

Arrow Contraction and Expansion in Tropical Diagrams

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ARROW CONTRACTION AND EXPANSION IN TROPICAL DIAGRAMMS

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ABSTRACT. Arrow contraction applied to a tropical diagram of probability spaces is a modification of the diagram, replacing one of the morphisms by an isomorphism, while preserving other parts of the diagram. It is related to the rate regions introduced by Ahlswede and Körner. In a companion article we use arrow contraction to derive information about the shape of the entropic cone. Arrow expansion is the inverse operation to the arrow contraction.

1. INTRODUCTION

In [MP18] we have initiated the theory of *tropical probability spaces* and in [MP19c] applied the techniques to derive a dimension-reduction result for the entropic cone of four random variables.

Two of the main tools used for the latter are what we call *arrow contraction* and *arrow expansion*. They are formulated for tropical commutative diagrams of probability spaces. Tropical diagrams are points in the asymptotic cone of the metric space of commutative diagrams of probability spaces endowed with the asymptotic entropy distance. Arrows in diagrams of probability spaces are (equivalence classes of) measure-preserving maps.

Arrow contraction and expansion take a commutative diagram of probability spaces as input, modify it, but preserve important properties of the diagram. The precise results are formulated as Theorems 3.1 and 3.3 in the main text. Their formulation requires language, notation and definitions that we review in Section 2.

However, to give an idea of the results in this paper, we now present two examples.

1.1. Two examples.

1.1.1. *Arrow contraction and expansion in a two-fan.* Suppose we are given a fan $\mathcal{Z} = (X \leftarrow Z \rightarrow Y)$ and we would like to complete it to a diamond

$$(1.1) \quad \mathcal{Z}_\diamond = \left(\begin{array}{ccccc} & & Z & & \\ & \swarrow & & \searrow & \\ X & & & & Y \\ & \searrow & & \swarrow & \\ & & V & & \end{array} \right)$$

such that the entropy of V , denoted by $[V]$, equals the mutual information $[X : Y]$ between X and Y . That is, we would like to realize the mutual

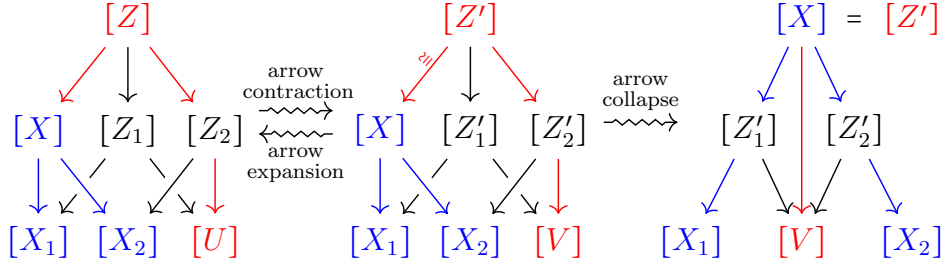


FIGURE 1. Arrow contraction and expansion in a Λ_3 -diagram. The fan $([X] \leftarrow [Z] \rightarrow [U])$ is admissible. Spaces $[Z_1]$, $[Z_2]$ and $[Z]$ belong to the co-ideal $[U]$.

information between X and U by a pair of reductions $X \rightarrow V$ and $U \rightarrow V$. This is not always possible, not even approximately.

Arrow contraction instead produces another fan $Z' = (X \leftarrow Z' \rightarrow V)$, such that the reduction $Z' \rightarrow X$ is an isomorphism and the relative entropy $[X[V]]$ of X given V equals $[X[U]]$. By collapsing this reduction we obtain as a diagram just the reduction $X \rightarrow V$. If need be, we can still keep the original spaces Z and U in the modified diagram obtaining the “broken diamond” diagram

$$\begin{array}{ccc} & Z & \\ X & \leftarrow & Y \\ & \searrow & \\ & V & \end{array}$$

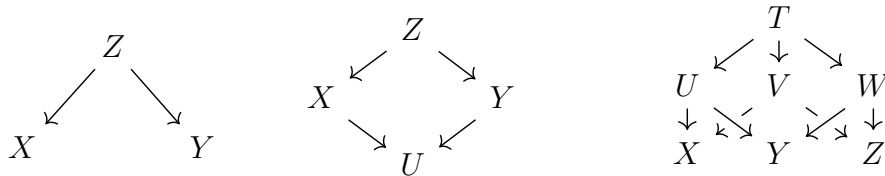
such that $[V] = [X : Y]$. Of course, no special technique is necessary to achieve this result, since it is easy to find a reduction from a tropical space $[X]$ to another tropical probability space with the prespecified entropy, as long as the Shannon inequality is not violated.

However, a similar operation becomes non-trivial and in fact impossible without passing to the tropical limit, if instead of a single space X there is a more complex sub-diagram as in the example in the next subsection.

To explain how arrow expansion works, let's start with the chain of reductions $Z \rightarrow X \rightarrow V$. Can we extend it to a diamond, as in (1.1) so that $[X : Y[V]] = 0$? This is again not possible, in general. However, if we pass to tropical diagrams, then such an extension always exists.

1.1.2. *One More Example of Arrow Expansion and Contraction.* Consider a diagram presented in Figure 1. Such a diagram is called a Λ_3 -diagram. We would like to find a reduction $X \rightarrow V$ so that $[\mathcal{X}[U]] = [\mathcal{X}[V]]$. It is not possible to achieve this within the realm of diagrams of classical probability spaces. But once we pass to the tropical limit, the reduction $[X] \rightarrow [V]$ can be found by contracting and then collapsing the arrow $[Z] \rightarrow [X]$, as shown in Figure 1.

Arrow contraction is closely related to the Shannon channel coding theorem. This is perhaps most obvious from the proof. Furthermore, arrow contraction has connections with rate regions as introduced by Ahlswede and Körner,



1. A fan 2. A diamond diagram 3. Full diagram on 3 spaces

FIGURE 2. Examples of diagrams of probability spaces

see [AK77, AK06]. These results by Ahlswede and Körner were applied by [MMRV02], resulting in a new non-Shannon information inequality. Moreover, in [MMRV02] a new proof was given of the results; this new proof is similar to the proof of the arrow contraction result in the present paper.

The main contribution of our work lies in the fact that we prove a much stronger preservation of properties of the diagram under arrow contraction.

2. PRELIMINARIES

2.1. Probability spaces and their diagrams. Our main objects of study will be *commutative diagrams of probability spaces*. A *finite probability space* X is a set with a probability measure on it, supported on a finite set. We denote by $|X|$ the cardinality of the support of the measure. The statement $x \in X$ means that point x is an atom with positive weight in X . For details see [MP18, MP19b, MP19a].

Examples of commutative diagrams of probability spaces are shown in Figure 2. The objects in such diagrams are finite probability spaces and morphisms are equivalence classes of measure-preserving maps. Two such maps are considered to be equivalent, if they coincide on a set of full measure. To record the combinatorial structure of a commutative diagram, i.e. the arrangement of spaces and morphisms, we use *indexing categories*, which are poset categories satisfying an additional property, that we describe below.

2.1.1. Indexing Categories. A *poset category* is a finite category such that there are at most one morphism between any two objects either way.

For a pair of objects k, l in a poset category $\mathbf{G} = \{i; \gamma_{ij}\}$, such that there is a morphism γ_{kl} in \mathbf{G} , we call k an ancestor of l and l a descendant of k . The set of all ancestors of an object k together with all the morphisms between them is itself a poset category and will be called a *co-ideal* generated by k and denoted by $[k]$. Similarly, a poset category consisting of all descendants of $k \in \mathbf{G}$ and morphisms between them will be called an *ideal* generated by k and denoted $[k]$.

An *indexing category* $\mathbf{G} = \{i; \gamma_{ij}\}$ used for indexing diagrams, is a poset category satisfying the following additional property: for any pair of objects $i_1, i_2 \in \mathbf{G}$ the intersection of co-ideals is also a co-ideal generated by some object $i_3 \in \mathbf{G}$,

$$[i_1] \cap [i_2] = [i_3]$$

In other words, for any pair of objects $i_1, i_2 \in \mathbf{G}$ there exists a *least common ancestor* i_3 , that is i_3 is an ancestor to both i_1 and i_2 and any other common ancestor is also an ancestor of i_3 . Any indexing category is *initial*, i.e. there is a (necessarily unique) *initial* object \hat{i} in it, which is the ancestor of any other object in \mathbf{G} , in other words $\mathbf{G} = [\hat{i}]$.

A *fan* in a category is a pair of morphisms with the same domain. A fan $(i \leftarrow k \rightarrow j)$ is called *minimal*, if for any other fan $(i \leftarrow l \rightarrow j)$ included in a commutative diagram

$$\begin{array}{ccc} & k & \\ & \swarrow & \searrow \\ i & & j \\ & \nwarrow & \nearrow \\ & l & \end{array}$$

the vertical morphism $(k \rightarrow l)$ must be an isomorphism. Any indexing category also satisfies the property that for any pair of objects in it there exists a minimal fan with target objects the given ones.

This terminology will also be applied to diagrams of probability spaces indexed by \mathbf{G} . Thus, given a space X in a \mathbf{G} -diagram, we can talk about its ancestors, descendants, co-ideal $\llbracket X \rrbracket$ and ideal $\lllbracket X \lllbracket$. We use square brackets to denote tropical diagrams and spaces in them. For the (co-)ideals in tropical diagrams, in order to unclutter notations, we will write

$$\llbracket X \rrbracket := \lllbracket [X] \lllbracket \quad \text{and} \quad \lllbracket X \lllbracket := \lllbracket \lllbracket [X] \lllbracket$$

2.1.2. *Diagrams.* For an indexing category $\mathbf{G} = \{i; \gamma_{ij}\}$ and a category \mathbf{Cat} , a commutative \mathbf{G} -diagram $\mathcal{X} = \{X_i; \chi_{ij}\}$ is a functor $\mathcal{X} : \mathbf{G} \rightarrow \mathbf{Cat}$. A diagram \mathcal{X} is called *minimal* if it maps minimal fans in \mathbf{G} to minimal fans in \mathbf{Cat} .

A *constant \mathbf{G} -diagram* denoted $X^{\mathbf{G}}$ is a diagram where all the objects equal to X and all morphisms are identities.

Important examples of indexing categories are a two-fan, a diamond category, a full category $\mathbf{\Lambda}_n$ on n spaces, chains \mathbf{C}_n . For detailed description and more examples, the reader is referred to the articles cited at the beginning of this section.

2.2. Tropical Diagrams.

2.2.1. *Intrinsic Entropy Distance.* For a fixed indexing category \mathbf{G} the space of commutative \mathbf{G} -diagrams will be denoted by $\mathbf{Prob} \langle \mathbf{G} \rangle$. Evaluating entropy on every space in a \mathbf{G} -diagram gives a map

$$\mathbf{Ent}_* : \mathbf{Prob} \langle \mathbf{G} \rangle \rightarrow \mathbb{R}^{\mathbf{G}}$$

where the target space $\mathbb{R}^{\mathbf{G}}$ is the space of real-valued functions on objects of \mathbf{G} . We endow this space with the ℓ^1 -norm. For a fan $\mathcal{F} = (\mathcal{X} \leftarrow \mathcal{Z} \rightarrow \mathcal{Y})$ of \mathbf{G} -diagrams we define the entropy distance between it terminal objects by

$$\text{kd}(\mathcal{F}) := \|\mathbf{Ent}_* \mathcal{Z} - \mathbf{Ent}_* \mathcal{X}\|_1 + \|\mathbf{Ent}_* \mathcal{Z} - \mathbf{Ent}_* \mathcal{Y}\|_1$$

and the intrinsic entropy distance between two arbitrary \mathbf{G} -diagrams by

$$\mathbf{k}(\mathcal{X}, \mathcal{Y}) := \inf \{\text{kd}(\mathcal{F}) : \mathcal{F} = (\mathcal{X} \leftarrow \mathcal{Z} \rightarrow \mathcal{Y})\}$$

The triangle inequality for \mathbf{k} and various its other properties are discussed in [MP18].

In the same article, also a useful estimate for the intrinsic entropy distance is proven, called the Slicing Lemma. The following corollary, [MP18, Corollary 3.10(1)], of the Slicing Lemma will be used in the next section.

Proposition 2.1. *Let \mathbf{G} be an indexing category, $\mathcal{X}, \mathcal{Y} \in \mathbf{Prob}\langle \mathbf{G} \rangle$ and $U \in \mathbf{Prob}$ included into a pair of two-fans*

$$\mathcal{X} \longleftarrow \tilde{\mathcal{X}} \longrightarrow U^{\mathbf{G}} \qquad U^{\mathbf{G}} \longleftarrow \tilde{\mathcal{Y}} \longrightarrow \mathcal{Y}$$

Then

$$\mathbf{k}(\mathcal{X}, \mathcal{Y}) \leq \int_U \mathbf{k}(\mathcal{X}|_u, \mathcal{Y}|_u) dp_U(u) + 2 \cdot \llbracket \mathbf{G} \rrbracket \cdot \text{Ent}(U)$$

□

2.2.2. *Tropical Diagrams.* Points in the asymptotic cone of $(\mathbf{Prob}\langle \mathbf{G} \rangle, \mathbf{k})$ are called tropical \mathbf{G} -diagrams and the space of all tropical \mathbf{G} -diagrams, denoted $\mathbf{Prob}[\mathbf{G}]$, is endowed with the *asymptotic entropy distance*. We explain this now in more detail and a more extensive description can be found in [MP19b].

To describe points in $\mathbf{Prob}[\mathbf{G}]$ we consider certain sequences $\bar{\mathcal{X}} := (\mathcal{X}(n) : n \in \mathbb{N})$ of \mathbf{G} -diagrams, that grow almost linearly and endow the space of all such sequences with the *asymptotic entropy distance* defined by

$$\kappa(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) := \lim_{n \rightarrow \infty} \frac{1}{n} \mathbf{k}(\mathcal{X}(n), \mathcal{Y}(n))$$

A tropical diagram $[\mathcal{X}]$ is defined to be an equivalence class of such sequences, where two sequences $\bar{\mathcal{X}}$ and $\bar{\mathcal{Y}}$ are equivalent, if $\kappa(\bar{\mathcal{X}}, \bar{\mathcal{Y}}) = 0$. The space $\mathbf{Prob}[\mathbf{G}]$ carries the asymptotic entropy distance and has the structure of a $\mathbb{R}_{\geq 0}$ -semi-module – one can take linear combinations with non-negative coefficients of tropical diagrams. The linear entropy functional $\text{Ent}_* : \mathbf{Prob}[\mathbf{G}] \rightarrow \mathbb{R}^{\mathbf{G}}$ is defined by

$$\text{Ent}_*[\mathcal{X}] := \lim_{n \rightarrow \infty} \frac{1}{n} \text{Ent}_* \mathcal{X}(n)$$

A detailed discussion about tropical diagrams can be found in [MP19b]. In the cited article we show that, the space $\mathbf{Prob}[\mathbf{G}]$ is metrically complete, and isometrically isomorphic to a closed convex cone in some Banach space.

For $\mathbf{G} = \mathbf{C}_k$ a *chain category*, containing k objects $\{1, \dots, k\}$ and unique morphism $i \rightarrow j$ for every pair $i \geq j$, we have shown that the space $\mathbf{Prob}[\mathbf{C}_k]$ is isomorphic to the following cone in $(\mathbb{R}^k, \|\cdot\|_1)$

$$\mathbf{Prob}[\mathbf{C}_k] \cong \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} : 0 \leq x_1 \leq \dots \leq x_k \right\}$$

The isomorphism is given by the entropy functional. Thus we can identify tropical probability spaces (elements in $\mathbf{Prob}[\mathbf{C}_1]$) with non-negative numbers via entropy. We will simply write $[X]$ to mean the entropy of the space $[X]$.

Along this line we also adopt the notations $[X|Y]$, $[X : Y]$ and $[X : Y|Z]$ for the relative entropy and mutual information for the tropical spaces included in some diagram.

2.3. Asymptotic Equipartition Property for Diagrams.

2.3.1. *Homogeneous diagrams.* A \mathbf{G} -diagram \mathcal{X} is called *homogeneous* if the automorphism group $\text{Aut}(\mathcal{X})$ acts transitively on every space in \mathcal{X} . Homogeneous probability spaces are uniform. For more complex indexing categories this simple description is not sufficient.

2.3.2. *Tropical Homogeneous Diagrams.* The subcategory of all homogeneous \mathbf{G} -diagrams will be denoted $\mathbf{Prob}\langle\mathbf{G}\rangle_{\mathbf{h}}$ and we write $\mathbf{Prob}\langle\mathbf{G}\rangle_{\mathbf{h},\mathbf{m}}$ for the category of minimal homogeneous \mathbf{G} -diagrams. These spaces are invariant under the tensor product, thus they are metric Abelian monoids.

Passing to the tropical limit we obtain spaces of tropical (minimal) homogeneous diagrams, that we denote $\mathbf{Prob}[\mathbf{G}]_{\mathbf{h}}$ and $\mathbf{Prob}[\mathbf{G}]_{\mathbf{h},\mathbf{m}}$.

2.3.3. *Asymptotic Equipartition Property.* In [MP18] the following theorem is proven

Theorem 2.2. *Suppose $\mathcal{X} \in \mathbf{Prob}\langle\mathbf{G}\rangle$ is a \mathbf{G} -diagram of probability spaces for some fixed indexing category \mathbf{G} . Then there exists a sequence $\mathcal{H} = (\mathcal{H}_n)_{n=0}^{\infty}$ of homogeneous \mathbf{G} -diagrams such that*

$$(2.1) \quad \frac{1}{n} \mathbf{k}(\mathcal{X}^n, \mathcal{H}_n) \leq C(|X_0|, \llbracket \mathbf{G} \rrbracket) \cdot \sqrt{\frac{\ln^3 n}{n}}$$

where $C(|X_0|, \llbracket \mathbf{G} \rrbracket)$ is a constant only depending on $|X_0|$ and $\llbracket \mathbf{G} \rrbracket$. \(\square\)

The approximating sequence of homogeneous diagrams is evidently quasi-linear with the defect bounded by the admissible function

$$\varphi(t) := 2C(|X_0|, \llbracket \mathbf{G} \rrbracket) \cdot t^{3/4} \geq 2C(|X_0|, \llbracket \mathbf{G} \rrbracket) \cdot t^{1/2} \cdot \ln^{3/2} t$$

Thus, Theorem 2.2 above states that $\mathbf{L}(\mathbf{Prob}\langle\mathbf{G}\rangle) \subset \mathbf{Prob}[\mathbf{G}]_{\mathbf{h}}$. On the other hand we have shown in [MP19b], that the space of linear sequences $\mathbf{L}(\mathbf{Prob}\langle\mathbf{G}\rangle)$ is dense in $\mathbf{Prob}[\mathbf{G}]$. Combining the two statements we get the following theorem.

Theorem 2.3. *For any indexing category \mathbf{G} , the space $\mathbf{Prob}[\mathbf{G}]_{\mathbf{h}}$ is dense in $\mathbf{Prob}[\mathbf{G}]$. Similarly, the space $\mathbf{Prob}[\mathbf{G}]_{\mathbf{h},\mathbf{m}}$ is dense in $\mathbf{Prob}[\mathbf{G}]_{\mathbf{m}}$. \(\square\)*

It is possible that the spaces $\mathbf{Prob}[\mathbf{G}]_{\mathbf{h}}$ and $\mathbf{Prob}[\mathbf{G}]$ coincide. At this time we have neither a proof nor a counterexample to this conjecture.

2.4. Conditioning in Tropical Diagrams. For a tropical \mathbf{G} -diagram $[\mathcal{X}]$ containing a space $[U]$ we defined a conditioned diagram $[\mathcal{X}[U]]$. It can be understood as the tropical limit of the sequence $(\mathcal{X}(n)[u_n])$, where $(\mathcal{X}(n))$ is the homogeneous approximation of $[\mathcal{X}]$, $U(n)$ is the space in $\mathcal{X}(n)$ that corresponds to $[U]$ under combinatorial isomorphism and u_n is any atom in $U(n)$.

We have shown in [MP19a] that operation of conditioning is Lipschitz-continuous with respect to the asymptotic entropy distance.

3. ARROW CONTRACTION

3.1. Arrow Collapse, Arrow Contraction and Arrow Expansion.

3.1.1. Prime Morphisms. A morphism $\gamma_{ij} : i \rightarrow j$ in an indexing category $\mathbf{G} = \{i; \gamma_{ij}\}$ will be called *prime* if it cannot be factored into a composition of two non-identity morphisms in \mathbf{G} . Morphism in a \mathbf{G} -diagram indexed by a prime morphism in \mathbf{G} will also be called *prime*.

3.1.2. Arrow Collapse. Suppose \mathcal{Z} is a \mathbf{G} -diagram such that for some pair $i, j \in \mathbf{G}$, the prime morphism $\zeta_{ij} : Z_i \rightarrow Z_j$ is an isomorphism. *Arrow collapse* applied to \mathcal{Z} results in a new diagram \mathcal{Z}' obtained from \mathcal{Z} by identifying Z_i and Z_j via the isomorphism ζ_{ij} . The combinatorial type of \mathcal{Z}' is different from that of \mathcal{Z} . The spaces Z_i and Z_j are replaced by a single space and the new space will inherit all the morphisms in \mathcal{Z} with targets and domains Z_i and Z_j .

3.1.3. Arrow Contraction and Expansion. *Arrow contraction and expansion* are two operations on tropical \mathbf{G} -diagrams. Roughly speaking, arrow contraction applied to a tropical \mathbf{G} -diagram $[\mathcal{Z}]$ results in another tropical \mathbf{G} -diagram $[\mathcal{Z}']$ such that one of the arrows become an isomorphism, while some parts of the diagram are not modified. Arrow expansion is an inverse operation to arrow contraction.

3.1.4. Admissible and Reduced Sub-fans. An *admissible fan* in a \mathbf{G} -diagram \mathcal{Z} is a minimal fan $X \leftarrow Z \rightarrow U$, such that Z is the initial space of \mathcal{Z} and any space in \mathcal{Z} belongs either to the co-ideal $[X]$ or ideal $[U]$.

An admissible fan will be called *reduced* if the morphism $Z \rightarrow X$ is an isomorphism.

3.2. The Contraction Theorem. Our aim is to prove the following theorem.

Theorem 3.1. *Let $([X] \leftarrow [Z] \rightarrow [U])$ be an admissible fan in some tropical \mathbf{G} -diagram $[\mathcal{Z}]$. Then for every $\varepsilon > 0$ there exists a \mathbf{G} -diagram $[\mathcal{Z}']$ containing an admissible fan $([X'] \leftarrow [Z'] \rightarrow [U'])$, corresponding to the original admissible fan through the combinatorial isomorphism, such that, with the notations $\mathcal{X} = [X]$ and $\mathcal{X}' = [X']$, the diagram $[\mathcal{Z}']$ satisfies*

- (i) $\kappa([\mathcal{X}'[U']], [\mathcal{X}[U]]) \leq \varepsilon$
- (ii) $\kappa(\mathcal{X}', \mathcal{X}) \leq \varepsilon$

$$(iii) [Z' \lfloor X'] \leq \varepsilon$$

□

It is not clear that constructing diagrams Z' as in the theorem above for a sequence of values of parameter ε decreasing to 0, we can obtain a convergent sequence in $\mathbf{Prob}[\mathbf{G}]$ with the limiting diagram satisfying conclusions of the theorem with $\varepsilon = 0$.

The proof of Theorem 3.1 is based on the following proposition, which will be proven in Section 5.

Proposition 3.2. \downarrow *Let $(X_0 \leftarrow Z_0 \rightarrow U)$ be an admissible fan in some homogeneous \mathbf{G} -diagram of probability spaces \mathcal{Z} . Then there exists \mathbf{G} -diagram Z' containing the admissible fan $(X'_0 \leftarrow Z'_0 \rightarrow U')$ such that, with the notations $\mathcal{X} := [X_0]$ and $\mathcal{X}' := [X'_0]$, it holds that*

- (1) $\mathcal{X}[u = \mathcal{X}'[u'$ for any $u \in U$ and $u' \in U'$.
- (2) $\kappa(\mathcal{X}, \mathcal{X}') \leq \mathbf{k}(\mathcal{X}, \mathcal{X}') \leq 20 \cdot \llbracket \mathbf{G} \rrbracket$
- (3) $[Z'_0 \lfloor X'_0] \leq 4 \ln \ln |X_0|$

□

Proof(of Theorem 3.1): First we assume that $[\mathcal{Z}]$ is a homogeneous tropical diagram. It means that it can be represented by a quasi-linear sequence $(\mathcal{Z}(n))_{n \in \mathbb{N}_0}$ of homogeneous diagrams, with defect of the sequence bounded by the function $\varphi(t) := C \cdot t^{3/4}$ for some $C > 0$. This means that for any $m, n \in \mathbb{N}$

$$\begin{aligned} \kappa(\mathcal{Z}(m) \otimes \mathcal{Z}(n), \mathcal{Z}(m+n)) &\leq \varphi(m+n) \\ \kappa(\mathcal{Z}^m(n), \mathcal{Z}(m \cdot n)) &\leq D_\varphi \cdot m \cdot \varphi(n) \end{aligned}$$

where D_φ is some constant depending on φ , see [MP19b].

Fix a number $n \in \mathbb{N}$ and apply Proposition 3.2 to the *homogeneous* diagram $\mathcal{Z}(n)$, containing the admissible fan $X_0(n) \leftarrow Z_0(n) \rightarrow U(n)$ and sub-diagram $\mathcal{X}(n) = [X_0(n)]$. As a result we obtain a diagram \mathcal{Z}'' containing the fan $X''_0 \leftarrow Z''_0 \rightarrow U''$ and the sub-diagram $\mathcal{X}'' = [X''_0]$, such that

$$(3.1) \quad \begin{aligned} \mathcal{X}''[u'' = \mathcal{X}(n)[u &\quad \text{for any } u'' \in U'' \text{ and } u \in U(n) \\ \kappa(\mathcal{X}'', \mathcal{X}(n)) &\leq 20 \llbracket \mathbf{G} \rrbracket \\ [Z''_0 \lfloor X''_0] &\leq 4 \ln \ln |X_0(n)| \end{aligned}$$

Define the two tropical diagrams

$$\begin{aligned} [\mathcal{Z}'] &:= \frac{1}{n} \overrightarrow{\mathcal{Z}''} \\ [\tilde{\mathcal{Z}}] &:= \frac{1}{n} \overrightarrow{\mathcal{Z}(n)} \end{aligned}$$

Since $\mathcal{X}''[u''$ does not depend on u'' and $\mathcal{X}(n)[u$ does not depend on u we have $[\mathcal{X}' \lfloor U'] = (1/n) \cdot \overrightarrow{(\mathcal{X}''[u'']})$ and $[\tilde{\mathcal{X}} \lfloor \tilde{U}] = (1/n) \cdot \overrightarrow{(\mathcal{X}(n)[u])}$. From (3.1), we

obtain

$$(3.2) \quad \begin{aligned} [\mathcal{X}'|U'] &= [\tilde{\mathcal{X}}|\tilde{U}] \\ \kappa([\mathcal{X}'], [\tilde{\mathcal{X}}]) &\leq \frac{20\llbracket \mathbf{G} \rrbracket}{n} \\ [Z'_0|X'_0] &\leq \frac{4 \ln \ln |X_0(n)|}{n} \end{aligned}$$

The distance between $[\tilde{\mathcal{Z}}]$ and $[\mathcal{Z}]$ can be bounded as follows

$$(3.3) \quad \begin{aligned} \kappa([\tilde{\mathcal{Z}}], [\mathcal{Z}]) &= \frac{1}{n} \kappa(\overrightarrow{\mathcal{Z}(n)}, n \cdot [\mathcal{Z}]) = \frac{1}{n} \lim_{m \rightarrow \infty} \frac{1}{m} \kappa(\mathcal{Z}^m(n), \mathcal{Z}(m \cdot n)) \\ &\leq \frac{1}{n} D_\varphi \cdot \varphi(n) \end{aligned}$$

This also implies

$$(3.4) \quad \kappa([\tilde{\mathcal{X}}], [\mathcal{X}]) \leq \frac{1}{n} D_\varphi \cdot \varphi(n)$$

Since conditioning is a Lipschitz continuous operation with Lipschitz constant 2, we also have

$$(3.5) \quad \kappa([\tilde{\mathcal{X}}|\tilde{U}], [\mathcal{X}|U]) \leq \frac{2}{n} D_\varphi \cdot \varphi(n)$$

Combining the estimates in (3.2), (3.3), (3.4) and (3.5) we obtain

$$\begin{aligned} \kappa([\mathcal{X}'|U'], [\mathcal{X}|U]) &\leq 2D_\varphi \cdot \frac{\varphi(n)}{n} \\ \kappa([\mathcal{X}'], [\mathcal{X}]) &\leq \frac{20\llbracket \mathbf{G} \rrbracket}{n} + D_\varphi \frac{\varphi(n)}{n} \\ [Z'_0|X'_0] &\leq \frac{4 \ln \ln |X_0(n)|}{n} + 2D_\varphi \frac{\varphi(n)}{n} \end{aligned}$$

Note that $|X_0(n)|$ grows at most exponentially (it is bounded by $e^{n([X_0]+C)}$ for some C) and φ is a strictly sub-linear function. Thus by choosing n sufficiently large depending on given $\varepsilon > 0$ we obtain $[\mathcal{Z}']$ satisfying conclusions of the theorem for $[\mathcal{Z}]$ homogeneous.

To prove the theorem in full generality observe that all the quantities on the right-hand side of the inequalities are Lipschitz-continuous. Since $\mathbf{Prob}[\mathbf{G}]_h$ is dense in $\mathbf{Prob}[\mathbf{G}]$ the theorem extends to any $[\mathcal{Z}]$ by first approximating it with any precision by a homogeneous configuration and applying the argument above. \square

3.3. The expansion Theorem. The following theorem is complimentary to Theorem 3.1. The expansion applied to a diagram containing a reduced admissible fan produces a diagram with an admissible fan, such that contraction of it is the original diagram. Thus, arrow expansion is a right inverse of the arrow contraction operation.

In general, contraction erases some information stored in the diagram, so there are many right inverses. We prove the theorem below by providing a simple construction of one such right inverse.

Theorem 3.3. *Let $([X] \leftarrow [Z'] \rightarrow [U'])$ be a reduced admissible fan in some tropical \mathbf{G} -diagram $[Z']$ and $\lambda > 0$. Let $[\mathcal{X}] := [X]$. Then there exists \mathbf{G} -diagram $[Z]$ containing the copy of $[\mathcal{X}]$, such that the corresponding admissible fan $([X] \leftarrow [Z] \rightarrow [U])$ has $[Z[X] = \lambda$ and $[\mathcal{X}[U] = [\mathcal{X}[U']$. \square*

Proof: Let $[W]$ be a tropical probability space with entropy equal to λ . For any reduction of tropical spaces $[A] \rightarrow [B]$, there are natural reductions

$$\begin{aligned} ([A] + [W]) &\rightarrow ([B] + [W]) \\ ([A] + [W]) &\rightarrow [W] \end{aligned}$$

We construct the diagram $[Z]$ by replacing every space $[V]$ in the ideal $[U']$ with $[U] + [W]$. Every morphisms $[V_1] \rightarrow [V_2]$ within $[U']$ is replaced by

$$([V_1] + [W]) \rightarrow ([V_2] + [W])$$

And any morphism from $[V]$ in $[U']$ to a space $[Y]$ in $[X]$ is replaced by a composition

$$([V] + [W]) \rightarrow [V] \rightarrow [Y]$$

Clearly the resulting diagram satisfies the conclusion of the theorem. \square

The rest of the article is devoted to the development of necessary tools and the proof of Proposition 3.2.

4. LOCAL ESTIMATE

In this section we derive a bound, very similar to Fano's inequality, on the intrinsic entropic distance between two diagrams of probability spaces with the same underlying diagram of sets. The bound will be in terms of total variation distance between two distributions corresponding to the diagrams of probability spaces. It will be used in the next section, to prove arrow contraction theorem.

4.1. Distributions.

4.1.1. *Distributions on sets.* For a finite set S we denote by ΔS the collection of all probability distributions on S and by $\|\pi_1 - \pi_2\|_1$ we denote the total variation distance between $\pi_1, \pi_2 \in \Delta S$.

4.1.2. *Distributions on Diagrams of Sets.* Let \mathbf{Set} denote the category of finite sets and surjective maps. For an indexing category \mathbf{G} , we denote by $\mathbf{Set}\langle\mathbf{G}\rangle$ the category of \mathbf{G} -diagrams in \mathbf{Set} . That is, objects in $\mathbf{Set}\langle\mathbf{G}\rangle$ are commutative diagrams of sets indexed by the category \mathbf{G} , the spaces in the such a diagram are finite sets and arrows represent surjective maps, subject to commutativity relations.

For a diagram of sets $\mathcal{S} = \{S_i; \sigma_{ij}\}$ we define the *space of distributions on the diagram \mathcal{S}* by

$$\Delta\mathcal{S} := \left\{ (\pi_i) \in \prod_i \Delta S_i : (\sigma_{ij})_* \pi_i = \pi_j \right\}$$

where $f_* : \Delta S \rightarrow \Delta S'$ is the affine map induced by a surjective map $f : S \rightarrow S'$. If S_0 is the initial space of \mathcal{S} , then there is an isomorphism

$$(4.1) \quad \begin{aligned} \Delta S_0 &\xrightarrow{\cong} \Delta\mathcal{S} \\ \Delta S_0 \ni \pi_0 &\mapsto \{(\sigma_{0i})_* \pi_0\} \in \Delta\mathcal{S} \\ \Delta S_0 \ni \pi_0 &\leftarrow \{\pi_i\} \in \Delta \end{aligned}$$

Using the isomorphism (4.1) we define total variation distance between two distributions $\pi, \pi' \in \Delta\mathcal{S}$ as

$$\|\pi - \pi'\|_1 := \|\pi_0 - \pi'_0\|_1$$

Given a \mathbf{G} -diagram of sets $\mathcal{S} = \{S_i; \sigma_{ij}\}$ and an element $\pi \in \Delta\mathcal{S}$ we can construct a \mathbf{G} -diagram of probability spaces $(\mathcal{S}, \pi) := \{(S_i, \pi_i); \sigma_{ij}\}$.

Below we give the estimate of the entropy distance between two \mathbf{G} -diagrams of probability spaces (\mathcal{S}, π) and (\mathcal{S}, π') in terms of the total variation distance $\|\pi - \pi'\|$ between distributions.

4.2. **The estimate.** The upper bound on the entropy distance, that we derive below, has two summands. One is linear in the total variation distance with the slope proportional to the log-cardinality of S_0 . The second one is super-linear in the total variation distance, but it does not depend on \mathcal{S} . So we have the following interesting observation: of course, the super-linear summand always dominates the linear one locally. However as the cardinality of \mathcal{S} becomes large it is the linear summand that starts playing the main role. This will be the case when we apply the bound in the next section.

For $\alpha \in [0, 1]$ consider a binary probability space with the weight of one of the atoms equal to α

$$\Lambda_\alpha := (\{\square, \blacksquare\}; p(\square) = 1 - \alpha, p(\blacksquare) = \alpha)$$

Proposition 4.1. *For an indexing category \mathbf{G} , consider a \mathbf{G} -diagram of sets $\mathcal{S} = \{S_i, \sigma_{ij}\} \in \mathbf{Set}\langle\mathbf{G}\rangle$. Let $\pi, \pi' \in \Delta\mathcal{S}$ be two probability distributions on \mathcal{S} . Denote $\mathcal{X} := (\mathcal{S}, \pi)$, $\mathcal{Y} := (\mathcal{S}, \pi')$ and $\alpha := \frac{1}{2} \|\pi - \pi'\|_1$. Then*

$$\mathbf{k}(\mathcal{X}, \mathcal{Y}) \leq 2\llbracket\mathbf{G}\rrbracket(\alpha \cdot \ln |S_0| + \text{Ent}(\Lambda_\alpha))$$

□

Proof: To prove the local estimate we decompose both π and π' into a convex combination of a common part $\hat{\pi}$ and rests π^+ and π'^+ . The coupling between the common parts gives no contribution to the distance, and the worst possible estimate on the other parts is still enough to get the bound in the lemma, by using Proposition 2.1.

Let S_0 be the initial set in the diagram \mathcal{S} . We will need the following obvious rough estimate of the entropy distance that holds for any $\pi, \pi' \in \Delta\mathcal{S}$:

$$(4.2) \quad \mathbf{k}(\mathcal{X}, \mathcal{Y}) \leq 2\llbracket \mathbf{G} \rrbracket \cdot \ln |S_0|$$

It can be obtained by taking a tensor product for the coupling between \mathcal{X} and \mathcal{Y} .

Our goal now is to write π and π' as the convex combination of three other distributions $\hat{\pi}$, π^+ and π'^+ as in

$$\begin{aligned} \pi &= (1 - \alpha) \cdot \hat{\pi} + \alpha \cdot \pi^+ \\ \pi' &= (1 - \alpha) \cdot \hat{\pi} + \alpha \cdot \pi'^+ \end{aligned}$$

with the smallest possible $\alpha \in [0, 1]$.

We could do it the following way. Let π_0 and π'_0 be the distributions on S_0 that correspond to π and π' under isomorphisms (4.1). Let $\alpha := \frac{1}{2} \|\pi - \pi'\|_1 = \frac{1}{2} \|\pi_0 - \pi'_0\|_1$. If $\alpha = 1$ then the proposition follows from the rough estimate (4.2), so from now on we assume that $\alpha < 1$. Define three probability distributions $\hat{\pi}_0$, π_0^+ and $\pi_0'^+$ on S_0 by setting for every $x \in S_0$

$$\begin{aligned} \hat{\pi}_0(x) &:= \frac{1}{1 - \alpha} \min \{ \pi_0(x), \pi'_0(x) \} \\ \pi_0^+ &:= \frac{1}{\alpha} (\pi_0 - (1 - \alpha)\hat{\pi}_0) \\ \pi_0'^+ &:= \frac{1}{\alpha} (\pi'_0 - (1 - \alpha)\hat{\pi}_0) \end{aligned}$$

Denote by $\hat{\pi}, \pi^+, \pi'^+ \in \Delta\mathcal{S}$ the distributions corresponding to $\hat{\pi}_0, \pi_0^+, \pi_0'^+ \in \Delta S_0$ under isomorphism (4.1). Thus we have

$$\begin{aligned} \pi &= (1 - \alpha)\hat{\pi} + \alpha \cdot \pi^+ \\ \pi' &= (1 - \alpha)\hat{\pi} + \alpha \cdot \pi'^+ \end{aligned}$$

Now we construct two fans of \mathbf{G} -diagrams

$$(4.3) \quad \begin{array}{ccc} & \tilde{\mathcal{X}} & \\ \mathcal{X} & \longleftarrow & \longrightarrow \Lambda_\alpha \\ & & \end{array} \quad \begin{array}{ccc} & \tilde{\mathcal{Y}} & \\ \Lambda_\alpha & \longleftarrow & \longrightarrow \mathcal{Y} \\ & & \end{array}$$

by setting

$$\begin{aligned} \tilde{X}_i &:= \left(S_i \times \underline{\Lambda}_\alpha; \tilde{\pi}_i(s, \square) = (1 - \alpha)\hat{\pi}_i(s), \tilde{\pi}_i(s, \blacksquare) = \alpha \cdot \pi_i^+(s) \right) \\ \tilde{Y}_i &:= \left(S_i \times \underline{\Lambda}_\alpha; \tilde{\pi}'_i(s, \square) = (1 - \alpha)\hat{\pi}_i(s), \tilde{\pi}'_i(s, \blacksquare) = \alpha \cdot \pi_i'^+(s) \right) \end{aligned}$$

and

$$\begin{aligned}\tilde{\mathcal{X}} &:= \{\tilde{X}_i; \sigma_{ij} \times \text{id}\} \\ \tilde{\mathcal{Y}} &:= \{\tilde{Y}_i; \sigma_{ij} \times \text{id}\}\end{aligned}$$

The reductions in the fans in (4.3) are given by coordinate projections. Note that the following isomorphisms hold

$$\begin{aligned}\mathcal{X}[\square] &\cong (\mathcal{S}, \hat{\pi}) \\ \mathcal{X}[\blacksquare] &\cong (\mathcal{S}, \pi^+) \\ \mathcal{Y}[\square] &\cong (\mathcal{S}, \hat{\pi}) \cong \mathcal{X}[\square] \\ \mathcal{Y}[\blacksquare] &\cong (\mathcal{S}, \pi'^+)\end{aligned}$$

Now we apply Proposition 2.1 along with the rough estimate in (4.2) to obtain the desired inequality

$$\begin{aligned}\mathbf{k}(\mathcal{X}, \mathcal{Y}) &\leq (1 - \alpha) \mathbf{k}(\mathcal{X}[\square], \mathcal{Y}[\square]) + \alpha \cdot \mathbf{k}(\mathcal{X}[\blacksquare], \mathcal{Y}[\blacksquare]) \\ &\quad + \sum_i [\text{Ent}(\Lambda_\alpha \lfloor X_i) + \text{Ent}(\Lambda_\alpha \lfloor Y_i)] \\ &\leq 2\llbracket \mathbf{G} \rrbracket (\alpha \cdot \ln |S_0| + \text{Ent}(\Lambda_\alpha))\end{aligned}$$

□

5. PROOF OF PROPOSITION 3.2

In this section we prove Proposition 3.2, which is shown below verbatim.

Proposition 3.2. \uparrow *Let $(X_0 \leftarrow Z_0 \rightarrow U)$ be an admissible fan in some homogeneous \mathbf{G} -diagram of probability spaces \mathcal{Z} . Then there exists \mathbf{G} -diagram \mathcal{Z}' containing the admissible fan $(X'_0 \leftarrow Z'_0 \rightarrow U')$ such that, with the notations $\mathcal{X} := [X_0]$ and $\mathcal{X}' := [X'_0]$, it holds that*

- (1) $\mathcal{X} \lfloor u = \mathcal{X}' \lfloor u'$ for any $u \in U$ and $u' \in U'$.
- (2) $\kappa(\mathcal{X}, \mathcal{X}') \leq \mathbf{k}(\mathcal{X}, \mathcal{X}') \leq 20 \cdot \llbracket \mathbf{G} \rrbracket$
- (3) $[Z'_0 \lfloor X'_0] \leq 4 \ln \ln |X_0|$

□

The proof consists of the construction in Section 5.1 and estimates in Propositions 5.4 and 5.5.

5.1. The construction. In this section we fix an indexing category \mathbf{G} , a minimal \mathbf{G} -diagram of probability spaces \mathcal{Z} with an admissible sub-fan $X_0 \leftarrow Z_0 \rightarrow U$. We denote $\mathcal{X} := [X_0]$ and by \mathbf{H} we denote the combinatorial type of $\mathcal{X} = \{X_i; \chi_{ij}\}$.

Instead of diagram \mathcal{Z} we consider an extended diagram, which is a two-fan of \mathbf{H} -diagrams

$$(5.1) \quad \mathcal{X} \longleftarrow_{\pi_1} \mathcal{Y} \longrightarrow U^{\mathbf{H}}$$

where $\mathcal{Y} = \{Y_i; v_{ij}\}$ consists of those spaces in \mathcal{Z} , that are initial spaces of two-fans with feet in U and in some space in \mathcal{X} . That is for every $i \in \mathbf{H}$ the space Y_i is defined to be the initial space in the minimal fan $X_i \leftarrow Y_i \rightarrow U$ in \mathcal{Z} . It may happen that for some pair of indices $i_1, i_2 \in \mathbf{H}$ the initial spaces of the fans with one feet U and the other X_{i_1} and X_{i_2} coincide in \mathcal{Z} . In \mathcal{Y} , however, they will be treated as separate spaces, so that the combinatorial type of \mathcal{Y} is \mathbf{H} . Starting with the diagram in (5.1) one can recover \mathcal{Z} by collapsing all the isomorphism arrows. The initial space of \mathcal{Y} will be denoted Y_0 .

We would like to construct a new fan $\mathcal{X}' \xleftarrow{\pi'_1} \mathcal{Y}' \rightarrow V^{\mathbf{H}}$, such that

$$(5.2) \quad \begin{cases} \mathcal{X}' \lfloor u = \mathcal{X}' \lfloor v & \text{for any } u \in U \text{ and } v \in V \\ \mathbf{k}(\mathcal{X}', \mathcal{X}) \leq 20 \llbracket \mathbf{G} \rrbracket \\ [Y'_0 \lfloor X'_0] \leq 4 \ln \ln |X_0| \end{cases}$$

Once this goal is achieved, we collapse all the isomorphisms to obtain \mathbf{G} -diagram satisfying conditions in the conclusion of Proposition 3.2.

We start with a general description of the idea behind the construction, followed by the detailed argument. To introduce the new space V we take its points to be N atoms in $u_1, \dots, u_N \in U$. Ideally we would like to choose the atoms in such a way that $X_0 \lfloor u_n$ are disjoint and cover the whole of X_0 . It is not always possible to achieve this exactly. However, when $|X_0|$ is large, N is taken slightly larger than $e^{\lfloor X_0:U \rfloor}$, and u_1, \dots, u_N are chosen at random, then with high probability the spaces $X_0 \lfloor u_n$ will overlap only little and will cover most of X_0 . The details of the construction follow.

We fix $N \in \mathbb{N}$ and construct several new diagrams. For each of the new diagrams we provide a verbal and formal description.

- The space U^N . Points in it are independent samples of length N of points in U .
- The space $V_N = (\{1, \dots, N\}, \text{unif})$. A point $n \in V_N$ should be interpreted as a choice of index in a sample $\bar{u} \in U^N$.
- The \mathbf{H} -diagram \mathcal{A} , where

$$\begin{aligned} \mathcal{A} &= \{A_i; \alpha_{ij}\} \\ A_i &= (\{(x, n, \bar{u}) : x \in X_i \lfloor u_n\}, \text{unif}) \\ \alpha_{ij} &= (\chi_{ij}, \text{id}, \text{id}) \end{aligned}$$

A point (x, n, \bar{u}) in A_i corresponds to the choice of a sample $\bar{u} \in U^N$, an independent choice of a member of the sample u_n and a point $x \in X_i \lfloor u_n$. Recall that the original diagram \mathcal{Z} was assumed to be homogeneous and, in particular, the distribution on $X_i \lfloor u_n$ is uniform. Due to the assumption on homogeneity of \mathcal{Z} , the space $X_i \lfloor u$ does not depend on $u \in U$. Since V_N is also equipped with the uniform distribution, it follows that the distribution on A_i will also be uniform.

- The **H**-diagram \mathcal{B} , where

$$\begin{aligned}\mathcal{B} &= \{B_i; \beta_{ij}\} \\ B_i &= \left(\left\{ (x, \bar{u}) : x \in \bigcup_{n=1}^N X_i \lfloor u_n \right\}, p_{B_i} \right) \\ \beta_{ij} &= (\chi_{ij}, \text{id})\end{aligned}$$

A point $(x, \bar{u}) \in B_i$ is the choice of a sample $\bar{u} \in U^N$ and a point x in one of the fibers $X_i \lfloor u_n$, $n = 1, \dots, N$. The distribution p_{B_i} on B_i is chosen so that the natural projection $A_i \rightarrow B_i$ is the reduction of probability spaces. Given a sample \bar{u} , if the fibers $X_i \lfloor u_n$ are not disjoint, then the distribution on $B_i \lfloor \bar{u}$ need not to be uniform. Below we will give an explicit description of $p_{\mathcal{B}}$ and study the dependence of $p_{\mathcal{B}}(\cdot \lfloor \bar{u})$ on the sample $\bar{u} \in U^N$.

These diagrams can be organized into a minimal diamond diagram of **H**-diagrams, where reductions are obvious projections.

$$(5.3) \quad \begin{array}{ccc} & \mathcal{A} & \\ & \swarrow \quad \searrow & \\ \mathcal{B} & & V_N \otimes U^N \\ & \searrow \quad \swarrow & \\ & U^N & \end{array}$$

To describe the probability distribution on \mathcal{B} first we define several relevant quantities:

$$\begin{aligned}\rho &:= \frac{|X_0 \lfloor u|}{|X_0|} = e^{-[X_0:U]} \\ N(x, \bar{u}) &:= \left| \{n \in V_N : x \in X_0 \lfloor u_n\} \right| \\ \nu(x, \bar{u}) &:= \frac{N(x, \bar{u})}{N} = p_{V_N} \{n \in V_N : x \in X_0 \lfloor u_n\}\end{aligned}$$

Recall that the distribution $p_{\mathcal{B}}$ is completely determined by the distribution p_{B_0} on the initial space of \mathcal{B} via isomorphism (4.1). From homogeneity of \mathcal{Z} it follows that distributions on both A_0 and $A \lfloor \bar{u}$ are uniform. Therefore

$$(5.4) \quad p_{B_0}(x \lfloor \bar{u}) := \frac{\nu(x, \bar{u})}{\rho \cdot |X_0|}$$

The desired fan ($\mathcal{X}' \leftarrow \mathcal{Y}' \rightarrow V^{\mathbf{H}}$) mentioned in the beginning of the section is obtained from the top fan in the diagram in (5.3) by conditioning on $\bar{u} \in U^N$. We will show later that for an appropriate choice of N and for most choices of \bar{u} , the fan we obtain in this way has the required properties.

First, we would like to make the following observations. Fix an arbitrary $\bar{u} \in U^N$. Then:

- (1) The underlying set of the probability space $B_0 \lfloor \bar{u} = X_0 \lfloor \bar{u}$ is \underline{X}_0 .

(2) The diagrams

$$\begin{aligned}\mathcal{Y}'_{\bar{u}} &:= \mathcal{A} \lfloor \bar{u} \\ \mathcal{X}'_{\bar{u}} &:= \mathcal{B} \lfloor \bar{u}\end{aligned}$$

are included into a two-fan of **H**-diagrams

$$\mathcal{X}'_{\bar{u}} \longleftarrow \mathcal{Y}'_{\bar{u}} \longrightarrow V_N$$

which is obtained by conditioning top fan in the diagram in (5.3).

The very important observation is that diagrams $\mathcal{X}'_{\bar{u}} \lfloor n$ and $\mathcal{X} \lfloor u$ are isomorphic for any choice of $n \in V_N$ and $u \in U$. The isomorphism is the composition of the following sequence of isomorphisms

$$\mathcal{X}'_{\bar{u}} \lfloor n \rightarrow \mathcal{B} \lfloor (\bar{u}, n) \rightarrow \mathcal{A} \lfloor (\bar{u}, n) \rightarrow \mathcal{X} \lfloor u_n \rightarrow \mathcal{X} \lfloor u$$

where the first isomorphism follows from the definition of $\mathcal{X}'_{\bar{u}}$, the second – from minimality of the fan $\mathcal{B} \leftarrow \mathcal{A} \rightarrow V_N$, the third – from the definition of \mathcal{A} and the fourth – from the homogeneity of \mathcal{Z} .

5.2. The estimates. We now claim and prove that one could choose a number N and \bar{u} in U^N such that

- (1) $\mathbf{k}(\mathcal{X}'_{\bar{u}}, \mathcal{X}) \leq 20 \llbracket \mathbf{H} \rrbracket$.
- (2) $[Y'_{\bar{u},0} \lfloor X'_{\bar{u},0}] \leq 4 \ln \ln |X_0|$, where $Y'_{\bar{u},0}$ and $X'_{\bar{u},0}$ are initial spaces in $\mathcal{X}'_{\bar{u}}$ and $\mathcal{Y}'_{\bar{u}}$, respectively.

5.2.1. Total Variation and Entropic Distance estimates. If we fix some $x_0 \in X_0$, then $\nu = \nu(x_0, \cdot)$ is a scaled binomially distributed random variable with parameters N and ρ , which means that $N \cdot \nu \sim \text{Bin}(N, \rho)$.

First we state the following bounds on the tails of a binomial distribution.

Lemma 5.2. *Let ν be a scaled binomial random variable with parameters N and ρ , then*

- (i) *for any $t \in [0, 1]$ holds*

$$\mathbb{P} \{ |\nu - \rho| > \rho \cdot t \} \leq 2 \cdot e^{-\frac{1}{3} \cdot N \cdot \rho \cdot t^2}$$

- (ii) *for any $t \in [0, 2]$ holds*

$$\mathbb{P} \left\{ \frac{\nu}{\rho} \ln \frac{\nu}{\rho} > t \right\} \leq e^{-\frac{1}{12} \cdot N \cdot \rho \cdot t^2}$$

□

The proof of Lemma 5.2 can be found at the end of this section.

Below we use the notation $\mathbb{P} := p_{U^N}$ for the probability distribution on U^N . For a pair of complete diagrams $\mathcal{C}, \mathcal{C}'$ with the same underlying diagram of sets and with initial spaces C_0, C'_0 , we will write $\alpha(\mathcal{C}, \mathcal{C}')$ for the halved total variation distance between their distributions

$$\alpha(\mathcal{C}, \mathcal{C}') := \frac{1}{2} \|p_{C_0} - p_{C'_0}\|_1$$

Proposition 5.3. *In the settings above, for $t \in [0, 1]$, the following inequality holds*

$$\mathbb{P}\{\bar{u} \in U^N : 2\alpha(\mathcal{X}'_{\bar{u}}, \mathcal{X}) > t\} \leq 2|X_0| \cdot e^{-\frac{1}{3}N \cdot \rho t^2}$$

□

Proof: Recall that by definition $\mathcal{X}'_{\bar{u}} = \mathcal{B}[\bar{u}]$. We use equation (5.4) to expand the left hand side of the inequality as follows

$$\begin{aligned} \mathbb{P}\{\bar{u} \in U^N : 2\alpha(\mathcal{B}[\bar{u}], \mathcal{X}) > t\} &= \mathbb{P}\left\{\bar{u} \in U^N : \sum_{x \in X_0} \left| \frac{\nu(x, \bar{u})}{\rho \cdot |X_0|} - \frac{1}{|X_0|} \right| > t\right\} \\ &= \mathbb{P}\left\{\bar{u} \in U^N : \sum_{x \in X_0} |\nu(x, \bar{u}) - \rho| > \rho \cdot |X_0| \cdot t\right\} \\ &\leq \mathbb{P}\{\bar{u} \in U^N : \text{there exists } x_0 \text{ such that } |\nu(x_0, \bar{u}) - \rho| > \rho \cdot t\} \\ &\leq \sum_{x \in X_0} \mathbb{P}\{\bar{u} \in U^N : |\nu(x, \bar{u}) - \rho| > \rho \cdot t\} \end{aligned}$$

Since by homogeneity of the original diagram all the summands are the same, we can fix some $x_0 \in X_0$ and estimate further:

$$\mathbb{P}\{\bar{u} \in U^N : 2\alpha(\mathcal{B}[\bar{u}], \mathcal{X}) > t\} \leq |X_0| \cdot \mathbb{P}\{\bar{u} \in U^N : |\nu(x_0, \bar{u}) - \rho| > \rho \cdot t\}$$

Applying Lemma 5.2(i) we obtain the required inequality. □

In the propositions below we assume that $|X_0|$ is sufficiently large (larger than e^{20}).

Proposition 5.4. *In the settings above and for any $\frac{10}{\ln |X_0|} \leq t \leq 1$ holds:*

$$\mathbb{P}\{\bar{u} \in U^N : \mathbf{k}(\mathcal{X}'_{\bar{u}}, \mathcal{X}) > t(2 \cdot \llbracket \mathbf{G} \rrbracket \cdot \ln |X_0|)\} \leq 2|X_0| \cdot e^{-\frac{1}{3}N \cdot \rho t^2}$$

□

Proof: We will use local estimate to bound entropy distance and then apply Proposition 5.3. To simplify notations we will write simply α for $\alpha(\mathcal{X}'_{\bar{u}}, \mathcal{X}) = \alpha(\mathcal{B}[\bar{u}], \mathcal{X})$.

$$\begin{aligned} &\mathbb{P}\{\bar{u} \in U^N : \mathbf{k}(\mathcal{B}[\bar{u}], \mathcal{X}) > (2 \cdot \llbracket \mathbf{G} \rrbracket \cdot \ln |X_0|)t\} \\ &\leq \mathbb{P}\{\bar{u} \in U^N : 2 \cdot \llbracket \mathbf{G} \rrbracket (\alpha \cdot \ln |X_0| + \text{Ent}(\Lambda_\alpha)) > (2 \cdot \llbracket \mathbf{G} \rrbracket \cdot \ln |X_0|)t\} \\ &\leq \mathbb{P}\{\bar{u} \in U^N : \alpha + \text{Ent}(\Lambda_\alpha) / \ln |X_0| > t\} \end{aligned}$$

Note that in the chosen regime, $t \geq 10 / \ln |X_0|$, the first summand on the left-hand side of the inequality is larger than the second, and thus

$$\begin{aligned} &\mathbb{P}\{\bar{u} \in U^N : \mathbf{k}(\mathcal{B}[\bar{u}], \mathcal{X}) > (2 \cdot \llbracket \mathbf{G} \rrbracket \cdot \ln |X_0|)t\} \\ &\leq \mathbb{P}\{\bar{u} \in U^N : 2\alpha > t\} \\ &\leq 2|X_0| \cdot e^{-\frac{1}{3}N \cdot \rho t^2} \end{aligned}$$

□

5.2.2. *The “height” estimate.* Recall that for given $N \in \mathbb{N}$ and $\bar{u} \in U^N$ we have constructed a two-fan of **H**-diagrams

$$\mathcal{X}'_{\bar{u}} \leftarrow \mathcal{Y}'_{\bar{u}} \rightarrow V_N^{\mathbf{H}}$$

We will now estimate the length of the arrow $Y'_{\bar{u},0} \rightarrow X'_{\bar{u},0}$.

Proposition 5.5. *In the settings above and for $t \in [0, 2]$*

$$\mathbb{P} \left\{ \bar{u} \in U^N : [Y'_{\bar{u},0} | X'_{\bar{u},0}] > \ln(N \cdot \rho) + t \right\} \leq |X_0| \cdot e^{-\frac{1}{12} N \cdot \rho \cdot t^2}$$

□

Proof: First we observe that the fiber of the reduction $Y'_{\bar{u},0} \rightarrow X'_{\bar{u},0}$ over a point $x \in X'_{\bar{u},0}$ is a homogeneous probability space of cardinality equal to $N(x, \bar{u})$, therefore its entropy is $\ln N(x, \bar{u})$.

$$\begin{aligned} & \mathbb{P} \left\{ \bar{u} \in U^N : [Y'_{\bar{u},0} | X'_{\bar{u},0}] > \ln(N \cdot \rho) + t \right\} \\ & \mathbb{P} \left\{ \bar{u} \in U^N : \int_{X'_{\bar{u},0}} [Y'_{\bar{u},0} | x] dp_{X'_{\bar{u},0}}(x) > \ln(N \cdot \rho) + t \right\} \\ & = \mathbb{P} \left\{ \bar{u} \in U^N : \sum_{x \in X_0} \frac{\nu(x, \bar{u})}{\rho \cdot |X_0|} \ln(N \cdot \nu(x, \bar{u})) > \ln(N \cdot \rho) + t \right\} \\ & \leq \mathbb{P} \left\{ \bar{u} \in U^N : \sum_{x \in X_0} \frac{\nu(x, \bar{u})}{\rho \cdot |X_0|} \ln\left(\frac{\nu(x, \bar{u})}{\rho}\right) > t \right\} \\ & \leq |X_0| \cdot \mathbb{P} \left\{ \bar{u} \in U^N : \frac{\nu(x_0, \bar{u})}{\rho} \ln\left(\frac{\nu(x_0, \bar{u})}{\rho}\right) > t \right\} \\ & \leq |X_0| \cdot e^{-\frac{1}{12} N \cdot \rho \cdot t^2} \end{aligned}$$

The last inequality above follows from Lemma 5.2(ii). □

5.3. Proof of Proposition 3.2. Let $\mathcal{X}'_{\bar{u}} \leftarrow \mathcal{Y}'_{\bar{u}} \rightarrow V_N$ be the fan constructed in Section 5.1. The construction is parameterized by number N and atom $\bar{u} \in U^N$. Below we will choose a particular value for N and apply estimates in Propositions 5.4 and 5.5 with particular choice of parameter t to show that there is $\bar{u} \in U^N$, so that the fan satisfies conclusions of Proposition 3.2.

Let

$$\begin{aligned} N & := \ln^3 |X_0| \cdot \rho^{-1} = \ln^3 |X_0| \cdot e^{[X_0:U]} \\ t & := \frac{10}{\ln |X_0|} \end{aligned}$$

With this choices of N and t Proposition 5.4 implies

$$\mathbb{P} \left\{ \bar{u} \in U^N : \mathbf{k}(\mathcal{X}'_{\bar{u}}, \mathcal{X}) > 20 \llbracket \mathbf{G} \rrbracket \right\} \leq \frac{1}{4}$$

while Proposition 5.5 gives

$$\mathbb{P}\{\bar{u} \in U^N : [Y'_{\bar{u},0} | X'_{\bar{u},0}] > 4 \ln \ln |X_0|\} \leq \frac{1}{4}$$

Therefore there is a choice of \bar{u} such that the fan

$$(\mathcal{X}' \leftarrow \mathcal{Y}' \rightarrow V) := (\mathcal{X}'_{\bar{u},0} \leftarrow \mathcal{Y}'_{\bar{u},0} \rightarrow V_N)$$

satisfies conditions in (5.2). As we have explained in the beginning of Section 5.1, by collapsing isomorphism arrows we obtain \mathbf{G} -diagram \mathcal{Z}' satisfying conclusions of Proposition 3.2. \square

5.4. Proof of Lemma 5.2. The Chernoff bound for the tail of a binomially distributed random variable $X \sim \text{Bin}(N, \rho)$ asserts that for any $0 \leq \delta \leq 1$ holds

$$\begin{aligned} \mathbb{P}\{X < (1 - \delta)N \cdot \rho\} &\leq e^{-\frac{1}{2}\delta^2 N \cdot \rho} \\ \mathbb{P}\{X > (1 + \delta)N \cdot \rho\} &\leq e^{-\frac{1}{3}\delta^2 N \cdot \rho} \end{aligned}$$

Applying the bound for upper and lower tail for the binomially distributed random variable $N \cdot \nu$ we obtain the inequality in (i).

The second assertion follows from the following estimate

$$\begin{aligned} \mathbb{P}\left\{\frac{\nu}{\rho} \ln \frac{\nu}{\rho} > t\right\} &\leq \mathbb{P}\left\{\frac{\nu}{\rho} \left(\frac{\nu}{\rho} - 1\right) > t\right\} \\ &= \mathbb{P}\left\{\nu > \rho \cdot \left(\frac{\sqrt{1+4t}-1}{2} + 1\right)\right\} \end{aligned}$$

For $0 \leq t \leq 2$ we have $\sqrt{1+4t}-1 \geq t$, therefore

$$\mathbb{P}\left\{\frac{\nu}{\rho} \ln \frac{\nu}{\rho} > t\right\} \leq \mathbb{P}\left\{\nu > \rho \cdot \left(\frac{t}{2} + 1\right)\right\}$$

By the Chernoff bound we have

$$\mathbb{P}\left\{\frac{\nu}{\rho} \ln \frac{\nu}{\rho} > t\right\} \leq e^{-\frac{1}{12}N \cdot \rho t^2}$$

\square

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