lemma 2.5). Consequently, it then holds that there exists a storage function $\mathcal{V}_\mathrm{p}$ such that

$$\mathcal{V}_\mathrm{p}(x(t_1), p(t_1)) - \mathcal{V}_\mathrm{p}(x(t_0), p(t_0)) \le \int_{t_0}^{t_1} z(t)^\top w(t) + w(t)^\top z(t)\, dt, \tag{A.15}$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_1 \ge t_0$ and $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$. Hence, it also holds that

$$0 \le \mathcal{V}_\mathrm{p}(x(T), p(T)) \le \int_0^T z(t)^\top w(t) + w(t)^\top z(t)\, dt + \mathcal{V}_\mathrm{p}(x(t_0), p(0)), \tag{A.16}$$

for all $T \ge 0$ and $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$. This implies that

$$\int_0^T 2z(t)^\top w(t)\, dt \ge -\mathcal{V}_\mathrm{p}(x(t_0), p(0)), \tag{A.17}$$

$$\int_0^T z(t)^\top w(t)\, dt \ge -\frac{1}{2}\mathcal{V}_\mathrm{p}(x(t_0), p(0)), \tag{A.18}$$

for all $T \ge 0$ and $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$, which is the passivity definition (see Definition 2.12), where $\zeta(x_0) = -\frac{1}{2}\mathcal{V}_\mathrm{p}(x_0, p(0))$.

Next, we show the derivation to obtain the LMI condition given in (2.39a). Based on the result of Theorem 2.5 for classical (Q, S, R) dissipativity, we have that a CT LPV system given by (2.26) is classically (Q, S, R) dissipative with $(Q, S, R) = (0, I, 0)$, if for all $p \in \mathcal{P}$ and $v \in \Pi$, it holds that

$$(\star)^\top \begin{bmatrix} \partial M(p, v) & M(p) \\ \star & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A(p) & B(p) \end{bmatrix} - (\star)^\top \begin{bmatrix} 0 & I \\ \star & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ C(p) & D(p) \end{bmatrix} \preceq 0, \tag{A.19}$$

which can be rewritten as

$$\begin{bmatrix} A(p)^\top M(p) + (\star)^\top + \partial M(p, v) & M(p)B(p) \\ \star & 0 \end{bmatrix} - \begin{bmatrix} 0 & C(p)^\top \\ \star & D(p) + (\star)^\top \end{bmatrix} \preceq 0, \tag{A.20}$$

$$\begin{bmatrix} A(p)^\top M(p) + (\star)^\top + \partial M(p, v) & M(p)B(p) - C(p)^\top \\ \star & -D(p) + (\star)^\top \end{bmatrix} \preceq 0, \tag{A.21}$$

which is equivalent to the condition in (2.39a).

Discrete-time

Similar as for the CT result, we can also show in DT that an LPV system given by (2.26) is passive, if it is classically (Q,S,R) dissipative with $(Q,S,R)=(0,I,0)$. This proof will not be repeated as it follows in a similar manner as in CT, see (A.15)–(A.18), but by using the DT dissipation inequality (2.35b).

Next, we show the derivation to obtain the LMI condition given in (2.39b). Based on the result of Theorem 2.5 for classical (Q,S,R) dissipativity, we have that a DT LPV system given by (2.26) is classically (Q,S,R) dissipative with $(Q,S,R)=(0,I,0)$ if for all $p\in\mathcal{P}$ and $v\in\Pi$ it holds that

$$(\star)^\top\begin{bmatrix}-M(p) & 0\\ \star & M(p+v)\end{bmatrix}\begin{bmatrix}I & 0\\ A(p) & B(p)\end{bmatrix}-(\star)^\top\begin{bmatrix}0 & I\\ \star & 0\end{bmatrix}\begin{bmatrix}0 & I\\ C(p) & D(p)\end{bmatrix}\preceq 0, \tag{A.22}$$

which can be rewritten as

$$\begin{bmatrix}(\star)^\top M(p+v)A(p)-M(p) & A(p)^\top M(p+v)B(p)\\ \star & (\star)^\top M(p+v)B(p)\end{bmatrix}-\begin{bmatrix}0 & C(p)^\top\\ \star & D(p)+(\star)^\top\end{bmatrix}\preceq 0, \tag{A.23}$$

$$\begin{bmatrix}-M(p) & -C(p)^\top\\ \star & -D(p)+(\star)^\top\end{bmatrix}+(\star)^\top M(p+v)^{-1}\begin{bmatrix}M(p+v)A(p) & M(p+v)B(p)\end{bmatrix}\preceq 0. \tag{A.24}$$

Through multiplication with -1 and through a Schur complement, this can be written as

$$\begin{bmatrix}M(p) & C(p)^\top & A(p)^\top M(p+v)\\ \star & D(p)+(\star)^\top & B(p)^\top M(p+v)\\ \star & \star & M(p+v)\end{bmatrix}\succeq 0. \tag{A.25}$$

Using a congruence transformation, this is equivalent to

$$(\star)^\top\begin{bmatrix}M(p) & C(p)^\top & A(p)^\top M(p+v)\\ \star & D(p)+(\star)^\top & B(p)^\top M(p+v)\\ \star & \star & M(p+v)\end{bmatrix}\begin{bmatrix}0 & \tilde{M}(p)^{-1} & 0\\ 0 & 0 & I\\ \tilde{M}(p+v)^{-1} & 0 & 0\end{bmatrix}\succeq 0, \tag{A.26}$$

resulting in

$$\begin{bmatrix}\bar{M}(p+v) & A(p)\bar{M}(p) & B(p)\\ \star & \bar{M}(p) & \bar{M}(p)C(p)^\top\\ \star & \star & D(p)+(\star)^\top\end{bmatrix}\succeq 0. \tag{A.27}$$

where $\bar{M}(p):=M(p)^{-1}\succ 0$ and $\bar{M}(p+v):=M(p+v)^{-1}\succ 0$, which is equivalent to the condition in (2.39b).

A.2.3 $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain

Continuous-time

We first proof that a CT LPV system given by (2.26) has a bounded $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain of γ, if it is classically (Q,S,R) dissipative for a storage function $\mathcal{V}_\mathrm{p}$ with

$(Q, S, R) = (\gamma, 0, 0)$ and it holds that

$$z(t)^\top z(t) \leq \gamma \mathcal{V}_\mathrm{p}(x(t), p(t)), \tag{A.28}$$

for all $t \geq 0$ and $(x, z) \in \pi_{\mathrm{x,z}} \mathfrak{B}_\mathrm{p}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$.

Consequently, it then holds that

$$\mathcal{V}_\mathrm{p}(x(t_1), p(t_1)) - \mathcal{V}_\mathrm{p}(x(t_0), p(t_0)) \leq \int_{t_0}^{t_1} \gamma(\star)^\top w(t)\, dt, \tag{A.29}$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_1 \geq t_0$ and $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$. Hence, it also holds that

$$0 \leq \mathcal{V}_\mathrm{p}(x(t), p(t)) \leq \int_0^t \gamma(\star)^\top w(\tau)\, d\tau + \mathcal{V}_\mathrm{p}(x(0), p(0)), \tag{A.30}$$

for all $t \geq 0$ and $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$. This implies that

$$\mathcal{V}_\mathrm{p}(x(t), p(t)) \leq \gamma \int_0^t \|w(\tau)\|^2\, d\tau + \mathcal{V}_\mathrm{p}(x(0), p(0)), \tag{A.31}$$

for all $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ and $p \in \mathcal{P}^{\mathbb{R}_0^+}$ with $w \in \mathcal{L}_{2\mathrm{e}}$. Combining this with (A.28), we obtain that,

$$\|z(t)\|^2 \leq \gamma^2 \int_0^t \|w(\tau)\|^2\, d\tau + \gamma \mathcal{V}_\mathrm{p}(x(0), p(0)), \tag{A.32}$$

for all $t \geq 0$ and $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$. Taking the supremum over $t \in [0,\, T]$ gives that for all $T \geq 0$

$$\|z\|_{\infty,T}^2 \leq \gamma^2 \|w\|_{2,T}^2 + \gamma \mathcal{V}_\mathrm{p}(x(0), p(0)), \tag{A.33}$$

for all $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ and $p \in \mathcal{P}^{\mathbb{R}_0^+}$ with $w \in \mathcal{L}_{2\mathrm{e}}$. Therefore,

$$\|z\|_{\infty,T} \leq \gamma \|w\|_{2,T} + \sqrt{\gamma \mathcal{V}_\mathrm{p}(x(0), p(0))}, \tag{A.34}$$

for all $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ and $p \in \mathcal{P}^{\mathbb{R}_0^+}$ with $w \in \mathcal{L}_{2\mathrm{e}}$, which is the $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain definition (see Definition 2.11), where $\zeta(x_0) = \sqrt{\gamma \mathcal{V}_\mathrm{p}(x_0, p(0))}$.

Next, we show the derivation to obtain the LMI conditions given in (2.40a). Based on the result of Theorem 2.5 for classical (Q, S, R) dissipativity, we have that a CT LPV system given by (2.26) is classically (Q, S, R) dissipative with $(Q, S, R) = (\gamma, 0, 0)$, if for all $p \in \mathcal{P}$ and $v \in \Pi$, it holds that

$$(\star)^\top \begin{bmatrix} \partial M(p, v) & M(p) \\ \star & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A(p) & B(p) \end{bmatrix} - (\star)^\top \begin{bmatrix} \gamma & 0 \\ \star & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ C(p) & D(p) \end{bmatrix} \preceq 0, \tag{A.35}$$

which can be rewritten as

$$\begin{bmatrix} A(p)^\top M(p) + (\star)^\top + \partial M(p, v) & M(p) B(p) \\ \star & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \star & \gamma \end{bmatrix} \preceq 0, \tag{A.36}$$

$$\begin{bmatrix} A(p)^\top M(p) + (\star)^\top + \partial M(p,v) & M(p)B(p) \\ \star & -\gamma \end{bmatrix} \preceq 0, \tag{A.37}$$

which is equivalent to the first LMI in (2.40a). Next, from (A.28), we have that[2]

$$(\star)^\top (C(p(t))x(t)) \leq \gamma\, x(t)^\top M(p(t))x(t), \tag{A.38}$$

for all $t \geq 0$ and $x \in \pi_{\mathrm{x}}\,\mathfrak{B}_{\mathrm{p}}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$. This holds if

$$(\star)^\top C(p)x \leq \gamma\, x^\top M(p)x, \tag{A.39}$$

for all $x \in \mathcal{X} \subseteq \mathbb{R}^{n_{\mathrm{x}}}$ and $p \in \mathcal{P}$, which holds if

$$M(p) - (\star)^\top(\gamma^{-1})C(p) \succeq 0, \tag{A.40}$$

for all $p \in \mathcal{P}$. Through a Schur complement, this is equivalent to

$$\begin{bmatrix} M(p) & C(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0, \tag{A.41}$$

which is equivalent to the second LMI in (2.40a).

Discrete-time

Similar as for the CT result, we can also show in DT that an LPV system given by (2.26) has a bounded ℓ_2-ℓ_∞-gain of γ, if it is classically (Q,S,R) dissipative with $(Q,S,R) = (\gamma,0,0)$ and (A.28) holds for all $t \geq 0$ and $(x,z) \in \pi_{\mathrm{x,z}}\,\mathfrak{B}_{\mathrm{p}}(p)$ for all $p \in \mathcal{P}^{\mathbb{N}_0}$. This proof will not be repeated as it follows in a similar manner as in CT, see (A.29)–(A.34), but by using the DT dissipation inequality (2.35b).

Next, we show the derivation to obtain the LMI condition given in (2.40b). Based on the result of Theorem 2.5 for classical (Q,S,R) dissipativity, we have that a DT LPV system given by (2.26) is classically (Q,S,R) dissipative with $(Q,S,R) = (\gamma,0,0)$, if for all $p \in \mathcal{P}$ and $v \in \Pi$, it holds that

$$(\star)^\top \begin{bmatrix} -M(p) & 0 \\ \star & M(p+v) \end{bmatrix} \begin{bmatrix} I & 0 \\ A(p) & B(p) \end{bmatrix} - (\star)^\top \begin{bmatrix} \gamma & 0 \\ \star & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ C(p) & D(p) \end{bmatrix} \preceq 0, \tag{A.42}$$

which can be rewritten as

$$\begin{bmatrix} (\star)^\top M(p+v)A(p) - M(p) & A(p)^\top M(p+v)B(p) \\ \star & (\star)^\top M(p+v)B(p) \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \star & \gamma \end{bmatrix} \preceq 0, \tag{A.43}$$

$$\begin{bmatrix} -M(p) & 0 \\ \star & -\gamma \end{bmatrix} + (\star)^\top M(p+v)^{-1} \begin{bmatrix} M(p+v)A(p) & M(p+v)B(p) \end{bmatrix} \preceq 0. \tag{A.44}$$

Through multiplication with -1 and through a Schur complement, this is equivalent to

$$\begin{bmatrix} M(p) & 0 & A(p)^\top M(p+v) \\ \star & \gamma & B(p)^\top M(p+v) \\ \star & \star & M(p+v) \end{bmatrix} \succeq 0. \tag{A.45}$$

[2]Note that for the $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain, $D(p) = 0$ for all $p \in \mathcal{P}$ is considered.

Using a congruence transformation, it then holds that

$$(\star)^\top \begin{bmatrix} M(p) & 0 & A(p)^\top M(p+v) \\ \star & \gamma & B(p)^\top M(p+v) \\ \star & \star & M(p+v) \end{bmatrix} \begin{bmatrix} 0 & M(p)^{-1} & 0 \\ 0 & 0 & I \\ M(p+v)^{-1} & 0 & 0 \end{bmatrix} \succeq 0, \tag{A.46}$$

resulting in

$$\begin{bmatrix} \bar{M}(p+v) & A(p)\bar{M}(p) & B(p) \\ \star & \bar{M}(p) & 0 \\ \star & \star & \gamma I \end{bmatrix} \succeq 0, \tag{A.47}$$

where $\bar{M}(p) := M(p)^{-1} \succ 0$ and $\bar{M}(p+v) := M(p+v)^{-1} \succ 0$, which is equivalent to the first LMI in (2.40b).

Next, from (A.28), we obtain through the steps in (A.38)–(A.41),

$$\begin{bmatrix} M(p) & C(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0. \tag{A.48}$$

Using a congruence transformation, this is equivalent to

$$(\star)^\top \begin{bmatrix} M(p) & C(p) \\ \star & \gamma I \end{bmatrix} \begin{bmatrix} M(p)^{-1} & 0 \\ 0 & I \end{bmatrix} \succeq 0, \tag{A.49}$$

resulting in

$$\begin{bmatrix} \bar{M}(p) & \bar{M}(p)C(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0, \tag{A.50}$$

where again $\bar{M}(p) = M(p)^{-1} \succ 0$, which is equivalent to the second LMI in (2.40b).

A.2.4 $\mathcal{L}_\infty$-gain

Continuous-time

We first proof that a CT LPV system given by (2.26) has a bounded $\mathcal{L}_\infty$-gain of γ, if there exists a storage function $\mathcal{V}_\mathrm{p}$ and $\alpha, \beta \in \mathbb{R}_0^+$ such that[3]

$$\frac{d}{dt}\mathcal{V}_\mathrm{p}(x(t), p(t)) \leq -\beta \mathcal{V}_\mathrm{p}(x(t), p(t)) + \alpha(\star)^\top w(t), \tag{A.51}$$

and

$$\gamma^{-1} z(t)^\top z(t) \leq \beta \mathcal{V}_\mathrm{p}(x(t), p(t)) + (\gamma - \alpha) w(t)^\top w(t), \tag{A.52}$$

for all $t \geq 0$ and $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$.

By Grönwall's Lemma (Khalil 2002, Lemma A.1), we have that (A.51) implies that

$$\mathcal{V}_\mathrm{p}(x(t), p(t)) \leq e^{-\beta t}\mathcal{V}_\mathrm{p}(x(0), p(0)) + \alpha \int_0^t e^{-\beta(t-\tau)} \left\| w(\tau) \right\|^2 d\tau, \tag{A.53}$$

[3] Note that this can be seen as strict classical (Q, S, R) dissipativity with $(Q, S, R) = (\alpha I, 0, 0)$, where the $\beta \mathcal{V}_\mathrm{p}$ term ensures strictness of the dissipation inequality.

for all $t \geq 0$ and $(x, w) \in \pi_{\mathrm{x,w}}\, \mathfrak{B}_{\mathrm{p}}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$.

Moreover, it holds that

$$\alpha \int_0^t e^{-\beta(t-\tau)} \left\|w(\tau)\right\|^2 \, d\tau \leq \alpha \left\|w\right\|_{\infty,t}^2 \int_0^t e^{-\beta(t-\tau)} \, d\tau \leq \frac{\alpha}{\beta} \left\|w\right\|_{\infty,t}^2, \tag{A.54}$$

for all $t \geq 0$ and $w \in \pi_{\mathrm{w}}\, \mathfrak{B}_{\mathrm{p}}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$. Furthermore, it holds that

$$e^{-\beta t} \mathcal{V}_{\mathrm{p}}(x(0), p(0)) \leq \mathcal{V}_{\mathrm{p}}(x(0), p(0)), \tag{A.55}$$

for all $t \geq 0$.

Therefore, combining (A.54) and (A.55) with (A.53), it holds that

$$\mathcal{V}_{\mathrm{p}}(x(t), p(t)) \leq \mathcal{V}_{\mathrm{p}}(x(0), p(0)) + \frac{\alpha}{\beta} \left\|w\right\|_{\infty,t}^2, \tag{A.56}$$

and

$$\beta \mathcal{V}_{\mathrm{p}}(x(t), p(t)) \leq \beta \mathcal{V}_{\mathrm{p}}(x(0), p(0)) + \alpha \left\|w\right\|_{\infty,t}^2, \tag{A.57}$$

for all $t \geq 0$ and $(x, w) \in \pi_{\mathrm{x,w}}\, \mathfrak{B}_{\mathrm{p}}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$.

Next, by (A.52), we have that it holds that

$$\gamma^{-1} \left\|z(t)\right\|^2 \leq \beta \mathcal{V}_{\mathrm{p}}(x(t), p(t)) + (\gamma - \alpha) \left\|w(t)\right\|^2, \tag{A.58}$$

for all $t \geq 0$ and $(x, w, z) \in \mathfrak{B}_{\mathrm{p}}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$. Using (A.57), it therefore also holds that

$$\gamma^{-1} \left\|z(t)\right\|^2 \leq \beta \mathcal{V}_{\mathrm{p}}(x(0), p(0)) + \alpha \left\|w\right\|_{\infty,t}^2 + (\gamma - \alpha) \left\|w(t)\right\|^2, \tag{A.59}$$

for all $t \geq 0$ and $(x, w, z) \in \mathfrak{B}_{\mathrm{p}}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$. By multiplication with γ and taking the supremum over $t \in [0, T]$, this gives that for all $T \geq 0$

$$\left\|z\right\|_{\infty,T}^2 \leq \gamma \beta \mathcal{V}_{\mathrm{p}}(x(0), p(0)) + \gamma^2 \left\|w\right\|_{\infty,T}^2, \tag{A.60}$$

for all $(x, w, z) \in \mathfrak{B}_{\mathrm{p}}(p)$ and $p \in \mathcal{P}^{\mathbb{R}_0^+}$ with $w \in \mathcal{L}_{\infty\mathrm{e}}$. Therefore,

$$\left\|z\right\|_{\infty,T} \leq \gamma \left\|w\right\|_{\infty,T} + \sqrt{\gamma \beta \mathcal{V}_{\mathrm{p}}(x(0), p(0))}, \tag{A.61}$$

for all $T \geq 0$, $(x, w, z) \in \pi\, \mathfrak{B}_{\mathrm{p}}(p)$, and $p \in \mathcal{P}^{\mathbb{R}_0^+}$ with $w \in \mathcal{L}_{\infty\mathrm{e}}$, which is the $\mathcal{L}_\infty$-gain definition (see Definition 2.11), where $\zeta(x_0) = \sqrt{\gamma \beta \mathcal{V}_{\mathrm{p}}(x_0, p(0))}$.

Next, we show the derivation to obtain the LMI conditions given in (2.41a). For a storage function of the form (2.36), (A.51) becomes

$$\begin{aligned} 2x(t)^\top M(p(t))\big(A(p(t))x(t) + B(p(t))w(t)\big) + (\star)^\top \partial M(p(t), \dot{p}(t))x(t) \leq \\ - \beta(\star)^\top M(p(t))x(t) + \alpha(\star)^\top w(t), \end{aligned} \tag{A.62}$$

which holds if

$$2x^\top M(p)\big(A(p)x + B(p)w\big) + (\star)^\top \partial M(p,v)x \le -\beta(\star)^\top M(p)x + \alpha(\star)^\top w, \quad (A.63)$$

$$(\star)^\top \begin{bmatrix} A(p)^\top M(p) + (\star)^\top + \beta M(p) + \partial M(p,v) & M(p)B(p) \\ \star & -\alpha I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \le 0, \quad (A.64)$$

for all $x \in \mathcal{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$, $w \in \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{w}}$, $p \in \mathcal{P}$, and $v \in \Pi$. This is implied if

$$\begin{bmatrix} A(p)^\top M(p) + (\star)^\top + \beta M(p) + \partial M(p,v) & M(p)B(p) \\ \star & -\alpha I \end{bmatrix} \preceq 0, \quad (A.65)$$

holds for all $p \in \mathcal{P}$, and $v \in \Pi$, which is equivalent to the first LMI in (2.41a).

Next, from (A.52), we have that

$$\gamma^{-1}(\star)^\top (C(p(t))x(t) + D(p(t))w(t)) \le \beta\, x(t)^\top M(p(t))x(t) + (\gamma - \alpha)w(t)^\top w(t), \quad (A.66)$$

for all $t \ge 0$ and $(x, w) \in \pi_{\mathrm{x,w}}\,\mathfrak{B}_\mathrm{p}(p)$ for all $p \in \mathcal{P}^{\mathbb{R}_0^+}$. This holds if

$$\gamma^{-1}(\star)^\top (C(p)x + D(p)w) \le \beta\, x(t)^\top M(p)x + (\gamma - \alpha)w^\top w, \quad (A.67)$$

or rewritten

$$(\star)^\top \left(\begin{bmatrix} \beta M(p) & 0 \\ \star & (\gamma - \alpha) \end{bmatrix} - (\star)^\top (\gamma^{-1} I) \begin{bmatrix} C(p) & D(p) \end{bmatrix} \right) \begin{bmatrix} x \\ w \end{bmatrix} \ge 0, \quad (A.68)$$

for all $x \in \mathcal{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$, $w \in \mathcal{W} \subseteq \mathbb{R}^{n_\mathrm{w}}$, and $p \in \mathcal{P}$, which holds if

$$\begin{bmatrix} \beta M(p) & 0 \\ \star & (\gamma - \alpha) \end{bmatrix} - (\star)^\top (\gamma^{-1} I) \begin{bmatrix} C(p) & D(p) \end{bmatrix} \succeq 0, \quad (A.69)$$

for all $p \in \mathcal{P}$. Through a Schur complement, this is equivalent to

$$\begin{bmatrix} \beta M(p) & 0 & C(p)^\top \\ \star & (\gamma - \alpha)I & D(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0, \quad (A.70)$$

which is equivalent to the second LMI in (2.41a).

Discrete-time

We first proof that a DT LPV system given by (2.26) has a bounded ℓ_∞-gain of γ, if there exists an $\alpha \in \mathbb{R}_0^+$, $\beta \in [0,\ 1]$, and storage function $\mathcal{V}_\mathrm{p}$ such that

$$\mathcal{V}_\mathrm{p}(x(t+1), p(t+1)) - \mathcal{V}_\mathrm{p}(x(t), p(t)) \le -\beta \mathcal{V}_\mathrm{p}(x(t), p(t)) + \alpha(\star)^\top w(t), \quad (A.71)$$

and (A.52) holds for all $t \ge 0$ and $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ for all $p \in \mathcal{P}^{\mathbb{N}_0}$.

We can rewrite (A.71) to

$$\mathcal{V}_\mathrm{p}(x(t+1), p(t+1)) \le (1 - \beta)\mathcal{V}_\mathrm{p}(x(t), p(t)) + \alpha(\star)^\top w(t), \quad (A.72)$$

by the DT version of Grönwall's Lemma, this implies that

$$\mathcal{V}_{\mathrm{p}}(x(t), p(t)) \leq (1-\beta)^t \mathcal{V}_{\mathrm{p}}(x(0), p(0)) + \alpha \sum_{\tau=0}^{t-1} (1-\beta)^{t-1-\tau} \|w(\tau)\|^2, \qquad \text{(A.73)}$$

for all $t \geq 0$ and $(x, w) \in \pi_{\mathrm{x,w}} \mathfrak{B}_{\mathrm{p}}(p)$ for all $p \in \mathcal{P}^{\mathbb{N}_0}$.

Moreover, it holds that

$$\alpha \sum_{\tau=0}^{t-1} (1-\beta)^{t-1-\tau} \|w(\tau)\|^2 \leq \alpha \|w\|_{\infty,t}^2 \sum_{\tau=0}^{t-1} (1-\beta)^{t-1-\tau} \leq \frac{\alpha}{\beta} \|w\|_{\infty,t}^2, \qquad \text{(A.74)}$$

for all $t \geq 0$ and $w \in \pi_{\mathrm{w}} \mathfrak{B}_{\mathrm{p}}(p)$ for all $p \in \mathcal{P}^{\mathbb{N}_0}$. Furthermore, it holds that

$$(1-\beta)^t \mathcal{V}_{\mathrm{p}}(x(0), p(0)) \leq \mathcal{V}_{\mathrm{p}}(x(0), p(0)), \qquad \text{(A.75)}$$

for all $t \geq 0$ and $\beta \in [0,\ 1]$.

Therefore, combining (A.74) and (A.75) with (A.73), it holds that

$$\mathcal{V}_{\mathrm{p}}(x(t), p(t)) \leq \mathcal{V}_{\mathrm{p}}(x(0), p(0)) + \frac{\alpha}{\beta} \|w\|_{\infty,t}^2, \qquad \text{(A.76)}$$

and

$$\beta \mathcal{V}_{\mathrm{p}}(x(t), p(t)) \leq \beta \mathcal{V}_{\mathrm{p}}(x(0), p(0)) + \alpha \|w\|_{\infty,t}^2, \qquad \text{(A.77)}$$

for all $t \geq 0$ and $(x, w) \in \pi_{\mathrm{x,w}} \mathfrak{B}_{\mathrm{p}}(p)$ for all $p \in \mathcal{P}^{\mathbb{N}_0}$.

Next, by (A.52), we have that it holds that

$$\gamma^{-1} \|z(t)\|^2 \leq \beta \mathcal{V}_{\mathrm{p}}(x(t), p(t)) + (\gamma - \alpha) \|w(t)\|^2, \qquad \text{(A.78)}$$

for all $t \geq 0$ and $(x, w, z) \in \mathfrak{B}_{\mathrm{p}}(p)$ for all $p \in \mathcal{P}^{\mathbb{N}_0}$. Using (A.77), it therefore also holds that

$$\gamma^{-1} \|z(t)\|^2 \leq \beta \mathcal{V}_{\mathrm{p}}(x(0), p(0)) + \alpha \|w\|_{\infty,t}^2 + (\gamma - \alpha) \|w(t)\|^2, \qquad \text{(A.79)}$$

for all $t \geq 0$ and $(x, w, z) \in \mathfrak{B}_{\mathrm{p}}(p)$ for all $p \in \mathcal{P}^{\mathbb{N}_0}$. By multiplication with γ and taking the supremum over $t \in [0,\ T]$, this gives that for all $T \geq 0$

$$\|z\|_{\infty,T}^2 \leq \gamma\beta \mathcal{V}_{\mathrm{p}}(x(0), p(0)) + \gamma^2 \|w\|_{\infty,T}^2, \qquad \text{(A.80)}$$

for all $(x, w, z) \in \pi\, \mathfrak{B}_{\mathrm{p}}(p)$ and $p \in \mathcal{P}^{\mathbb{N}_0}$ with $w \in \mathcal{L}_{\infty\mathrm{e}}$. Therefore,

$$\|z\|_{\infty,T} \leq \gamma \|w\|_{\infty,T} + \sqrt{\gamma\beta \mathcal{V}_{\mathrm{p}}(x(0), p(0))}, \qquad \text{(A.81)}$$

for all $T \geq 0$, $(x, w, z) \in \mathfrak{B}_{\mathrm{p}}(p)$, and $p \in \mathcal{P}^{\mathbb{N}_0}$ with $w \in \mathcal{L}_{\infty\mathrm{e}}$, which is the $\mathcal{L}_\infty$-gain definition (see Definition 2.11), where $\zeta(x_0) = \sqrt{\gamma\beta \mathcal{V}_{\mathrm{p}}(x_0, p(0))}$.

Next, we show the derivation to obtain the LMI conditions given in (2.41b). For a storage function of the form (2.36), (A.71) becomes

$$\begin{aligned}(\star)^\top M\big(p(t)+v(t)\big)\big(A(p(t))x(t) + B(p(t))w(t)\big) - (\star)^\top M(p(t))x(t) \leq \\ -\beta(\star)^\top M(p(t))x(t) + \alpha(\star)^\top w(t),\end{aligned} \qquad \text{(A.82)}$$

which holds if

$$(\star)^\top M(p+v)\big(A(p)x+B(p)w\big)-(\star)^\top M(p)x \leq -\beta(\star)^\top M(p)x+\alpha(\star)^\top w, \tag{A.83}$$

$$(\star)^\top\left(\begin{bmatrix}-(1-\beta)M(p) & 0\\ \star & -\alpha I\end{bmatrix}+\right.$$
$$\left.(\star)^\top M(p+v)^{-1}\begin{bmatrix}M(p+v)A(p) & M(p+v)B(p)\end{bmatrix}\right)\begin{bmatrix}x\\ w\end{bmatrix}\leq 0, \tag{A.84}$$

for all $x\in\mathcal{X}\subseteq\mathbb{R}^{n_\mathrm{x}}$, $w\in\mathcal{W}\subseteq\mathbb{R}^{n_\mathrm{w}}$, $p\in\mathcal{P}$, and $v\in\Pi$, which is implied if

$$\begin{bmatrix}(1-\beta)M(p) & 0\\ \star & \alpha I\end{bmatrix}-(\star)^\top M(p+v)^{-1}\begin{bmatrix}M(p+v)A(p) & M(p+v)B(p)\end{bmatrix}\succeq 0, \tag{A.85}$$

holds for all $p\in\mathcal{P}$, and $v\in\Pi$. Using a Schur complement, this is equivalent to

$$\begin{bmatrix}(1-\beta)M(p) & 0 & A(p)^\top M(p+v)\\ \star & \alpha I & B(p)^\top M(p+v)\\ \star & \star & M(p+v)\end{bmatrix}\succeq 0. \tag{A.86}$$

Using a congruence transformation, this is equivalent to

$$(\star)^\top\begin{bmatrix}(1-\beta)M(p) & 0 & A(p)^\top M(p+v)\\ \star & \alpha I & B(p)^\top M(p+v)\\ \star & \star & M(p+v)\end{bmatrix}\begin{bmatrix}0 & M(p)^{-1} & 0\\ 0 & 0 & I\\ M(p+v)^{-1} & 0 & 0\end{bmatrix}\succeq 0, \tag{A.87}$$

resulting, in

$$\begin{bmatrix}\bar{M}(p+v) & A(p)\bar{M}(p) & B(p)\\ \star & (1-\beta)\bar{M}(p) & 0\\ \star & \star & \alpha I\end{bmatrix}\succeq 0, \tag{A.88}$$

where $\bar{M}(p):=M(p)^{-1}\succ 0$ and $\bar{M}(p+v):=M(p+v)^{-1}\succ 0$, which is equivalent to the first condition in (2.41b).

Next, from (A.52), we obtain through the steps in (A.66)–(A.70),

$$\begin{bmatrix}\beta M(p) & 0 & C(p)^\top\\ \star & (\gamma-\alpha)I & D(p)^\top\\ \star & \star & \gamma I\end{bmatrix}\succeq 0. \tag{A.89}$$

Using a congruence transformation, this is equivalent to

$$(\star)^\top\begin{bmatrix}\beta M(p) & 0 & C(p)^\top\\ \star & (\gamma-\alpha)I & D(p)^\top\\ \star & \star & \gamma I\end{bmatrix}\begin{bmatrix}M(p)^{-1} & 0 & 0\\ 0 & I & 0\\ 0 & 0 & I\end{bmatrix}\succeq 0, \tag{A.90}$$

resulting in

$$\begin{bmatrix}\beta\bar{M}(p) & 0 & \bar{M}(p)C(p)^\top\\ \star & (\gamma-\alpha)I & D(p)^\top\\ \star & \star & \gamma I\end{bmatrix}\succeq 0, \tag{A.91}$$

where again $\bar{M}(p)=M(p)^{-1}\succ 0$, which is equivalent to the second condition in (2.41b).

Remark A.1. Note that the $\mathcal{L}_\infty/\ell_\infty$ analysis conditions are not LMIs due to the multiplication of β and M, which are both unknown variables. In practice, this is often solved by performing a line search over β. For each fixed β in the line search, (A.65), (A.70), (A.88), and (A.91) are LMIs.

Remark A.2. As mentioned in Section 2.5.2, in the LPVcore Toolbox (Boef et al. 2021), the analysis algorithms have been implemented for LPV systems given by (2.26) with affine scheduling dependency. In that case, the matrix function M is also parameterized as an affine matrix function, i.e., of the form $M(p) = M_0 + \sum_{i=1}^{n_\mathrm{p}} M_i p_i$, where $M_i \in \mathbb{S}^{n_\mathrm{x}}$ for $i = 0, \ldots, n_\mathrm{p}$. In that case, $\partial M(p, v) = \sum_{i=1}^{n_\mathrm{p}} \frac{\partial M(p)}{\partial p_i} v_i = \sum_{i=1}^{n_\mathrm{p}} M_i v_i$, and $M(p + v) = M_0 + \sum_{i=1}^{n_\mathrm{p}} M_i(p_i + v_i) = M(p) + \sum_{i=1}^{n_\mathrm{p}} M_i v_i$.

A.3 Derivations of Synthesis Results

A.3.1 (Q, S, R) performance

Continuous-time

By Theorem 2.5, the closed-loop LPV system given by (2.44) is classically (Q, S, R) dissipative, if there exists a positive definite matrix function $M : \mathcal{P} \to \mathbb{S}^{n_\mathrm{x}+n_{\mathrm{x_k}}}$, such that, in CT, for all $p \in \mathcal{P}$ and $v \in \Pi$ it holds that

$$(\star)^\top \begin{bmatrix} \partial M(p, v) & M(p) \\ \star & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\mathrm{cl}(p) & B_\mathrm{cl}(p) \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\mathrm{cl}(p) & D_\mathrm{cl}(p) \end{bmatrix} \preceq 0. \tag{A.92}$$

As both M and the controller matrices, part of $A_\mathrm{cl}, \ldots, D_\mathrm{cl}$ (see (2.45)), are unknown, (A.92) is not an LMI. Next, we will derive an LMI condition which is equivalent to (A.92). Let us take $n_{\mathrm{x_k}} = n_\mathrm{x}$ and we partition M as follows

$$M(p) = \begin{bmatrix} M_\mathrm{x}(p) & U(p) \\ \star & * \end{bmatrix}, \tag{A.93}$$

where $M_\mathrm{x} : \mathcal{P} \to \mathbb{S}^{n_\mathrm{x}}$ and $U : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$, here $*$ indicates that this entry is not relevant for our derivation. Moreover, the inverse of M is partitioned as

$$M(p)^{-1} = \begin{bmatrix} M_\mathrm{y}(p) & V(p) \\ \star & * \end{bmatrix}, \tag{A.94}$$

where $M_\mathrm{y} : \mathcal{P} \to \mathbb{S}^{n_\mathrm{x}}$ and $V : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$. As by definition $M(p)M(p)^{-1} = I$, we have from (A.93) and (A.94) that $M_\mathrm{x}(p)M_\mathrm{y}(p) + U(p)V(p)^\top = I$. Next, we introduce the matrix

$$N(p) := \begin{bmatrix} M_\mathrm{y}(p) & I \\ V(p)^\top & 0 \end{bmatrix}, \tag{A.95}$$

and

$$P(p) := N(p)^\top M(p) = \begin{bmatrix} I & 0 \\ M_\mathrm{x}(p) & U(p) \end{bmatrix}. \tag{A.96}$$

Based on these matrices, we apply the following congruence transformation to (A.92)

$$\begin{bmatrix} N(p)^\top & 0 \\ 0 & I \end{bmatrix} \left((\star)^\top \begin{bmatrix} \partial M(p,v) & M(p) \\ \star & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_{\mathrm{cl}}(p) & B_{\mathrm{cl}}(p) \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C_{\mathrm{cl}}(p) & D_{\mathrm{cl}}(p) \end{bmatrix} \right) \begin{bmatrix} N(p) & 0 \\ 0 & I \end{bmatrix} \preceq 0, \tag{A.97}$$

resulting in

$$\begin{bmatrix} N(p)^\top & 0 \\ 0 & I \end{bmatrix} \left(\begin{bmatrix} \partial M(p,v) + M(p)A_{\mathrm{cl}}(p) + (\star)^\top & M(p)B_{\mathrm{cl}}(p) \\ \star & 0 \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C_{\mathrm{cl}}(p) & D_{\mathrm{cl}}(p) \end{bmatrix} \right) \begin{bmatrix} N(p) & 0 \\ 0 & I \end{bmatrix} \preceq 0, \tag{A.98}$$

$$\begin{bmatrix} (\star)^\top(\partial M(p,v))N(p) + (\star)^\top M(p)A_{\mathrm{cl}}(p)N(p) + (\star)^\top & N(p)^\top M(p)B_{\mathrm{cl}}(p) \\ \star & 0 \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C_{\mathrm{cl}}(p)N(p) & D_{\mathrm{cl}}(p) \end{bmatrix} \preceq 0. \tag{A.99}$$

We have that $A_{\mathrm{cl}}, \ldots, D_{\mathrm{cl}}$ in (2.45) can be written as

$$A_{\mathrm{cl}}(p) = \begin{bmatrix} A(p) & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_{\mathrm{u}}(p) \\ I & 0 \end{bmatrix} \begin{bmatrix} A_{\mathrm{k}}(p) & B_{\mathrm{k}}(p) \\ C_{\mathrm{k}}(p) & D_{\mathrm{k}}(p) \end{bmatrix} \begin{bmatrix} 0 & I \\ C_{\mathrm{y}}(p) & 0 \end{bmatrix}, \tag{A.100a}$$

$$B_{\mathrm{cl}}(p) = \begin{bmatrix} B_{\mathrm{w}}(p) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B_{\mathrm{u}}(p) \\ I & 0 \end{bmatrix} \begin{bmatrix} A_{\mathrm{k}}(p) & B_{\mathrm{k}}(p) \\ C_{\mathrm{k}}(p) & D_{\mathrm{k}}(p) \end{bmatrix} \begin{bmatrix} 0 \\ D_{\mathrm{yw}}(p) \end{bmatrix}, \tag{A.100b}$$

$$C_{\mathrm{cl}}(p) = \begin{bmatrix} C_{\mathrm{z}}(p) & 0 \end{bmatrix} + \begin{bmatrix} 0 & D_{\mathrm{zu}}(p) \end{bmatrix} \begin{bmatrix} A_{\mathrm{k}}(p) & B_{\mathrm{k}}(p) \\ C_{\mathrm{k}}(p) & D_{\mathrm{k}}(p) \end{bmatrix} \begin{bmatrix} 0 & I \\ C_{\mathrm{y}}(p) & 0 \end{bmatrix}, \tag{A.100c}$$

$$D_{\mathrm{cl}}(p) := D_{\mathrm{zw}}(p) + \begin{bmatrix} 0 & D_{\mathrm{zu}}(p) \end{bmatrix} \begin{bmatrix} A_{\mathrm{k}}(p) & B_{\mathrm{k}}(p) \\ C_{\mathrm{k}}(p) & D_{\mathrm{k}}(p) \end{bmatrix} \begin{bmatrix} 0 \\ D_{\mathrm{yw}}(p) \end{bmatrix}. \tag{A.100d}$$

Next, we use these relations and (A.93)–(A.96), to rewrite the terms in (A.99). First, we focus on the term $(\star)^\top(\partial M(p,v))N(p) = N(p)^\top(\partial M(p,v))N(p)$. We have by (A.96), dropping dependency on p and v for brevity, that

$$\partial P = (\partial N)^\top M + N^\top(\partial M), \tag{A.101}$$

therefore

$$N^\top(\partial M) = \partial P - (\partial N)^\top M, \tag{A.102}$$

and

$$\begin{aligned} N^\top(\partial M)N &= (\partial P)N - (\partial N)^\top MN, \\ &= (\partial P)N - (\partial N)^\top P^\top. \end{aligned} \tag{A.103}$$

Filling in (A.93)–(A.96), we obtain that

$$\begin{aligned}(\star)^\top(\partial M)N &= \begin{bmatrix} 0 & 0 \\ \partial M_\mathrm{x} & \partial U \end{bmatrix}\begin{bmatrix} M_\mathrm{y} & I \\ V^\top & 0 \end{bmatrix} - \begin{bmatrix} \partial M_\mathrm{y} & \partial V \\ 0 & 0 \end{bmatrix}\begin{bmatrix} I & M_\mathrm{x} \\ 0 & U^\top \end{bmatrix}, \\ &= \begin{bmatrix} -\partial M_\mathrm{y} & -(\partial M_\mathrm{y})M_\mathrm{x} - (\partial V)U^\top \\ (\partial M_\mathrm{x})M_\mathrm{y} + (\partial U)V^\top & \partial M_\mathrm{x} \end{bmatrix}.\end{aligned} \tag{A.104}$$

Note that this matrix is symmetric (as ∂M is symmetric), which means that $\big((\partial M_\mathrm{x})M_\mathrm{y} + (\partial U)V^\top\big)^\top = -(\partial M_\mathrm{y})M_\mathrm{x} - (\partial V)U^\top$.

Next, we take a look at $(\star)^\top(\partial M)N + (\star)^\top M A_\mathrm{cl} N + (\star)^\top$, which using (A.96) can be written as

$$N^\top(\partial M)N + PA_\mathrm{cl}N + (\star)^\top. \tag{A.105}$$

Filling in (A.95), (A.96), (A.100), and (A.104), results in

$$\begin{aligned}&\begin{bmatrix} -\partial M_\mathrm{y} & -(\partial M_\mathrm{y})M_\mathrm{x} - (\partial V)U^\top \\ (\partial M_\mathrm{x})M_\mathrm{y} + (\partial U)V^\top & \partial M_\mathrm{x} \end{bmatrix} + \\ &\begin{bmatrix} I & 0 \\ M_\mathrm{x} & U \end{bmatrix}\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix}\begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix}\begin{bmatrix} 0 & I \\ C_\mathrm{y} & 0 \end{bmatrix}\right)\begin{bmatrix} M_\mathrm{y} & I \\ V^\top & 0 \end{bmatrix} + (\star)^\top.\end{aligned} \tag{A.106}$$

This can be written as follows

$$\begin{aligned}&\begin{bmatrix} -\partial M_\mathrm{y} & -(\partial M_\mathrm{y})M_\mathrm{x} - (\partial V)U^\top \\ (\partial M_\mathrm{x})M_\mathrm{y} + (\partial U)V^\top & \partial M_\mathrm{x} \end{bmatrix} + \left(\begin{bmatrix} AM_\mathrm{y} & A \\ M_\mathrm{x}AM_\mathrm{y} & M_\mathrm{x}A \end{bmatrix} + \right. \\ &\qquad \left.\begin{bmatrix} 0 & B_\mathrm{u} \\ U & M_\mathrm{x}B_\mathrm{u} \end{bmatrix}\begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix}\begin{bmatrix} V^\top & 0 \\ C_\mathrm{y}M_\mathrm{y} & C_\mathrm{y} \end{bmatrix}\right) + (\star)^\top,\end{aligned} \tag{A.107}$$

$$\begin{aligned}&\begin{bmatrix} AM_\mathrm{y} + (\star)^\top - \partial M_\mathrm{y} & A \\ \star & M_\mathrm{x}A + (\star)^\top + \partial M_\mathrm{x} \end{bmatrix} + \\ &\qquad \begin{bmatrix} 0 & \star \\ M_\mathrm{x}AM_\mathrm{y} + (\partial M_\mathrm{x})M_\mathrm{y} + (\partial U)V^\top & 0 \end{bmatrix} + \\ &\left(\begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix}\begin{bmatrix} U & M_\mathrm{x}B_\mathrm{u} \\ 0 & I \end{bmatrix}\begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix}\begin{bmatrix} V^\top & 0 \\ C_\mathrm{y}M_\mathrm{y} & I \end{bmatrix}\begin{bmatrix} I & 0 \\ 0 & C_\mathrm{y} \end{bmatrix}\right) + (\star)^\top,\end{aligned} \tag{A.108}$$

$$\begin{aligned}&\begin{bmatrix} AM_\mathrm{y} + (\star) - \partial M_\mathrm{y} & A \\ \star & M_\mathrm{x}A + (\star) + \partial M_\mathrm{x} \end{bmatrix} + \\ &\qquad \begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix}\left(\begin{bmatrix} U & M_\mathrm{x}B_\mathrm{u} \\ 0 & I \end{bmatrix}\begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix}\begin{bmatrix} V^\top & 0 \\ C_\mathrm{y}M_\mathrm{y} & I \end{bmatrix} + \right. \\ &\qquad \left.\begin{bmatrix} M_\mathrm{x}AM_\mathrm{y} + (\partial M_\mathrm{x})M_\mathrm{y} + (\partial U)V^\top & 0 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} I & 0 \\ 0 & C_\mathrm{y} \end{bmatrix} + (\star)^\top.\end{aligned} \tag{A.109}$$

Let us define

$$\begin{bmatrix} \mathcal{A}_\mathrm{k} & \mathcal{B}_\mathrm{k} \\ \mathcal{C}_\mathrm{k} & \mathcal{D}_\mathrm{k} \end{bmatrix} := \\ \begin{bmatrix} U & M_\mathrm{x} B_\mathrm{u} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} V^\top & 0 \\ C_\mathrm{y} M_\mathrm{y} & I \end{bmatrix} + \begin{bmatrix} M_\mathrm{x} A M_\mathrm{y} + (\partial M_\mathrm{x}) M_\mathrm{y} + (\partial U) V^\top & 0 \\ 0 & 0 \end{bmatrix}, \tag{A.110}$$

where $\mathcal{A}_\mathrm{k} : \mathcal{P} \times \Pi \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$, $\mathcal{B}_\mathrm{k} : \mathcal{P} \times \Pi \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{y}}$, $\mathcal{C}_\mathrm{k} : \mathcal{P} \times \Pi \to \mathbb{R}^{n_\mathrm{u} \times n_\mathrm{x}}$, and $\mathcal{D}_\mathrm{k} : \mathcal{P} \times \Pi \to \mathbb{R}^{n_\mathrm{u} \times n_\mathrm{y}}$.

This allows us to write (A.109), and hence, $(\star)^\top (\partial M) N + (\star)^\top M A_\mathrm{cl} N + (\star)^\top$, as

$$\begin{bmatrix} A M_\mathrm{y} + (\star) - \partial M_\mathrm{y} & A \\ \star & M_\mathrm{x} A + (\star) + \partial M_\mathrm{x} \end{bmatrix} + \begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix} \begin{bmatrix} \mathcal{A}_\mathrm{k} & \mathcal{B}_\mathrm{k} \\ \mathcal{C}_\mathrm{k} & \mathcal{D}_\mathrm{k} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_\mathrm{y} \end{bmatrix} + (\star)^\top, \tag{A.111}$$

$$\begin{bmatrix} (A M_\mathrm{y} + B_\mathrm{u} \mathcal{C}_\mathrm{k}) + (\star)^\top - \partial M_\mathrm{y} & A + \mathcal{A}_\mathrm{k}^\top + B_\mathrm{u} \mathcal{D}_\mathrm{k} C_\mathrm{y} \\ \star & (M_\mathrm{x} A + \mathcal{B}_\mathrm{k} C_\mathrm{y}) + (\star)^\top + \partial M_\mathrm{x} \end{bmatrix}. \tag{A.112}$$

We then define

$$\mathcal{A}_\mathrm{cl} := \begin{bmatrix} A M_\mathrm{y} + B_\mathrm{u} \mathcal{C}_\mathrm{k} - \frac{1}{2} \partial M_\mathrm{y} & A + B_\mathrm{u} \mathcal{D}_\mathrm{k} C_\mathrm{y} \\ \mathcal{A}_\mathrm{k} & M_\mathrm{x} A + \mathcal{B}_\mathrm{k} C_\mathrm{y} + \frac{1}{2} \partial M_\mathrm{x} \end{bmatrix}, \tag{A.113}$$

such that (A.112) can be expressed as $\mathcal{A}_\mathrm{cl} + \mathcal{A}_\mathrm{cl}^\top$.

Let us now take a look at the term $N^\top M B_\mathrm{cl} = P B_\mathrm{cl}$ in (A.99). Filling in (A.96) and (A.100) result in

$$\begin{bmatrix} I & 0 \\ M_\mathrm{x} & U \end{bmatrix} \left(\begin{bmatrix} B_\mathrm{w} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} 0 \\ D_\mathrm{yw} \end{bmatrix} \right), \tag{A.114}$$

which can be rewritten as follows

$$\begin{bmatrix} B_\mathrm{w} \\ M_\mathrm{x} B_\mathrm{w} \end{bmatrix} + \begin{bmatrix} 0 & B_\mathrm{u} \\ U & M_\mathrm{x} B_\mathrm{w} \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} 0 \\ D_\mathrm{yw} \end{bmatrix}, \tag{A.115}$$

$$\begin{bmatrix} B_\mathrm{w} \\ M_\mathrm{x} B_\mathrm{w} \end{bmatrix} + \begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix} \begin{bmatrix} U & M_\mathrm{x} B_\mathrm{u} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} V^\top & 0 \\ C_\mathrm{y} M_\mathrm{y} & I \end{bmatrix} \begin{bmatrix} 0 \\ D_\mathrm{yw} \end{bmatrix}, \tag{A.116}$$

$$\begin{bmatrix} B_\mathrm{w} \\ M_\mathrm{x} B_\mathrm{w} \end{bmatrix} + \begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix} \left(\begin{bmatrix} U & M_\mathrm{x} B_\mathrm{u} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} V^\top & 0 \\ C_\mathrm{y} M_\mathrm{y} & I \end{bmatrix} + \\ \begin{bmatrix} M_\mathrm{x} A M_\mathrm{y} + (\partial M_\mathrm{x}) M_\mathrm{y} + (\partial U) V^\top & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ D_\mathrm{yw} \end{bmatrix}. \tag{A.117}$$

Using (A.110), we can write this, and hence, $N^\top M B_\mathrm{cl}$, as

$$\begin{bmatrix} B_\mathrm{w} \\ M_\mathrm{x} B_\mathrm{w} \end{bmatrix} + \begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix} \begin{bmatrix} \mathcal{A}_\mathrm{k} & \mathcal{B}_\mathrm{k} \\ \mathcal{C}_\mathrm{k} & \mathcal{D}_\mathrm{k} \end{bmatrix} \begin{bmatrix} 0 \\ D_\mathrm{yw} \end{bmatrix}, \tag{A.118}$$

which can be rewritten to

$$\begin{bmatrix} B_\mathrm{w} + B_\mathrm{u}\mathcal{D}_\mathrm{k}D_\mathrm{yw} \\ M_\mathrm{x}B_\mathrm{w} + B_\mathrm{k}D_\mathrm{yw} \end{bmatrix} =: \mathcal{B}_\mathrm{cl}. \tag{A.119}$$

Next, we take a look at the term $C_\mathrm{cl}N$ in (A.99). Filling in (A.95) and (A.100) results in

$$\left(\begin{bmatrix} C_\mathrm{z} & 0 \end{bmatrix} + \begin{bmatrix} 0 & D_\mathrm{zu} \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\mathrm{y} & 0 \end{bmatrix} \right) \begin{bmatrix} M_\mathrm{y} & I \\ V^\top & 0 \end{bmatrix}, \tag{A.120}$$

which can be rewritten as follows

$$\begin{bmatrix} C_\mathrm{z}M_\mathrm{y} & C_\mathrm{z} \end{bmatrix} + \begin{bmatrix} 0 & D_\mathrm{zu} \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} V^\top & 0 \\ C_\mathrm{y}M_\mathrm{y} & C_\mathrm{y} \end{bmatrix}, \tag{A.121}$$

$$\begin{bmatrix} C_\mathrm{z}M_\mathrm{y} & C_\mathrm{z} \end{bmatrix} + \begin{bmatrix} 0 & D_\mathrm{zu} \end{bmatrix} \begin{bmatrix} U & M_\mathrm{x}B_\mathrm{u} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} V^\top & 0 \\ C_\mathrm{y}M_\mathrm{y} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_\mathrm{y} \end{bmatrix}, \tag{A.122}$$

$$\begin{aligned} \begin{bmatrix} C_\mathrm{z}M_\mathrm{y} & C_\mathrm{z} \end{bmatrix} + \begin{bmatrix} 0 & D_\mathrm{zu} \end{bmatrix} \Bigg(&\begin{bmatrix} U & M_\mathrm{x}B_\mathrm{u} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} V^\top & 0 \\ C_\mathrm{y}M_\mathrm{y} & I \end{bmatrix} + \\ &\begin{bmatrix} M_\mathrm{x}AM_\mathrm{y} + (\partial M_\mathrm{x})M_\mathrm{y} + (\partial U)V^\top & 0 \\ 0 & 0 \end{bmatrix} \Bigg) \begin{bmatrix} I & 0 \\ 0 & C_\mathrm{y} \end{bmatrix}. \end{aligned} \tag{A.123}$$

Similarly as for the other terms, using (A.110), we can then write this, and hence, $C_\mathrm{cl}N$, as

$$\begin{bmatrix} C_\mathrm{z}M_\mathrm{y} & C_\mathrm{z} \end{bmatrix} + \begin{bmatrix} 0 & D_\mathrm{zu} \end{bmatrix} \begin{bmatrix} \mathcal{A}_\mathrm{k} & \mathcal{B}_\mathrm{k} \\ \mathcal{C}_\mathrm{k} & \mathcal{D}_\mathrm{k} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_\mathrm{y} \end{bmatrix}, \tag{A.124}$$

which can be rewritten to

$$\begin{bmatrix} C_\mathrm{z}M_\mathrm{y} + D_\mathrm{zu}\mathcal{C}_\mathrm{k} & C_\mathrm{z} + D_\mathrm{zu}\mathcal{D}_\mathrm{k}C_\mathrm{y} \end{bmatrix} := \mathcal{C}_\mathrm{cl}. \tag{A.125}$$

Finally, we focus on D_cl in (A.99). Using (A.100), we have that

$$D_\mathrm{cl} = D_\mathrm{zw} + \begin{bmatrix} 0 & D_\mathrm{zu} \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} 0 \\ D_\mathrm{yw} \end{bmatrix} = D_\mathrm{zw} + D_\mathrm{zu}D_\mathrm{k}D_\mathrm{yw}. \tag{A.126}$$

By (A.110), we have that $\mathcal{D}_\mathrm{k} = D_\mathrm{k}$, hence,

$$D_\mathrm{cl} = D_\mathrm{zw} + D_\mathrm{zu}\mathcal{D}_\mathrm{k}D_\mathrm{yw} =: \mathcal{D}_\mathrm{cl}. \tag{A.127}$$

Combining the results (A.113), (A.119), (A.125), and (A.127), we can rewrite (A.99) as

$$\begin{bmatrix} \mathcal{A}_\mathrm{cl}(p,v) + (\star)^\top & \mathcal{B}_\mathrm{cl}(p) \\ \star & 0 \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & I \\ \mathcal{C}_\mathrm{cl}(p) & \mathcal{D}_\mathrm{cl}(p) \end{bmatrix} \preceq 0, \tag{A.128}$$

to hold for all $(p,v) \in \mathcal{P} \times \Pi$. Note that (A.128) is not an LMI, as it is quadratic in $\mathcal{C}_\mathrm{cl}(p)$ and $\mathcal{D}_\mathrm{cl}(p)$. However, for specific choices of (Q,S,R) we can rewrite this to an LMI, see e.g. Appendices A.3.2 to A.3.5.

Finally, to ensure M is a positive definite matrix function, it needs to hold for all $p \in \mathcal{P}$ that

$$M(p) \succ 0. \tag{A.129}$$

Using a congruence transformation, this holds if

$$N(p)^\top M(p) N(p) \succ 0, \tag{A.130}$$

which results, using the definitions of N and P in (A.95) and (A.96), respectively, in

$$P(p)N(p) \succ 0, \tag{A.131}$$

$$\begin{bmatrix} I & 0 \\ M_\mathrm{x}(p) & U(p) \end{bmatrix} \begin{bmatrix} M_\mathrm{y}(p) & I \\ V(p)^\top & 0 \end{bmatrix} \succ 0, \tag{A.132}$$

$$\begin{bmatrix} M_\mathrm{y}(p) & I \\ M_\mathrm{x}(p)M_\mathrm{y}(p) + U(p)V(p)^\top & M_\mathrm{x}(p) \end{bmatrix} \succ 0, \tag{A.133}$$

and using that $M_\mathrm{x}(p)M_\mathrm{y}(p) + U(p)V(p)^\top = I$, this becomes

$$\mathcal{M}(p) := \begin{bmatrix} M_\mathrm{y}(p) & I \\ I & M_\mathrm{x}(p) \end{bmatrix} \succ 0. \tag{A.134}$$

Once a solution has been found for (A.128) and (A.134), i.e., matrix functions $\mathcal{A}_\mathrm{k}, \ldots \mathcal{D}_\mathrm{k}$, M_x, M_y have been found such that (A.128) and (A.134) hold for all $(p, v) \in \mathcal{P} \times \Pi$, we can reconstruct $A_\mathrm{k}, \ldots D_\mathrm{k}$, i.e., the state-space matrices of the LPV controller. Namely, by (A.110), we have, omitting dependence on p and v for brevity, that

$$\begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} = \\ \begin{bmatrix} U & M_\mathrm{x} B_\mathrm{u} \\ 0 & I \end{bmatrix}^{-1} \left(\begin{bmatrix} \mathcal{A}_\mathrm{k} - M_\mathrm{x} A M_\mathrm{y} - (\partial M_\mathrm{x}) M_\mathrm{y} - (\partial U) V^\top & \mathcal{B}_\mathrm{k} \\ \mathcal{C}_\mathrm{k} & \mathcal{D}_\mathrm{k} \end{bmatrix} \right) \begin{bmatrix} V^\top & 0 \\ C_\mathrm{y} M_\mathrm{y} & I \end{bmatrix}^{-1}, \tag{A.135}$$

where U and V are arbitrary solutions to $M_\mathrm{x}(p)M_\mathrm{y}(p) + U(p)V(p)^\top = I$.

Remark A.3. Note that ∂M_x and ∂U appear in (A.135), which depend on p and v. Therefore, the state-space matrices of the controller, $A_\mathrm{k}, \ldots, D_\mathrm{k}$, also depend on both p and v. This means that the synthesized controller will depend on the scheduling-variable and its derivative, i.e., p and $\dot{p}$. However, this can be avoided by taking M_x and ensuring U to be parameter independent during the synthesis procedure, i.e., $M_\mathrm{x} \in \mathbb{S}^{n_\mathrm{x}}$ and $U \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$. In that case, $\partial M_\mathrm{x} = 0$ and $\partial U = 0$, which results in the dependency on v to drop out of (A.135). This is also described in (Apkarian and Adams 1998). However, due to the restriction of M_x and U, this comes at the cost of increased conservatism of the solution to the synthesis problem.

Remark A.4. In order to obtain a controller with affine scheduling dependency, we require $A_\mathrm{k}, \ldots, D_\mathrm{k}$ to have an affine dependency. To ensure this, we also require the generalized plant to have affine scheduling dependency and we require B_u and C_y to be parameter independent. Moreover, for synthesis, one needs to take M_x

and M_y as parameter independent matrices (therefore U and V are also parameter independent). Under these considerations, constructing the matrices $A_\mathrm{k}, \ldots, D_\mathrm{k}$ through (A.135) results in $A_\mathrm{k}, \ldots, D_\mathrm{k}$ to have affine scheduling dependency.

Under the considerations in Remark A.3, (A.128), (A.134), and (A.135) become (2.47a), (2.47b), and (2.49), respectively.

Furthermore, note that in case the variation of $v = \dot{p}$ is unbounded, one must take M_x and M_y (and therefore U and V) as parameter independent matrices, as otherwise the terms ∂M_y etc., that dependent on v, also become unbounded, making the synthesis conditions infeasible. In this case, that means that one should take $M_\mathrm{x}, M_\mathrm{y} \in \mathbb{S}^{n_\mathrm{x}}$ and $U, V \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$. This is equivalent to taking M (A.93) to be a constant positive-definite matrix.

Discrete-time

By Theorem 2.5, the closed-loop LPV system given by (2.44) is classically (Q, S, R) dissipative, if there exists a positive definite matrix function $M : \mathcal{P} \to \mathbb{S}^{n_\mathrm{x}+n_{\mathrm{x}_\mathrm{k}}}$, such that, in DT, for all $p \in \mathcal{P}$ and $v \in \Pi$, it holds that

$$(\star)^\top \begin{bmatrix} -M(p) & M(p+v) \\ \star & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\mathrm{cl}(p) & B_\mathrm{cl}(p) \end{bmatrix} - (\star)^\top \begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\mathrm{cl}(p) & D_\mathrm{cl}(p) \end{bmatrix} \preceq 0. \tag{A.136}$$

Like in CT, as both M and the controller matrices, part of $A_\mathrm{cl}, \ldots, D_\mathrm{cl}$ (see (2.45)), are unknown, (A.136) is not an LMI. Next, we will derive an LMI condition which is equivalent to (A.136). Let us first rewrite (A.136) as follows, omitting dependence on p and v for brevity:

$$\begin{bmatrix} -M - (\star)^\top R C_\mathrm{cl} & -C_\mathrm{cl}^\top S - C_\mathrm{cl}^\top R D_\mathrm{cl} \\ \star & -Q - (S D_\mathrm{cl} + (\star)^\top) - (\star)^\top R D_\mathrm{cl} \end{bmatrix} + (\star)^\top M_+ \begin{bmatrix} A_\mathrm{cl} & B_\mathrm{cl} \end{bmatrix} \preceq 0, \tag{A.137}$$

where M_+ denotes $M(p+v)$. Let us rewrite this further as follows

$$\begin{bmatrix} M + (\star)^\top R C_\mathrm{cl} & C_\mathrm{cl}^\top S + C_\mathrm{cl}^\top R D_\mathrm{cl} \\ \star & Q + (S D_\mathrm{cl} + (\star)^\top) + (\star)^\top R D_\mathrm{cl} \end{bmatrix} - (\star)^\top M_+^{-1} \begin{bmatrix} M_+ A_\mathrm{cl} & M_+ B_\mathrm{cl} \end{bmatrix} \succeq 0, \tag{A.138}$$

which using a Schur complement is equivalent to

$$\begin{bmatrix} M + (\star)^\top R C_\mathrm{cl} & C_\mathrm{cl}^\top S + C_\mathrm{cl}^\top R D_\mathrm{cl} & A_\mathrm{cl}^\top M_+ \\ \star & Q + (S D_\mathrm{cl} + (\star)^\top) + (\star)^\top R D_\mathrm{cl} & B_\mathrm{cl}^\top M_+ \\ \star & \star & M_+ \end{bmatrix} \succeq 0. \tag{A.139}$$

Using a congruence transformation, this is equivalent to

$$(\star)^\top \begin{bmatrix} M + (\star)^\top R C_\mathrm{cl} & {C_\mathrm{cl}}^\top S + {C_\mathrm{cl}}^\top R D_\mathrm{cl} & {A_\mathrm{cl}}^\top M_+ \\ \star & Q + (S D_\mathrm{cl} + (\star)^\top) + (\star)^\top R D_\mathrm{cl} & {B_\mathrm{cl}}^\top M_+ \\ \star & \star & M_+ \end{bmatrix} \begin{bmatrix} 0 & M^{-1} & 0 \\ 0 & 0 & I \\ M_+^{-1} & 0 & 0 \end{bmatrix} \succeq 0, \tag{A.140}$$

resulting in

$$\begin{bmatrix} \bar{M}_+ & A_{\mathrm{cl}}\bar{M} & B_{\mathrm{cl}} \\ \star & \bar{M} + (\star)^\top R C_{\mathrm{cl}}\bar{M} & \bar{M}C_{\mathrm{cl}}^\top S + \bar{M}C_{\mathrm{cl}}^\top R D_{\mathrm{cl}} \\ \star & \star & Q + (SD_{\mathrm{cl}} + (\star)^\top) + (\star)^\top R D_{\mathrm{cl}} \end{bmatrix} \succeq 0, \tag{A.141}$$

where $\bar{M} = M^{-1}$ and $\bar{M}_+ = M_+^{-1}$. Inspired by (De Caigny et al. 2012), this condition is equivalent to, considering $R \preceq 0$: there exists a $G : \mathcal{P} \to \mathbb{R}^{(n_{\mathrm{x}}+n_{\mathrm{x_k}})\times(n_{\mathrm{x}}+n_{\mathrm{x_k}})}$ such that

$$\begin{bmatrix} \bar{M}_+ & A_{\mathrm{cl}}G & B_{\mathrm{cl}} \\ \star & G + G^\top - \bar{M} + (\star)^\top R C_{\mathrm{cl}}G & GC_{\mathrm{cl}}^\top S + GC_{\mathrm{cl}}^\top R D_{\mathrm{cl}} \\ \star & \star & Q + (SD_{\mathrm{cl}} + (\star)^\top) + (\star)^\top R D_{\mathrm{cl}} \end{bmatrix} \succeq 0. \tag{A.142}$$

It is clear that (A.141) implies (A.142) by taking $G = \bar{M}$ for (A.142). By (A.142), we have that

$$G + G^\top - \bar{M} + (\star)^\top R C_{\mathrm{cl}} G \succeq 0, \tag{A.143}$$

which under the assumption that $R \preceq 0$ implies that

$$G + G^\top \succeq \bar{M} \succ 0, \tag{A.144}$$

therefore, G is non-singular (for all $p \in \mathcal{P}$). Moreover, as $\bar{M}$ is positive definite, it holds that

$$(\bar{M} - G)^\top \bar{M}^{-1}(\bar{M} - G) \succ 0, \tag{A.145}$$

$$G^\top \bar{M}^{-1} G \succ G + G - \bar{M}. \tag{A.146}$$

Combining this and (A.142), it therefore holds that

$$\begin{bmatrix} \bar{M}_+ & A_{\mathrm{cl}}G & B_{\mathrm{cl}} \\ \star & (\star)^\top \bar{M}^{-1} G + (\star)^\top R C_{\mathrm{cl}}G & G^\top C_{\mathrm{cl}}^\top S + G^\top C_{\mathrm{cl}}^\top R D_{\mathrm{cl}} \\ \star & \star & Q + (SD_{\mathrm{cl}} + (\star)^\top) + (\star)^\top R D_{\mathrm{cl}} \end{bmatrix} \succeq 0. \tag{A.147}$$

Through a congruence transformation, this is equivalent to

$$(\star)^\top \begin{bmatrix} \bar{M}_+ & A_{\mathrm{cl}}G & B_{\mathrm{cl}} \\ \star & (\star)^\top \bar{M}^{-1} G + (\star)^\top R C_{\mathrm{cl}}G & G^\top C_{\mathrm{cl}}^\top S + G^\top C_{\mathrm{cl}}^\top R D_{\mathrm{cl}} \\ \star & \star & Q + (SD_{\mathrm{cl}} + (\star)^\top) + (\star)^\top R D_{\mathrm{cl}} \end{bmatrix} \cdot \begin{bmatrix} I & 0 & 0 \\ 0 & G^{-1}\bar{M} & 0 \\ 0 & 0 & I \end{bmatrix} \succeq 0, \tag{A.148}$$

which gives us (A.141), hence, (A.142) implies (A.141). Therefore, (A.141) and (A.142) are equivalent. We can then rewrite (A.142) as

$$\begin{bmatrix} \bar{M}_+ & A_{\mathrm{cl}}G & B_{\mathrm{cl}} \\ \star & G + G^\top - \bar{M} & 0 \\ \star & \star & 0 \end{bmatrix} + (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & C_{\mathrm{cl}}G & D_{\mathrm{cl}} \end{bmatrix} \succeq 0. \tag{A.149}$$

Next, let us take $n_{\mathrm{x_k}} = n_{\mathrm{x}}$ and we partition G as follows

$$G(p) = \begin{bmatrix} G_{\mathrm{x}}(p) & G_1(p) \\ U(p) & G_2(p) \end{bmatrix}, \tag{A.150}$$

where $G_{\mathrm{x}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$, $G_1 : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$, $G_2 : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$, $U : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$. Moreover, the inverse of G is partitioned as

$$G(p)^{-1} = \begin{bmatrix} G_{\mathrm{y}}(p)^\top & G_3(p) \\ V(p)^\top & G_4(p) \end{bmatrix}, \tag{A.151}$$

where $G_{\mathrm{y}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$, $G_3 : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$, $G_4 : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$, $V : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$. As by definition $G(p)G(p)^{-1} = I$, we have from (A.150) and (A.151) that $G_{\mathrm{x}}(p)G_{\mathrm{y}}(p)^\top + G_1(p)V(p)^\top = I$ and $U(p)G_{\mathrm{y}}(p)^\top + G_2(p)V(p)^\top = 0$. Next, we introduce the matrix

$$N(p) := \begin{bmatrix} I & G_{\mathrm{y}}(p)^\top \\ 0 & V(p)^\top \end{bmatrix}, \tag{A.152}$$

and

$$P(p) := G(p)N(p) = \begin{bmatrix} G_{\mathrm{x}}(p) & G_{\mathrm{x}}(p)G_{\mathrm{y}}(p)^\top + G_1(p)V(p)^\top \\ U(p) & U(p)G_{\mathrm{y}}(p)^\top + G_2(p)V(p)^\top \end{bmatrix} = \begin{bmatrix} G_{\mathrm{x}}(p) & I \\ U(p) & 0 \end{bmatrix}. \tag{A.153}$$

Based on these matrices, we apply a congruence transformation to (A.149), again omitting dependence on p and v for brevity,

$$(\star)^\top \left(\begin{bmatrix} \bar{M}_+ & A_{\mathrm{cl}}G & B_{\mathrm{cl}} \\ \star & G + G^\top - \bar{M} & 0 \\ \star & \star & 0 \end{bmatrix} + (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & C_{\mathrm{cl}}G & D_{\mathrm{cl}} \end{bmatrix} \right) \begin{bmatrix} N_+ & 0 & 0 \\ 0 & N & 0 \\ 0 & 0 & I \end{bmatrix} \succeq 0, \tag{A.154}$$

where N_+ denotes $N(p+v)$ resulting in

$$\begin{bmatrix} (\star)^\top \bar{M}_+ N_+ & N_+^\top A_{\mathrm{cl}}GN & N_+^\top B_{\mathrm{cl}} \\ \star & T & 0 \\ \star & \star & 0 \end{bmatrix} + (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & C_{\mathrm{cl}}GN & D_{\mathrm{cl}} \end{bmatrix} \succeq 0, \tag{A.155}$$

where $T = (\star)^\top GN + (\star)^\top - (\star)^\top \bar{M}N$. Next, similar to the CT case, we use (A.150)–(A.153) and (A.100) to rewrite the terms in (A.155). First, we focus on the term $(\star)^\top \bar{M}N$ and $(\star)^\top \bar{M}_+ N_+$, the former which we parameterize as follows

$$N(p)^\top \bar{M}(p)N(p) = \begin{bmatrix} M_{\mathrm{x}}(p) & M_{\mathrm{y}}(p) \\ \star & M_{\mathrm{z}}(p) \end{bmatrix} =: \mathcal{M}(p), \tag{A.156}$$

where $M_{\mathrm{x}} : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$, $M_{\mathrm{y}} : \mathcal{P} \to \mathbb{R}^{n_{\mathrm{x}} \times n_{\mathrm{x}}}$, and $M_{\mathrm{z}} : \mathcal{P} \to \mathbb{S}^{n_{\mathrm{x}}}$, such that

$$N_+^\top \bar{M}_+ N_+ = N(p+v)^\top \bar{M}(p+v)N(p+v) = \begin{bmatrix} M_{+,\mathrm{x}} & M_{+,\mathrm{y}} \\ \star & M_{+,\mathrm{z}} \end{bmatrix} = \mathcal{M}(p+v). \tag{A.157}$$

where $M_{+,\mathrm{x}}$, $M_{+,\mathrm{y}}$, and $M_{+,\mathrm{z}}$, denote $M_{\mathrm{x}}(p+v)$, $M_{\mathrm{y}}(p+v)$, and $M_{\mathrm{z}}(p+v)$, respectively. Next, we focus on the term $(\star)^\top GN$, which using (A.153) is equivalent to $N^\top P$, filling in (A.152) and (A.153), we get, again omitting dependence on p and v for brevity,

$$N^\top P = \begin{bmatrix} I & 0 \\ G_\mathrm{y} & V \end{bmatrix} \begin{bmatrix} G_\mathrm{x} & I \\ U & 0 \end{bmatrix} = \begin{bmatrix} G_\mathrm{x} & I \\ J & G_\mathrm{y} \end{bmatrix} := \mathcal{G}, \tag{A.158}$$

where $J := G_\mathrm{y} G_\mathrm{x} + UV$ with $J : \mathcal{P} \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$.

Next, we take a look at the term $N_+^\top A_\mathrm{cl} G N = N_+^\top A_\mathrm{cl} P$ in (A.155). Filling in (A.100), (A.152), and (A.153) results in

$$\begin{bmatrix} I & 0 \\ G_{+,\mathrm{y}} & V_+ \end{bmatrix} \left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\mathrm{y} & 0 \end{bmatrix} \right) \begin{bmatrix} G_\mathrm{x} & I \\ U & 0 \end{bmatrix}, \tag{A.159}$$

where $G_{+,\mathrm{y}}$ and V_+ denote $G_\mathrm{y}(p+v)$ and $V_+(p+v)$, respectively. We can then rewrite (A.159) as follows

$$\begin{bmatrix} A G_\mathrm{x} & A \\ G_{+,\mathrm{y}} A G_\mathrm{x} & G_{+,\mathrm{y}} A \end{bmatrix} + \begin{bmatrix} 0 & B_\mathrm{u} \\ V_+ & G_{+,\mathrm{y}} B_\mathrm{u} \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} U & 0 \\ C_\mathrm{y} G_\mathrm{x} & C_\mathrm{y} \end{bmatrix}, \tag{A.160}$$

$$\begin{aligned} &\begin{bmatrix} A G_\mathrm{x} & A \\ 0 & G_{+,\mathrm{y}} A \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ G_{+,\mathrm{y}} A G_\mathrm{x} & 0 \end{bmatrix} + \\ &\qquad \begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix} \begin{bmatrix} V_+ & G_{+,\mathrm{y}} B_\mathrm{u} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} U & 0 \\ C_\mathrm{y} G_\mathrm{x} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_\mathrm{y} \end{bmatrix}, \end{aligned} \tag{A.161}$$

$$\begin{aligned} &\begin{bmatrix} A G_\mathrm{x} & A \\ 0 & G_{+,\mathrm{y}} A \end{bmatrix} + \\ &\begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix} \left(\begin{bmatrix} V_+ & G_{+,\mathrm{y}} B_\mathrm{u} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} U & 0 \\ C_\mathrm{y} G_\mathrm{x} & I \end{bmatrix} + \begin{bmatrix} G_{+,\mathrm{y}} A G_\mathrm{x} & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ 0 & C_\mathrm{y} \end{bmatrix}. \end{aligned} \tag{A.162}$$

Let us define

$$\begin{bmatrix} \mathcal{A}_\mathrm{k} & \mathcal{B}_\mathrm{k} \\ \mathcal{C}_\mathrm{k} & \mathcal{D}_\mathrm{k} \end{bmatrix} := \begin{bmatrix} V_+ & G_{+,\mathrm{y}} B_\mathrm{u} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} U & 0 \\ C_\mathrm{y} G_\mathrm{x} & I \end{bmatrix} + \begin{bmatrix} G_{+,\mathrm{y}} A G_\mathrm{x} & 0 \\ 0 & 0 \end{bmatrix}, \tag{A.163}$$

where $\mathcal{A}_\mathrm{k} : \mathcal{P} \times \Pi \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$, $\mathcal{B}_\mathrm{k} : \mathcal{P} \times \Pi \to \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{y}}$, $\mathcal{C}_\mathrm{k} : \mathcal{P} \times \Pi \to \mathbb{R}^{n_\mathrm{u} \times n_\mathrm{x}}$, and $\mathcal{D}_\mathrm{k} : \mathcal{P} \times \Pi \to \mathbb{R}^{n_\mathrm{u} \times n_\mathrm{y}}$. This allows us to write (A.162), and hence, $N_+^\top A_\mathrm{cl} G N$, as

$$\begin{bmatrix} A G_\mathrm{x} & A \\ 0 & G_{+,\mathrm{y}} A \end{bmatrix} + \begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix} \begin{bmatrix} \mathcal{A}_\mathrm{k} & \mathcal{B}_\mathrm{k} \\ \mathcal{C}_\mathrm{k} & \mathcal{D}_\mathrm{k} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_\mathrm{y} \end{bmatrix}, \tag{A.164}$$

$$\begin{bmatrix} A G_\mathrm{x} + B_\mathrm{u} \mathcal{C}_\mathrm{k} & A + B_\mathrm{u} \mathcal{D}_\mathrm{k} C_\mathrm{y} \\ \mathcal{A}_\mathrm{k} & G_{+,\mathrm{y}} A + \mathcal{B}_\mathrm{k} C_\mathrm{y} \end{bmatrix} := \mathcal{A}_\mathrm{cl}. \tag{A.165}$$

Next, we consider the term $N_+^\top B_\mathrm{cl}$ in (A.155). Filling in (A.100) and (A.152), results in,

$$\begin{bmatrix} I & 0 \\ G_{+,\mathrm{y}} & V_+ \end{bmatrix} \left(\begin{bmatrix} B_\mathrm{w} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & B_\mathrm{u} \\ I & 0 \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} 0 \\ D_\mathrm{yw} \end{bmatrix} \right), \tag{A.166}$$

which we can rewrite as follows

$$\begin{bmatrix} B_{\mathrm{w}} \\ G_{+,\mathrm{y}}B_{\mathrm{w}} \end{bmatrix} + \begin{bmatrix} 0 & B_{\mathrm{u}} \\ V_{+} & G_{+,\mathrm{y}}B_{\mathrm{u}} \end{bmatrix} \begin{bmatrix} A_{\mathrm{k}} & B_{\mathrm{k}} \\ C_{\mathrm{k}} & D_{\mathrm{k}} \end{bmatrix} \begin{bmatrix} 0 \\ D_{\mathrm{yw}} \end{bmatrix}, \tag{A.167}$$

$$\begin{bmatrix} B_{\mathrm{w}} \\ G_{+,\mathrm{y}}B_{\mathrm{w}} \end{bmatrix} + \begin{bmatrix} 0 & B_{\mathrm{u}} \\ I & 0 \end{bmatrix} \begin{bmatrix} V_{+} & G_{+,\mathrm{y}}B_{\mathrm{u}} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\mathrm{k}} & B_{\mathrm{k}} \\ C_{\mathrm{k}} & D_{\mathrm{k}} \end{bmatrix} \begin{bmatrix} U & 0 \\ C_{\mathrm{y}}G_{\mathrm{x}} & I \end{bmatrix} \begin{bmatrix} 0 \\ D_{\mathrm{yw}} \end{bmatrix}, \tag{A.168}$$

$$\begin{bmatrix} B_{\mathrm{w}} \\ G_{+,\mathrm{y}}B_{\mathrm{w}} \end{bmatrix} + \begin{bmatrix} 0 & B_{\mathrm{u}} \\ I & 0 \end{bmatrix} \left(\begin{bmatrix} V_{+} & G_{+,\mathrm{y}}B_{\mathrm{u}} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\mathrm{k}} & B_{\mathrm{k}} \\ C_{\mathrm{k}} & D_{\mathrm{k}} \end{bmatrix} \begin{bmatrix} U & 0 \\ C_{\mathrm{y}}G_{\mathrm{x}} & I \end{bmatrix} + \begin{bmatrix} G_{+,\mathrm{y}}AG_{\mathrm{x}} & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} 0 \\ D_{\mathrm{yw}} \end{bmatrix}. \tag{A.169}$$

Using (A.163), we can write this, and hence, $N_{+}^{\top}B_{\mathrm{cl}}$, as

$$\begin{bmatrix} B_{\mathrm{w}} \\ G_{+,\mathrm{y}}B_{\mathrm{w}} \end{bmatrix} + \begin{bmatrix} 0 & B_{\mathrm{u}} \\ I & 0 \end{bmatrix} \begin{bmatrix} \mathcal{A}_{\mathrm{k}} & \mathcal{B}_{\mathrm{k}} \\ \mathcal{C}_{\mathrm{k}} & \mathcal{D}_{\mathrm{k}} \end{bmatrix} \begin{bmatrix} 0 \\ D_{\mathrm{yw}} \end{bmatrix}, \tag{A.170}$$

which can be written as

$$\begin{bmatrix} B_{\mathrm{w}} + B_{\mathrm{u}}\mathcal{D}_{\mathrm{k}}D_{\mathrm{yw}} \\ G_{+,\mathrm{y}}B_{\mathrm{w}} + \mathcal{B}_{\mathrm{k}}D_{\mathrm{yw}} \end{bmatrix} := \mathcal{B}_{\mathrm{cl}}. \tag{A.171}$$

Next, we take a look at the term $C_{\mathrm{cl}}GN = C_{\mathrm{cl}}P$ in (A.155). Filling in (A.100) and (A.153) results in

$$\left(\begin{bmatrix} C_{\mathrm{z}} & 0 \end{bmatrix} + \begin{bmatrix} 0 & D_{\mathrm{zu}} \end{bmatrix} \begin{bmatrix} A_{\mathrm{k}} & B_{\mathrm{k}} \\ C_{\mathrm{k}} & D_{\mathrm{k}} \end{bmatrix} \begin{bmatrix} 0 & I \\ C_{\mathrm{y}} & 0 \end{bmatrix} \right) \begin{bmatrix} G_{\mathrm{x}} & I \\ U & 0 \end{bmatrix}, \tag{A.172}$$

which can be rewritten as follows

$$\begin{bmatrix} C_{\mathrm{z}}G_{\mathrm{x}} & C_{\mathrm{z}} \end{bmatrix} + \begin{bmatrix} 0 & D_{\mathrm{zu}} \end{bmatrix} \begin{bmatrix} A_{\mathrm{k}} & B_{\mathrm{k}} \\ C_{\mathrm{k}} & D_{\mathrm{k}} \end{bmatrix} \begin{bmatrix} U & 0 \\ C_{\mathrm{y}}G_{\mathrm{x}} & C_{\mathrm{y}} \end{bmatrix}, \tag{A.173}$$

$$\begin{bmatrix} C_{\mathrm{z}}G_{\mathrm{x}} & C_{\mathrm{z}} \end{bmatrix} + \begin{bmatrix} 0 & D_{\mathrm{zu}} \end{bmatrix} \begin{bmatrix} V_{+} & G_{+,\mathrm{y}}B_{\mathrm{u}} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\mathrm{k}} & B_{\mathrm{k}} \\ C_{\mathrm{k}} & D_{\mathrm{k}} \end{bmatrix} \begin{bmatrix} U & 0 \\ C_{\mathrm{y}}G_{\mathrm{x}} & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_{\mathrm{y}} \end{bmatrix}, \tag{A.174}$$

$$\begin{bmatrix} C_{\mathrm{z}}G_{\mathrm{x}} & C_{\mathrm{z}} \end{bmatrix} + \begin{bmatrix} 0 & D_{\mathrm{zu}} \end{bmatrix} \left(\begin{bmatrix} V_{+} & G_{+,\mathrm{y}}B_{\mathrm{u}} \\ 0 & I \end{bmatrix} \begin{bmatrix} A_{\mathrm{k}} & B_{\mathrm{k}} \\ C_{\mathrm{k}} & D_{\mathrm{k}} \end{bmatrix} \begin{bmatrix} U & 0 \\ C_{\mathrm{y}}G_{\mathrm{x}} & I \end{bmatrix} + \begin{bmatrix} G_{+,\mathrm{y}}AG_{\mathrm{x}} & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} I & 0 \\ 0 & C_{\mathrm{y}} \end{bmatrix}. \tag{A.175}$$

Again, using (A.163), we can write this, and hence, $C_{\mathrm{cl}}GN$, as

$$\begin{bmatrix} C_{\mathrm{z}}G_{\mathrm{x}} & C_{\mathrm{z}} \end{bmatrix} + \begin{bmatrix} 0 & D_{\mathrm{zu}} \end{bmatrix} \begin{bmatrix} \mathcal{A}_{\mathrm{k}} & \mathcal{B}_{\mathrm{k}} \\ \mathcal{C}_{\mathrm{k}} & \mathcal{D}_{\mathrm{k}} \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & C_{\mathrm{y}} \end{bmatrix}, \tag{A.176}$$

which we can rewrite as

$$\begin{bmatrix} C_\mathrm{z}G_\mathrm{x} + D_\mathrm{zu}\mathcal{C}_\mathrm{k} & C_\mathrm{z} + D_\mathrm{zu}\mathcal{D}_\mathrm{k}C_\mathrm{y} \end{bmatrix} := \mathcal{C}_\mathrm{cl}. \tag{A.177}$$

Finally, we take a look at D_cl in (A.155). Using (A.100), we have that

$$D_\mathrm{cl} = D_\mathrm{zw} + \begin{bmatrix} 0 & D_\mathrm{zu} \end{bmatrix} \begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} \begin{bmatrix} 0 \\ D_\mathrm{yw} \end{bmatrix} = D_\mathrm{zw} + D_\mathrm{zu}D_\mathrm{k}D_\mathrm{yw}, \tag{A.178}$$

By (A.163), we have that $\mathcal{D}_\mathrm{k} = D_\mathrm{k}$, hence,

$$D_\mathrm{cl} = D_\mathrm{zw} + D_\mathrm{zu}\mathcal{D}_\mathrm{k}D_\mathrm{yw} =: \mathcal{D}_\mathrm{cl}. \tag{A.179}$$

Combining the results (A.156)–(A.158), (A.165), (A.171), (A.177), and (A.179), we can rewrite (A.155) as

$$\begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_\mathrm{cl}(p,v) & \mathcal{B}_\mathrm{cl}(p,v) \\ \star & \mathcal{G}(p) + (\star)^\top - \mathcal{M}(p) & 0 \\ \star & \star & 0 \end{bmatrix} + (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & \mathcal{C}_\mathrm{cl}(p) & \mathcal{D}_\mathrm{cl}(p) \end{bmatrix} \succeq 0, \tag{A.180}$$

Note that (A.180) is not an LMI, as it is quadratic in $\mathcal{C}_\mathrm{cl}(p)$ and $\mathcal{D}_\mathrm{cl}(p)$. However, for specific choices of (Q, S, R) we can rewrite this to an LMI, see e.g. Appendices A.3.2 to A.3.5.

Finally to ensure M is a positive definite matrix function we use (A.156), resulting in the condition:

$$\mathcal{M}(p) = \begin{bmatrix} M_\mathrm{x}(p) & M_\mathrm{y}(p) \\ \star & M_\mathrm{z}(p) \end{bmatrix} \succ 0, \tag{A.181}$$

for all $p \in \mathcal{P}$.

Once a solution has been found for (A.180) and (A.181), i.e., matrix functions $\mathcal{A}_\mathrm{k}, \dots \mathcal{D}_\mathrm{k}$, M_x, M_y, M_z, G_x, G_y, and J have been found such that (A.180) and (A.181) hold for all $(p, v) \in \mathcal{P} \times \Pi$, we can reconstruct $A_\mathrm{k}, \dots D_\mathrm{k}$, i.e., the state-space matrices of the LPV controller. Namely, by (A.163), we have, omitting dependence on p and v for brevity, that

$$\begin{bmatrix} A_\mathrm{k} & B_\mathrm{k} \\ C_\mathrm{k} & D_\mathrm{k} \end{bmatrix} = \begin{bmatrix} V_+ & G_{+,\mathrm{y}}B_\mathrm{u} \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{A}_\mathrm{k} - G_{+,\mathrm{y}}AG_\mathrm{x} & \mathcal{B}_\mathrm{k} \\ \mathcal{C}_\mathrm{k} & \mathcal{D}_\mathrm{k} \end{bmatrix} \begin{bmatrix} U & 0 \\ C_\mathrm{y}G_\mathrm{x} & I \end{bmatrix}^{-1}, \tag{A.182}$$

where U and V are arbitrary solutions to $J = G_\mathrm{y}G_\mathrm{x} + UV$.

Remark A.5. Note that as $G_{+,\mathrm{y}}$ and V_+ appear in (A.182), which depend on p and v, the controller matrices $A_\mathrm{k}, \dots, D_\mathrm{k}$ depend on both $p(t)$ and $v(t) = p(t+1) - p(t)$. This means that the synthesized controller depends on the scheduling-variable at the current time and the scheduling one step in the future, i.e., $p(t)$ and $p(t+1)$, respectively. However, this can be avoided by taking G_y and ensuring V to be parameter independent matrices during the synthesis procedure, i.e., $G_\mathrm{y} \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$ and $V \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$. In that case, $G_\mathrm{y} = G_{+,\mathrm{y}}$ and $V = V_+$, which results in the dependency on $p(t+1)$ to drop out of (A.182). However, due to the restriction of G_y and V, this comes at the cost of increased conservatism of the solution to the synthesis problem, similar as we had in CT, see Remark A.3.

Remark A.6. In order to obtain a controller with affine scheduling dependency, we require $A_\mathrm{k}, \dots, D_\mathrm{k}$ to have an affine dependency. To ensure this, we also require the generalized plant to have affine scheduling dependency and we require B_u and C_y to be parameter independent. Moreover, for synthesis, one needs to take G_x, G_y, and J as parameter independent matrices (therefore U and V are also parameter independent). Under these considerations, constructing the matrices $A_\mathrm{k}, \dots, D_\mathrm{k}$ through (A.182) results in $A_\mathrm{k}, \dots, D_\mathrm{k}$ to have affine scheduling dependency. Note that unlike in the CT case (see Remark A.4), we do not require the matrices of the storage function, i.e., M_x, M_y, M_z to be parameter independent. This is a result of the introduction of the variable G in the synthesis conditions, which allows us to decouple the controller matrices from the storage matrices.

Under the considerations in Remark A.5, (A.180)–(A.182) become (2.50a), (2.50b), and (2.54), respectively.

Moreover, similar as in CT, note that in case the variation of $v(t) = p(t+1) - p(t)$ is unbounded, one must take M_x, M_y, M_z, G_x, G_y, and J as parameter independent matrices, as otherwise the terms $G_{+,\mathrm{y}}$ etc., that dependent on v, also become unbounded, making the synthesis conditions infeasible. In this case, that means that one should take $M_\mathrm{x}, M_\mathrm{z} \in \mathbb{S}^{n_\mathrm{x}}$ and $M_\mathrm{y}, G_\mathrm{x}, G_\mathrm{y}, J \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$. This is equivalent to taking M and G to be constant matrices.

A.3.2 $\mathcal{L}_2$-gain

Continuous-time

In Appendix A.2.1, we have shown that an LPV system given by (2.26) has a bounded $\mathcal{L}_2$-gain of γ, if it is classically (Q, S, R) dissipative for the tuple $(Q, S, R) = (\gamma I, 0, -\gamma^{-1} I)$. Combining this with the results for CT LPV controller synthesis for (Q, S, R) performance in Appendix A.3.1, we get that (A.128) becomes

$$\begin{bmatrix} \mathcal{A}_\mathrm{cl}(p,v) + (\star)^\top & B_\mathrm{cl}(p) \\ \star & 0 \end{bmatrix} - (\star)^\top \begin{bmatrix} \gamma I & 0 \\ \star & -\gamma^{-1} I \end{bmatrix} \begin{bmatrix} 0 & I \\ \mathcal{C}_\mathrm{cl}(p) & \mathcal{D}_\mathrm{cl}(p) \end{bmatrix} \preceq 0. \quad \text{(A.183)}$$

This can be rewritten as follows

$$\begin{bmatrix} \mathcal{A}_\mathrm{cl}(p,v) + (\star)^\top & B_\mathrm{cl}(p) \\ \star & -\gamma I \end{bmatrix} - (\star)^\top (-\gamma^{-1} I) \begin{bmatrix} \mathcal{C}_\mathrm{cl}(p) & \mathcal{D}_\mathrm{cl}(p) \end{bmatrix} \preceq 0, \quad \text{(A.184)}$$

which through a Schur complement is equivalent to

$$\begin{bmatrix} \mathcal{A}_\mathrm{cl}(p,v) + (\star)^\top & \mathcal{B}_\mathrm{cl}(p) & \mathcal{C}_\mathrm{cl}(p)^\top \\ \star & -\gamma I & \mathcal{D}_\mathrm{cl}(p)^\top \\ \star & \star & -\gamma I \end{bmatrix} \preceq 0. \quad \text{(A.185)}$$

This gives us the LMI in (2.55a).

Discrete-time

Like in CT, also in DT, classically (Q, S, R) dissipativity for the tuple $(Q, S, R) = (\gamma I, 0, -\gamma^{-1} I)$ corresponds to an ℓ_2-gain of γ. Combining this with the results for DT LPV controller synthesis for (Q, S, R) performance in Appendix A.3.1, we get that (A.180) becomes

$$\begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_{\text{cl}}(p,v) & \mathcal{B}_{\text{cl}}(p,v) \\ \star & \mathcal{G}(p) + (\star)^\top - \mathcal{M}(p) & 0 \\ \star & \star & 0 \end{bmatrix} + (\star)^\top \begin{bmatrix} \gamma I & 0 \\ \star & -\gamma^{-1} I \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & \mathcal{C}_{\text{cl}}(p) & \mathcal{D}_{\text{cl}}(p) \end{bmatrix} \succeq 0. \quad \text{(A.186)}$$

This can be rewritten as follows

$$\begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_{\text{cl}}(p,v) & \mathcal{B}_{\text{cl}}(p,v) \\ \star & \mathcal{G}(p) + (\star)^\top - \mathcal{M}(p) & 0 \\ \star & \star & \gamma I \end{bmatrix} - (\star)^\top (\gamma^{-1} I) \begin{bmatrix} 0 & \mathcal{C}_{\text{cl}}(p) & \mathcal{D}_{\text{cl}}(p) \end{bmatrix} \succeq 0, \quad \text{(A.187)}$$

which through a Schur complement is equivalent to

$$\begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_{\text{cl}}(p,v) & \mathcal{B}_{\text{cl}}(p,v) & 0 \\ \star & \mathcal{G}(p) + (\star)^\top - \mathcal{M}(p) & 0 & \mathcal{C}_{\text{cl}}(p)^\top \\ \star & \star & \gamma I & \mathcal{D}_{\text{cl}}(p)^\top \\ \star & \star & \star & \gamma I \end{bmatrix} \succeq 0. \quad \text{(A.188)}$$

This gives us the LMI in (2.55b).

A.3.3 Passivity

Continuous-time

In Appendix A.2.2, we have shown that an LPV system given by (2.26) is passive, if it is classically (Q, S, R) dissipative for the tuple $(Q, S, R) = (0, I, 0)$. Combining this with the results for CT LPV controller synthesis for (Q, S, R) performance in Appendix A.3.1, we get that (A.128) becomes

$$\begin{bmatrix} \mathcal{A}_{\text{cl}}(p,v) + (\star)^\top & B_{\text{cl}}(p) \\ \star & 0 \end{bmatrix} - (\star)^\top \begin{bmatrix} 0 & I \\ \star & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ \mathcal{C}_{\text{cl}}(p) & \mathcal{D}_{\text{cl}}(p) \end{bmatrix} \preceq 0. \quad \text{(A.189)}$$

This can be rewritten as follows

$$\begin{bmatrix} \mathcal{A}_{\text{cl}}(p,v) + (\star)^\top & \mathcal{B}_{\text{cl}}(p) - \mathcal{C}_{\text{cl}}(p)^\top \\ \star & -\mathcal{D}_{\text{cl}}(p) + (\star)^\top \end{bmatrix} \preceq 0. \quad \text{(A.190)}$$

This gives us the LMI in (2.56a).

Discrete-time

Like in CT, also in DT, classically (Q, S, R) dissipativity for the tuple $(Q, S, R) = (0, I, 0)$ corresponds to passivity. Combining this with the results for DT LPV controller synthesis for (Q, S, R) performance in Appendix A.3.1, we get that (A.180) becomes

$$\begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_{\mathrm{cl}}(p,v) & \mathcal{B}_{\mathrm{cl}}(p,v) \\ \star & \mathcal{G}(p)+(\star)-\mathcal{M}(p) & 0 \\ \star & \star & 0 \end{bmatrix} + \\ (\star)^\top \begin{bmatrix} 0 & I \\ \star & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & \mathcal{C}_{\mathrm{cl}}(p) & \mathcal{D}_{\mathrm{cl}}(p) \end{bmatrix} \succeq 0. \quad \text{(A.191)}$$

This can be rewritten as follows

$$\begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_{\mathrm{cl}}(p,v) & \mathcal{B}_{\mathrm{cl}}(p,v) \\ \star & \mathcal{G}(p)+(\star)^\top-\mathcal{M}(p) & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & \star & \mathcal{D}_{\mathrm{cl}}+(\star)^\top \end{bmatrix} \succeq 0. \quad \text{(A.192)}$$

This gives us the LMI in (2.56b).

A.3.4 $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain

Continuous-time

In Appendix A.2.3, we have shown that an LPV system given by (2.26) has a bounded $\mathcal{L}_2$-$\mathcal{L}_\infty$-gain of γ, if it is classically (Q, S, R) dissipative for the tuple $(Q, S, R) = (\gamma I, 0, 0)$ and (A.41) holds for all $p \in \mathcal{P}$. Combining this with the results for CT LPV controller synthesis for (Q, S, R) performance in Appendix A.3.1, we get that (A.128) becomes

$$\begin{bmatrix} \mathcal{A}_{\mathrm{cl}}(p,v)+(\star)^\top & B_{\mathrm{cl}}(p) \\ \star & 0 \end{bmatrix} - (\star)^\top \begin{bmatrix} \gamma I & 0 \\ \star & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ \mathcal{C}_{\mathrm{cl}}(p) & \mathcal{D}_{\mathrm{cl}}(p) \end{bmatrix} \preceq 0. \quad \text{(A.193)}$$

This can be rewritten as follows

$$\begin{bmatrix} \mathcal{A}_{\mathrm{cl}}(p,v)+(\star)^\top & B_{\mathrm{cl}}(p) \\ \star & -\gamma I \end{bmatrix} \preceq 0. \quad \text{(A.194)}$$

This gives us the first LMI in (2.57a).

For controller synthesis, the considered LPV system is the closed-loop system (2.44), hence, (A.41) becomes

$$\begin{bmatrix} M(p) & C_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0, \quad \text{(A.195)}$$

where C_{cl} is given in (A.100). Using a congruence transformation, this is equivalent to

$$(\star)^\top \begin{bmatrix} M(p) & C_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \begin{bmatrix} N(p) & 0 \\ 0 & I \end{bmatrix} \succeq 0, \quad \text{(A.196)}$$

where N is given in (A.95), which then results in

$$(\star)^\top \begin{bmatrix} (\star)^\top M(p)N(p) & N(p)^\top C_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0. \tag{A.197}$$

Using the results of (A.120)–(A.125) and (A.130)–(A.134), this can be expressed as

$$\begin{bmatrix} \mathcal{M}(p) & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0. \tag{A.198}$$

This gives us the second LMI in (2.57a).

Discrete-time

Like in CT, also in DT, classically (Q, S, R) dissipativity for the tuple $(Q, S, R) = (\gamma I, 0, 0)$ together with (A.50) corresponds to an ℓ_2-ℓ_∞-gain of γ. Combining this with the results for DT LPV controller synthesis for (Q, S, R) performance in Appendix A.3.1, we get that (A.180) becomes

$$\begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_{\mathrm{cl}}(p,v) & \mathcal{B}_{\mathrm{cl}}(p,v) \\ \star & \mathcal{G}(p) + (\star) - \mathcal{M}(p) & 0 \\ \star & \star & 0 \end{bmatrix} + \\ (\star)^\top \begin{bmatrix} \gamma I & 0 \\ \star & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & I \\ 0 & \mathcal{C}_{\mathrm{cl}}(p) & \mathcal{D}_{\mathrm{cl}}(p) \end{bmatrix} \succeq 0. \tag{A.199}$$

This can be rewritten as follows

$$\begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_{\mathrm{cl}}(p,v) & \mathcal{B}_{\mathrm{cl}}(p,v) \\ \star & \mathcal{G}(p) + (\star) - \mathcal{M}(p) & 0 \\ \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{A.200}$$

This gives us the first LMI in (2.57b).

For controller synthesis, the considered LPV system is the closed-loop system (2.44), hence, (A.50) becomes

$$\begin{bmatrix} \bar{M}(p) & \bar{M}(p)C_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0, \tag{A.201}$$

where $\bar{M}(p) = M(p)^{-1}$ and C_{cl} is given in (A.100). Similar to how we have shown that (A.141) and (A.142) are equivalent, we can show that (A.201) and there existing a matrix function G such that

$$\begin{bmatrix} G(p) + (\star)^\top - \bar{M}(p) & G(p)C_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0, \tag{A.202}$$

are equivalent. Namely, (A.201) implies (A.202) by taking $G = \bar{M}$, and (A.202) implies (A.201) by substituting $G(p) + (\star)^\top - \bar{M}(p)$ for $(\star)^\top \bar{M}(p)^{-1} G(p)$ in (A.202) (by (A.146)). Performing a congruence transformation on the resulting matrix inequality with $\mathrm{diag}(G(p)^{-1}M(p), I)$ results in (A.201).

Using a congruence transformation, (A.202) is equivalent to

$$(\star)^\top \begin{bmatrix} G(p) + (\star)^\top - \bar{M}(p) & G(p)C_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \begin{bmatrix} N(p) & 0 \\ 0 & I \end{bmatrix} \succeq 0, \tag{A.203}$$

where N is given in (A.152), which then results in

$$\begin{bmatrix} (\star)^\top G(p)N(p) + (\star)^\top - (\star)^\top \bar{M}(p)N(p) & N(p)^\top G(p)C_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0. \tag{A.204}$$

Using the results of (A.172)–(A.177) and (A.156) and (A.158), this can be expressed as

$$\begin{bmatrix} \mathcal{G} + (\star)^\top - \mathcal{M}(p) & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & \gamma I \end{bmatrix} \succeq 0. \tag{A.205}$$

This gives us the second LMI in (2.57b).

A.3.5 $\mathcal{L}_\infty$-gain

Continuous-time

In Appendix A.2.3, we have shown that an LPV system given by (2.26) has a bounded $\mathcal{L}_\infty$-gain of γ, if it satisfies (A.65) and (A.70) for all $p \in \mathcal{P}$. For controller synthesis, the considered LPV system is the closed-loop system (2.44), hence, (A.65) becomes

$$\begin{bmatrix} A_{\mathrm{cl}}(p)^\top M(p) + (\star)^\top + \beta M(p) + \partial M(p, v) & M(p)B_{\mathrm{cl}}(p) \\ \star & -\alpha I \end{bmatrix} \preceq 0, \tag{A.206}$$

where A_{cl} and B_{cl} are given in (A.100). Using a congruence transformation, this is equivalent to

$$(\star)^\top \begin{bmatrix} A_{\mathrm{cl}}(p)^\top M(p) + (\star)^\top + \beta M(p) + \partial M(p, v) & M(p)B_{\mathrm{cl}}(p) \\ \star & -\alpha I \end{bmatrix} \begin{bmatrix} N(p) & 0 \\ 0 & I \end{bmatrix} \preceq 0, \tag{A.207}$$

where N is given in (A.95), which then results in, omitting dependence on p and v for brevity,

$$\begin{bmatrix} (\star)^\top A_{\mathrm{cl}}^\top MN + (\star)^\top + \beta(\star)^\top MN + (\star)^\top \partial MN & N^\top MB_{\mathrm{cl}} \\ \star & -\alpha I \end{bmatrix} \preceq 0. \tag{A.208}$$

Using the results of (A.101)–(A.112), (A.114)–(A.119), and (A.130)–(A.134), this is equivalent to

$$\begin{bmatrix} \mathcal{A}_{\mathrm{cl}}(p, v) + (\star)^\top + \beta \mathcal{M}(p) & \mathcal{B}_{\mathrm{cl}}(p) \\ \star & -\alpha I \end{bmatrix} \preceq 0. \tag{A.209}$$

This gives us the first LMI in (2.58a).

Next, we focus on (A.70), which for controller synthesis becomes

$$\begin{bmatrix} \beta M(p) & 0 & C_{\mathrm{cl}}(p)^\top \\ \star & (\gamma - \alpha)I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0, \tag{A.210}$$

where C_{cl} and D_{cl} are given in (A.100). Using a congruence transformation, this is equivalent to

$$(\star)^\top \begin{bmatrix} \beta M(p) & 0 & C_{\text{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & D_{\text{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \begin{bmatrix} N(p) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \succeq 0, \tag{A.211}$$

resulting in

$$\begin{bmatrix} \beta(\star)^\top M(p)N(p) & 0 & N(p)^\top C_{\text{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & D_{\text{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{A.212}$$

Using the results of (A.120)–(A.125), (A.126)–(A.127), and (A.130)–(A.134), this can be expressed as

$$\begin{bmatrix} \beta \mathcal{M}(p) & 0 & \mathcal{C}_{\text{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & \mathcal{D}_{\text{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{A.213}$$

This gives us the second LMI in (2.58a).

Discrete-time

Like in CT, also in DT, we have shown in Appendix A.2.3 that an LPV system given by (2.26) has a bounded ℓ_∞-gain of γ, if it satisfies (A.88) and (A.91) for all $p \in \mathcal{P}$. For controller synthesis, the considered LPV system is the closed-loop system (2.44), hence, (A.88) becomes

$$\begin{bmatrix} \bar{M}(p+v) & A_{\text{cl}}(p)\bar{M}(p) & B_{\text{cl}}(p) \\ \star & (1-\beta)\bar{M}(p) & 0 \\ \star & \star & \alpha I \end{bmatrix} \succeq 0, \tag{A.214}$$

where $\bar{M}(p) = M(p)^{-1}$ and A_{cl} and B_{cl} are given in (A.100). Again, similar to how we have shown that (A.141) and (A.142) are equivalent, we can show that (A.214) and there existing a matrix function G such that

$$\begin{bmatrix} \bar{M}(p+v) & A_{\text{cl}}(p)G(p) & B_{\text{cl}}(p) \\ \star & (1-\beta)(G(p)+(\star)^\top - \bar{M}(p)) & 0 \\ \star & \star & \alpha I \end{bmatrix} \succeq 0, \tag{A.215}$$

are equivalent. Namely, (A.214) implies (A.215) by taking $G = \bar{M}$, and (A.215) implies (A.214) by substituting $G(p) + (\star)^\top - \bar{M}(p)$ for $(\star)^\top \bar{M}(p)^{-1} G(p)$ in (A.215) (by (A.146)). Performing a congruence transformation on the resulting matrix inequality with $\operatorname{diag}(I, G(p)^{-1}M(p), I)$ results in (A.214).

Using a congruence transformation, (A.215) is equivalent to

$$(\star)^\top \begin{bmatrix} \bar{M}(p+v) & A_{\text{cl}}(p)\bar{G}(p) & B_{\text{cl}}(p) \\ \star & (1-\beta)(G(p)+(\star)^\top - \bar{M}(p)) & 0 \\ \star & \star & \alpha I \end{bmatrix} \begin{bmatrix} N(p+v) & 0 & 0 \\ 0 & N(p) & 0 \\ 0 & 0 & I \end{bmatrix} \succeq 0, \tag{A.216}$$

resulting in

$$\begin{bmatrix} (\star)^\top \bar{M}(p+v)N(p+v) & N(p+v)^\top A_{\mathrm{cl}}(p)\bar{G}(p)N(p) & N(p+v)^\top B_{\mathrm{cl}}(p) \\ \star & T(p,v) & 0 \\ \star & \star & \alpha I \end{bmatrix} \succeq 0, \tag{A.217}$$

where $T(p,v) = (1-\beta)(\star)^\top (G(p)+(\star)^\top - \bar{M}(p))N(p)$. Using the results of (A.159)–(A.165), (A.166)–(A.171), and (A.156)–(A.158), this can be expressed as,

$$\begin{bmatrix} \mathcal{M}(p+v) & \mathcal{A}_{\mathrm{cl}}(p,v) & \mathcal{B}_{\mathrm{cl}}(p,v) \\ \star & (1-\beta)(\mathcal{G}(p)+(\star)^\top - \mathcal{M}(p)) & 0 \\ \star & \star & \alpha I \end{bmatrix} \succeq 0. \tag{A.218}$$

This gives us the first LMI in (2.58b).

Next, we focus on (A.91), which for controller synthesis becomes

$$\begin{bmatrix} \beta \bar{M}(p) & 0 & \bar{M}(p)C_{\mathrm{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0, \tag{A.219}$$

where again $\bar{M}(p) = M(p)^{-1}$, and C_{cl} and D_{cl} are given in (A.100). Similar as before, we can show that (A.219) and there existing a matrix function G such that

$$\begin{bmatrix} \beta(G(p)+(\star)^\top - \bar{M}(p)) & 0 & G(p)C_{\mathrm{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0, \tag{A.220}$$

are equivalent. Namely, (A.219) implies (A.220) by taking $G = \bar{M}$, and (A.220) implies (A.219) by substituting $G(p)+(\star)^\top - \bar{M}(p)$ for $(\star)^\top \bar{M}(p)^{-1}G(p)$ in (A.220) (by (A.146)). Performing a congruence transformation on the resulting matrix inequality with $\mathrm{diag}(G(p)^{-1}M(p), I, I)$ results in (A.219).

Using a congruence transformation, (A.220) is equivalent to

$$(\star)^\top \begin{bmatrix} \beta(G(p)+(\star)^\top - \bar{M}(p)) & 0 & G(p)C_{\mathrm{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \begin{bmatrix} N(p) & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \succeq 0, \tag{A.221}$$

resulting in

$$\begin{bmatrix} \beta\cdot(\star)^\top (G(p)+(\star)^\top - \bar{M}(p))N(p) & 0 & N(p)^\top G(p)C_{\mathrm{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & D_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{A.222}$$

Using the results of (A.172)–(A.177), (A.179), (A.156) and (A.158), this can be expressed as,

$$\begin{bmatrix} \beta\cdot(\star)^\top (\mathcal{G}(p)+(\star)^\top - \mathcal{M}(p))N(p) & 0 & \mathcal{C}_{\mathrm{cl}}(p)^\top \\ \star & (\gamma-\alpha)I & \mathcal{D}_{\mathrm{cl}}(p)^\top \\ \star & \star & \gamma I \end{bmatrix} \succeq 0. \tag{A.223}$$

This gives us the second LMI in (2.58b).

Remark A.7. Note that similar to the $\mathcal{L}_\infty/\ell_\infty$ based analysis conditions (see Remark A.1), also the above $\mathcal{L}_\infty/\ell_\infty$ controller synthesis conditions are not LMIs due to the multiplication of β with $\mathcal{M}$ and $\mathcal{G}$. In practice, this is often solved by performing a line search over β for which the conditions do become LMIs.

B

Mathematical Proofs

B.1 Proofs of Chapter 2

Proof of Lemma 2.6 (LPV behavioral embedding)

As the *Linear Parameter-Varying* (LPV) system is a global LPV embedding of the nonlinear system on the region $\mathcal{X} \times \mathcal{W} = \mathcal{X} \times \mathcal{W}$ for any trajectory $(x, w, z) \in \mathfrak{B}$, we also have that $(x, w, z) \in \mathfrak{B}_{\mathrm{p}}(\eta(x, w))$. Moreover, as $\mathcal{X} \times \mathcal{W} = \mathcal{X} \times \mathcal{W}$ and $\eta(\mathcal{X}, \mathcal{W}) \subseteq \mathcal{P}$, we get the following relation

$$\mathfrak{B} = \bigcup_{(x,w)\in\pi_{\mathrm{x,w}}\mathfrak{B}} \mathfrak{B}_{\mathrm{p}}(\eta(x, w)) \subseteq \bigcup_{(x,w)\in(\mathcal{X},\mathcal{W})^{\mathcal{T}}} \mathfrak{B}_{\mathrm{p}}(\eta(x, w)) \subseteq \bigcup_{p\in\mathcal{P}^{\mathcal{T}}} \mathfrak{B}_{\mathrm{p}}(p) = \breve{\mathfrak{B}}_{\mathrm{p}}. \tag{B.1}$$

Proof of Theorem 2.5 (Classical (Q, S, R) dissipativity conditions for LPV systems)

Left and right multiplication of the inequalities in (2.37) by $\mathrm{col}(x, w)^\top$ and $\mathrm{col}(x, w)$, respectively, we obtain that

$$\begin{aligned}(\star)^\top + M(p)\,(A(p)x + B(p)w) + \partial M(p, v) \leq \\ (\star)^\top Q w + 2w^\top S\,(C(p)x + D(p)w) + (\star)^\top R\,(C(p)x + D(p)w)\,,\end{aligned} \tag{B.2a}$$

and

$$\begin{aligned}(\star)^\top M(p+v)\,(A(p)x + B(p)w) - M(p) \leq \\ (\star)^\top Q w + 2w^\top S\,(C(p)x + D(p)w) + (\star)^\top R\,(C(p)x + D(p)w)\,,\end{aligned} \tag{B.2b}$$

for all $p \in \mathcal{P}$, $v \in \Pi$, and $(x, w) \in (\mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}})$. This implies

$$
\begin{aligned}
(\star)^\top + M(p(t))\left(A(p(t))x(t) + B(p(t))w(t)\right) + \partial M(p(t), \dot{p}(t)) \leq \\
(\star)^\top Q w(t) + 2w(t)^\top S\left(C(p(t))x(t) + D(p)w(t)\right) + \\
(\star)^\top R\left(C(p(t))x(t) + D(p(t))w(t)\right),
\end{aligned}
\tag{B.3a}
$$

and

$$
\begin{aligned}
(\star)^\top M(p(t+1))\left(A(p(t))x(t) + B(p(t))w(t)\right) - M(p(t)) \leq \\
(\star)^\top Q w(t) + 2w(t)^\top S\left(C(p(t))x(t) + D(p(t))w(t)\right) + \\
(\star)^\top R\left(C(p(t))x(t) + D(p(t))w(t)\right),
\end{aligned}
\tag{B.3b}
$$

for all $t \in \mathcal{T}$ and $(x, w, z) \in \mathfrak{B}_\mathrm{p}(p)$ for all $p \in \mathcal{P}^\mathcal{T}$. Through integration/summation from t_0 to t_1, this implies condition (2.35), where $\mathcal{V}_\mathrm{p}$ is given by (2.36) and s_v by (2.25), meaning the LPV system is classically (Q, S, R) dissipative.

B.2 Proofs of Chapter 4

Proof of Theorem 4.1 (Universal shifted Lyapunov stability)

For every $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$, the function $V : x \mapsto V_{\mathrm{s}}(x, w_*)$ satisfies the conditions for a Lyapunov function for the equilibrium point x_*, as $V = (x \mapsto V_{\mathrm{s}}(x, w_*)) \in \mathcal{Q}_{x_*}$ for every $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$, see Theorem 2.2. Consequently, by (4.6), it then holds for every $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$ that

$$\frac{\partial}{\partial t} V(x(t)) \leq 0, \tag{B.4}$$

for all $t \in \mathbb{R}_0^+$ and $x \in \pi_{\mathrm{x}} \mathfrak{B}_{\mathrm{w}}(w \equiv w_*)$. Hence, by Theorem 2.2, the system is stable at each equilibrium point $(x_*, w_*, z_*) \in \mathscr{E}$, meaning, by definition, it is universally shifted stable. Similarly, when (4.6) holds, but with a strict inequality except when $x(t) = x_*$, this implies that (B.4) holds, but with a strict inequality except when $x(t) = x_*$. Therefore, by Theorem 2.2, asymptotic stability of the nonlinear system follows at each equilibrium point $(x_*, w_*, z_*) \in \mathscr{E}$.

Proof of Lemma 4.1 (Condition for universal shifted dissipativity)

If the nonlinear system given by (4.1) is universally shifted dissipative for every $(x_*, w_*, z_*) \in \mathscr{E}$, condition (4.7) holds for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_1 \geq t_0$ and $(x, w, z) \in \mathfrak{B}$. If $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{C}_1$ for every $w_* \in \mathscr{W}$, this is equivalent to requiring that the following relation holds for every $(x_*, w_*, z_*) \in \mathscr{E}$:

$$\frac{\partial}{\partial t} \mathcal{V}_{\mathrm{s}}(x(t), w_*) \leq s_{\mathrm{s}}(w(t), w_*, z(t), z_*)\, dt, \tag{B.5}$$

for all $t \in \mathbb{R}_0^+$ and $(x, w, z) \in \mathfrak{B}$. This is equivalent to the following relation requiring to hold for every $(x_*, w_*, z_*) \in \mathscr{E}$

$$\nabla_x \mathcal{V}_{\mathrm{s}}(x(t), w_*) f(x(t), w(t)) \leq s_{\mathrm{s}}(w(t), w_*, h(x(t), w(t)), z_*), \tag{B.6}$$

for all $t \in \mathbb{R}_0^+$ and $(x, w) \in \pi_{\mathrm{x,w}} \mathfrak{B}$, which holds if, for every $(x_*, w_*, z_*) \in \mathscr{E}$, (4.8) holds for all $x \in \mathcal{X}$ and $w \in \mathcal{W}$.

Proof of Theorem 4.2 (Induced classical dissipativity)

If (4.1) is universally shifted (Q, S, R) dissipative, there exists a storage function $\mathcal{V}_{\mathrm{s}} : \mathscr{X} \times \mathscr{W} \to \mathbb{R}_0^+$ with $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{C}_0$ and $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{Q}_{x_*}$ for every $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$, such that it holds that

$$\mathcal{V}_{\mathrm{s}}(x(t_1), w_*) - \mathcal{V}_{\mathrm{s}}(x(t_0), w_*) \leq \int_{t_0}^{t_1} \begin{bmatrix} w - w_* \\ z - z_* \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w - w_* \\ z - z_* \end{bmatrix} dt, \tag{B.7}$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_1 \geq t_0$ and $(x, w, z) \in \mathfrak{B}$. As we assume that $(0,0,0) \in \mathscr{E}$, it also holds that

$$\mathcal{V}_\mathrm{s}(x(t_1), 0) - \mathcal{V}(x(t_0), 0) \leq \int_{t_0}^{t_1} \begin{bmatrix} w \\ z \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} dt, \tag{B.8}$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_1 \geq t_0$ and $(x, w, z) \in \mathfrak{B}$. Defining $\mathcal{V}(x) = \mathcal{V}_\mathrm{s}(x, 0)$, which satisfies that $\mathcal{V} \in \mathcal{Q}_0$, we have that

$$\mathcal{V}(x(t_1)) - \mathcal{V}_\mathrm{s}(x(t_0)) \leq \int_{t_0}^{t_1} \begin{bmatrix} w \\ z \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w \\ z \end{bmatrix} dt, \tag{B.9}$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_1 \geq t_0$ and $(x, w, z) \in \mathfrak{B}$, which is the condition for classical (Q, S, R) dissipativity, see also Definition 2.13.

Proof of Lemma 4.2 ($\mathcal{L}_{\mathrm{s}2}$-gain based on universal shifted dissipativity)

If the system is universally shifted (Q, S, R) dissipative with $(Q, S, R) = (\gamma^2 I, 0, -I)$ it holds that there exists a $\mathcal{V}_\mathrm{s}$ such that for every $(x_*, w_*, z_*) \in \mathscr{E}$

$$\mathcal{V}_\mathrm{s}(x(t_1), w_*) - \mathcal{V}_\mathrm{s}(x(t_0), w_*) \leq \int_{t_0}^{t_1} \gamma^2 (\star)^\top (w(t) - w_*) - (\star)^\top (z(t) - z_*)\, dt, \tag{B.10}$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_1 \geq t_0$ and $(x, w, z) \in \mathfrak{B}$. Consequently, it also holds for every $(x_*, w_*, z_*) \in \mathscr{E}$ that

$$0 \leq \mathcal{V}_\mathrm{s}(x(t), w_*) \leq \int_0^T \gamma^2 \left\| w(t) - w_* \right\| - \left\| z(t) - z_* \right\|\, dt + \mathcal{V}_\mathrm{s}(x_0, w_*), \tag{B.11}$$

for all $T \geq 0$ and $(x, w, z) \in \mathfrak{B}$. Using the $\mathcal{L}_{2\mathrm{e}}$ signal norm definition, see Definition 2.10, and using that $w_* = \kappa(x_*)$, this is equivalent to

$$\left\| z - w_* \right\|_{2,T}^2 \leq \gamma^2 \left\| w - z_* \right\|_{2,T}^2 + \mathcal{V}_\mathrm{s}(x_0, \kappa(x_*)), \tag{B.12}$$

holding for all $T \geq 0$ and $(x, w, z) \in \mathfrak{B}$ with $w \in \mathcal{L}_{2\mathrm{e}}$. Next, we take the square root on both sides of (B.12), which gives us

$$\left\| z - \tilde{z} \right\|_{2,T} \leq \sqrt{\gamma^2 \left\| w - \tilde{w} \right\|_{2,T}^2 + \mathcal{V}_\mathrm{i}(x_0, \tilde{x}(0))} \leq \gamma \left\| w - \tilde{w} \right\|_{2,T} + \sqrt{\mathcal{V}_\mathrm{s}(x_0, \kappa(x_*))}, \tag{B.13}$$

which is equivalent to (4.10) with $\zeta_\mathrm{s}(x_0, x_*) = \sqrt{\mathcal{V}_\mathrm{s}(x_0, \kappa(x_*))}$.

Proof of Theorem 4.3 (Universal shifted stability from universal shifted dissipativity)

If, the system given by (4.1) is universally shifted dissipative w.r.t. a supply function s_s, for which for every $(x_*, w_*, z_*) \in \mathscr{E}$, (4.12) holds for all $z \in \mathcal{Z}$, then it holds for every $(x_*, w_*, z_*) \in \mathscr{E}$ that

$$\mathcal{V}_\mathrm{s}(x(t_1), w_*) - \mathcal{V}_\mathrm{s}(x(t_0), w_*) \leq \int_{t_0}^{t_1} s_\mathrm{s}(w_*, w_*, z(t), z_*)\, dt \leq 0, \tag{B.14}$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_0 \leq t_1$ and $(x, z) \in \pi_{\mathrm{x,z}}\, \mathfrak{B}_\mathrm{w}(w \equiv w_*)$. As $\mathcal{V}_\mathrm{s}(\cdot, w_*) \in \mathcal{C}_1$ for all $w_* \in \mathscr{W}$, this means that for every $(x_*, w_*, z_*) \in \mathscr{E}$ it holds that

$$\frac{d}{dt}\mathcal{V}_\mathrm{s}(x(t), w_*) \leq 0, \tag{B.15}$$

for all $t \in \mathbb{R}_0^+$ and $x \in \pi_\mathrm{x}\, \mathfrak{B}_\mathrm{w}(w \equiv w_*)$. The storage function $\mathcal{V}_\mathrm{s}$ satisfies the conditions for the universal shifted Lyapunov function V_s in Theorem 4.1. Hence, (B.15) implies (4.6), which by Theorem 4.1 implies universal shifted stability.

In case of universal shifted asymptotic stability, the supply function satisfies (4.12), but with strict inequality for all $z \neq z_*$. Moreover, as the system is assumed to be observable, this means that $z \in \pi_\mathrm{z}\, \mathfrak{B}_\mathrm{w}(w \equiv w_*)$ for which $z(t) = z_* \in \pi_{\mathrm{z}_*}\, \mathscr{E}$ for all $t \in \mathbb{R}_0^+$, implies that $x(t) = x_* \in \pi_{\mathrm{x}_*}\, \mathscr{E}$, with $(x_*, z_*) \in \pi_{\mathrm{x}_*,\mathrm{z}_*}\, \mathscr{E}$. Consequently, we have that (B.15) holds, but with strict inequality except when $x(t) = x_*$, which by Theorem 4.1 implies universal shifted asymptotic stability.

Proof of Lemma 4.3 (Condition for velocity dissipativity)

We have that the system given by (4.1) is velocity dissipative if there exists a storage function $\mathcal{V}_\mathrm{v}$ s.t. (4.16) holds, which, for $\mathcal{V}_\mathrm{v} \in \mathcal{C}_1$, is equivalent to

$$\frac{\partial}{\partial t}\mathcal{V}_\mathrm{v}(\dot{x}(t)) \leq s_\mathrm{v}(\dot{w}(t), \dot{z}(t)), \tag{B.16}$$

holding for all $t \in \mathbb{R}_0^+$ and $(\dot{x}, \dot{w}, \dot{z}) \in \mathfrak{B}_\mathrm{v}$. Using the dynamics of the velocity form given by (4.15) and expression for $\dot{x}$ given by (4.1a), (B.16) equivalently holds if

$$\begin{aligned}\nabla\mathcal{V}_\mathrm{v}\big(f(x(t), w(t))\big)\big(A_\mathrm{v}(x(t), w(t))f(x(t), w(t)) + B_\mathrm{v}(x(t), w(t))\dot{w}(t)\big) \leq \\ s_\mathrm{v}\big(\dot{w}(t), C_\mathrm{v}(x(t), w(t))f(x(t), w(t)) + D_\mathrm{v}(x(t), w(t))\dot{w}(t)\big),\end{aligned} \tag{B.17}$$

for all $t \in \mathbb{R}_0^+$, $\dot{w} \in \pi_{\dot{\mathrm{w}}}\, \mathfrak{B}_\mathrm{v}$, and corresponding $(x, w) \in \pi_{\mathrm{x,w}}\, \mathfrak{B}_\mathrm{c}$. Consequently, if (4.18) holds for all values $w_\mathrm{v} \in \mathbb{R}^{n_\mathrm{w}}$, $x \in \mathcal{X}$, and $w \in \mathcal{W}$, then, (B.17) holds for all $t \in \mathbb{R}_0^+$, $\dot{w} \in \pi_{\dot{\mathrm{w}}}\, \mathfrak{B}_\mathrm{v}$. This means that the system is velocity dissipative w.r.t. the supply function s_v.

Proof of Theorem 4.4 (Velocity (Q, S, R) dissipativity condition)

If (4.20) holds for all $(x, w) \in \mathcal{X} \times \mathcal{W}$, we have by pre- and post multiplication of (4.20) with $\mathrm{col}(f(x, w), w_\mathrm{v})^\top$ and $\mathrm{col}(f(x, w), w_\mathrm{v})$, respectively, that

$$\begin{aligned}2(f(x, w)^\top M(f(x, w))\,(A_\mathrm{v}(x, w)f(x, w) + B_\mathrm{v}(x, w)w_\mathrm{v}) - \\ w_\mathrm{v}^\top Q w_\mathrm{v} - 2w_\mathrm{v}^\top S\big(C_\mathrm{v}(x, w)f(x, w) + D_\mathrm{v}(x, w)w_\mathrm{v}\big) - \\ (\star)^\top R\big(C_\mathrm{v}(x, w)f(x, w) + D_\mathrm{v}(x, w)w_\mathrm{v}\big) \leq 0,\end{aligned} \tag{B.18}$$

all $w_{\mathrm{v}} \in \mathbb{R}^{n_{\mathrm{w}}}$ and $(x, w) \in \mathcal{X} \times \mathcal{W}$. This corresponds to the condition in Lemma 4.3 with the storage function given by (4.19) and a (Q, S, R) supply function (4.17). Hence, by Lemma 4.3, the system given by (4.1) is velocity (Q, S, R) dissipative.

Proof of Theorem 4.5 (Implied universal shifted stability)

For each equilibrium point $(x_*, w_*, z_*) \in \mathscr{E}$ consider

$$V_{\mathrm{s}}(x(t), w_*) := V_{\mathrm{v}}(f(x(t), w_*)) = V_{\mathrm{v}}(\dot{x}(t)). \tag{B.19}$$

For each $(x_*, w_*, z_*) \in \mathscr{E}$, this choice implies that $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{Q}_{x_*}$ and $\mathcal{V}_{\mathrm{s}}(\cdot, w_*) \in \mathcal{C}_1$ as $\mathcal{V}_{\mathrm{v}} \in \mathcal{Q}_0$ and $\mathcal{V}_{\mathrm{v}} \in \mathcal{C}_1$ (and $f \in \mathcal{C}_1$). Note that this requires uniqueness of the equilibrium points (see Assumption 4.1), as otherwise there exists multiple x_* for which $V_{\mathrm{s}}(x_*, w_*) = 0$. By this choice of V_{s}, we have that for each $(x_*, w_*, z_*) \in \mathscr{E}$

$$\frac{d}{dt} V_{\mathrm{s}}(x(t), w_*) = \frac{d}{dt} V_{\mathrm{v}}(\dot{x}(t)) \leq 0, \tag{B.20}$$

for all $t \in \mathbb{R}_0^+$ and $\dot{x} \in \pi_{\dot{\mathrm{x}}}\, \mathfrak{B}_{\mathrm{v,w}}(w \equiv w_*)$ and correspondingly for all $x \in \pi_{\mathrm{x}}\, \mathfrak{B}_{\mathrm{c,w}}(w \equiv w_*)$. This implies that (4.6) holds for all $x \in \pi_{\mathrm{x}}\, \mathfrak{B}_{\mathrm{c,w}}(w \equiv w_*)$ and for all equilibrium points $(x_*, w_*) \in \pi_{\mathrm{x}_*, \mathrm{w}_*}\, \mathscr{E}$. Hence, by Theorem 4.1, the system is universally shifted stable. The asymptotic stability version follows similarly by changing (B.20) to a strict inequality.

Proof of Theorem 4.6 (Universal shifted stability from velocity dissipativity)

If the system given by (4.1) is velocity dissipative w.r.t. a supply function s_{v} which satisfies (4.22) for all $z_{\mathrm{v}} \in \mathbb{R}^{n_{\mathrm{z}}}$, then it holds that

$$\mathcal{V}_{\mathrm{v}}(\dot{x}(t_1)) - \mathcal{V}_{\mathrm{v}}(\dot{x}(t_0)) \leq \int_{t_0}^{t_1} s_{\mathrm{v}}(0, \dot{z}(t))\, dt \leq 0, \tag{B.21}$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_1 \geq t_0$ and $(\dot{x}, \dot{z}) \in \pi_{\dot{\mathrm{x}}, \dot{\mathrm{z}}}\, \mathfrak{B}_{\mathrm{v}, \mathscr{W}}$. As $\mathcal{V}_{\mathrm{v}} \in \mathcal{C}_1$, this means that

$$\frac{d}{dt} \mathcal{V}_{\mathrm{v}}(\dot{x}(t)) \leq 0, \tag{B.22}$$

for all $t \in \mathbb{R}_0^+$ and $\dot{x} \in \pi_{\dot{\mathrm{x}}}\, \mathfrak{B}_{\mathrm{v}, \mathscr{W}}$, which implies through Theorem 4.5 universal shifted stability, where $\mathcal{V}_{\mathrm{v}} = V_{\mathrm{v}}$.

For universal shifted asymptotic stability, s_{v} satisfies (4.22), but with a strict inequality when $z_{\mathrm{v}} \neq 0$. As it is also assumed that the system is observable, this means that $\dot{z}(t) = 0$ for all $t \in \mathbb{R}_0^+$, corresponding to $z \in \pi_{\mathrm{z}}\, \mathfrak{B}_{\mathrm{c,w}}(w \equiv w_*)$ for which $z(t) = z_* \in \pi_{\mathrm{z}_*}\, \mathscr{E}$ for all $t \in \mathbb{R}_0^+$, implies that $x(t) = x_* \in \pi_{\mathrm{x}_*}\, \mathscr{E}$, with $(x_*, z_*) \in \pi_{\mathrm{x}_*, \mathrm{z}_*}\, \mathscr{E}$ corresponding to $\dot{x}(t) = 0$ for all $t \in \mathbb{R}_0^+$. Consequently, we have that (B.22) where the inequality is strict except when $\dot{x}(t) = 0$ is satisfied. This implies by Theorem 4.5 universal shifted asymptotic stability of the system.

Proof of Theorem 4.7 (Universal shifted performance from velocity dissipativity)

If the nonlinear system is velocity dissipative w.r.t. the supply function $s_\mathrm{v}(\dot{w},\dot{z}) = \dot{w}^\top Q\dot{w} + \dot{z}^\top R\dot{z}$, there exists a storage function $\mathcal{V}_\mathrm{v}$, such that for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_1 \geq t_0$

$$\mathcal{V}_\mathrm{v}(\dot{x}(t_1)) - \mathcal{V}_\mathrm{v}(\dot{x}(t_0)) \leq \int_{t_0}^{t_1} \dot{w}(t)^\top Q\dot{w}(t) + \dot{z}(t)^\top R\dot{z}(t)\,dt, \tag{B.23}$$

for all $(\dot{x},\dot{w},\dot{z}) \in \mathfrak{B}_\mathrm{v}$, corresponding to $(x,w,z) \in \mathfrak{B}_\mathrm{c}$. Hence, as $\mathcal{V}_\mathrm{v}(\dot{x}(0)) = \mathcal{V}_\mathrm{v}(0) = 0$ and $\mathcal{V}_\mathrm{v}(x_\mathrm{v}) > 0$, $\forall\, x_\mathrm{v} \in \mathbb{R}^{n_\mathrm{x}} \backslash \{0\}$ this implies that

$$0 < \int_0^T \dot{w}(t)^\top Q\dot{w}(t) + \dot{z}(t)^\top R\dot{z}(t)\,dt, \tag{B.24}$$

for all $T > 0$ and $(\dot{x},\dot{w},\dot{z}) \in \mathfrak{B}_\mathrm{v}$. Defining $\tilde{Q} := \frac{1}{\|Q\|}Q$ and $\tilde{R} := \frac{1}{\|Q\|}R$, it also holds that

$$0 < \int_0^T \dot{w}(t)^\top \tilde{Q}\dot{w}(t) + \dot{z}(t)^\top \tilde{R}\dot{z}(t)\,dt, \tag{B.25}$$

Next, using (4.23)–(4.25), we have that, omitting dependence on time for brevity,

$$\begin{aligned}
\dot{z}^\top \tilde{R}\dot{z} &= \dot{x}^\top C^\top \tilde{R}C\dot{x}, && \text{(B.26a)}\\
&= (\star)^\top \tilde{R}C(f(x) + Bw), && \text{(B.26b)}\\
&= (\star)^\top \tilde{R}C(f(x) + Bw \underbrace{-(f(x_*) + Bw_*)}_{=0}), && \text{(B.26c)}\\
&= (\star)^\top \tilde{R}C(f(x) - f(x_*) + B(w - w_*)). && \text{(B.26d)}
\end{aligned}$$

Through, Lemma C.1.1, we have that

$$\begin{aligned}
f(x) - f(x_*) &= \left(\int_0^1 \frac{\partial f}{\partial x}(x_* + \lambda(x - x_*))\,d\lambda\right)(x - x_*),\\
&= \underbrace{\left(\int_0^1 A_\mathrm{v}(x_* + \lambda(x - x_*))\,d\lambda\right)}_{\bar{A}(x,x_*)}(x - x_*).
\end{aligned} \tag{B.27}$$

Combining this with Assumption 4.2, we can write (B.26d) as

$$\dot{z}^\top \tilde{R}\dot{z} = (\star)^\top \tilde{R}C\bar{A}(x,x_*)(x - x_*). \tag{B.28}$$

Next, by satisfying Assumption 4.3 for $T = \tilde{R} \preceq 0$, we have that for every $x_* \in \mathscr{X}$

$$\dot{z}^\top \tilde{R}\dot{z} = (\star)^\top \tilde{R}C\bar{A}(x,x_*)(x - x_*) \leq \alpha^{-1}(\star)^\top \tilde{R}C(x - x_*) = \alpha^{-1}(\star)^\top \tilde{R}(z - z_*). \tag{B.29}$$

Moreover, by Assumption 4.4, we have that, for a given $(x_*, w_*, z_*) \in \mathscr{E}$,

$$\dot{w}^\top \tilde{Q}\dot{w} = (\star)^\top \tilde{Q}A_\mathrm{w}(w - w_*) \leq \beta^2(\star)^\top \tilde{Q}(w - w_*), \tag{B.30}$$

where $w \in \mathfrak{W}$ and $0 \preceq \tilde{Q} \preceq I$. Combining (B.25), (B.29), and (B.30), we obtain that, for every $(x_*, w_*, z_*) \in \mathscr{E}$,

$$\int_0^T \beta^2 (\star)^\top \tilde{Q}(w(t) - w_*) + \alpha^{-1} (\star)^\top \tilde{R}(z(t) - z_*)\, dt > 0, \tag{B.31}$$

for all $T > 0$ and $(w, z) \in \pi_{\mathrm{w,z}} \mathfrak{B}_{\mathrm{c}}$ with $w \in \mathfrak{W}$. Hence, also

$$\int_0^T \beta^2 (\star)^\top Q(w(t) - w_*) + \alpha^{-1} (\star)^\top R(z(t) - z_*)\, dt > 0, \tag{B.32}$$

for all $T > 0$ and $(w, z) \in \pi_{\mathrm{w,z}} \mathfrak{B}_{\mathrm{c}}$ with $w \in \mathfrak{W}$.

Proof of Corollary 4.1 (Bounded $\mathcal{L}_{\mathrm{s}2}$-gain from velocity dissipativity)

From Theorem 4.7 with $(Q, S, R) = (\gamma^2 I, 0, -I)$, we have that there exists a function $\zeta_{\mathrm{s}} : \mathcal{X} \times \mathscr{X} \to \mathbb{R}$, such that, for every $(x_*, w_*, z_*) \in \mathscr{E}$,

$$\int_0^T \gamma^2 \beta^2 (\star)^\top (w(t) - w_*) - \alpha^{-1} (\star)^\top (z(t) - z_*)\, dt + \zeta_{\mathrm{s}}(x_0, x_*) \geq 0, \tag{B.33}$$

for all $T > 0$ and $(w, z) \in \pi_{\mathrm{w,z}} \mathfrak{B}_{\mathrm{c}}$, with $w \in \mathfrak{W}$. This is equivalent to

$$\alpha^{-1} \int_0^T \|z(t) - z_*\|^2 \, dt \leq \gamma^2 \beta^2 \int_0^T \|w(t) - w_*\|^2 \, dt + \zeta_{\mathrm{s}}(x_0, x_*), \tag{B.34}$$

$$\|z - z_*\|_{2,T}^2 \leq \alpha \gamma^2 \beta^2 \|w - w_*\|_{2,T}^2 + \alpha \zeta_{\mathrm{s}}(x_0, x_*), \tag{B.35}$$

for all $T > 0$ and $(w, z) \in \pi_{\mathrm{w,z}} \mathfrak{B}_{\mathrm{c}}$ with $w \in \mathfrak{W}$. Hence, this implies that for every $(x_*, w_*, z_*) \in \mathscr{E}$

$$\|z - z_*\|_{2,T} \leq \tilde{\gamma} \|w - w_*\|_{2,T} + \alpha \zeta_{\mathrm{s}}(x_0, x_*), \tag{B.36}$$

for all $T > 0$ and $(w, z) \in \pi_{\mathrm{w,z}} \mathfrak{B}_{\mathrm{c}}$ with $w \in \mathfrak{W}$ where $\tilde{\gamma} = \alpha \gamma^2 \beta^2$, corresponding to the $\mathcal{L}_{\mathrm{s}2}$-gain definition (see Definition 4.3).

Proof of Lemma 4.4 (VPV behavioral embedding)

The LPV representation (4.30) is a *Velocity Parameter-Varying* (VPV) embedding of the system given by (4.1) on the region $\mathcal{X} \times \mathcal{W} = \mathcal{X} \times \mathcal{W}$. Consequently, for any trajectory $(\dot{x}, \dot{w}, \dot{z}) \in \mathfrak{B}_{\mathrm{v}}$, we also have that $(\dot{x}, \dot{w}, \dot{z}) \in \mathfrak{B}_{\mathrm{p}}(\eta(x, w))$. Moreover, as $\mathcal{X} \times \mathcal{W} = \mathcal{X} \times \mathcal{W}$ and $\eta(\mathcal{X}, \mathcal{W}) \subseteq \mathcal{P}$, we get the following relation

$$\mathfrak{B}_{\mathrm{v}} = \bigcup_{(x,w) \in \pi_{\mathrm{x,w}} \mathfrak{B}} \mathfrak{B}_{\mathrm{p}}(\eta(x, w)) \subseteq \bigcup_{(x,w) \in (\mathcal{X}, \mathcal{W})^{\mathbb{R}_0^+}} \mathfrak{B}_{\mathrm{p}}(\eta(x, w)) \subseteq \bigcup_{p \in \mathcal{P}^{\mathbb{R}_0^+}} \mathfrak{B}_{\mathrm{p}}(p) = \breve{\mathfrak{B}}_{\mathrm{p}}. \tag{B.37}$$

Proof of Theorem 4.8 (Velocity dissipativity analysis through the LPV framework)

As the LPV representation (4.30) is a VPV embedding of the nonlinear system given by (4.1) on the region $\mathcal{X} \times \mathcal{W} = \mathcal{X} \times \mathcal{W}$, we have by Lemma 4.4 that the LPV representation describes the full behavior of the velocity form (4.15), i.e., $\mathfrak{B}_{\mathrm{v}} \subseteq \breve{\mathfrak{B}}_{\mathrm{p}}$. Consequently, if the LPV representation (4.30) is classically dissipative for all trajectories in $\breve{\mathfrak{B}}_{\mathrm{p}}$, we have that the velocity form is classically dissipative for all trajectories in $\mathfrak{B}_{\mathrm{v}}$, which corresponds to the nonlinear system being velocity dissipative, see Definition 4.6.

Proof of Theorem 4.9 (Closed-loop velocity form)

W.l.o.g., we can omit w and z and assume P is given by (dependence on t is omitted for brevity):

$$\dot{x} = f(x, u); \tag{B.38a}$$

$$y = h_{\mathrm{y}}(x, u); \tag{B.38b}$$

and K is given by (4.34). P and K are interconnected such that $u_{\mathrm{k}} = y$ and $u = y_{\mathrm{k}}$. We assume that the interconnection is well-posed, i.e., there exists a $\mathcal{C}_1$ function $\breve{h}$, such that $u = h_{\mathrm{k}}(x_{\mathrm{k}}, h_{\mathrm{y}}(x, u))$ can be expressed as $u = \breve{h}(x, x_{\mathrm{k}})$. The closed-loop is then given by

$$\dot{x} = f(x, \breve{h}(x, x_{\mathrm{k}})); \tag{B.39a}$$

$$\dot{x}_{\mathrm{k}} = f_{\mathrm{k}}(x_{\mathrm{k}}, h_{\mathrm{y}}(x, \breve{h}(x, x_{\mathrm{k}}))). \tag{B.39b}$$

The velocity form of (B.39) is

$$\ddot{x} = \frac{\partial f}{\partial x}(x, u)\dot{x} + \frac{\partial f}{\partial u}(x, u)\left(\frac{\partial \breve{h}}{\partial x}(x, x_{\mathrm{k}})\dot{x} + \frac{\partial \breve{h}}{\partial x_{\mathrm{k}}}(x, x_{\mathrm{k}})\dot{x}_{\mathrm{k}}\right); \tag{B.40a}$$

$$\begin{aligned}\ddot{x}_{\mathrm{k}} = \frac{\partial f_{\mathrm{k}}}{\partial x_{\mathrm{k}}}(x_{\mathrm{k}}, u_{\mathrm{k}})\dot{x}_{\mathrm{k}} + \frac{\partial f_{\mathrm{k}}}{\partial u_{\mathrm{k}}}(x_{\mathrm{k}}, u_{\mathrm{k}})\Bigg(&\frac{\partial h_{\mathrm{y}}}{\partial x}(x, u)\dot{x} + \frac{\partial h_{\mathrm{y}}}{\partial u}(x, u)\cdot \\ &\left(\frac{\partial \breve{h}}{\partial x}(x, x_{\mathrm{k}})\dot{x} + \frac{\partial \breve{h}}{\partial x_{\mathrm{k}}}(x, x_{\mathrm{k}})\dot{x}_{\mathrm{k}}\right)\Bigg);\end{aligned} \tag{B.40b}$$

where $u_{\mathrm{k}} = h_{\mathrm{y}}(x, \breve{h}(x, x_{\mathrm{k}}))$. The velocity form of P, P_{v}, is given by

$$\ddot{x} = \frac{\partial f}{\partial x}(x, u)\dot{x} + \frac{\partial f}{\partial u}(x, u)\dot{u}; \tag{B.41a}$$

$$\dot{y} = \frac{\partial h_{\mathrm{y}}}{\partial x}(x, u)\dot{x} + \frac{\partial h_{\mathrm{y}}}{\partial u}(x, u)\dot{u}; \tag{B.41b}$$

and the velocity form of K, K_{v}, is given by

$$\ddot{x}_{\mathrm{k}} = \frac{\partial f_{\mathrm{k}}}{\partial x_{\mathrm{k}}}(x_{\mathrm{k}}, u_{\mathrm{k}})\dot{x}_{\mathrm{k}} + \frac{\partial f_{\mathrm{k}}}{\partial u_{\mathrm{k}}}(x_{\mathrm{k}}, u_{\mathrm{k}})\dot{u}_{\mathrm{k}}; \tag{B.42a}$$

$$\dot{y}_{\mathrm{k}} = \frac{\partial h_{\mathrm{k}}}{\partial x_{\mathrm{k}}}(x_{\mathrm{k}}, u_{\mathrm{k}})\dot{x}_{\mathrm{k}} + \frac{\partial h_{\mathrm{k}}}{\partial u_{\mathrm{k}}}(x_{\mathrm{k}}, u_{\mathrm{k}})\dot{u}_{\mathrm{k}}. \tag{B.42b}$$

Interconnecting these in a similar manner as P and K, i.e., $\dot{u}_{\mathrm{k}} = \dot{y}$ and $\dot{u} = \dot{y}_{\mathrm{k}}$, results for (B.42b) in

$$\dot{u} = \frac{\partial h_{\mathrm{k}}}{\partial x_{\mathrm{k}}}(x_{\mathrm{k}}, u_{\mathrm{k}})\dot{x}_{\mathrm{k}} + \frac{\partial h_{\mathrm{k}}}{\partial u_{\mathrm{k}}}(x_{\mathrm{k}}, u_{\mathrm{k}})\left(\frac{\partial h_{\mathrm{y}}}{\partial x}(x,u)\dot{x} + \frac{\partial h_{\mathrm{y}}}{\partial u}(x,u)\dot{u}\right). \tag{B.43}$$

By the well-posedness assumption, we know that $u = h_{\mathrm{k}}(x_{\mathrm{k}}, h_{\mathrm{y}}(x,u))$ can be expressed as $u = \breve{h}(x, x_{\mathrm{k}})$, hence, in the velocity form, (B.43) can equivalently be expressed as $\dot{u} = \frac{\partial \breve{h}}{\partial x}(x, x_{\mathrm{k}})\dot{x} + \frac{\partial \breve{h}}{\partial x_{\mathrm{k}}}(x, x_{\mathrm{k}})\dot{x}_{\mathrm{k}}$. Combining this with (B.41) and (B.42) allows us to express the interconnection of P_{v} and K_{v} as (B.40). Note that by writing (B.43) as

$$\dot{u} = \overbrace{h_{\mathrm{v}}(x, x_{\mathrm{k}})\frac{\partial h_{\mathrm{k}}}{\partial u_{\mathrm{k}}}(x_{\mathrm{k}}, u_{\mathrm{k}})\frac{\partial h_{\mathrm{y}}}{\partial x}(x,u)}^{\frac{\partial \breve{h}}{\partial x}(x, x_{\mathrm{k}})}\dot{x} + \overbrace{h_{\mathrm{v}}(x, x_{\mathrm{k}})\frac{\partial h_{\mathrm{k}}}{\partial x_{\mathrm{k}}}(x_{\mathrm{k}}, u_{\mathrm{k}})}^{\frac{\partial \breve{h}}{\partial x_{\mathrm{k}}}(x, x_{\mathrm{k}})}\dot{x}_{\mathrm{k}}, \tag{B.44}$$

where[1] $h_{\mathrm{v}}(x, x_{\mathrm{k}}) = \left(I - \frac{\partial h_{\mathrm{k}}}{\partial u_{\mathrm{k}}}(x_{\mathrm{k}}, u_{\mathrm{k}})\frac{\partial h_{\mathrm{y}}}{\partial u}(x,u)\right)^{-1}$, we get conditions on the existence of $\breve{h}$. Namely, if $h_{\mathrm{v}}(x, x_{\mathrm{k}})$ exists for some point $(x, x_{\mathrm{k}}) \in \mathcal{X} \times \mathbb{R}^{n_{\mathrm{x_k}}}$, then, by the implicit function theorem, there exists a neighborhood around this point for which $\breve{h}(x, x_{\mathrm{k}})$ exists (in that neighborhood) and is in $\mathcal{C}_1$.

Proof of Theorem 4.10 (Velocity closed-loop $\mathcal{L}_2$-gain)

As P_{vpv} is a VPV embedding on the region $\mathcal{X} \times \mathcal{U} \subseteq \mathcal{X} \times \mathcal{U}$, we have through Lemma 4.4 that $\mathfrak{B}_{\mathrm{v},\mathcal{XU}} \subseteq \breve{\mathfrak{B}}_{\mathrm{p}}$. Consequently, through Theorem 4.8, we have that $\mathcal{F}_{\mathrm{l}}(P_{\mathrm{v}}, K_{\mathrm{v}})$ with $p = \eta(x,u)$ for K_{v} is classically dissipative and has an $\mathcal{L}_2$-gain bound $\leq \gamma$ for all $(\dot{x}, \dot{u}) \in \pi_{\dot{\mathrm{x}},\dot{\mathrm{u}}}\, \mathfrak{B}_{\mathrm{v},\mathcal{XU}}$.

Proof of Theorem 4.11 (Velocity behavior inclusion)

We introduce the operators $\xi = \frac{d}{dt}$ and $\xi^{-1}w(t) = \int_0^t w(\tau)\, d\tau$. In order to integrate the inputs and differentiate the outputs of the nonlinear system given by (4.1), we introduce the new input and output $\hat{w}$ and $\hat{z}$, respectively, with the following relations:

$$w(t) = \xi^{-1}\hat{w}(t), \qquad \hat{z}(t) = \xi z(t), \tag{B.45}$$

which exist, as the solutions $(x, w, z) \in \mathfrak{B}_{\mathrm{c}}$. Obviously, multiplication with ξ or ξ^{-1} is not commutative. We can then write (4.1) as

$$\xi x(t) = f(x(t), \xi^{-1}\hat{w}(t)); \tag{B.46a}$$
$$\xi^{-1}\hat{z}(t) = h(x(t), \xi^{-1}\hat{w}(t)); \tag{B.46b}$$

[1] Note again that $u_{\mathrm{k}} = h_{\mathrm{y}}(x, \breve{h}(x, x_{\mathrm{k}}))$ and $u = \breve{h}(x, x_{\mathrm{k}})$.

which, by multiplication with ξ in order to obtain the new output $\hat{z}$, results in

$$\xi^2 x(t) = \xi f(x(t), \xi^{-1}\hat{w}(t)); \tag{B.47a}$$
$$\hat{z}(t) = \xi h(x(t), \xi^{-1}\hat{w}(t)); \tag{B.47b}$$

$$\xi^2 x(t) = \frac{\partial f(x(t), \xi^{-1}\hat{w}(t))}{\partial x}\xi x(t) + \frac{\partial f(x(t), \xi^{-1}\hat{w}(t))}{\partial w}\xi\left(\xi^{-1}\hat{w}(t)\right); \tag{B.48a}$$
$$\hat{z}(t) = \frac{\partial h(x(t), \xi^{-1}\hat{w}(t))}{\partial x}\xi x(t) + \frac{\partial h(x(t), \xi^{-1}\hat{w}(t))}{\partial w}\xi\left(\xi^{-1}\hat{w}(t)\right); \tag{B.48b}$$

$$\xi^2 x(t) = \frac{\partial f(x(t), w(t))}{\partial x}\xi x(t) + \frac{\partial f(x(t), w(t))}{\partial w}\hat{w}(t); \tag{B.49a}$$
$$\hat{z}(t) = \frac{\partial h(x(t), w(t))}{\partial x}\xi x(t) + \frac{\partial h(x(t), w(t))}{\partial w}\hat{w}(t). \tag{B.49b}$$

By using the definitions in (B.45), we can express (B.49) as

$$\ddot{x}(t) = \frac{\partial f(x(t), w(t))}{\partial x}\dot{x}(t) + \frac{\partial f(x(t), w(t))}{\partial w}\dot{w}(t); \tag{B.50a}$$
$$\dot{z}(t) = \frac{\partial h(x(t), w(t))}{\partial x}\dot{x}(t) + \frac{\partial h(x(t), w(t))}{\partial w}\dot{w}(t). \tag{B.50b}$$

From this, it is clear that we obtain the velocity form of (4.1) given by (4.15).

Proof of Theorem 4.12 (Universal shifted controller realization)

First, the primal form of the controller is realized. To realize the primal form we differentiate the input to the velocity controller $u_{\mathrm{v,k}}$ and integrate the output $y_{\mathrm{v,k}}$ of the velocity controller K_{v} given by (4.41). Therefore, the following relations hold (omitting time dependency for brevity)

$$\xi y_{\mathrm{k}} = y_{\mathrm{v,k}}, \qquad \xi u_{\mathrm{k}} = u_{\mathrm{v,k}}, \tag{B.51}$$

where again $\xi = \frac{d}{dt}$ and $\xi^{-1}w(t) = \int_0^t w(\tau)\,d\tau$. By simply rewriting (4.41), we get

$$\xi x_{\mathrm{v,k}} = A_{\mathrm{k}}(p)x_{\mathrm{v,k}} + B_{\mathrm{k}}(p)\xi u_{\mathrm{k}}; \tag{B.52a}$$
$$\xi y_{\mathrm{k}} = C_{\mathrm{k}}(p)x_{\mathrm{v,k}} + D_{\mathrm{k}}(p)\xi u_{\mathrm{k}}; \tag{B.52b}$$

Through (B.52a), we have the following equalities

$$\xi x_{\mathrm{v,k}} = A_{\mathrm{k}}(p)x_{\mathrm{v,k}} + B_{\mathrm{k}}(p)\xi u_{\mathrm{k}} + (\xi B_{\mathrm{k}}(p))u_{\mathrm{k}} - (\xi B_{\mathrm{k}}(p))u_{\mathrm{k}}, \tag{B.53}$$
$$\xi x_{\mathrm{v,k}} = A_{\mathrm{k}}(p)x_{\mathrm{v,k}} + \xi\left(B_{\mathrm{k}}(p)u_{\mathrm{k}}\right) - (\xi B_{\mathrm{k}}(p))u_{\mathrm{k}}, \tag{B.54}$$
$$\xi\left(x_{\mathrm{v,k}} - B_{\mathrm{k}}(p)u_{\mathrm{k}}\right) = A_{\mathrm{k}}(p)x_{\mathrm{v,k}} - (\xi B_{\mathrm{k}}(p))u_{\mathrm{k}}, \tag{B.55}$$

we then define $\tilde{x}_{\mathrm{k}} = x_{\mathrm{v,k}} - B_{\mathrm{k}}(p)u_{\mathrm{k}}$, resulting in

$$\xi\tilde{x}_{\mathrm{k}} = A_{\mathrm{k}}(p)\tilde{x}_{\mathrm{k}} + A_{\mathrm{k}}(p)B_{\mathrm{k}}(p)u_{\mathrm{k}} - (\xi B_{\mathrm{k}}(p))u_{\mathrm{k}}, \tag{B.56}$$

which can be rewritten to

$$\dot{\tilde{x}}_{\mathrm{k}} = A_{\mathrm{k}}(p)\tilde{x}_{\mathrm{k}} + \left(A_{\mathrm{k}}(p)B_{\mathrm{k}}(p) - \partial B_{\mathrm{k}}(p,\dot{p})\right)u_{\mathrm{k}}, \tag{B.57}$$

where $\partial B_{\mathrm{k}}(p,\dot{p}) = \sum_{i=1}^{n_{\mathrm{p}}} \frac{\partial B_{\mathrm{k}}(p)}{\partial p_i}\dot{p}_i$.

A similar procedure can be used to rewrite (B.52b), resulting in

$$\dot{\hat{x}}_{\mathrm{k}} = C_{\mathrm{k}}(p)\tilde{x}_{\mathrm{k}} + \left(C_{\mathrm{k}}(p)B_{\mathrm{k}}(p) - \partial D_{\mathrm{k}}(p,\dot{p})\right)u_{\mathrm{k}}, \tag{B.58}$$

where $\hat{x}_{\mathrm{k}} = y_{\mathrm{k}} - D_{\mathrm{k}}(p)u_{\mathrm{k}}$ and $\partial D_{\mathrm{k}}(p,\dot{p}) = \sum_{i=1}^{n_{\mathrm{p}}} \frac{\partial D_{\mathrm{k}}(p)}{\partial p_i}\dot{p}_i$. Due to the definition of $\hat{x}_{\mathrm{k}}$, we then have that $y_{\mathrm{k}} = \hat{x}_{\mathrm{k}} + D_{\mathrm{k}}(p)u_{\mathrm{k}}$. Combining these results gives us the primal realization K:

$$\left[\begin{array}{c} \dot{\tilde{x}}_{\mathrm{k}} \\ \dot{\hat{x}}_{\mathrm{k}} \\ \hdashline y_{\mathrm{k}} \end{array}\right] = \left[\begin{array}{cc:c} A_{\mathrm{k}}(p) & 0 & A_{\mathrm{k}}(p)B_{\mathrm{k}}(p) - \partial B_{\mathrm{k}}(p,\dot{p}) \\ C_{\mathrm{k}}(p) & 0 & C_{\mathrm{k}}(p)B_{\mathrm{k}}(p) - \partial D_{\mathrm{k}}(p,\dot{p}) \\ \hdashline 0 & I & D_{\mathrm{k}}(p) \end{array}\right] \left[\begin{array}{c} \tilde{x}_{\mathrm{k}} \\ \hat{x}_{\mathrm{k}} \\ \hdashline u_{\mathrm{k}} \end{array}\right]. \tag{B.59}$$

From which (4.42) can be constructed by introducing $\breve{x}_{\mathrm{k}} = \mathrm{col}(\tilde{x}_{\mathrm{k}}, \hat{x}_{\mathrm{k}})$ and using (4.43). Note that the realization of the controller (B.59) is not necessarily state minimal. However, in the literature techniques exists which can be applied in order to construct a minimal state-space realization of an LPV system, see e.g. (Petreczky, Tóth, et al. 2017).

As $u_{\mathrm{v,k}}, y_{\mathrm{v,k}}, p \in \mathcal{C}_1$, we have by Theorem 4.11 that the controller given by (4.42) with its inputs integrated and outputs differentiated is equal to its velocity from (4.41). Note, that to be conform with the initial form of K given in (4.41) the scheduling-variable is also considered an input to the controller. Substituting the relations of (B.51) in the first row of (B.59) and rewriting it results in

$$\xi\tilde{x}_{\mathrm{k}} = A_{\mathrm{k}}(p)\tilde{x}_{\mathrm{k}} + A_{\mathrm{k}}(p)B_{\mathrm{k}}(p)\xi^{-1}u_{\mathrm{v,k}} - \left(\xi B_{\mathrm{k}}(p)\right)\xi^{-1}u_{\mathrm{v,k}}, \tag{B.60}$$

for which we can write

$$\begin{aligned} \xi\tilde{x}_{\mathrm{k}} =& A_{\mathrm{k}}(p)\tilde{x}_{\mathrm{k}} + A_{\mathrm{k}}(p)B_{\mathrm{k}}(p)\xi^{-1}u_{\mathrm{v,k}} - \\ & \underbrace{\left(\xi B_{\mathrm{k}}(p)\right)\xi^{-1}u_{\mathrm{v,k}} - B_{\mathrm{k}}(p)u_{\mathrm{v,k}}}_{\xi\left(B_{\mathrm{k}}(p)\xi^{-1}u_{\mathrm{v,k}}\right)} + B_{\mathrm{k}}(p)u_{\mathrm{v,k}}, \end{aligned} \tag{B.61}$$

$$\xi\left(\tilde{x}_{\mathrm{k}} + B_{\mathrm{k}}(p)\xi^{-1}u_{\mathrm{v,k}}\right) = A_{\mathrm{k}}(p)\left(\tilde{x}_{\mathrm{k}} + B_{\mathrm{k}}(p)\xi^{-1}u_{\mathrm{v,k}}\right) + B_{\mathrm{k}}(p)u_{\mathrm{v,k}}, \tag{B.62}$$

then, defining $x_{\mathrm{v,k}} = \tilde{x}_{\mathrm{k}} + B_{\mathrm{k}}(p)\xi^{-1}u_{\mathrm{v,k}}$, results in

$$\xi x_{\mathrm{v,k}} = A_{\mathrm{k}}(p)x_{\mathrm{v,k}} + B_{\mathrm{k}}(p)u_{\mathrm{v,k}}. \tag{B.63}$$

Similarly, based on the second row of (B.59), we can find that

$$\xi\left(\hat{x}_{\mathrm{k}} + D_{\mathrm{k}}(p)\xi^{-1}u_{\mathrm{v,k}}\right) = C_{\mathrm{k}}(p)\left(\tilde{x}_{\mathrm{k}} + B_{\mathrm{k}}(p)\xi^{-1}u_{\mathrm{v,k}}\right) + D_{\mathrm{k}}(p)u_{\mathrm{v,k}}. \tag{B.64}$$

Now, we can use that $\hat{x}_{\mathrm{k}} + D_{\mathrm{k}}(p)\xi^{-1}u_{\mathrm{v,k}} = \hat{x}_{\mathrm{k}} + D_{\mathrm{k}}(p)u = y_{\mathrm{k}}$ based on the third row of (B.59) and using $x_{\mathrm{v,k}} = \tilde{x}_{\mathrm{k}} + B_{\mathrm{k}}(p)\xi^{-1}u_{\mathrm{v,k}}$ to rewrite (B.64) to

$$\xi y_{\mathrm{k}} = C_{\mathrm{k}}(p)x_{\mathrm{v,k}} + D_{\mathrm{k}}(p)u_{\mathrm{v,k}}. \tag{B.65}$$

Finally, by combining (B.63) and (B.65), we arrive at the velocity form of the controller

$$\dot{x}_{\mathrm{v,k}} = A_{\mathrm{k}}(p)x_{\mathrm{v,k}} + B_{\mathrm{k}}(p)u_{\mathrm{v,k}}; \tag{B.66a}$$
$$y_{\mathrm{v,k}} = C_{\mathrm{k}}(p)x_{\mathrm{v,k}} + D_{\mathrm{k}}(p)u_{\mathrm{v,k}}; \tag{B.66b}$$

which is equivalent with (4.41).

Proof of Theorem 4.13 (Closed-loop universal shifted stability and performance)

For our generalized plant P given by (4.33) with behavior $\mathfrak{B}_{\mathrm{c}}$ and velocity form P_{v} given by (4.38), we have by Theorem 4.10 that K_{v} given in (4.41) ensures classical dissipativity and a bounded $\mathcal{L}_2$-gain of γ of the closed-loop $\mathcal{F}_{\mathrm{l}}(P_{\mathrm{v}}, K_{\mathrm{v}})$ on $\mathcal{X} \times \mathcal{U}$. Moreover, we consider the set $\tilde{\mathcal{W}} \subseteq \mathcal{W}$, for which $\mathcal{X}_{\mathrm{cl}} = \mathcal{X} \times \mathcal{X}_{\mathrm{k}}$ is invariant, meaning that for any $w \in \tilde{\mathcal{W}}^{\mathbb{R}_0^+}$, the resulting $(x(t), u(t)) \in \mathcal{X} \times \mathcal{U}$, $\forall\, t \in \mathbb{R}_0^+$. Hence, we will remain in the design set on which classical dissipativity and a bounded $\mathcal{L}_2$-gain of the velocity form is ensured.

By Theorem 4.12, we have that the velocity form of K (4.42) is given by K_{v}. Consequently, by Theorem 4.9, the velocity form of $\mathcal{F}_{\mathrm{l}}(P, K)$ is given $\mathcal{F}_{\mathrm{l}}(P_{\mathrm{v}}, K_{\mathrm{v}})$. Hence, $\mathcal{F}_{\mathrm{l}}(P, K)$ is velocity (Q, S, R) dissipative with $(Q, S, R) = (\gamma^2 I, 0, -I)$ for $(x, u) \in \mathfrak{B}_{\mathrm{c},\mathcal{X}\mathcal{U}}$, which by Proposition 4.1 implies that $\mathcal{F}_{\mathrm{l}}(P, K)$ is universally shifted (Q, S, R) dissipative for $(Q, S, R) = (\gamma^2 I, 0, -I)$ for all $w \in \tilde{\mathcal{W}}^{\mathbb{R}_0^+} \cap \mathcal{L}_{2\mathrm{e}}$ and any $w_* \in \mathscr{W} \cap \tilde{\mathcal{W}}$. Hence, by Lemma 4.2, the (primal form of the) closed-loop system has a bounded $\mathcal{L}_{\mathrm{s}2}$-gain of γ and, by Theorem 4.3, it is universally shifted (asymptotically) stable for all $w \in \tilde{\mathcal{W}}^{\mathbb{R}_0^+} \cap \mathcal{L}_{2\mathrm{e}}$ and any $w_* \in \mathscr{W} \cap \tilde{\mathcal{W}}$.

Proof of Corollary 4.2 (Universal shifted realization with integral action)

For the realization of the controller in Theorem 4.12, the input to the velocity controller is time differentiated, which can be seen as appending a differentiator to the input of the velocity controller, see also Figure 4.3. The integration filter M, given by $M(s) = \frac{s+\alpha}{s}$, is also connected to the input of the controller as depicted in Figure 4.4. As differentiation in time can be expressed in the Laplace domain as s, we have that the interconnection of weighting filter and differentiator is given by $s \cdot M(s) = s\frac{s+\alpha}{s} = s + \alpha$. Hence, as $y = u_{\mathrm{k}}$, in the proof of Theorem 4.12, $s \cdot u_{\mathrm{k}} = \xi u_{\mathrm{k}} = u_{\mathrm{v,k}}$ in (B.51) becomes $(s+\alpha)u_{\mathrm{k}} = \xi u_{\mathrm{k}} + \alpha u_{\mathrm{k}} = u_{\mathrm{v,k}}$. Consequently, (B.52a) becomes

$$\xi x_{\mathrm{v,k}} = A_{\mathrm{k}}(p)x_{\mathrm{v,k}} + B_{\mathrm{k}}(p)(\xi u_{\mathrm{k}} + \alpha u_{\mathrm{k}}), \tag{B.67}$$

and we can write, similarly as (B.55), that

$$\xi\left(x_{\mathrm{v,k}} - B_{\mathrm{k}}(p)u_{\mathrm{k}}\right) = A_{\mathrm{k}}(p)x_{\mathrm{v,k}} - (\xi B_{\mathrm{k}}(p))u_{\mathrm{k}} + (B_{\mathrm{k}}(p)\alpha)u_{\mathrm{k}}. \tag{B.68}$$

Defining again that $\tilde{x}_{\mathrm{k}} = x_{\mathrm{v,k}} - B_{\mathrm{k}}(p)u_{\mathrm{k}}$, we obtain

$$\dot{\tilde{x}}_{\mathrm{k}} = A_{\mathrm{k}}(p)\tilde{x}_{\mathrm{k}} + \left(A_{\mathrm{k}}(p)B_{\mathrm{k}}(p) + B_{\mathrm{k}}(p)\alpha I - \partial B_{\mathrm{k}}(p, \dot{p})\right)u_{\mathrm{k}}, \tag{B.69}$$

Similarly, (B.58) becomes

$$\dot{\tilde{x}}_{\mathrm{k}} = C_{\mathrm{k}}(p)\tilde{x}_{\mathrm{k}} + \left(C_{\mathrm{k}}(p)B_{\mathrm{k}}(p) + D_{\mathrm{k}}(p)\alpha I - \partial D_{\mathrm{k}}(p,\dot{p})\right)u_{\mathrm{k}}. \tag{B.70}$$

Then, along the same lines as in the proof of Theorem 4.12, we obtain $\breve{A}_{\mathrm{k}}$, $\breve{C}_{\mathrm{k}}$, and $\breve{D}_{\mathrm{k}}$ as given in (4.43), and $\breve{B}_{\mathrm{k}}$ given by (4.44).

B.3 Proofs of Chapter 5

Proof of Theorem 5.1 (Differential (Q, S, R) dissipativity condition)

By Definition 5.3, the primal form (5.1) is differentially dissipative, if the differential form (5.7) is classically dissipative. Hence, it suffices to show that if (5.13) holds, the differential form is classically dissipative with storage function (5.11) and supply function (5.12). Note that (5.11) is differentiable. Therefore, we start with substituting (5.11) and (5.12) into the differentiated differential dissipation inequality, i.e., (5.10) differentiated w.r.t. t, resulting in requiring

$$\frac{d}{dt}\Big(x_\delta(t)^\top M(\bar{x}(t))x_\delta(t)\Big) \leq \begin{bmatrix} w_\delta(t) \\ z_\delta(t) \end{bmatrix}^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w_\delta(t) \\ z_\delta(t) \end{bmatrix}. \tag{B.71}$$

to hold for all $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,w}}\mathfrak{B}$ and $t \in \mathbb{R}_0^+$. By (Willems 1972), (B.71) is satisfied for all possible trajectories of (5.7) if and only if (B.71) holds for all values $(x_\delta(t), w_\delta(t), z_\delta(t)) \in \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}} \times \mathbb{R}^{n_\mathrm{z}}$, and $\bar{x}(t) \in \mathcal{X}$. Which, writing out (B.71), using the differential dynamics (5.7), and through the assumption that $\dot{x}(t) \in \mathcal{D}$, (hence, also $\dot{\bar{x}}(t) \in \mathcal{D}$) for all $t \in \mathbb{R}_0^+$, yields in requiring for all $(x_\delta, w_\delta) \in \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}}$, $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$, and $\bar{x}_\mathrm{v} \in \mathcal{D}$, that

$$\begin{aligned} 2x_\delta^\top M(\bar{x})\big(A_\delta(\bar{x},\bar{w})x_\delta + B_\delta(\bar{x},\bar{w})w_\delta\big) + x_\delta^\top \partial M(\bar{x},\bar{x}_\mathrm{v})x_\delta \leq \\ w_\delta^\top Q w_\delta + 2w_\delta^\top S\big(C_\delta(\bar{x},\bar{w})x_\delta + D_\delta(\bar{x},\bar{w})w_\delta\big) \\ + (\star)^\top R\big(C_\delta(\bar{x},\bar{w})x_\delta + D_\delta(\bar{x},\bar{w})w_\delta\big), \end{aligned} \tag{B.72}$$

where $\partial M(\bar{x},\bar{x}_\mathrm{v}) = \sum_{i=1}^{n_\mathrm{x}} \frac{\partial M(\bar{x})}{\partial \bar{x}_i}\bar{x}_{\mathrm{v},i}$ and $A_\delta, \ldots, D_\delta$ as in (5.8). Here $\bar{x}_\mathrm{v}$ corresponds to the values taken by $\dot{\bar{x}}(t) \in \mathcal{D}$. It is trivial to see that (B.72) is equivalent to the pre- and post multiplication of (5.13) with $\mathrm{col}(x_\delta, w_\delta)^\top$ and $\mathrm{col}(x_\delta, w_\delta)$, respectively. Consequently, requiring (B.72) to hold for all $(x_\delta, w_\delta) \in \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{w}}$, $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$, and $\bar{x}_\mathrm{v} \in \mathcal{D}$, is equivalent to require the condition in (5.13) to hold for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$ and $\bar{x}_\mathrm{v} \in \mathcal{D}$, which proves the statement.

Proof of Theorem 5.2 (Induced incremental dissipativity)

By writing out the λ-dependence in (5.10) for differential dissipativity, allows to integrate it over λ:

$$\begin{aligned} \int_0^1 \Big[\mathcal{V}_\delta\big(\bar{x}(t_1,\lambda), x_\delta(t_1,\lambda)\big) - \mathcal{V}_\delta\big(\bar{x}(t_0,\lambda), x_\delta(t_0,\lambda)\big) - \\ \int_{t_0}^{t_1} s_\delta\big(w_\delta(\tau,\lambda), z_\delta(\tau,\lambda)\big)\, d\tau\Big]\, d\lambda \leq 0. \end{aligned} \tag{B.73}$$

We compute the integral of the storage terms first. We define the following minimum energy path between x and $\tilde{x}$ by

$$\chi_{(x,\tilde{x})}(\lambda) := \underset{\hat{x}\in\Gamma_{\mathcal{X}}(x,\tilde{x})}{\arg\inf} \int_0^1 \mathcal{V}_\delta\left(\hat{x}(\lambda), \frac{\partial \hat{x}(\lambda)}{\partial\lambda}\right) d\lambda. \tag{B.74}$$

As $V_\delta(\bar{x}, x_\delta) = x_\delta^\top M(\bar{x})x_\delta$, the path $\chi_{(x,\tilde{x})}$ can be seen as the geodesic connecting x and $\bar{x}$ corresponding to the Riemannian metric $M(\bar{x})$, see also (Manchester and Slotine 2018; Reyes-Báez 2019). By the Hopf-Rinow theorem, this implies for any $x, \bar{x} \in \mathcal{X}$, $\chi_{(x,\tilde{x})}$ is a unique, smooth function (Manchester and Slotine 2017; Manchester and Slotine 2018). Next, we define

$$\mathcal{V}_\mathrm{i}(x,\tilde{x}) := \int_0^1 \mathcal{V}_\delta\left(\chi_{(x,\tilde{x})}(\lambda), \frac{\partial\chi_{(x,\tilde{x})}(\lambda)}{\partial\lambda}\right) d\lambda, \tag{B.75}$$

which will be our incremental storage function. Note that $\mathcal{V}_\delta(\bar{x},\cdot) \in \mathcal{Q}_0$, $\forall\, \bar{x} \in \mathcal{X}$. Therefore, $\mathcal{V}_\mathrm{i}(x,x) = 0$ for all $x \in \mathcal{X}$ as $\chi_{(x,x)}(\lambda) = x$, hence, $\frac{\partial\chi_{(x,\tilde{x})}(\lambda)}{\partial\lambda} = 0$ and by definition $\mathcal{V}_\delta(\cdot, 0) = 0$. Moreover, for all $x, \tilde{x} \in \mathcal{X}$ for which $x \neq \tilde{x}$, we have that $\mathcal{V}_\mathrm{i}(x,\tilde{x}) > 0$, as in that case there exists a set of $\lambda \in [0, 1]$ for which $\frac{\partial\chi_{(x,\tilde{x})}(\lambda)}{\partial\lambda} \in \mathbb{R}\backslash\{0\}$ (as it can only be zero for all λ if $x = \tilde{x}$) and by definition $\mathcal{V}_\delta(\bar{x}, x_\delta) > 0$, $\forall\, x_\delta \in \mathbb{R}^{n_\mathrm{x}}\backslash\{0\}$. Consequently, we have that $\mathcal{V}_\mathrm{i} \in \mathcal{Q}_\mathrm{i}$.

Using this incremental storage function, we have that

$$\mathcal{V}_\mathrm{i}(x(t_1), \tilde{x}(t_1)) \leq \int_0^1 \mathcal{V}_\delta\big(\bar{x}(t_1,\lambda), x_\delta(t_1,\lambda)\big)\, d\lambda, \tag{B.76}$$

for any $(\lambda \mapsto \bar{x}(t_1,\lambda)) \in \Gamma_{\mathcal{X}}(x(t_1), \tilde{x}(t_1))$ with $x(t_1), \tilde{x}(t_1) \in \mathcal{X}$, $t_1 \in \mathbb{R}_0^+$, and $(t \mapsto \bar{x}(t,\lambda)) \in \pi_\mathrm{x}\mathfrak{B}$ for any $\lambda \in [0,\, 1]$. Furthermore, we take as parametrization for our initial condition $\bar{x}(t_0,\lambda) = \bar{x}_0(\lambda) = \chi_{(x_0,\tilde{x}_0)}(\lambda)$. Hence, we have that

$$-\mathcal{V}_\mathrm{i}(x(t_0), \tilde{x}(t_0)) = -\int_0^1 \mathcal{V}_\delta\big(\bar{x}(t_0,\lambda), x_\delta(t_0,\lambda)\big)\, d\lambda. \tag{B.77}$$

Combining (B.76) and (B.77) gives that

$$\begin{aligned}\mathcal{V}_\mathrm{i}\big(x(t_1), \tilde{x}(t_1)\big) - \mathcal{V}_\mathrm{i}\big(x(t_0), \tilde{x}(t_0)\big) \leq \\ \int_0^1 \mathcal{V}_\delta\big(\bar{x}(t_1,\lambda), x_\delta(t_1,\lambda)\big) - \mathcal{V}_\delta\big(\bar{x}(t_0,\lambda), x_\delta(t_0,\lambda)\big)\, d\lambda.\end{aligned} \tag{B.78}$$

This together with (B.73) implies

$$\mathcal{V}_\mathrm{i}\big(x(t_1), \tilde{x}(t_1)\big) - \mathcal{V}_\mathrm{i}\big(x(t_1), \tilde{x}(t_0)\big) \leq \int_0^1 \int_{t_0}^{t_1} s_\delta\big(w_\delta(\tau,\lambda), z_\delta(\tau,\lambda)\big)\, d\tau\, d\lambda. \tag{B.79}$$

We now consider the right-hand side of the inequality (B.79). Substituting (Q, S, R) supply function (5.12) and hanging the order of integration gives

$$\int_{t_0}^{t_1} \int_0^1 (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w_\delta(\tau,\lambda) \\ z_\delta(\tau,\lambda) \end{bmatrix} d\lambda\, d\tau. \tag{B.80}$$

We now solve the individual terms in the inner integral,

$$\int_0^1 (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w_\delta(\tau,\lambda) \\ z_\delta(\tau,\lambda) \end{bmatrix} d\lambda = \int_0^1 (\star)^\top Q w_\delta(\tau,\lambda)\, d\lambda + \\ 2\int_0^1 w_\delta(\tau,\lambda)^\top S z_\delta(\tau,\lambda)\, d\lambda + \int_0^1 (\star)^\top R z_\delta(\tau,\lambda)\, d\lambda. \quad \text{(B.81)}$$

Taking $\bar{w}(t,\lambda) = \tilde{w}(t) + \lambda(w(t) - \tilde{w}(t))$ as a parametrization, we obtain $w_\delta(t) = \frac{\partial \bar{w}(t,\lambda)}{\partial \lambda} = w(t) - \tilde{w}(t)$. Hence, the first term in (B.81) resolves to $(\star)^\top Q(w(\tau) - \tilde{w}(\tau))$, while the second term gives

$$\begin{aligned} 2(w(\tau) - \tilde{w}(\tau))^\top S \int_0^1 \frac{\partial \bar{z}(\tau,\lambda)}{\partial \lambda}\, d\lambda &= 2(w(\tau) - \tilde{w}(\tau))^\top S\, (\bar{z}(\tau,1) - \bar{z}(\tau,0)), \\ &= 2(w(\tau) - \tilde{w}(\tau))^\top S\, (z(\tau) - \tilde{z}(\tau)). \end{aligned} \quad \text{(B.82)}$$

For the third term in (B.81) where $R \preceq 0$, i.e., $-R \succeq 0$, we use Lemma C.4.1 in Appendix C.4 to obtain an upper bound:

$$\int_0^1 (\star)^\top R \frac{\partial \bar{z}(\tau,\lambda)}{\partial \lambda}\, d\lambda \leq (\star)^\top R \left(\int_0^1 \frac{\partial \bar{z}(\tau,\lambda)}{\partial \lambda}\, d\lambda \right) = \\ (\star)^\top R\, (\bar{z}(\tau,1) - \bar{z}(\tau,0)) = (\star)^\top R\, (z(\tau) - \tilde{z}(\tau)). \quad \text{(B.83)}$$

Combining these results, yields

$$\int_{t_0}^{t_1} (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w(\tau) - \tilde{w}(\tau) \\ z(\tau) - \tilde{z}(\tau) \end{bmatrix} d\tau, \quad \text{(B.84)}$$

as an upper bound for (B.80). Thus, if (5.10) holds, we know that (B.73) holds, which in turn implies, considering a supply function (5.12) with $R \preceq 0$, that

$$\mathcal{V}_\mathrm{i}\big(x(t_1), \tilde{x}(t_1)\big) - \mathcal{V}_\mathrm{i}\big(x(t_0), \tilde{x}(t_0)\big) \leq \int_{t_0}^{t_1} (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w(\tau) - \tilde{w}(\tau) \\ z(\tau) - \tilde{z}(\tau) \end{bmatrix} d\tau, \quad \text{(B.85)}$$

via the upper bound (B.84) and using (B.75) together with (B.78). Hence, if the system is differentially (Q, S, R) dissipative w.r.t. the supply function (5.12) with $R \preceq 0$, then the system is incrementally (Q, S, R) dissipative w.r.t. the equally parametrized supply function (5.14).

Proof of Lemma 5.1 (Induced incremental storage function)

Based on the proof for Theorem 5.2, starting with (B.73), we need to compute the terms

$$\int_0^1 \mathcal{V}_\delta(\bar{x}(t_1,\lambda), x_\delta(t_1,\lambda))\, d\lambda, \quad \text{(B.86a)}$$

and

$$-\int_0^1 \mathcal{V}_\delta(\bar{x}(t_0,\lambda), x_\delta(t_0,\lambda))\, d\lambda. \quad \text{(B.86b)}$$

Based on Assumption 5.1, we can decompose $M(\bar{x})$ into

$$M(\bar{x}) = N(\bar{x})^\top P N(\bar{x}), \tag{B.87}$$

where $P \in \mathbb{S}^{n_\mathrm{x}}$ with $P \succ 0$ and, because of Condition 5.1, $N(\bar{x}(t,\lambda)) \in \mathbb{R}^{n_\mathrm{x} \times n_\mathrm{x}}$ is invertible on $\mathcal{X}$, i.e., $\det N(\bar{x}) \neq 0$, $\forall\, \bar{x} \in \mathcal{X}$. Furthermore, by Assumption 5.1, there exists a diffeomorphism $\mu : \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}^{n_\mathrm{x}}$ such that $\frac{d\mu}{d\bar{x}}(\bar{x}) = N(\bar{x})$, $\forall\, \bar{x} \in \mathcal{X}$. Next, define $\bar{\nu}(t,\lambda) := \mu(\bar{x}(t,\lambda))$, resulting in

$$\nu_\delta(t,\lambda) = \frac{\partial}{\partial \lambda} \bar{\nu}(t,\lambda) = N(\bar{x}(t,\lambda)) x_\delta(t,\lambda). \tag{B.88}$$

This allows to rewrite (5.11) as

$$\mathcal{V}_\delta(\bar{x}, x_\delta) = x_\delta^\top M(\bar{x}) x_\delta = x_\delta^\top N^\top(\bar{x}) P N(\bar{x}) x_\delta = \nu_\delta^\top P \nu_\delta. \tag{B.89}$$

Using this relation, the first term (B.86a) can be written as

$$\int_0^1 \nu_\delta(t_1,\lambda)^\top P \nu_\delta(t_1,\lambda)\, d\lambda. \tag{B.90}$$

Applying Lemma C.4.1, see Appendix C.4, to (B.90) results in

$$\left(\int_0^1 \nu_\delta(t_1,\lambda)\, d\lambda \right)^\top P \left(\int_0^1 \nu_\delta(t_1,\lambda)\, d\lambda \right) \leq \int_0^1 \nu_\delta(t_1,\lambda)^\top P \nu_\delta(t_1,\lambda)\, d\lambda.$$

Hence,

$$\begin{aligned} (\star)^\top P\big(\mu(x(t_1)) - \mu(\tilde{x}(t_1))\big) = (\star)^\top P\big(\bar{\nu}(t_1,1) - \bar{\nu}(t_1,0)\big) \leq \\ \int_0^1 \nu_\delta(t_1,\lambda)^\top P \nu_\delta(t_1,\lambda)\, d\lambda = \\ \int_0^1 x_\delta(t_1,\lambda)^\top M(\bar{x}(t_1,\lambda)) x_\delta(t_1,\lambda)\, d\lambda. \end{aligned} \tag{B.91}$$

Before looking at the second term, i.e., (B.86b), let us recall some definitions. As aforementioned, the parametrized initial condition $\bar{x}(t_0,\lambda) = \bar{x}_0(\lambda)$ can be taken as any smooth parametrization $\bar{x}_0 \in \Gamma_\mathcal{X}(x_0, \tilde{x}_0)$. Recall that μ is a diffeomorphism, implying that $\mu^{-1} : \mathbb{R}^{n_\mathrm{x}} \to \mathbb{R}^{n_\mathrm{x}}$ exists and $\mu, \mu^{-1} \in \mathcal{C}_1$. Hence, w.l.o.g. we take

$$\bar{x}_0(\lambda) = \mu^{-1}(\bar{\nu}(t_0,\lambda)), \tag{B.92}$$

where $\bar{\nu}(t_0,\lambda) = \mu(\tilde{x}_0) + \lambda(\mu(x_0) - \mu(\tilde{x}_0))$. Note that this choice of $\bar{x}_0(\lambda)$ satisfies the aforementioned conditions. Consequently, we have that

$$\nu_\delta(t_0,\lambda) = \frac{\partial}{\partial \lambda} \bar{\nu}(t_0,\lambda) = \mu(x_0) - \mu(\tilde{x}_0). \tag{B.93}$$

Using this result and (B.89), the second term (B.86b) gives

$$\int_0^1 (\mu(x_0) - \mu(\tilde{x}_0))^\top P(\mu(x_0) - \mu(\tilde{x}_0))\, d\lambda = (\mu(x_0) - \mu(\tilde{x}_0))^\top P(\mu(x_0) - \mu(\tilde{x}_0)). \tag{B.94}$$

Combining the results of (B.91) and (B.94), it holds that

$$(\star)^\top P(\mu(x(t_1)) - \mu(\tilde{x}(t_1))) - (\star)^\top P(\mu(x(t_0)) - \mu(\tilde{x}(t_0))) \leq \\ \int_0^1 \mathcal{V}_\delta(\bar{x}(t_1,\lambda), x_\delta(t_1,\lambda)) - \mathcal{V}_\delta(\bar{x}(t_0,\lambda), x_\delta(t_0,\lambda))\, d\lambda, \quad \text{(B.95)}$$

where $\mathcal{V}_\delta$ is given by (5.11). Combining this result with (B.84) gives

$$\mathcal{V}_\mathrm{i}(x(t_1),\tilde{x}(t_1)) - \mathcal{V}_\mathrm{i}(x(t_0),\tilde{x}(t_0)) \leq \int_{t_0}^{t_1} (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w(\tau) - \tilde{w}(\tau) \\ z(\tau) - \tilde{z}(\tau) \end{bmatrix} d\tau, \quad \text{(B.96)}$$

where $\mathcal{V}_\mathrm{i}$ is according to (5.15). Hence, (5.15) qualifies as an incremental storage function for (5.1).

In case $M(\bar{x}) = M \in \mathbb{S}^{n_\mathrm{x}}$ for all $\bar{x} \in \mathcal{X}$ with $M \succ 0$, the decomposition in (B.87) simplifies to $N = I$ and $P = M$, hence, $\mu(x) = x$ and we obtain (5.16). Note that the same result is obtained when solving (B.74) and (B.75) directly for $\mathcal{V}_\delta(\bar{x}, x_\delta) = x_\delta^\top M x_\delta$, as in that case $\chi_{(x,\tilde{x})} = \tilde{x} + \lambda(x - \tilde{x})$ and hence $\mathcal{V}_\delta$ is given by (5.16).

Proof of Theorem 5.3 (Induced dissipativity)

If the system is incrementally (Q, S, R) dissipative w.r.t. the supply function (5.14) under the storage function $\mathcal{V}_\mathrm{i}$, then it holds that

$$\mathcal{V}_\mathrm{i}\big(x(t_1),\tilde{x}(t_1)\big) - \mathcal{V}_\mathrm{i}\big(x(t_0),\tilde{x}(t_0)\big) \leq \int_{t_0}^{t_1} (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w(t) - \tilde{w}(t) \\ z(t) - \tilde{z}(t) \end{bmatrix} dt,$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_0 \leq t_1$. Let the trajectory $(\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$ be equal to the equilibrium point $(x_*, w_*, z_*) \in \mathscr{E}$ for all $t \in \mathbb{R}_0^+$, i.e., $(\tilde{x}(t), \tilde{w}(t), \tilde{z}(t)) = (x_*, w_*, z_*)$, $\forall\, t \in \mathbb{R}_0^+$. Hence, for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_0 \leq t_1$

$$\mathcal{V}_\mathrm{i}\big(x(t_1), x_*\big) - \mathcal{V}_\mathrm{i}\big(x(t_0), x_*\big) \leq \int_{t_0}^{t_1} (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w(t) - w_* \\ z(t) - z_* \end{bmatrix} dt. \quad \text{(B.97)}$$

By Assumption 4.1, we have that for each $w_* \in \mathscr{W} = \pi_{\mathrm{w}_*} \mathscr{E}$ there is a unique $x_* \in \mathscr{X} = \pi_{\mathrm{x}_*} \mathscr{E}$, i.e., there exists a bijective map $\kappa : \mathscr{W} \to \mathscr{X}$ s.t. $x_* = \kappa(w_*)$. Defining

$$\mathcal{V}_\mathrm{s}(x, w_*) := \mathcal{V}_\mathrm{i}(x, \kappa(w_*)), \quad \text{(B.98)}$$

which satisfies that $\mathcal{V}_\mathrm{s}(\cdot, w_*) \in \mathcal{Q}_{x_*}$ for every $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$, as $\mathcal{V}_\mathrm{i} \in \mathcal{Q}_\mathrm{i}$. Substituting (B.98) in (B.97) results in holding for every $(x_*, w_*, z_*) \in \mathscr{E}$ that for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_1 \geq t_0$,

$$\mathcal{V}_\mathrm{s}(x(t_1), w_*) - \mathcal{V}_\mathrm{s}(x(t_0), w_*) \leq \int_{t_0}^{t_1} (\star)^\top \begin{bmatrix} Q & S \\ \star & R \end{bmatrix} \begin{bmatrix} w(t) - w_* \\ z(t) - z_* \end{bmatrix} dt, \quad \text{(B.99)}$$

for all $(x, w, z) \in \mathfrak{B}$, meaning that the system is universally shifted (Q, S, R) dissipative, see Definition 4.2. Note that Assumption 4.1 is satisfied if the system is incrementally asymptotically stable, see Definition 5.4, as each trajectory has to converge towards one another as $t \to \infty$ (for the same input). Hence, there cannot exist a $x_{*,1}, x_{*,2} \in \pi_{x_*} \mathscr{E}$ s.t. $(x_{*,1}, w_*), (x_{*,2}, w_*) \in \pi_{x_*,w_*} \mathscr{E}$, as $x_{*,1}$ and $x_{*,2}$ do not converge towards one another (as they are by definition equilibrium points).

By Theorem 4.2, we have that universal shifted (Q, S, R) dissipativity implies (Q, S, R) classical dissipativity, assuming that $(0, 0, 0) \in \mathscr{E}$, thereby completing the proof.

Proof of Theorem 5.5 (Incremental stability implied by incremental dissipativity)

If the system given by (5.1) is incrementally dissipative w.r.t. a supply function s_{i} for which (5.20) holds for all $w \in \mathcal{W}$ and all $z, \tilde{z} \in \mathcal{Z}$, then, it holds that

$$\mathcal{V}_{\mathrm{i}}\big(x(t_1), \tilde{x}(t_1)\big) - \mathcal{V}_{\mathrm{i}}\big(x(t_0), \tilde{x}(t_0)\big) \leq \int_{t_0}^{t_1} s_{\mathrm{i}}(w(t), w(t), z(t), \tilde{z}(t))\, dt \leq 0, \quad \text{(B.100)}$$

for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_0 \leq t_1$ and $(x, z), (\tilde{x}, \tilde{z}) \in \pi_{\mathrm{x,z}} \mathfrak{B}_{\mathrm{w}}(w)$ for all measurable and bounded $w \in \mathcal{W}^{\mathbb{R}_0^+}$. As $\mathcal{V}_{\mathrm{i}} \in \mathcal{C}_1$, this also means that

$$\frac{d}{dt}\mathcal{V}_{\mathrm{i}}\big(x(t), \tilde{x}(t)\big) \leq 0, \quad \text{(B.101)}$$

for all $t \in \mathbb{R}_0^+$ and $(x, z), (\tilde{x}, \tilde{z}) \in \pi_{\mathrm{x,z}} \mathfrak{B}_{\mathrm{w}}(w)$ for all measurable and bounded $w \in \mathcal{W}^{\mathbb{R}_0^+}$. Moreover $\mathcal{V}_{\mathrm{i}}$ also satisfies the conditions for V_{i} in Theorem 5.4, we consequently have that (5.19) is satisfied. For incremental asymptotic stability, we have that the supply function also satisfies (5.20), but with strict inequality when $z \neq \tilde{z}$. As we assume that the system is observable, see Definition 2.2, this means that $z, \tilde{z} \in \pi_{\mathrm{z}} \mathfrak{B}_{\mathrm{w}}(w)$ with $w \in \mathcal{W}^{\mathbb{R}_0^+}$, for which $z = \tilde{z}$, implies that $x = \tilde{x} \in \pi_{\mathrm{x}} \mathfrak{B}_{\mathrm{w}}(w)$. Hence, we have that (B.101) with the inequality sign being strict holds for all $t \in \mathbb{R}_0^+$ and $x, \tilde{x} \in \pi_{\mathrm{x}} \mathfrak{B}_{\mathrm{w}}(w)$ for which $x \neq \tilde{x}$.

Proof of Corollary 5.2 ($\mathcal{L}_{\mathrm{i}2}$-gain bound)

First, we show that if for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$, (5.26) holds with $M \succ 0$, the system given by (5.1) is incrementally dissipative w.r.t. the supply function

$$s_{\mathrm{i}}(w, \tilde{w}, z, \tilde{z}) = \gamma^2 w^\top w - z^\top z, \quad \text{(B.102)}$$

and storage function

$$\mathcal{V}_{\mathrm{i}}(x, \tilde{x}) = (x - \tilde{x})^\top M (x - \tilde{x}), \quad M \succ 0. \quad \text{(B.103)}$$

Applying the Schur-complement on (5.26), we obtain

$$(\star)^\top \begin{bmatrix} 0 & M \\ \star & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\delta(\bar{x}, \bar{w}) & B_\delta(\bar{x}, \bar{w}) \end{bmatrix} - (\star)^\top \begin{bmatrix} \gamma I & 0 \\ 0 & -\gamma^{-1} I \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\delta(\bar{x}, \bar{w}) & D_\delta(\bar{x}, \bar{w}) \end{bmatrix} \preceq 0, \quad \text{(B.104)}$$

and by multiplying by γ,

$$(\star)^\top \begin{bmatrix} 0 & \tilde{M} \\ \star & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ A_\delta(\bar{x},\bar{w}) & B_\delta(\bar{x},\bar{w}) \end{bmatrix} - (\star)^\top \begin{bmatrix} \gamma^2 I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} 0 & I \\ C_\delta(\bar{x},\bar{w}) & D_\delta(\bar{x},\bar{w}) \end{bmatrix} \preceq 0, \quad \text{(B.105)}$$

where $\tilde{M} = \gamma M \succ 0$. Therefore, we have the (Q, S, R)-triple $(\gamma^2 I, 0, -I)$, for which $R = -I \prec 0$, see Theorem 5.1. Hence, from Corollary 5.1, we have that the system given by (5.1) is incrementally dissipative w.r.t. the storage function (B.103), and supply function

$$s_\mathrm{i}(w, \tilde{w}, z, \tilde{z}) = \begin{bmatrix} w - \tilde{w} \\ z - \tilde{z} \end{bmatrix}^\top \begin{bmatrix} \gamma^2 I & 0 \\ \star & -I \end{bmatrix} \begin{bmatrix} w - \tilde{w} \\ z - \tilde{z} \end{bmatrix}, \quad \text{(B.106)}$$

which is equivalent to the supply function (B.102). Therefore, if (5.26) holds with $M \succ 0$, the system given by (5.1) is incrementally dissipative w.r.t. supply function (B.102). Among the lines of (Van der Schaft 2017, Prop. 3.1.7), we now show that this implies a bounded $\mathcal{L}_{\mathrm{i}2}$-gain of γ for (5.1).

Note that if the system is incrementally dissipative w.r.t. (B.102), meaning there exists a $\mathcal{V}_\mathrm{i}$ such that for all $t \geq 0$ and $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$ it holds that

$$-\mathcal{V}_\mathrm{i}(x(0), \tilde{x}(0)) \leq \mathcal{V}_\mathrm{i}(x(t), \tilde{x}(t)) - \mathcal{V}_\mathrm{i}(x(0), \tilde{x}(0)) \leq \int_0^t \gamma^2 \left\| w(\tau) - \tilde{w}(\tau) \right\|^2 - \left\| z(\tau) - \tilde{z}(\tau) \right\|^2 \, d\tau. \quad \text{(B.107)}$$

which is equivalent to

$$\int_0^t \left\| z(\tau) - \tilde{z}(\tau) \right\|^2 \, d\tau \leq \gamma^2 \int_0^t \left\| w(\tau) - \tilde{w}(\tau) \right\|^2 \, d\tau + \mathcal{V}_\mathrm{i}(x(0), \tilde{x}(0)), \quad \text{(B.108)}$$

holding for all $t \geq 0$ and $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$. Using the $\mathcal{L}_{2\mathrm{e}}$ signal norm definition, see Definition 2.10, this is equivalent to

$$\left\| z - \tilde{z} \right\|_{2,T}^2 \leq \gamma^2 \left\| w - \tilde{w} \right\|_{2,T}^2 + \mathcal{V}_\mathrm{i}(x(0), \tilde{x}(0)), \quad \text{(B.109)}$$

holding for all $T \geq 0$ and $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$ with $w, \tilde{w} \in \mathcal{L}_{2\mathrm{e}}$. Taking the square root on both sides, we obtain

$$\left\| z - \tilde{z} \right\|_{2,T} \leq \sqrt{\gamma^2 \left\| w - \tilde{w} \right\|_{2,T}^2 + \mathcal{V}_\mathrm{i}(x(0), \tilde{x}(0))} \leq \gamma \left\| w - \tilde{w} \right\|_{2,T} + \sqrt{\mathcal{V}_\mathrm{i}(x(0), \tilde{x}(0))}, \quad \text{(B.110)}$$

which is equivalent to (5.25) with $\zeta_\mathrm{i}(x_0, \tilde{x}_0) = \sqrt{\mathcal{V}_\mathrm{i}(x_0, \tilde{x}_0)}$.

Proof of Corollary 5.3 ($\mathcal{L}_{\mathrm{i}\infty}$-gain bound)

The concept of this proof is based on (Scherer 2000, Section 10.3) and (Scherer and Weiland 2015, Section 3.3.5). Pre- and post-multiplying (5.27a) with $\mathrm{col}(x_\delta, w_\delta)^\top$ and $\mathrm{col}(x_\delta, w_\delta)$, respectively, yields that, omitting dependence on t, λ for brevity,

$$\dot{x}_\delta x^\top M x_\delta + x_\delta^\top M \dot{x}_\delta + \beta x_\delta^\top M x_\delta - \alpha w_\delta^\top w_\delta \leq 0, \quad \text{(B.111)}$$

For a storage function $\mathcal{V}_\delta$ of the form (5.11) (and by abusing notation by omitting the $\bar{x}$ argument), (B.111) is equivalent to

$$\frac{d}{dt}\Big(\mathcal{V}_\delta(x_\delta(t,\lambda))\Big) + \beta\mathcal{V}_\delta(x_\delta(t,\lambda)) - \alpha\left\|w_\delta(t,\lambda)\right\|^2 \leq 0. \tag{B.112}$$

Next, we have that (B.112), by Grönwall's Lemma (Khalil 2002, Lemma A.1), implies that

$$\mathcal{V}_\delta(x_\delta(t,\lambda)) \leq e^{-\beta t}\mathcal{V}_\delta(x_\delta(0,\lambda)) + \alpha\int_0^t e^{-\beta(t-\tau)}\left\|w_\delta(\tau,\lambda)\right\|^2 d\tau. \tag{B.113}$$

Moreover, we can define an upper bound for the integral in (B.113) as follows,

$$\alpha\int_0^t e^{-\beta(t-\tau)}\left\|w_\delta(\tau,\lambda)\right\|^2 d\tau \leq \alpha\left\|w_\delta(\lambda)\right\|_{\infty,t}^2\int_0^t e^{-\beta(t-\tau)}\,d\tau \leq \frac{\alpha}{\beta}\left\|w_\delta(\lambda)\right\|_{\infty,t}^2. \tag{B.114}$$

Furthermore, it holds that,

$$e^{-\beta t}\mathcal{V}_\delta(x_\delta(0,\lambda)) \leq \mathcal{V}_\delta(x_\delta(0,\lambda)), \tag{B.115}$$

for all $t \geq 0$ and $\lambda \in [0,\, 1]$. Therefore, by combining the result of (B.114) and (B.115), it holds that for all $t \geq 0$ and $\lambda \in [0,\, 1]$

$$\beta\mathcal{V}_\delta(x_\delta(t,\lambda)) \leq \beta\mathcal{V}_\delta(x_\delta(0,\lambda)) + \alpha\left\|w_\delta(\lambda)\right\|_{\infty,t}^2. \tag{B.116}$$

As we have that

$$x_\delta(0,\lambda) = \frac{\partial\bar{x}_0(\lambda)}{\partial\lambda} = \frac{\partial}{\partial\lambda}(\tilde{x}_0 + \lambda(x_0 - \tilde{x}_0)) = x_0 - \tilde{x}_0, \tag{B.117}$$

and as $\mathcal{V}_\delta$ is of the form (5.11), we have that

$$\mathcal{V}_\delta(x_\delta(0,\lambda)) = x_\delta(0,\lambda)^\top M x_\delta(0,\lambda) = (x_0 - \tilde{x}_0)^\top M(x_0 - \tilde{x}_0) = \mathcal{V}_{\mathrm{i}}(x_0,\tilde{x}_0). \tag{B.118}$$

Moreover, as $\beta\mathcal{V}_\delta(x_\delta(t,\lambda)) \geq 0$, it then follows from (B.116) that the following is true,

$$\begin{aligned}\int_0^1 \beta\mathcal{V}_\delta(x_\delta(t,\lambda))\,d\lambda = \beta\int_0^1 x_\delta(t,\lambda)^\top M x_\delta(t,\lambda)\,d\lambda \leq \\ \int_0^1 \beta\mathcal{V}_\delta(x_\delta(0,\lambda)) + \alpha\left\|w_\delta(\lambda)\right\|_\infty^2 d\lambda = \beta\mathcal{V}_{\mathrm{i}}(x_0,\tilde{x}_0) + \alpha\left\|w - \tilde{w}\right\|_{\infty,t}^2.\end{aligned} \tag{B.119}$$

The latter equality comes from (B.118) and the fact that $w_\delta(\lambda) = \frac{\partial\bar{w}}{\partial\lambda} = w - \tilde{w}$. We will use the bound in (B.119) later in the proof.

Consider now the second inequality (5.27b), which can be rewritten using the Schur complement as

$$\gamma^{-1}(\star)^\top\begin{bmatrix} C_\delta(\bar{x},\bar{w}) & D(\bar{x},\bar{w})\end{bmatrix} \preceq \begin{bmatrix}\beta M & 0 \\ 0 & (\gamma-\alpha)I\end{bmatrix}.$$

Pre- and post-multiplying the latter with $\mathrm{col}(x_\delta, w_\delta)^\top$ and $\mathrm{col}(x_\delta, w_\delta)$, respectively, gives that,

$$\gamma^{-1} z_\delta(t,\lambda)^\top z_\delta(t,\lambda) \leq \beta\, x_\delta(t,\lambda)^\top M x_\delta(t,\lambda) + (\gamma-\alpha) w_\delta(t,\lambda)^\top w_\delta(t,\lambda)^\top. \quad \text{(B.120)}$$

As (B.120) only consists of positive terms, we know the following is true as well,

$$\int_0^1 \gamma^{-1} z_\delta(t,\lambda)^\top z_\delta(t,\lambda)\, d\lambda \leq \int_0^1 \beta\; x_\delta(t,\lambda)^\top M x_\delta(t,\lambda) + (\gamma-\alpha) w_\delta(t,\lambda)^\top w_\delta(t,\lambda)\, d\lambda. \quad \text{(B.121)}$$

Using Lemma C.4.1, we know that the left-hand side of the inequality in (B.121), is bounded from below by

$$\gamma^{-1} \left(\int_0^1 z_\delta(t,\lambda)\, d\lambda\right)^\top \left(\int_0^1 z_\delta(t,\lambda)\, d\lambda\right) = \gamma^{-1}(\star)^\top (z(t) - \tilde{z}(t)). \quad \text{(B.122)}$$

Again due to the selection of the path parametrization of $\bar{w}(\lambda) = \tilde{w} + \lambda(w - \tilde{w})$, we have that the last term in (B.121) is not dependent on λ, as $w_\delta(t,\lambda) = w(t) - \tilde{w}(t)$. Hence, we have that (B.121) can be rewritten to

$$\gamma^{-1} \left\| z(t) - \tilde{z}(t) \right\|^2 \leq \beta \int_0^1 x_\delta(t,\lambda)^\top M x_\delta(t,\lambda)\, d\lambda + (\gamma-\alpha) \left\| w(t) - \tilde{w}(t) \right\|^2. \quad \text{(B.123)}$$

By substituting the inequality (B.119) in the inequality (B.123), we obtain the following relationship,

$$\gamma^{-1} \left\| z(t) - \tilde{z}(t) \right\|^2 \leq \beta \mathcal{V}_\mathrm{i}(x_0, \tilde{x}_0) + \alpha \left\| w - \tilde{w} \right\|_{\infty,t}^2 + (\gamma - \alpha) \left\| w(t) - \tilde{w}(t) \right\|^2.$$

By taking the supremum over all $t \in [0, T]$ for $T > 0$, we infer

$$\left\| z - \tilde{z} \right\|_{\infty,T}^2 \leq \gamma^2 \left\| w - \tilde{w} \right\|_{\infty,T}^2 + \gamma\beta \mathcal{V}_\mathrm{i}(x_0, \tilde{x}_0).$$

Therefore, for all $T \geq 0$ and $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$ with $w, \tilde{w} \in \mathcal{L}_{\infty\mathrm{e}}$

$$\left\| z - \tilde{z} \right\|_{\infty,T} \leq \gamma \left\| w - \tilde{w} \right\|_{\infty,T} + \sqrt{\gamma\beta \mathcal{V}_\mathrm{i}(x_0, \tilde{x}_0)}.$$

Which gives us the definition of the incremental $\mathcal{L}_\infty$-gain, see Definition 5.5, with $\zeta_\mathrm{i}(x_0, \tilde{x}_0) = \sqrt{\gamma\beta\mathcal{V}_\mathrm{i}(x_0, \tilde{x}_0)}$, proving the claim.

Proof of Corollary 5.5 ($\mathcal{L}_{\mathrm{i}2}$-$\mathcal{L}_{\mathrm{i}\infty}$-gain bound)

The concept of this proof is based on (Scherer and Weiland 2015, Section 3.3.4). We have that the first matrix inequality (5.30a) is equivalent to the condition for differential dissipativity of a system with the supply function

$$s_\delta(w_\delta(t,\lambda), z_\delta(t,\lambda)) = \gamma \left\| w_\delta(t,\lambda) \right\|^2 = \gamma \left\| \frac{\partial \bar{w}(t,\lambda)}{\partial \lambda} \right\|^2, \quad \text{(B.124)}$$

as a result from Theorem 5.1. Therefore, it holds from (5.10) for all $t \geq 0$ that

$$\underbrace{x_\delta(t,\lambda)^\top M x_\delta(t,\lambda)}_{=\mathcal{V}_\delta(x_\delta(t,\lambda))} - \underbrace{x_\delta(0,\lambda)^\top M x_\delta(0,\lambda)}_{=\mathcal{V}_\delta(x_\delta(0,\lambda))} \leq \gamma \int_0^t \|w(\tau) - \tilde{w}(\tau)\|^2 \, d\tau, \tag{B.125}$$

using the fact that $\frac{\partial \bar{u}(t,\lambda)}{\partial \lambda} = \frac{\partial}{\partial \lambda}(\tilde{w}(t) + \lambda(w(t) - \tilde{w}(t))) = w(t) - \tilde{w}(t)$. Moreover, as here also (B.117) holds, we have that (B.118) holds. Furthermore, as $\mathcal{V}_\delta(x_\delta(t,\lambda)) \geq 0$, we have from (B.125) that the following holds true as well

$$\int_0^1 \mathcal{V}_\delta(x_\delta(t,\lambda)) \, d\lambda \leq \int_0^1 \left(\mathcal{V}_\delta(x_\delta(0,\lambda)) + \gamma \int_0^t \|w(\tau) - \tilde{w}(\tau)\|^2 \, d\tau \right) d\lambda,$$

that is, using (B.118),

$$\int_0^1 \mathcal{V}_\delta(x_\delta(t,\lambda)) \, d\lambda \leq \mathcal{V}_\mathrm{i}(x_0, \tilde{x}_0) + \gamma \int_0^t \|w(\tau) - \tilde{w}(\tau)\|^2 \, d\tau. \tag{B.126}$$

Consider now the second matrix inequality (5.30b), which can be rewritten using the Schur complement, such that for $\gamma > 0$ it holds that

$$M - \gamma^{-1} C_\delta(\bar{x}, \bar{w})^\top C_\delta(\bar{x}, \bar{w}) \succeq 0. \tag{B.127}$$

The matrix inequality (B.127) can be rewritten to

$$C_\delta(\bar{x}, \bar{w})^\top C_\delta(\bar{x}, \bar{w}) \preceq \gamma M. \tag{B.128}$$

Pre- and post-multiplication of (B.128) with $x_\delta(t,\lambda)^\top$ and $x_\delta(t,\lambda)$, respectively, gives

$$z_\delta(t,\lambda)^\top z_\delta(t,\lambda) \leq \gamma \mathcal{V}_\delta(x_\delta(t,\lambda)), \tag{B.129}$$

as $D_\delta(\bar{x}, \bar{w}) = 0$ for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$. Clearly, if (B.129) holds, we know that

$$\int_0^1 z_\delta(t,\lambda)^\top z_\delta(t,\lambda) \, d\lambda \leq \gamma \int_0^1 \mathcal{V}_\delta(x_\delta(t,\lambda)) \, d\lambda, \tag{B.130}$$

holds. As in the proof for the $\mathcal{L}_{\mathrm{i}\infty}$-gain, we use Lemma C.4.1 to obtain a lower bound for the left-hand side of (B.130). The lower bound is

$$\left(\int_0^1 z_\delta(t,\lambda) \, d\lambda \right)^\top \left(\int_0^1 z_\delta(t,\lambda) \, d\lambda \right) = (z(t) - \tilde{z}(t))^\top (z(t) - \tilde{z}(t)) = \|z(t) - \tilde{z}(t)\|^2,$$

and gives that (B.130) implies that for all $t \geq 0$

$$\|z(t) - \tilde{z}(t)\|^2 \leq \gamma \int_0^1 \mathcal{V}_\delta(x_\delta(t,\lambda)) \, d\lambda. \tag{B.131}$$

Combining the result of (B.126) and (B.131) gives that for all $t \geq 0$

$$\|z(t) - \tilde{z}(t)\|^2 \leq \gamma \mathcal{V}_\mathrm{i}(x_0, \tilde{x}_0) + \gamma^2 \int_0^t \|w(\tau) - \tilde{w}(\tau)\|^2 \, d\tau. \tag{B.132}$$

Taking the supremum over $t \in [0, T]$ gives that for all $T \geq 0$

$$\|z - \tilde{z}\|_{\infty,T}^2 \leq \gamma \mathcal{V}_\mathrm{i}(x_0, \tilde{x}_0) + \gamma^2 \|w - \tilde{w}\|_{2,T}^2 . \tag{B.133}$$

Therefore, from (B.133), it also follows that

$$\|z - \tilde{z}\|_{\infty,T} \leq \gamma \|w - \tilde{w}\|_{2,T} + \sqrt{\gamma \mathcal{V}_\mathrm{i}(x_0, \tilde{x}_0)},$$

for all $T \geq 0$ and $(x, w, z), (\tilde{x}, \tilde{w}, \tilde{z}) \in \mathfrak{B}$ with $w, \tilde{w} \in \mathcal{L}_{2\mathrm{e}}$, giving us the definition of $\mathcal{L}_{\mathrm{i}2}$-$\mathcal{L}_{\mathrm{i}\infty}$-gain where $\zeta_\mathrm{i}(x_0, \tilde{x}_0) = \sqrt{\gamma \mathcal{V}_\mathrm{i}(x_0, \tilde{x}_0)}$, corresponding to incremental generalized $\mathcal{H}_2$ performance, proving the statement.

Proof of Lemma 5.3 (DPV behavioral embedding)

As the LPV representation (5.31) is a *Differential Parameter-Varying* (DPV) embedding of the system given by (5.1) on the region $\mathcal{X} \times \mathscr{W} = \mathcal{X} \times \mathcal{W}$, for any trajectory $(x_\delta, w_\delta, z_\delta) \in \mathfrak{B}_\delta(\bar{x}, \bar{w})$ and any $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,w}} \mathfrak{B}$, we also have that $(x_\delta, w_\delta, z_\delta) \in \mathfrak{B}_\mathrm{p}(\eta(\bar{x}, \bar{w}))$. Moreover, as $\mathcal{X} \times \mathscr{W} = \mathcal{X} \times \mathcal{W}$ and $\eta(\mathcal{X}, \mathscr{W}) \subseteq \mathcal{P}$, we get the following relation

$$\breve{\mathfrak{B}}_\delta = \bigcup_{(\bar{x},\bar{w}) \in \pi_{\mathrm{x,w}} \mathfrak{B}} \mathfrak{B}_\delta(\bar{x}, \bar{w}) \subseteq \bigcup_{(\bar{x},\bar{w}) \in (\mathcal{X}, \mathcal{W})^\mathbb{R}} \mathfrak{B}_\mathrm{p}(\eta(\bar{x}, \bar{w})) \subseteq \bigcup_{p \in \mathcal{P}^\mathbb{R}} \mathfrak{B}_\mathrm{p}(p) = \breve{\mathfrak{B}}_\mathrm{p}. \tag{B.134}$$

Proof of Theorem 5.7 (Incremental dissipativity through the LPV framework)

Through the DPV embedding (5.31) on the region $\mathcal{X} \times \mathscr{W} = \mathcal{X} \times \mathcal{W}$ and by Lemma 5.3 we have that we describe the full behavior of the differential form of (5.1), given by (5.7). Hence, if the LPV representation (5.31) is classically dissipative, i.e. for all $p \in \mathcal{P}^\mathbb{R}$, then the differential form (5.7) is classically dissipative, i.e., for all $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,w}} \mathfrak{B}$, corresponding to system given by (5.1) being differential (Q, S, R) dissipative. As $R \preceq 0$ this then implies by Theorem 5.2 incremental (Q, S, R) dissipativity of (5.1).

B.4 Proofs of Chapter 6

Proof of Theorem 6.1 (Closed-loop differential form)

The proof follows in the same manner as the proof of Theorem 4.9 regarding the velocity form, where $\dot{x}$, $\dot{x}_\mathrm{k}$, $\dot{u}$, $\dot{u}_\mathrm{k}$, $\dot{y}$, and $\dot{y}_\mathrm{k}$ then become x_δ, $x_{\delta,\mathrm{k}}$, u_δ, $u_{\delta,\mathrm{k}}$, y_δ, and $y_{\delta,\mathrm{k}}$, respectively, and P_v and K_v become P_δ and K_δ. Therefore, the proof is not repeated.

Proof of Theorem 6.2 (Differential closed-loop $\mathcal{L}_2$-gain)

By synthesis, we obtain a controller K_δ (6.9) such that the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P_\mathrm{dpv}, K_\delta)$ has a bounded $\mathcal{L}_2$-gain of γ for all $p \in \mathcal{P}^\mathbb{R}$. Moreover, as P_dpv is a DPV embedding on the region $\mathcal{X} \subseteq \mathcal{X}$ we have through Lemma 5.3 (see also Remark 5.6) that $\breve{\mathfrak{B}}_{\delta,\mathcal{X}} \subseteq \breve{\mathfrak{B}}_\mathrm{p}$. Consequently, these two results imply that $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$ with $p = \eta(\bar{x})$ for K_δ is $\mathcal{L}_2$-gain stable with a $\mathcal{L}_2$-gain $\leq \gamma$ for all $\bar{x} \in \pi_\mathrm{x}\, \mathfrak{B}_\mathcal{X}$.

Proof of Theorem 6.3 (Incremental controller realization)

Based on the definition of the differential variables[2] we have that $x_{\delta,\mathrm{k}}(t) = \frac{\partial}{\partial\lambda}\bar{x}_\mathrm{k}(t,\lambda)$, $u_{\delta,\mathrm{k}}(t,\lambda) = \frac{\partial}{\partial\lambda}\bar{u}_\mathrm{k}(t,\lambda)$, $y_{\delta,\mathrm{k}}(t,\lambda) = \frac{\partial}{\partial\lambda}\bar{y}_\mathrm{k}(t,\lambda)$. The family of parameterized trajectories is defined as $(\bar{x}_\mathrm{k}(\lambda), \bar{u}_\mathrm{k}(\lambda), \bar{y}_\mathrm{k}(\lambda))$ with $\lambda \in [0,\,1]$ such that $(\bar{x}_\mathrm{k}(1), \bar{u}_\mathrm{k}(1), \bar{y}_\mathrm{k}(1)) = (x_\mathrm{k}, u_\mathrm{k}, y_\mathrm{k})$ is the current trajectory and $(\bar{x}_\mathrm{k}(0), \bar{u}_\mathrm{k}(0), \bar{y}_\mathrm{k}(0)) = (x_\mathrm{k}^*, u_\mathrm{k}^*, y_\mathrm{k}^*)$ is the steady-state trajectory. Consequently,

$$y_\mathrm{k}(t) = y_\mathrm{k}^*(t) + \int_0^1 \frac{\partial}{\partial\lambda}\bar{y}_\mathrm{k}(t,\lambda)\, d\lambda, \tag{B.135a}$$

$$= y_\mathrm{k}^*(t) + \int_0^1 y_{\delta,\mathrm{k}}(t,\lambda)\, d\lambda. \tag{B.135b}$$

Based on K_δ, in terms of (6.9b), we get

$$y_\mathrm{k}(t) = y_\mathrm{k}^*(t) + \int_0^1 C_\mathrm{k}(\eta(\bar{x}(t,\lambda))x_{\delta,\mathrm{k}}(t,\lambda) + D_\mathrm{k}(\eta(\bar{x}(t,\lambda))u_{\delta,\mathrm{k}}(t,\lambda)\, d\lambda. \tag{B.136}$$

The closed-loop differential storage function is $V(x_\mathrm{cl}, x_{\delta,\mathrm{cl}}) = x_{\delta,\mathrm{cl}}^\top M x_{\delta,\mathrm{cl}}$ with $M \succ 0$, corresponding to a constant Riemannian metric. Hence, the homotopy path connecting $x_\mathrm{cl}(t)$ and $x_\mathrm{cl}^*(t)$ can be considered a straight line, i.e., by

$$\bar{x}_\mathrm{cl}(t,\lambda) = x_\mathrm{cl}^*(t) + \lambda(x_\mathrm{cl}(t) - x_\mathrm{cl}^*(t)), \tag{B.137}$$

see (Manchester and Slotine 2018) and Lemma 5.1. Therefore, $\bar{x}(t,\lambda) = x^*(t) + \lambda(x(t) - x^*(t))$ and $\bar{x}_\mathrm{k}(t,\lambda) = x_\mathrm{k}^*(t) + \lambda(x_\mathrm{k}(t) - x_\mathrm{k}^*(t))$. This implies that $x_{\delta,\mathrm{k}}(t,\lambda) = \frac{\partial}{\partial\lambda}\bar{x}_\mathrm{k}(t,\lambda) = \frac{\partial}{\partial\lambda}\left(x_\mathrm{k}^*(t) + \lambda(x_\mathrm{k}(t) - x_\mathrm{k}^*(t))\right) = x_\mathrm{k}(t) - x_\mathrm{k}^*(t) = x_{\Delta,\mathrm{k}}(t)$ and similarly

[2] See Chapter 5 for more details.

$x_\delta(t,\lambda) = x(t) - x^*(t)$. Furthermore, define the parameterized trajectory $\bar{w}(t,\lambda) = w^*(t) + \lambda(w(t) - w^*(t))$, such that $w_\delta(t,\lambda) := \frac{\partial}{\partial\lambda}\bar{w}(t,\lambda) = w(t) - w^*(t)$. Hence, as $u_{\delta,\mathrm{k}}(t,\lambda) = y_\delta(t,\lambda) = C_\mathrm{y} x_\delta(t,\lambda) + D_{\mathrm{yw}} w_\delta(t,\lambda)$ is linear in x_δ and w_δ, and as $w_\delta(t,\lambda) = w(t) - w^*(t)$ and $x_\delta(t,\lambda) = x(t) - x^*(t)$, we obtain $u_{\delta,\mathrm{k}}(t,\lambda) = u_\mathrm{k}(t) - u_\mathrm{k}^*(t) = u_{\Delta,\mathrm{k}}(t)$. Using these relations for the differential state equation of K_δ in (6.9a), filling these relations in (B.136), and using that $\bar{p}(t,\lambda) = \eta(\bar{x}(t,\lambda))$, result in

$$\dot{x}_{\Delta,\mathrm{k}}(t) = \left(\int_0^1 A_\mathrm{k}(\bar{p}(t,\lambda))\,d\lambda\right) x_{\Delta,\mathrm{k}}(t) + \left(\int_0^1 B_\mathrm{k}(\bar{p}(t,\lambda))\,d\lambda\right) u_{\Delta,\mathrm{k}}(t); \quad \text{(B.138a)}$$

$$y_\mathrm{k}(t) = y_\mathrm{k}^*(t) + \left(\int_0^1 C_\mathrm{k}(\bar{p}(t,\lambda))\,d\lambda\right) x_{\Delta,\mathrm{k}}(t) + \left(\int_0^1 D_\mathrm{k}(\bar{p}(t,\lambda))\,d\lambda\right) u_{\Delta,\mathrm{k}}(t); \quad \text{(B.138b)}$$

giving us K (6.10). Next, it is shown that the differential form of K (6.10) is K_δ (6.9). Based on (B.138) define:

$$\dot{\bar{x}}_\mathrm{k}(t,\lambda) = \dot{x}_\mathrm{k}^*(t) + \left(\int_0^\lambda A_\mathrm{k}(\bar{p}(t,\lambda))\,d\lambda\right) x_{\Delta,\mathrm{k}}(t) + \left(\int_0^\lambda B_\mathrm{k}(\bar{p}(t,\lambda))\,d\lambda\right) u_{\Delta,\mathrm{k}}(t); \quad \text{(B.139a)}$$

$$\bar{y}_\mathrm{k}(t,\lambda) = y_\mathrm{k}^*(t) + \left(\int_0^\lambda C_\mathrm{k}(\bar{p}(t,\lambda))\,d\lambda\right) x_{\Delta,\mathrm{k}}(t) + \left(\int_0^\lambda D_\mathrm{k}(\bar{p}(t,\lambda))\,d\lambda\right) u_{\Delta,\mathrm{k}}(t); \quad \text{(B.139b)}$$

Differentiating (B.139) w.r.t. λ, we obtain

$$\frac{\partial}{\partial\lambda}\dot{\bar{x}}_\mathrm{k}(t,\lambda) = A_\mathrm{k}(\bar{p}(t,\lambda))x_{\delta,\mathrm{k}}(t) + B_\mathrm{k}(\bar{p}(t,\lambda))u_{\delta,\mathrm{k}}(t); \quad \text{(B.140a)}$$

$$\frac{\partial}{\partial\lambda}\bar{y}_\mathrm{k}(t,\lambda) = C_\mathrm{k}(\bar{p}(t,\lambda))x_{\delta,\mathrm{k}}(t) + D_\mathrm{k}(\bar{p}(t,\lambda))u_{\delta,\mathrm{k}}(t). \quad \text{(B.140b)}$$

Then, using that $u_{\delta,\mathrm{k}}(t) = u_{\delta,\mathrm{k}}(t,\lambda)$, $x_{\delta,\mathrm{k}}(t) = x_{\delta,\mathrm{k}}(t,\lambda)$, $x_{\delta,\mathrm{k}}(t,\lambda) = \frac{\partial}{\partial\lambda}\bar{x}_\mathrm{k}(t,\lambda)$ and $y_{\delta,\mathrm{k}}(t,\lambda) = \frac{\partial}{\partial\lambda}\bar{y}_\mathrm{k}(t,\lambda)$, we get

$$\delta\dot{x}(t) = A_\mathrm{k}(p(t))x_{\delta,\mathrm{k}}(t) + B_\mathrm{k}(p(t))u_{\delta,\mathrm{k}}(t); \quad \text{(B.141a)}$$

$$\delta y(t) = C_\mathrm{k}(p(t))x_{\delta,\mathrm{k}}(t) + D_\mathrm{k}(p(t))u_{\delta,\mathrm{k}}(t); \quad \text{(B.141b)}$$

which is K_δ (6.9), completing the proof.

Proof of Theorem 6.4 (Closed-loop $\mathcal{L}_{\mathrm{i}2}$-gain stability)

By Theorem 6.2, it holds that K_δ ensures $\mathcal{L}_2$-gain stability with a bounded $\mathcal{L}_2$-gain γ for $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$ for all $\bar{x} \in \pi_\mathrm{x}\,\mathfrak{B}_\mathcal{X}$. Furthermore, by Theorem 6.3, the differential form of K given by (6.10) is equal to K_δ given by (6.9). Consequently, by Theorem 6.1, the differential form of $\mathcal{F}_\mathrm{l}(P,K)$ is given by $\mathcal{F}_\mathrm{l}(P_\delta, K_\delta)$. Moreover, we consider the set $\tilde{\mathcal{W}} \subseteq \mathcal{W}$, for which $\mathcal{X}_\mathrm{cl} = \mathcal{X} \times \mathcal{X}_\mathrm{k}$ is invariant, meaning that for any $w, w^* \in \tilde{\mathcal{W}}^{\mathbb{R}_0^+}$, the resulting $x(t), x^*(t) \in \mathcal{X},\ \forall t \in \mathbb{R}_0^+$. Hence, we will remain in the design set on which $\mathcal{L}_2$-gain stability of the differential form is ensured. Based on Theorem 5.2, this then implies that $\mathcal{F}_\mathrm{l}(P, K)$ is incrementally dissipative w.r.t. the $\mathcal{L}_{\mathrm{i}2}$-gain supply function, hence, there exists a function $\zeta_\mathrm{i} : \mathcal{X}_\mathrm{cl} \times \mathcal{X}_\mathrm{cl} \to \mathbb{R}$ s.t.

$$\left\|\mathcal{F}_\mathrm{l}(P,K)(w, x_{\mathrm{cl},0}) - \mathcal{F}_\mathrm{l}(P,K)(w^*, x^*_{\mathrm{cl},0})\right\|_{2,T} \leq \gamma \left\|w - w^*\right\|_{2,T} + \zeta_\mathrm{i}(x_{\mathrm{cl},0}, x^*_{\mathrm{cl},0}), \tag{B.142}$$

for all $T \geq 0$, $x_{\mathrm{cl},0}, \tilde{x}_{\mathrm{cl},0} \in \mathcal{X}_\mathrm{cl}$ and any $w, w^* \in \tilde{\mathcal{W}}^{\mathbb{R}_0^+}$ with $w - w^* \in \mathcal{L}_{2\mathrm{e}}$, which implies (6.14).

As $\mathcal{F}_\mathrm{l}(P, K)$ is incrementally dissipative (for the $\mathcal{L}_{\mathrm{i}2}$-gain supply function), it is also incrementally stable based on Theorem 5.5. The differential storage function is given by $\mathcal{V}_\delta(x_\mathrm{cl}, x_{\delta,\mathrm{cl}}) = x_{\delta,\mathrm{cl}}^\top M x_{\delta,\mathrm{cl}}$, which implies that the incremental storage function is given by $\mathcal{V}_\mathrm{i}(x_\mathrm{cl}, x^*_\mathrm{cl}) = (x_\mathrm{cl} - x^*_\mathrm{cl})^\top M (x_\mathrm{cl} - x^*_\mathrm{cl})$, see Lemma 5.1. The latter also qualifies as an incremental Lyapunov function, see Theorem 5.5.

Moreover, the (desired) steady-state trajectory $\vartheta \in \mathfrak{B}_\mathcal{X}$ is a valid solution of P with corresponding $(x^*_\mathrm{cl}, w^*, z^*) \in (\mathcal{X}_\mathrm{cl} \times \mathcal{W} \times \mathcal{Z})^{\mathbb{R}_0^+}$ due to the well-posedness of $\mathcal{F}_\mathrm{l}(P, K)$. Consequently, this implies by (Rüffer et al. 2013, Theorem 11) that all solutions converge towards $(x^*_\mathrm{cl}, w^*, z^*)$. Meaning, for all $w \in \tilde{\mathcal{W}}^{\mathbb{R}_0^+}$, when $w(t) \to w^*(t)$ as $t \to \infty$, $(x_\mathrm{cl}(t), w(t), z(t)) \to (x^*_\mathrm{cl}(t), w^*(t), z^*(t))$ as $t \to \infty$.

Proof of Theorem 6.5 (Nonlinear observer)

Define $F_\mathrm{e}(x_\mathrm{e}, \hat{x}_\mathrm{e}, u, w_\mathrm{m}) = f_\mathrm{e}(\hat{x}_\mathrm{e}, u, w_\mathrm{m}) + LC_\mathrm{e}(x_\mathrm{e}(t) - \hat{x}_\mathrm{e}(t))$ and $H_\mathrm{e}(x_\mathrm{e}, \hat{x}_\mathrm{e}, w_\mathrm{m}) = C_\mathrm{e}\hat{x}_\mathrm{e}(t) + D_\mathrm{yw}w_\mathrm{m}(t)$. As $F_\mathrm{e}(x_\mathrm{e}, x_\mathrm{e}, u, w_\mathrm{m}) = f_\mathrm{e}(x_\mathrm{e}, u, w_\mathrm{m})$ and $H_\mathrm{e}(x_\mathrm{e}, x_\mathrm{e}, w_\mathrm{m}) = C_\mathrm{e}x_\mathrm{e}(t) + D_\mathrm{yw}w_\mathrm{m}(t))$, we have that (6.17) is a virtual system of (6.16), see also (W. Wang and Slotine 2005; Jouffroy and Fossen 2010). The virtual system given by (6.17) is virtually contractive, meaning that $\hat{x}_\mathrm{e}(t) \to x_\mathrm{e}(t)$ for $t \to \infty$, see (W. Wang and Slotine 2005; Reyes-Báez 2019), if

$$\dot{\hat{x}}_{\delta,\mathrm{e}}(t) = \left(\frac{\partial f_\mathrm{e}}{\partial \hat{x}_\mathrm{e}}(\hat{x}_\mathrm{e}(t), u(t), w_\mathrm{m}(t))\right)\hat{x}_{\delta,\mathrm{e}}(t) - LC_\mathrm{e}\hat{x}_{\delta,\mathrm{e}}(t), \tag{B.143}$$

is asymptotically stable. The differential form of the virtual system given by (B.143) can be written as

$$\dot{\hat{x}}_{\delta,\mathrm{e}}(t) = \big(A_{\delta,\mathrm{e}}(\hat{x}_\mathrm{e}(t), u(t), w_\mathrm{m}(t)) - LC_\mathrm{e}\big)\hat{x}_{\delta,\mathrm{e}}(t). \tag{B.144}$$

The system given by (B.144) is asymptotically stable with (differential) Lyapunov function $V_\delta(\hat{x}_{\delta,\mathrm{e}}) = \hat{x}_{\delta,\mathrm{e}}^\top P \hat{x}_{\delta,\mathrm{e}}$ if (6.18) holds for all $(x_\mathrm{e}, u, w_\mathrm{m}) \in \mathcal{X}_\mathrm{e} \times \mathcal{U} \times \pi_{w_\mathrm{m}} \mathcal{W}$.

B.5 Proofs of Chapter 7

Proof of Theorem 7.1 (DT differential (Q, S, R) dissipativity condition)

The proof follows similarly as the *Continuous-Time* (CT) variant of Theorem 5.1. Namely, the system given by (7.1) is differentially dissipative w.r.t. a supply function s_δ and for a storage function $\mathcal{V}_\delta$, if (7.9) holds for all $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,u}} \mathfrak{B}$ and for all $t_0, t_1 \in \mathbb{N}_0$ with $t_0 \leq t_1$. This condition is equivalent with

$$\mathcal{V}_\delta\big(\bar{x}(t+1), x_\delta(t+1)\big) - \mathcal{V}_\delta\big(\bar{x}(t), x_\delta(t)\big) \leq s_\delta\big(w_\delta(t), z_\delta(t)\big), \tag{B.145}$$

holding for all $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,u}} \mathfrak{B}$ and for all $t \in \mathbb{N}_0$. Substituting the differential dynamics (7.7), the considered supply function (7.10), and storage function (7.11) in (B.145) results in

$$\begin{aligned}(\star)^\top M(\bar{x}(t+1))\big(A_\delta(\bar{x}(t), \bar{w}(t))x_\delta(t) + B_\delta(\bar{x}(t), \bar{w}(t))w_\delta(t)\big) - \\ x_\delta(t)^\top M(\bar{x}(t))x_\delta(t) \leq w_\delta(t)^\top Q w_\delta(t) + 2w_\delta(t)^\top S\big(C_\delta(\bar{x}(t), \bar{w}(t))x_\delta(t) + \\ D_\delta(\bar{x}(t), \bar{w}(t))w_\delta(t)\big) + (\star)^\top R\big(C_\delta(\bar{x}(t), \bar{w}(t))x_\delta(t) + D_\delta(\bar{x}(t), \bar{w}(t))w_\delta(t)\big),\end{aligned} \tag{B.146}$$

holding for all $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,u}} \mathfrak{B}$ and for all $t \in \mathbb{N}_0$. If it holds for all $(\bar{x}, \bar{w}) \in \mathcal{X} \times \mathcal{W}$, $x_\mathrm{v} \in \mathcal{D}$, $x_\delta \in \mathbb{R}^{n_\mathrm{x}}$, and $w_\delta \in \mathbb{R}^{n_\mathrm{w}}$ that

$$\begin{aligned}(\star)^\top M(\bar{x} + \bar{x}_\mathrm{v})\big(A_\delta(\bar{x}, \bar{w})x_\delta + B_\delta(\bar{x}, \bar{w})w_\delta\big) - \\ x_\delta^\top M(\bar{x})x_\delta \leq w_\delta^\top Q w_\delta + 2w_\delta^\top S\big(C_\delta(\bar{x}, \bar{w})x_\delta + \\ D_\delta(\bar{x}, \bar{w})w_\delta\big) + (\star)^\top R\big(C_\delta(\bar{x}, \bar{w})x_\delta + D_\delta(\bar{x}, \bar{w})w_\delta\big),\end{aligned} \tag{B.147}$$

then, (B.146) holds. Finally, (7.12) is equivalent to (B.147) by pre- and post multiplication of (7.12) with $\mathrm{col}(x_\delta, w_\delta)^\top$ and $\mathrm{col}(x_\delta, w_\delta)$, respectively.

Proof of Theorem 7.2 (Induced DT incremental dissipativity)

Similar to the CT proof for Theorem 5.2 in Appendix B.3, we start with the differential dissipativity condition, which writing out the λ-dependence and integrating over λ results in

$$\begin{aligned}\int_0^1 \Big[\mathcal{V}_\delta\big(\bar{x}(t_1+1, \lambda), x_\delta(t_1+1, \lambda)\big) - \mathcal{V}_\delta\big(\bar{x}(t_0, \lambda), x_\delta(t_0, \lambda)\big) - \\ \sum_{t=t_0}^{t_1} s_\delta\big(w_\delta(t, \lambda), z_\delta(t, \lambda)\big)\Big]\, d\lambda \leq 0.\end{aligned} \tag{B.148}$$

holding for all $(\bar{x}, \bar{w}) \in \pi_{\mathrm{x,u}} \mathfrak{B}$, $\lambda \in [0,\, 1]$, and for all $t_0, t_1 \in \mathbb{N}_0$ with $t_0 \leq t_1$. We use the CT results, specifically (B.74)–(B.77), to bound the storage function part, i.e.,

$$\begin{aligned}&\mathcal{V}_{\mathrm{i}}\big(x(t_1+1),\tilde{x}(t_1+1)\big)-\mathcal{V}_{\mathrm{i}}\big(x(t_0),\tilde{x}(t_0)\big)\leq\\&\qquad\int_0^1\mathcal{V}_\delta\big(\bar{x}(t_1+1,\lambda),x_\delta(t_1+1,\lambda)\big)-\mathcal{V}_\delta\big(\bar{x}(t_0,\lambda),x_\delta(t_0,\lambda)\big)\,d\lambda.\end{aligned}\tag{B.149}$$

where $\mathcal{V}_{\mathrm{i}}$ is given by (B.75). Next, we consider the supply function part of (B.148), which, by changing summation and integration operations, is given by

$$\sum_{t=t_0}^{t_1}\int_0^1(\star)^\top\begin{bmatrix}Q & S\\ \star & R\end{bmatrix}\begin{bmatrix}w_\delta(t,\lambda)\\ z_\delta(t,\lambda)\end{bmatrix}d\lambda.\tag{B.150}$$

Again, using the CT results, specifically (B.81)–(B.83), results in

$$\sum_{t=t_0}^{t_1}\int_0^1(\star)^\top\begin{bmatrix}Q & S\\ \star & R\end{bmatrix}\begin{bmatrix}w_\delta(t,\lambda)\\ z_\delta(t,\lambda)\end{bmatrix}d\lambda\leq\sum_{t=t_0}^{t_1}(\star)^\top\begin{bmatrix}Q & S\\ \star & R\end{bmatrix}\begin{bmatrix}w(t)-\tilde{w}(t)\\ z(t)-\tilde{z}(t)\end{bmatrix}.\tag{B.151}$$

Combining (B.149) and (B.151) with (B.148) results in

$$\mathcal{V}_{\mathrm{i}}\big(x(t_1+1),\tilde{x}(t_1+1)\big)-\mathcal{V}_{\mathrm{i}}\big(x(t_0),\tilde{x}(t_0)\big)\leq\sum_{t=t_0}^{t_1}s_{\mathrm{i}}\big(w(t),\tilde{w}(t),z(t),\tilde{z}(t)\big),\tag{B.152}$$

for all $t_0,t_1\in\mathbb{N}_0$ with $t_0\leq t_1$ and any two trajectories $(x,w,z),(\tilde{x},\tilde{w},\tilde{z})\in\mathfrak{B}$ with $\mathcal{V}_{\mathrm{i}}$ is given by (B.75), which is the condition for incremental dissipativity in Definition 7.1.

Proof of Corollary 7.2 ($\ell_{\mathrm{i}2}$-gain analysis)

Using a Schur complement and a congruence transformation we have that (7.14) is equivalent to

$$\begin{aligned}&(\star)^\top\begin{bmatrix}-\bar{M} & 0\\ 0 & \bar{M}\end{bmatrix}\begin{bmatrix}I & 0\\ A_\delta(\bar{x},\bar{w}) & B_\delta(\bar{x},\bar{w})\end{bmatrix}-\\&\qquad(\star)^\top\begin{bmatrix}\gamma^2 I & 0\\ 0 & -I\end{bmatrix}\begin{bmatrix}0 & I\\ C_\delta(\bar{x},\bar{w}) & D_\delta(\bar{x},\bar{w})\end{bmatrix}\preceq 0,\end{aligned}\tag{B.153}$$

where $\bar{M}=\gamma M^{-1}$. This corresponds to (7.1) being differentially (Q,S,R) dissipative with $Q=\gamma^2 I$, $S=0$ and $R=-I$ and storage function $\mathcal{V}_\delta(\bar{x},x_\delta)=x_\delta^\top\bar{M}x_\delta$. By Theorem 7.2, this hence implies existence of a storage function $\mathcal{V}_{\mathrm{i}}$ s.t. that for all $T\in\mathbb{N}_0$, $(x,w,z),(\tilde{x},\tilde{w},\tilde{z})\in\mathfrak{B}$

$$\begin{aligned}&\mathcal{V}_{\mathrm{i}}(x(T+1),\tilde{x}(T+1))-\mathcal{V}_{\mathrm{i}}(x_0,\tilde{x}_0)\leq\\&\qquad\sum_{t=0}^{T}\gamma^2(\star)^\top(w(t)-\tilde{w}(t))-(\star)^\top(z(t)-\tilde{z}(t)).\end{aligned}\tag{B.154}$$

This implies that

$$\gamma^2 \|w-\tilde{w}\|_{2,T}^2 - \|z-\tilde{z}\|_{2,T}^2 + \mathcal{V}_\mathrm{i}(x_0,\tilde{x}_0) \geq \mathcal{V}_\mathrm{i}(x(T+1),\tilde{x}(T+1)) \geq 0, \quad \text{(B.155)}$$

for all $T \in \mathbb{N}_0$, $(x,w,z),(\tilde{x},\tilde{w},\tilde{z}) \in \mathfrak{B}$ with $(w-\tilde{w}) \in \ell_{\mathrm{i}2}$. Consequently, we have that

$$\|z-\tilde{z}\|_{2,T}^2 \leq \gamma^2 \|w-\tilde{w}\|_{2,T}^2 + \mathcal{V}_\mathrm{i}(x_0,\tilde{x}_0), \quad \text{(B.156)}$$

which by taking the square root on both sides[3] results in holding for all $T \in \mathbb{N}_0$, $(x,w,z),(\tilde{x},\tilde{w},\tilde{z}) \in \mathfrak{B}$ with $(w-\tilde{w}) \in \ell_{\mathrm{i}2}$

$$\|z-\tilde{z}\|_{2,T} \leq \gamma \|w-\tilde{w}\|_{2,T} + \sqrt{\mathcal{V}_\mathrm{i}(x_0,\tilde{x}_0)}, \quad \text{(B.157)}$$

which is equivalent to (7.13), for $p=q=2$, with $\zeta_\mathrm{i}(x_0,\tilde{x}_0) = \sqrt{\mathcal{V}_\mathrm{i}(x_0,\tilde{x}_0)}$.

Proof of Corollary 7.3 (DT incremental passivity analysis)

According to Definition 7.4, a system of the form (7.1) is incrementally passive if it is incrementally dissipative with respect to the supply function s_i given by (7.15). This supply function can also be written in (Q,S,R) form, see (7.5), by taking $Q=0$, $S=I$ and $R=0$. By using the results of Corollary 7.1, and filling in $Q=0$, $S=I$ and $R=0$ in condition (7.12), where the matrix function M is constant, it can simply be rewritten into (7.16) by taking a Schur complement and congruence transformation.

Proof of Theorem 7.4 (Differential closed-loop ℓ_2-gain)

Through the DPV embedding P_dpv given by (7.30) and Lemma 5.3, we have that $\breve{\mathfrak{B}}_{\delta,\mathcal{X}} \subseteq \breve{\mathfrak{B}}_\mathrm{p}$. Consequently, if $\mathcal{F}_\mathrm{l}(P_\mathrm{dpv},K_\delta)$ is ℓ_2-gain stable and has an ℓ_2-gain of γ for all $p \in \mathcal{P}^{\mathbb{N}_0}$, then $\mathcal{F}_\mathrm{l}(P_\delta,K_\delta)$ is ℓ_2-gain stable and its ℓ_2-gain is bounded by γ for all $\bar{x} \in \mathcal{X}^{\mathbb{N}_0}$, hence, this also holds for all $\bar{x} \in \mathfrak{B}_\mathcal{X}$.

Proof of Theorem 7.6 (Closed-loop $\ell_{\mathrm{i}2}$-gain)

This proof follows similarly as the proof of Theorem 6.4. Theorem 7.4 shows that the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P_\delta,K_\delta)$ of differential form of the generalized plant P_δ given by (7.29) and LPV controller K_δ given by (7.31) is ℓ_2-gain stable and its ℓ_2-gain is bounded by γ for all $\bar{x} \in \pi_{\bar{\mathrm{x}}}\mathfrak{B}_\mathcal{X}$ if the closed-loop interconnection $\mathcal{F}_\mathrm{l}(P_\mathrm{dpv},K_\delta)$ of the DPV embedding P_dpv given by (7.30) and the LPV controller K_δ is ℓ_2-gain stable and has a bounded ℓ_2-gain of γ for all $p \in \mathcal{P}^{\mathbb{N}_0}$. Theorem 7.5 shows, assuming Assumption 6.1, that for the controller (7.32) its differential form is given by (7.31). Therefore, based on Theorem 7.2, this implies that $\mathcal{F}_\mathrm{l}(P,K)$ is

[3] Also using that for $a,b>0$, $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$.

incrementally dissipative w.r.t. the $\ell_{\mathrm{i}2}$-gain supply function, hence, there exists a function $\zeta_{\mathrm{i}} : \mathcal{X}_{\mathrm{cl}} \times \mathcal{X}_{\mathrm{cl}} \to \mathbb{R}$ s.t.

$$\left\| \mathcal{F}_{\mathrm{l}}(P,K)(w, x_{\mathrm{cl},0}) - \mathcal{F}_{\mathrm{l}}(P,K)(w^*, x^*_{\mathrm{cl},0}) \right\|_{2,T} \leq \gamma \left\| w - w^* \right\|_{2,T} + \zeta_{\mathrm{i}}(x_{\mathrm{cl},0}, x^*_{\mathrm{cl},0}), \tag{B.158}$$

for all $T \geq 0$, $x_{\mathrm{cl},0}, \tilde{x}_{\mathrm{cl},0} \in \mathcal{X}_{\mathrm{cl}}$ and any $w, w^* \in \mathcal{W}^{\mathbb{R}_0^+}$ with $w - w^* \in \mathcal{L}_{2\mathrm{e}}$ for which $x, \tilde{x} \in \mathfrak{B}_{\mathcal{X}}$. Here $\mathcal{F}_{\mathrm{l}}(P,K)(w, x_{\mathrm{cl},0})$ denotes the output $z \in \mathcal{Z}^{\mathbb{N}_0}$ of $\mathcal{F}_{\mathrm{l}}(P,K)$ for an input $w \in \mathcal{W}^{\mathbb{N}_0}$ and initial condition $x_{\mathrm{cl},0} = \mathrm{col}(x(0), x_{\mathrm{k}}(0)) \in \mathcal{X}_{\mathrm{cl}}$. We have that (B.158) implies that $\mathcal{F}_{\mathrm{l}}(P,K)$ has an ℓ_2-gain bound of γ, see (7.13). Furthermore, as the closed-loop is incrementally (Q,S,R) dissipative for a supply function corresponding to the $\ell_{\mathrm{i}2}$-gain, it is incrementally asymptotically stable, see Theorem 5.5. This means that all trajectories converge towards each other, and as the steady-state trajectory $\vartheta = (x^*, w^*, u^*, z^*, y^*) \in \mathfrak{B}_{\mathcal{X}}$ is by design of the controller (7.28) a feasible trajectory, all trajectories $(x, w, u, z, y) \in \mathfrak{B}_{\mathcal{X}}$ will converge towards $(x^*, w^*, u^*, z^*, y^*) \in \mathfrak{B}_{\mathcal{X}}$, i.e. $(x, w, u, z, y) \to (x^*, w^*, u^*, z^*, y^*)$ as $t \to \infty$, for $w(t) \to w^*(t)$ as $t \to \infty$.

B.6 Proofs of Chapter 8

Proof of Lemma 8.1 (Condition for DT velocity dissipativity)

If the system given by (8.1) is velocity dissipative, it holds that

$$\mathcal{V}_{\mathrm{v}}(x_\Delta(t+1)) - \mathcal{V}_{\mathrm{v}}(x_\Delta(t)) \leq s_{\mathrm{v}}(w_\Delta(t), z_\Delta(t)), \tag{B.159}$$

for all $t \in \mathbb{N}_0$ and $(x_\Delta, w_\Delta, z_\Delta) \in \mathfrak{B}_{\mathrm{v}}$. Using the dynamics of the velocity form (8.10) and using that $x_\Delta(t) = x(t+1) - x(t)$, condition (B.159) equivalently holds if

$$\begin{aligned}
&\mathcal{V}_{\mathrm{v}}\Big(\bar{A}_{\mathrm{v}}\big(x(t+1), x(t), w(t+1), w(t)\big)\big(x(t+1) - x(t)\big)+ \\
&\quad \bar{B}_{\mathrm{v}}\big(x(t+1), x(t), w(t+1), w(t)\big)\big(w(t+1) - w(t)\big)\Big) - \mathcal{V}_{\mathrm{v}}\big(x(t+1) - x(t)\big) \leq \\
&\quad s_{\mathrm{v}}\Big(w(t+1) - w(t), \bar{C}_{\mathrm{v}}\big(x(t+1), x(t), w(t+1), w(t)\big)\big(x(t+1) - x(t)\big)+ \\
&\qquad \bar{D}_{\mathrm{v}}\big(x(t+1), x(t), w(t+1), w(t)\big)\big(w(t+1) - w(t)\big)\Big),
\end{aligned} \tag{B.160}$$

for all $t \in \mathbb{N}_0$ and $(x, w, z) \in \mathfrak{B}$. Consequently, if (8.14) holds for all values $w_+, w \in \mathcal{W}$ and $x \in \mathcal{X}$, then (B.160) holds for all for all $t \in \mathbb{N}_0$ and $(x, w, z) \in \mathfrak{B}$, which means the system is velocity dissipative w.r.t. the supply function s_{v}.

Proof of Theorem 8.2 (DT velocity (Q, S, R) dissipativity condition)

If (8.17) holds for all $(x, w) \in \mathcal{X} \times \mathcal{W}$, we have by pre- and post multiplication of (8.17) with $\mathrm{col}(x_\Delta, w_\Delta)^\top$ and $\mathrm{col}(x_\Delta, w_\Delta)$, respectively, that

$$\begin{aligned}
&(\star)^\top M(A_{\mathrm{v}}(x, w)x_\Delta + B_{\mathrm{v}}(x, w)w_\Delta) - x_\Delta^\top M x_\Delta - \\
&\quad w_\Delta^\top Q w_\Delta - 2w_\Delta^\top S\big(C_{\mathrm{v}}(x, w)x_\Delta + D_{\mathrm{v}}(x, w)w_\Delta\big) - \\
&\qquad (\star)^\top R\big(C_{\mathrm{v}}(x, w)x_\Delta + D_{\mathrm{v}}(x, w)w_\Delta\big) \leq 0,
\end{aligned} \tag{B.161}$$

all $x_\Delta \in \mathbb{R}^{n_{\mathrm{x}}}$, $w_\Delta \in \mathbb{R}^{n_{\mathrm{w}}}$ and $(x, w) \in \mathcal{X} \times \mathcal{W}$. Consequently, it also holds for any $\lambda \in [0, 1]$ and $x_+, x \in \mathcal{X}$ and $w_+, w \in \mathcal{W}$ that

$$\begin{aligned}
&(\star)^\top M(A_{\mathrm{v}}(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + B_{\mathrm{v}}(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta) - x_\Delta^\top M x_\Delta - \\
&\quad w_\Delta^\top Q w_\Delta - 2w_\Delta^\top S\big(C_{\mathrm{v}}(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_{\mathrm{v}}(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\big) - \\
&\qquad (\star)^\top R\big(C_{\mathrm{v}}(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_{\mathrm{v}}(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\big) \leq 0,
\end{aligned} \tag{B.162}$$

where $\bar{x}(\lambda) = x + \lambda(x_+ - x)$ and $\bar{w}(\lambda) = w + \lambda(w_+ - w)$. Hence, we also have by integration over λ that

$$\begin{aligned}
&\int_0^1 (\star)^\top M(A_{\mathrm{v}}(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + B_{\mathrm{v}}(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta) - x_\Delta^\top M x_\Delta - \\
&\quad w_\Delta^\top Q w_\Delta - 2w_\Delta^\top S\big(C_{\mathrm{v}}(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_{\mathrm{v}}(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\big) - \\
&\qquad (\star)^\top R\big(C_{\mathrm{v}}(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_{\mathrm{v}}(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\big)\, d\lambda \leq 0,
\end{aligned} \tag{B.163}$$

for any $x_+, x \in \mathcal{X}$, $w_+, w \in \mathcal{W}$, $x_\Delta \in \mathbb{R}^{n_\mathrm{x}}$ and $w_\Delta \in \mathbb{R}^{n_\mathrm{w}}$. By Lemma C.4.1, as $M \succ 0$, we have that

$$(\star)^\top M\left(\int_0^1 A_\mathrm{v}(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + B_\mathrm{v}(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta \, d\lambda\right) \leq \\ \int_0^1 (\star)^\top M(A_\mathrm{v}(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + B_\mathrm{v}(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta) \, d\lambda, \quad \text{(B.164)}$$

and similarly, as $R \preceq 0$, we have that

$$(\star)^\top (-R)\left(\int_0^1 C_\mathrm{v}(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_\mathrm{v}(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta \, d\lambda\right) \leq \\ \int_0^1 (\star)^\top (-R)\big(C_\mathrm{v}(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_\mathrm{v}(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta\big) \, d\lambda. \quad \text{(B.165)}$$

Note that $A_\mathrm{v} = \frac{\partial f}{\partial x}$, $B_\mathrm{v} = \frac{\partial f}{\partial w}$, $C_\mathrm{v} = \frac{\partial h}{\partial x}$, $D_\mathrm{v} = \frac{\partial h}{\partial w}$. Hence, using the definition of $\bar{A}_\mathrm{v}, \ldots, \bar{D}_\mathrm{v}$ in (8.11), we have that

$$\int_0^1 A_\mathrm{v}(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + B_\mathrm{v}(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta \, d\lambda = \\ \bar{A}_\mathrm{v}(x_+, x, w_+, w)x_\Delta + \bar{B}_\mathrm{v}(x_+, x, w_+, w)w_\Delta, \quad \text{(B.166)}$$

$$\int_0^1 C_\mathrm{v}(\bar{x}(\lambda), \bar{w}(\lambda))x_\Delta + D_\mathrm{v}(\bar{x}(\lambda), \bar{w}(\lambda))w_\Delta \, d\lambda = \\ \bar{C}_\mathrm{v}(x_+, x, w_+, w)x_\Delta + \bar{D}_\mathrm{v}(x_+, x, w_+, w)w_\Delta. \quad \text{(B.167)}$$

Combining (B.164)–(B.167) with (B.163), we obtain that

$$(\star)^\top M\left(\bar{A}_\mathrm{v}(x_+, x, w_+, w)x_\Delta + \bar{B}_\mathrm{v}(x_+, x, w_+, w)w_\Delta\right) - x_\Delta^\top M x_\Delta - \\ w_\Delta^\top Q w_\Delta - 2w_\Delta^\top S\big(\bar{C}_\mathrm{v}(x_+, x, w_+, w)x_\Delta + \bar{D}_\mathrm{v}(x_+, x, w_+, w)w_\Delta\big) - \\ (\star)^\top R\big(\bar{C}_\mathrm{v}(x_+, x, w_+, w)x_\Delta + \bar{D}_\mathrm{v}(x_+, x, w_+, w)w_\Delta\big) \leq 0, \quad \text{(B.168)}$$

for any $x_+, x \in \mathcal{X}$, $w_+, w \in \mathcal{W}$, $x_\Delta \in \mathbb{R}^{n_\mathrm{x}}$ and $w_\Delta \in \mathbb{R}^{n_\mathrm{w}}$. Substituting $x_\Delta = x_+ - x$, $w_\Delta = w_+ - w$, and $x_+ = f(x, w)$ in (B.168), we obtain the inequality (8.14) where $\mathcal{V}_\mathrm{v}$ is given by (8.16) and s_v is given by (8.15). By Lemma 8.1, this then implies velocity (Q, S, R) dissipativity.

Proof of Theorem 8.3 (Implied universal shifted stability)

The proofs follows in a similar manner as the proof of Theorem 4.5. Namely, for each equilibrium point $(x_*, w_*, z_*) \in \mathscr{E}$, consider

$$V_\mathrm{s}(x(t), w_*) := V_\mathrm{v}(f(x(t), w_*) - x(t)) = V_\mathrm{v}(x_\Delta(t)). \quad \text{(B.169)}$$

For each $(x_*, w_*, z_*) \in \mathscr{E}$, this choice implies that $\mathcal{V}_\mathrm{s}(\cdot, w_*) \in \mathcal{Q}_{x_*}$, as $\mathcal{V}_\mathrm{v} \in \mathcal{Q}_0$. Note that this requires uniqueness of the equilibrium points (see Assumption 4.1), as otherwise there exists multiple x_* for which $V_\mathrm{s}(x_*, w_*) = 0$. By this choice of V_s, we have that for each $(x_*, w_*, z_*) \in \mathscr{E}$,

$$V_\mathrm{s}(x(t+1), w_*) - V_\mathrm{s}(x(t), w_*) = V_\mathrm{v}(x_\Delta(t+1)) - V_\mathrm{v}(x_\Delta(t)) \leq 0, \tag{B.170}$$

for all $t \in \mathbb{N}_0$ and $x_\Delta \in \pi_{\mathrm{x}_\Delta} \mathfrak{B}_{\mathrm{v,w}}(w \equiv w_*)$ and correspondingly all $x \in \pi_\mathrm{x} \mathfrak{B}_\mathrm{w}(w \equiv w_*)$. This implies that (8.6) holds for all $x \in \pi_\mathrm{x} \mathfrak{B}_\mathrm{w}(w \equiv w_*)$ and for all equilibrium points $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*} \mathscr{E}$. Hence, by Theorem 8.1, (8.1) is universally shifted stable. The asymptotic stability version follows similarly by changing (B.170) to a strict inequality.

Proof of Theorem 8.4 (Universal shifted performance from velocity dissipativity)

The proofs follows in a similar manner to the proof of Theorem 4.7. If the nonlinear system given by (8.19) is velocity dissipative w.r.t. the supply function $s_\mathrm{v}(w_\Delta, z_\Delta) = w_\Delta^\top Q w_\Delta + z_\Delta^\top R z_\Delta$, there exists a storage function $\mathcal{V}_\mathrm{v}$, such that for all $t_0, t_1 \in \mathbb{R}_0^+$ with $t_1 \geq t_0$

$$\mathcal{V}_\mathrm{v}(x_\Delta(t_1+1)) - \mathcal{V}_\mathrm{v}(x_\Delta(t_0)) \leq \sum_{t=t_0}^{t_1} w_\Delta(t)^\top Q w_\Delta(t) + z_\Delta(t)^\top R z_\Delta(t), \tag{B.171}$$

for all $(x_\Delta, w_\Delta, z_\Delta) \in \mathfrak{B}_\mathrm{v}$, corresponding to $(x, w, z) \in \mathfrak{B}$. Hence, as $\mathcal{V}_\mathrm{v}(x_\Delta(0)) = \mathcal{V}_\mathrm{v}(0) = 0$ and $\mathcal{V}_\mathrm{v}(x_\mathrm{v}) > 0$, $\forall\, x_\mathrm{v} \in \mathbb{R}^{n_\mathrm{x}} \backslash \{0\}$, this implies that

$$0 < \sum_{t=0}^{T} w_\Delta(t)^\top Q w_\Delta(t) + z_\Delta(t)^\top R z_\Delta(t), \tag{B.172}$$

for all $T > 0$ and $(x_\Delta, w_\Delta, z_\Delta) \in \mathfrak{B}_\mathrm{v}$. Defining $\tilde{Q} := \frac{1}{\|Q\|} Q$ and $\tilde{R} := \frac{1}{\|Q\|} R$, it also holds that

$$0 < \sum_{t=0}^{T} w_\Delta(t)^\top \tilde{Q} w_\Delta(t) + z_\Delta(t)^\top \tilde{R} z_\Delta(t), \tag{B.173}$$

Next, using (8.19)–(8.21) and as $x_\Delta(t) = x(t+1) - x(t)$, we have that, omitting dependence on time for brevity,

$$\begin{aligned}
z_\Delta^\top \tilde{R} z_\Delta &= x_\Delta^\top C^\top \tilde{R} C x_\Delta, && \text{(B.174a)}\\
&= (\star)^\top \tilde{R} C(f(x) + Bw - x), && \text{(B.174b)}\\
&= (\star)^\top \tilde{R} C(f(x) + Bw - x + \underbrace{x_* - (f(x_*) + Bw_*)}_{=0}), && \text{(B.174c)}\\
&= (\star)^\top \tilde{R} C(f(x) - f(x_*) - (x - x_*) + B(w - w_*)). && \text{(B.174d)}
\end{aligned}$$

Through Lemma C.1.1, we have that

$$
\begin{aligned}
f(x) - f(x_*) &= \left(\int_0^1 \frac{\partial f}{\partial x}(x_* + \lambda(x - x_*))\, d\lambda \right)(x - x_*), \\
&= \underbrace{\left(\int_0^1 A_{\mathrm{v}}(x_* + \lambda(x - x_*))\, d\lambda \right)}_{\bar{A}_{\mathrm{v}}(x, x_*)}(x - x_*),
\end{aligned}
\tag{B.175}
$$

hence,

$$
f(x) - f(x_*) - (x - x_*) = (\bar{A}_{\mathrm{v}}(x, x_*) - I)(x - x_*). \tag{B.176}
$$

Combining this with Assumption 4.2, we can write (B.174d) as

$$
z_\Delta^\top \tilde{R} z_\Delta = (\star)^\top \tilde{R}\, C(\bar{A}_{\mathrm{v}}(x, x_*) - I)(x - x_*). \tag{B.177}
$$

Next, by satisfying Assumption 8.1 for $T = \tilde{R} \preceq 0$, we have that, for every $x_* \in \mathscr{X}$,

$$
z_\Delta^\top \tilde{R} z_\Delta = (\star)^\top \tilde{R} C(\bar{A}_{\mathrm{v}}(x, x_*) - I)(x - x_*) \leq \alpha^{-1} (\star)^\top \tilde{R} C(x - x_*) = \alpha^{-1} (\star)^\top \tilde{R}(z - z_*). \tag{B.178}
$$

Moreover, by Assumption 8.2 and using that $w_\Delta(t) = w(t+1) - w(t)$, we have that, for a given $(x_*, w_*, z_*) \in \mathscr{E}$,

$$
\begin{aligned}
w(t+1) &= A_{\mathrm{w}}(w(t) - w_*) + w_*, \\
w(t+1) - w(t) + w(t) &= A_{\mathrm{w}}(w(t) - w_*) + w_*, \\
w(t+1) - w(t) &= A_{\mathrm{w}}(w(t) - w_*) - (w(t) - w_*), \\
w_\Delta(t) &= (A_{\mathrm{w}} - I)(w(t) - w_*),
\end{aligned}
\tag{B.179}
$$

and hence,

$$
w_\Delta(t)^\top \tilde{Q} w_\Delta(t) = (\star)^\top \tilde{Q}(A_{\mathrm{w}} - I)(w(t) - w_*) \leq \beta^2 (\star)^\top \tilde{Q}(w(t) - w_*), \tag{B.180}
$$

where $w \in \mathfrak{W}$ and $0 \preceq \tilde{Q} \preceq I$. Combining (B.173), (B.178), and (B.180), we obtain that, for every $(x_*, w_*, z_*) \in \mathscr{E}$,

$$
\sum_{t=0}^{T} \beta^2 (\star)^\top \tilde{Q}(w(t) - w_*) + \alpha^{-1} (\star)^\top \tilde{R}(z(t) - z_*) > 0, \tag{B.181}
$$

for all $T > 0$ and $(w, z) \in \pi_{\mathrm{w,z}} \mathfrak{B}$ with $w \in \mathfrak{W}$. Hence, also

$$
\sum_{t=0}^{T} \beta^2 (\star)^\top Q(w(t) - w_*) + \alpha^{-1} (\star)^\top R(z(t) - z_*) > 0, \tag{B.182}
$$

for all $T > 0$ and $(w, z) \in \pi_{\mathrm{w,z}} \mathfrak{B}$ with $w \in \mathfrak{W}$.

Proof of Theorem 8.6 (DT universal shifted controller realization)

For the proof, we first use part of the results of the proof Theorem 7.6 in Appendix B.5. For simplicity, we assume in the proof that solutions are defined on

$\mathbb{Z}$. First, based on Theorem 7.5, the differential form of the incremental controller realization (7.32) is given by (7.31), equivalent to (8.34). Consequently (see also Theorem 7.6), based on Theorem 7.2, this implies that $\mathcal{F}_\mathrm{l}(P,K)$ is incrementally dissipative w.r.t. the $\ell_{\mathrm{i}2}$-gain supply function. Under Assumption 6.1, this (and through Lemma 5.1) means there exists an $M \succ 0$ such that

$$(\star)^\top M(x_\mathrm{cl}(t_1) - x^*_\mathrm{cl}(t_1)) - (\star)^\top M(x_\mathrm{cl}(t_0) - x^*_\mathrm{cl}(t_0)) \leq \\ \sum_{t=t_0}^{t_1} \gamma^2(\star)^\top(w(t) - w^*(t)) - (\star)^\top(z(t) - z^*(t)). \quad \text{(B.183)}$$

for all $t_0, t_1 \in \mathbb{Z}$, $t_1 \geq t_0$, and $(x_\mathrm{cl}, w, z), (x^*_\mathrm{cl}, w^*, z^*) \in \mathfrak{B}_{\mathrm{cl},\mathcal{X}}$ (where $\mathfrak{B}_{\mathrm{cl},\mathcal{X}}$ is the behavior associated with closed-loop $\mathcal{F}_\mathrm{l}(P,K)$ for which $x \in \mathfrak{B}_\mathcal{X}$). Note that this corresponds to the incremental storage function $\mathcal{V}_\mathrm{i}(x,\tilde{x}) = (\star)^\top M(x - \tilde{x})$.

Now, consider the restriction of the trajectory ϑ to

$$\begin{aligned} \vartheta(t) &= \big(x^*(t), w^*(t), u^*(t), z^*(t), y^*(t)\big), \\ &= \big(x(t-1), w(t-1), u(t-1), z(t-1), y(t-1)\big), \end{aligned} \quad \text{(B.184)}$$

i.e., the (current) trajectory delayed by one time instance. For this restriction, the incremental controller realization (7.32) becomes

$$\begin{aligned} x_{\Delta,\mathrm{k}}(t+1) &= \bar{A}_\mathrm{k}(x(t), x(t-1))x_{\Delta,\mathrm{k}}(t) + \\ &\quad \bar{B}_\mathrm{k}(x(t), x(t-1))(u_\mathrm{k}(t) - u_\mathrm{k}(t-1)); \quad \text{(B.185a)} \\ y_\mathrm{k}(t) &= y_\mathrm{k}(t-1) + \bar{C}_\mathrm{k}(x(t), x(t-1))x_{\Delta,\mathrm{k}}(t) + \\ &\quad \bar{D}_\mathrm{k}(x(t), x(t-1))(u_\mathrm{k}(t) - u_\mathrm{k}(t-1)); \quad \text{(B.185b)} \end{aligned}$$

with $x_{\Delta,\mathrm{k}}(t) \in \mathbb{R}^{n_{\mathrm{x_k}}}$ and $\bar{A}_\mathrm{k}, \ldots, \bar{D}_\mathrm{k}$ as given (8.37). By defining $\tilde{x}_\mathrm{k}(t) = u_\mathrm{k}(t-1)$ and $\hat{x}_\mathrm{k}(t) = y_\mathrm{k}(t-1)$, we get that

$$\tilde{x}_\mathrm{k}(t+1) = u_\mathrm{k}(t), \quad \text{(B.186)}$$

then, (B.185a) becomes

$$\begin{aligned} x_{\Delta,\mathrm{k}}(t+1) &= \bar{A}_\mathrm{k}(x(t), x(t-1))x_{\Delta,\mathrm{k}}(t) + \\ &\quad \bar{B}_\mathrm{k}(x(t), x(t-1))(u_\mathrm{k}(t) - \tilde{x}_\mathrm{k}(t-1)), \end{aligned} \quad \text{(B.187)}$$

and from (B.185b) we get that

$$\begin{aligned} y_\mathrm{k}(t) &= \hat{x}_\mathrm{k}(t) + \bar{C}_\mathrm{k}(x(t), x(t-1))x_{\Delta,\mathrm{k}}(t) + \\ &\quad \bar{D}_\mathrm{k}(x(t), x(t-1))(u_\mathrm{k}(t) - \tilde{x}_\mathrm{k}(t)), \end{aligned} \quad \text{(B.188)}$$

and

$$\begin{aligned} \hat{x}_\mathrm{k}(t+1) &= \hat{x}_\mathrm{k}(t) + \bar{C}_\mathrm{k}(x(t), x(t-1))x_{\Delta,\mathrm{k}}(t) + \\ &\quad \bar{D}_\mathrm{k}(x(t), x(t-1))(u_\mathrm{k}(t) - \tilde{x}_\mathrm{k}(t)). \end{aligned} \quad \text{(B.189)}$$

Combining (B.186)–(B.189), and defining $\breve{x}_{\mathrm{k}} = \mathrm{col}(\tilde{x}_{\mathrm{k}}, x_{\Delta,\mathrm{k}}, \hat{x}_{\mathrm{k}})$, we can express the dynamics of (B.185) as (8.35) with matrices given by (8.36) and (8.37).

Moreover, through the restriction of ϑ given by (B.184), we have from (B.183) that

$$(\star)^\top M(x_{\mathrm{cl}}(t_1) - x_{\mathrm{cl}}(t_1 - 1)) - (\star)^\top M(x_{\mathrm{cl}}(t_0) - x_{\mathrm{cl}}(t_0 - 1)) \leq \sum_{t=t_0}^{t_1} \gamma^2 (\star)^\top (w(t) - w(t-1)) - (\star)^\top (z(t) - z(t-1)). \quad \text{(B.190)}$$

for all $t_0, t_1 \in \mathbb{Z}$, $t_1 \geq t_0$, and $(x_{\mathrm{cl}}, w, z), (x^*_{\mathrm{cl}}, w^*, z^*) \in \mathfrak{B}_{\mathrm{cl}}$ for which $x \in \mathfrak{B}_{\mathcal{X}}$. Hence, we also have that

$$(\star)^\top M(x_{\mathrm{cl}}(t_1 + 1) - x_{\mathrm{cl}}(t_1)) - (\star)^\top M(x_{\mathrm{cl}}(t_0 + 1) - x_{\mathrm{cl}}(t_0)) \leq \sum_{t=t_0}^{t_1} \gamma^2 (\star)^\top (w(t+1) - w(t)) - (\star)^\top (z(t+1) - z(t)). \quad \text{(B.191)}$$

By using that $z_\Delta(t) = z(t+1) - z(t)$ and $w_\Delta(t) := w(t+1) - w(t)$, and defining $x_{\Delta,\mathrm{cl}}(t) = x_{\mathrm{cl}}(t_1 + 1) - x_{\mathrm{cl}}(t_1)$, we get that

$$(\star)^\top M x_{\Delta,\mathrm{cl}}(t_1) - (\star)^\top M x_{\Delta,\mathrm{cl}}(t_0) \leq \sum_{t=t_0}^{t_1} \gamma^2 (\star)^\top w_\Delta(t) - (\star)^\top z_\Delta(t), \quad \text{(B.192)}$$

for all $t_0, t_1 \in \mathbb{Z}$, $t_1 \geq t_0$, and $(x_{\Delta,\mathrm{cl}}, w_\Delta, z_\Delta) \in \Delta\mathfrak{B}_{\mathrm{cl},\mathcal{X}}$. This is the condition for velocity (Q, S, R) dissipativity with $(Q, S, R) = (\gamma^2 I, 0, -I)$ and $\mathcal{V}_{\mathrm{v}}(x_{\Delta,\mathrm{cl}}) = x_{\Delta,\mathrm{cl}}^\top M x_{\Delta,\mathrm{cl}}$, see Definition 8.4.

Proof of Corollary 8.2 (Universal shifted realization with integral action)

For the realization of the controller in the proof of Theorem 8.6, we have that the time-difference $u_{\mathrm{k}}(t) - u_{\mathrm{k}}(t-1)$ is taken, see (B.185a). This operation can also be expressed as filtering of u_{k} by the filter $\frac{q-1}{q}$, where q is the discrete time-shift operator. The integration filter M that is considered is given by $M(q) = \frac{q+\alpha}{q-1}$, which is also connected to the input controller the controller as depicted in Figure 4.4. Consequently, the interconnection of the integration filter M and the filter $\frac{q-1}{q}$, representing the time difference, can be simplified as $\frac{q-1}{q} \cdot \frac{q+\alpha}{q-1} = \frac{q+\alpha}{q}$. Hence, as $y = u_{\mathrm{k}}$, in the proof of Theorem 8.6, $\frac{q-1}{q} u_{\mathrm{k}}(t) = u_{\mathrm{k}}(t) - u_{\mathrm{k}}(t-1)$ in (B.185a) becomes $\frac{q+\alpha}{q} u_{\mathrm{k}}(t) = u_{\mathrm{k}}(t) + \alpha u_{\mathrm{k}}(t-1)$. Consequently, (B.187)–(B.189) become

$$x_{\Delta,\mathrm{k}}(t+1) = \bar{A}_{\mathrm{k}}(x(t), x(t-1)) x_{\Delta,\mathrm{k}}(t) + \bar{B}_{\mathrm{k}}(x(t), x(t-1))(u_{\mathrm{k}}(t) + \alpha \tilde{x}_{\mathrm{k}}(t)), \quad \text{(B.193)}$$

$$y_{\mathrm{k}}(t) = \hat{x}_{\mathrm{k}}(t) + \bar{C}_{\mathrm{k}}(x(t), x(t-1)) x_{\Delta,\mathrm{k}}(t) + \bar{D}_{\mathrm{k}}(x(t), x(t-1))(u_{\mathrm{k}}(t) + \alpha \tilde{x}_{\mathrm{k}}(t)), \quad \text{(B.194)}$$

and

$$\begin{aligned}\hat{x}_{\mathrm{k}}(t+1) = \hat{x}_{\mathrm{k}}(t) + \bar{C}_{\mathrm{k}}(x(t), x(t-1))x_{\Delta,\mathrm{k}}(t) + \\ \bar{D}_{\mathrm{k}}(x(t), x(t-1))(u_{\mathrm{k}}(t) + \alpha\tilde{x}_{\mathrm{k}}(t)),\end{aligned} \tag{B.195}$$

respectively. Then, along the same lines as in the proof of Theorem 8.6, we obtain $\breve{B}_{\mathrm{k}}$ and $\breve{D}_{\mathrm{k}}$ as given in (8.36) and $\breve{A}_{\mathrm{k}}$ and $\breve{C}_{\mathrm{k}}$ given by (8.38).

C

Other Technical Results

C.1 Function Factorization

This appendix is based on a result given in (Koelewijn and Tóth 2021a).

We provide the following useful lemma:

Lemma C.1.1 (Function factorization)**.** *For a function $g : \mathbb{R}^n \to \mathbb{R}^m$ with $g \in \mathcal{C}_1$, it holds for all $\varsigma, \tilde{\varsigma} \in \mathbb{R}^n$ that*

$$g(\varsigma) - g(\tilde{\varsigma}) = \left(\int_0^1 \frac{dg}{d\varsigma}(\tilde{\varsigma} + \lambda(\varsigma - \tilde{\varsigma}))\, d\lambda\right)(\varsigma - \tilde{\varsigma}), \tag{C.1}$$

where $\frac{dg}{d\varsigma}$ denotes the Jacobian of g and the integral of the Jacobian is taken element-wise.

Proof. Denote $g(\varsigma) = \begin{bmatrix} g_1(\varsigma) & \cdots & g_m(\varsigma) \end{bmatrix}$, where $g_i(\varsigma) : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ and $\varsigma \in \mathbb{R}^n$. For $\varsigma, \tilde{\varsigma} \in \mathbb{R}^n$, define the functions $\bar{g}_i(\lambda) = g_i(\tilde{\varsigma} + \lambda(\varsigma - \tilde{\varsigma}))$ for $i = 1, \ldots, m$, with $\bar{g}_i : [0, 1] \to \mathbb{R}$. Using the second fundamental theorem of calculus, it holds that

$$\bar{g}_i(1) - \bar{g}_i(0) = \int_0^1 \frac{d\bar{g}_i}{d\lambda}(\bar{\lambda})\, d\bar{\lambda}, \quad \text{for } i = 1, \ldots m, \tag{C.2}$$

which results in

$$\begin{aligned} g_i(\varsigma) - g_i(\tilde{\varsigma}) &= \int_0^1 \left(\frac{dg_i}{d\varsigma}(\tilde{\varsigma} + \bar{\lambda}(\varsigma - \tilde{\varsigma}))\right)(\varsigma - \tilde{\varsigma})\, d\bar{\lambda}, && \text{(C.3)} \\ &= \left(\int_0^1 \frac{dg_i}{d\varsigma}(\tilde{\varsigma} + \bar{\lambda}(\varsigma - \tilde{\varsigma}))\, d\bar{\lambda}\right)(\varsigma - \tilde{\varsigma}), && \text{(C.4)} \end{aligned}$$

for $i = 1, \dots m$. Consequently, we have that

$$g(\varsigma) - g(\tilde{\varsigma}) = \begin{bmatrix} g_1(\varsigma) - g_1(\tilde{\varsigma}) \\ \vdots \\ g_m(\varsigma) - g_m(\tilde{\varsigma}) \end{bmatrix}, \tag{C.5}$$

$$= \begin{bmatrix} \left(\int_0^1 \frac{dg_1}{d\varsigma}(\tilde{\varsigma} + \bar{\lambda}(\varsigma - \tilde{\varsigma}))\, d\bar{\lambda}\right)(\varsigma - \tilde{\varsigma}) \\ \vdots \\ \left(\int_0^1 \frac{dg_m}{d\varsigma}(\tilde{\varsigma} + \bar{\lambda}(\varsigma - \tilde{\varsigma}))\, d\bar{\lambda}\right)(\varsigma - \tilde{\varsigma}) \end{bmatrix}, \tag{C.6}$$

$$= \left(\int_0^1 \frac{dg}{d\varsigma}(\tilde{\varsigma} + \bar{\lambda}(\varsigma - \tilde{\varsigma}))\, d\bar{\lambda}\right)(\varsigma - \tilde{\varsigma}), \tag{C.7}$$

where $\frac{dg}{d\varsigma}$ denotes the Jacobian of g and the integral of the Jacobian is taken element-wise. $\square$

C.2 Coarser System Structure through Filtering

C.2.1 Conversion of System to Coarser Structure

Consider the nonlinear system

$$\dot{x}(t) = f(x(t), w(t)); \tag{C.8a}$$
$$z(t) = h(x(t), w(t)); \tag{C.8b}$$

where $t \in \mathbb{R}_0^+$ is time, $x(t) \in \mathbb{R}^{n_\mathrm{x}}$ the state, $w(t) \in \mathbb{R}^{n_\mathrm{w}}$ is the input of the system, and $z(t) \in \mathbb{R}^{n_\mathrm{z}}$ is the output of the system.

Consider the following (low-pass) filters, denoted by F_i

$$\dot{x}_{\mathrm{F},i}(t) = -\Omega_i \, x_{\mathrm{F},i}(t) + \Omega_i \, u_{\mathrm{F},i}(t); \tag{C.9a}$$
$$y_{\mathrm{F},i}(t) = x_{\mathrm{F},i}(t); \tag{C.9b}$$

for $i = 1, 2$ and where $\Omega_i = \mathrm{diag}(\omega_{1,i}, \ldots, \omega_{n_{\mathrm{F},i}})$ with $\omega_{i,j} > 0$ for $j = 1, \ldots, n_{\mathrm{F},i}$ and $x_{\mathrm{F},i}(t) \in \mathbb{R}^{n_{\mathrm{F},i}}$.

Connecting F_1 and F_2 as given by (C.9) to (C.8), such that $w = y_{\mathrm{F},1}$ and $u_{\mathrm{F},2} = z$, results in

$$\dot{x}_{\mathrm{F},1}(t) = -\Omega_1 x_{\mathrm{F},1}(t) + \Omega_1 \hat{w}(t); \tag{C.10a}$$
$$\dot{x}_{\mathrm{F},2} = -\Omega_2 x_{\mathrm{F},2}(t) + \Omega_2 h(x(t), x_{\mathrm{F},1}(t)); \tag{C.10b}$$
$$\dot{x}(t) = f(x(t), x_{\mathrm{F},1}(t)); \tag{C.10c}$$
$$\hat{z}(t) = x_{\mathrm{F},2}(t). \tag{C.10d}$$

The system (C.10) with input $\hat{w}$, output $\hat{z}$, and state $\mathrm{col}(x_{\mathrm{F},1}, x_{\mathrm{F},2}, x)$ is of the form

$$\dot{x}(t) = f(x(t)) + Bw(t); \tag{C.11a}$$
$$y(t) = Cx(t). \tag{C.11b}$$

For (C.10) we have that $B = \begin{bmatrix} \Omega_1^\top & 0 & 0 \end{bmatrix}^\top$ and $C = \begin{bmatrix} 0 & I & 0 \end{bmatrix}$, and consequently $CB = 0$.

C.2.2 Conversion of Generalized Plant to Coarser Structure

This section is based on (Koelewijn, Tóth, Nijmeijer, et al. 2022, Appendix V).

Consider a nonlinear dynamical system, describing a generalized plant, of the form

$$\dot{x}(t) = f\left(x(t), u(t)\right) + B_\mathrm{w} w(t); \tag{C.12a}$$
$$z(t) = h_\mathrm{z}(x(t), u(t)) + D_\mathrm{zw} w(t); \tag{C.12b}$$
$$y(t) = h_\mathrm{y}(x(t), u(t)) + D_\mathrm{yw} w(t); \tag{C.12c}$$

where $t \in \mathbb{R}_0^+$ is time, $x(t) \in \mathbb{R}^{n_\mathrm{x}}$ is the state, $w(t) \in \mathbb{R}^{n_\mathrm{w}}$ is the generalized disturbance, $z(t) \in \mathbb{R}^{n_\mathrm{z}}$ the generalized performance, $u(t) \in \mathbb{R}^{n_\mathrm{u}}$ the control input, and $y(t) \in \mathbb{R}^{n_\mathrm{y}}$ the measured output.

Moreover, consider a more restrictive class of (C.12), given by

$$\dot{x}(t) = f(x(t)) + B_\mathrm{w} w(t) + B_\mathrm{u} u(t); \tag{C.13a}$$

$$z(t) = h_\mathrm{z}(x(t)) + D_\mathrm{zw} w(t) + D_\mathrm{zu} u(t); \tag{C.13b}$$

$$y(t) = C_\mathrm{y} x(t) + D_\mathrm{yw} w(t). \tag{C.13c}$$

While (C.13) may seem restrictive, the general class of nonlinear plants (C.12) can be expressed as (C.13) by the use of appropriate filters. Connecting F_1 and F_2 given by (C.9) to (C.12), such that $u = y_{\mathrm{F},1}$ and $u_{\mathrm{F},2} = y$, results in

$$\dot{x}(t) = f\left(x(t), x_{\mathrm{F},1}(t)\right) + B_\mathrm{w} w(t); \tag{C.14a}$$

$$\dot{x}_{\mathrm{F},1}(t) = -\Omega_1 x_{\mathrm{F},1}(t) + \Omega_1 \hat{u}(t); \tag{C.14b}$$

$$\dot{x}_{\mathrm{F},2}(t) = \Omega_2\, h_\mathrm{y}(x(t), x_{\mathrm{F},1}(t)) - \Omega_2\, x_{\mathrm{F},2}(t) + \Omega_2\, D_\mathrm{yw} w(t); \tag{C.14c}$$

$$z(t) = h_\mathrm{z}\left(x(t), x_{\mathrm{F},1}(t)\right) + D_\mathrm{zw} w(t); \tag{C.14d}$$

$$\hat{y} = x_{\mathrm{F},2}(t); \tag{C.14e}$$

where $\hat{u} = u_{\mathrm{F},1}$ is the new 'u'-input and $\hat{y} = y_{\mathrm{F},2}$ is the new 'y'-output. Note that the system (C.14) is of the form (C.13). Moreover, if $\omega_{i,j}$ is taken large enough (e.g., $5\times$ the intended bandwidth) then the desired closed-loop performance is not affected by the conversion.

C.3 Results on Connecting Velocity and Universal Shifted Dissipativity

In this appendix, we highlight some additional results that were obtained during the research into connection of velocity dissipativity and universal shifted dissipativity, as also discussed in Chapter 4.

In this appendix, we consider nonlinear systems of the form

$$\dot{x}(t) = f(x(t), w(t)); \tag{C.15a}$$
$$z(t) = h(x(t)). \tag{C.15b}$$

Similarly to what is considered in Chapter 4, we will denote its behavior by $\mathfrak{B}$, set of equilibrium points by $\mathscr{E}$, etc.

For a nonlinear system (C.15), its equilibrium points $(x_*, w_*, z_*) \in \mathscr{E}$ satisfy

$$0 = f(x_*, w_*); \tag{C.16a}$$
$$z_* = h(x_*); \tag{C.16b}$$

and the velocity form of (C.15) is given by

$$\ddot{x}(t) = A_\mathrm{v}(x(t), w(t))\dot{x}(t) + B_\mathrm{v}(x(t), w(t))\dot{w}(t); \tag{C.17a}$$
$$\dot{z}(t) = C_\mathrm{v}(x(t))\dot{x}(t), \tag{C.17b}$$

where $A_\mathrm{v} = \frac{\partial f}{\partial x}$, $B_\mathrm{v} = \frac{\partial f}{\partial w}$, and $C_\mathrm{v} = \frac{\partial h}{\partial x}$.

Before discussing some of our results, we consider the following assumption, similar to Assumption 4.3:

Assumption C.3.1. *Given a symmetric matrix $R \preceq 0$, there exists an $\alpha > 0$, such that for every $(x_*, w_*) \in \pi_{\mathrm{x}_*,\mathrm{w}_*}\mathscr{E}$, it holds that*

$$(\star)^\top R\, C(x)\bar{A}(x, x_*, w_*)(x - x_*) \leq \alpha^{-1}(\star)^\top R\, \bar{C}(x, x_*)(x - x_*), \tag{C.18}$$

for all $x \in \mathcal{X}$, where $\bar{A}(x, x_*, w_*) = \int_0^1 A_\mathrm{v}(\bar{x}(\lambda), w_*)d\lambda$, $\bar{C}(x, x_*) = \int_0^1 C_\mathrm{v}(\bar{x}(\lambda))d\lambda$ with $\bar{x}(\lambda) = x_* + \lambda(x - x_*)$.

Consider the nonlinear system (C.15) for which the condition (4.18) in Lemma 4.3 holds[1] w.r.t. a (Q, S, R) supply function (4.17) for which $Q \succeq 0$ and $R \preceq 0$. Moreover, we assume that, for this considered R, Assumption C.3.1 holds.

For a nonlinear system (C.15), the condition in (4.18) for the considered supply function becomes

$$\nabla\mathcal{V}_\mathrm{v}(f(x, w))\,(A_\mathrm{v}(x, w)f(x, w) + B_\mathrm{v}(x, w)w_\mathrm{v}) \leq \\ w_\mathrm{v}^\top Q w_\mathrm{v} + (\star)^\top R\, C_\mathrm{v}(x)f(x, w), \tag{C.19}$$

[1] Which implies velocity dissipativity by Lemma 4.3.

which holds for all $w_\mathrm{v} \in \mathbb{R}^{n_\mathrm{w}}$, $x \in \mathcal{X}$, and $w \in \mathcal{W}$. As (C.19) holds for all $w_\mathrm{v} \in \mathbb{R}^{n_\mathrm{w}}$, $x \in \mathcal{X}$, and $w \in \mathcal{W}$, it also holds for every $w = w_* \in \mathscr{W}$ and $w_\mathrm{v} = 0$, which results in

$$\nabla \mathcal{V}_\mathrm{v}(f(x,w_*))A_\mathrm{v}(x,w_*)f(x,w_*) \leq (\star)^\top R\,C(x)f(x,w_*), \tag{C.20}$$

for all $x \in \mathcal{X}$. By Lemma C.1.1, we have that

$$f(x,w_*) = f(x,w_*)\underbrace{-f(x_*,w_*)}_{=0} = \bar{A}(x,x_*,w_*)(x-x_*), \tag{C.21}$$

where $\bar{A}(x,x_*,w_*) = \int_0^1 A_\mathrm{v}(\bar{x}(\lambda),w_*)d\lambda$ with $\bar{x}(\lambda) = x_* + \lambda(x-x_*)$. Hence, the righthand side of (C.20) can be expressed as

$$(\star)^\top R\,C(x)f(x,w_*) = (\star)^\top R\,C(x)\bar{A}(x,x_*,w_*)(x-x_*). \tag{C.22}$$

Similarly, by Lemma C.1.1, we have that

$$z - z_* = h(x) - h(x_*) = \bar{C}(x,x_*)(x-x_*), \tag{C.23}$$

where $\bar{C}(x,x_*) = \int_0^1 C_\mathrm{v}(\bar{x}(\lambda))d\lambda$. Combining this with Assumption C.3.1, we hence have that $\exists\,\alpha > 0$ such that

$$(\star)^\top R\,C(x)\bar{A}(x,x_*,w_*)(x-x_*) \leq \alpha^{-1}(\star)^\top R\,\bar{C}(x,x_*)(x-x_*) = \alpha^{-1}(\star)^\top R\,(z-z_*). \tag{C.24}$$

Hence, together with (C.20), (C.22) and (C.23), this gives that for any $(x_*,w_*,z_*) \in \mathscr{E}$

$$\nabla \mathcal{V}_\mathrm{v}(f(x,w_*))A_\mathrm{v}(x,w_*)f(x,w_*) \leq \alpha^{-1}(\star)^\top R(z-z_*), \tag{C.25}$$

for all $x \in \mathcal{X}$. This also implies that for any $(x_*,w_*,z_*) \in \mathscr{E}$

$$\nabla \mathcal{V}_\mathrm{v}(f(x,w_*))A_\mathrm{v}(x,w_*)f(x,w_*) \leq \underbrace{\alpha^{-1}(\star)^\top Q(w-w_*)}_{\geq 0} + \alpha^{-1}(\star)^\top R(z-z_*), \tag{C.26}$$

for all $(x,w) \in \mathcal{X} \times \mathcal{W}$. Let us define:

$$\mathcal{V}_\mathrm{s}(x,w_*) := \alpha\mathcal{V}_\mathrm{v}(f(x,w_*)), \tag{C.27}$$

which gives us that

$$\nabla_x \mathcal{V}_\mathrm{s}(x,w_*) = \alpha\nabla\mathcal{V}_\mathrm{v}(f(x,w_*))A_\mathrm{v}(x,w_*). \tag{C.28}$$

Substituting this in (C.26), we get that for any $(x_*,w_*,z_*) \in \mathscr{E}$

$$\nabla_x \mathcal{V}_\mathrm{s}(x,w_*)f(x,w_*) \leq (\star)^\top Q(w-w_*) + (\star)^\top R(z-z_*), \tag{C.29}$$

for all $(x,w) \in \mathcal{X} \times \mathcal{W}$. This condition is close, but not equal, to the condition for universal shifted (Q,S,R) dissipativity given in Lemma 4.1. Namely, for $\mathcal{V}_\mathrm{s}$ given by (C.27), and a (Q,S,R) supply function for which $S = 0$, (4.8) in Lemma 4.1 becomes

$$\nabla_x \mathcal{V}_\mathrm{s}(x,w_*)f(x,w) \leq (\star)^\top Q(w-w_*) + (\star)^\top R(z-z_*), \tag{C.30}$$

where the difference is highlighted in orange, i.e., instead of f being evaluated at w in (C.30), it is evaluated at w_* in (C.29).

In conclusion, while these results again highlight there is a close connection between velocity dissipativity and universal shifted dissipativity, they are not able to show if velocity dissipativity implies universal shifted dissipativity.

C.4 Norm Integral Inequality

This appendix is based on the appendix in (Koelewijn and Tóth 2021b).

We give the following useful lemma:

Lemma C.4.1 (Norm Integral Inequality). *Given a positive definite $M \in \mathbb{S}^n$, i.e., $M \succ 0$, and a continuous function $\phi : [0,1] \to \mathbb{R}^n$, then*

$$\left(\int_0^1 \phi(t)\, dt\right)^\top M \left(\int_0^1 \phi(t)\, dt\right) \leq \int_0^1 \phi(t)^\top M \phi(t)\, dt \tag{C.31}$$

Proof. As M is positive definite, we can define the Euclidean vector space with norm: $\|v\| := \sqrt{v^\top M v}$, where $v \in \mathbb{R}^n$. By the Cauchy-Schwarz inequality, for a continuous function $\phi : [0,1] \to \mathbb{R}^n$

$$\left\|\int_0^1 \phi(t)\, dt\right\| \leq \int_0^1 \|\phi(t)\|\ dt, \tag{C.32}$$

see (Rudin 1976). Furthermore, it also holds that for a function $\psi : [0,1] \to \mathbb{R}$

$$\left|\int_0^1 \psi(t)\, dt\right|^2 \leq \left(\int_0^1 1\, dt\right)\left(\int_0^1 |\psi(t)|^2\, dt\right) = \left(\int_0^1 |\psi(t)|^2\, dt\right). \tag{C.33}$$

Hence, using (C.32) and (C.33), with $\psi(t) = \|\phi(t)\|$, we get

$$\left\|\int_0^1 \phi(t)\, dt\right\|^2 \leq \left(\int_0^1 \|\phi(t)\|\ dt\right)^2 \leq \int_0^1 \|\phi(t)\|^2\ dt. \tag{C.34}$$

Using the norm definition, this results in (C.31). □

C.5 Decomposition of a Positive Definite Matrix Function that is Continuously Differentiable

C.5.1 Introduction

When studying nonlinear problems in a quadratic form, it can be of interest to decompose a continuously differentiable, positive definite matrix function A, which maps from $\mathcal{X} \subseteq \mathbb{R}^n$ to $\mathbb{R}^{n\times n}$ into a constant matrix that is pre- and post multiplied with another matrix function that maps from $\mathcal{X} \subseteq \mathbb{R}^n$ to $\mathbb{R}^{n\times n}$, i.e., for all $x \in \mathcal{X}$ we have $A(x) = B(x)^\top Q B(x)$. In this appendix, we show that this can be accomplished under certain assumptions.

This appendix is based on the work in (Verhoek, Koelewijn, et al. 2022).

C.5.2 Problem definition

Consider the matrix function $A : \mathcal{X} \to \mathbb{R}^{n\times n}$, where $\mathcal{X} \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$, and assume that A admits the following properties

- $\forall x \in \mathcal{X}$, we have that $A(x)$ is real, symmetric and bounded.
- $A(x)$ is a $\mathcal{C}_1$ function, i.e., $A(x)$ and $\frac{\partial A(x)}{\partial x}$ are continuous functions.
- There exist two constants $0 < c_1 \leq c_2$ such that $\forall x \in \mathcal{X}$, $c_1 I \preceq A(x) \preceq c_2 I$.

Note that the last property ensures that $A(x)$ is positive definite for all $x \in \mathcal{X}$. The problem is to show that $A(x)$ can always be decomposed as

$$A(x) = B(x)^\top Q B(x),$$

where the matrix Q is positive definite, i.e., $0 \prec Q \in \mathbb{R}^{n\times n}$, and the matrix function $B : \mathcal{X} \to \mathbb{R}^{n\times n}$ is a $\mathcal{C}_1$ function which is non-singular for all $x \in \mathcal{X}$, i.e., $\forall x \in \mathcal{X}$, $\det(B(x)) \neq 0$.

C.5.3 Result

We now show that this decomposition is possible under the assumptions in Appendix C.5.2. First, denote the set of all real, symmetric matrices by $\mathbb{S}$ and the set of all positive definite matrices by $\mathbb{P}$. Note that $\mathbb{P}$ is open in the $\frac{n(n+1)}{2}$–dimensional real vector space $\mathbb{S}$. The positive definite square-root function $g(A) = A^{\frac{1}{2}}$ is well-defined on $\mathbb{P}$ (Koeber and Schäfer 2006). In fact, given A, its positive definite square-root is uniquely determined by the Lagrange interpolation polynomial that maps each distinct eigenvalue λ of A to $\sqrt{\lambda}$.

Now, let A_0 be any positive definite matrix and let $B_0 = A_0^{\frac{1}{2}}$. Note that B_0 is positive definite. Define $f(B) = B^2$ on $\mathbb{S}$. Then f is $\mathcal{C}_1$ and its Fréchet derivative

$\mathrm{D}f : X \mapsto BX + XB$ is non-singular at B_0. Note that X is the argument of $\mathrm{D}f$ here, and serves as a dummy variable. As the Fréchet derivative of f is non-singular at B_0, by the inverse function theorem, f has a $\mathcal{C}_1$ local inverse defined on some neighbourhood $\mathbb{W}$ containing A_0. Since $\mathbb{P}$ is open, we may assume that $\mathbb{W} \subseteq \mathbb{P}$. As this local inverse of f gives a positive definite square-root function on $\mathbb{W}$, it must agree with g. Hence, g is $\mathcal{C}_1$ on $\mathbb{W}$. It follows that g is $\mathcal{C}_1$ on $\mathbb{P}$ because A_0 is arbitrary.

Finally, when A is a $\mathcal{C}_1$ function of x, its square root $A(x)^{\frac{1}{2}} = g(A(x))$ is also $\mathcal{C}_1$, by the chain rule. Therefore, the matrix function $A(x)$ can always be decomposed as $A(x) = B(x)^\top QB(x)$ by taking $B(x) := A(x)^{\frac{1}{2}}$ and $Q = I$.

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C.6 Automatic Grid-based LPV Embedding of Nonlinear Systems

C.6.1 Introduction

The *Linear Parameter-Varying* (LPV) framework offers systematic tools for the analysis and control of LPV models. By embedding nonlinear systems in an LPV representation, the tools from the LPV framework can be applied to nonlinear systems for convex analysis and controller synthesis. However, this embedding step is often ignored but nonetheless very important. While a few automatic embedding procedures exist, see (Tóth 2010; Abbas, Tóth, Petreczky, Meskin, and Mohammadpour Velni 2014) and references therein, these methods still require expert knowledge from the user in order to be applied. In this appendix, we propose an automated grid-based LPV embedding procedure for nonlinear systems which only uses the nonlinear model as input and a set of grid-points.

This appendix is based on the work in (Koelewijn and Tóth 2021a).

C.6.2 Problem

Assume we have a nonlinear dynamical system of the form

$$\begin{aligned} \xi x(t) &= f(x(t), u(t)); \\ y(t) &= h(x(t), u(t)); \end{aligned} \tag{C.35}$$

where ξ is the differential operator in the continuous time case, i.e. $\xi = \frac{d}{dt}$ and $\xi x(t) = \frac{d}{dt}x(t)$, and where ξ is the shift operator in discrete time case, i.e. $\xi = q$ and $\xi x(t) = x(t+1)$. The functions $f : \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{u}} \to \mathbb{R}^{n_\mathrm{x}}$ and $h : \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{u}} \to \mathbb{R}^{n_\mathrm{y}}$ are assumed to be continuously differentiable once, i.e, $f, h \in \mathcal{C}_1$, and are assumed such that $f(0,0) = 0$ and $h(0,0) = 0$. We want to embed this nonlinear system in an LPV representation of the form

$$\begin{aligned} \xi x(t) &= A(p(t))x(t) + B(p(t))u(t); \\ y(t) &= C(p(t))x(t) + D(p(t))u(t); \end{aligned} \tag{C.36}$$

where $p(t) = \eta(x(t), u(t))$ is the scheduling-variable and $\eta : \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{u}} \to \mathbb{R}^{n_\mathrm{p}}$ is the scheduling-map. Based on Definition 2.14, the LPV model (C.36) is an global LPV embedding of the nonlinear system (C.35) on the region $\mathcal{X} \times \mathcal{U}$, if there exists a scheduling-map η such that $f(x,u) = A(\eta(x,u))x + B(\eta(x,u))u$ and $h(x,u) = C(\eta(x,u))x + D(\eta(x,u))u$ for all $x \in \mathcal{X} \subseteq \mathbb{R}^{n_\mathrm{x}}$ and $u \in \mathcal{U} \subseteq \mathbb{R}^{n_\mathrm{u}}$. We say a *grid-based* LPV model is an embedding of the nonlinear system (C.35) on the region $\mathcal{X} \times \mathcal{U}$, if there exists a scheduling-map η such that $f(x,u) = A(\eta(x,u))x + B(\eta(x,u))u$ and $h(x,u) = C(\eta(x,u))x + D(\eta(x,u))u$ for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$, where $\mathbb{X}$ and $\mathbb{U}$ are finite sets with $\mathbb{X} \times \mathbb{U} \subset \mathcal{X} \times \mathcal{U}$ and such that $\mathbb{X} \times \mathbb{U}$ sufficiently covers $\mathcal{X} \times \mathcal{U}$.

C.6.3 Main results

To perform the embedding, we use Lemma C.1.1, see Appendix C.1, which allows us to state the following theorem:

Theorem C.6.1 (Factorized LPV embedding)**.** *The LPV system (C.36) with*

$$\begin{aligned} A(p) &= \bar{A}(x,u), & B(p) &= \bar{B}(x,u), \\ C(p) &= \bar{C}(x,u), & D(p) &= \bar{D}(x,u), \end{aligned} \tag{C.37}$$

where $p = \eta(x,u)$ and,

$$\begin{aligned} \bar{A}(x,u) &= \int_0^1 \frac{\partial f}{\partial x}(\lambda x, \lambda u)\, d\lambda, & \bar{B}(x,u) &= \int_0^1 \frac{\partial f}{\partial u}(\lambda x, \lambda u)\, d\lambda, \\ \bar{C}(x,u) &= \int_0^1 \frac{\partial h}{\partial x}(\lambda x, \lambda u)\, d\lambda, & \bar{D}(x,u) &= \int_0^1 \frac{\partial h}{\partial u}(\lambda x, \lambda u)\, d\lambda, \end{aligned} \tag{C.38}$$

is an LPV embedding of the nonlinear system (C.35) on any region $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{u}}$.

Proof. Applying Lemma C.1.1 to f and h of the nonlinear system (C.35), we obtain that

$$f(x,u) \underbrace{-f(0,0)}_{=0} = \left(\int_0^1 \frac{\partial f}{\partial x}(\lambda x, \lambda u)\right) x + \left(\int_0^1 \frac{\partial f}{\partial u}(\lambda x, \lambda u)\right) u, \tag{C.39a}$$

$$h(x,u) \underbrace{-h(0,0)}_{=0} = \left(\int_0^1 \frac{\partial h}{\partial x}(\lambda x, \lambda u)\right) x + \left(\int_0^1 \frac{\partial h}{\partial u}(\lambda x, \lambda u)\right) u, \tag{C.39b}$$

for all $(x,u) \in \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{u}}$. Taking the scheduling-map as[2] $\eta(x,u) = \begin{bmatrix} x^\top & u^\top \end{bmatrix}^\top$, using the definitions for $\bar{A}, \ldots, \bar{D}$ in (C.38), and using (C.37), we obtain our result. Moreover, as (C.39) holds for all $x \in \mathbb{R}^{n_\mathrm{x}}$ and $u \in \mathbb{R}^{n_\mathrm{u}}$, the embedding can be taken on any region $\mathcal{X} \times \mathcal{U} \subset \mathbb{R}^{n_\mathrm{x}} \times \mathbb{R}^{n_\mathrm{u}}$. □

If we can analytically compute the integrals in (C.39), the above approach can even be used to obtain LPV embeddings with a specific scheduling dependency, such as an affine or rational dependency. We briefly demonstrate the approach in the following example:

[2] Based on the dependency of the Jacobians on x and u, only certain elements of x and u might be required in the scheduling-map.

Example C.1 *(LPV embedding of an unbalanced disk system)*. Consider the dynamics of an unbalanced disk system[3] given by

$$\dot{x}_1(t) = x_1(t); \tag{C.40a}$$

$$\dot{x}_2(t) = \tfrac{Mgl}{J}\sin(x_1(t)) - \tfrac{1}{\tau}x_2(t) + \tfrac{K_m}{\tau}u(t); \tag{C.40b}$$

$$y(t) = x_1(t); \tag{C.40c}$$

where x_1 represents the angle of the disk, x_2 represent its angular velocity, u the input voltage to the motor, y is the output of the system, and M, g, l, J, τ, and K_m are physical parameters of the system. Applying Theorem C.6.1 to (C.40), we obtain

$$\dot{x}(t) = \overbrace{\begin{bmatrix} 0 & 1 \\ \frac{Mgl}{J}\int_0^1 \cos(\lambda x_1(t))\,d\lambda & -\frac{1}{\tau} \end{bmatrix}}^{\bar{A}(x,u)} + \overbrace{\begin{bmatrix} 0 \\ \frac{K_m}{\tau} \end{bmatrix}}^{\bar{B}(x,u)} u(t); \tag{C.41a}$$

$$y(t) = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\bar{C}(x,u)} x(t); \tag{C.41b}$$

where $x = \mathrm{col}(x_1, x_2)$. As $\int_0^1 \cos(\lambda x_1(t))\,d\lambda = \frac{\sin(x_1(t))}{x_1(t)} = \mathrm{sinc}(x_1(t))$, we can obtain the following affine LPV representation:

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ \frac{Mgl}{J}p(t) & -\frac{1}{\tau} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{K_m}{\tau} \end{bmatrix} u(t); \tag{C.42a}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t); \tag{C.42b}$$

where $p(t) = \mathrm{sinc}(x_1(t))$, which is an LPV embedding of (C.40) .

C.6.4 Grid-Based LPV Models

While the Jacobians of f and h can often easily be automatically computed using various symbolic computation packages such as found in MATLAB (The MathWorks, Inc. 2022) and Mathematica (Wolfram Research, Inc. 2022), computation of the integrals of the Jacobians of f and h is often more difficult (although in some cases it can still be performed, e.g., as in Example C.1). However, for grid-based LPV models, it is not necessary to have an analytical expression of the LPV state-space matrices, as we only require the values of the matrices at the chosen grid-points. In that case, we can use numerical integration techniques to compute the value of the matrices in (C.38) at different points x and u. In fact, an analytical expression for the Jacobian is also not necessary, as *Automatic Differentiation* (AD) techniques can be used to evaluate the Jacobians of f and h at different point required for numerical integration. Numerical integration techniques and AD techniques have been implemented in various software tools, such as in Python (Maclaurin et al.

[3]See Example 4.2 for more details.

2019) and Julia (Revels et al. 2016). Hence, we can rely on numerical methods for computation of the grid-based LPV state-space matrices using the embedding method described above.

More formally, assume that we want to obtain a grid-based LPV model of the nonlinear system (C.35) at the grid-points $x \in \mathbb{X} \subset \mathcal{X}$ and $u \in \mathbb{U} \subset \mathcal{U}$, with $\mathbb{X} = \{x_1, \ldots, x_\mathrm{N}\}$ and $\mathbb{U} = \{u_1, \ldots, u_\mathrm{M}\}$, where $N, M \in \mathbb{N}$. We can then evaluate using numerical integration (and if necessary using AD), for all $x \in \mathbb{X}$ and $u \in \mathbb{U}$, the matrices in (C.38). As the scheduling-variable is $p = \mathrm{col}(x, u)$, we obtain set of $N \cdot M$ grid-points for p. This results in a grid-based LPV model on the compact region $\mathcal{X} \times \mathcal{U}$ (at the points $x \in \mathbb{X}$ and $u \in \mathbb{U}$) of the form (C.36) for (C.35).

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List of Abbreviations

AD	Automatic Differentiation
AE	Autoencoder
ANN	Artifical Neural Network
CMG	Control Momement Gyroscope
CT	Continuous-Time
DNN	Deep (Artifical) Neural Network
DOF	Degree of Freedom
DPV	Differential Parameter-Varying
DT	Discrete-Time
e.g.	exempli gratia (for example)
etc.	et cetera (and so forth)
GP	Gaussian Process
GPRV	Generic Parafoil Return Vehicle
i.e.	id est (that is)
IO	Input-Output
IQC	Integral Quadratic Constraint
KPCA	Kernel Principal Component Analysis
KYP	Kalman-Yakubovich-Popov
LFR	Linear Fractional Representation
LMI	Linear Matrix Inequality
LPV	Linear Parameter-Varying
LQG	Linear-Quadratic-Gaussian
LQR	Linear-Quadratic Regulator

LTI	Linear Time-Invariant
MIMO	Multi-Input-Multi-Output
MPC	Model Predictive Control
MRAC	Model Reference Adaptive Control
NMPC	Nonlinear Model Predictive Control
PCA	Principal Component Analysis
PID	Proportional–Integral–Derivative
PWA	Piecewise-Affine
ReLU	Rectified Linear Unit
s.t.	such that
SDP	Semidefinite Program
SDR	Scheduling Dimension Reduction
SISO	Single-Input-Single-Output
SOS	Sum-of-Squares
SQP	Sequential Quadratic Programming
STC	Self-Tuning Control
SVD	Singular Value Decomposition
TS	Takagi-Sugeno
TSK	Takagi-Sugeno-Kang
VPV	Velocity Parameter-Varying
w.l.o.g.	without loss of generality
w.r.t.	with respect to

List of Symbols

Fields and spaces

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of positive real numbers
$\mathbb{R}_0^+$	Set of non-negative real numbers
$\mathbb{N}$	Set of natural numbers (excluding zero)
$\mathbb{N}_0$	Set of natural numbers including zero
$\mathbb{Z}$	Set of integers
$\mathbb{I}$	Subset of ordered integers
$\mathbb{S}$	Set of real symmetric matrices
$\mathcal{T}$	Time-axis
$\mathcal{X}$	State space
$\mathcal{D}$	Space of the derivative of the state
$\mathcal{U}$	(Control) input space
$\mathcal{W}$	Generalized disturbance space
$\mathcal{Y}$	(Measured) output space
$\mathcal{Z}$	Generalized performance space
$\mathcal{P}$	Scheduling space
Π	Time derivative/difference scheduling space
$\mathfrak{A}$	Function space of LPV matrix functions
$\mathscr{E}$	Set of equilibrium points
$\mathscr{X}, \mathscr{W}, \mathscr{Z}$	Set of state, generalized disturbance, and generalized performace equilibrium points

$\mathcal{X}$, $\mathcal{W}$, $\mathcal{U}$	Embedding space w.r.t. state, input/generalized disturbance, and control input, respectively
$\mathcal{L}_p/\ell_p$	Space of integrable/summable functions with finite p-norm
$\mathcal{L}_{pe}/\ell_{pe}$	Extended $\mathcal{L}_p/\ell_p$ space
$\mathcal{C}_n$	Class of n-times continuously differentiable functions
$\mathcal{K}$	Class of strictly increasing functions
$\mathcal{Q}_\bullet$	Class of positive definite and decrescent functions

Operators

$\circ$	Function composition operator
q	Discrete time-shift operator
$\dot{x}$	Time derivative of signal x
ξ	Time dynamics operator, equal to $\frac{\partial}{\partial t}$ for CT systems and equal to q for DT systems
∂	Derivative operator on matrix function or behavior
$\equiv$	Equivalent to, on the whole function domain
$\lvert\bullet\rvert$	Absolute value
$\lVert\bullet\rVert$	Euclidian vector norm and spectral matrix norm
$\lVert\bullet\rVert_{\mathrm{F}}$	Frobenius matrix norm
det	Determinant
vec	Matrix vectorization
diag	Diagonal concatenation
col	Columnwise concatenation
π_{a}	Set projection w.r.t. a
co	Convex hull
min	Minimum
max	Maximum
arg	Argument of
inf	Infimum
sup	Supremum
ess sup	Essential supremum

∇_a	Gradient w.r.t. a
$\bullet^\top$	Transposition
$\bullet^{-1}$	Inverse
$\succ$, $(\succeq)$	Positive (semi-)definite
$\prec$, $(\preceq)$	Negative (semi-)definite
$\sum$	Summation operator
lim	Limit

Dynamical systems

G	Plant
P	Generalized plant
K	Controller
$W_\bullet$	Weight
$\mathcal{F}_\mathrm{l}(P, K)$	Closed-loop interconnection of P and K
G_v, P_v, K_v	Velocity form of a plant, a geneneralized plant, and a controller, respectively
P_vpv	VPV embedding of a generalized plant
G_δ, P_δ, K_δ	Differential form of a plant, a geneneralized plant, and a controller, respectively
P_dpv	DPV embedding of generalized plant

Behaviors

$\mathfrak{B}$	Behavior of a dynamical system
$\mathfrak{B}_\bullet$	Behavior of a dynamical system restricted to a specifc subset
$\mathfrak{B}_\mathrm{w}(\bullet)$	Behavior of a dynamical system for a given input trajectory
$\mathfrak{B}_0$	Behavior of dynamical system for the zero input
$\mathfrak{B}_\mathrm{p}(\bullet)$	Behavior of an LPV system for a given scheduling trajectory
$\mathfrak{B}_{\mathrm{p},0}(\bullet)$	Behavior of an LPV system for the zero input and for a given scheduling trajectory
$\breve{\mathfrak{B}}_\mathrm{p}$	Full behavior of an LPV system

$\mathfrak{B}_\mathrm{c}$	Subset of the behavior of a system which is continuously differentiable
$\mathfrak{B}_\mathrm{v}$	Behavior of the velocity form of a system
$\mathfrak{B}_{\mathrm{v,w}}(\bullet)$	Behavior of the velocity form of a system for a given input trajectory
$\mathfrak{B}_{\mathrm{v},\bullet}$	Behavior of the velocity form of a system for a space of input trajectories
$\mathfrak{B}_\delta(\bullet)$	Behavior of the differential form of a system along a given trajectory
$\breve{\mathfrak{B}}_\delta$	Full behavior of the differential form of a system

Signals and variables

x	State associated with state-space representation
u	(Control) input
w	Input and/or generalized disturbance
y	(Measured) output
z	Output and/or generalized performance
p	Scheduling-variable associated with an LPV representation
v	Time derivative (in CT) or time difference (in DT) of a scheduling-variable
$n_\bullet$	Dimension of a specific variable
x_k, u_k, y_k	Signals associated with a controller
x_cl	State associated with the closed-loop state-space representation
x_*, u_*, w_*, y_*, z_*, p_*	Variables associated with an equilibrium point of a system
x_v, u_v, w_v, y_v, z_v	Signals associated with the velocity form of a system
$\tilde{x}$, $\tilde{u}$, $\tilde{w}$, $\tilde{y}$, $\tilde{z}$	Signals associated with another (arbitrary) trajectory of a system
$\bar{x}$, $\bar{u}$, $\bar{w}$, $\bar{y}$, $\bar{z}$	Signals associated with parameterized famility of trajectories of a system
x_δ, u_δ, w_δ, y_δ, z_δ	Signals associated with the differential form of a system
x^*, u^*, w^*, y^*, z^*	Signals associated with the target/desired trajectory of a system

x_Δ, u_Δ, w_Δ, y_Δ, z_Δ	Signals associated with the DT velocity form of a system
φ	(Reduced) scheduling-variable associated with reduced LPV model
$l_{(\bullet)}$	Signal associated with a layer of an ANN

Functions

f	State transition function of a nonlinear state-space representation
h	Output function of a nonlinear state-space representation
ϕ_x	State transition map
$\zeta_\bullet$	Initial condition function
V	Lyapunov function
$\mathcal{V}$	Storage function
s	Supply function
A, B, C, D	Matrix functions (or matrices) associated with an LPV (or LTI) state-space representation
A_k, B_k, C_k, D_k	Matrix functions (or matrices) associated with an LPV (or LTI) state-space controller
η	Scheduling-map
V_p, $\mathcal{V}_\mathrm{p}$	Lyapunov and storage functions for dissipativity of an LPV system
M	Positive-definite matrix (function) of a quadratic Lyapunov/storage function
V_s, $\mathcal{V}_\mathrm{s}$, s_s	Lyapunov, storage, and supply functions for universal shifted stability and dissipativity
V_v, $\mathcal{V}_\mathrm{v}$, s_v	Lyapunov, storage, and supply functions for velocity dissipativity
A_v, B_v, C_v, D_v	Matrix functions associated with the velocity form of a system
V_i, $\mathcal{V}_\mathrm{i}$, s_i	Lyapunov, storage, and supply functions for incremental stability and dissipativity
V_δ, $\mathcal{V}_\delta$, s_δ	Lyapunov, storage, and supply functions for differential dissipativity
A_δ, B_δ, C_δ, D_δ	Matrix functions associated with the differential form of a system
$\hat{A}$, $\hat{B}$, $\hat{C}$, $\hat{D}$	Matrix functions associated with a scheduling dimension reduced LPV state-space representation

L, $\hat{L}$	LPV matrix functions for scheduling dimension reduction
μ	Scheduling reduction map
$\mathcal{N}$	Row-wise matrix normalization function
σ	Activation function of an ANN

Coefficients and constants

γ	Induced $\mathcal{L}_p$-$\mathcal{L}_q$-gain of system
Q, S, R	Matrices corresponding to (Q, S, R) dissipativity
T_s	Sampling time
N	Number of samples
$W^{[\bullet]}$	Weight matrix in a layer of an ANN
$b^{[\bullet]}$	Bias vector in a layer of an ANN

Data sets and matrices

Γ, Γ_n, $\hat{\Gamma}$	Data matrix of scheduling trajectories
$\mathfrak{D}$	Data set of scheduling trajectories
$\mathfrak{D}_\mathrm{xu}$	Data set of state and input trajectories

Publiekssamenvatting

Analyse en Regeling van Niet-Lineaire Systemen met Stabiliteits- en Prestatiegaranties

Een Lineaire Parameter-Variërende Benadering

Bij technische systemen, zoals die in de lucht- en ruimtevaart of de mechatronica-industrie, wordt voortdurend gestreefd naar betere prestaties. Bijvoorbeeld de verbetering van de manoeuvreerbaarheid en betrouwbaarheid van lucht- en ruimtevaartuigen of de verbetering van de nauwkeurigheid en toepasbaarheid van robots. Dit heeft geleid tot complexere systeemontwerpen, die gericht zijn op betere prestaties, wat ook leidt tot complexer dynamisch gedrag van deze systemen.

In de afgelopen decennia is een groot aantal methodes ontwikkeld voor het analyseren en ontwerpen van regelalgoritmen voor dynamische systemen, zelfs voor zeer complexe systemen. Desondanks, gebruiken ingenieurs in de industrie meestal methodes die ervan uitgaan dat het onderliggende systeem lineair en tijdinvariant is. Voor lineaire dynamische systemen bestaat namelijk een uitgebreid, systematisch regeltechnisch-raamwerk dat relatief gemakkelijk te gebruiken is en waarmee ingenieurs de prestaties van systemen gemakkelijk kunnen afstellen. Bovendien wordt het ook uitgebreid ondersteund door krachtige computerprogramma's. Naarmate we echter hogere prestaties nastreven, wordt het gedrag van de systemen gedomineerd door niet-lineairiteiten en zijn deze methodes niet meer toereikend.

Hoewel voor niet-lineaire dynamische systemen er ook veel theorie bestaat voor prestatieanalyse en regelaarontwerp, zijn de daarmee gepaard gaande methodes vaak te complex voor gebruik in de techniek, zijn ze rekenkrachtig te duur, of missen ze de mogelijkheid om de prestatie af te stellen. Om dit probleem aan te pakken, hebben we in dit proefschrift een raamwerk ontwikkeld voor systematische en rekenkundig efficiënte prestatieanalyse en regeling van niet-lineaire dynamische systemen in de techniek. Bovendien kunnen met het ontwikkelde raamwerk de prestaties van systemen eenvoudig worden afgesteld.

Om deze doorbraak te bereiken hebben wij eerst systematische en rekenkundig efficiënte wiskundige analysemethodes ontwikkeld voor niet-lineaire dynamische systemen. Met de ontwikkelde methodes kunnen wij een vrij algemene analyse maken van de veiligheid en de prestaties van de operatie van het systeem, ongeacht de referentiepunten waar het systeem moet opereren of het gewenste referentiegedrag dat het moet volgen. Dit is bijvoorbeeld nuttig om te analyseren of een ruimtevaartuig veilig werkt en voldoet aan de gewenste prestatie-eisen gedurende verschillende vliegroutes.

Vervolgens hebben wij systematische en rekenkundig efficiënte regelalgoritmen ontwikkeld die deze veiligheids- en prestatieconcepten ook kunnen garanderen voor niet-lineaire dynamische systemen. Belangrijk is dat het afstellen van de prestaties van de ontwikkelde regelalgoritmen vergelijkbaar is met dat van bestaande lineaire methoden. Dit maakt het afstemmen van de regelalgoritmen voor complexe niet-lineaire systemen om de gewenste prestaties te bereiken eenvoudig voor de gebruiker. Bovendien maakt de rekenkundige efficiëntie van de ontwikkelde regeltechnieken het mogelijk om snel te prototypen, wat leidt tot een snellere ontwikkeling van nieuwe en betere producten.

Tenslotte hebben wij een methode ontwikkeld om de complexiteit van dynamische systeemmodellen te verminderen. In combinatie met de ontwikkelde analyse- en regelalgoritmen stelt dit ons in staat zeer complexe systemen aan te pakken.

Concluderend, met het ontwikkelde raamwerk kunnen we systematisch en efficiënt het opereren van complexe systemen analyseren en er regelaars voor ontwerpen. Deze gereedschappen zullen ingenieurs in staat stellen hun systemen naar hogere prestatieniveaus te tillen en tegelijkertijd een veilige en robuuste werking van deze systemen te garanderen.

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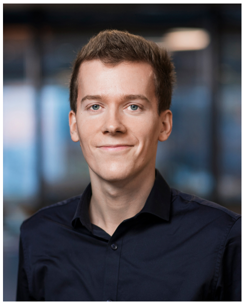
About the Author



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