Semicontinuity of capacity under pointed intrinsic flat convergence

SEMICONTINUITY OF CAPACITY UNDER POINTED INTRINSIC FLAT CONVERGENCE

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Abstract. The concept of the capacity of a compact set in $\mathbb{R}^n$ generalizes readily to noncompact Riemannian manifolds and, with more substantial work, to metric spaces (where multiple natural definitions of capacity are possible). Motivated by analytic and geometric considerations, and in particular Jauregui's definition of capacity-volume mass and Jauregui and Lee's results on the lower semicontinuity of the ADM mass and Huisken's isoperimetric mass, we investigate how the capacity functional behaves when the background spaces vary. Specifically, we allow the background spaces to consist of a sequence of local integral current spaces converging in the pointed Sormani–Wenger intrinsic flat sense. For the case of volume-preserving ($\mathcal{VF}$) convergence, we prove two theorems that demonstrate an upper semicontinuity phenomenon for the capacity: one version is for balls of a fixed radius centered about converging points; the other is for Lipschitz sublevel sets. Our approach is motivated by Portegies' investigation of the semicontinuity of eigenvalues under $\mathcal{VF}$ convergence. We include examples to show the semicontinuity may be strict, and that the volume-preserving hypothesis is necessary. Finally, there is a discussion on how capacity and our results may be used towards understanding the general relativistic total mass in non-smooth settings.

1. Introduction

The capacity (or "harmonic" or "electrostatic" or "Newtonian" capacity) of a compact set $K \subset \mathbb{R}^n$, $n \geq 3$, is defined as

$$\operatorname{cap}(K) = \inf_{\phi} \left\{ \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} |\nabla \phi|^2 dV \ : \ \phi \text{ is Lipschitz with compact support, and } \phi \equiv 1 \text{ on } K \right\},$$

where ω_{n-1} is the hypersurface area of the unit $(n-1)$-sphere. If ∂K is sufficiently regular (e.g., a C^1 hypersurface), then there exists a unique harmonic function on $\mathbb{R}^n \setminus K$, equaling 1 on ∂K and approaching 0 at infinity, such that

$$\operatorname{cap}(K) = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n \setminus K} |\nabla u|^2 dV = -\frac{1}{(n-2)\omega_{n-1}} \int_S \frac{\partial u}{\partial \nu} dA,$$

for any surface S enclosing ∂K. For example, a ball of radius r has capacity equal to r^{n-2}. Capacity is monotone under set inclusion and enjoys nice measure-theoretic properties, such as inner and outer regularity [8]. Geometrically, it can be bounded below by the volume radius of K and, if ∂K is convex, bounded above in terms of the total mean curvature of ∂K [25].

Capacity also makes sense with an analogous definition in complete Riemannian manifolds, such as asymptotically flat manifolds. (Without some control on the asymptotics, however, the capacity could be zero for every compact set.) It is natural to ask how capacity behaves along a converging sequence of Riemannian manifolds. For example, it is not difficult to show that if M is a smooth manifold equipped with a sequence of complete Riemannian metrics g_i, and g_i converges uniformly (i.e, in C^0) on compact sets to a Riemannian metric g, then for any compact set $K \subset M$,

$$\limsup_{i \to \infty} \operatorname{cap}_{g_i}(K) \leq \operatorname{cap}_g(K).$$

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More generally, an analogous statement holds for a sequence of complete Riemannian manifolds converging in the pointed C^0 Cheeger–Gromov sense. In fact, strict inequality can hold. This upper semicontinuity of capacity contrasts sharply with two other natural notions of the "size" of K, the volume and perimeter, which are of course continuous under C^0 convergence of the background metrics. This different behavior of capacity springs from its non-local nature, specifically its dependence on the geometry "at infinity."

The aim of this paper is to study capacity in lower regularity background spaces and in particular to analyze its behavior under lower-regularity convergence.

While capacity in lower regularity (such as in metric measure spaces) has received significant attention in the analysis literature (see below), we are also interested in capacity and its continuity properties for geometric reasons. For example a recent paper by Jauregui [18] suggests a definition of total mass in general relativity for asymptotically flat 3-manifolds that is based on the capacity–volume inequality, generalizes the well-known ADM mass, and is inspired by Huisken's isoperimetric mass [16, 17]. Several important open problems in general relativity related to the total mass seem to naturally involve Sormani–Wenger intrinsic flat ("$\mathcal{F}$") limits (we refer the reader to [29] and [21], for example), so the behavior of capacity under such convergence is of interest. For instance, Jauregui and Lee showed a lower semicontinuity Huisken's isoperimetric mass [16, 17] under pointed $\mathcal{VF}$-convergence [21] (where "$\mathcal{VF}$" refers to volume-preserving intrinsic flat convergence). The definition of mass in [18] involves capacity with a negative sign, so the upper semicontinuity we prove here is supportive of lower semicontinuity of that mass. We continue this discussion to section 5.

Our approach to establishing the upper semicontinuity of capacity is inspired by Portegies' proof that certain min-max values of the Laplacian on a compact Riemannian manifold M are upper semicontinuous under $\mathcal{VF}$ convergence [26]. Recall that, for example, the first such eigenvalue,

$$\lambda_1 = \inf_{f \in C^\infty(M)} \left\{ \int_M |\nabla f|^2 dV \ : \ \int_M f^2 dV = 1, \int_M f dV = 0 \right\},$$

varies continuously with respect to C^2 convergence of Riemannian metrics, but may only be upper semicontinuous for weaker types of convergence such as measured Gromov–Hausdorff convergence [10]. Since capacity is only interesting in noncompact spaces, we will specifically study the behavior of capacity under *pointed* $\mathcal{VF}$-convergence. In Section 4, we provide examples to illustrate that there are essentially two distinct reasons for the capacity to jump under a $\mathcal{VF}$-limit, and both jumps "go the same way." One reason is non-uniform control at infinity, which may be seen even under smooth convergence; the other is an effect of the relatively coarse nature of $\mathcal{VF}$ convergence.

Below we state one of our main theorems, that the capacity of closed balls of a fixed radius about converging points in a sequence of converging spaces cannot jump down in a limit. The natural setting for intrinsic flat convergence is the integral current spaces of Sormani and Wenger [30], which are constructed using the integral currents on metric spaces of Ambrosio and Kirchheim [1]. We will use local versions of these spaces (building on Lang and Wenger's locally integral currents [24]) and pointed convergence and make use of the definition of Dirichlet energy in these spaces appearing in [26] to define capacity. The relevant definitions will be given in Section 2.

Theorem 1. *Let $N_i = (X_i, d_i, T_i)$ and $N_\infty = (X_\infty, d_\infty, T_\infty)$ be local integral current spaces of dimension $m \geq 2$, such that $N_i \to N_\infty$ in the pointed volume-preserving intrinsic flat sense with respect to $p_i \in X_i$ and $p_\infty \in X_\infty$. Suppose the closed ball $\overline{B}(p_\infty, r)$ in X is compact for some $r > 0$. Then:*

$$\limsup_{i \to \infty} \operatorname{cap}_{N_i}(\overline{B}(p_i, r)) \leq \operatorname{cap}_{N_\infty}(\overline{B}(p_\infty, r)). \tag{1}$$

The other main theorem, Theorem 18, is a version of Theorem 1 that replaces closed balls with sublevel sets of Lipschitz functions. Both of these theorems will be proved using the technical result Theorem 19, an extrinsic version of the theorems which itself is inspired by Theorem 6.2 in [26].

We note that there is a body of literature on capacities of sets in metric measure spaces. While we make no attempt at a comprehensive account, we discuss this partially here, referring the reader to the books by Björn and Björn [5] and Heinonen, Koskela, Shanmugalingam, and Tyson [14] for example. One approach begins with the definition of Sobolev functions on a metric measure space, due to Hajłasz [13], based on the Hardy–Littlewood maximal operator. Kinnunen and Martio used Hajłasz's definition to develop a Sobolev p-capacity [22]. (Sobolev p-capacity, in contrast to the capacity we consider here, is found by minimizing the Sobolev norm, i.e., it includes the L^p norm of the test functions.) A different approach is to consider, instead of Hajłasz's Sobolev spaces, the Sobolev spaces defined by Shanmugalingam [27], called Newtonian spaces, an approach to Sobolev spaces using weak upper gradients. This is explored in detail in [5], where the version of capacity based only on the L^p norm of the weak upper gradient (as we consider in this paper) is called the variational capacity. Since the norm of the tangential differential is $\|T\|$-almost-everywhere equal to the minimal relaxed gradient (see Theorem 28 in the appendix, cf. [26, Theorem 5.2]), this latter capacity is equivalent to that which we use in this paper. We continue this discussion in the appendix and also refer the reader to [12] for additional discussion on capacities in metric measure spaces.

Outline. Section 2 covers the essential background material, including Ambrosio–Kirchheim currents on metric spaces, flat convergence, integral current spaces, and Sormani–Wenger intrinsic flat convergence, before moving on to local integral current spaces and pointed $\mathcal{F}$- and $\mathcal{VF}$-convergence. We also recall the definition of the Dirichlet energy of a Lipschitz function using the tangential differential (where some details are deferred to the appendix), and use that to define the capacity in a local integral current space. The main results are presented and proved in Section 3, and several examples are given in Section 4 to demonstrate how capacity may "jump up" and show that volume-preserving convergence is necessary. Section 5 includes further discussion regarding how capacity on integral current spaces may be of interest for problems involving mass in general relativity.

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2. Definitions and basic objects

The theory of m-dimensional currents on $\mathbb{R}^n$ originated with de Rham [7] and was developed by Whitney [34] and especially Federer–Fleming [9]. This area of geometric measure theory has been instrumental for attacking a wide variety of problems in geometric analysis. However, there is interest in generalizing this theory to ambient spaces that are not smooth manifolds. This was accomplished by Ambrosio–Kirchheim by following an idea of de Giorgi: rather than viewing a current as a functional on the space of smooth differential forms, Ambrosio and Kirchheim defined an m-dimensional current on a metric space X as a functional on $(m+1)$-tuples of Lipschitz functions $(f, \pi_1, \ldots, \pi_m)$, satisfying some additional properties [1]. Special types of classical currents, such as normal and integral currents, generalize to the metric space setting.

Inspired by the Gromov–Hausdorff distance between compact metric spaces and seeking an analog of Whitney's flat convergence to be defined for metric spaces, Sormani and Wenger produced a definition of *integral current spaces*, which are essentially metric spaces equipped with an integral current in the sense of Ambrosio–Kirchheim. For example, compact, connected oriented Riemannian manifolds can naturally be viewed as integral current spaces. Moreover, Sormani and Wenger defined the *intrinsic flat distance* between integral current spaces, using the flat distance on metric spaces due to Wenger [32].

Our primary goal is to study the capacity of compact sets in an integral current space, and in particular the continuity behavior of capacity. We certainly want the theory to include ambient spaces of infinite mass (volume), such as $\mathbb{R}^n$, yet Sormani–Wenger integral current spaces (built using Ambrosio–Kirchheim currents) by definition have finite mass. To work around this, we will use Lang–Wenger's extension [24] of Ambrosio–Kirchheim [1] integral currents on metric spaces to so-called *locally integral currents*, then generalize Sormani–Wenger's definition of integral current space accordingly. We recall the details of locally integral currents in Section 2.1. In Section 2.2, we recall local flat, weak, and flat convergence. Next, in Section 2.3, we recall the details of Sormani–Wenger intrinsic flat convergence and its pointed version. Finally, in Section 2.4, we give the definition of capacity in local integral current spaces. This definition involves a differential of Lipschitz functions on a metric space, and there are a number of ideas from [2] that will be recalled in the appendix. This approach is inspired by the work of Portegies on studying the behavior of eigenvalues of the Laplacian under $\mathcal{VF}$-convergence [26].

2.1. **Locally integral currents.** The goal of this section is to arrive at the definition of a locally integral current. Below we are essentially summarizing the parts of [24] that will get us to that point, including only the minimal details, with no proofs.

First, we recall metric functionals and how they produce an outer measure. Currents of locally finite mass will be those metric functionals whose measure behaves well from the inside and outside. Locally integer rectifiable currents are defined next, and then finally locally integral currents.

Given a metric space Z, define:

- $\mathrm{Lip}(Z)$ as the vector space of Lipschitz functions $Z \to \mathbb{R}$,
- $\mathrm{Lip}_{\mathrm{loc}}(Z)$ as the vector space of functions $Z \to \mathbb{R}$ that are Lipschitz on bounded sets,
- $\mathrm{Lip}^{\mathrm{b}}(Z)$ as the vector space of Lipschitz functions $Z \to \mathbb{R}$ that are bounded, and
- $\mathrm{Lip}_{\mathrm{B}}(Z)$ as the vector space of Lipschitz functions $Z \to \mathbb{R}$ that are bounded with bounded support.

The Lipschitz constant of a function $f : Z \to \mathbb{R}$ will be denoted by $\mathrm{Lip}(f)$.

For an integer $m \geq 0$, an m-dimensional metric functional T will act on $(m+1)$-tuples of functions in $\mathrm{Lip}_{\mathrm{B}}(Z) \times [\mathrm{Lip}_{\mathrm{loc}}(Z)]^m$ and produce a real number. Such an m-tuple $(f, \pi_1, \ldots, \pi_m)$ (sometimes denoted more briefly by (f, π)) should be conceptually thought of as the differential form "$f d\pi_1 \wedge \cdots \wedge d\pi_m$", so metric functionals will generalize the idea of currents. The precise definition of T being a *metric functional* is that

(i) T is multilinear.
(ii) (continuity) Suppose $f \in \mathrm{Lip}_{\mathrm{B}}(Z)$ and $(\pi_1, \ldots, \pi_m) \in \mathrm{Lip}_{\mathrm{loc}}(Z)^m$, and that we have m sequences $\{\pi_i^j\}_{j=1}^\infty$ in $\mathrm{Lip}_{\mathrm{loc}}(Z)$ such that $\pi_i^j \to \pi_i$ pointwise as $j \to \infty$ for each $i = 1, \ldots, m$, and the Lipschitz constants of π_i^j are uniformly bounded in j on any bounded subset of Z. Then
$$\lim_{j\to\infty} T(f, \pi_1^j, \ldots, \pi_m^j) = T(f, \pi_1, \ldots, \pi_m).$$
(iii) (locality) Consider $(f, \pi_1, \ldots, \pi_m) \in \mathrm{Lip}_{\mathrm{B}}(Z) \times [\mathrm{Lip}_{\mathrm{loc}}(Z)]^m$. Suppose that one of the π_i is constant on the δ-neighborhood of $\mathrm{spt}(f)$ for some $\delta > 0$. Then
$$T(f, \pi_1, \ldots, \pi_m) = 0.$$

Metric functionals as defined here are natural analogs of the metric functionals considered by Ambrosio and Kirchheim [1] (in that case, f was a bounded Lipschitz function and the π_i were required to be (globally) Lipschitz).

If X and Y are metric spaces and $\varphi \in \mathrm{Lip}_{\mathrm{loc}}(X, Y)$ with $\varphi^{-1}(A)$ bounded for any bounded set $A \subseteq Y$, a metric functional T on X can be pushed forward to a metric functional on Y of the same dimension as follows:
$$(\varphi_\# T)(f, \pi_1, \ldots, \pi_m) = T(f \circ \varphi, \pi_1 \circ \varphi, \ldots, \pi_m \circ \varphi).$$

The *boundary* of an m-dimensional metric functional T, $m \geq 1$, is an $(m-1)$-dimensional metric functional ∂T defined by
$$\partial T(f, \pi_1, \ldots, \pi_m) = T(\sigma, f, \pi_1, \ldots, \pi_m),$$
where σ is any bounded Lipschitz function with bounded support that is identically 1 on the support of f. In [24] it is shown this definition is independent of the choice of σ and defines a metric functional. The boundary satisfies nice properties, such as commuting with the push-forward and $\partial(\partial T) = 0$.

To define currents, we require the notion of the mass measure $\|T\|$ associated to a metric functional T. Ambrosio–Kirchheim's definition requires finite mass, but the approach we take following Lang–Wenger will require only locally finite mass.

Given an m-dimensional metric functional T and an open set $V \subseteq Z$, the *mass of T in V* is defined to be

$$\mathbf{M}_V(T) = \sup \sum_{\lambda} T(f^\lambda, \pi^\lambda),$$

where the supremum is taken over all finite collections $f^\lambda \in \mathrm{Lip}_{\mathrm{B}}(Z)$, $\pi^\lambda = (\pi_1^\lambda, \ldots, \pi_m^\lambda) \in \mathrm{Lip}(Z)$ such that $\mathrm{Lip}(\pi_i^\lambda) \leq 1$, f^λ is supported in V, and $\sum_\lambda |f^\lambda| \leq 1$ everywhere. This definition serves as the mass of T for any open set. For an arbitrary subset $A \subseteq Z$, we define

$$\|T\|(A) = \inf\{\mathbf{M}_V(T) \mid V \supseteq A \text{ is open}\},$$

so certainly $\|T\|(A) = \mathbf{M}_A(T)$ if A is open.

Definition 2 ([24])**.** An *m-dimensional metric current on Z with locally finite mass*, $m \geq 0$, is an m-dimensional metric functional T on Z such that given any $\epsilon > 0$ and any bounded open set $U \subseteq Z$, we have $\mathbf{M}_U(T) < \infty$ and that there exists a compact set $C \subseteq Z$ such that $\mathbf{M}_{U \setminus C}(T) < \epsilon$.

The set of such objects will be denoted $\mathbf{M}_{\mathrm{loc},m}(Z)$. For $T \in \mathbf{M}_{\mathrm{loc},m}(Z)$, $\|T\|$ defined above is a Borel regular outer measure on Z that is concentrated on a σ-compact set [24, Proposition 2.4].

We also recall the definition of two ways of identifying where the m-dimensional metric functional T "lives": the support and the canonical set:

$$\mathrm{spt}(T) = \{z \in Z \ : \ \|T\|(B(z,r)) > 0 \text{ for all } r > 0\}$$
$$\mathrm{set}(T) = \{z \in Z \ : \ \liminf_{r \to 0} \frac{\|T\|(B(z,r))}{r^m} > 0\}.$$

The latter definition was used by Sormani and Wenger, particularly in their definition of integral current space.

Recall that metric functionals T are defined as functions $\mathrm{Lip}_{\mathrm{B}}(Z) \times [\mathrm{Lip}_{\mathrm{loc}}(Z)]^m \to \mathbb{R}$. However, it is shown in [24] that if $T \in \mathbf{M}_{\mathrm{loc},m}(Z)$, then T has a natural extension to the larger space of $(m+1)$-tuples (f, π) such that f is a bounded Borel function with bounded support (and the π are as before). This allows us to restrict $T \in \mathbf{M}_{\mathrm{loc},m}(Z)$ to a bounded Borel set $A \subseteq Z$ as follows:

$$(T \llcorner A)(f, \pi) := T(f\chi_A, \pi).$$

Furthermore, $T \llcorner A \in \mathbf{M}_{\mathrm{loc},m}(Z)$ and $\|T \llcorner A\| = \|T\| \llcorner A$.

There is nice compatibility between Ambrosio–Kirchheim currents and the metric currents with locally finite mass; see [24, Section 2.5].

Our main interest is in the definition of locally integral currents, so we build up to that now.

A subset $C \subseteq Z$ is a *compact m-rectifiable set* if there exist compact sets $K_1, \ldots, K_n \subset \mathbb{R}^m$ and Lipschitz maps $f_i : K_i \to Z$ such that

$$C = \bigcup_{i=1}^n f_i(K_i).$$

Definition 3 ([24])**.** An m-dimensional metric functional T on Z is a *locally integer rectifiable current* if:

(a) Given $\epsilon > 0$ and any bounded open subset U of Z, there is a compact m-rectifiable set $C \subseteq Z$ such that $\mathbf{M}_U(T) < \infty$ and $\mathbf{M}_{U \setminus C}(T) < \epsilon$.

(b) For every bounded Borel set $B \subseteq Z$ and every Lipschitz map $\varphi : Z \to \mathbb{R}^m$, we have

$$\varphi_\#(T \llcorner B)(f, \pi) = \int_{\mathbb{R}^m} \theta f \, d\pi_1 \wedge \ldots \wedge d\pi_m,$$

for some integrable $\theta : \mathbb{R}^m \to \mathbb{Z}$.

The abelian group of locally integer rectifiable m-currents on Z will be denoted by $\mathcal{I}_{\mathrm{loc},m}(Z)$.

By (a) above, $\mathcal{I}_{\mathrm{loc},m}(Z) \subseteq \mathbf{M}_{\mathrm{loc},m}(Z)$.

2.2. Flat and weak convergence; integral currents.

Definition 4 ([24])**.** An m-dimensional locally integer rectifiable current T is a *locally integral current* if $m = 0$, or, for $m \geq 1$, if $\partial T \in \mathcal{I}_{\mathrm{loc},m-1}(Z)$. (By [24, Theorem 2.2], it is sufficient to assume $\partial T \in \mathbf{M}_{\mathrm{loc},m-1}(Z)$.) The abelian group of locally integral currents on Z will be denoted by $\mathbf{I}_{\mathrm{loc},m}(Z)$.

Definition 5 ([24])**.** A sequence T_i in $\mathbf{I}_{\mathrm{loc},m}(Z)$ converges to $T \in \mathbf{I}_{\mathrm{loc},m}(Z)$ in the *local flat topology* if, for every closed, bounded set $B \subseteq Z$ there exists a sequence S_i in $\mathbf{I}_{\mathrm{loc},m+1}(Z)$ such that

$$\|T - T_i - \partial S_i\|(B) + \|S_i\|(B) \to 0$$

as $i \to \infty$.

Analogous to the corresponding statement for classical currents, if $T_i \to T$ in the local flat topology, then $T_i \to T$ weakly (pointwise as functionals). Moreover, under weak convergence, $\liminf_{i\to\infty} \|T_i\|(U) \geq \|T\|(U)$ for all bounded open sets U. We also require the following lemma, generalizing [26, Lemma 2.7] to the local case.

Lemma 6. *Suppose $T_i \to T$ weakly in $\mathbf{I}_{loc,m}(Z)$, and suppose there exists a bounded open set $U \subseteq Z$ such that $\|T_i\|(U) \to \|T\|(U)$ as $i \to \infty$. Then $\|T_i\| \llcorner U \to \|T\| \llcorner U$ weakly as a bounded sequence of finite Borel measures. That is, for every bounded, continuous function $f : Z \to \mathbb{R}$ supported in U,*

$$\int_Z f d\|T_i\| \to \int_Z f d\|T\|$$

as $i \to \infty$. In particular, for every closed set $C \subset U$,

$$\limsup_{i\to\infty} \|T_i\|(C) \leq \|T\|(C).$$

This follows immediately from the proof of [26, Lemma 2.7], which uses the portmanteau theorem.

We conclude this section by recalling Wenger's flat distance for currents of finite mass. First, define:

$$\mathbf{M}_m(Z) = \{T \in \mathbf{M}_{\mathrm{loc},m}(Z) \ : \ \|T\|(Z) < \infty\}.$$

In Section 2.5 of [24] it is shown that $\mathbf{M}_m(Z)$ may be identified with the set of currents on Z as originally defined by Ambrosio and Kirchheim in [1]. It is straightforward to verify that

$$\mathbf{I}_m(Z) = \{T \in \mathbf{I}_{\mathrm{loc},m}(Z) \ : \ \|T\|(Z) < \infty\}$$

may be identified with the set of integral currents on Z as defined in [1]. (While currents and integral currents in [1] were only defined on complete spaces, it is pointed out in Section 2.2 of [24] that the completeness restriction can be avoided.) We therefore may take the above equations as definitions of currents and integral currents on Z.

Given two integral m-currents T_1 and T_2 on Z, now taken to be a complete metric space, Wenger defined their *flat distance* [32]:

$$d_Z^F(T_1, T_2) = \inf_{A \in \mathbf{I}_m(Z), B \in \mathbf{I}_{m+1}(Z)} \{\mathbf{M}(A) + \mathbf{M}(B) \ : \ T_2 - T_1 = A + \partial B\},$$

where we use $\mathbf{M}(A)$ to mean $\|A\|(Z)$, etc. This will be used below in the definition of Sormani–Wenger intrinsic flat distance in the next section.

2.3. **Local integral current spaces and pointed $\mathcal{F}$-convergence.** We first recall Sormani–Wenger's definition of an integral current space:

Definition 7 ([30])**.** An *integral current space* $N = (X, d, T)$ is a metric space (X, d) equipped with an integral current T on the completion $(\overline{X}, \overline{d})$, such that $\mathrm{set}(T) = X$. The *dimension* of N is the dimension of T.

Definition 8 ([30])**.** The *Sormani–Wenger intrinsic flat distance* between integral current spaces $N_1 = (X_1, d_1, T_1)$ and $N_2 = (X_2, d_2, T_2)$ of dimension m is:

$$d_{\mathcal{F}}(N_1, N_2) = \inf_{Z,\varphi_1,\varphi_2} \{d_Z^F(\varphi_{1\#}(T_1), \varphi_{2\#}(T_2))\},$$

where the infimum is taken over all complete metric spaces Z and isometric embeddings $\varphi_1 : X_1 \to Z$ and $\varphi_2 : X_2 \to Z$.

Note that the φ_i canonically extend to $\overline{X}_i$ as isometric embeddings, so the push-forwards are well-defined.

In [30] it is shown that $d_{\mathcal{F}}$ defines a distance on the set of equivalence classes of precompact integral current spaces of dimension m (where $d_{\mathcal{F}}(N_1, N_2) = 0$ if and only if there exists an isometry $\varphi : X_1 \to X_2$ so that $\varphi_\#(T_1) = T_2$).

Definition 9 ([30], [29])**.** A sequence of m-dimensional integral current spaces N_i *$\mathcal{F}$-converges* to an m-dimensional integral current space N (written $N_i \xrightarrow{\mathcal{F}} N$) if

$$d_{\mathcal{F}}(N_i, N) \to 0$$

and *$\mathcal{VF}$-converges* to N (written $N_i \xrightarrow{\mathcal{VF}} N$) if

$$d_{\mathcal{F}}(N_i, N) + |\mathbf{M}(N_i) - \mathbf{M}(N)| \to 0,$$

as $i \to \infty$, where $\mathbf{M}(\cdot)$ denotes the mass of the underlying integral current.

In general, if $N_i \xrightarrow{\mathcal{F}} N$, lower semicontinuity of mass holds: $\liminf_{i\to\infty} \mathbf{M}(N_i) \geq \mathbf{M}(N)$. Thus, the volume-preserving hypothesis in $\mathcal{VF}$-convergence simply assures there is no mass drop in the limit.

We also recall an indispensable result of Sormani and Wenger that establishes the existence of a single "common space" into which an $\mathcal{F}$-converging sequence embeds:

Theorem 10 (Theorem 4.2 of [30])**.** *Suppose $N_i = (X_i, d_i, T_i)$ $\mathcal{F}$-converges to $N = (X, d, T)$. Then there exists a complete metric space Z and isometric embeddings $\varphi_i : X_i \to Z$ and $\varphi : X \to Z$ such that $d_Z^F(\varphi_{i\#}(T_i), \varphi_\#(T)) \to 0$. Furthermore, by applying the Kuratowski embedding theorem, one may assume without loss of generality that Z is a w^*-separable Banach space, i.e., $Z = G^*$, the dual space of G, where G is a separable Banach space.*

Although the underlying spaces in an $\mathcal{F}$-converging sequence $N_i = (X_i, d_i, T_i) \xrightarrow{\mathcal{F}} N = (X, d, T)$ may all be distinct, Sormani [28] defines the convergence of points $x_i \in X_i$ to $x \in (\overline{X}, \overline{d})$ as the existence of Z and isometric embeddings φ_i and φ as in Theorem 10 for which

$$\varphi_i(x_i) \to \varphi(x) \text{ in } Z \text{ as } i \to \infty, \tag{2}$$

where φ denotes the canonical extension to $\overline{X}$. This concept will be used in the definition of pointed convergence below.

Next, observe that the definition of integral current space readily generalizes to the local setting:

Definition 11 ([21])**.** A *local integral current space* of dimension $m \geq 0$ is a triple $N = (X, d, T)$ in which (X, d) is a metric space and T is a locally integral m-current on $(\overline{X}, \overline{d})$ such that $\mathrm{set}(T) = X$.

For example, any complete, connected, oriented Riemannian m-manifold forms a local integral current space with the usual distance function, where T is given as integration [24, Section 2.8]:

$$T(f,\pi_1,\dots,\pi_m)=\int_M f d\pi_1\wedge\cdots\wedge d\pi_m.$$

In this case, $\|T\|$ agrees with the Riemannian volume measure (see [30, Example 2.32] and [21, Lemma 22(a)]).

We gather some basic known results regarding local integral current spaces:

Proposition 12. *Let $N=(X,d,T)$ be a local integral current space of dimension m.*

(a) Given $q\in X$, for almost all $r>0$, $T\llcorner B(q,r)$ is an integral m-current on X, so in particular, $N\llcorner B(q,r):=(\mathrm{set}(T\llcorner B(q,r)),d,T\llcorner B(q,r))$ forms an integral current space; see [28, Lemma 2.34] *and* [21, Lemma 13]*. Throughout the paper, the ball in the notation "$T\llcorner B(q,r)$" refers to a ball in the completion of X.)*

(b) $\|T\|$ is a Borel measure on X that is finite on bounded sets [24].

(c) (X,d) is countably $\mathcal{H}^m$-rectifiable, that is, there exist at most a countable number of Lipschitz maps $\varphi_i:A_i\subset\mathbb{R}^m\to X$ such that $\mathcal{H}^m(X\setminus\bigcup_i\varphi_i(A_i))=0$. If desired, the A_i can be taken to be compact and the $\varphi_i(A_i)$ pairwise disjoint. (Claim (c) follows from (a) and [30, Remark 2.36]*.)*

(d) There exists a w^*-separable Banach space Y such that (X,d) embeds isometrically in Y. (Proof below.)

To see (d), recall that $\mathrm{spt}(T)$ is a σ-compact set [24, Proposition 2.4], and therefore separable. By the Kuratowski embedding theorem, we may isometrically embed $\mathrm{spt}(T)$ into ℓ^∞, which the dual of ℓ^1. Since $\mathrm{spt}(T)\supseteq\mathrm{set}(T)=X$, the claim follows.

The following definition is taken from [21], suitably generalized to possibly incomplete spaces. We also refer the reader to the paper of Takeuchi [31] giving a different approach to pointed $\mathcal{F}$-convergence (cf. [21, Remark 1]).

Definition 13. A sequence $N_i=(X_i,d_i,T_i)$ of local integral current spaces of dimension m converges to a local integral current space $N=(X,d,T)$ of dimension m in the *pointed Sormani–Wenger intrinsic flat sense* or "*pointed $\mathcal{F}$-sense*" (respectively, *pointed volume-preserving intrinsic flat sense* or "*pointed $\mathcal{VF}$-sense*") with respect to $p_i\in X_i$ and $q\in\overline{X}$ if for any $r_0>0$, there exists $r\geq r_0$ such that $N\llcorner B(q,r)$ and $N_i\llcorner B(p_i,r)$ are precompact integral current spaces (for all i sufficiently large), and $N_i\llcorner B(p_i,r)\xrightarrow{\mathcal{F}} N\llcorner B(q,r)$ (respectively, $N_i\llcorner B(p_i,r)\xrightarrow{\mathcal{VF}} N\llcorner B(q,r)$), and if $p_i\to q$ as in (2).

We verify that pointed $\mathcal{F}$-convergence is a reasonable notion, in that limits are unique:

Proposition 14. *Suppose $N_i=(X_i,d_i,T_i)$ is a sequence of local integral current spaces of dimension m that converges in the pointed $\mathcal{F}$-sense to a local integral current space of dimension m, $N=(X,d,T)$, with respect to $p_i\in X$ and $q\in\overline{X}$. Suppose that N_i also converges in the pointed $\mathcal{F}$-sense to another local integral current space of dimension m, $N'=(X',d',T')$, with respect to $p_i\in X$ and $q'\in\overline{X}'$. Then there exists an isometry Φ from the completion of (X,d) to the completion of (X',d') such that $\Phi_\#(T)=T'$ and $\Phi(q)=q'$.*

A similar result appears in [31, Proposition 3.7].

Proof. We will first inductively define an increasing sequence of radii $\{r_k^*\}$, diverging to infinity, and a sequence of isometries $\Phi_k:B(q,r_k^*)\to B(q',r_k^*)$ that satisfy $\Phi_k(q)=q'$ and that are current-preserving in the sense that $(\Phi_k)_\#(T\llcorner B(q,r_k^*))=T'\llcorner B(q',r_k^*)$. Afterwards, we will use a diagonal argument to create a current-preserving isometry $\Phi:X\to X'$.

Let $r_1>0$ be arbitrary. By definition of pointed $\mathcal{F}$-convergence, there exist $r>r_1$ and $r'>r$ such that $N\llcorner B(q,r)$, $N_i\llcorner B(p_i,r)$, $N'\llcorner B(q',r')$, and $N_i\llcorner B(p_i,r')$ are all precompact integral current spaces for all i

sufficiently large, and that

$$N_i \llcorner B(p_i, r) \xrightarrow{\mathcal{F}} N \llcorner B(q, r)$$

with $p_i \to q$ and

$$N_i \llcorner B(p_i, r') \xrightarrow{\mathcal{F}} N' \llcorner B(q', r')$$

with $p_i \to q'$.

By [28, Lemma 4.1], we may pass to a subsequence such that for almost all radii $\leq r$ we have convergence in the first equation above, and for almost all radii $\leq r'$, we have convergence in the second equation above. In particular, there exists $r_1^* > r_1$ such that

$$N_i \llcorner B(p_i, r_1^*) \xrightarrow{\mathcal{F}} N \llcorner B(q, r_1^*)$$

and

$$N_i \llcorner B(p_i, r_1^*) \xrightarrow{\mathcal{F}} N' \llcorner B(q', r_1^*)$$

as sequences of precompact integral current spaces, with $p_i \to q$ and $p_i \to q'$, respectively. That is, there exist complete metric spaces Z and Z' and isometric embeddings $\varphi : B(q, r_1^*) \to Z$, $\varphi_i : B(p_i, r_1^*) \to Z$, $\varphi' : B(q', r_1^*) \to Z'$, and $\varphi_i' : B(p_i, r_1^*) \to Z'$ and such that

$$d_F^Z((\varphi_i)_\#(T_i \llcorner B(p_i, r_1^*)), \varphi_\#(T \llcorner B(q, r_1^*))) \to 0$$

and $\varphi_i(p_i) \to \varphi(q)$ in Z, and

$$d_F^{Z'}((\varphi_i')_\#(T_i \llcorner B(p_i, r_1^*), (\varphi')_\#(T' \llcorner B(q', r_1^*)))) \to 0$$

and $\varphi_i'(p_i) \to \varphi'(q')$ in Z'.

We apply [24, Proposition 1.1] to the sequence of complete metric spaces given by the closed balls $\overline{B}(p_i, r_1^*)$ in the completions $\overline{X}_i$, the points p_i, the integral currents $T_i \llcorner B(p_i, r_1^*)$, and the isometric embeddings φ_i and φ_i' (which canonically extend to $\overline{B}(p_i, r_1^*)$). This produces an isometry

$$\Phi_1 : \{q\} \cup \operatorname{spt}(T \llcorner B(q, r_1^*)) \to \{q'\} \cup \operatorname{spt}(T' \llcorner B(q', r_1^*))$$

that satisfies $\Phi_1(q) = q'$ and is current-preserving: $(\Phi_1)_\#(T \llcorner B(q, r_1^*)) = T' \llcorner B(q', r_1^*)$. We claim that the open ball $B_{\overline{X}}(q, r_1^*)$ in the completion is a subset of $\operatorname{spt}(T \llcorner B(q, r_1^*))$. Let $x \in B_{\overline{X}}(q, r_1^*)$. There exists a sequence $\{x_j\}$ in X that converges in $\overline{X}$ to x. By the triangle inequality, $\overline{d}(x_j, q) < r_1^*$ for j sufficiently large. Now, since $x_j \in X = \operatorname{set}(T)$, it follows that $x_j \in \operatorname{set}(T \llcorner B(q, r_1^*)) \subseteq \operatorname{spt}(T \llcorner B(q, r_1^*))$. Since the support is a closed set, we have $x \in \operatorname{spt}(T \llcorner B(q, r_1^*))$, which shows the claim. Thus, Φ_1 restricts to an isometry $B_{\overline{X}}(q, r_1^*) \to B_{\overline{X}'}(q', r_1^*)$, which then extends to an isometry (also denoted Φ_1) $\overline{B}(q, r_1^*) \to \overline{B}(q', r_1^*)$ (where again $\overline{B}$ refers to the closed ball in the completion). Moreover, Φ_1 remains base-point-preserving and current-preserving.

Suppose r_k^* and Φ_k have been defined for some $k \in \mathbb{N}$. Now take some $r_{k+1} > r_k^* + 1$ and in the same way as above find an $r_{k+1}^* > r_{k+1}$ and a current-preserving isometry $\Phi_{k+1} : B(q, r_{k+1}^*) \to B(q', r_{k+1}^*)$ that maps q to q'. Clearly the r_k^* diverge to infinity.

Now that we have created a sequence of base point- and current-preserving isometries $\{\Phi_k\}$ we will perform a diagonal argument to create a base point- and current-preserving isometry $\Phi : \overline{X} \to \overline{X}'$. For each k, we can restrict the maps Φ_k to maps from $\overline{B}(q, r_1^*) \to \overline{B}(q', r_1^*)$. Note that $\overline{B}(q, r_1^*)$ and $\overline{B}(q', r_1^*)$ are compact metric spaces by hypothesis, so by the Arzela–Ascoli theorem for functions defined on compact metric spaces taking values in compact metric spaces [23, Theorem 7.17], there exists a subsequence $n^{(1)}$ indexed by ℓ such that $\Phi_{n_\ell^{(1)}} : \overline{B}(q, r_1^*) \to \overline{B}(q', r_1^*)$ converges uniformly on $\overline{B}(q, r_1^*)$ as $\ell \to \infty$ to an isometry and clearly the resulting maps takes q to q'. If for some $j \in \mathbb{N}$, the subsequence $n^{(j)}$ has been defined, we may select a subsequence $n^{(j+1)}$ of $n^{(j)}$ such that the restriction of the sequence of functions $\Phi_{n_\ell^{(j+1)}}$ to isometries $\overline{B}(q, r_{j+1}^*) \to \overline{B}(q', r_{j+1}^*)$ converges uniformly on $\overline{B}(q, r_{j+1}^*)$ as $\ell \to \infty$ to an isometry.

Finally, we consider the diagonal subsequence $\Phi_{n_\ell^{(\ell)}}$. This subsequence converges uniformly on compact sets to some map $\Phi : \overline{X} \to \overline{X}'$. It is straightforward to see that Φ is an isometry and $\Phi(q) = q'$.

We still need to show that $\Phi_\# T = T'$. It suffices to show that for every $k \in \mathbb{N}$,

$$\Phi_\#(T \llcorner B(q, r_k^*)) = T' \llcorner B(q', r_k^*).$$

This follows as for every ℓ sufficiently large we have

$$(\Phi_{n_\ell^{(\ell)}})_\#(T \llcorner B(q, r_k^*)) = T' \llcorner B(q', r_k^*).$$

and the isometries $\Phi_{n_\ell^{(\ell)}}$ converge uniformly to Φ. Here, we are using continuity properties of metric currents with locally finite mass: see (2.4) in [24]. $\square$

2.4. **Dirichlet energy and capacity.** For integral currents on a Banach space, Portegies gave a definition of the Dirichlet energy of a Lipschitz function based on Ambrosio–Kirchheim's definition of the tangential differential of a Lipschitz function on a rectifiable set in a Banach space. By embedding a metric space in a Banach space, this allowed him to define the Dirichlet energy of a Lipschitz function on an integral current space. To make this precise, we recall from [2] (with the necessary details included in the appendix) the notion of the tangential differential $d_x^S f$ of a Lipschitz function f on a $\mathcal{H}^m$-countably rectifiable set S. We then proceed to generalize the Dirichlet energy to a local integral current space $N = (X, d, T)$. Given a Lipschitz function $f : X \to \mathbb{R}$ with bounded support, define (following [26]):

$$E_N(f) = \int_X |d_x^X f|^2 d\|T\|(x),$$

where $d_x^X f$ is the tangential differential when X is embedded in some appropriate Banach space. This is well-defined and independent of the embedding. We also note $|d_x^X f| \leq \mathrm{Lip}(f)$ and $\|T\|$ is finite on bounded sets, so that $E_N(f) < \infty$.

This is consistent with the usual definition in the smooth case, again following [26]:

Proposition 15. *Let (M, g) be a complete, connected, oriented Riemannian manifold, and let N be the associated local integral current space. Then for a Lipschitz function f of M with compact support,*

$$E_N(f) = \int_M |\nabla f|^2 dV,$$

where the gradient norm and volume measure are taken with respect to g.

Now, let $N = (X, d, T)$ denote a local integral current space of dimension $m \geq 2$. Let $K \subset X$ be a closed, bounded subset. We define

$$\mathrm{cap}_N(K) = \frac{1}{\gamma_m} \inf\{E_N(f) \ : \ f \in \mathrm{Lip}_B(X), f \equiv 1 \text{ on a neighborhood of } K\}, \tag{3}$$

where $\gamma_m = (m-2)\omega_{m-1}$ for $m \geq 3$ and $\gamma_2 = 2\pi$. It is clear that $\mathrm{cap}_N(K) \in [0, \infty)$ and that $K_1 \subset K_2$ implies $\mathrm{cap}_N(K_1) \leq \mathrm{cap}_N(K_2)$. In Euclidean space, this agrees with the usual definition of capacity stated in the introduction. (The latter required merely that $f \equiv 1$ on K itself, but in $\mathbb{R}^m$ the distinction is immaterial.) For example, a ball of radius r in $\mathbb{R}^m$ has capacity r^{m-2} for $m \geq 3$. Note that capacity is only interesting (i.e., not identically zero) in the case in which X is unbounded, and even then it is possible $\mathrm{cap}_N \equiv 0$ depending on the behavior of X and $\|T\|$ "at infinity."

Remark 1. Given a competitor f for the capacity, replacing f with its truncation between values of 0 and 1 produces another competitor whose Dirichlet energy has not increased. Thus, we may restrict to functions f in (3) satisfying $0 \leq f \leq 1$.

Remark 2. One can similarly define the p-capacity, for $p \geq 1$, by replacing $|d_x^X f|^2$ in the definition of E_N with $|d_x^X f|^p$.

3. Semicontinuity of capacity

The main results of this paper are Theorems 17 and 18 below, where Theorem 17 is a restatement of Theorem 1 from the introduction. In both cases we assume pointed $\mathcal{VF}$-convergence of local integral current spaces and establish the upper semicontinuity of the capacity of sets in the spaces. In the first case, the sets are balls centered around converging points; in the second, the sets are defined as Lipschitz sublevel sets. Both theorems will ultimately be consequences of the main technical result, Theorem 19.

3.1. **Corresponding regions.** Before presenting the theorems, we will recall the construction in [21] of "corresponding regions." Let $N_i = (X_i, d_i, T_i) \to N = (X, d, T)$ in the pointed $\mathcal{F}$-sense as local integral current spaces of dimension $m \geq 2$ with respect to $p_i \in X_i$ and $p \in \overline{X}$. Suppose $K \subsetneq X$ is nonempty and compact. The corresponding regions, to be defined, will be subsets K_i of X_i.

Fix a function $u : X \to \mathbb{R}$ with $\{u \leq 0\} = K$, $\mathrm{Lip}(u) = 1$ and

$$u(x) = d(x, K) \text{ for } x \in X \setminus K. \tag{4}$$

Such function u will be called a *defining function* for K. For example, $u(x) = d(x, K) \geq 0$ for $x \in X$ is such a function, but the definition allows for, for example, a signed distance function to ∂K if it can be well-defined and if it is a 1-Lipschitz function.

We fix r_0 so that $B(p, r_0) \supset K$ in X, and apply the definition of pointed $\mathcal{F}$-convergence to obtain $r \geq r_0$ such that $N_i \llcorner B(p_i, r) \xrightarrow{\mathcal{F}} N \llcorner B(p, r)$. Consequently, by Theorem 10, there exists a w^*-separable Banach space Y and isometric embeddings $\varphi_i : \mathrm{set}(T_i \llcorner B(p_i, r)) \to Y$ and $\varphi : \mathrm{set}(T \llcorner B(p, r)) \to Y$ such that the pushed-forward integral currents flat-converge in Y and such that $\varphi_i(p_i) \to \varphi(p)$ in Y.

Let $U : Y \to \mathbb{R}$ be the standard 1-Lipschitz extension[1] of $u \circ \varphi^{-1}$, where the latter is defined on the image of φ. We then define $u_i = U \circ \varphi_i$, a 1-Lipschitz function on $\mathrm{set}(T_i \llcorner B(p_i, r))$ for each i. Given a sequence of nonnegative real numbers $\{\alpha_i\}$ converging to 0, the sets

$$K_i = u_i^{-1}(-\infty, \alpha_i] \subseteq X_i \tag{5}$$

will be called a sequence of *corresponding regions*; they depend on the choices of u, r_0, the space Y, the embeddings, and the sequence α_i. They were essentially defined in [21], where it was proved that (roughly — see the proof of Proposition 16 below) $N_i \llcorner K_i$ subsequentially $\mathcal{F}$-converges to $N \llcorner K$. (We are generalizing the definition in [21] slightly by allowing the α_i to depend on i as well as working in *local* integral current spaces, though we are restricting the form of u.)

Let us try to provide some intuition for the sets K_i. We claim that the intersection of $\varphi_i(\mathrm{set}(T_i \llcorner B(p_i, r)))$ with the closed α_i-tubular neighborhood of $\varphi(K)$ is contained in $\varphi_i(K_i)$. Indeed let $x \in \mathrm{set}(T_i \llcorner B(p_i, r))$, define $y := \varphi_i(x)$ and assume that $d_Y(y, \varphi(K)) \leq \alpha_i$. Then

$$\begin{aligned} u_i(x) &= \inf_{a \in \varphi(\mathrm{set}(T \llcorner B(p,r)))} [u \circ \varphi^{-1}(a) + d_Y(a, y)] \leq \inf_{a \in \varphi(K)} [u \circ \varphi^{-1}(a) + d_Y(a, y)] \\ &\leq \inf_{a \in \varphi(K)} d_Y(a, y) = d_Y(y, \varphi(K)) \leq \alpha_i, \end{aligned}$$

so $x \in K_i$; in the second inequality we used that u is non-positive on K. In case the defining function is chosen to be $u := d_X(\cdot, K)$, then the sets $\varphi_i(K_i)$ are *exactly* the intersections of $\varphi_i(\mathrm{set}(T_i \llcorner B(p_i, r)))$ with the closed α_i-neighborhood of $\varphi(K)$.

[1] When we refer to the "standard Lipschitz extension" of a Lipschitz function $f : A \subset Y \to \mathbb{R}$, we mean the extension $\tilde{f} : Y \to \mathbb{R}$ given by

$$\tilde{f}(y) := \inf_{a \in A} \left(f(a) + \mathrm{Lip}(f) d_Y(a, y) \right),$$

which has $\mathrm{Lip}(\tilde{f}) = \mathrm{Lip}(f)$.

We now establish in Proposition 16 conditions to assure that the sets K_i are not "too small" — a priori they could even be empty. Note that we include (b) to accommodate applications that may involve a signed distance function.

Proposition 16. *As in the above setup, consider a nonempty compact set $K \subsetneq X$ with a defining function u and choices of $r \geq r_0 > 0$ and isometric embeddings.*

(a) There exists a subsequence N_{i_k} and a positive sequence $\alpha_{i_k} \searrow 0$ such that the corresponding regions K_{i_k} for this subsequence satisfy
$$\liminf_{k\to\infty} \|T_{i_k}\|(K_{i_k}) \geq \|T\|(K).$$

(b) With the choice $\alpha_i = 0$, there exists a subsequence N_{i_k} such that the corresponding regions K_{i_k} for this subsequence satisfy:
$$\liminf_{k\to\infty} \|T_{i_k}\|(K_{i_k}) \geq \|T\|(\{u < 0\}).$$

Proof. (a). For $\delta \in \mathbb{R}$, let $K^\delta = \{u \leq \delta\}$. By [21, Lemma 27], we may pass to a subsequence (keeping the same indexing) such that the restriction of T_i to $K_i^\delta = \{u_i \leq \delta\}$ $\mathcal{F}$-converges to $T \llcorner K^\delta \neq 0$. By lower semicontinuity of mass under $\mathcal{F}$-convergence, we have for $\delta \geq 0$:
$$\liminf_{i\to\infty} \|T_i\|(K_i^\delta) \geq \|T\|(K^\delta) \geq \|T\|(K). \tag{6}$$
Select $\delta_1 \in (2^{-1}, 2^0)$ for which (6) holds with $\delta = \delta_1$. Then there exists $i_1 \geq 1$ such that
$$\|T_{i_1}\|(K_{i_1}^{\delta_1}) \geq \|T\|(K)\left(1 - 2^{-1}\right).$$
Using (6) repeatedly, we iteratively select $\delta_k \in (2^{-k}, 2^{-k+1})$ and $i_k > i_{k-1}$ such that
$$\|T_{i_k}\|(K_{i_k}^{\delta_k}) \geq \|T\|(K)\left(1 - 2^{-k}\right).$$
Since $K_{i_k}^{\delta_k} = u_{i_k}^{-1}(-\infty, \delta_k]$, the claim follows with $\alpha_{i_k} = \delta_k$.

(b) Given $\epsilon > 0$, we may select $\delta_0 < 0$ such that
$$\|T\|(\{u < \delta_0\}) \geq \|T\|(\{u < 0\}) - \epsilon. \tag{7}$$
As in the proof of (a), by [21, Lemma 27], we may pass to a subsequence (keeping the same indexing) such that the restriction of T_i to $K_i^\delta = \{u_i \leq \delta\}$ $\mathcal{F}$-converges to $T \llcorner K^\delta \neq 0$ for almost all $\delta \in (\delta_0, 0)$. By lower semicontinuity of mass, we have
$$\liminf_{i\to\infty} \|T_i\|(K_i^\delta) \geq \|T\|(K^\delta)$$
However, for K_i defined using the sequence $\alpha_i = 0$, $K_i \supseteq K_i^\delta$ and $K^\delta \supseteq \{u < \delta\}$, so by (7)
$$\liminf_{i\to\infty} \|T_i\|(K_i) \geq \|T\|(\{u < 0\}) - \epsilon.$$
From this and (7), the claim follows by letting $\epsilon \searrow 0$ and applying a diagonal argument. □

Remark 3. We point out there is a uniform (i.e., independent of r_0) method to define corresponding regions in the case in which $N_i \to N$ in the pointed $\mathcal{F}$-sense "with a common space" with respect to $p_i \in X_i$ and $p \in \overline{X}$. By this, we mean when there exist a w^*-separable Banach space Y and isometric embeddings $\varphi_i : X_i \to Y$ and $\varphi : X \to Y$ such that $\varphi_{i\#}(T_i) \to \varphi_\#(T)$ in the local flat topology (as in Definition 5), and for which $\varphi_i(p_i) \to \varphi(p)$. In this case, we still define $u_i = U \circ \varphi_i$ (but note that now u_i is defined on the whole X_i and not only on $\mathrm{set}(T_i \llcorner B(p_i, r))$, and let K_i be as in (5). This construction will be used in the proof of Theorem 18.

3.2. **Main results and proofs.** The following two theorems are our main results. The first is simply a restatement of Theorem 1 from the introduction.

Theorem 17. *Let $N_i = (X_i, d_i, T_i)$ and $N = (X, d, T)$ be local integral current spaces of dimension $m \geq 2$, such that $N_i \to N$ in the pointed $\mathcal{VF}$ sense with respect to $p_i \in X_i$ and $p \in X$. Suppose for some $r > 0$ that the closed ball $\overline{B}(p, r)$ in X is compact. Then:*

$$\limsup_{i\to\infty} \operatorname{cap}_{N_i}(\overline{B}(p_i, r)) \leq \operatorname{cap}_N(\overline{B}(p, r)). \tag{8}$$

In the second main theorem, given a fixed set K in the limit space X, we consider a sequence of corresponding regions. Since it is not clear in general if a "common space" exists as in Remark 3, we rely on Lang–Wenger's compactness theorem [24] to produce such a space for a subsequence.

Theorem 18. *Let $N_i = (X_i, d_i, T_i)$ and $N = (X, d, T)$ be local integral current spaces of dimension $m \geq 2$, such that $N_i \to N$ in the pointed $\mathcal{VF}$ sense with respect to $p_i \in X_i$ and $p \in \overline{X}$. Assume that for all $r > 0$,*

$$\sup_{i\in\mathbb{N}} \|\partial T_i\|(B(p_i, r)) < \infty. \tag{9}$$

Let $K \subsetneq X$ be nonempty and compact. Then, passing to a subsequence of N_i that we do not relabel, one can define corresponding regions $K_i \subseteq X_i$ as in Remark 3, and

$$\limsup_{i\to\infty} \operatorname{cap}_{N_i}(K_i) \leq \operatorname{cap}_N(K). \tag{10}$$

Note the hypothesis (9) is trivially satisfied if $\partial T_i = 0$ for each i, or, more generally, if the boundary masses are uniformly bounded.

We will first prove an extrinsic version of these theorems, i.e. for a sequence of locally integral currents all on a fixed Banach space. Theorem 19's proof follows many of the ideas in the proof of [26, Theorem 6.2].

Theorem 19. *Let Y be a w^*-separable Banach space, and let $T \neq 0$ be a locally integral m-dimensional current on Y, $m \geq 2$. Note $N_\infty = (S, d_Y, T)$ is an m-dimensional local integral current space, where $S = \operatorname{set}(T)$ and d_Y is (the restriction of) the distance on Y. Let $K \subsetneq S$ be a nonempty compact set. Let $u : S \to \mathbb{R}$ be a defining function for K (as in (4)), and let $U : Y \to \mathbb{R}$ be the standard 1-Lipschitz extension of u. Let $f \in \operatorname{Lip_B}(S)$ be given, with $0 \leq f \leq 1$, $f \equiv 1$ on a neighborhood of K, and $\operatorname{spt}(f) \subseteq B(z_0, r_0)$, where we fix any $z_0 \in Y$ and take $r_0 > 3 \operatorname{diam}(\hat{K}) + d_Y(K, S \setminus K)$, where $\hat{K} = K \cup \{z_0\}$.*

Let T_i be a sequence of locally integral m-dimensional currents on Y such that $T_i \to T$ weakly (pointwise as functionals). Assume that for any $r > 0$, there exists a bounded open set $V \supseteq B(z_0, r)$ such that $\|T_i\|(V) \to \|T\|(V)$. Let $S_i = \operatorname{set}(T_i)$, and note $N_i = (S_i, d_Y, T_i)$ are m-dimensional local integral current spaces. Let $\{\alpha_i\}$ be a sequence of nonnegative real numbers that converges to 0. Define

$$K_i = U^{-1}(-\infty, \alpha_i] \cap S_i. \tag{11}$$

Then each $K_i \subseteq S_i$ is a closed and bounded subset of S_i, and there exists a sequence $f_i \in \operatorname{Lip_B}(S_i)$, $0 \leq f_i \leq 1$, with $f_i \equiv 1$ on a neighborhood of K_i (for i sufficiently large), $\operatorname{Lip}(f_i) \leq 1 + 3 \operatorname{Lip}(f)$, and $\operatorname{spt}(f_i) \subseteq B(z_0, r_0 + 3)$ such that

$$\limsup_{i\to\infty} E_{N_i}(f_i) \leq E_{N_\infty}(f). \tag{12}$$

In particular,

$$\limsup_{i\to\infty} \operatorname{cap}_{N_i}(K_i) \leq \operatorname{cap}_{N_\infty}(K). \tag{13}$$

A few remarks: 1) In many cases $d_Y(K, S \setminus K)$ is zero, but may be positive, e.g. if K is a connected component of S. 2) The K_i defined in (11) are the same as the corresponding regions in Remark 3, where

here the embeddings are simply the inclusion maps. 3) Allowing for a sequence $\alpha_i \to 0$ in the definition of the corresponding regions K_i, we can first derive the upper semicontinuity of capacity for a sequence of sets $C_i \subset X_i$ that are such that every tubular neighborhood of $\varphi(K)$ contains the set $\varphi_i(C_i)$, for i large enough, and then use the monotonicity of the capacity with respect to set inclusion.

Proof of Theorem 19. We first verify the claim that each K_i (as defined in the statement of the theorem) is bounded. Choose a constant $\alpha > \sup_i \alpha_i$. Let $p \in K_i$, so $U(p) \le \alpha_i$. Then by definition of $U(p)$, there exists $s \in S$ such that $u(s) + d_Y(s,p) \le \alpha$. If $u(s) \ge 0$, we obtain that $d_Y(s,p) \le \alpha$. If $u(s) \le 0$, since K is compact, $u|_K$ is bounded below by $-L$ for some $L > 0$. Then in both cases $d_Y(s,p) \le \alpha + L$. For any $\delta > 0$ we define the closed and bounded subsets K^δ of S by

$$K^\delta := u^{-1}(-\infty, \delta] \subseteq S.$$

Note that we have $u(s) \le \alpha$, so $s \in K^\alpha$. Then choosing $k \in K$ to minimize the distance from K to s,

$$\begin{aligned} d_Y(p, z_0) &\le d_Y(p,s) + d_Y(s,k) + d_Y(k, z_0) \\ &\le \alpha + L + \alpha + \operatorname{diam}(\hat{K}) < \infty, \end{aligned}$$

showing K_i is bounded.

Now for a fixed $\epsilon > 0$, we show that there exists a sequence $f_i^\epsilon \in \mathrm{Lip}_{\mathrm{B}}(S_i)$, $0 \le f_i^\epsilon \le 1$, with $f_i^\epsilon \equiv 1$ on a neighborhood of K_i (for i sufficiently large, depending on ϵ), $\mathrm{Lip}(f_i^\epsilon) \le 1 + 3\,\mathrm{Lip}(f)$, and $\mathrm{spt}(f_i^\epsilon) \subseteq B(z_0, r_0 + 3)$ such that

$$\limsup_{i\to\infty} E_{N_i}(f_i^\epsilon) \le E_{N_\infty}(f) + \epsilon. \tag{14}$$

This will be enough to conclude the main claim, (13). For clarity of notation, we drop the index ϵ in what follows. At the very end of the proof, we show how to obtain (12) from (14).

Let $f : S \to \mathbb{R}$ be the Lipschitz function considered in the statement of the theorem and let $\Lambda = \mathrm{Lip}(f)$. Let $O \subset S$ denote the given neighborhood of K on which $f \equiv 1$. The following lemma allows $f : S \to \mathbb{R}$ to be extended as a Lipschitz function to a larger subset of Y that intersects S inside of O.

Lemma 20. *For $\gamma > 0$ sufficiently small, the extension of f from S to $S \cup U^{-1}(-\infty, \gamma]$, defined as 1 in $U^{-1}(-\infty,\gamma] \setminus S$, is a Lipschitz function, bounded between 0 and 1, with Lipschitz constant $\le 2\Lambda$. (We will also call the extension f, so now $\mathrm{Lip}(f) \le 2\Lambda$.)*

Proof. Since K is compact, we can choose $\gamma > 0$ small enough to ensure $K^{3\gamma} = u^{-1}(-\infty, 3\gamma] \subset O$.

The extension $f : S \cup U^{-1}(-\infty,\gamma] \to \mathbb{R}$ obviously satisfies $0 \le f \le 1$. We now prove it is Lipschitz. Let $p, q \in S \cup U^{-1}(-\infty, \gamma]$. If $p, q \in S$, then $|f(p) - f(q)|$ is bounded above by $\mathrm{Lip}(f|_S) d_Y(p,q) = \Lambda d_Y(p,q)$. If $p, q \in (U^{-1}(-\infty,\gamma] \setminus S) \cup O$, then $f(p) - f(q) = 1 - 1 = 0$. Therefore, the only case left to consider is that in which (say) $p \in (U^{-1}(-\infty,\gamma] \setminus S) \cup O$ and $q \in S$. In fact, since $O \subset S$ we may assume that $p \in U^{-1}(-\infty,\gamma] \setminus S$ and $q \in S \setminus O$.

By the definition of defining function, since $q \notin K$, for any $\eta > 0$, there exists $k \in K$ with $d_Y(k,q) \le u(q) + \eta$. Then since $f(p) = 1 = f(k)$ and $f|_S$ is Lipschitz, we obtain:

$$\begin{aligned} |f(p) - f(q)| &= |f(k) - f(q)| \\ &\le \mathrm{Lip}(f|_S) d_Y(k,q) \\ &\le \Lambda(u(q) + \eta). \end{aligned}$$

We now bound $u(q) = U(q)$ in terms of $d_Y(p,q)$. Since U is 1-Lipschitz, we have

$$U(q) - U(p) \le |U(q) - U(p)| \le d_Y(p,q),$$

so that

$$u(q) \le d_Y(p,q) + \gamma.$$

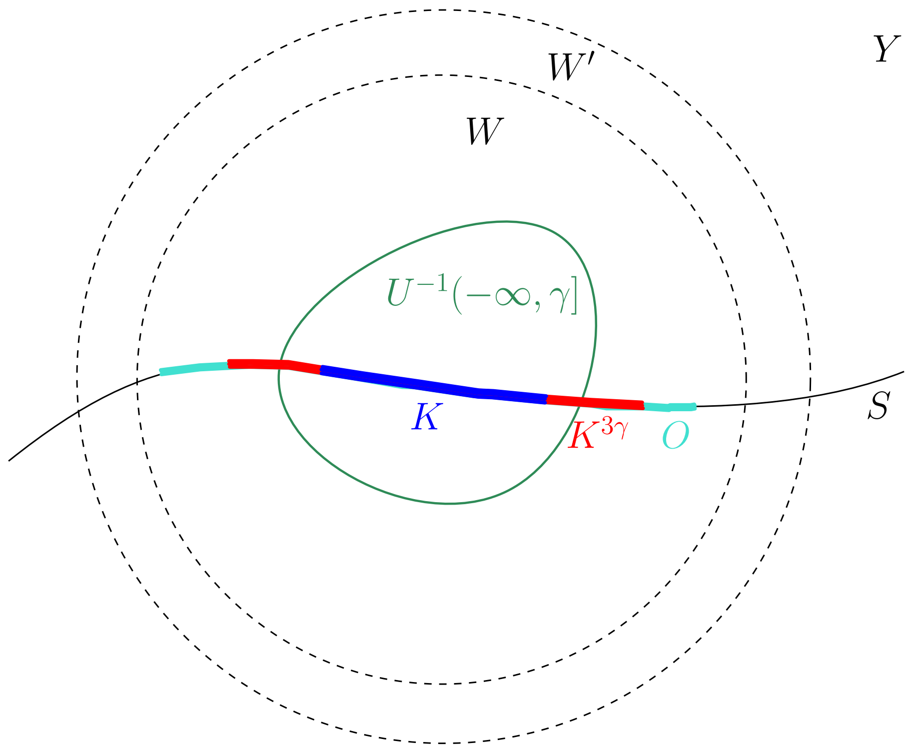

FIGURE 1. The sets $K \subset O \subset S$ are shown, along with the balls $W \subset W'$. The number $\gamma > 0$ is chosen sufficiently small so that $K^{3\gamma} \subset O$. The function $f : S \to \mathbb{R}$ is identically 1 on O and in Lemma 20, f is extended by 1 to $U^{-1}(-\infty, \gamma]$.

We now show that $\gamma < d_Y(p, q)$. From the definition of $U(p) \leq \gamma$, there exists $s \in S$ such that

$$u(s) + d_Y(s, p) < 2\gamma. \tag{15}$$

Since $q \in S \setminus O$, we have $u(q) > 3\gamma$, and thus

$$3\gamma - u(s) < u(q) - u(s) \leq |u(q) - u(s)| \leq d_Y(q, s) \leq d_Y(p, q) + d_Y(s, p) < d_Y(p, q) + 2\gamma - u(s),$$

having used that u is 1-Lipschitz and (15) to bound $d_Y(s, p)$. From this, it follows that $\gamma < d_Y(p, q)$. Thus,

$$|f(p) - f(q)| \leq \Lambda(2d_Y(p, q) + \eta),$$

since η can be arbitrarily small, the proof is complete. □

The following technical lemma constructs many of the objects used in the rest of the proof of Theorem 19. Please refer to Figures 1 and 2 for an illustration of some aspects of this setup.

Lemma 21. *Let $\epsilon_1 > 0$ be given and fixed. There exist open balls $W, W' \subset Y$ about z_0 of radius in $(r_0, r_0 + 1)$ and $(r_0 + 2, r_0 + 3)$, respectively, so that for $S' = S \cap W'$,*

$$T \llcorner S' \in \mathbf{I}_m(Y), \quad \|T\|(\partial W) = 0, \quad \text{and} \quad \|T\|(\partial W') = 0. \tag{16}$$

There exists $\delta > 0$ sufficiently small so that $U^{-1}(-\infty, \delta] \subseteq W$, and moreover $K^\delta = u^{-1}(-\infty, \delta]$ satisfies

$$\|T\|(K^\delta \setminus K) < \epsilon_2, \quad T \llcorner K^\delta \in \mathbf{I}_m(Y), \quad \text{and} \quad T \llcorner (S' \setminus K^\delta) \in \mathbf{I}_m(Y), \tag{17}$$

where $\epsilon_2 = (1 + 3\Lambda)^{-2}\epsilon_1$.

Letting $\epsilon_3 = \epsilon_1 \left(\|T\|(W')\right)^{-1} > 0$ and $\epsilon_4 = (1 + 3\Lambda)^{-2}\epsilon_1$, there exists a finite collection of compact sets $A_\ell \subset S' \setminus K^\delta$ for $\ell = 1, \ldots, N$ such that

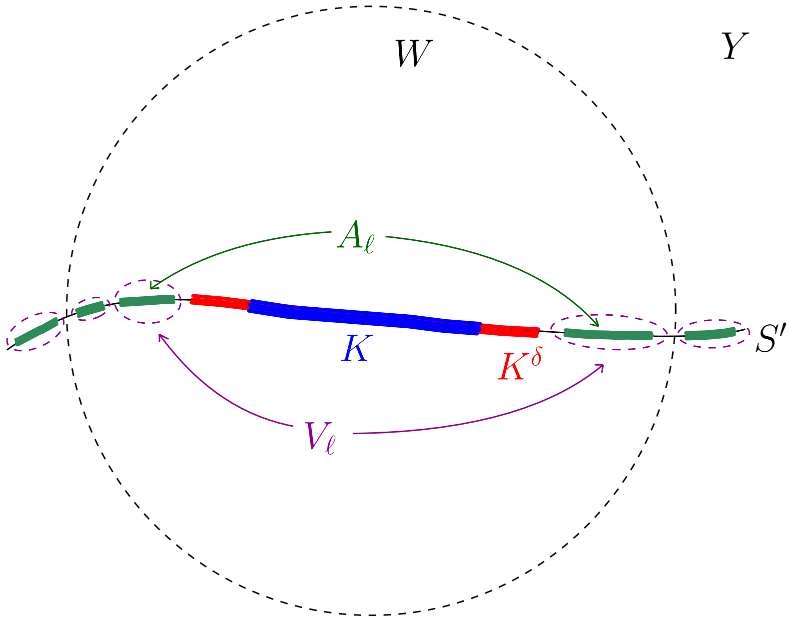

FIGURE 2. The finitely many sets A_ℓ are subsets of $S' \setminus K^\delta$ and cover all of this set except for $\|T\|$-measure $< \epsilon_4$. The V_ℓ are disjoint neighborhoods of the A_ℓ, each of which lies in W or $Y \setminus \overline{W}$.

- *Each A_ℓ is the bi-Lipschitz image of a compact subset of $\mathbb{R}^m$,*
- *Each A_ℓ is either a subset of W or of $Y \setminus \overline{W}$,*
- *The A_ℓ are pairwise disjoint,*
- *For any $\gamma \in (0, \frac{\delta}{2})$, we have $d_Y(A_\ell, U^{-1}(-\infty, \gamma]) > \delta/2$ for all ℓ*
- *Letting $\cup_\ell$ denote $\cup_{\ell=1}^N$ henceforth,*
$$\|T\|((S' \setminus K^\delta) \setminus \cup_\ell A_\ell) < \epsilon_4, \tag{18}$$
- *For all $x \in A_\ell$*
$$(c_\ell)^2 - \epsilon_3 \leq |d_x^S f|^2 \leq (c_\ell)^2, \tag{19}$$
where $c_\ell = \mathrm{Lip}(f|_{A_\ell}) \leq \Lambda$.

Furthermore, there exists $b > 0$ sufficiently small so that the open $b/10$-neighborhoods of each A_ℓ in Y, denoted V_ℓ, satisfy

- *Each V_ℓ are subsets of W',*
- *Each V_ℓ is either a subset of W or of $Y \setminus \overline{W}$,*
- *The V_ℓ are pairwise disjoint,*
- *$d_Y(V_\ell, U^{-1}(-\infty, \gamma]) > 9b/10$ for all ℓ,*
- *For $\epsilon_5 = \Lambda^{-2}\epsilon_1$,*
$$\|T\| \left(\cup_\ell \overline{V}_\ell \setminus \cup A_\ell\right) < \epsilon_5, \tag{20}$$
(If $\Lambda = 0$, we take $\epsilon_5 = 1$).

Proof. We note that since $\|T\|$ is a Borel measure that is finite on bounded open sets, it follows that $\|T\|$ is zero on almost all metric spheres about a given point. Choose an open ball $W \subset Y$ about z_0 of radius in $(r_0, r_0 + 1)$, with the radius chosen so that $\|T\|(\partial W) = 0$.

We claim that for $\delta > 0$ sufficiently small, $U^{-1}(-\infty, \delta] \subseteq W$. The proof is similar to the proof of each K_i being bounded: Let $p \in U^{-1}(-\infty, \delta]$, i.e., $U(p) \leq \delta$. Then given $\eta > 0$, there exists $s \in S$ such that $u(s) + d_Y(s,p) \leq \delta + \eta$. In particular, $u(s) \leq \delta + \eta$, i.e. $s \in K^{\delta+\eta}$.

We consider two cases according to the sign of $u(s)$. If $u(s) \geq 0$, we obtain that $d_Y(s,p) \leq \delta + \eta$. Using $s \in K^{\delta+\eta}$, take $k \in K$ to minimize the distance to s, obtaining

$$d_Y(z_0, s) \leq d_Y(z_0, k) + d_Y(k, s) \leq \operatorname{diam}(\hat{K}) + \delta + \eta.$$

Then by the triangle inequality

$$d_Y(z_0, p) \leq d_Y(z_0, s) + d_Y(s, p) \leq \operatorname{diam}(\hat{K}) + 2\delta + 2\eta.$$

Choosing δ and η sufficiently small, we have $\operatorname{diam}(\hat{K}) + 2\delta + 2\eta \leq 3\operatorname{diam}(\hat{K}) < r_0$. So $p \in W$ in this case.

In the other case, if $u(s) < 0$, then $s \in K$ and so $d_Y(z_0, s) \leq \operatorname{diam}(\hat{K})$. We also have $d_Y(s,p) \leq \delta + \eta + |u(s)|$. Let $q \in K$ and $q' \in S \setminus K$ achieve the minimum distance between K and $S \setminus K$, within η:

$$d_Y(q, q') \leq d_Y(K, S \setminus K) + \eta.$$

Since $u(q') > 0$,

$$|u(s)| \leq |u(q') - u(s)| \leq d_Y(q', s) \leq d_Y(q', q) + d_Y(q, s) \leq d_Y(K, S \setminus K) + \eta + \operatorname{diam}(K).$$

Thus,

$$d_Y(s,p) \leq \delta + \eta + |u(s)| \leq \operatorname{diam}(K) + d_Y(K, S \setminus K) + \delta + 2\eta.$$

Again, by the triangle inequality and choosing δ and η sufficiently small, we obtain

$$d_Y(z_0, p) \leq d_Y(z_0, s) + d_Y(s, p) \leq \operatorname{diam}(\hat{K}) + \operatorname{diam}(K) + d_Y(K, S \setminus K) + \delta + 2\eta < r_0,$$

so $p \in W$. It follows that $U^{-1}(-\infty, \delta] \subseteq W$.

We now choose $\delta > 0$ as small as needed so that $U^{-1}(-\infty, \delta] \subset W$ and that K^δ satisfies

$$\|T\|(K^\delta \setminus K) < \epsilon_2 \quad \text{and} \quad T \llcorner K^\delta \in \mathbf{I}_m(Y),$$

where $\epsilon_2 = (1 + 3\Lambda)^{-2}\epsilon_1$ (cf. [21, Lemma 24], using Sormani's argument in [28, Lemma 2.34]).

Now fix another open ball $W' \subset Y$, centered about z_0, of radius in $(r_0 + 2, r_0 + 3)$, so that in particular $\overline{W} \subset W'$. Let $S' = S \cap W'$; adjusting the radius if necessary, we may ensure $T \llcorner S' \in \mathbf{I}_m(Y)$ and

$$\|T\|(\partial W') = 0.$$

So $T \llcorner (S' \setminus K^\delta) \in \mathbf{I}_m(Y)$ as well.

Apply Lemmas 3.2 and 6.1 of [26] to the integral m-current $T \llcorner (S' \setminus K^\delta)$ and the restriction of f to $S' \setminus K^\delta$ with $\epsilon_3 = \epsilon_1 \left(\|T\|(W')\right)^{-1} > 0$, thereby obtaining a sequence of compact sets $A_\ell \subset S' \setminus K^\delta$ for $\ell = 1, 2, \ldots$ such that

- Each A_ℓ is the bi-Lipschitz image of a compact subset of $\mathbb{R}^m$.
- The A_ℓ are pairwise disjoint.
- $\cup_{\ell=1}^{\infty} A_\ell$ has zero co-measure in $S' \setminus K^\delta$ with respect to $\|T\|$.
- For all $x \in A_\ell$
$$(c_\ell)^2 - \epsilon_3 \leq |d_x^S f|^2 \leq (c_\ell)^2,$$
where $c_\ell = \operatorname{Lip}(f|_{A_\ell}) \leq \Lambda$.

Since $\|T\|$ is Borel regular and $\|T\|(\partial W) = 0$, we may assume without loss of generality that each A_ℓ is either a subset of W or of $Y \setminus \overline{W}$.

Now, choose a finite subset of $\{A_\ell\}$, call it $A_1, \ldots, A_N$, such that

$$\|T\|((S' \setminus K^\delta) \setminus \cup_\ell A_\ell) < \epsilon_4,$$

where, for the rest of this proof, $\cup_\ell$ will denote $\cup_{\ell=1}^N$. Let $\gamma \in (0, \frac{\delta}{2})$. We claim that the distance from any A_ℓ to $U^{-1}(-\infty, \gamma]$ in Y is at least $\delta - \gamma > \frac{\delta}{2}$. Let $q \in A_\ell$. Since A_ℓ is disjoint from K^δ we have $U(q) = u(q) > \delta$. Then since U is 1-Lipschitz, if $z \in U^{-1}(-\infty, \gamma]$,

$$d_Y(q, z) \geq |U(q) - U(z)| > \delta - \gamma,$$

which proves the claim.

We know that these finitely many A_ℓ are pairwise disjoint and compact; so let $a > 0$ be the minimum pairwise distance between them. Let b be a positive real number less than $\min\{a, \frac{\delta}{2}\} > 0$ and V_ℓ be the open $b/10$-neighborhood of A_ℓ in Y, so $V_1, \ldots, V_N$ are pairwise disjoint and their distance to $U^{-1}(-\infty, \gamma]$ is greater than $9b/10$. Since the A_ℓ are compact subsets of W', we may shrink $b > 0$ if necessary to ensure the V_ℓ are subsets of W'. We can also shrink $b > 0$ again to guarantee each V_ℓ is either a subset of W or of $Y \setminus \overline{W}$. Furthermore, since $\|T\|$ is regular, we may shrink $b > 0$ if necessary to ensure

$$\|T\| \left(\cup_\ell \overline{V}_\ell \setminus \cup A_\ell\right) < \epsilon_5. \qquad \square$$

Following as in (6.14)–(6.17) of [26], we claim:

Lemma 22. *Under the assumptions of Lemma 21 and $\gamma > 0$ small enough so that Lemma 20 holds, there exists a Lipschitz function $\hat{f} : Y \to \mathbb{R}$ such that*

(a) $\mathrm{Lip}(\hat{f}) \leq 1 + 3\Lambda$
(b) $0 \leq \hat{f} \leq 1$
(c) $\hat{f}$ *agrees with* f *on* $\cup_\ell A_\ell$ *and on* $U^{-1}(-\infty, \gamma]$ *(in particular,* $\hat{f}|_{U^{-1}(-\infty,\gamma]} = 1$*)*
(d) $\mathrm{Lip}(\hat{f}|_{V_\ell}) \leq \mathrm{Lip}(f|_{A_\ell})$ *for* $\ell = 1, \ldots, N$
(e) $\hat{f} \equiv 1$ *on a neighborhood of* K_i *for all* i *sufficiently large*
(f) $\mathrm{spt}(\hat{f}) \subseteq \overline{W'}$.

In particular, $f_i := \hat{f}|_{S_i}$ is Lipschitz, bounded, has bounded support, and is 1 on a neighborhood of K_i (for all i sufficiently large), i.e., f_i is an allowable test function for $\mathrm{cap}_{N_i}(K_i)$.

Again, refer to Figures 1 and 2 for an illustration of some aspects of this setup.

Note that in Lemma 20 we extend $f : S \to \mathbb{R}$ to a function $f : S \cup U^{-1}(-\infty, \gamma] \to \mathbb{R}$. We now construct $\hat{f}$ by prescribing its values on $W' \setminus U^{-1}(-\infty, \gamma]$. We remark that $\hat{f}$ will not generally be an extension of f; they may differ on $(\cup V_\ell \setminus \cup A_\ell) \cap S$.

Proof. First, we define for each ℓ the standard Lipschitz extension of $f|_{A_\ell}$ to V_ℓ:

$$f^\ell(x) = \inf_{a \in A_\ell} \left(f(a) + c_\ell d_Y(a, x)\right), \qquad x \in V_\ell,$$

where, again, $c_\ell = \mathrm{Lip}(f|_{A_\ell})$. Note that $\mathrm{Lip}(f^\ell) \leq c_\ell$. Truncate these functions by defining $\hat{f}^\ell := \max\{\min\{f^\ell, 1\}, 0\}$, (recalling $0 \leq f \leq 1$), and note $\mathrm{Lip}(\hat{f}^\ell) \leq c_\ell$.

Subsequently, define the function $\hat{f} : (\cup_\ell V_\ell) \cup U^{-1}(-\infty, \gamma] \to \mathbb{R}$ by

$$\hat{f}(x) = \begin{cases} \hat{f}^\ell(x) & \text{if } x \in V_\ell \\ 1 & \text{if } x \in U^{-1}(-\infty, \gamma], \end{cases}$$

which satisfies $0 \leq \hat{f} \leq 1$.

Let us prove that $\mathrm{Lip}(\hat{f}) \leq 3\Lambda$. There are only two nontrivial cases. First, if $x \in V_{\ell_1}$, $y \in V_{\ell_2}$, $\ell_1 \neq \ell_2$, there exist $x_0 \in A_{\ell_1}$ and $y_0 \in A_{\ell_2}$ such that $d_Y(x, x_0) < \frac{b}{10}$ and $d_Y(y, y_0) < \frac{b}{10}$. By the triangle inequality, since $d_Y(x_0, y_0) \geq a$, we find $d_Y(x, y) \geq \frac{4}{5}b$. Therefore

$$\begin{aligned}
|\hat{f}(x) - \hat{f}(y)| &\leq |\hat{f}(x) - \hat{f}(x_0)| + |\hat{f}(x_0) - \hat{f}(y_0)| + |\hat{f}(y_0) - \hat{f}(y)| \\
&\leq c_{\ell_1} d_Y(x, x_0) + \mathrm{Lip}(f) d_Y(x_0, y_0) + c_{\ell_2} d_Y(y_0, y) \\
&\leq \mathrm{Lip}(f) \left(d_Y(x, x_0) + d_Y(x_0, y_0) + d_Y(y_0, y) \right) \\
&\leq \mathrm{Lip}(f) \left(\frac{b}{10} + d_Y(x_0, x) + d_Y(x, y) + d_Y(y, y_0) + \frac{b}{10} \right) \\
&\leq \mathrm{Lip}(f) \left(\frac{4b}{10} + d_Y(x, y) \right) \\
&\leq \mathrm{Lip}(f) \left(\frac{3}{2} d_Y(x, y) \right) \\
&= 3\Lambda d_Y(x, y).
\end{aligned}$$

Second, assume that $x \in V_\ell$, $y \in U^{-1}(-\infty, \gamma]$, so $d_Y(x, y) \geq \frac{9b}{10}$. There exists $x_0 \in A_\ell$ such that $d_Y(x, x_0) < \frac{b}{10}$ and therefore

$$\begin{aligned}
|\hat{f}(x) - \hat{f}(y)| &\leq |\hat{f}(x) - \hat{f}(x_0)| + |\hat{f}(x_0) - \hat{f}(y)| \\
&\leq c_{\ell_1} d_Y(x, x_0) + \mathrm{Lip}(f) d_Y(x_0, y) \\
&\leq \mathrm{Lip}(f) \left(d_Y(x, x_0) + d_Y(x_0, y) \right) \\
&\leq \mathrm{Lip}(f) \left(d_Y(x, x_0) + d_Y(x_0, x) + d_Y(x, y) \right) \\
&\leq \mathrm{Lip}(f) \left(\frac{2b}{10} + d_Y(x, y) \right) \\
&\leq \mathrm{Lip}(f) \left(\frac{11}{9} d_Y(x, y) \right) \\
&\leq 3\Lambda d_Y(x, y).
\end{aligned}$$

Consequently, $\hat{f} : (\cup_\ell V_\ell) \cup U^{-1}(-\infty, \gamma] \to \mathbb{R}$ has Lipschitz constant at most 3Λ. We extend it to a Lipschitz function on all Y in the standard way (with the same Lipschitz constant), truncate at values of 0 and 1, and call the result $\hat{f}$.

However, $\hat{f}$ will not have bounded support, so we will modify $\hat{f}$ using a cutoff function while ensuring claims (a)–(f) will hold. Let $0 \leq \rho \leq 1$ be a Lipschitz function on Y that is identically one on $\overline{W}$ and is supported in W'. Since the radii of W and W' differ by more than 1, we may assume without loss of generality that

$$\mathrm{Lip}(\rho) \leq 1. \tag{21}$$

We claim $\rho\hat{f}$ is the desired function. The function $\rho\hat{f}$ clearly has bounded support in $\overline{W'}$ and is bounded between 0 and 1, i.e. (b) and (f) hold. We see $\rho\hat{f}$ is Lipschitz as well:

$$\begin{aligned}
|\rho(x)\hat{f}(x) - \rho(y)\hat{f}(y)| \leq& |\rho(x) - \rho(y)||\hat{f}(x)| + |\hat{f}(x) - \hat{f}(y)||\rho(y)| \\
&\leq \left(\mathrm{Lip}(\rho) + \mathrm{Lip}(\hat{f}) \right) d_Y(x, y) \\
&\leq (1 + 3\Lambda) \, d_Y(x, y),
\end{aligned}$$

having used (21). That is, (a) holds.

Next, we show (c): first suppose $x \in A_\ell$. If $A_\ell \subset W$, then $\rho(x) = 1$ and it follows that $\rho(x)\hat{f}(x) = f^\ell(x) = f(x)$. Otherwise, $A_\ell \subset Y \setminus \overline{W}$, which implies $f|_{A_\ell} = 0$, as W contains the support of f. Then $c_\ell = 0$, so $\hat{f}(x) = f^\ell(x) = 0$. Then $\rho\hat{f}$ and f both vanish on A_ℓ. Next, suppose $x \in U^{-1}(-\infty, \gamma]$. Since $W \supset U^{-1}(-\infty, \gamma]$ and $\rho \equiv 1$ on W, it follows that $\rho(x)\hat{f}(x) = 1 = f(x)$.

To address, (d), consider $\rho\hat{f}|_{V_\ell}$. If V_ℓ is a subset of W, then $\rho \equiv 1$ on V_ℓ, so $\rho\hat{f}|_{V_\ell} = \hat{f}|_{V_\ell} = f^\ell$, whose Lipschitz constant is at most $c_\ell = \mathrm{Lip}(f|_{A_\ell})$. On the other hand if V_ℓ is a subset of $Y \setminus \overline{W}$, then $\rho\hat{f}|_{V_\ell} = 0$. But since $A_\ell \subset Y \setminus \overline{W}$ and $\mathrm{spt}(f) \subseteq W$, we have $f|_{A_\ell} = 0$, which implies $f^\ell = 0$. So in this case as well, $\mathrm{Lip}(\rho\hat{f}|_{V_\ell}) \le \mathrm{Lip}(f|_{A_\ell})$ (both are zero).

To show (e), restrict to i sufficiently large so that $\alpha_i < \gamma$. Then $U^{-1}(-\infty, \alpha_i] \subset U^{-1}(-\infty, \gamma)$. By (c), $\hat{f} \equiv 1$ on $U^{-1}(-\infty, \gamma]$. Since $U^{-1}(-\infty, \gamma] \subset W$ and $\rho \equiv 1$ on W, it follows that $\rho\hat{f} \equiv 1$ on $U^{-1}(-\infty, \gamma]$ as well. In particular, $\rho\hat{f}$ is identically 1 on the neighborhood $U^{-1}(-\infty, \gamma)$ of $K_i = U^{-1}(-\infty, a_i] \cap S_i$, i.e. (e) holds.

For simplicity, we will henceforth refer to $\rho\hat{f}$ as simply $\hat{f}$. $\square$

We now establish the energy estimate (14), which will be sufficient to obtain (13).

Note that from our hypotheses and Lemma 6, for every closed, bounded set $C \subset Y$, we have

$$\limsup_{i\to\infty} \|T_i\|(C) \le \|T\|(C). \tag{22}$$

We begin with:

$$\begin{aligned} E_{N_i}(f_i) &= \int_Y |d_x^{S_i} f_i|^2 d\|T_i\| \\ &= \int_{Y\setminus W'} |d_x^{S_i} \hat{f}|^2 d\|T_i\| + \int_{\cup_\ell V_\ell} |d_x^{S_i} \hat{f}|^2 d\|T_i\| + \int_{W'\setminus \cup_\ell V_\ell} |d_x^{S_i} \hat{f}|^2 d\|T_i\|. \end{aligned} \tag{23}$$

The first term is zero, since $\hat{f}$ vanishes outside W'. For the second term in (23), by Lemma 22(d) and (19) we get

$$\begin{aligned} \int_{\cup_{\ell=1}^N V_\ell} |d_x^{S_i} \hat{f}|^2 d\|T_i\| &\le \sum_{\ell=1}^N \mathrm{Lip}(\hat{f}|_{V_\ell})^2 \|T_i\|(V_\ell) \\ &\le \sum_{\ell=1}^N (c_\ell)^2 \|T_i\|(V_\ell). \end{aligned}$$

We replace V_ℓ with its closure in the above and take the limsup to obtain, by (22), (20), and (19):

$$\begin{aligned} \limsup_{i\to\infty} \int_{\cup_\ell V_\ell} |d_x^{S_i} \hat{f}|^2 d\|T_i\| &\le \limsup_{i\to\infty} \sum_{\ell=1}^N (c_\ell)^2 \|T_i\|(\overline{V}_\ell) \\ &\le \sum_{\ell=1}^N (c_\ell)^2 \|T\|(\overline{V}_\ell) \\ &\le \sum_{\ell=1}^N (c_\ell)^2 \|T\|(A_\ell) + \Lambda^2 \epsilon_5 \\ &\le \int_{\cup_\ell A_\ell} \left(|d_x^S f|^2 + \epsilon_3\right) d\|T\| + \epsilon_1 \\ &\le E_{N_\infty}(f) + \|T\|(W')\epsilon_3 + \epsilon_1 \\ &\le E_{N_\infty}(f) + \epsilon_1 + \epsilon_1. \end{aligned} \tag{24}$$

For the third term in (23) we may omit integration over the open set $U^{-1}(-\infty,\gamma)$, since $\hat{f}\equiv 1$ there. It follows that:

$$\int_{W'\setminus\cup_\ell V_\ell} |d_x^{S_i}\hat{f}|^2 d\|T_i\| \le \operatorname{Lip}(\hat{f})^2\|T_i\|\left(\overline{W'}\setminus\left(\cup_\ell V_\ell\cup U^{-1}(-\infty,\gamma)\right)\right).$$

The set on the right is closed and bounded, so by (22):

$$\limsup_{i\to\infty}\int_{W'\setminus\cup_\ell V_\ell} |d_x^{S_i}\hat{f}|^2 d\|T_i\| \le \operatorname{Lip}(\hat{f})^2\|T\|\left(\overline{W'}\setminus\left(\cup_\ell V_\ell\cup U^{-1}(-\infty,\gamma)\right)\right).$$

Now, by considering intersections and set subtractions with S' and with K^δ, we find

$$\left(\overline{W'}\setminus U^{-1}(-\infty,\gamma)\right)\setminus\cup_\ell V_\ell \subseteq (\overline{W'}\setminus S')\cup\Big((S'\setminus K^\delta)\setminus\cup V_\ell\Big)\cup(K^\delta\setminus K).$$

Thus, using (18) and (17),

$$\begin{aligned}\limsup_{i\to\infty}&\int_{W'\setminus\cup_\ell V_\ell} |d_x^{S_i}\hat{f}|^2 d\|T_i\| \\ &\le \operatorname{Lip}(\hat{f})^2\left[\|T\|\left(\overline{W'}\setminus S'\right)+\|T\|\Big((S'\setminus K^\delta)\setminus\cup V_\ell\Big)+\|T\|\Big(K^\delta\setminus K\Big)\right]\\ &\le \operatorname{Lip}(\hat{f})^2\left[\|T\|\left(W'\setminus S'\right)+\|T\|(\partial W')+\epsilon_4+\epsilon_2\right]. \end{aligned}\tag{25}$$

Note that $\|T\|\,(W'\setminus S')\le\|T\|\,(Y\setminus S)$, where we recall $S=\operatorname{set}(T)$. It is shown in [1, Theorem 4.6] that an integral current's measure is concentrated on its canonical set. The same goes for locally integral currents, i.e. $\|T\|\,(Y\setminus S)=0$. The next term, $\|T\|(\partial W')$, vanishes by (16).

Combining (23), (24), and (25), we have (using the definition of ϵ_2 and ϵ_4):

$$\begin{aligned}\limsup_{i\to\infty} E_{N_i}(f_i) &\le E_{N_\infty}(f)+2\epsilon_1+(1+3\Lambda)^2(\epsilon_4+\epsilon_2)\\ &= E_{N_\infty}(f)+4\epsilon_1.\end{aligned}$$

So (14) follows.

To then show (13), given $\epsilon_1>0$, choose $f\in\operatorname{Lip_B}(S)$, with $0\le f\le 1$ and $f\equiv 1$ on a neighborhood of K, such that

$$E_{N_\infty}(f)\le\gamma_m\operatorname{cap}_{N_\infty}(K)+\epsilon_1.\tag{26}$$

Then apply the above argument to produce functions f_i (that are valid test functions for the capacity of K_i for i sufficiently large), so that

$$\limsup_{i\to\infty}\gamma_m\operatorname{cap}_{N_i}(K_i)\le\limsup_{i\to\infty}E_{N_i}(f_i)\le E_{N_\infty}(f)+4\epsilon_1.$$

Combining this with (26), (13) follows, since ϵ_1 was arbitrary.

We conclude by establishing (12) (though this is not needed for the proof of (13)). By (14), we may assume that for every $j\in\mathbb{N}$ we have a sequence $\left(f_i^{1/j}\right)_i$ such that $f_i^{1/j}\in\operatorname{Lip_B}(S_i)$, $0\le f_i^{1/j}\le 1$, with $f_i^{1/j}\equiv 1$ on a neighborhood of K_i (for i sufficiently large, depending on $1/j$), $\operatorname{Lip}(f_i^{1/j})\le 1+3\operatorname{Lip}(f)$, and $\operatorname{spt}(f_i^{1/j})\subseteq B(z_0,r_0+3)$ such that

$$\limsup_{i\to\infty}E_{N_i}(f_i^{1/j})\le E_{N_\infty}(f)+1/j.$$

Then we may construct a monotonically increasing sequence $n:\mathbb{N}\to\mathbb{N}$ such that $n_1=1$ and for all $j\in\mathbb{N}\setminus\{1\}$ it holds that for all $i\ge n_j$, $f_i^{1/j}\equiv 1$ on a neighborhood of K_i and

$$E_{N_i}(f_i^{1/j})\le E_{N_\infty}(f)+2/j.$$

We then define a new non-decreasing sequence $m : \mathbb{N} \to \mathbb{N}$ by

$$m_j = \max\{k \mid n_k \leq j\}.$$

Note that the maximum is well-defined as (n_k) is monotonically increasing. Moreover, $m_j \to \infty$ as $j \to \infty$ since for every $M_0 \in \mathbb{N}$, if we define $j_0 := n_{M_0}$, we have $m_{j_0} \geq M_0$ (in fact $m_{j_0} = M_0$ since $n : \mathbb{N} \to \mathbb{N}$ is monotonically increasing). The sequence (f_i) defined by $f_i := f_i^{1/m_i}$ then satisfies the properties mentioned in the theorem: in particular, since by construction $i \geq n_{m_i}$ for all $i \in \mathbb{N}$, we know that for all $i \in \mathbb{N} \setminus \{1\}$ $f_i^{1/m_i} \equiv 1$ on a neighborhood of K_i, and

$$E_{N_i}\left(f_i^{1/m_i}\right) \leq E_{N_\infty}(f) + 2/m_i$$

whereas $m_i \to \infty$ as $i \to \infty$. $\square$

Now we may prove the first main theorem, Theorem 17.

Proof of Theorem 17. Suppose the sequence $N_i = (X_i, d_i, T_i)$ converges to $N = (X, d, T)$ in the pointed $\mathcal{VF}$ sense as local integral current spaces of dimension $m \geq 2$, with respect to $p_i \in X_i$ and $p \in X$. Assume for some $r > 0$ that the closed ball $K = \overline{B}(p, r)$ in X is compact. In the first part of the proof, we assume $K \neq X$.

Given $\epsilon > 0$, take a function $f \in \mathrm{Lip}_B(X)$, $0 \leq f \leq 1$, with $f \equiv 1$ in a neighborhood of K and

$$E_N(f) \leq \gamma_m \operatorname{cap}_N(K) + \epsilon. \tag{27}$$

Choose $r_0 > 3\operatorname{diam}(K) + d(K, X \setminus K)$ sufficiently large so that $K \subseteq \operatorname{spt}(f) \subseteq B(p, r_0)$, and

$$B(p, r_0) \cap (X \setminus K) \neq \emptyset. \tag{28}$$

Using Definition 13, choose $R > r_0 + 4$ such that $N_i \llcorner B(p_i, R) \to N \llcorner B(p, R)$ in the $\mathcal{VF}$ sense as integral current spaces. Let $S_i = \operatorname{set}(T_i \llcorner B(p_i, R)) \subseteq X_i$ and $S = \operatorname{set}(T \llcorner B(p, R)) \subseteq X$. Since $p \in X = \operatorname{set}(T)$, it follows $p \in \operatorname{set}(T \llcorner B(p, R)) = S$. Thus $S \neq \emptyset$ and $T \llcorner B(p, R) \neq 0$. It is straightforward to see $K \subseteq S$. That K is a proper subset of S follows from (28).

By Theorem 10 there exists a w^*-separable Banach space Y and isometric embeddings $\varphi_i : S_i \to Y$ and $\varphi : S \to Y$ such that the integral currents $\varphi_{i\#}(T_i \llcorner B(p_i, R))$ converge to $\varphi_\#(T \llcorner B(p, R))$ in the flat d_Z^F sense (and therefore in the weak sense) and the masses converge,

$$\mathbf{M}(\varphi_{i\#}(T_i \llcorner B(p_i, R))) \to \mathbf{M}(\varphi_\#(T \llcorner B(p, R))), \tag{29}$$

and finally that $\varphi_i(p_i) \to \varphi(p)$ as $i \to \infty$.

Select $u(x) = d(K, x)$ as a defining function for K on S. Since φ is an isometric embedding, $u \circ \varphi^{-1} : \varphi(S) \to \mathbb{R}$ is equal to $d_Y(\varphi(K), \cdot)$. It is elementary to verify that the 1-Lipschitz extension U of $u \circ \varphi^{-1}$ to Y is simply given by $d_Y(\varphi(K), \cdot)$.

We apply Theorem 19 to the Banach space Y; the nonzero integral current $\varphi_\#(T \llcorner B(p, R))$ on Y, whose canonical set is $\varphi(S)$; the nonempty compact set $\varphi(K) \subsetneq \varphi(S)$ in Y; the defining function $u \circ \varphi^{-1}$ of $\varphi(K)$ with standard 1-Lipschitz extension $U : Y \to \mathbb{R}$; the Lipschitz function $f \circ \varphi^{-1} : \varphi(S) \to \mathbb{R}$; the point $z_0 = \varphi(p)$; the value r_0; the weakly converging sequence of integral currents $\varphi_{i\#}(T_i \llcorner B(p_i, R))$ on Y, whose canonical sets are $\varphi_i(S_i)$; and the sequence $\alpha_i = d_Y(\varphi_i(p_i), \varphi(p))$, which converges to 0 as $i \to \infty$.

A few hypotheses require verification in order to apply Theorem 19. First, we claim $r_0 > 3\operatorname{diam}(\varphi(K)) + d_Y(\varphi(K), \varphi(S) \setminus \varphi(K))$. By our choice of r_0 and the fact that isometries preserve diameter, it suffices to show $d(K, X \setminus K) \geq d_Y(\varphi(K), \varphi(S) \setminus \varphi(K))$. Given $\eta > 0$, there exists $k \in K$ and $x \in X \setminus K$ such that

$$d(k, x) \leq d(K, X \setminus K) + \eta.$$

Then $\varphi(k) \in \varphi(K)$, and we claim that $x \in S$ (if η was chosen sufficiently small). By the triangle inequality,

$$\begin{aligned} d(p,x) &\le d(p,k) + d(k,x) \\ &\le \operatorname{diam}(K) + d(K, X \setminus K) + \eta, \end{aligned}$$

which is $< R$ if η is sufficiently small. Then $x \in B(p,R)$. Since $x \in X = \operatorname{set}(T)$, we have $x \in \operatorname{set}(T \llcorner B(p,R)) = S$. Then $\varphi(x) \in \varphi(S) \setminus \varphi(K)$, so

$$d_Y(\varphi(K), \varphi(S) \setminus \varphi(K)) \le d_Y(\varphi(k), \varphi(x)) = d(k,x) \le d(K, X \setminus K) + \eta.$$

Since $\eta > 0$ can be arbitrarily small, the proof of the claim is complete.

Second, $f \circ \varphi^{-1}$ is clearly Lipschitz, bounded between 0 and 1, with $f \circ \varphi^{-1} \equiv 1$ in a neighborhood of $\varphi(K)$. Since $\operatorname{spt}(f) \subseteq B(p, r_0)$ and φ is an isometric embedding, we have $\operatorname{spt}(f \circ \varphi^{-1}) \subseteq B(\varphi(p), r_0) = B(z_0, r_0)$.

Third, the mass convergence hypothesis holds by (29), with V taken to be any ball about z_0 of radius greater than R.

By Theorem 19, for each i sufficiently large, there exists a Lipschitz function $f_i : \varphi_i(S_i) \to \mathbb{R}$, $0 \le f_i \le 1$, with $f_i \equiv 1$ on a neighborhood of K_i (where $K_i = U^{-1}(-\infty, \alpha_i] \cap \varphi_i(S_i)$), and $\operatorname{spt}(f_i) \subseteq B(\varphi(p), r_0 + 3)$. Moreover,

$$\limsup_{i \to \infty} E_{\varphi_{i\#}(N_i \llcorner B(p_i, R))}(f_i) \le E_{\varphi_\#(N \llcorner B(p,R))}(f \circ \varphi^{-1}), \tag{30}$$

where the push-forward of an integral current space under an isometric embedding is defined in the natural way; cf. [30, Lemma 2.39].

Consider $f_i \circ \varphi_i : S_i \to \mathbb{R}$, which is Lipschitz, bounded between 0 and 1, equalling 1 on a neighborhood of $\varphi^{-1}(K_i)$. To control the support, we have that for i large, $d_Y(\varphi_i(p_i), \varphi(p)) < 1$. From this it follows $\operatorname{spt}(f_i) \subseteq B(\varphi_i(p_i), r_0 + 4)$, and so

$$\operatorname{spt}(f_i \circ \varphi_i) \subseteq B(p_i, r_0 + 4) \subseteq B(p_i, R).$$

Thus, we may extend $f_i \circ \varphi_i$ by 0 on $X_i \setminus B(p_i, R)$ to produce a Lipschitz function on X_i with the same Dirichlet energy; call it $\hat{f}_i$, which is a valid test function for the capacity of K_i.

Using (30) on the fourth line below,

$$\begin{aligned} \limsup_{i \to \infty} \gamma_m \operatorname{cap}_{N_i}(\varphi^{-1}(K_i)) &\le \limsup_{i \to \infty} E_{N_i}(\hat{f}_i) \\ &= \limsup_{i \to \infty} E_{N_i \llcorner B(p_i, R)}(f_i \circ \varphi_i) \\ &= \limsup_{i \to \infty} E_{\varphi_{i\#}(N_i \llcorner B(p_i, R))}(f_i) \\ &\le E_{\varphi_\#(N \llcorner B(p,R))}(f \circ \varphi^{-1}) \\ &= E_{N \llcorner B(p,R)}(f) \\ &= E_N(f) \\ &\le \gamma_m \operatorname{cap}_N(K) + \epsilon, \end{aligned}$$

having used (27) on the last line.

Since ϵ was arbitrary, the proof will now follow from the monotonicity of capacity by showing $\overline{B}(p_i, r) \subseteq \varphi_i^{-1}(K_i)$. To see this, first observe that $U^{-1}(-\infty, \alpha_i] = \overline{B}_Y(\varphi(p), r_0 + \alpha_i)$, which follows from Y being a Banach space and φ being an isometric embedding. Intersecting both sides of this equality with $\varphi_i(S_i)$ leads to

$$K_i = \overline{B}_Y(\varphi(p), r_0 + \alpha_i) \cap \varphi_i(S_i).$$

Noting $\overline{B}_Y(\varphi_i(p_i), r_0) \subseteq \overline{B}_Y(\varphi(p), r_0 + \alpha_i)$, we apply φ_i^{-1} to obtain

$$\varphi_i^{-1}(K_i) \supseteq \varphi_i^{-1}\left(\overline{B}_Y(\varphi_i(p_i), r_0) \cap \varphi_i(S_i)\right).$$

It is straightforward to show directly that the right-hand side is simply $\overline{B}(p_i, r_0)$.

The proof of the theorem is complete in the case $K = \overline{B}(p, r_0) \neq X$.

To complete the proof, we consider the case $\overline{B}(p, r) = X$. In particular, X is bounded and the function $f \equiv 1$ on X is a valid test function for the capacity, i.e. $\mathrm{cap}_N(\overline{B}(p, r)) = 0$.

Now, choose $R > r$ so that $N_i \llcorner B(p_i, R)$ and $N_i \llcorner B(p_i, R+1)$ converge in the $\mathcal{VF}$ sense as integral current spaces to $N \llcorner B(p, R) = N = N \llcorner B(p, R+1)$. By mass convergence,

$$\|T_i\|(A(p_i, R, R+1)) \to 0 \text{ as } i \to \infty,$$

where $A(p_i, R, R+1)$ is the annulus $B(p_i, R+1) \setminus B(p_i, R)$. Let f_i be a function on X_i that equals 1 on $\overline{B}(p_i, R)$, 0 on $X_i \setminus B(p_i, R+1)$ and has $\mathrm{Lip}(f_i) \leq 1$. (Such a function can easily be constructed as a radial function of $d_i(p_i, \cdot)$.) Then we have

$$\begin{aligned}
\mathrm{cap}_{N_i}(\overline{B}(p_i, r)) &\leq \mathrm{cap}_{N_i}(\overline{B}(p_i, R)) \\
&\leq \frac{1}{\gamma_m} \int_{X_i} |d_x^{X_i} f_i|^2 d\|T_i\|(x) \\
&\leq \frac{1}{\gamma_m} \mathrm{Lip}(f_i)^2 \|T_i\|(A(p_i, R, R+1)).
\end{aligned}$$

It follows that $\limsup_{i\to\infty} \mathrm{cap}_{N_i}(\overline{B}(p_i, r)) = 0$, completing the proof. □

We conclude this section by proving the other main result, Theorem 18.

Proof of Theorem 18. Recall that by definition each T_i is a locally integral current defined on $(\overline{X}_i, \overline{d}_i)$. So, we can apply Theorem 1.1 in [24] to the currents T_i and points $p_i \in X_i$. The hypothesis

$$\sup_{i\in\mathbb{N}} \Big(\|T_i\|(B(p_i, r)) + \|\partial T_i\|(B(p_i, r)) \Big) < \infty$$

for each $r > 0$ holds by (9) and the hypothesis of pointed $\mathcal{VF}$ convergence. Thus, by Theorem 1.1 in [24], there exist a subsequence of N_i (note: in this proof we will not relabel subsequences), a complete metric space (Z, d_Z), a point $z \in Z$, and isometric embeddings $\varphi_i : \overline{X}_i \to Z$ such that $\varphi_i(p_i) \to z$ in Z and $\varphi_{i\#}(T_i) \to T'$ in the local flat topology, for some locally integral current T' on Z of dimension m. We point out that Z can be taken to be a w^*-separable Banach space: first, recall that integral current spaces are separable [30, Remark 2.36], and the same goes for local integral current spaces. The construction of Z in [24] comes directly from Proposition 5.2 in [33]. There, Z is constructed as the completion of a countable union of the X_i and is therefore separable. Thus, we can apply Kuratowski's embedding theorem and by replacing Z with $\ell^\infty(Z)$ we may assume that Z is a w^*-separable Banach space.

Let $N' = (\overline{\mathrm{set}(T'), d_Z, T' \llcorner \mathrm{set}(T')})$. If we show that $N_i \to N'$ in the pointed $\mathcal{VF}$ sense with respect to $p_i \in X_i$ and $z \in \overline{\mathrm{set}(T')}$, then by uniqueness of pointed $\mathcal{F}$ limits (Proposition 14), we would get that $N \cong N'$. Once this is done, the result would follow by applying Theorem 19.

Let

$$G = \{r > 0 \ : \ N_i \llcorner B(p_i, r) \xrightarrow{\mathcal{F}} N \llcorner B(p, r) \text{ with } p_i \to p \text{ as } i \to \infty\}.$$

By Definition 13, G is unbounded. Using the slicing argument in [28, Lemma 4.1], we pass to a subsequence so that $\mathbb{R}^+ \setminus G$ has measure zero. Applying a similar argument in Z, we may pass to a further subsequence and replace G with a subset, still with $\mathbb{R}^+ \setminus G$ having measure zero, such that

$$\varphi_{i\#}(T_i \llcorner B(p_i, r)) \to T' \llcorner B(z, r) \tag{31}$$

as integral currents in the flat sense in Z for all $r \in G$.

We now verify that $z \in \overline{\mathrm{set}(T')}$ using Theorem 2.9 in [15] as follows. Let $r_1 < r_2$ belong to G. Take $N_i \llcorner B(p_i, r_2)$ as "M_i" and $B(p_i, r_2)$ as "V_i" in the theorem. Then $M_i \llcorner V_i$ to $N \llcorner B(p, r_2)$ with $p_i \to p$. Thus, condition (1) of that theorem is satisfied, with $N \llcorner B(p, r_2)$ playing the role of "N_∞" and p as "x_∞." Condition (2) follows, taking $\delta = r_1$. Since we also have $M_i = N_i \llcorner B(p_i, r_2) \to N' \llcorner B(z, r_2)$ in the flat sense in Z, with $\varphi_i(p_i) \to z$, Theorem 2.9 in [15] guarantees a subsequence such that $\varphi_i(p_i)$ converges to some $x' \in \overline{\mathrm{set}(T' \llcorner B(z,r_2))}$ in Z. But we know that $\varphi_i(p_i) \to z$ in Z. Hence, $z = x'$, and $x' \in \overline{\mathrm{set}(T')}$.

Finally, we prove that $N_i \to N'$ in the pointed $\mathcal{VF}$ sense with respect to $p_i \in X_i$ and $z \in \overline{\mathrm{set}(T')}$. Let $r_0 > 0$. There exists $r \geq r_0$ with $r \in G$. Thus, from (31), $N_i \llcorner B(p_i, r) \xrightarrow{\mathcal{F}} N' \llcorner B(z, r)$. Since $\varphi_i(p_i) \to z$, we have shown the claim.

Putting this all together, we obtain the conclusion of the theorem. □

4. Examples

In this section we give examples to demonstrate that a) the capacity upper semicontinuity can be strict, i.e., the capacity can jump up in a limit (Examples 1–3), and b) *volume-preserving* convergence is necessary to guarantee upper semicontinuity (Example 4). We find there are essentially two independent reasons for the upper semicontinuity phenomenon. First, even under smooth local convergence the capacity of a set can jump up due to non-uniform control at infinity, e.g. a change in the end geometry of the manifold. Second, under $\mathcal{VF}$-convergence the capacity can also jump up, even with uniform control on the geometry at infinity.

Example 1: transition from cylindrical to Euclidean end geometry. Consider rotationally symmetric smooth Riemannian metrics on $\mathbb{R}^n$, $n \geq 2$ of the form

$$g_i = ds^2 + f_i(s)^2 d\sigma^2,$$

where each $f_i : [0, \infty) \to \mathbb{R}$, $i = 1, 2, \dots$ is smooth, with $f_i(0) = 0$, $f_i(s) > 0$ for $s > 0$, and $d\sigma^2$ is the standard metric on the unit $(n-1)$-sphere. If we assume

$$f_i(s) = \begin{cases} s, & 0 \leq s \leq i \\ i+1, & s \geq i+1 \end{cases},$$

then the corresponding Riemannian manifold $(\mathbb{R}^n, g_i)$ is isometric to a Euclidean ball for $s \leq i$ and to a cylinder (sphere-line product) of radius $i+1$ for $s \geq i+1$. The capacity of every compact set in $(\mathbb{R}^n, g_i)$ is zero, due to the cylindrical end (explained below). However, this sequence of Riemannian manifolds converges smoothly on compact sets, and hence in the pointed $\mathcal{VF}$ sense, to Euclidean space (where of course there exist compact sets of positive capacity). This example shows we cannot expect the capacity to behave continuously even for smooth local convergence.

To verify that the capacity vanishes identically with respect to g_i, given i, consider a radial Lipschitz function $\varphi_L(s)$ on $\mathbb{R}^n$ with

$$\varphi_L(s) = \begin{cases} 1, & s \leq L \\ 2 - \frac{s}{L}, & L < s \leq 2L \\ 0, & 2L < s \end{cases}$$

for a parameter L. Taking $L > i+1$, we have

$$\int_{\mathbb{R}^n} |\nabla \varphi_L|^2 dV_{g_i} = \omega_{n-1} \int_L^{2L} \frac{1}{L^2} ds = \frac{\omega_{n-1}}{L},$$

which can be made arbitrarily small by taking L large. Moreover, by taking L large, we can arrange $\varphi_L = 1$ on any compact set.

Example 2: formation of a new end. Let $M = \mathbb{R} \times S^2$ be equipped with a rotationally symmetric Riemannian metric

$$g = ds^2 + f(s)^2 d\sigma^2,$$

where $f > 0$ is smooth, even function. Further, assume f^{-2} is integrable on $\mathbb{R}$. Let K be the compact subset $\{0\} \times S^2$. We compute the capacity of K in (M, g) as follows.

It is elementary to verify that the function

$$\psi(s) = \begin{cases} \int_0^s f(r)^{-2} dr & s > 0 \\ \int_s^0 f(r)^{-2} dr & s < 0 \end{cases}$$

is g-harmonic on $M \backslash K$, equalling zero on K and approaching a positive constant $C = \int_0^\infty f(r)^{-2} dr$ at $\pm\infty$. In particular, $\varphi = 1 - \frac{1}{C}\psi$ is a minimizer for the capacity of K. (Although φ is not 1 on a neighborhood of K, this discrepancy may be neglected: it is straightforward to modify φ near K so that it is 1 on a neighborhood of K and such that the Dirichlet energy changes by an arbitrary small amount.) From this, we can verify that the capacity of K in (M, g) equals $\frac{2}{C}$:

$$\begin{aligned} \operatorname{cap}(K) &= \frac{1}{4\pi} \int_M |\nabla \varphi|^2 dV \\ &= \frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{1}{C^2} f(s)^{-4} (4\pi f(s)^2) ds \\ &= \frac{2}{C}. \end{aligned}$$

Now, consider a sequence of smooth, positive functions $f_i : [-2i, \infty) \to \mathbb{R}$ such that $f_i(s) = f(s)$ for $s \geq -i$, and such that $g_i = ds^2 + f(s)^2 d\sigma^2$ is a smooth Riemannian metric with a pole at $s = -2i$, i.e. $f_i(-2i) = 0$, so the underlying manifold is diffeomorphic to $\mathbb{R}^3$.

Then for every i, the capacity of K with respect to g_i equals $1/C$, i.e. is half the capacity of K in (M, g). This can be seen by observing the capacity of K in (M, g_i) is achieved by the function that is 1 for $-2i \leq s \leq 0$ and otherwise agreeing with φ above.

But the g_i converge smoothly on compact sets to g, and the set K has capacity $2/C$ in the limit space.

Example 3: capacity jump with $\mathcal{VF}$-convergence. Let Y be Euclidean 3-space, and let X be the $z = 0$ subspace. Then X naturally becomes a local integral current space N of dimension 2 with the Euclidean metric, where the locally integral current is given by integration, oriented up. Obviously X is isometrically embedded in Y.

Let $K = \{(x, y, 0) \mid x^2 + y^2 \leq 1\}$. For each $i = 1, 2, \ldots$, define

$$X_i = K \cup \{(x, y, 1/i) \ : \ x^2 + y^2 \geq 1\}.$$

Letting X_i have the induced Euclidean metric, X_i is obviously isometrically embedded in Y. X_i may also be equipped with the locally integral 2-current given by integration, oriented up, producing a local integral current space, N_i. See Figure 3.

Observe that N_i converges to N in the pointed $\mathcal{VF}$-sense as $i \to \infty$, where all the points are chosen to be the origin in Y. This can easily be seen from the fact that $X_i \to X$ in the usual local flat sense in Y.

Let u be the defining function for K in X given as the signed distance in X to ∂K, negative inside of K, and let $U : Y \to \mathbb{R}$ be the standard Lipschitz extension. Consider $K_i = U^{-1}(-\infty, 0] \cap X_i$, a sequence of corresponding regions as in section 3. We claim $K_i = K$. If $x \in K$, then $u(x) = U(x) \leq 0$. Since $K \subset X_i$, we have $x \in K_i$. On the other hand, suppose $p \in K_i$, so $U(p) \leq 0$. Then there exists $x \in X$ such that

$$u(x) + d_Y(x, p) \leq 0.$$

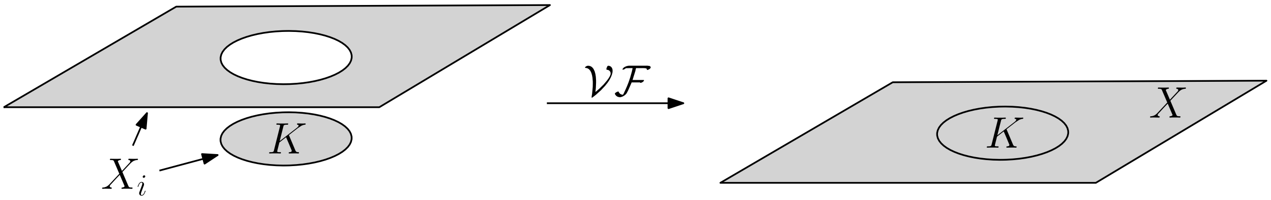

FIGURE 3. In Example 3, the space X_i is the union of a unit disk K and a plane minus a disk sitting at height $\frac{1}{i}$ above K. X_i, naturally viewed as a local integral current space, pointed $\mathcal{VF}$-converges to X with respect to some sequence of points, which is a Euclidean plane. The corresponding regions for K are simply $K_i = K$.

Clearly $u(x) \leq 0$, i.e. $x \in K$. We can see the defining function is given by $u(x) = d_Y(0, x) - 1$. With the triangle inequality, we have

$$d_Y(p, 0) \leq 1,$$

i.e., x belongs to the closed unit ball in Y about p. The latter only intersects X_i at K, so $p \in K$.

Now, the capacity of K in X is positive, but the capacity of K_i in X_i is zero for all i. This is easy to see because X_i is disconnected: the function that equals 1 on K and vanishes on $X_i \setminus K$ is Lipschitz and is a valid test function for the capacity, with zero Dirichlet energy. Thus, in (10) of Theorem 18, we have strict inequality (without needing to take a subsequence).

If desired, one can arrange a similar example with the X_i connected, as follows. Join the two connected components of X_i with a thin "strip" of area of $O(1/i^3)$ and length $O(1/i)$. Then with a Lipschitz test function f_i equalling 1 on K, with $\mathrm{Lip}(f_i)$ of $O(i)$ on the strip, and 0 elsewhere, the Dirichlet energy of f_i would be $O(1/i)$, i.e., the capacity of K in the connected space X_i would still converge to 0.

Example 4: cancellation and necessity of volume-preserving $\mathcal{F}$ convergence. Here, we demonstrate that upper semicontinuity of capacity may fail for pointed $\mathcal{F}$-convergence, without assuming $\mathcal{VF}$-convergence. We exploit the "cancellation" phenomenon of intrinsic flat convergence as in [30, Example A.19].

Let Y be Euclidean 3-space, and let X_i be the union of the $z = 0$ plane and an annulus sitting slightly above:

$$X_i = \{(x, y, z) \mid z = 0\} \cup \{(x, y, 1/i) \mid 1 \leq x^2 + y^2 \leq 4\}.$$

Equip X_i with the induced metric, so that X_i is isometrically embedded in Y. Let T_i be the locally integral current on X_i given by integration, oriented up on the $z = 0$ plane and down on the annulus. X_i with the induced metric, equipped with T_i, produces a sequence of local integral current spaces, N_i. Letting K be the unit disk

$$\{(x, y, 0) \mid x^2 + y^2 \leq 1\},$$

we have $K \subset X_i$, and the capacity of K in X_i is a positive constant independent of i.

Now, N_i converges in the pointed $\mathcal{F}$-sense (but not $\mathcal{VF}$-sense) to

$$X = K \cup \{(x, y, 0) \mid x^2 + y^2 \geq 4\},$$

with the induced metric and integral current given by integration, oriented up. See Figure 4. Here, all the base points are chosen to be the origin. Since K is a compact component of X, we have $\mathrm{cap}_N(K) = 0$. Using $r = 1$, we have a violation of Theorem 17 if $\mathcal{VF}$-convergence is not assumed.

5. Asymptotically flat local integral current spaces and general relativistic mass

Asymptotically flat (AF) Riemannian manifolds are of particular interest in the study of general relativity. These spaces are characterized by their metric tensors (and derivatives) decaying in a precise sense to the Euclidean metric in some appropriate coordinate chart that covers all but a compact set. The ADM mass is a numerical geometric invariant of an AF manifold that is of both significant physical and

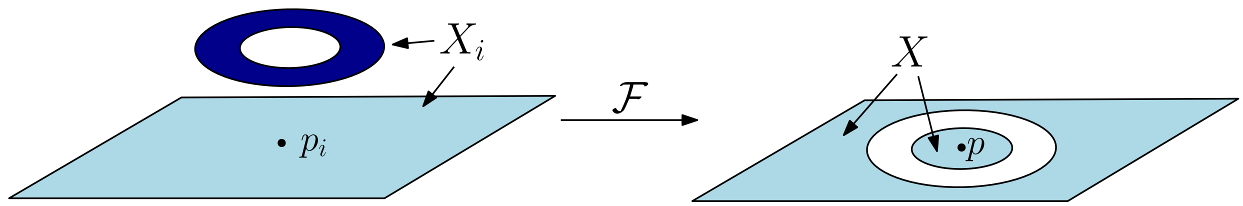

FIGURE 4. In Example 4, the space X_i is the union of a plane with an oppositely-oriented annulus sitting above at height $\frac{1}{i}$. X_i, naturally viewed as a local integral current space, converges in the pointed $\mathcal{F}$-sense (but not $\mathcal{VF}$) to X which is a Euclidean plane minus an annulus representing where the cancellation occurred.

geometric interest [4]. As described in the introduction, a number of open problems seem to necessitate an understanding of asymptotic flatness and ADM mass for spaces that are neither smooth nor Riemannian (again, we refer the reader to [29] and [21], for example).

In this section, we give a possible definition of asymptotic flatness for local integral current spaces and describe two possible definitions of general relativistic mass for such spaces.

We begin with a generalization of asymptotic flatness to metric spaces:

Definition 23. We define a metric space (X, d) to be *asymptotically flat of dimension* $n \geq 3$ if for any $\epsilon > 0$, there exists a compact set $K \subset X$ and a bijective map Φ from $X \setminus K$ to $\mathbb{R}^n \setminus B$ (for a closed ball $B \subset \mathbb{R}^n$) that is bi-Lipschitz when $X \setminus K$ and $\mathbb{R}^n \setminus B$ are endowed with the restricted distance of d and of the Euclidean distance function, respectively, such that

$$\mathrm{Lip}(\Phi), \mathrm{Lip}(\Phi^{-1}) \leq 1 + \epsilon.$$

It is possible to show that any AF Riemannian manifold of dimension n (in the usual sense) is an AF metric space of dimension n with its natural distance function.

Similarly, we define a metric measure space (X, d, μ) to be AF of dimension $n \geq 3$ if the above properties hold for (X, d) and also if

$$(1 + \epsilon)^{-n} \mathcal{L}^n \leq \Phi_{\#}(\mu) \leq (1 + \epsilon)^n \mathcal{L}^n$$

as Borel measures on $\mathbb{R}^n \setminus B$, where $\mathcal{L}^n$ is the Lebesgue measure. For example, if (X, d) is an AF metric space of dimension n, then equipped with Hausdorff n-measure, it becomes an AF metric measure space.

Now we can define a local integral current space (X, d, T) of dimension n to be AF if $(X, d, \|T\|)$ is an asymptotically flat metric measure space of dimension n. (We note other reasonable definitions are possible.) In this setting, the capacity of compact sets is well defined, as is the boundary mass of balls for almost all radii [28, Lemma 2.34].

We now proceed to discuss the concept of general relativistic mass for asymptotically flat local integral current spaces (not to be confused with the mass measure). The standard definition of ADM mass involves derivatives of the Riemannian metric coefficients and so is unsuitable for metric spaces.

A well-known approach to a "weak" understanding of ADM mass is due to Huisken [16, 17]: his so-called *isoperimetric mass* uses only volumes and areas (perimeters) in its formulation. In dimension three, with nonnegative scalar curvature, it is known to equal the ADM mass in the smooth asymptotically flat case [6, 16, 17, 20]. In [21], Jauregui and Lee gave a definition of asymptotically flat local integral current space (more restrictive than that which we use here, essentially requiring the complement of a compact set to be a smooth manifold with a C^0 Riemannian metric), and used Huisken's isoperimetric mass as a substitute for ADM mass. Since the perimeters of compact sets are well defined even for C^0 Riemannian metrics, it was clear that Huisken's definition was well defined.

Huisken's isoperimetric mass, m_{iso}, is typically defined for asymptotically flat Riemannian 3-manifolds. We can generalize this concept to any 3-dimensional asymptotically flat local integral current space, using boundary mass in place of perimeter:

$$m_{iso}(X,d,T) = \sup_{\{K_j\}} \limsup_{j\to\infty} \frac{2}{\mathbb{M}(\partial(T\llcorner K_j))}\left[\|T\|(K_j) - \frac{1}{6\sqrt{\pi}}\mathbb{M}(\partial(T\llcorner K_j))^{\frac{3}{2}}\right] \qquad \in [-\infty,\infty],$$

where $\{K_j\}$ is an exhaustion of X by compact sets. Note that if $\mathbb{M}(\partial(T\llcorner K_j)) = \infty$, the expression inside the $\limsup$ is $-\infty$. In particular, we may restrict to exhaustions such that $\mathbb{M}(\partial(T\llcorner K_j))$ is finite, which is equivalent to saying $T\llcorner K_j$ is an integral n-current on (X,d).

Huisken's definition was inspired by the isoperimetric inequality: far out in the AF end, the inequality almost holds, and the ADM mass can be detected through the deficit. Jauregui proposed a corresponding definition of mass based on the isocapacitary inequality (that the capacity of a compact set of a given volume in $\mathbb{R}^n$ is minimized by balls) [18]. This definition of "capacity-volume mass" was for AF manifolds, including C^0 AF manifolds. However, it can be generalized to AF local integral current spaces of dimension 3 as follows:

$$m_{CV}(X,d,T) = \sup_{\{K_j\}} \limsup_{j\to\infty} \frac{1}{4\pi\,\mathrm{cap}(K_j)^2}\left[\|T\|(K_j) - \frac{4\pi}{3}\,\mathrm{cap}(K_j)^3\right],$$

where the capacity is defined as in (3). In [18], strong evidence was given for m_{CV} recovering the ADM mass in the smooth case with nonnegative scalar curvature (and hence serving as a weak stand-in for the ADM mass). Furthermore, it was observed that capacity is in some ways better behaved than perimeter or boundary mass — for example, capacity is less sensitive to perturbations, and as confirmed by our main theorems and discussed below, has a favorable semicontinuity property — so m_{CV} may ultimately be easier to work with in low-regularity ADM mass problems than m_{iso}.

To connect this discussion of mass with our main theorems, we conclude with a discussion of the lower semicontinuity of total mass in general relativity. In [19, 20] it was shown that the ADM mass functional (and more generally, Huisken's isoperimetric mass) is lower semicontinuous on an appropriate class of asymptotically flat 3-manifolds of nonnegative scalar curvature, for pointed C^2, and more generally, for pointed C^0 Cheeger–Gromov convergence. This was further generalized to pointed $\mathcal{VF}$ convergence, under natural hypotheses, using Huisken's isoperimetric mass as a stand-in for the mass of the (potentially non-smooth) limit space [21]. Below, we argue that Theorem 18 supports lower semicontinuity of m_{CV} in dimension three.

To simplify the discussion, we recall from the appendix of [18] that m_{CV} may alternatively be written:

$$m_{CV}(X,d,T) = \sup_{\{K_j\}} \limsup_{j\to\infty} \left[\left(\frac{\|T\|(K_j)}{4\pi}\right)^{1/3} - \mathrm{cap}(K_j)\right].$$

Now consider the inner expression $\left(\frac{\|T\|(K)}{4\pi}\right)^{1/3} - \mathrm{cap}(K)$ as a functional on compact sets K. To have any hope of showing m_{CV} is lower semicontinuous under pointed $\mathcal{VF}$ convergence, it seems necessary to know that "volume radius minus capacity" itself is lower semicontinuous. Since volume is by definition continuous in $\mathcal{VF}$, this amounts to the statement that capacity is *upper* semicontinuous. We demonstrated this in Theorem 18 for example. In other words, the results of this paper are supportive of m_{CV} itself being lower semicontinuous under pointed $\mathcal{VF}$-convergence, though a full proof of this is more subtle, requiring, for example, an unproven analog of the ADM mass estimate [20, Theorem 17], but for the capacity-volume mass in place of the isoperimetric mass.

Appendix: tangential differential, Dirichlet energy, Sobolev spaces, and capacity

In this section we first review the definition of Dirichlet energy for a Lipschitz function defined on the canonical set of a current as was done by Portegies [26], which we use in the definition of capacity. This will require the concepts of metric and w^*-differentials, approximate tangent spaces, and the tangential differential. After that, we relate the latter to the minimal relaxed gradient and briefly discuss several notions of Sobolev spaces on metric spaces. We conclude with a comparison of the definition of capacity we employ in this paper and other definitions appearing in the literature.

A *w^*-separable Banach space* Z is by definition a dual space $Z = G^*$, and hence Banach, of a separable Banach space G. The function $d_w : Z \times Z \to \mathbb{R}$ given by

$$d_w(x,y) := \sum_{j=0}^{\infty} 2^{-j} |\langle x - y, g_j \rangle|, \quad \text{for } x, y \in Z,$$

where $\{g_j\}_{j=0}^{\infty}$ is a countable dense subset in the unit ball in G, is a distance; d_w induces the w^*-topology on bounded subsets of Z, and (Z, d_w) is a separable space. One of the main examples of a w^*-separable Banach space is the space $\ell^\infty = (\ell^1)^*$.

Definition 24 ([2, Definitions 3.1 and 3.4])**.** Let Z be a metric space and $g : \mathbb{R}^n \to Z$ a function.

- We say that g is *metrically differentiable* at $x \in \mathbb{R}^n$ if there is a seminorm $md_x g : \mathbb{R}^n \to \mathbb{R}$ such that
$$d(g(y), g(x)) - md_x g(y - x) = o(|y - x|), \quad y \to x.$$
We call $md_x g$ the *metric differential* of g at x.
- If Z is a w^*-separable Banach space, we say that g is *w^*-differentiable* at $x \in \mathbb{R}^n$ if there is a linear map $wd_x g : \mathbb{R}^n \to Z$ such that
$$\lim_{y \to x} \frac{g(y) - g(x) - wd_x g(y - x)}{|y - x|} = 0,$$
where the limit is understood in the w^*-sense. The map $wd_x g$ is called the *w^*-differential* of g at x.

For Lipschitz maps the following is known.

Theorem 25 ([2, Theorems 3.2 and 3.5])**.** *If Z is a metric space, then any Lipschitz function $g : \mathbb{R}^n \to Z$ is metrically differentiable $\mathcal{L}^n$-a.e. If additionally, Z is a w^*-separable Banach space, then g is also w^*-differentiable $\mathcal{L}^n$-a.e, and the metric and weak differential satisfy*

$$md_x g(v) = \|wd_x g(v)\|, \qquad \textit{for all } v \in \mathbb{R}^n \textit{ and } \mathcal{L}^n\textit{-a.e.}\, x \in \mathbb{R}^n.$$

A subset S of a metric space Z is *countably $\mathcal{H}^n$-rectifiable* if there exist Lipschitz functions $g_j : A_j \subset \mathbb{R}^n \to Z$, $j \in \mathbb{N}$, defined on Borel sets A_j such that

$$\mathcal{H}^n \left(S \setminus \bigcup_{j=1}^{\infty} g_j(A_j) \right) = 0.$$

If Z is a w^*-separable Banach space, the *approximate tangent space* to S at a point x is defined as

$$\mathrm{Tan}^{(\mathrm{n})}(S, x) = wd_y g_j(\mathbb{R}^n),$$

whenever $y = g_j^{-1}(x)$ and g_j is metrically and w^*-differentiable at y, with $J_n(wd_y g_j) > 0$, where for any linear function $L : V \to W$ between two Banach spaces, with $n = \dim V$,

$$J_n(L) = \frac{\omega_n}{\mathcal{H}^n\{v \in V \,:\, ||L(v)|| \leq 1\}}$$

denotes the n-Jacobian of L. By [2], $\mathrm{Tan}^{(\mathrm{n})}(S, x)$ is well defined for $\mathcal{H}^n$-almost all $x \in S$. A finite Borel measure μ is called *n-rectifiable* if $\mu = \theta \mathcal{H}^n \llcorner S$ for a countably $\mathcal{H}^n$-rectifiable set S and a Borel function $\theta : S \to (0, \infty)$.

The next theorem shows the existence of tangential differentials of Lipschitz functions on rectifiable sets.

Theorem 26 ([2, Theorem 8.1])**.** *Let Z and Z' be two w^*-separable Banach spaces, $S \subset Z$ an $\mathcal{H}^n$-countably rectifiable subset and $f : Z \to Z'$ a Lipschitz function. Let $\theta : S \to (0,\infty)$ be an $\mathcal{H}^n$-integrable function and denote by $\mu = \theta \mathcal{H}^n \llcorner S$ the corresponding n-rectifiable measure.*

Then for $\mathcal{H}^n$-almost every $x \in S$, there exist a Borel set $S^x \subset S$ such that the upper n-dimensional density of $\mu \llcorner S^x$ equals zero,

$$\Theta^*_n(\mu \llcorner S^x, x) = 0,$$

and a linear and w^*-continuous map $L : Z \to Z'$ so that

$$\lim_{y \in S \setminus S^x \to x} \frac{d_w(f(y), f(x) + L(y-x))}{|y-x|} = 0.$$

$\mathrm{Tan}^{(\mathrm{n})}(S,x)$ *exists and L is uniquely determined on* $\mathrm{Tan}^{(\mathrm{n})}(S,x)$ *and its restriction to* $\mathrm{Tan}^{(\mathrm{n})}(S,x)$ *is called the tangential differential to S at x and is denoted by*

$$d^S_x f : \mathrm{Tan}^{(\mathrm{n})}(S,x) \to Z'.$$

Furthermore, the tangential differential is characterized by the property that for any Lipschitz map $g : A \subset \mathbb{R}^n \to S$,

$$wd_y(f \circ g) = d^S_{g(y)} f \circ wd_y g, \qquad \text{for } \mathcal{L}^n\text{-a.e. } y \in A.$$

Note that if $d^S_x f$ is defined, then its dual norm satisfies $|d^S_x f| \le \mathrm{Lip}(f)$.

If S is an arbitrary separable, countably $\mathcal{H}^n$-rectifiable metric space, we isometrically embed S into a w^*-separable Banach space Z, $\iota : S \to Z$. Then for $\mathcal{H}^n$-almost every $x \in S$ we can define the *approximate tangent space* of S at x as

$$\mathrm{Tan}^{(\mathrm{n})}(S,x) := \mathrm{Tan}^{(\mathrm{n})}(\iota(S), \iota(x)).$$

Even if we have chosen a particular isometric embedding, $\mathrm{Tan}^{(\mathrm{n})}(S,x)$ is uniquely determined $\mathcal{H}^n$-a.e. up to linear isometries [2]. If additionally θ and μ are as in the previous theorem, then we define

$$|d^S_x f| = |d^{\iota(S)}_{\iota(x)}(f \circ \iota^{-1})|,$$

for μ-a.e. $x \in S$ and where the right hand side denotes the dual norm of $d^{\iota(S)}_{\iota(x)}(f \circ \iota^{-1})$. This quantity is also well defined, independent of the isometric embedding.

Definition 27 (Definition 3.8 of [26])**.** Let X be a complete metric space, and let $T \in \mathbf{I}_n(X)$. Let $S = \mathrm{set}(T)$, and let $f : S \to \mathbb{R}$ be a Lipschitz function. Then the *(Dirichlet) energy* of f is given by

$$E_T(f) := \int_X |d^S_x f|^2 \, d\|T\|(x).$$

The energy of f is invariant under isometric embeddings, and for any compact oriented Riemannian manifold (M,g) we have that the energy is given by $\int_M |\nabla f|^2 dV$, where the gradient and volume measure are taken with respect to g.

We next mention the relationship between $|d^S_x f|$ and the minimal relaxed gradient.

Let X be a complete metric space, $T \in \mathbf{I}_n(X)$ and $S = \mathrm{set}(T)$. The space $L^2(\|T\|)$ is the Hilbert space of equivalence classes of functions on X that are square-integrable with respect to $\|T\|$ with inner product

$$\langle f, g \rangle_{L^2(\|T\|)} := \int_X f g \, d\|T\|.$$

The space $W^{1,2}(\|T\|)$ is the completion of the set of bounded Lipschitz functions on $\mathrm{spt}\, T$ with respect to the norm $\|.\|_{W^{1,2}}$ given by

$$\|f\|^2_{W^{1,2}} = \int_X f^2 d\|T\| + \int_X |d^S_x f|^2 d\|T\|(x).$$

By definition, every f in $W^{1,2}(\|T\|)$ can be represented by a Cauchy sequence f_i of bounded Lipschitz functions. The limit of $d_x^S f_i$ in $\mathcal{T}_2^*(\|T\|)$,where S denotes set(T), the Banach space of equivalence classes of covector fields endowed with the norm

$$\|\psi\|^2_{\mathcal{T}_2^*(\|T\|)} := \int_X |\psi(x)|^2_{[\mathrm{Tan}^{(n)}(S,x)]^*} d\|T\|(x),$$

is denoted by $d_x f$.

Theorem 28 (Theorem 5.2 [26])**.** *Let X be a complete metric space, $T \in \mathbf{I}_n(X)$ and $f \in L^2(\|T\|)$. Then f has a relaxed gradient in the sense of* [3, Definition 4.2] *if and only if $f \in W^{1,2}(||T||)$. Moreover, the minimal relaxed gradient equals $|d_x f|$ for $||T||$-a.e. $x \in X$.*

We conclude with a discussion of Sobolev spaces and capacity. In [5,14] metric measure spaces (X, d, m), where (X, d) is separable and m is a locally finite Borel regular measure on X, are considered and Newtonian spaces of functions $N^{1,p}(X, d, m)$, $1 \leq p < \infty$, are defined. Originally defined by Shanmugalingam in her PhD thesis and subsequent paper [27], these are a type of Sobolev space. Then the *p-capacity* of a set $E \subset X$ (what was called the Sobolev p-capacity in the introduction) is defined as

$$\begin{aligned}\mathrm{cap}_p(E) = \inf \Big\{ \int |u|^p dm + \int \rho_u^p dm \\ : u \in N^{1,p}(X, d, m), u \geq 1 \text{ on } E \text{ outside a } p\text{-exceptional set of measure zero} \Big\},\end{aligned}$$

where ρ_u denotes the minimal p-weak upper gradient of u. It is also shown that this is equivalent to [14, Lemma 7.2.6]

$$\mathrm{cap}_p(E) = \inf \left\{ \int |u|^p dm + \int \rho_u^p dm \; : \; u \in N^{1,p}(X, d, m), \; 0 \leq u \leq 1 \text{ and } u = 1 \text{ on } E \right\}.$$

Other types of Sobolev spaces on metric spaces have been defined, see Theorem 10.5.1—10.5.3 in [14]. For $1 < p < \infty$, the Cheeger space, $W^{1,p}_{Ch}$, and Newtonian space, $N^{1,p}$, are equal (up to representatives) and both norms coincide. Provided m is a doubling measure and X satisfies a q-Poincaré inequality, $1 \leq q < 2$, several Sobolev spaces coincide

$$M^{1,2} = P^{1,2} = KS^{1,2} = N^{1,2} = W^{1,2}_{Ch},$$

though some norms are only comparable. Here $M^{1,2}$ is the Hajłasz Sobolev space [13], $P^{1,2}$ is the Poincaré Sobolev space, and $KS^{1,2}$ is the Korevaar–Schoen Sobolev space. If m is only a doubling measure then $M^{1,2} \subseteq P^{1,2} \subseteq KS^{1,2} \subseteq N^{1,2} = W^{1,2}_{Ch}$.

For complete and separable metric measure spaces (X, d, m), in [3, Theorem 6.2] (cf. [11, Theorem 2.2.28]) it was shown that $W^{1,2}_{Ch}$ and the Sobolev space $W^{1,2}(X, d, m)$ using the minimal relaxed gradient in the sense of [3, Definition 4.2] (and their norms) are the same. Furthermore, any $f \in W^{1,2}(X, d, m)$ can be approximated by functions in $\mathrm{Lip}(X) \cap L^2(m)$.

The main difference between the capacity we use in this paper and the definition given in [14], is that in our definition of capacity we only integrate the gradient term. However, it is possible to bound these capacities in terms of each other. For some constant C, we immediately have:

$$C||T||(K) + \mathrm{cap}(K) \leq \mathrm{cap}_2(K).$$

On the other hand, if the space admits a Poincaré inequality, one can obtain $\mathrm{cap}_2(K) \leq C' \,\mathrm{cap}(K)$ for a constant C' (see [5, Theorem 6.16]).

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